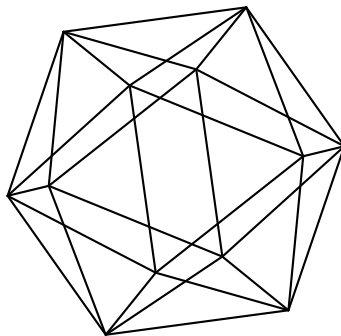# Max-Planck-Institut für Mathematik Bonn

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by

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# HARISH-CHANDRA MODULES OVER INVARIANT SUBALGEBRAS IN A SKEW-GROUP RING

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ABSTRACT. We construct a new class of algebras resembling enveloping algebras and generalizing orthogonal Gelfand-Zeitlin algebras and rational Galois algebras studied by [EMV, FGRZ, RZ, Har]. The algebras are defined via a geometric realization in terms of sheaves of functions invariant under an action of a finite group. A natural class of modules over these algebra can be constructed via a similar geometric realization. In the special case of a local reflection group, these modules are shown to have an explicit basis, generalizing similar results for orthogonal Gelfand-Zeitlin algebras from [EMV] and for rational Galois algebras from [FGRZ]. We also construct a family of canonical simple Harish-Chandra modules and give sufficient conditions for simplicity of some modules.

## 1. INTRODUCTION

In the last decade there was a significant progress in understanding infinite dimensional simple modules over the Lie algebra $\mathfrak{gl}_n$, see e.g. [FGR, Ni1, Ni2, EMV] and references therein. An essential part of this progress is related to the study of so-called *Gelfand-Zeitlin modules* which originate from [DOF] based on [GZ] (see [EMV] for a detailed literature overview on Gelfand-Zeitlin modules). Various approaches to the study of Gelfand-Zeitlin modules rely on different realizations of the universal enveloping algebras which led to a number of generalizations of such algebras. These include *orthogonal Gelfand-Zeitlin algebras* introduced in [Ma] and *Galois algebras* introduced in [FO]. These generalizations include also finite W-algebras of type $A$, see [Ar, Har], and were studied in, in particular, [EMV, Har, FGRZ, RZ]. The recent preprint [KTWWY] establishes a relation between orthogonal Gelfand-Zeitlin algebras and Khovanov-Lauda-Rouquier algebras from [KL, Ro] and, in particular, leads to a (not very explicit) classification of simple Gelfand-Zeitlin modules over orthogonal Gelfand-Zeitlin algebras.

In the present paper we define and study a simultaneous geometric generalization of orthogonal Gelfand-Zeitlin algebras and Galois algebras. Both our construction and methods of study are inspired by the geometric approach of [Vi1, Vi2] to singular Gelfand-Zeitlin modules and is formulated in elementary sheaf-theoretic terms. To any semidirect product $G \ltimes V$ of a finite group $G$ and a complex-analytic or linear algebraic group $V$, we associate the corresponding skew-group ring $\mathcal{S}$. We denote by $\mathcal{O}$ the sheaf of holomorphic or polynomial functions on $V$. There is a natural action of $G$ on $\mathcal{O}$ and it is natural to consider the sheaf $\mathcal{O}^G$ of $G$-invariants in $\mathcal{O}$. Main protagonists of the present paper are subalgebras in $\mathcal{S}$ that preserve the sheaf $\mathcal{O}^G$. Orthogonal Gelfand-Zeitlin algebras, Galois algebras, finite W-algebras and Galois orders, studied in [EMV, FGRZ, RZ, Har], are all special cases of our construction. In a special case which we call *standard algebras of type A*, we give an explicit description of our algebras as subalgebras in the *universal ring* as introduced in [Vi2]. Our geometric approach also naturally provides a construction of a large

family of (simple) modules over our algebras, generalizing [Vi1, Vi2, EMV]. We note that, the general case of our construction seems to be outside the scope of *Harish-Chandra subalgebras and Gelfand-Zeitlin modules* as defined in [DFO]. However, it still fits into the general Harish-Chandra setup which studies modules over some algebra on which a certain subalgebra acts locally finite. In particular, our results significantly generalize and simplify many results from [FGRZ].

The paper is organized as follows: Section 2 contains a description of our setup and preliminaries. Section 3 defines and provides basic structure results for our algebras. Sections 4, 5 and 7 study in detail the spacial case of rational Galois orders. Sections 6 describes applications of our approach to the study of Gelfand-Zeitlin modules. Finally, in Sections 8 we construct canonical simple Harish-Chandra modules over our algebras and give a sufficient condition for simplicity of these modules.

## 2. Preliminaries

2.1. **Skew-group ring.** Throughout the paper we work over the field $\mathbb{C}$ of complex numbers. Let $G$ and $V$ be two complex-analytic or linear algebraic Lie groups such that $G$ acts on $V$. Let $G \ltimes V$ be the corresponding semidirect product. To simplify notations we will write $G$ and $V$ for subgroups $G \times \{e\}$ and $\{e\} \times V$ in $G \ltimes V$, respectively. On $V$ we have a free transitive action of $V$ by left translations $\phi_\xi$, where $\xi \in V$, and an action of $G$ given by $v \mapsto g \cdot v = gvg^{-1}$. Both actions are assumed to be holomorphic or algebraic. Note that $e \in V$ is a fixed point of the action of $G$. We denote by $\beth$ a fixed subgroup in $V$.

Denote by $\mathbb{C}\beth$ the group algebra of $\beth$. Consider the vector space of global meromorphic (or rational) sections of the trivial vector bundle $V \times \mathbb{C}\beth \to V$. We will denote this vector space by $\mathcal{S}(\beth)$ or simply by $\mathcal{S}$, if $\beth$ is clear from the context. We assume that any section of $\mathcal{S}(\beth)$ has the form $f = \sum\limits_{i=1}^{s} f_i \phi_{\xi_i}$, where $f_i$ are meromorphic (or rational) functions on $V$, $\xi_i \in \beth$ and $s < \infty$. The vector space $\mathcal{S}(\beth)$ has the natural structure of a skew-group ring defined in the following way:

$$\sum_i f_i \phi_{\xi_i} \circ \sum_j f'_j \phi_{\xi'_j} = \sum_{i,j} f_i \phi_{\xi_i}(f'_j) \phi_{\xi_i \circ \xi'_j}.$$

Here, by definition, $\phi_{\xi_i}(f'_j)(x) := f'_j(\xi_i^{-1}(x))$ for any $x \in V$. Clearly, for any subgroup $\beth' \in \beth$ the ring $\mathcal{S}(\beth')$ can be viewed as a subring in $\mathcal{S}(\beth)$ in the obvious way.

The action of $G$ on $V$ induces an action of $G$ on $\mathcal{S}(V)$ and also an action of $G$ on $S(\beth)$ provided that $\beth$ is $G$-invariant. More precisely, $g \cdot f\phi_\xi = (g \cdot f)\phi_{g\xi g^{-1}}$ and $g \cdot f$ is a function on $V$ defined as follows $g \cdot f(v) = f(g^{-1} \cdot v)$ for $v \in V$. Let $\beth$ be $G$-invariant. Then we have the subring $\mathcal{S}(\beth)^G$ of $G$-invariant sections of $\mathcal{S}(\beth)$. Denote by $\mathcal{M}$ and by $\mathcal{O}$ the sheaves of meromorphic (or rational) and holomorphic (or polynomial) functions on $V$, respectively. For any $v \in V$, we denote by $\mathcal{M}_v$ and

$\mathcal{O}_v$ the corresponding algebras of germs of meromorphic and holomorphic functions at $v$. We put

$$\mathfrak{M} := \bigoplus_{x \in V} \mathcal{M}_x, \quad \mathfrak{O} := \bigoplus_{x \in V} \mathcal{O}_x.$$

If $W \subset V$ is a subset, we set $\mathfrak{M}|_W := \bigoplus_{x \in W} \mathcal{M}_x$ and $\mathfrak{O}|_W := \bigoplus_{x \in W} \mathcal{O}_x$.

The ring $\mathcal{S}(V)$ acts on the vector space $\mathfrak{M}$ in the following way:

$$f\phi_\xi : \mathcal{M}_v \to \mathcal{M}_{\xi(v)}, \quad F_v \mapsto (f\phi_\xi(F_v))_{\xi(v)}.$$

Consequently, the ring $\mathcal{S}(\beth)$ acts on the vector space $\mathfrak{M}|_{\beth \cdot v}$, where $v \in V$ and $\beth \cdot v$ is the $\beth$-orbit of $v$. Note that, in general, we do not have any action of $\mathcal{S}$ on $\mathfrak{O}$, since sections of $\mathcal{S}$ are assumed to be meromorphic (resp. rational) and not holomorphic (resp. polynomial).

In case we need to distinguish complex-analytic and algebraic categories, we will use the subscripts $\mathsf{C}$ and $\mathsf{A}$, respectively. For example, we will write $\mathcal{S}_\mathsf{C}$ and $\mathcal{O}_\mathsf{C}$ to specify that we are working with meromorphic sections of our skew-ring and with holomorphic functions on $V$.

## 2.2. Example: the classical Gelfand-Zeitlin operators.

For $n \geq 2$, denote by $V$ the vector space

$$V = \mathbb{C}^{n(n+1)/2} = \{(v_{ki}) \mid 1 \leq i \leq k \leq n\}.$$

An element of $V$ is called a *Gelfand-Zeitlin tableaux*. Let $\beth \simeq \mathbb{Z}^{n(n-1)/2}$ be the subgroup of $V$ generated by Kronecker vectors $\delta^{st} = (\delta^{st}_{ki})$, where $k$ and $i$ are as above, $1 \leq t \leq s \leq n-1$ and $\delta^{st}_{ki} = 1$, if $k = s$ and $i = t$, and $\delta^{st}_{ki} = 0$ otherwise. The product $G = S_1 \times S_2 \times \cdots \times S_n$ of symmetric groups acts on $V$ in the following way: the element $s = (s_1, \ldots, s_n) \in G$ acts on $v = (v_{ki})$ via $(s(v))_{ki} = v_{ks_k(i)}$. For $a \in \mathbb{C}$, set $\xi^a_k = a\delta^{k1}$. Consider the following elements in $\mathcal{S}(\beth)^G$:

$$E_{k,k+1} = \sum_{g \in G} g \cdot \frac{\prod_{j=1}^{k+1}(v_{k1} - v_{k+1,j})}{\prod_{j=2}^{k}(v_{k1} - v_{kj})} \phi_{\xi^1_k}; \quad E_{k+1,k} = \sum_{g \in G} g \cdot \frac{\prod_{j=1}^{k-1}(v_{k1} - v_{k-1,j})}{\prod_{j=2}^{k}(v_{k1} - v_{kj})} \phi_{\xi^1_k}^{-1};$$

$$E_{kk} = \sum_{i=1}^{k}(v_{ki} + i - 1) - \sum_{i=1}^{k-1}(v_{k-1,i} + i - 1).$$

The subalgebra $U \subset \mathcal{S}(\beth)^G$ generated by $E_{st}$ is isomorphic to universal enveloping algebra $\mathcal{U}(\mathfrak{gl}_n(\mathbb{C}))$, see e.g. [DFO, Ma] for details.

## 2.3. Orthogonal Gelfand-Zeitlin algebras.

Orthogonal Gelfand-Zeitlin algebras are generalizations of $\mathcal{U}(\mathfrak{gl}_n(\mathbb{C}))$ introduced in [Ma]. Fix a positive integer $n \geq 2$ and let $n_k$, where $k = 1, \ldots, n$, be positive integers. Denote by $V$ the following vector space

$$V = \mathbb{C}^{\sum_k n_k} = \{v = (v_{ki}) \mid 1 \leq i \leq n_k, \ 1 \leq k \leq n\}.$$

Let $\beth \simeq \mathbb{Z}^{\sum_k n_k}$ be the subgroup of $V$ generated by $\delta^{st} = (\delta^{st}_{ki})$, where $1 \leq t \leq n_k$, $1 \leq s \leq n$, as in Subsection 2.2. The group $G = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_n}$ acts on $V$ as in Subsection 2.2 which defines the ring $\mathcal{S}$ and its subring $\mathcal{S}^G$.

With $\xi_k^1$ defined as in Subsection 2.2, an *orthogonal Gelfand-Zeitlin algebra* is a subalgebra in $\mathcal{S}^G$ generated by all $G$-invariant polynomials on $V$ and by the elements

$$
E_k = \sum_{g \in G} g \cdot \frac{\prod\limits_{j=1}^{n_{k+1}} (v_{k1} - v_{k+1,j})}{\prod\limits_{j=2}^{n_k} (v_{k1} - v_{kj})} \phi_{\xi_k^1}; \quad F_k = \sum_{g \in G} g \cdot \frac{\prod\limits_{j=1}^{n_{k-1}} (v_{k1} - v_{k-1,j})}{\prod\limits_{j=2}^{n_k} (v_{k1} - v_{kj})} \phi_{\xi_k^1}^{-1}.
$$

The algebra $\mathcal{U}(\mathfrak{gl}_n(\mathbb{C}))$ from Subsection 2.2 is just a special case of this construction, for $n_k = k$.

