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Abstract

We consider 3-dimensional Sklyanin algebras S, which are quadratic algebras over a field $\mathbb{K}$ given by generators x, y, z and relations $pxy+qyx+rzz = 0$, $pyz+qzy+rxx = 0$ and $pzx+qxz+ryy = 0$, where $p, q, r \in \mathbb{K}$. This class of algebras has enjoyed much attention. In particular, using tools from algebraic geometry, Feigin, Odesskii [9], and Artin, Tate and Van Den Bergh [3], showed that if at least two of the parameters p, q and r are non-zero and at least two of three numbers p^3, q^3 and r^3 are distinct, then S is Artin–Schelter regular. More specifically, S is Koszul and has the same Hilbert series as the algebra of commutative polynomials in 3 indeterminates. The authors [7] have previously proved the same result using only combinatorial and algebraic techniques. However our proof was no less complicated than the one based on algebraic geometry. In this paper we exhibit a linear substitution after which it becomes possible to determine the leading monomials of a reduced Gröbner basis for the ideal of relations of S (without passing to a suitable S-module as was the case in our previous proof). We also find out explicitly (in terms of parameters) which Sklyanin algebras are isomorphic. The only drawback of the new technique is that it fails for characteristic 3.

MSC: 17A45, 16A22

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1 Introduction

Throughout this paper $\mathbb{K}$ is an arbitrary field of characteristic different from 3. If B is a graded algebra, let B_m denotes the m^{th} graded component of the algebra B. If V is an n-dimensional vector space over $\mathbb{K}$, then $F = F(V)$ is the tensor algebra of V. For any choice of a basis $x_1, \dots, x_n$ in V, F is naturally identified with the free $\mathbb{K}$-algebra with the generators $x_1, \dots, x_n$. We always consider the degree grading on the free algebra F: the m^{th} graded component of F is $V^m = V^{\otimes m}$. If R is a subspace of the n^2-dimensional space $V^2 = V \otimes V$, then the quotient of F by the ideal I generated by R is called a *quadratic algebra* and denoted $A(V, R)$. This is a standard notation, used for example, in [10]. Since each quadratic algebra A is degree graded, we can consider associated generating function, - its Hilbert series

$$H_A(t) = \sum_{j=0}^{\infty} \dim_{\mathbb{K}} A_j \, t^j.$$

Another concept playing an important role in this paper is Koszulity. For a quadratic algebra $A = A(V, R)$, the augmentation map $A \to \mathbb{K}$ equips $\mathbb{K}$ with the structure of a commutative graded A-bimodule. The algebra A is called *Koszul* if $\mathbb{K}$ as a graded right A-module has a free resolution $\dots \to M_m \to \dots \to M_1 \to A \to \mathbb{K} \to 0$ with the second last arrow being the augmentation map and the matrices of the maps $M_m \to M_{m-1}$ with respect to some free bases consisting of homogeneous elements of degree 1.

One of the important features of Sklyanin algebras, as of many other algebras, originated in physics, is that they are potential algebras (or in other terminology vacualgebras, or Jacobi algebras). The notion of a *noncommutative potential* was introduced by Kontsevich in [8, 5]. We make use of an equivalent definition from [4]. An element F of $\mathbb{K}\langle x_1,\dots,x_n\rangle$ is called *cyclicly invariant* if F is invariant for the linear map $C:\mathbb{K}\langle x_1,\dots,x_n\rangle\to\mathbb{K}\langle x_1,\dots,x_n\rangle$ defined by its action on monomials as follows: $C(1)=1$ and $C(x_ju)=ux_j$ for every j and every monomial u. The symbol $\mathbb{K}^{\rm cyc}\langle x_1,\dots,x_n\rangle$ stands for the vector space of all cyclicly invariant elements of $\mathbb{K}\langle x_1,\dots,x_n\rangle$. We also consider the linear maps $\frac{\delta}{\delta x_j}:\mathbb{K}\langle x_1,\dots,x_n\rangle\to\mathbb{K}\langle x_1,\dots,x_n\rangle$ defined by their action on monomials u as follows: $\frac{\delta u}{\delta x_j}=0$ if u does not start with x_j and $\frac{\delta u}{\delta x_j}=v$ if $u=x_jv$. A *potential algebra* A_F defined by the potential $F\in\mathbb{K}^{\rm cyc}\langle x_1,\dots,x_n\rangle$ is a $\mathbb{K}$-algebra given by the generators $x_1,\dots,x_n$ and the relations $\frac{\delta F}{\delta x_j}=0$ for $1\leqslant j\leqslant n$. For the sake of convenience, we consider the onto linear map $G\mapsto G^{\circlearrowright}$ from $\mathbb{K}\langle x_1,\dots,x_n\rangle$ to $\mathbb{K}^{\rm cyc}\langle x_1,\dots,x_n\rangle$ defined by its action on homogeneous elements by $u^{\circlearrowright}=C(u)+\dots+C^du$, where d is the degree of u. For example, $x^{4\circlearrowright}=4x^4$ and $x^2y^{\circlearrowright}=x^2y+xyx+yx^2$. There is a number of generalizations of potential algebras, see, for instance, [4].

Recall that if $(p,q,r)\in\mathbb{K}^3$, the *Sklyanin algebra* $Q^{p,q,r}$ is the quadratic algebra over $\mathbb{K}$ with generators x,y,z given by 3 relations

$$pyz+qzy+rxx=0,\quad pzx+qxz+ryy=0,\quad pxy+qyx+rzz=0.$$

and it is a potential algebra with the potential $r(x^3+y^3+z^3)+pxyz^{\circlearrowright}+qxzy^{\circlearrowright}$.

Artin, Tate and Van den Bergh [3, 2], and Feigin, Odesskii [9], considered a certain family of infinite dimensional representations of Sklyanin algebra. Namely, they used graded modules with all graded components being one-dimensional, known as pointed modules. The geometric interpretation of the space of such modules is in the core for most of their arguments. Artin, Tate and Van den Bergh showed that if at least two of the parameters p, q and r are non-zero and the equality $p^3=q^3=r^3$ fails, then $Q^{p,q,r}$ is Artin–Shelter regular. More specifically, $Q^{p,q,r}$ is Koszul and has the same Hilbert series as the algebra of commutative polynomials in three variables.

It became commonly accepted that it is impossible to obtain the same results by purely algebraic and combinatorial means like the Gröbner basis technique, see, for instance, comments in [13]. We have dispelled this notion in [7]. However our previous proof is rather complicated. It uses a construction of a Gröbner basis in a suitable one-sided module over $Q^{p,q,r}$ and has quite a number of cases to consider. In this paper we exhibit a linear substitution after which it becomes possible to determine the leading monomials of a reduced Gröbner basis for the ideal of relations of S itself.

In section 3 we explicitly classify all Sklyanin algebras (in terms of parameters) up to isomorphism.

The only drawback of the new technique is that it fails if the characteristic of the ground field equals 3.

We say that a Sklyanin algebra $S^{p,q,r}$ is *degenerate* if either $p^3=q^3=r^3$ or there are at least two zeros among p, q and r.

Theorem 1.1. *The algebra $A=Q^{p,q,r}$ is Koszul for any $(p,q,r)\in\mathbb{K}^3$. The equality $H_A=(1-t)^{-3}$ holds if and only if the Sklyanin algebra A is non-degenerate.*

We stress again that the above theorem is essentially one of the main results in [3] with a different proof provided in [7]. Our new proof is however much easier and shorter. Concerning another result of this paper, where we give an explicit description in terms of parameters, which Sklyanin algebras are isomorphic as graded algebras. The question when two different Sklyanin algebras are isomorphic is touched in [3]. The authors associate to each Sklyanin algebra certain geometric data (an elliptic curve and its automorphism) and observe that Sklyanin algebras are isomorphic precisely, when the same happens to geometric data associated to them. This certainly may be useful on many occasions, however this characterization of isomorphic Sklyanin algebras is hard to use when the only data we have are just the triples of parameters (possible, of course, but rather inconvenient). In this section we find out explicitly in terms of just the values of parameters which Sklyanin algebras are isomorphic to each other. Note that according to Proposition 2.3 below, the answer does not change if we allow all algebra isomorphisms instead of just graded (=linear substitutions in our case) ones. Keeping this in mind, we just deal with isomorphisms in the category of graded algebras. The next theorem yields a complete description of isomorphic Sklyanin algebras. Note that monomiality of degenerate Sklyanin algebras was first noticed by Smith [12], while their Koszulity was noticed in [13].

Theorem 1.2. *Let $\mathbb{K}$ be a field such that there is $\theta\in\mathbb{K}$ satisfying $\theta^3=1\neq\theta$ (a non-trivial cubic root of 1). Then the following statements hold true:*

(I1) *A degenerate Sklyanin algebra $A=Q^{p,q,r}$ is isomorphic to one of the following three pairwise non-isomorphic monomial algebras $Q^{0,0,0}=\mathbb{K}\langle x,y,z\rangle$, $Q^{1,0,0}$ given by the relations $xy=zx=yz=0$ or $Q^{0,0,1}$ given by the relations $xx=yy=zz=0$. Furthermore,*

  (I1.1) *A is isomorphic to $Q^{0,0,0}$ if and only if $p=q=r=0$;*

  (I1.2) *A is isomorphic to $Q^{1,0,0}$ if and only if either $r=pq=0$ and $(p,q)\neq(0,0)$ or $p^3=q^3=r^3\neq 0$ and $p\neq q$;*

  (I1.3) *A is isomorphic to $Q^{0,0,1}$ if and only if either $p=q=0$ and $r\neq 0$ or $p=q\neq 0$ and $p^3=r^3$.*

  Finally, a degenerate Sklyanin algebra is never isomorphic to a non-degenerate one.

(I2) *A non-degenerate Sklyanin algera $A=Q^{p,q,r}$ is isomorphic to a quantum polynomial algebra B^α given by the relations $xy=\alpha yx$, $zx=\alpha xz$ and $yz=\alpha zy$ with $\alpha\in\mathbb{K}^*$ if and only if $r=0$ or $(p+q)^3+r^3=0$. More specifically, A is isomorphic to B^α with given $\alpha\in\mathbb{K}^*$ if and only if either $r=0$ and $\alpha\in\{-\frac{p}{q},-\frac{q}{p}\}$, or $(p+q)^3+r^3=0$ and $\alpha\in\{\theta\frac{p-\theta^2q}{p-\theta q},\theta^2\frac{p-\theta q}{p-\theta^2 q}\}$. Furthermore, B^α and B^β are isomorphic if and only if either $\alpha=\beta$ or $\alpha\beta=1$.*

(I3) *A non-degenerate Sklyanin algera $A=Q^{p,q,r}$ non-isomorphic to any quantum polynomial algebra B^α with $\alpha\in\mathbb{K}^*$ satisfies $r\neq 0$ and $(p+q)^3+r^3\neq 0$ and therefore up to scaling of the parameters (dividing by r), (p,q,r) turns into $(a,b,1)$ with $(a,b)\in M$, where*

$$M=\{(a,b)\in\mathbb{K}^2:(a,b)\neq(0,0),\ (a+b)^3+1\neq 0,\ (a^3-1,b^3-1)\neq(0,0)\}.$$

If both (a,b) and (a',b') belong to M, then the Sklyanin algebras $Q^{a,b,1}$ and $Q^{a',b',1}$ are isomorphic if and only if (a,b) and (a',b') are in the same orbit of the group action on M generated by two maps

$$(a,b)\mapsto(\theta a,\theta b)\ \text{and}\ (a,b)\mapsto\Big(\tfrac{\theta a+\theta^2b+1}{a+b+1},\tfrac{\theta^2a+\theta b+1}{a+b+1}\Big).$$

This group is finite, consists of 24 elements (thus if $\mathbb{K}$ is infinite, for generic $(a,b)\in M$, there are exactly 23 other elements of M giving rise to an isomorphic Sklyanin algebra) and is isomorphic to $SL_2(\mathbb{Z}_3)$. The complete list of pairs $(a',b')\in M$ such that for a given $(a,b)\in M$, $Q^{a,b,1}$ and $Q^{a',b',1}$ are isomorphic is as follows:

(θ^ja,θ^jb) and (θ^jb,θ^ja) with $j\in\{0,1,2\}$;

$\Big(\frac{\theta^ja+\theta^kb+\theta^m}{a+b+\theta^n},\frac{\theta^ka+\theta^jb+\theta^m}{a+b+\theta^n}\Big)$ with $n\in\{0,1,2\}$ and $\{j,k,m\}=\{0,1,2\}$.