Note that the generators $E_k$ and $F_k$ of the orthogonal Gelfand-Zeitlin algebra are rational (and not polynomial), however, it was shown in [EMV, Proposition 1] that the operators $E_k$ and $F_k$ preserve the vector space $H^0(V, \mathcal{O}^G)$. By [KTWWY], orthogonal Gelfand-Zeitlin algebras are related to shifted Yangians and generalized $W$-algebras in type $A$.

### 2.4. Standard algebras of type $\mathbb{A}$.
Let $V$ and $G$ be as in Section 2.3. An element $A$ in $\mathcal{S}^G$ is called *standard* if $A = \sum\limits_{g \in G} g \cdot (f \phi_{\xi_k^a})$, where $a \in \mathbb{C}$.

**Definition 1.** *A subalgebra $\mathcal{A} \subset \mathcal{S}(V)^G$ is called standard of type $\mathbb{A}$ if $\mathcal{A}$ is generated by linear combinations of standard elements.*

Orthogonal Gelfand-Zeitlin algebras are examples of standard algebras of type $\mathbb{A}$. Other examples of such algebras are: finite W-algebras of type $A$ and, more general, standard Galois orders of type $A$, see [FGRZ, Section 8] or [Har] for definition. In Section 5 we will show that standard algebras of type $\mathbb{A}$ that preserve the vector space $\mathfrak{O}^G$ are exactly standard Galois orders of type $A$.

### 2.5. Harish-Chandra modules.
In this paper we will study modules which fit into the general philosophy of *Harish-Chandra modules*. Let $\mathcal{A} \subset \mathcal{S}^G$ be a subalgebra containing, as a subalgebra, the algebra $\mathcal{B}$ of all global $G$-invariant functions on $V$.

**Definition 2.** *We say that an $\mathcal{A}$-module $M$ is a Harish-Chandra module provided that the action of $\mathcal{B}$ on $M$ is locally finite.*

Gelfand-Zeitlin modules for orthogonal Gelfand-Zeitlin algebras and Galois orders, studied in [EMV, FGRZ, Vi1, Vi2] are examples of Harish-Chandra modules.

### 3. Algebras preserving the vector space $\mathfrak{O}^G$ and their modules

### 3.1. A fibration corresponding to the sheaf of invariant functions.
Consider a semidirect product $G \ltimes V$, where $G$ is a finite group and $V$ is a complex-analytic or linear algebraic group. As above we denote by $\mathcal{O}$ the structure sheaf of the complex-analytic (or algebraic) variety $V$. In other words, we assume that all sections of $\mathcal{O}$ are holomorphic or polynomial functions on $V$, respectively. We now define the sheaf $\mathcal{O}^G$ of $G$-invariant holomorphic (or polynomial) functions on $V/G$. For a $G$-invariant open set $U$ in $V$, we let $\mathcal{O}^G(U/G)$ be the algebra of $G$-invariant holomorphic (or polynomial) functions on $U$. Below we will consider the algebra $\mathcal{O}_v^G$ of germs of functions at a point $v \in V$. By definition, $\mathcal{O}_v^G$ is the algebra of

germs $f \in \mathcal{O}_v$ such that there exists a $G$-invariant function $F \in \mathcal{O}_{G.v}$ that has the germ $f$ at the point $v$. We put

$$\mathfrak{M}^G := \bigoplus_{\bar{x} \in V/G} \mathcal{M}_{\bar{x}}^G, \quad \mathfrak{O}^G := \bigoplus_{\bar{x} \in V/G} \mathcal{O}_{\bar{x}}^G.$$

If $W \subset V$ is a $G$-invariant subset, we set $\mathfrak{M}^G|_W := \bigoplus_{\bar{x} \in W/G} \mathcal{M}_{\bar{x}}$ and $\mathfrak{O}^G|_W := \bigoplus_{\bar{x} \in W/G} \mathcal{O}_{\bar{x}}$.

If $v$ is a fixed point of the action of $G$, then the algebra $\mathcal{O}_v^G$ is invariant with respect to the action of $G$. The group $G$ has at least one fixed point, namely, the identity $e \in V$. Consider the algebra $\mathcal{O}_e$ of germs of functions at the point $e$ and its $G$-invariant subalgebra $\mathcal{O}_e^G \subset \mathcal{O}_e$. Denote by $\mathcal{J}_e$ the ideal in $\mathcal{O}_e$ generated by functions from $\mathcal{O}_e^G$ that are equal to $0$ at $e$. As above, we denote by $\phi_\xi : \mathcal{O}_x \to \mathcal{O}_{\xi x}$, $f \mapsto \phi_\xi(f) = f \circ \xi^{-1}$, the left translation by $\xi \in V$.

**Lemma 3.** *Let $G$ be a finite group, $\xi \in V$, $G_\xi$ the stabilizer of $\xi$ and $\xi G \xi^{-1} \subset G \ltimes V$ the group obtained from $G$ by conjugation with $\xi$. We have*

$$\phi_\xi(\mathcal{O}_e^{G_\xi}) = \mathcal{O}_\xi^G \quad and \quad \phi_\xi(\mathcal{O}_e^G) = \mathcal{O}_\xi^{\xi G \xi^{-1}}.$$

*In particular, we have*

$$\phi_\xi(\mathcal{O}_e^{G_\xi}/(\mathcal{O}_e^{G_\xi} \cap \mathcal{J}_e)) = \mathcal{O}_\xi^G/\langle \mathcal{O}_\xi^G \cap (\mathcal{O}_\xi^{\xi G \xi^{-1}})^+\rangle,$$

*where the superscript $+$ means that we consider all functions from $\mathcal{O}_\xi^{\xi G \xi^{-1}}$ that are equal to $0$ at $\xi$ and $\langle \mathcal{O}_\xi^G \cap (\mathcal{O}_\xi^{\xi G \xi^{-1}})^+\rangle$ denotes the ideal in $\mathcal{O}_\xi^G$ generated by $\mathcal{O}_\xi^G \cap (\mathcal{O}_\xi^{\xi G \xi^{-1}})^+$.*

*Proof.* First of all, we note that $\mathcal{O}_\xi^G = \mathcal{O}_\xi^{G_\xi}$. Indeed, if $f \in \mathcal{O}_\xi^G$, then, clearly, $f \in \mathcal{O}_\xi^{G_\xi}$. Further, if $f \in \mathcal{O}_\xi^{G_\xi}$, then the sum of germs $\sum_{g \in G} g(f)$ is an element of $\bigoplus_{g \in G} \mathcal{O}_{g \cdot \xi}^G$. Therefore $f \in \mathcal{O}_\xi^G$. Furthermore, the sheaf isomorphism $\phi_\xi : \mathcal{O}_e \to \mathcal{O}_\xi$ is $G_\xi$-equivariant. Therefore, $\phi_\xi(\mathcal{O}_e^{G_\xi}) = \mathcal{O}_\xi^{G_\xi}$. The second and the third statements are clear, details are left to the reader. $\square$

The above defines the vector space $\mathcal{O}_e/\mathcal{J}_e$ and its subspaces $\mathcal{O}_e^{G_\xi}/(\mathcal{O}_e^{G_\xi} \cap \mathcal{J}_e)$, for any $\xi \in V$. Consider the following correspondence:

$$V \ni \xi \longmapsto \mathbb{E}_\xi := \mathcal{O}_\xi^G/\langle \mathcal{O}_\xi^G \cap (\mathcal{O}_\xi^{\xi G \xi^{-1}})^+\rangle = \phi_\xi(\mathcal{O}_e^{G_\xi}/(\mathcal{O}_e^{G_\xi} \cap \mathcal{J}_e)).$$

This correspondence defines a fibration $\mathbb{E} = (\mathbb{E}_\xi)_{\xi \in V}$ of vector spaces over $V$. The group $G$ acts naturally on the fibration $\mathbb{E}$. Indeed, if $f \in \mathcal{O}_\xi^G$, then $g \cdot f \in \mathcal{O}_{g \cdot \xi}^G$, and if $f \in (\mathcal{O}_\xi^{\xi G \xi^{-1}})^+$, then $g \cdot f \in (\mathcal{O}_{g \cdot \xi}^{g \cdot \xi G(g \cdot \xi)^{-1}})^+$. Now we can define the fibration $\mathbb{E}^G = (\mathbb{E}_{\bar{\xi}}^G)_{\bar{\xi} \in V/G}$ on $V/G$ in the following way: $\mathbb{E}_{\bar{\xi}}^G$ is the vector space of all $G$-invariant elements from $\bigoplus_{\xi' \in \bar{\xi}} \mathbb{E}_{\xi'}$. We set

$$\mathfrak{E} := \bigoplus_{x \in V} \mathbb{E}_x, \quad \mathfrak{E}|_{W'} := \bigoplus_{x \in W'} \mathbb{E}_x, \quad \mathfrak{E}^G := \bigoplus_{\bar{x} \in V/G} \mathbb{E}_{\bar{x}}^G, \quad \mathfrak{E}^G|_W := \bigoplus_{\bar{x} \in W/G} \mathbb{E}_{\bar{x}}^G$$

for a subset $W' \subset V$ and for a $G$-invariant subset $W \subset V$.

3.2. **Action of elements in $\mathcal{S}$ on $\mathfrak{E}$.** Let $\mathbf{J}$ be a $G$-invariant subgroup in $V$ and $v \in V$. Consider the $G$-invariant subset $(G \ltimes \mathbf{J}) \cdot v$. Then $\mathfrak{E}^G|_{(G \ltimes \mathbf{J}) \cdot v/G}$ is defined. Take an element $A$ in $\mathcal{S}$ preserving the vector space $\mathfrak{O}^G|_{(G \ltimes \mathbf{J}) \cdot v/G}$. Any such $A$ has the following form on $\mathfrak{O}^G|_{(G \ltimes \mathbf{J}) \cdot v/G}$:

$$(1) \qquad A|_{\mathfrak{O}^G|_{(G \ltimes \mathbf{J}) \cdot v/G}} = \sum_i \sum_{h \in G} h \cdot (f_i \phi_{\xi_i}).$$

Note that $A$ may be meromorphic. Also, we do not assume that $A(\mathfrak{O}) \subset \mathfrak{O}$.

**Theorem 4.** *Assume that $G$ is a finite group and $A$ sends $\mathfrak{O}^G|_{(G \ltimes \mathbf{J}) \cdot v/G}$ to itself. Then the action of $A$ on $\mathfrak{O}^G|_{(G \ltimes \mathbf{J}) \cdot v/G}$ induces an action of $A$ on $\mathfrak{E}^G|_{(G \ltimes \mathbf{J}) \cdot v/G}$.*

*Proof.* The element $A \in \mathcal{S}$ acts on $\mathfrak{O}^G|_{(G \ltimes \mathbf{J}) \cdot v/G}$. We need to show that this action induces an action on $\mathfrak{E}^G|_{(G \ltimes \mathbf{J}) \cdot v/G}$, or, equivalently, that it induces an action on the vector space

$$\bigoplus_{\bar{\xi} \in (G \ltimes \mathbf{J}) \cdot v/G} \left[ \bigoplus_{\xi' \in \bar{\xi}} \mathcal{O}^G_{\xi'} / \phi_{\xi'} (\mathcal{O}^{G_{\xi'}}_e \cap \mathcal{J}_e) \right]^G.$$

In other words, we need to show that $A(F)$, where $F \in \left[ \bigoplus_{g \in G} \phi_{g \cdot \xi} (\mathcal{O}^{G_{g \cdot \xi}}_e \cap \mathcal{J}_e) \right]^G$, is a sum of elements from $\left[ \bigoplus_{g \in G} \phi_{g \cdot \xi'} (\mathcal{O}^{G_{g \cdot \xi'}}_e \cap \mathcal{J}_e) \right]^G$ for various $\xi'$. Let us take $F$ such that there exists $F' \in \mathcal{O}^G_e$ and $X \in \mathcal{O}^{G_{g \cdot \xi}}_e$ with

$$F = \sum_{g \in G} (F' \circ g \cdot \xi^{-1})[g \cdot (X \circ \xi^{-1})].$$

Note that $F'$ is either in the ideal $\mathcal{J}_e$ or is an invertible $G$-invariant element. We have

$$A(F) = \sum_i \sum_{h,g \in G} (h \cdot f_i) F' \circ (g \cdot \xi^{-1} \circ h \cdot \xi_i^{-1})[g \cdot (X) \circ g \cdot \xi^{-1} \circ h \cdot \xi_i^{-1}].$$

This is a sum of $G$-invariant germs supported at the points $h \cdot \xi_i \circ g \cdot \xi$. Consider, for example, the germ of $A(F)$ at the point $\eta := h_0 \cdot \xi_{i_0} \circ \xi$:

$$(2) \qquad \begin{aligned} A(F)_\eta &= \sum_{(g,h,i) \in \Lambda} (h \cdot f_i) F' \circ (g \cdot \xi^{-1} \circ h \cdot \xi_i^{-1})[g \cdot (X) \circ \eta^{-1}] = \\ &\qquad F' \circ \eta^{-1} \sum_{(g,h,i) \in \Lambda} (h \cdot f_i)[g \cdot (X) \circ \eta^{-1}], \end{aligned}$$

where $\Lambda = \{(g,h,i) \mid (h \cdot \xi_i) \circ (g \cdot \xi) = \eta\}$. We see that the product of a meromorphic function

$$H := \sum_{(g,h,i) \in \Lambda} (h \cdot f_i) g \cdot (X) \circ \eta^{-1}$$

and a holomorphic function $F' \circ \eta^{-1}$ is holomorphic, since $A(F)_\eta$ is holomorphic. This holds for any $F' \in \mathcal{O}^G_e$, in particular, for constant $F'$. The latter implies that $H$ is holomorphic at $\eta$. Similarly, we conclude that $H$ is in $\mathcal{O}^G_\eta$. Summing up, we have $F' \circ \eta^{-1} \in \mathcal{O}^{\eta G \eta^{-1}}_\eta$ and $H \in \mathcal{O}^G_\eta$. Note that, from $F' \in \mathcal{J}_e$, it follows that $F' \circ \eta^{-1} \in (\mathcal{O}^{\eta G \eta^{-1}}_\eta)^+$. Now the assertion of the theorem follows from Lemma 3.   $\square$

3.3. **$\mathcal{A}$-modules corresponding to $\mathfrak{E}$.** For convenience we put

$$(3) \qquad M(G, (G \ltimes \beth) \cdot v) = \mathfrak{E}|_{\bar{\xi} \in (G \ltimes \beth) \cdot v / G}.$$

Denote by $M^*(G, (G \ltimes \beth) \cdot v)$ the vector space

$$(4) \qquad M^*(G, (G \ltimes \beth) \cdot v) := \bigoplus_{\bar{\xi} \in (G \ltimes \beth) \cdot v / G} (\mathbb{E}_{\bar{\xi}}^G)^*.$$

Note that, in general, $M^*(G, (G \ltimes \beth) \cdot v) \subsetneq \left(\mathfrak{E}|_{\bar{\xi} \in (G \ltimes \beth) \cdot v / G}\right)^*$. We will need the following lemma.