Throughout this paper, we **always** use the **left-to-right degree-lexicographical order** on monomials in x, y and z assuming $x>y>z$. We recall some general well-known facts on quadratic algebras in Section 2. Elementary observations concerning Sklyanin algebras are collected in Section 3. Most of them are borrowed from our paper [7]. We prove Theorem 1.1 in Section 4. Section 5 is devoted to the proof of Theorem 1.2, while the final Section 6 contains some extra comments.

2 General background

We shall use the following well-known facts, all of which can be found in [10]. A quadratic algebra $A = A(V, R)$ is called PBW if there are linear bases $x_1, \dots, x_n$ in V and $r_1, \dots, r_m$ in R and a well-ordering of monomials in x_j compatible with multiplication such that $r_1, \dots, r_m$ form a Gröbner basis of the ideal of relations of A. Here we follow notation from [10]. As a matter of warning, the term PBW is overused in the literature: we are aware of at least four pairwise non-equivalent definitions of PBW algebras. This is probably because all these notions generalizes in a different directions the PBW property of the universal enveloping of Lie algebra.

Note that every monomial quadratic algebra $A = A(V, R)$ is a PBW-algebra, in our sense (see also [10]), which we will fix once and for all.

Let us pick a basis $x_1, \dots, x_n$ in V, we get a bilinear form b on the free algebra $F = F(V)$ defined by $b(u, v) = \delta_{u,v}$ for every monomials u and v in the variables $x_1, \dots, x_n$. The algebra $A^! = A(V, R^\perp)$, where $R^\perp = \{u \in V^2 : b(r, u) = 0 \text{ for each } r \in R\}$, is called the *dual algebra* of A. Clearly, $A^!$ is a quadratic algebra in its own right. One easily sees that up to a graded algebra isomorphism, $A^!$ does not depend on the choice of a basis in V. We recall that

$$\begin{aligned}
&\text{every PBW-algebra is Koszul;}\\
&A \text{ is Koszul} \iff A^! \text{ is Koszul;}\\
&\text{if } A \text{ is Koszul, then } H_A(-t)H_{A^!}(t) = 1.
\end{aligned}$$

The latter property of A, when the Hilbert series satisfies the property: $H_A(-t)H_{A^!}(t) = 1$, we will call *numeric Koszulity.*

The following lemma proved in [7] allows us to prove Koszulity of Sklyanin algebras once we have computed their Hilbert series. We say that $u \in A = A(V, R)$ is a *right annihilator* if $Vu = \{0\}$ in A. A right annihilator u is *non-trivial* if $u \neq 0$.

Lemma 2.1. *Let $A = A(V, R)$ be a quadratic algebra such that $A^!_4 = \{0\}$, $A^!_3$ is one-dimensional and $wA^!_2 \neq \{0\}$ for every non-zero $w \in A^!_1$. Then the following statements are equivalent:*

(2.1.1) *A is Koszul;*

(2.1.2) *A has no non-trivial right annihilators and $H_A(-t)H_{A^!}(t) = 1$.*

Next, we stress upon a well-known fact that neither Koszulity nor the Hilbert series of a quadratic algebra $A = A(V, R)$ is sensitive to changing the ground field.

Remark 2.2. Fix the bases $x_1, \dots, x_n$ and $r_1, \dots, r_m$ in V and R respectively. Then $A = A(V, R)$ is given by generators $x_1, \dots, x_n$ and relations $r_1, \dots, r_m$. Let $\mathbb{K}_0$ be the subfield of $\mathbb{K}$ generated by the coefficients in the relations $r_1, \dots, r_m$ and B be the $\mathbb{K}_0$-algebra defined by the exact same generators $x_1, \dots, x_n$ and the exact same relations $r_1, \dots, r_m$. Then A is Koszul if and only if B is Koszul (see, for instance, [10]) and the Hilbert series of A and of B coincide. The latter follows from the fact that the Hilbert series depends only on the set of leading monomials of the Gröbner basis. In particular, replacing the original field $\mathbb{K}$ by its algebraic closure or by an even bigger field does not change the Hilbert series or Koszulity of A. On the other hand, the PBW-property is sensitive to changing the ground field [10].

Since our secondary task is to describe all isomorphic Sklyanin algebras, a natural question arises. Do we deal with isomorphism in the category of $\mathbb{K}$-algebras or consider only isomorphisms in the category of graded $\mathbb{K}$-algebras. Formally speaking, the answer may depend on the choice of the category. It does not do so in our case. This follows from the next elementary and very general result.