**Lemma 5.** *Assume that $A$ sends the vector space $\mathfrak{O}^G|_{(G \ltimes \beth) \cdot v / G}$ to itself. Then the action of $A$ on $\mathfrak{O}^G|_{(G \ltimes \beth) \cdot v / G}$ induces an action of $A$ on $M^*(G, (G \ltimes \beth) \cdot v)$.*

*Proof.* Let us take $A = \sum_i \sum_{h \in G} h \cdot (f_i \phi_{\xi_i})$, $\alpha \in (\mathbb{E}_{\bar{\eta}}^G)^*$ and $\sum_{g \in G} g \cdot F \in \mathcal{O}_{\bar{\xi}}$. We have

$$[A(\alpha)]\left(\sum_{g \in G} g \cdot F\right) = \alpha\left(\sum_i \sum_{h,g \in G} h \cdot (f_i)(g \cdot F) \circ h \cdot \xi_i^{-1}\right).$$

Inside the brackets on the right hand side we have a sum of $G$-invariant germs supported at the points from the finite set $\{h \cdot \xi_i \circ g \cdot \xi \mid g, h \in G\}$. Therefore, $[A(\alpha)]\left(\sum_{g \in G} g \cdot F\right) = 0$, if $\bar{\eta} \notin \{h \cdot \xi_i \circ g \cdot \xi \mid g, h \in G\}/G$. In other words,

$$A(\alpha) \subset \bigoplus_{\bar{\xi}' \in \Lambda / G} (\mathbb{E}_{\bar{\xi}}^G)^*, \quad \text{where} \quad \Lambda = \{h \cdot \xi_i^{-1} \circ g \cdot \eta \mid g, h \in G\}$$

and the proof is complete. $\qquad \square$

As a consequence of Theorem 4 and Lemma 5, we have the following statement.

**Corollary 6.** *Let $\mathcal{A}$ be a subalgebra in $\mathcal{S}(\beth)$ that preserves the vector space $\mathfrak{O}^G|_{(G \ltimes \beth) \cdot v}$. Then both $M(G, (G \ltimes \beth) \cdot v)$ and $M^*(G, (G \ltimes \beth) \cdot v)$ are $\mathcal{A}$-modules.*

In the next sections we will consider the case when $G$ acts locally as a reflection group. In this case all vector spaces $\mathbb{E}_{\bar{\xi}}^G$ are finite dimensional of dimension $|G|$ by Chevalley-Shephard-Todd Theorem.

3.4. **Construction of new $\mathcal{A}$-modules.** Recall that $\beth$ is a $G$-invariant subgroup in $V$. Let $\mathcal{A}$ be a subalgebra in $\mathcal{S}(\beth)$ that preserves the vector space $\mathfrak{O}^G|_{(G \ltimes \beth) \cdot v / G}$, where $v \in V$ is a fixed point. Denote by $G_{\beth \cdot v}$ the stabilizer in $G$ of the orbit $\beth \cdot v$. Let $W := (G \ltimes \beth) \cdot v \setminus \beth \cdot v$. In other words, $W \subset V$ is the union of all orbits of $\beth$ in $(G \ltimes \beth) \cdot v$ except for $\beth \cdot v$. By definition, the group $G_{\beth \cdot v}$ acts on $\beth \cdot v$. Therefore, $G_{\beth \cdot v}$ acts on $W$ too.

Further, we have a natural projection $\pi_G : \mathfrak{O}^G|_{(G \ltimes \beth) \cdot v / G} \to \mathfrak{O}^{G_{\beth \cdot v}}|_{\beth \cdot v / G_{\beth \cdot v}}$ defined by the following formula:

$$(5) \quad \mathfrak{O}^G|_{(G \ltimes \beth) \cdot v / G} \ni F = \sum_{g \in G_{\beth \cdot v}} g \cdot f_\xi + \sum_{g \in L} g \cdot f_\xi \longmapsto \sum_{g \in G_{\beth \cdot v}} g \cdot f_\xi \in \mathfrak{O}^{G_{\beth \cdot v}}|_{\beth \cdot v / G_{\beth \cdot v}},$$

where $\xi \in \beth \cdot v$, $f_\xi \in \mathcal{O}_\xi^{G_\xi}$ is a $G_\xi$-invariant germ and

$$L := G \setminus G_{\beth \cdot v} = \{g \in G \mid g \cdot \xi \notin \beth \cdot v\}.$$

Note that, for any such $F$, there exists $f_\xi$ with $\xi \in \beth \cdot v$ and the map (5) is independent of the choice of $\xi \in \beth \cdot v$.

**Lemma 7.** *The map $\pi_G$ is a bijection.*

*Proof.* Assume that $\pi_G(F) = 0$. Then $f_\xi = 0$ and hence $F' := \sum\limits_{g \in G_{\beth \cdot v}} g \cdot f_\xi = 0$.

Further, let us take $F' = \sum\limits_{g \in G_{\beth \cdot v}} g \cdot f_\xi \in \mathcal{O}^{G_{\beth \cdot v}}|_{\beth \cdot v / G_{\beth \cdot v}}$. Then

$$F' = \pi_G\Big( \sum_{g \in G_{\beth \cdot v}} g \cdot f_\xi + \sum_{g \in L} g \cdot f_\xi \Big).$$

Explicitly, the map $\pi_G^{-1}$ is given by

$$\pi_G^{-1}\Big( \sum_{g \in G_{\beth \cdot v}} g \cdot f_\xi \Big) = \frac{1}{|G_{\beth \cdot v}|} \sum_{g' \in G} g' \cdot \Big( \sum_{g \in G_{\beth \cdot v}} g \cdot f_\xi \Big).$$

$\square$

We will need the following proposition.

**Proposition 8.** *Let $\mathcal{A}$ be a subalgebra in $\mathcal{S}(\beth)$ that preserves the vector space $\mathfrak{O}^G|_{(G \ltimes \beth) \cdot v / G}$. Then $\mathcal{A}$ also preserves $\mathfrak{O}^{G_{\beth \cdot v}}|_{\beth \cdot v / G_{\beth \cdot v}}$ and the map $\pi$ is an isomorphism of $\mathcal{A}$-modules.*

*Proof.* Let $A \in \mathcal{A}$ be as in (1). We apply $A$ to a germ $F = \sum\limits_{g \in G} g \cdot f_\xi \in \mathcal{O}_{\bar{\xi}}^G$, where $f_\xi \in \mathcal{O}_\xi^{G_\xi}$, $\bar{\xi} = G \cdot \xi$ and $\xi \in \beth \cdot v$. We get

$$A(F) = \sum_i \sum_{h,g \in G} (h \cdot f_i)[(g \cdot f_\xi) \circ h \cdot \xi_i^{-1}].$$

Note that, if $(h \cdot \xi_i) \circ (g \cdot \xi) \in \beth \cdot v$, then $g \cdot \xi \in \beth \cdot v$ and hence $g \in G_{\beth \cdot v}$.

Now we compute the germ of $A(F)$ at the point $\eta := (h_0 \cdot \xi_{i_0}) \circ (g_0 \cdot \xi) \in \beth \cdot v$:

$$A(F)_\eta = \sum_{(g,h,i) \in \Lambda} (h \cdot f_i)[(g \cdot f_\xi) \circ h \cdot \xi_i^{-1}],$$

where $\Lambda = \{(g,h,i) \mid (h \cdot \xi_i) \circ (g \cdot \xi) = \eta\}$. If $(h \cdot \xi_i) \circ (g \cdot \xi) = \eta$, then $(g,h,i) \in \Lambda$ and we have $g \cdot \xi = \phi_{h \cdot \xi_i^{-1}}(\eta)$ implying $g \in G_{\beth \cdot v}$. As a consequence of these observations, we obtain

$$\pi(A(F)) = A(\pi(F)).$$

In particular, this equality implies that $A(\pi(F))$ is holomorphic and therefore $A$ preserves $\mathfrak{O}^{G_{\beth \cdot v}}|_{\beth \cdot v / G_{\beth \cdot v}}$. It also implies that $\pi$ is a homomorphism of $\mathcal{A}$-modules. The proof is complete. $\square$

Here comes yet another construction of $\mathcal{A}$-modules. Let $\mathcal{A}$ be as above and $H$ a subgroup of $G$ such that $\mathcal{A}$ preserves the vector space $\mathfrak{O}^H|_{(H \ltimes \beth) \cdot v / H}$. For the pair $H \subset G$, we have the obvious inclusion $P_H^G : \mathcal{O}^G|_{(G \ltimes \beth) \cdot v / G} \hookrightarrow \mathcal{O}^H|_{(G \ltimes \beth) \cdot v / H}$.

**Lemma 9.** *Assume that $\mathcal{A}$ preserves the vector spaces $\mathfrak{O}^G|_{(G \ltimes \beth) \cdot v / G}$ and $\mathfrak{O}^H|_{(H \ltimes \beth) \cdot v / H}$. Then the diagram*

$$
\begin{array}{ccc}
\mathfrak{O}^G|_{(G \ltimes \beth) \cdot v / G} & \xrightarrow{\ \pi_G\ } & \mathfrak{O}^{G_{\beth \cdot v}}|_{\beth \cdot v / G_{\beth \cdot v}} \\
{\scriptstyle P_H^G} \downarrow & & \downarrow {\scriptstyle P_{H_{\beth \cdot v}}^{G_{\beth \cdot v}}} \\
\mathfrak{O}^H|_{(G \ltimes \beth) \cdot v / G} & \xrightarrow{\ \pi_H\ } & \mathfrak{O}^{H_{\beth \cdot v}}|_{\beth \cdot v / H_{\beth \cdot v}},
\end{array}
$$

*in which all maps are homomorphisms of $\mathcal{A}$-modules, commutes.*

*Proof.* This follows directly from the definitions. $\square$

The above leads us to the following theorem.

**Theorem 10.** *Assume that $\mathcal{A}$ preserves the vector spaces $\mathfrak{O}^G|_{(G \ltimes \mathbf{J}) \cdot v / G}$ and $\mathfrak{O}^H|_{(H \ltimes \mathbf{J}) \cdot v / H}$. Then we have the following commutative diagram of $\mathcal{A}$-modules:*

$$
\begin{array}{ccc}
M(G, (G \ltimes \mathbf{J}) \cdot v) & \xrightarrow{\tilde{\pi}_G} & M(G_{\mathbf{J} \cdot v}, \mathbf{J} \cdot v) \\
{\scriptstyle \mathbf{P}_H^G} \downarrow & & \downarrow {\scriptstyle \mathbf{P}_{H_{\mathbf{J} \cdot v}}^{G_{\mathbf{J} \cdot v}}} \\
M(H, (G \ltimes \mathbf{J}) \cdot v) & \xrightarrow{\tilde{\pi}_H} & M(H_{\mathbf{J} \cdot v}, \mathbf{J} \cdot v),
\end{array}
$$

*where $\tilde{\pi}_G$ and $\tilde{\pi}_H$ are induced by $\pi_G$ and $\pi_H$ from Proposition 8, respectively. Moreover, the map*

$$
\Upsilon = \mathbf{P}_H^G \circ \pi_G^{-1} : \mathfrak{O}^{G_{\mathbf{J} \cdot v}}|_{\mathbf{J} \cdot v / G_{\mathbf{J} \cdot v}} \longrightarrow M(H, (G \ltimes \mathbf{J}) \cdot v)
$$

*is also a homomorphism of $\mathcal{A}$-modules.*

*Proof.* Theorem 4 defines all involved $\mathcal{A}$-module structures. Let us argue, for example, that the morphism $\tilde{\pi}_G$ of $\mathcal{A}$-modules induced by $\pi_G$ is well-defined. This follows from the fact that, to obtain the module $M(G, (G \ltimes \mathbf{J}) \cdot v)$, we factor out by the ideal generated by $G$-invariants and, to obtain the module $M(G_{\mathbf{J} \cdot v}, \mathbf{J} \cdot v)$, we factor out by the ideal generated by $G_{\mathbf{J} \cdot v}$-invariants. As we obviously have $G_{\mathbf{J} \cdot v} \subset G$, the necessary statement is obtained by the standard factorization argument. The commutativity of the diagram follows from Lemma 9. $\qquad \square$

3.5. **The vector space $(\mathcal{O}_\mathsf{C} / \mathcal{J}_\mathsf{C})_e$ is finite dimensional.** In this section we show that the vector space $(\mathcal{O}_\mathsf{C} / \mathcal{J}_\mathsf{C})_e$ is finite dimensional. In particular, this implies that the fibration $\mathbb{E}$ has finite dimensional fibers. Several observations of this section were pointed out to us by D. Timashev.