Proposition 2.3. *Let $A = A(V_1, R_1)$ and $B = A(V_2, R_2)$ be quadratic algebras over the same ground field $\mathbb{K}$ (any field of any characteristic is allowed here). Assume also that A and B are isomorphic as $\mathbb{K}$-algebras. Then they are isomorphic as graded $\mathbb{K}$-algebras.*

Proof. Let $A = A(V_1, R_1)$ and $B = A(V_2, R_2)$ be quadratic algebras over the same ground field $\mathbb{K}$. Let $x_1, \dots, x_n$ be a linear basis in V_1 and $y_1, \dots, y_m$ be a linear basis in V_2. Pick a linear basis $f_1, \dots, f_s$ in R_1 and a linear basis $g_1, \dots, g_t$ in R_2. Let $\varphi : A \to B$ be a $\mathbb{K}$-algebra isomorphism. The proof will be complete if we construct another algebra isomorphism $\psi : A \to B$ such that its restriction to V_1 is a linear isomorphism from V_1 onto V_2 (then ψ is a graded algebra isomorphism). Since $f_j \in V_1^2$, $f_j = \sum\limits_{1\leqslant k,p\leqslant n} a_{k,p}^{(j)} x_k x_p$ with $a_{k,p}^{(j)} \in \mathbb{K}$. Let $u^{(j)} = \varphi(x_j)$ for $1 \leqslant j \leqslant n$. As usual, for an element u of a graded algebra, u_j stands for degree j homogeneous component of u. Since φ is an algebra isomorphism, $u^{(j)}$ must generate B. Since the first component of every element of B in the subalgebra generated by $u^{(j)}$ is in the linear span of $u_1^{(j)}$, the said span must coincide with V_2. In particular, $n \geqslant m$. The same argument with the roles of A and B switched yields $m \geqslant n$. Hence $m = n$ and $u_1^{(j)}$ for $1 \leqslant j \leqslant n$ form a linear basis in V_2. Since φ is an algebra homomorphism, we have

$$\sum_{1\leqslant k,p\leqslant n} a_{k,p}^{(j)} u^{(k)} u^{(p)} = 0 \ \text{ in } B \text{ for } 1 \leqslant j \leqslant s. \tag{2.1}$$

The zero component of the above equation reads

$$\sum_{1\leqslant k,p\leqslant n} a_{k,p}^{(j)} u_0^{(k)} u_0^{(p)} = 0 \ \text{ in } \mathbb{K} \text{ for } 1 \leqslant j \leqslant s. \tag{2.2}$$

The degree 1 component of (2.1) looks like

$$\sum_{1\leqslant k,p\leqslant n} a_{k,p}^{(j)} (u_0^{(k)} u_1^{(p)} + u_0^{(p)} u_1^{(k)}) = 0 \ \text{ in } V_2 \text{ for } 1 \leqslant j \leqslant s. \tag{2.3}$$

Since $u_1^{(j)}$ are linearly independent, the left-hand sides in (2.3) are zero as polynomials in $u_1^{(j)}$. Hence,

$$\sum_{1\leqslant k,p\leqslant n} a_{k,p}^{(j)} (u_0^{(k)} u^{(p)} + u_0^{(p)} u^{(k)}) = 0 \ \text{ in } B \text{ for } 1 \leqslant j \leqslant s. \tag{2.4}$$

Combining (2.1), (2.2) and (2.4) we get

$$\sum_{1\leqslant k,p\leqslant n} a_{k,p}^{(j)} (u^{(k)} - u_0^{(k)})(u^{(p)} - u_0^{(p)}) = 0 \ \text{ in } B \text{ for } 1 \leqslant j \leqslant s. \tag{2.5}$$

The second degree component of (2.5) now is

$$\sum_{1\leqslant k,p\leqslant n} a_{k,p}^{(j)} u_1^{(k)} u_1^{(p)} = 0 \ \text{ in } B \text{ for } 1 \leqslant j \leqslant s. \tag{2.6}$$

Since f_j are linearly independent and $u_1^{(j)}$ are linearly independent, (2.6) provides s linearly independent quadratic relations in B. In particular, $s \leqslant t$. The same argument with the roles of A and B reversed, gives $t \leqslant s$. Hence $s = t$ and the linear span of the left-hand sides of (2.6) must coincide with R_2. It follows that the map $x_i \mapsto u_1^{(i)}$ extends to an algebra isomorphism of A and B. Clearly, this is a graded isomorphism, we were after. □

3 Elementary observations

The following elementary facts are proved in [7]. The next lemma is proved by a direct computation of the reduced Gröbner basis for the ideal of relations of the dual of Sklyanin algebra (using the usual left-to-right degree lexicographical ordering on monomials assuming $x > y > z$. The last statement is verified directly since knowing the Gröner basis we know the multiplication table in the relevant finite dimensional (in this case) algebra $A^!$.

Lemma 3.1. *Let $(p,q,r) \in \mathbb{K}^3$ and $A = Q^{p,q,r}$. Then the Hilbert series of $A^!$ is given by*

$$H_{A^!}(t) = \begin{cases} 1+3t & \text{if } p=q=r=0; \\ \frac{1+2t}{1-t} & \text{if } p^3=q^3=r^3\neq 0 \text{ or exactly two of } p,\ q \text{ and } r \text{ equal } 0; \\ (1+t)^3 & \text{otherwise.} \end{cases} \tag{3.1}$$

Moreover, $wA^!_2 \neq \{0\}$ for each non-zero $w \in A^!_1$ provided $H_{A^!}(t) = (1+t)^3$.

Note [10] that for every quadratic algebra $A = A(V,R)$ (Koszul or otherwise), the power series $H_A(t)H_{A^!}(-t) - 1$ starts with t^k with $k \geqslant 4$. This allows to determine $\dim A_3$ provided we know $\dim A^!_j$ for $j \leqslant 3$. This observation together with (3.1) yield the following fact.

Corollary 3.2. *Let $(p,q,r) \in \mathbb{K}^3$ and $A = Q^{p,q,r}$. Then*

$$\dim A_3 = \begin{cases} 27 & \text{if } p=q=r=0; \\ 12 & \text{if } p^3=q^3=r^3\neq 0 \text{ or exactly two of } p,\ q \text{ and } r \text{ equal } 0; \\ 10 & \text{otherwise.} \end{cases} \tag{3.2}$$

The following two lemmas can be found in [7]. Since their proof is very short, we present it for the sake of reader's convenience.