Let $V$ be a complex-analytic or linear algebraic Lie group. Any linear algebraic group is a complex-analytic Lie group, see [Hum]. Recall that we emphasize by the subscripts $\mathsf{C}$ and $\mathsf{A}$ objects in the complex-analytic and the algebraic category, respectively. For example, we denote by $\mathcal{O}_\mathsf{C}$ and by $\mathcal{O}_\mathsf{A}$ the sheaves of complex-analytic (holomorphic) and algebraic (polynomial) functions, respectively.

Let $V$ be a linear algebraic group. Note that we can choose coordinates $(x_i)$ in a neighborhood $U$ of the identity $e \in V$ such that $e$ is the origin and the vector space $W = \langle x_1, \ldots, x_n \rangle$ is $G$-invariant. Indeed, denote by $\mathfrak{m}_e$ the maximal ideal in $(\mathcal{O}_\mathsf{A})_e$. Then $\mathfrak{m}_e^2$ is a $G$-invariant subspace in $\mathfrak{m}_e$. We choose any coordinates $\{y_1, \ldots, y_n\}$ in $U$. Let $W'$ be the $\mathbb{C}$-span of $\{g \cdot y_i \mid i = 1, \ldots, n, g \in G\}$. Then $W'$ and $W' \cap \mathfrak{m}_e^2$ are $G$-invariant. Since $G$ is finite, there exists $G$-invariant subspace $W$ such that $W' = W \oplus (W' \cap \mathfrak{m}_e^2)$. Let $x_1, \ldots, x_n$ be a basis in $W$. If $f \in (\mathcal{O}_\mathsf{C}^G)_e$, then there exists a decomposition $f = \sum\limits_{k=0}^{\infty} f_k$, where $f_k$ are $G$-invariant homogeneous polynomials in $(x_i)$ of degree $k$. If $V$ is complex analytic but not algebraic, we mean by $(\mathcal{O}_\mathsf{A})_e$ the algebra of germs of polynomial functions in $(x_i)$.

A classical fact from the invariant theory is that the extension $(\mathcal{O}_\mathsf{A}^G)_e \subset (\mathcal{O}_\mathsf{A})_e$ of rings is integral. Indeed, any polynomial $f \in (\mathcal{O}_\mathsf{A})_e$ is integral over $(\mathcal{O}_\mathsf{A}^G)_e$ since it is a root of the polynomial $\prod\limits_{g \in G} (t - g \cdot f)$. In particular, $f^{|G|}$ is a linear combination of $f^p$, where $p < |G|$, with coefficients from $(\mathcal{O}_\mathsf{A}^G)_e$.

**Lemma 11.** *We have that $(\mathcal{O}_\mathsf{A})_e$ is a finitely generated $(\mathcal{O}_\mathsf{A}^G)_e$-module and the minimal number of generators is less than or equal to $|G|^{\dim V}$.*

*Proof.* The proof follows from the fact that $x_i^{|G|}$ is a linear combination of $x_i^p$, where $p < |G|$, with coefficients from $(\mathcal{O}_A^G)_e$.                                                               $\square$

**Corollary 12.** *The vector space $(\mathcal{O}_A/\mathcal{J}_A)_e$ is finite dimensional and its dimension is less than or equal to $|G|^{\dim V}$.*

**Theorem 13.** *Let $V$ be a complex analytic or linear algebraic group and $G$ a finite group acting on $V$. Then*

$$(\mathcal{O}_A/\mathcal{J}_A)_e \simeq (\mathcal{O}_C/\mathcal{J}_C)_e.$$

*In particular, $(\mathcal{O}_C/\mathcal{J}_C)_e$ is finite dimensional and its dimension is less than or equal to $|G|^{\dim V}$.*

*Proof.* We have the obvious map

$$(6) \qquad (\mathcal{O}_A/\mathcal{J}_A)_e \longrightarrow (\mathcal{O}_C/\mathcal{J}_C)_e, \quad f \mapsto f + (\mathcal{J}_C)_e.$$

Let us show that this map is a bijection.

*Step 1.* Let us first show that the map (6) is injective. To start with, assume that $f \in (\mathcal{O}_A)_e \cap (\mathcal{J}_C)_e$. Then $f = \sum\limits_{j=1}^{s} f_{1j} f_{2j}$, where $f_{1j} = \sum\limits_{k=0}^{\infty} f_k^{j1} \in (\mathcal{O}_C)_e$, $f_{2j} = \sum\limits_{p=1}^{\infty} f_p^{j2} \in (\mathcal{O}_C^G)_e$, $f_k^{j1}$ are homogeneous polynomials in $(x_i)$ of degree $k$ and $f_p^{j2}$ are homogeneous $G$-invariant polynomials in $(x_i)$ of degree $p$. We see that the polynomial $f = \sum\limits_{j=1}^{s} \sum\limits_{k=0}^{\infty} \sum\limits_{p=1}^{\infty} f_k^{j1} f_p^{j2}$ is an element in $(\mathcal{J}_A)_e$.

*Step 2.* Let us now show that the map (6) is surjective. Denote by $z_1, \ldots, z_p$ a system of generators for the $(\mathcal{O}_A^G)_e$-module $(\mathcal{O}_A)_e$ and set $N = \max\limits_{s}\{\deg z_s\}$. Let us take $f \in \mathfrak{m}_e^{N+1}$, where $\mathfrak{m}_e$ is the maximal ideal in $(\mathcal{O}_C)_e$.

Assume first that $f = \sum\limits_{i=N+1}^{t} f_i$, where $f_i$ is a homogeneous polynomial of degree $i$, is a polynomial. The polynomial $\prod\limits_{g \in G} (t - g \cdot f)$, considered above, is homogeneous. Hence we can assume that $z_j$ are homogeneous and $f_i = \sum\limits_{j} f_{ij} z_j$ is a decomposition with homogeneous $G$-invariant coefficients. Since $\deg f_i > N$, we conclude that $f \in (\mathcal{J}_C)_e$.

Further, let us take $f = \sum\limits_{i=N+1}^{\infty} f_i \in (\mathcal{O}_C)_e$, where $f_i$ are homogeneous polynomials in $(x_i)$ of degree $i$. Assume that $f$ is not identically equal to zero on the $x_n$-axis (we may ensure this by a linear change of coordinates). By the Weierstrass preparation theorem, we have $f = Pf_1$, where $P = x_n^r + a_{r-1}x_n^{r-1} + \cdots + a_1 x_n + a_0$ is a Weierstrass polynomial and $f_1$ is a unit. Here $a_i$ is a holomorphic function in $x_1, \ldots, x_{n-1}$, for any $i$. Since $f_1$ is a unit, $P = ff_1^{-1} \in \mathfrak{m}_e^{N+1}$. Note that in the Taylor expansions of $a_\alpha x_n^\alpha$ and $a_\beta x_n^\beta$ in a neighborhood of $e$, where $\alpha \neq \beta$, we do not have equal summands. Therefore, $a_\alpha x_n^\alpha \in \mathfrak{m}_e^{N+1}$ for any $\alpha$. Similarly, we apply the Weierstrass preparation theorem to $a_\alpha$ and proceed inductively. We obtain a polynomial in $\mathfrak{m}_e^{N+1}$ that, by the above, belongs to $(\mathcal{J}_C)_e$. Now, assume, by induction, that $a_\alpha x_n^\alpha \in (\mathcal{J}_C)_e$. Hence $P \in (\mathcal{J}_C)_e$ and therefore $f = Pf_1 \in (\mathcal{J}_C)_e$.

Now we can show that the map (6) is surjective. Indeed, by the above, any element $F \in (\mathcal{O}_C/\mathcal{J}_C)_e$ has a polynomial representative. This completes the proof.          $\square$

Let $\mathcal{A}$ be as in Theorem 10 and $\mathcal{B} \subset \mathcal{S}$ be the algebra of $G$-invariant functions. Assume, in addition, that $\mathcal{B} \subset \mathcal{A}$

**Proposition 14.** *The $\mathcal{A}$-modules constructed in Theorem 10 are Harish-Chandra modules.*

*Proof.* This follows from Theorem 13. $\qquad\square$

## 4. Rational Galois orders and their modules

4.1. **Reflection groups and divided difference operators.** Let $V_\mathbb{R}$ be a vector space over $\mathbb{R}$ equipped with a non degenerate symmetric bilinear form $( , )$. Set $V = \mathbb{C} \otimes_\mathbb{R} V_\mathbb{R}$ and denote the corresponding to $( , )$ inner product on $V$ by the same symbol. For $v \in V$, the *reflection $\sigma_v$ with respect to $v$* is the linear transformation of $V$ that fixes the hyperplane $\{w \in V \mid (w, v) = 0\}$ and maps $v$ to $-v$. It is given by the formula $\sigma_v(x) = x - \frac{2(x,v)}{(v,v)} v$. A *root system* $\Phi$ is a finite subset in $V_\mathbb{R} \setminus \{0\}$ that satisfies the following properties:

(I) If $x, y \in \Phi$, then $\sigma_x(y) \in \Phi$.

(II) If $x$ and $kx$ in $\Phi$, for some $k \in \mathbb{R}$, then $k = \pm 1$.

For a root system $\Phi$, the corresponding *reflection group* $G \subset GL(V)$ is the group generated by all reflections $\alpha_v$, where $v \in \Phi$. A *system of simple roots* or a *basis* of $\Phi$ is a linearly independent subset in $\Phi$ such that every $x \in \Phi$ can be written as a linear combination of elements from $\Psi$ with all non-negative or all non-positive coefficients. Any root system $\Phi$ has a basis. If a basis $\Psi \in \Phi$ is fixed, we get a partition $\Phi = \Phi^+ \cup \Phi^-$, where $\Phi^+$ is the system of positive roots and $\Phi^-$ is the system of negative ones. Here a root $x$ is called *positive* (resp. *negative*) with respect to $\Psi$, if it is a linear combination of vectors from $\Psi$ with all non-negative (resp. non-positive) coefficients. We denote by $\Theta$ the set of *simple reflections*, that is reflections corresponding to elements in $\Psi$.

Let $G$ be a reflection group, $\Psi$ be a system of simple roots and $\Theta$ be the corresponding system of simple reflections. For any $x \in V$, we have a unique $\gamma_x \in V^*$ such that $\gamma_x(y) = (x, y)$, for all $y \in V$. Further, for any simple reflection $\sigma_x \in \Theta$, we define the corresponding *divided difference operator* $\partial_{\sigma_x}$ on the set of holomorphic (or meromorphic, or rational or polynomial) functions on $V$ via

$$\partial_{\sigma_x} \cdot f := \frac{f - \sigma_x \cdot f}{\gamma_x}.$$

For any $w \in G$, we set $\partial_w = \partial_{\sigma_1} \circ \cdots \circ \partial_{\sigma_p}$, where $w = \sigma_1 \circ \cdots \circ \sigma_p$ is a reduced expression. By [BGG, Page 5], we have $\partial_w = 0$, if the expression $w = \sigma_1 \circ \cdots \circ \sigma_p$ is not reduced. Moreover, the operator $\partial_w$ is independent of the choice of a reduced expression.

4.2. **Rational Galois orders.** Rational Galois orders is a large class of algebras introduced in [Har, Section 4]. This class includes, for instance, orthogonal Gelfand-Zeitlin algebras, finite W-algebras of type $A$ and, as we will see in Section 5, standard algebras of type $\mathbb{A}$ that preserve the vector space $\mathfrak{O}^G$. Note that a particular case of rational Galois orders was considered earlier in [Vi1, Vi2]. In the terminology of [Vi1, Vi2], these are finitely generated over $H^0(V, \mathcal{O}^G)$ subalgebras in the so-called *universal ring*.

Let $G$ be a reflection group in $V$ as in Subsection 4.1 (note that the definition of a rational Galois order was given in [Har] for a more general case of a pseudo-reflection group or a complex reflection group $G$). Let $\chi : G \to \mathbb{C}^\times$ be a character. The space of relative invariants

$$H^0(V, \mathcal{O})^G_\chi := \{f \in H^0(V, \mathcal{O}) \mid g \cdot f = \chi(g)f \ \text{ for all } \ g \in G\}$$

is, naturally, an $H^0(V, \mathcal{O})^G$-module. This module is free of rank 1 and is generated by

$$d_\chi = \prod_{H \in A(G)} (\gamma_H)^{a_H},$$

where $A(G)$ is the set of all hyperplanes $H$ that are fixed by a certain element $\sigma_H$ in $G$, $\gamma_H \in V^*$ with $\ker \gamma_H = H$ and $a_H$ is the minimal non-negative integer such that $\chi(\sigma_H) = \det(\sigma_H^*)^{a_H}$. If $G$ is a reflection group, then $a_H = 0$ or 1, see [Ter, Section 2] for details.

**Definition 15.** [Har, Definition 4.3] *A* rational Galois order *is a subalgebra $\mathcal{R}$ in $\mathcal{S}(V)^G$ that contains $H^0(V, \mathcal{O}^G)$ and that is generated by a finite number of elements $X \in \mathcal{S}(V)^G$ such that, for any such $X$, there exists a character $\chi$ of $G$ such that $d_\chi X$ is holomorphic in $V$.*

In [Har, Theorem 4.2] it was shown that a rational Galois order preserves $H^0(V, \mathcal{O}^G)$. In the following Lemma we prove a more general result: a rational Galois order preserves the vector space $\mathfrak{O}^G$.

**Lemma 16.** *Let $X$ be a generator of a rational Galois order. Then $X(\mathfrak{O}^G) \subset \mathfrak{O}^G$.*

*Proof.* Let $\chi$ be a character of $G$ such that $d_\chi X$ is holomorphic in $V$. We take $F_{G \cdot \xi} \in \mathcal{O}^G_{G \cdot \xi}$ and consider a germ $P_\eta$ of $P = X(F_{G \cdot \xi})$ at a point $\eta \in V$. Denote by $d_\eta$ the product of all divisors $\gamma_H$ of $d_\chi$ such that $\gamma_H(\eta) = 0$. The corresponding reflections $\sigma_H$ generate the group $G_\eta$. Then $P_\eta = P'_\eta / \chi_\eta$, where $P'_\eta$ is a holomorphic function at $\eta$. We see that $P'_\eta$ is a relative invariant for the character $\chi_\eta$, where $\chi_\eta(h) = (h \cdot d_\eta)/d_\eta$, $h \in G_\eta$. By [Ter, Section 2], we have $P'_\eta = d_\eta P''_\eta$, where $P''_\eta$ is holomorphic at $\eta$. Therefore, $P_\eta$ is also holomorphic at $\eta$. $\qquad\square$

Here is an example.

**Example 17.** Assume that we are in the setup of Subsection 2.2. Let $n \geq 4$ and consider for example the classical Gelfand-Zeitlin operator $E_{34}$. We will now show explicitly that $E_{34}(F)$ is holomorphic, where $F = \sum_{g \in G} g \cdot (f_{\xi_3^1}) \in \bigoplus_{g \in G} \mathcal{O}^G_{g \cdot \xi_3^1}$.

We compute, for example, the germ of $E_{34}(F)$ at the point $\eta := \xi_3^1 + \xi'$, where $\xi' = (\delta_{32}^{ki})$. We have

$$E_{34}(F)_\eta = \frac{\prod\limits_{j=1}^{4}(v_{31} - v_{4j})}{(v_{31} - v_{32})(v_{31} - v_{33})} f_{\xi_3^1} \circ (\xi')^{-1} + \frac{\prod\limits_{j=1}^{4}(v_{32} - v_{4j})}{(v_{32} - v_{31})(v_{32} - v_{33})} f_{\xi'} \circ (\xi_3^1)^{-1} =$$

$$\frac{(v_{31} - v_{33}) \prod\limits_{j=1}^{4}(v_{32} - v_{4j}) f_{\xi'} \circ (\xi_3^1)^{-1} - (v_{32} - v_{33}) \prod\limits_{j=1}^{4}(v_{31} - v_{4j}) f_{\xi_3^1} \circ (\xi')^{-1}}{(v_{32} - v_{33})(v_{32} - v_{31})(v_{31} - v_{33})}.$$

We see that the polynomial in the numerator changes the sign, if we permute $v_{32}$ and $v_{31}$. Therefore the factor $v_{32} - v_{31}$ cancels and the fraction is a holomorphic function at $\eta$. Another important observation here is that we have to consider

the holomorphic category instead of the algebraic one. Indeed, the rational operator $E_{34}$ sends a polynomial germ $F$ to the holomorphic germ $E_{34}(F)_\eta$ plus other holomorphic summands.

Representation theory of rational Galois orders was developed in [FGRZ]. In this paper, we generalize some of the constructions from [FGRZ] for any finite group, see Section 6.

4.3. **Bases in some modules over rational Galois orders.** Assume that there is a $G$-invariant neighborhood $U$ of $e \in V$ such that $G$ acts as a reflection group in $U$. In this case, we will call $G$ a *local reflection group*. An example of this situation is $G = S_n$ and $V \simeq \mathbb{C}^n$, where $S_n$ acts via its permutation representation. Another example is $G = S_n$ and $V = \mathbb{C}^n/\mathbb{Z}^n$. More generally, $G$ is a generalized Weyl group acting on $\mathbb{C}^n$ and $V = \mathbb{C}^n/\beth'$, where $\beth'$ is a $G$-invariant discrete lattice in $\mathbb{C}^n$.

In this subsection we will describe the finite dimensional vector spaces $\mathbb{E}^*_{\bar{\xi}}$ using divided difference operators. If $G$ is a local reflection group, by Chevalley-Shephard-Todd Theorem, the factor space $\mathcal{O}_e/\mathcal{J}_e$ is finite dimensional and has dimension $|G|$. Denote by $\Delta(\Psi)$ the product of all $\alpha_x$, where $x \in \Phi^+$. For any $g \in G$, we put $\mathcal{P}_g := \partial_{g^{-1}w_0}\Delta(\Psi)$. The obtained polynomials are called *Schubert polynomials* and their images in $\mathcal{O}_e/\mathcal{J}_e$ form there a basis. Note that $\mathcal{P}_w(e) = 0$ if $w \neq e$ and $\mathcal{P}_e$ is a non-zero constant. Now we can easily construct the dual basis. Consider

(7) $$B(\Theta) := \langle ev_e \circ \partial_w \,|\, w \in G \rangle,$$

where $ev_e$ is the evaluation at $e \in V$. To show that $B(\Theta)$ is a basis of $(\mathcal{O}_e/\mathcal{J}_e)^*$, we note that $ev_e \circ \partial_w(\mathcal{P}_g)$ is 0, if and only if $g \neq w$. If $\Theta'$ is another system of simple reflections in $G$ and $\rho(\Theta) = \Theta'$, then

$$B(\Theta') = \langle ev_e \circ \rho \circ \partial_w \circ \rho^{-1} \,|\, w \in G \rangle$$

is another basis of $(\mathcal{O}_e/\mathcal{J}_e)^*$. We note also that a basis of

$$(\mathcal{O}_e^{G_\xi}/\mathcal{O}_e^{G_\xi} \cap \mathcal{J}_e)^* \subset (\mathcal{O}_e/\mathcal{J}_e)^*$$

is given by $\langle ev_e \circ \partial_w \,|\, w \in (G/G_\xi)^{short} \rangle$, where $(G/G_\xi)^{short}$ denotes the set of shortest coset representatives.

Assume that $\Theta$ is fixed. In any class $\bar{\xi} \in V/G$, we can choose a representative $\tilde{\xi}$ such that $G_{\tilde{\xi}}$ is parabolic with respect to $\Theta$. A description of the basis in $(\mathbb{E}^G_{\bar{\xi}})^*$ corresponding to $B(\Theta)$ is given in the following straightforward statement:

**Lemma 18.** *Let $\Theta$ be a system of simple roots, $\tilde{\xi}$ be as above and $B(\Theta)$ be the corresponding basis of $(\mathcal{O}_e/\mathcal{J}_e)^*$. Then $\{ev_e \circ \partial_w \circ \phi_{\tilde{\xi}}, \ w \in (G/G_{\tilde{\xi}})^{short}\}$ is a basis of $(\mathbb{E}^G_{\bar{\xi}})^*$.*

We summarize the above results in the following theorem.

**Theorem 19.** *Let $G$ be a local reflection group, $\Theta$ be a system of simple reflections and $\mathcal{A}$ be a subalgebra in the skew-ring $\mathcal{S}$ that preserves the vector space $\mathfrak{O}^G|_{(G \ltimes \beth) \cdot v}$, for a subgroup $\beth \subset V$. Then*

$$\bigcup_{\tilde{\xi} \in \beth/G} \{ev_e \circ \partial_w \circ \phi_{\tilde{\xi}}, \ w \in (G/G_{\tilde{\xi}})^{short}\},$$

*is basis of the $\mathcal{A}$-module $M^*(G, (G \ltimes \beth) \cdot v)$.*

*Proof.* The statement follows from Corollary 6 and Lemmata 5 and 18. □

For instance, we have Theorem 19 for all rational Galois orders.

## 5. Characterization of rational Galois orders

Let $V$ and $G$ be as in Subsections 2.3 and 2.4. Denote by $(x_{ki})$ the standard dual basis in $V^*$, that is, $x_{ki}(v) = v_{ki}$, where $v = (v_{st}) \in V$.

**Theorem 20.** *Let $A = \sum\limits_{s=1}^{p} \sum\limits_{g \in G} g \cdot (f_s \phi_{\xi_{i_s}^{a_s}}) \in \mathcal{S}(V)^G$ and assume that $A$ preserves the vector space $\mathfrak{O}^G$. Then $A$ is a generator of a rational Galois order $\mathcal{D}$ (cf. Definition 15).*

*Proof. Step 1.* We start by reducing the statement to the case $p = 1$. For this, we show that $B_s := \sum\limits_{g \in G} g \cdot (f_s \phi_{\xi_{i_s}^{a_s}})$ also preserves the vector space $\mathfrak{O}^G$, for any $s = 1, \ldots, p$. Denote by $S_t$ the $G$-invariant polynomial

$$\sum_{g \in G} g \cdot x_{i_t,1} = \frac{|G|}{n_{i_t}} \sum_{j=1}^{n_{i_t}} x_{i_t,j},$$

where $t \in \{1, \ldots, p\}$. Consider the operator $S_t id \in \mathcal{S}(V)^G$ and the following composition of operators

$$A \circ S_t id = S_t \sum_{s=1}^{p} \sum_{g \in G} g \cdot (f_s \phi_{\xi_{i_s}^{a_s}}) - \frac{a_{i_t}|G|}{n_{i_t}} \sum_{g \in G} g \cdot (f_t \phi_{\xi_{i_t}^{a_t}}) = S_t A - \frac{a_{i_t}|G|}{n_{i_t}} B_t.$$

The operators $A \circ S_t id$, $S_t id$ and $S_t A$ all preserve $\mathfrak{O}^G$. Hence the element $B_t$ also preserves $\mathfrak{O}^G$, in case $a_t \neq 0$.

Consider now the case $a_t = 0$. Let us rewrite the operator $A$:

$$A = \sum_{a_s \neq 0} \sum_{g \in G} g \cdot (f_s \phi_{\xi_{i_s}^{a_s}}) + \sum_{g \in G} g \cdot (f_s \phi_{\xi_{i_s}^{0}}) = \sum_{a_s \neq 0} \sum_{g \in G} g \cdot (f_s \phi_{\xi_{i_s}^{a_s}}) + H id,$$

where $H$ is $G$-invariant. Since $A$ and the first summand preserve $\mathfrak{O}^G$, we deduce that $H id$ also preserves $\mathfrak{O}^G$.

Therefore to prove our theorem it is enough to show that, if $C := \sum\limits_{g \in G} g \cdot (f \phi_{\xi_i^{a}})$ preserves the vector space $\mathfrak{O}^G$, then $C \in \mathcal{D}$.

*Step 2.* Assume that $C = \sum\limits_{g \in G} g \cdot (f \phi_{\xi_i^{a}})$ preserves the vector space $\mathfrak{O}^G$. Let us show that every function $g \cdot f$ is holomorphic in any Weyl chamber. In other words, we want to show that the function $g \cdot f$ is holomorphic at any point $w \in V$ such that $w = (w_{ki})$, where $w_{ki} \neq w_{kj}$, for any $k$ and $i \neq j$.

First of all we note that, if $a = 0$, then the operator $C$ is holomorphic at any point $v \in V$. Indeed, in this case $C = H id$, where $H$ is a $G$-invariant meromorphic function. Let us take $\sum\limits_{h \in G} h \cdot c \in \mathcal{O}_{\bar{v}}^G$, where $c \in \mathbb{C} \setminus \{0\}$. Then

$$C(\sum_{h \in G} h \cdot c) = H \sum_{h \in G} h \cdot c \in \mathcal{O}_{\bar{v}}^G,$$

where $\bar{v} = G \cdot v$. Therefore, $cH$ is holomorphic at any $h \cdot v$. Hence $H$ is holomorphic on $V$.