Lemma 3.3. *Assume that $(p,q,r) \in \mathbb{K}^3$ and $\theta \in \mathbb{K}$ is such that $\theta^3 = 1$ and $\theta \neq 1$. Then the graded algebras $Q^{p,q,r}$ and $Q^{p,q,\theta r}$ are isomorphic.*

Proof. The relations of $Q^{p,q,r}$ in the variables u, v, w given by $x = u$, $y = v$ and $z = \theta^2 w$ read $puv + qvu + \theta rww = 0$, $pwu + quw + \theta rvv = 0$ and $pvw + qwv + \theta ruu = 0$. Thus this change of variables provides an isomorphism between $Q^{p,q,r}$ and $Q^{p,q,\theta r}$. □

Lemma 3.4. *Assume that $(p,q,r) \in \mathbb{K}^3$ and $\theta \in \mathbb{K}$ is such that $\theta^3 = 1$ and $\theta \neq 1$. Then the graded algebras $Q^{p,q,r}$ and $Q^{p',q',r'}$ are isomorphic, where $p' = \theta^2 p + \theta q + r$, $q' = \theta p + \theta^2 q + r$ and $r' = p + q + r$.*

Proof. A direct computation shows that the space of the quadratic relations of $Q^{p,q,r}$ in the variables u, v, w given by $x = u + v + w$, $y = u + \theta v + \theta^2 w$ and $z = u + \theta^2 v + \theta w$ (the matrix of this change of variables is non-degenerate) is spanned by $p'uv + q'vu + r'ww = 0$, $p'wu + q'uw + r'vv = 0$ and $p'vw + q'wv + r'uu = 0$. Thus $Q^{p,q,r}$ and $Q^{p',q',r'}$ are isomorphic. □

Lemma 3.5. *Assume that $\mathbb{K}$ contains an element θ such that $\theta^3 = 1 \neq \theta$. Let $A = Q^{p,q,r}$ be a degenerate Sklyanin algebra. Then*

- *$A = \mathbb{K}\langle x,y,z\rangle \iff p = q = r = 0$;*
- *A is isomorphic to the monomial algebra $Q^{1,0,0}$ given by the relations $xy = zx = yz = 0$ if and only if either $r = 0$, $pq = 0$ and $(p,q) \neq (0,0)$ or $p^3 = q^3 = r^3 \neq 0$ and $p \neq q$;*
- *A is isomorphic to the monomial algebra $Q^{0,0,1}$ given by the relations $xx = yy = zz = 0$ if and only if either $p = q = 0$ and $r \neq 0$ or $p = q \neq 0$ and $p^3 = r^3$;*

Furthermore, the algebras $\mathbb{K}\langle x,y,z\rangle$, $Q^{1,0,0}$ and $Q^{0,0,1}$ are pairwise non-isomorphic and a degenerate Sklyanin algebra can not be isomorphic to a non-degenerate one.

Proof. The statement $A = \mathbb{K}\langle x,y,z\rangle \iff p = q = r = 0$ is obvious. If exactly two of p, q and r are zero, then up to scaling we have three options for (p,q,r): $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Note that swapping x and y, while leaving z as it is, provides an isomorphism between $Q^{1,0,0}$ and $Q^{0,1,0}$. It remains to deal with the case $p^3 = q^3 = r^3 \neq 0$. If $p = q$, then by scaling we can make $p = q = 1$. Then $r^3 = 1$. By Lemma 3.3, $Q^{p,q,r}$ is isomorphic to $Q^{1,1,1}$. By Lemma 3.4, $Q^{1,1,1}$ is isomorphic to $Q^{p',q',r'}$, where $p' = \theta^2 + \theta + 1 = 0$, $q' = \theta + \theta^2 + 1 = 0$ and $r' = 1 + 1 + 1 = 3 \neq 0$. Hence $Q^{p,q,r}$ is isomorphic to $Q^{0,0,1}$. It remains to deal with the situation $p^3 = q^3 = r^3$ and $p \neq q$. By scaling, we can make $p = 1$. Then $r^3 = q^3 = 1$ and $q \neq 1$. By Lemma 3.3, $Q^{p,q,r}$ is isomorphic to Q^{1,q,q^2}. By Lemma 3.4, Q^{1,q,q^2} is isomorphic to $Q^{p',q',r'}$, where $p' = q + q^3 + q^2 = 0$, $q' = q^2 + q^2 + q^2 = 3q^2 \neq 0$ and $r' = 1 + q + q^2 = 0$. Thus $Q^{p,q,r}$ is isomorphic to $Q^{1,0,0}$.

Clearly, $\mathbb{K}\langle x,y,z\rangle$ is not isomorphic to any of $Q^{1,0,0}$ and $Q^{0,0,1}$ (for example, $\mathbb{K}\langle x,y,z\rangle$ has no non-trivial zero divisors, while $Q^{1,0,0}$ and $Q^{0,0,1}$ have non-trivial zero divisors provided by the defining relations). The algebras $Q^{1,0,0}$ and $Q^{0,0,1}$ are non-isomorphic since the only one-dimensional representation of $Q^{0,0,1}$ is the augmentation map, while the map sending x to 1 and y and z to 0 extends to a one-dimensional representation of $Q^{1,0,0}$.