Assume now that $a \neq 0$. Let us take $\sum\limits_{h \in G} h \cdot F \in \mathcal{O}_{\bar{v}}^G$, where $F = e \cdot F \in \mathcal{O}_v^G$. Then $C(\sum\limits_{h \in G} h \cdot F) \in \mathfrak{O}^G$ is a sum of $G$-invariant germs supported at the points from the set

$$T = \{h \cdot v + g \cdot \xi_i^a \mid g, h \in G\}.$$

Let us show that, from the fact that $h \cdot v + g \cdot \xi_i^a = h' \cdot v + g' \cdot \xi_i^a$ is a point in a Weyl chamber, it follows that $h \cdot v = h' \cdot v$ and $g \cdot \xi_i^a = g' \cdot \xi_i^a$.

Take $w = (w_{kj}) = h \cdot v + g \cdot \xi_i^a \in T$, a point from a Weyl chamber. Assume that there is $w' = (w'_{kj}) = h' \cdot v + g' \cdot \xi_i^a \in T$ such that $w' = w$. First of all, from $w = w'$, it follows that $w_{kj} = w'_{kj}$, for any $k \neq i$ and for any $j$. Further, we have two possibilities: $v_{ij} + a = v_{ip} + a$ or $v_{ij} + a = v_{ip}$, for some $p$. In the first case, we have $v_{ij} = v_{ip}$. Using that $w$ is in a Weyl chamber, we conclude that $h = id$ or $h$ is the transposition that sends $v_{ij}$ to $v_{ip}$. In particular, $h \cdot v = h' \cdot v$. Consider the case $v_{ij} + a = v_{ip}$, where $p \neq j$. In this case we have a contradiction with the assumption that $w$ is in a Weyl chamber. Summing up, we have $h \cdot v = h' \cdot v$, and hence $g \cdot \xi_i^a = g' \cdot \xi_i^a$.

Now consider the summand

$$(8) \quad \sum_{h_1 \in G_v} (hh_1 \cdot F) \circ (g \cdot \xi_i^a)^{-1} \sum_{g_1 \in G_{\xi_i^a}} (gg_1 \cdot f) = \alpha[(h \cdot F) \circ (g \cdot \xi_i^a)^{-1}](g \cdot f) \in \mathcal{O}_w^G,$$

where $\alpha \in \mathbb{C} \setminus \{0\}$, from $C(\sum\limits_{h \in G} g \cdot F)$, supported at the point $w = h \cdot v + g \cdot \xi_i^a$ from a Weyl chamber. Note that, to obtain (8), we use the fact that $G_F = G_v$ and $G_f = G_{\xi_i^a}$. Further, putting $F = const \neq 0$, we see that $g \cdot f$ is holomorphic at $w$.

*Step 3.* Our goal now is to show that $C \in \mathcal{D}$. Take $w = (w_{kj}) = h \cdot v + g \cdot \xi_i^a \in T$ such that the stabilizer of $w$ has order 2. We have two possibilities:

(1) $v_{ks} = v_{kt}$, for some $s \neq t$, $G_w = \{id, \sigma\}$, where $\sigma$ is the transposition that swaps the point $v_{ks}$ and $v_{kt}$;

(2) $v_{ij} + a = v_{ip}$, for some $j \neq p$, $G_w = \{id, \tau\}$, where $\tau$ is the transposition that swaps the point $v_{ij} + a$ and $v_{ip}$.

In the first case, as in Step 2, we get that $h \cdot f$ is holomorphic at $w$. Consider the second possibility. The summand from $C(\sum\limits_{h \in G} h \cdot F)$ supported at the point $w$ is

$$(9) \quad \begin{aligned} &\sum_{h_1 \in G_x} (hh_1 \cdot F) \circ (g \cdot \xi_i^a)^{-1} \sum_{g_1 \in G_{\xi_i^a}} (gg_1 \cdot f)+ \\ &\tau[\sum_{h_1 \in G_v} (hh_1 \cdot F) \circ (g \cdot \xi_i^a)^{-1} \sum_{g_1 \in G_{\xi_i^a}} (gg_1 \cdot f)] \in \mathcal{O}_w^G. \end{aligned}$$

Let $F = c \in \mathbb{C} \setminus \{0\}$. From (9), we get that $g \cdot f + \tau(g \cdot f) \in \mathcal{O}_w^G$. We put $z_1 := x_{ij} - x_{ip}$ and $z_2 := x_{ij} + x_{ip}$. Then $(z_1, z_2, x_{kt})$, where $(kt) \neq (ij), (ip)$, form a new coordinate system. Moreover, $z_2$ and $x_{kt}$ are $\tau$-invariant and $\tau(z_1) = -z_1$.

From Step 2 it follows that $g \cdot f$ is a holomorphic function in a neighborhood of $w$, except for points $y$ with $z_1(y) = 0$. Any such function possesses a Hartogs-Laurent series, see [Sh, Section 8]. Let this series be $g \cdot f = \sum\limits_{s=q}^{\infty} H_s z_1^s$, where $H_s$ are holomorphic functions in $z_2$ and all $x_{kt}$. We have

$$g \cdot f + \tau(g \cdot f) = \sum_{s=q}^{\infty} (1 + (-1)^s) H_s z_1^s \in \mathcal{O}_w^G.$$

We obtain that $H_s = 0$, for all $s = 2r < 0$.

Further, we note that $G_{h \cdot v} = \{id\}$ or $G_{h \cdot v} = \{id, \theta\}$, where $\theta$ is an involution that swaps $v_{ij}$ with some $v_{ij'}$, where $j' \neq p$. In the first case, set $h \cdot F = z_1 \in \mathcal{O}_{h \cdot v}^{G_{h \cdot v}}$. In the second case, set $h \cdot F = z_1 + \theta(z_1) \in \mathcal{O}_{h \cdot v}^{G_{h \cdot v}}$. In both cases, using (9), we obtain

$$z_1 \left( \sum_{s=q}^{\infty} H_s z_1^s - \sum_{s=q}^{\infty} (-1)^s H_s z_1^s \right) \in \mathcal{O}_w^G.$$

This is possible only if $H_s = 0$, for $s < 1$. Therefore $g \cdot f$ has only a simple pole at $w$.

Denote by $\Delta$ the product of all $x_{ki} - x_{kj}$, where $i \neq j$. Summing up, above we proved that $f$ is holomorphic in any Weyl chamber and it has a simple pole or it is holomorphic at all points with the stabilizer of order 2. This implies that $f\Delta$ is holomorphic at all point with the stabilizer of order 1 or 2. By the Riemann extension theorem, see e.g. [Dem, Corollary 6.4], singularities of codimension at least 2 are removable. It follows that $f\Delta = H$ is homomorphic in $V$. The proof is complete. $\square$

**Corollary 21.** *Let $\mathcal{A} \subset \mathcal{S}(V)^G$ be a finitely generated over $H^0(V, \mathcal{O}^G)$ standard algebra of type $\mathbb{A}$ that preserves the vector space $\mathfrak{O}^G$. Then $\mathcal{A}$ is a rational Galois order.*

This description of standard algebras of type $\mathbb{A}$ that preserve $\mathfrak{O}^G$ is surprising. It would be interesting to prove an analog of this result (or to find a counter-example) for other reflection groups.

## 6. APPLICATIONS OF THEOREM 4 TO GELFAND-ZEITLIN MODULES

Let $\mathcal{A}$ be a subalgebra in $\mathcal{S}(V)$ that preserves the vector space $\mathfrak{O}^G|_{(G \ltimes \beth) \cdot v}$, for some $v \in V$, and $\mathcal{B}$ be the algebra of global $G$-invariant functions on $V$. Then, by Corollary 6 and Proposition 14, $M(G, (G \ltimes \beth) \cdot v)$ and $M^*(G, (G \ltimes \beth) \cdot v)$ are $\mathcal{A}$-modules. These $\mathcal{A}$-modules and their submodules were studied, for some special cases, simultaneously and independently in [RZ] (the case of $\mathcal{A} = U(\mathfrak{gl}_n(\mathbb{C}))$) and in [EMV] (the case of $\mathcal{A}$ being an orthogonal Gelfand-Zeitlin algebra). The case when $\mathcal{A}$ is a rational Galois orders corresponding to any reflection group was later considered in [FGRZ]. In this section, we show how to obtain [RZ, Section 5.6, Theorem], [EMV, Theorem 10] and [FGRZ, Theorem 7.4] using Corollary 6, Theorem 10 and Proposition 14.

6.1. **The case of orthogonal Gelfand-Zeitlin algebras.** Let $V$, $\beth$ and $G$ be as in Subsection 2.3. The classical Gelfand-Zeitlin operators $E_{st}$ and the generators of the orthogonal Gelfand-Zeitlin algebra $E_k$ and $F_k$ are rational, however as it was shown in Lemma 16, we have $E_k(\mathfrak{O}^G) \subset \mathfrak{O}^G$ and $F_k(\mathcal{O}^G) \subset \mathcal{O}^G$. Clearly, the same holds for $E_{st}$. Further let us take $v' \in V$. It is easy to see that there exists $v \in \beth \cdot v'$ such that $G_v$ includes all stabilizers $G_w$, where $w \in \beth \cdot v'$.

**Lemma 22.** *We have $G_v = G_{\beth \cdot v}$.*

*Proof.* For $v = (v_{ki})$, the following holds: if $v_{ki} - v_{kj} \in \mathbb{Z}$, then $v_{ki} = v_{kj}$. Further, it is clear that $G_v \subset G_{\beth \cdot v}$. If $g \in G_{\beth \cdot v}$, then $g \cdot v \in \beth \cdot v$ or, equivalently, $g \cdot v - v \in \beth$. Hence $(g \cdot v)_{ki} - v_{ki} \in \mathbb{Z}$ and thus $(g \cdot v)_{ki} = v_{ki}$, implying $g \in G_v$. $\square$

By Lemma 22 and Proposition 8, we get that $M(G_v, \rfloor \cdot v)$ and $M^*(G_v, \rfloor \cdot v)$ are $\mathcal{A}$-modules. From Proposition 14 it follows that these modules are Harish-Chandra modules and therefore Gelfand-Zeitlin modules. This recovers the corresponding results from [RZ] and [EMV].

6.2. **The case of rational Galois orders.** Let $V, \rfloor$ and $G$ be as in Subsection 4.2. Take $v \in V$ and let $H$ be a subgroup in $G$ that contains all stabilizers $G_w$, where $w \in \rfloor \cdot v$. Then it is easy to check (we refer to Theorem 25 for details) that a rational Galois order $\mathcal{A}$ preserves the vector space $\mathfrak{O}^H|_{(H \ltimes \rfloor) \cdot v}$. By Lemma 16, the algebra $\mathcal{A}$ preserves also the vector space $\mathfrak{O}^G|_{(G \ltimes \rfloor) \cdot v}$. Therefore we may apply Theorem 10 to obtain a family of the corresponding modules. In the case when $H$ is a reflection group and satisfies some other conditions (it has to be parabolic with respect to a fixed system of simple roots), the $\mathcal{A}$-module $Im(\Upsilon^*)$, cf Theorem 10, was constructed in [FGRZ, Theorem 7.4]. This recovers the corresponding result of [FGRZ].

## 7. Structure theorem for rational Galois order

7.1. **Further examples of algebras that preserve the vector space $\mathfrak{O}^G$.** In this section we assume that $G$ is a reflection group on $V \simeq \mathbb{C}^n$. Let us fix a system $\Psi$ of simple roots and let $\Theta$ be the set of the corresponding simple reflections. Our goal now is to define two classes of algebras preserving the vector space $\mathfrak{O}^G$. As above, we denote by $\circ$ composition of operators or the product in $G \ltimes V$ and we use $\cdot$ to denote the action of $G$. For example, if $g \in G$ and $\xi \in V$, then $g \cdot \xi = g \circ \xi \circ g^{-1}$ and $g \cdot \phi_\xi = g \circ \phi_\xi \circ g^{-1}$.

*Algebras of type I.* These are subalgebras of $\mathcal{S}$ generated by elements of the form $\sum_i \partial_{w_i} \circ p_i \phi_{v_i}$, where, for each $i$, the stabilizer $G_{v_i}$ of $v_i \in V$ is parabolic with respect to $\Theta$, the function $p_i$ is $G_{v_i}$-invariant and holomorphic (or meromorphic, or rational or polynomial) and $w_i$ is the longest element in $(G/G_{v_i})^{short}$.

*Algebras of type II.* These are subalgebras of $\mathcal{S}$ generated by elements in the form $\sum_i \partial_{w_i} \cdot p_i \phi_{v_i}$, where $v_i$, $p_i$ and $w_i$ are as in type I (note the difference of using $\cdot$ in type $II$ instead of $\circ$ in type $I$).

Let **A** be an algebra of type $I$. Denote by $\rfloor$ the subgroup of $V$ generated by all possible $g \cdot v_i$, where $g \in G$ and $v_i$ appears in a generator of **A**, see above.

**Proposition 23.** *Let $E = \sum_i \partial_{w_i} \circ p_i \phi_{v_i}$ be a generator of the algebra **A**. If all $p_i$ are holomorphic in $V$, then*

$$E(\mathfrak{O}^G) \subset \mathfrak{O}^G.$$

*Proof.* Take a simple reflection $\tau \in \Theta$. Then

$$(10) \qquad (id - \tau) \circ \partial_{w_i} \circ p_i \phi_{v_i} = \gamma_\tau \partial_\tau \circ \partial_{w_i} \circ p_i \phi_{v_i}.$$

Since $w_i$ is the longest element in $(G/G_{v_i})^{short}$, the operator $\partial_\tau \circ \partial_{w_i}$ is either zero or can be written as $\partial_u \circ \partial_s$, where $\partial_s \in G_{v_i}$. Therefore the right hand side of (10) is identically zero on $\mathfrak{M}^G$. Hence, for any $F \in \mathfrak{M}^G$ and $g \in G$, we have $g \circ \partial_{w_i} \circ p_i \phi_{v_i}(F) = \partial_{w_i} \circ p_i \phi_{v_i}(F)$ implying $\partial_{w_i} \circ p_i \phi_{v_i}(\mathfrak{M}^G) \subset \mathfrak{M}^G$.

Further, we have $\partial_\tau(\mathfrak{O}) \subset \mathfrak{O}$. Indeed, let us take $f_x \in \mathcal{O}_x$ and consider $\partial_\tau(f_x)$. If $\tau(x) = x$, then $\gamma_\tau(x) = 0$. In this case $\gamma_\tau$ is a divisor of $f_x - \tau(f_x) \in \mathcal{O}_x$.

Therefore, $\partial_\tau(f_x) \in \mathcal{O}_x$. If $\tau(x) \neq x$, then $\gamma_\tau(x) \neq 0$. Hence $f_x/\gamma_\tau \in \mathcal{O}_x$ and $\tau(f_x)/\gamma_\tau \in \mathcal{O}_{\tau(x)}$.                                                                       $\square$

Let $\mathbf{A}$ be an algebra of type $I$ and $\mathbf{B}$ be an algebra of type $II$. Assume that for each generator $E = \sum_i \partial_{w_i} \circ p_i \phi_{v_i}$ of $\mathbf{A}$ there is a generator $E' = \sum_i \partial_{w_i} \cdot p_i \phi_{v_i}$ of $\mathbf{B}$ and vice versa. The next lemma describes when the actions of $E$ and $E'$ coincide.

**Lemma 24.** *Assume that all $p_i$ are holomorphic. We have the equality of operators*

$$E|_{\mathfrak{O}^G} = E'|_{\mathfrak{O}^G}.$$

*Therefore, the actions of algebras $\mathbf{A}$ and $\mathbf{B}$ as above on $\mathfrak{O}^G$ coincide.*

*Proof.* Consider first the operators $\partial_\rho \circ f\phi_x$ and $\partial_\rho \cdot f\phi_x$, where $f \in \mathcal{M}$ is any meromorphic function and $\rho \in G$ is any (not necessary longest) element with reduced expression $\rho = \tau_1 \tau_2 \cdots \tau_k$. Let us prove, by induction on $k$, that

$$\partial_\rho \circ f\phi_x|_{\mathfrak{M}^G} = \partial_\rho \cdot f\phi_x|_{\mathfrak{M}^G}.$$

For $k = 1$, the claim is obvious. To establish the induction step, we have

$$\partial_{\tau_1} \circ \cdots \circ \partial_{\tau_k} \circ f\phi_x|_{\mathfrak{M}^G} = \partial_{\tau_1} \circ \cdots \circ \partial_{\tau_{k-1}} \circ (\partial_{\tau_k} \cdot f\phi_x)|_{\mathfrak{M}^G} =$$
$$\partial_{\tau_1} \circ \cdots \circ \partial_{\tau_{k-1}} \circ (f/\gamma_{\tau_k}\phi_x - (\tau_k \cdot f)/\gamma_{\tau_k}\tau_k \cdot \phi_x)|_{\mathfrak{M}^G} =$$
$$\partial_{\tau_1} \circ \cdots \circ \partial_{\tau_{k-1}} \cdot (f/\gamma_{\tau_k}\phi_x - (\tau_k \cdot f)/\gamma_{\tau_k}\tau_k \cdot \phi_x)|_{\mathfrak{M}^G} =$$
$$\partial_{\tau_1} \cdots \partial_{\tau_k} \cdot f\phi_x|_{\mathfrak{M}^G}.$$

The result now follows from Proposition 23.                                     $\square$

7.2. **Structure theorem for rational Galois order.** In this section we assume that $G$ is a reflection group on $V \simeq \mathbb{C}^n$, where $\Phi$ is a root system with basis $\Psi$ and $\Theta$ is the set of corresponding simple reflections (cf. Subsection 4.1). We have the decomposition $\Phi = \Phi^+ \cup \Phi^-$ corresponding to $\Psi$. Consider the following product of linear functions on $V$:

$$\Delta := \prod_{x \in \Phi^+} \gamma_x,$$

where $\gamma_x(v) = (x, v)$, for any $v \in V$, see Section 4.1. We have $\sigma_x \cdot \Delta = -\Delta$, for any simple reflection $\sigma_x \in \Theta$. If $G = S_n$, then $\Delta$ may be identified with the classical Vandermonde determinant.

Let us take $v \in V$ such that the stabilizer $G_v$ is parabolic in $G$ with respect to $\Theta$. Denote by $\Delta'$ the product of $\gamma_x$, where $\sigma_x$ is a reflection in $G_v$. Let us take also a polynomial (or a holomorphic function) $p'$ and let $w$ be the longest element in $(G/G_v)^{short}$.

Consider an element of the form $\sum_{\tau \in G} \tau \cdot (\frac{p'}{\Delta}\phi_v)$ from a rational Galois order $\mathcal{A}$, see Subsection 4.2. We always can choose $p'$ such that it satisfies

$$\tau \cdot p' = \chi(\tau)p', \quad \text{where} \quad \chi(\tau) := \frac{\tau \cdot \Delta}{\Delta}, \quad \text{for} \quad \tau \in G_v.$$

Therefore we have $p' = \Delta'p$, where $p$ is a $G_v$-invariant polynomial or a holomorphic function (cf. Subsection 4.2). In other words, if $\Phi' \subset \Phi$ is the root subsystem corresponding to $G_v$, then

$$\Delta' := \prod_{x \in \Phi'^+} \gamma_x,$$

where $\Phi'^+$ is the subsystem of positive roots generated by $\Psi \cap \Phi'$. If $w_0$ is the longest element in $G$, then we have the following equality on global rational functions

$$(11) \qquad \partial_{w_0} = \sum_{\tau \in G} \tau \cdot \frac{1}{\Delta},$$

see [Hill, Section IV, Proposition 1.6]. From Dedekind's Theorem it follows that the operators (11) are equal as elements in $\mathcal{S}$. Therefore we have

$$(12) \qquad \partial_{w_0}|_{\mathfrak{O}} = \sum_{\tau \in G} \tau \cdot \frac{1}{\Delta}|_{\mathfrak{O}},$$

The following theorem generalizes [EMV, Proposition 7].

**Theorem 25** (Structure Theorem)**.**

(a) *We have*

$$(13) \qquad \sum_{\tau \in G} \tau \cdot \frac{\Delta'}{\Delta} p\phi_v\big|_{\mathfrak{O}^G} = a\partial_w \circ p\phi_v\big|_{\mathfrak{O}^G},$$

*where $a \neq 0$ is a scalar.*

(b) *Let $G'$ be any subgroup in $G$ which is parabolic with respect to $\Theta$. Then*

$$(14) \qquad \sum_{\tau \in G} \tau \cdot \frac{\Delta'}{\Delta} p\phi_v\big|_{\mathfrak{O}^G} = \sum_{s=1}^{k} \partial_{w_s} \circ t_s \phi_{v_s}|_{\mathfrak{O}^G},$$

*where $w_s \in G'/G'_{v_s}$ is the longest reduced element and $t_s$ are rational functions defined in Weyl chambers and at $\ker \gamma_x$, where $x \in \Phi^+$ and $\sigma_x \in G'$.*

*Proof.* Note that we always can choose $v$ such that $G_v$ is parabolic with respect to $\Theta$. Using (12), we have

$$\sum_{\tau \in G} \tau \cdot \frac{\Delta'}{\Delta} p\phi_v\big|_{\mathfrak{O}^G} = \sum_{\tau \in G} \tau \cdot \frac{1}{\Delta} \Delta' p\phi_v|_{\mathfrak{O}^G} = \partial_{w_0} \circ \Delta' p\phi_v|_{\mathfrak{O}^G} =$$

$$\partial_w \circ \partial_{w_0'} \circ \Delta' p\phi_v|_{\mathfrak{O}^G} = \partial_w \circ p\phi_v \partial_{w_0'} \circ (\Delta')|_{\mathfrak{O}^G} = a\partial_w \circ p\phi_v|_{\mathfrak{O}^G},$$

where $w_0'$ is the longest element in $G_v$. This implies claim (a).

To prove claim (b), let $G'$ be a subgroup in $G$ which is parabolic with respect to $\Theta$. We have

$$\sum_{\tau \in G} \tau \cdot \frac{\Delta'}{\Delta} p\phi_v = \sum_{\tau \in G'} \tau \cdot \sum_{s=1}^{k} \tau_s \cdot \frac{\Delta'}{\Delta} p\phi_v.$$

Here $\tau_s \in G' \backslash G$ is a coset representative and $k = |G' \backslash G|$. Note that we can choose the representatives $\tau_s$ such that $\phi_{v_s} := \tau_s \cdot \phi_v$ has a parabolic stabilizer $G'_{v_s}$ with respect to $\Theta$.

Denote by $\tilde{\Delta}$ the product of $\gamma_x$, where $x \in \Phi^+$ and $\sigma_x \in G'$, and by $\tilde{\Delta}_s$ the product of $\gamma_x$, where $x \in \Phi^+$ and $\sigma_x \in G'_{v_s}$. Clearly, $\tilde{\Delta}$ is a divisor of $\Delta$ and $\tilde{\Delta}_s$ is a divisor of $\tau_s \cdot \Delta'$. Denote by $l(\tau_s)$ the length of $\tau_s$. We have

$$\tau_s \cdot \frac{\Delta'}{\Delta} p\phi_v = \frac{(-1)^{l(\tau_s)} \tau_s \cdot \Delta'}{\Delta} p_s \phi_{v_s} = \frac{\tilde{\Delta}_s}{\tilde{\Delta}} \frac{(-1)^{l(\tau_s)} (\tau_s \cdot \Delta')/\tilde{\Delta}_s}{\Delta/\tilde{\Delta}} p_s \phi_{v_s},$$

where $p_s := \tau_s \cdot p$. We put

$$t_s := \frac{(-1)^{l(\tau_s)} (\tau_s \cdot \Delta')/\tilde{\Delta}_s}{\Delta/\tilde{\Delta}} p_s.$$

We see that $t_s$ are rational functions defined in Weyl chambers and at $\ker \gamma_x$, where $x \in \Phi^+$ and $\sigma_x \in G'$.

Using (a), we obtain

$$\sum_{s=1}^{k} \sum_{\tau \in G'} \tau \cdot \frac{\tilde{\Delta}_s}{\tilde{\Delta}} t_s \phi_{v_s}|_{\mathfrak{O}^G} = \sum_{s=1}^{k} a_s \partial_{w_s} \circ t_s \phi_{v_s}|_{\mathfrak{O}^G},$$

where $a_s \neq 0$ are scalars and $w_s \in G'/G'_{v_s}$ are longest element in the set of shortest coset representatives. $\qquad\square$

In the case $G = S_n$, formula (13) was conjectured by the second author in [Vi3] and later independently proved in [RZ, EMV]. It was extended to an arbitrary reflection group in [FGRZ] where it was also shown that it plays a crucial role in construction and study of simple Gelfand-Zeitlin modules for rational Galois orders.

Consider a rational Galois order $\mathcal{A}$ as above. Fix $v \in V$ and denote by $H$ the subgroup in $G$ generated by all stabilizers $G_v$, where $v \in \mathbf{J} \cdot v$. In the proof of Theorem 25 we obtained the following expression

$$A = \sum_{\tau \in G} \tau \cdot \frac{\Delta'}{\Delta} p \phi_v = \sum_{s=1}^{k} \sum_{\tau \in H} \tau \cdot \frac{\tilde{\Delta}_s}{\tilde{\Delta}} t_s \phi_{v_s},$$

where $\tilde{\Delta}$ is the product of $\gamma_x$, for $x \in \Phi^+$ and $\sigma_x \in H$, and $\tilde{\Delta}_s$ the product of $\gamma_x$, for $x \in \Phi^+$ and $\sigma_x \in H_{v_s}$. By Lemma 16, the operator $A$ preserves the vector spaces $\mathfrak{O}^G$ and $\mathfrak{O}^H$. Therefore we have the families of modules given by Theorem 10. In particular, we have the $\mathcal{A}$-modules $M^*(G, (G \ltimes \mathbf{J}) \cdot v)$ and $M^*(H, \mathbf{J} \cdot v)$. A basis of these modules is constructed in Theorem 19. Using Theorem 25, we get the following fairly explicit result.