Finally, by Corollary 3.2, a degenerate and a non-degenerate Sklyanin algebras always have different dimensions of their third graded components. Hence a degenerate Sklyanin algebra can not be isomorphic to a non-degenerate one. □

Lemma 3.6. *Assume that $\mathbb{K}$ contains an element θ such that $\theta^3 = 1 \neq \theta$. A non-degenerate Sklyanin algebra $A = Q^{p,q,r}$ is isomorphic to a quantum polynomial algebra B^α given by the relations $xy = \alpha yx$, $zx = \alpha xz$ and $yz = \alpha zy$ with $\alpha \in \mathbb{K}^*$ if and only if $r = 0$ or $(p+q)^3 + r^3 = 0$. More specifically, A is isomorphic to B^α with given $\alpha \in \mathbb{K}^*$ if and only if either $r = 0$ and $\alpha \in \{-\frac{p}{q}, -\frac{q}{p}\}$, or $(p+q)^3 + r^3 = 0$ and $\alpha \in \{\theta\frac{p-\theta^2 q}{p-\theta q}, \theta^2\frac{p-\theta q}{p-\theta^2 q}\}$. Furthermore, B^α and B^β are isomorphic if and only if either $\alpha = \beta$ or $\alpha\beta = 1$.*

Proof. If $r = 0$, then obviously $A = Q^{p,q,r}$ is isomorphic to B^α with $\alpha = -\frac{q}{p}$ (these algebras coincide). Assume now that $(p+q)^3 + r^3 = 0$ and $r \neq 0$. Clearly, $Q^{p,q,r} = Q^{s,t,1}$, where $s = \frac{p}{r}$ and $t = \frac{q}{r}$. Since $(p+q)^3 + r^3 = 0$, we have $(-s-t)^3 = 1$. By Lemma 3.3, $Q^{s,t,1}$ is isomorphic to $Q^{s,t,-s-t}$. By Lemma 3.4, $Q^{s,t,-s-t}$ is isomorphic to $Q^{p',q',r'}$, where $p' = (\theta^2-1)s + (\theta-1)t$, $q' = (\theta-1)s + (\theta^2-1)t$ and $r' = 0$. Note that the assumptions yield that $Q^{p,q,r}$ is non-degenerate and therefore so is $Q^{p',q',r'}$. Hence $p'q' \neq 0$. Then $Q^{p',q',r'} = Q^{1,-\alpha,0}$ with $\alpha = -\frac{(\theta-1)s+(\theta^2-1)t}{(\theta^2-1)s+(\theta-1)t}$. After easy simplifications, one gets $\alpha = \theta\frac{p-\theta^2 q}{p-\theta q} \in \mathbb{K}^*$. Thus A is isomorphic to B^α with $\alpha = \theta\frac{p-\theta^2 q}{p-\theta q}$. Since swapping x and y, while leaving z as it is, provides an isomorphism between B^α and $B^{\alpha^{-1}}$, we see that A is isomorphic to B^α if either $r = 0$ and $\alpha \in \{-\frac{p}{q}, -\frac{q}{p}\}$, or $(p+q)^3 + r^3 = 0$ and $\alpha \in \{\theta\frac{p-\theta^2 q}{p-\theta q}, \theta^2\frac{p-\theta q}{p-\theta^2 q}\}$.

Next, we shall verify that A is not isomorphic to any B^α provided $r \neq 0$ and $(p+q)^3 + r^3 \neq 0$. Indeed, assume that $r \neq 0$ and $(p+q)^3 + r^3 \neq 0$. We shall verify that A has no one-dimensional representations except for the augmentation map. Let $\varphi : A \to \mathbb{K}$ be a representation (=an algebra homomorphism) and let $a = \varphi(x)$, $b = \varphi(y)$ and $c = \varphi(z)$. We have to prove that $a = b = c = 0$. From the defining relations of A it follows that $(p+q)ab = -rc^2$, $(p+q)bc = -ra^2$ and $(p+q)ac = -rb^2$. Multiplying these equalities, we get $a^2b^2c^2((p+q)^3 + r^3) = 0$. Since $(p+q)^3 + r^3 \neq 0$, we have $abc = 0$. Without loss of generality, we may assume that $a = 0$. From $(p+q)ab = -rc^2$ and $(p+q)ac = -rb^2$ we now get $rb = rc = 0$. Since $r \neq 0$, we obtain $a = b = c = 0$. On the other hand, the map $x \mapsto 1$, $y \mapsto 0$ and $z \mapsto 0$ extends to a one-dimensional representation for each B^α. Thus each B^α has more than one one-dimensional representations. Thus A is not isomorphic to any B^α provided $r \neq 0$ and $(p+q)^3 + r^3 \neq 0$.

The proof will be complete if we verify that B^α and B^β are isomorphic if and only if either $\alpha = \beta$ or $\alpha\beta = 1$. Since $B^1 = \mathbb{K}[x,y,z]$ is the only commutative algebra among B^α, it is non-isomorphic to any B^α with $\alpha \neq 1$. Thus it remains to deal with the case $\alpha \neq 1$ and $\beta \neq 1$. Let $\alpha, \beta \in \mathbb{K}^*$, $\alpha \neq 1$

and $\beta \neq 1$. Assume that there is a graded algebra isomorphism $\varphi : B^\alpha \to B^\beta$. Let x, y, z be the usual generators of B^β, while u, v and w be the images under φ of the usual generators of B^α. Then u, v and w form a linear basis in the degree 1 component $B^\beta_1 = V$ of B^β and satisfy $uv = \alpha vu$, $wu = \alpha uw$ and $vw = \alpha wv$. Furthermore, the linear span of $uv-\alpha vu$, $wu-\alpha uw$ and $vw-\alpha wv$ treated as elements of $\mathbb{K}\langle x,y,z\rangle$ must coincide with the linear span of $xy-\beta yx$, $zx-\beta xz$ and $yz-\beta zy$. In particular, $uv-\alpha vu$, $wu-\alpha uw$ and $vw-\alpha wv$ must be 'square-free': when written in terms of x, y and z they should not contain any of xx, yy and zz. Since $\alpha \neq 1$, it immediately follows that each of u, v and w, when written in terms of x, y and z must be a scalar multiple of a single element of $\{x,y,z\}$. That is, the matrix of our linear substitution φ is a product of a diagonal matrix and a permutation matrix (of size 3×3). If the permutation in question is even, one easily sees that $uv-\alpha vu$, $wu-\alpha uw$ and $vw-\alpha wv$ are scalar multiples of $xy-\alpha yx$, $zx-\alpha xz$ and $yz-\alpha zy$ (in some order), while if the permutation is odd $uv-\alpha vu$, $wu-\alpha uw$ and $vw-\alpha wv$ are scalar multiples of $\alpha xy-yx$, $\alpha zx-xz$ and $\alpha yz-\alpha zy$ (in some order). Thus, the spans of the triples $uv-\alpha vu$, $wu-\alpha uw$ and $vw-\alpha wv$ and $xy-\beta yx$, $zx-\beta xz$ and $yz-\beta zy$ can coincide only if $\alpha=\beta$ or $\alpha\beta=1$, which completes the proof. □

Lemma 3.7. *Degenerate Sklyanin algebras $A = Q^{p,q,r}$ are Koszul and have the Hilbert series $H_A = (1-3t)^{-1}$ if $p=q=r=0$ and $H_A = \frac{1+t}{1-2t}$ otherwise. Non-degenerate Sklyanin algebras $A = Q^{p,q,r}$ satisfying either $r=0$, or $(p+q)^3+r^3=0$, or $p^3=q^3$ are Koszul and have the Hilbert series $H_A = (1-t)^{-3}$.*

Proof. By Remark 2.2, by passing to a field extension, we can, without loss of generality, assume that $\mathbb{K}$ is algebraically closed. Since $\operatorname{char}\mathbb{K}\neq 3$ and $\mathbb{K}$ is algebraically closed, we can find $\theta\in\mathbb{K}$ such that $\theta^3 = 1\neq\theta$. Now by Lemmas 3.5 and 3.6 in all cases except for $p^3=q^3\neq 0$ and $r\neq 0$, A is PBW and therefore Koszul. Indeed, A is either isomorphic to a monomial algebra or to quantum polynomials. The computation of the Hilbert series is straightforward and very easy.

It remains to consider the case $p^3=q^3\neq 0$ and $r\neq 0$. By normalizing, we can without loss of generality assume $p=1$. Then $q^3=1$ and the defining relations take form $xx = -\frac{1}{r}yz-\frac{q}{r}zy$, $xy=-qyx-rzz$ and $xz=-\frac{1}{q}zx-\frac{r}{q}yy$. A direct computation shows that the reduced Gröber basis of the ideal of relations comprises the defining relations $rxx+yz+qzy$, $xy+qyx+rzz$, $qxz+ryy+zx$ together with $yyz-q^2zyy$ and $yzz-q^2zzy$ (the basis is finite; one has to use the equality $q^3=1$). Now the normal words (the words, which do not contain any of the leading monomials xx, xy, xz, yyz, yzz of the basis as submonomials) are exactly $z^k(yz)^my^lx^\varepsilon$, where k,m,l are non-negative integers and $\varepsilon\in\{0,1\}$. As there are precisely $\frac{(n+1)(n+2)}{2}$ normal words of degree n, we have $H_A=(1-t)^{-3}$. Since the set of normal words is closed under multiplication by z on the left, the map $u\mapsto zu$ from A to itself is injective. In particular, A has no non-trivial right annihilators. By Lemma 3.1 $H_{A^!} = (1+t)^3$ and $wA^!_2\neq\{0\}$ for every non-zero $w\in A^!_1$. Now Lemma 3.2 implies that A is Koszul. □

We need another observation made in [7].

Lemma 3.8. *For every $p,q,r\in\mathbb{K}$, the Sklyanin algebra $A=Q^{p,q,r}$ satisfies $\dim A_n \geqslant \frac{(n+1)(n+2)}{2}$ for every $n\in\mathbb{Z}_+$.*

4 Proof of Theorem 1.1

Throughout this section $p,q,r\in\mathbb{K}$ and $A=Q^{p,q,r}$. By Remark 2.2, by passing to a field extension, we can, without loss of generality, assume that $\mathbb{K}$ is algebraically closed. Since $\operatorname{char}\mathbb{K}\neq 3$ and $\mathbb{K}$ is algebraically closed, we can find $\theta\in\mathbb{K}$ such that $\theta^3=1\neq\theta$. If $p^3=q^3$ or $r=0$ or $(p+q)^3+r^3=0$, the conclusion of Theorem 1.1 follows from Lemma 3.7. Thus for the rest of the proof, we can assume that none of these equalities holds. Moreover, if $p+q=0$, one easily sees that a substitution from Lemma 3.4 breaks this equality. Thus we can additionally assume that $p+q\neq 0$. Since $r\neq 0$, by scaling the relations, we can without loss of generality assume that $r=1$. Then $(p,q)\neq(0,0)$, $(p^3-1,q^3-1)\neq(0,0)$, $p+q\neq 0$ and $(p+q)^3+1\neq 0$.

Our proof hinges on finding a convenient linear substitution. The main objective is to make specific monomials (namely, xx, xy and yz) into leading monomials of defining relations with respect to the standard left-to-right degree-lexicographical ordering assuming $x>y>z$. We perform the substitution in a number of steps: each of the steps is a linear sub itself and the resulting substitution is their composition. We keep the same letters x,y,z for both old and new variables. We introduce a substitution by showing by which linear combination of (new) x,y,z must the (old) variables be replaced. For example, if we write $x\to x+y+z$, $y\to z-y$ and $z\to 7z$, this means that all occurrences of x (in the relations, potential etc.) are replaced by $x+y+z$, all occurrences of y are replaced by $z-y$, while z is swapped for $7z$.