**Corollary 26.** *With respect to the basis of Theorem 19, the action of $\mathcal{A}$ on the modules $M^*(G, (G \ltimes \mathbf{J}) \cdot v)$ or $M^*(H, \mathbf{J} \cdot v)$ can be computed using the following formula:*

$$(ev_0 \circ \partial_w \circ \phi_\xi) \circ A|_{\mathfrak{O}^G} = (ev_0 \circ \partial_w \circ \phi_\xi) \circ (\partial_w \circ p\phi_v)|_{\mathfrak{O}^G} =$$

(15)
$$\sum_{s=1}^{k} a_s ev_0 \circ \partial_w \circ \partial_{w_s} \circ (\phi_\xi \cdot t_s) \phi_{\xi \circ v_s}|_{\mathfrak{O}^G},$$

*where $a_s \in \mathbb{C} \setminus \{0\}$, cf. Theorem 25. Here $t_s$ and $v_s$ correspond to $G' = G_\xi$.*

## 8. A CONSTRUCTION OF SIMPLE MODULES AND SUFFICIENT CONDITIONS FOR SIMPLICITY

8.1. **Canonical simple Harish-Chandra modules.** In this section we construct a family of simple modules which we will call *canonical Harish-Chandra modules*. This construction generalizes the corresponding constructions from [EMV] and [Har]. Assume that $V$ is a complex-analytic Lie group, $G$ is a finite group, $\mathbf{J} \subset V$ is a subgroup and $v \in V$. Let $\mathcal{A} \subset S(V)^G$ be a subalgebra containing $H^0(V, \mathcal{O}^G)$, which preserves the vector space $\mathfrak{O}^G|_{(G \ltimes \mathbf{J}) \cdot v}$. Consider the $\mathcal{A}$-module $M(G, (G \ltimes \mathbf{J}) \cdot v)$. Denote by $N_{\bar{w}}$, where $\bar{w} = G \cdot w$ for some $w \in (G \ltimes \mathbf{J}) \cdot v$, the submodule in $M(G, (G \ltimes \mathbf{J}) \cdot v)$ generated by $\tilde{1}_{\bar{w}} \in \mathbb{E}_{\bar{w}}^G$, where $\tilde{1}_{\bar{w}}$ is the class generated by the constant function 1.

**Proposition 27.** *Assume that $H^0(V, \mathcal{O}^G)$ separates orbits in $(G \ltimes \mathbf{J}) \cdot v$ and that the $H^0(V, \mathcal{O}^G)$-module $\mathbb{E}_{\bar{w}}^G$ is generated by $\tilde{1}_{\bar{w}}$. Then $N_{\bar{w}}$ has a unique maximal submodule.*

*Proof.* The unique maximal submodule is the sum of all submodules $N'$ in $N_{\bar{w}}$ such that $N' \cap \mathbb{E}_{\bar{w}}^G \subset \mathfrak{n}_{\bar{w}} \cdot \tilde{1}_{\bar{w}}$, where $\mathfrak{n}_{\bar{w}} \subset H^0(V, \mathcal{O}^G)$ is the ideal of all $G$-invariant functions that are equal to 0 at $\bar{w}$. $\qquad\square$

The quotient of $N_{\bar{w}}$ by its unique maximal submodule is denoted $L_{\bar{w}}$ and is called the *canonical simple Harish-Chandra module associated to* $\bar{w}$.

8.2. **Standard algebras of type** $\mathbb{A}$. Let $V$ and $G$ be as in Subsection 2.3 and 2.4. As we have seen in Corollary 21, a finitely generated over $H^0(V, \mathcal{O}^G)$ standard algebra of type $\mathbb{A}$ that preserves the vector space $\mathfrak{O}^G$ is a rational Galois order. Consider a special case of such algebras, the algebra $\mathcal{A}$ that is generated by $H^0(V, \mathcal{O}^G)$ and by the following elements:

$$E_i = \sum_{g \in G} g \cdot (\frac{\Delta' H_i^E}{\Delta} \phi_{\xi_i^a}), \quad F_i = \sum_{g \in G} g \cdot (\frac{\Delta' H_i^F}{\Delta} \phi_{\xi_i^{-a}}), \quad i = 1, \ldots, n,$$

where $a \in \mathbb{C} \setminus \{0\}$, $H_i^E, H_i^F$ are holomorphic functions in $V$ such that we have $G_{H_i^E} = G_{H_i^F} = G_{\xi_i^a}$, for $i = 1, \ldots, n$, and $\Delta$ and $\Delta'$ are as in Subsection 7.2.

Let $\beth$ be a subgroup in $V$ generated by $\xi_i^a$, where $i = 1, \ldots, n$, and $v' \in V$ be any point. In this case, for $G_{\beth \cdot v'}$ we have an analogue of Lemma 22. That is, there exists $v \in \beth \cdot v'$ such that $G_{\beth \cdot v} = G_v$. The module $M^*(G_{\beth \cdot v}, \beth \cdot v) = M^*(G_v, \beth \cdot v)$ was studied in [EMV, Theorem 11]. More precisely, in [EMV] the following theorem was proved.

**Theorem 28.** [EMV, Theorem 11] *Assume that* $H_i^E, H_i^F$, *where* $i = 1, \ldots, n$, *have no zeros on* $\beth \cdot v$. *Then the* $\mathcal{A}$-*module* $M^*(G_v, \beth \cdot v)$ *is irreducible.*

In [EMV], this theorem was proved only for a special choice of functions $H_i^E, H_i^F$. However exactly the same proof as in [EMV] works for any functions $H_i^E, H_i^F$. This fact was noticed in [FGRZ, Theorem 8.5], where the result [EMV, Theorem 11] was discussed in detail.

8.3. **Regular modules.** Assume that $V$ is a complex-analytic Lie group, $\beth \subset V$ is a subgroup and $v \in V$. Let $\mathcal{A} \subset S(V)$ be a finitely generated, over $H^0(V, \mathcal{O})$, subalgebra which preserves the vector space $\mathfrak{O}|_{\beth \cdot v} = \mathfrak{O}^e|_{\beth \cdot v}$. We denote by $\Gamma$ an oriented graph that is defined in the following way. The vertices of $\Gamma$ are all points from $\beth \cdot v$ and we connect $x$ and $y$ with an arrow $x \to y$ if there exists $A = \sum f_i \phi_{\xi_i} \in \mathcal{A}$ and $i_0$ such that $\phi_{\xi_{i_0}}(x) = y$ and $f_{i_0}(y) \neq 0$. Note that, in this case, all $f_i$ are holomorphic in $\beth \cdot v$.

**Proposition 29.** *Assume that* $\mathcal{A} \subset S(V)$ *is a finitely generated over* $H^0(V, \mathcal{O})$ *subalgebra that preserves the vector space* $\mathfrak{O}|_{\beth \cdot v} = \mathfrak{O}^e|_{\beth \cdot v}$, $H^0(V, \mathcal{O})$ *separates points of* $\beth \cdot v$ *and* $\Gamma$ *is connected as an oriented graph. Then the* $\mathcal{A}$-*module* $M(\{e\}, \beth \cdot v)$ *is irreducible.*

*Proof.* First of all, we note that the $\mathcal{A}$-module $M(\{e\}, \beth \cdot v)$ is a direct sum of $\mathbb{E}_\xi = \phi_\xi(\mathcal{O}_e^{\{e\}}/(\mathcal{O}_e^{\{e\}} \cap \mathcal{J}_e)) \simeq \mathbb{C}$. In other words, $M(\{e\}, \beth \cdot v)$ is a vector space of all finite linear combinations of points $v_s \in \beth \cdot v$.

Let $\sum a_s v_s$ be an element in a submodule $N$. Since $H^0(V, \mathcal{O})$ separates points of $\beth \cdot v$, we see that $v_s \in N$ for any $s$. Let us take a submodule $N' \subset M(\{e\}, \beth \cdot v)$ that contains a point $x \in \beth \cdot v$. Further let $y \in \beth \cdot v$. Since $\Gamma$ is connected as an oriented graph, there exists a sequence

$$x_0 = x, x_1, \ldots, x_{n-1}, x_n = y$$

such that the path $x_0 \to x_1 \to \cdots \to x_n$ connects $x$ and $y$. Assume, by induction, that we proved that $x_{s-1} \in N$. From our assumptions, there exists $A = \sum f_i \phi_{\xi_i} \in \mathcal{A}$ and $i_0$ such that $\phi_{\xi_{i_0}}(x_{s-1}) = x_s$ and $f_{i_0}(x_s) \neq 0$. We have $A(x_{s-1}) = \sum f_i(x_s)\phi_{\xi_i}(x_{s-1}) \in N$. Therefore $x_s \in N'$.    $\square$

Assume that $\mathcal{A}$ is generated by $H^0(\beth \cdot v, \mathcal{O})$ and, additionally, by a finite set of elements $E_i = \sum f_{ij}\phi_{\xi_{ij}}$. Let $\beth$ be the group generated by all $\xi_{ij}$. Denote by $Q(\xi_{ij})$ the monoid generated by all $\xi_{ij}$.

**Proposition 30.** *Assume that*

(i) $H^0(V, \mathcal{O})$ *separates points of* $\beth \cdot v$;

(ii) $Q(\xi_{ij}) = \beth$;

(iii) *every $f_{ij}$ has no zeros at $\beth \cdot v$.*

*Then the $\mathcal{A}$-module $M(\{e\}, \beth \cdot v)$ is irreducible.*

*Proof.* Due to assumptions (i) and (iii), to be able to use Proposition 29, we only need to show that $\Gamma$ is connected. The latter, however, follows directly from assumption (ii). Therefore the claim follows from Proposition 29    $\square$

8.4. **Singular modules.** Assume that $V$ is a complex-analytic Lie group, $\beth \subset V$ is a subgroup and $v \in V$. Let $\mathcal{A} \subset S(V)^{G_{\beth \cdot v}}$ be a finitely generated over $H^0(V, \mathcal{O}^{G_{\beth \cdot v}})$ subalgebra which preserves the vector space $\mathfrak{O}^{G_{\beth \cdot v}}|_{\beth \cdot v}$. Assume that $H^0(V, \mathcal{O}^{G_{\beth \cdot v}})$ separates $G_{\beth \cdot v}$-orbits in $\beth \cdot v$ and that the $H^0(V, \mathcal{O}^{G_{\beth \cdot v}})$-module $\mathbb{E}_{\xi}^{G_{\beth \cdot v}}$, see (3), is generated by a non-trivial constant $c \in \mathbb{C} \setminus \{0\}$, for any $\xi \in \beth \cdot v$.

We denote by $\Gamma$ the oriented graph defined as follows:

- the vertices of $\Gamma$ are all $G_{\beth \cdot v}$-orbits in $\beth \cdot v$;

- for two orbits $\bar{\xi}$ to $\bar{\eta}$, there is an oriented arrow from $\bar{\xi}$ to $\bar{\eta}$, if there exists $A = \sum f_i \phi_{\xi_i} \in \mathcal{A}$ such that the function $H := \sum\limits_{(g,h,i)\in\Lambda} h \cdot f_i$, cf. (2) for $X = 1$, exists and is not equal to 0 at $\eta$. (Note that the function $H$ depends on the orbits $\bar{\xi}$ and $\bar{\eta}$ and on the element $A$.)

**Theorem 31.** *In the above situation, we have:*

(i) *For every $\bar{\xi}$, the module $M(G_{\beth \cdot v}, \beth \cdot v)$ has a unique submodule $N(\bar{\xi})$ which is maximal, with respect to inclusions, among all submodules of $M(G_{\beth \cdot v}, \beth \cdot v)$ that do not contain $\mathbb{E}_{\bar{\xi}}^{G_{\beth \cdot v}}$.*

(ii) *If $\Gamma$ is connected as an oriented graph, then $M(G_{\beth \cdot v}, \beth \cdot v)$ is generated by the class of a non-trivial constant function and also has a unique maximal submodule.*

*Proof.* Claim (i) follows from Proposition 29. Further we have

$$M(G_{\beth \cdot v}, \beth \cdot v) = \bigoplus_{\bar{\xi} \in \beth \cdot v / G_{\beth \cdot v}} \mathbb{E}_{\bar{\xi}}^{G_{\beth \cdot v}},$$

see (3). Denote by $N$ the $\mathcal{A}$-submodule generated by the class $c_{\bar{\xi}} \in \mathbb{E}_{\bar{\xi}}^{G_{\beth \cdot v}}$ of a non-trivial constant function $c$. Let $\bar{y} \subset \beth \cdot v$. Since $\Gamma$ is connected as an oriented graph, there exists a sequence

$$\bar{x}_0 = \bar{x}, \bar{x}_1, \ldots, \bar{x}_{n-1}, \bar{x}_n = \bar{y}$$

such that the path $\bar{x}_0 \to \bar{x}_1 \to \cdots \to \bar{x}_n$ connects $\bar{x}$ and $\bar{y}$. Assume, by induction, that we proved that $1_{\bar{x}_{s-1}} \in N$. From our assumptions, there is $A = \sum f_i \phi_{\xi_i} \in \mathcal{A}$ that sends $1_{\bar{x}_{s-1}}$ to $a_{\bar{x}_{s-1}}$ with a constant non-trivial representative $a \neq 0$, see (2). This implies the first part of claim (ii) and the second part of claim (ii) follows from the first part of claim (ii) and claim (i). $\qquad\square$

Let $M(\bar{\xi})$ denote the $\mathcal{A}$-submodule of $M(G_{\beth \cdot v}, \beth \cdot v)$ generated by $\mathbb{E}^{G_{\beth \cdot v}}_{\bar{\xi}}$. The simple quotient $M(\bar{\xi})/N(\bar{\xi})$, whose existence is guaranteed by Theorem 31(i), is the *canonical* Harish-Chandra $\mathcal{A}$-module associated to $\bar{\xi}$.

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