Note that our Sklyanin algebra $Q^{p,q,1}$ is potential with the potential

$$F_{p,q}=x^3+y^3+z^3+pxyz^{\circlearrowleft}+qxzy^{\circlearrowleft}.$$

First, we perform the sub $x\to-\frac{x}{p+q}$, $y\to y$ and $z\to z$. As a result, we see that $A=Q^{p,q,1}$ is isomorphic to the potential algebra with the potential

$$F'_{p,q}=-\tfrac{(p+q)^3+1}{(p+q)^3}x^3+(x^3+y^3+z^3)-\tfrac{p}{p+q}xyz^{\circlearrowleft}-\tfrac{q}{p+q}xzy^{\circlearrowleft}.$$

Note that we have used the condition $p+q\neq 0$. Next, we do the sub $x\to x+y+z$, $y\to x+\theta^2y+\theta z$ and $z\to x+\theta y+\theta^2 z$. As a result, A is isomorphic to the potential algebra with the potential

$$F''_{p,q}=-\tfrac{(p+q)^3+1}{(p+q)^3}(x+y+z)^3+\tfrac{3((1-\theta)p+(1-\theta^2)q)}{p+q}xyz^{\circlearrowleft}+\tfrac{3((1-\theta^2)p+(1-\theta)q)}{p+q}xzy^{\circlearrowleft}.$$

Note that the $(x+y+z)^3$-coefficient in $F''_{p,q}$ is non-zero since $(p+q)^3+1\neq 0$. Since multiplying a potential by a non-zero constant has no effect on the corresponding potential algebra, A is isomorphic to the potential algebra with the potential

$$G_{a,b}=(x+y+z)^3+axyz^{\circlearrowleft}+bxzy^{\circlearrowleft},$$

where

$$a=\tfrac{3(p+q)^2((\theta-1)p+(\theta^2-1)q)}{(p+q)^3+1},\quad b=\tfrac{3(p+q)^2((\theta^2-1)p+(\theta-1)q)}{(p+q)^3+1}.$$

Note that $a=0\iff p=\theta^2q$, $b=0\iff p=\theta q$ and $a=b\iff p=q$. Since $p^3\neq q^3$, we have $ab(a-b)\neq 0$. Furthermore, $a+b=-\frac{9(p+q)^3}{(p+q)^3+1}$ and therefore $a+b\neq 0$ as well.

Now we perform the sub $x\to\frac{a}{a+b}x$, $y\to\frac{b}{a+b}x+y-z$ and $z\to z$. We need the fact that $a(a+b)\neq 0$ for it to be non-degenerate. As a result, A is isomorphic to the potential algebra with the potential

$$G'_{a,b}=(x+y)^3+\tfrac{ab}{a+b}xxz^{\circlearrowleft}+\tfrac{a^2}{a+b}xyz^{\circlearrowleft}+\tfrac{ab}{a+b}xzy^{\circlearrowleft}-axzz^{\circlearrowleft}.$$

By this point, we have already reached our main objective. One easily sees that the leading monomials of the defining relations of the last algebra are indeed xx, xy and yz. However, we shall perform one final sub (with a triangular matrix) in order to simplify the potential. Namely, we use the sub $x\to x-\frac{a-b}{a}y+\frac{(a+b)^2+a^2b}{(a-b)^3}z$, $y\to\frac{a-b}{a}y-\frac{(a+b)^2+ab^2}{(a-b)^3}z$ and $z\to-\frac{a+b}{(a-b)^2}z$, which is non-degenerate since $ab(a+b)(a-b)\neq 0$. As a result, A is isomorphic to the potential algebra with the potential

$$P_{\alpha,\gamma}=x^3-xyz^{\circlearrowleft}+yyz^{\circlearrowleft}+\alpha yzz^{\circlearrowleft}-(\gamma-\alpha^2)z^3, \tag{4.1}$$

where

$$\alpha=-\tfrac{(a+b)^3+ab(a^2+b^2)}{(a-b)^4},\quad \gamma=-\tfrac{(a+b)^4(a^2-ab+b^2)+ab(a+b)^3(2a^2+2b^2-3ab)+a^2b^2(a^4+b^4+a^2b^2-a^3b-ab^3)}{(a-b)^8}.$$

Remark 4.1. There is no black magic to this string of substitutions. There is no visible pattern to the leading monomials of the Gröbner basis of the ideal of relations of the Sklyanin algebras in their original form. We had to change something and the most drastic change ensues from altering the leading monomials of the defining relations. However for generic Sklyanin, it is impossible to get rid of the square xx of the largest generator. It is equally impossible to lose both xy and xz, so we might as well keep xy. The only freedom we have now is to find a sub, which will change the smallest leading term xz and this can not possibly go below yz. So we set on changing it into yz. We use elementary linear algebra to find necessary and sufficient conditions on a potential for the corresponding potential algebra to have xy, xz and yz as the leading terms of defining relations. These have the form of equations on the coefficients of the potential. Each of the first few of the above subs were designed to alter the potential in such a way that one equation is satisfied without spoiling the previously obtained ones. The last sub is there for the sake of neatness.

The case $\alpha = \gamma = 0$ does not occur (does not come from a Sklyanin algebra). One can see it directly using the above formulas for α and γ. However there is an indirect way. The potential algebra with the potential $P_{0,0}$ enjoys a rank 1 quadratic relation $xy = yy$ (we mean the usual rank in $V \otimes V$). On the other hand, it is elementary to see that no such thing exists for any non-degenerate Sklyanin algebra. We shall discuss the potential $P_{0,0}$ later since it is peculiar indeed.

Thus A is isomorphic to the potential algebra B defined by the potential $P_{\alpha,\gamma}$ of (4.1) with $\alpha, \gamma \in \mathbb{K}$, $(\alpha, \gamma) \neq (0, 0)$. Since A and B are isomorphic, the proof will be complete if we show that B is Koszul and $H_B = (1 - t)^{-3}$. A direct computation shows that B is presented by generators x, y, z and relations

$$xx - zx + zy + \alpha zz = 0, \quad xy - yy - \alpha zx + \gamma zz = 0, \quad yz - zx + zy + \alpha zz = 0 \tag{4.2}$$

(observe the leading monomials xx, xy and yz). Let I be the right ideal in B generated by y and z: $I = yB + zB$. For $f, g \in B$, we write $f = g \,(\mathrm{mod}\, I)$ if $f - g \in I$. For each $k \in \mathbb{Z}_+$ consider the following property:

(Π_k) there exist $a_k, b_k \in \mathbb{K}$ such that $xz^k x = a_k xz^{k+1} \,(\mathrm{mod}\, I)$ and $xz^k y = b_k xz^{k+1} \,(\mathrm{mod}\, I)$.

Note that according to (4.2), Π_0 is satisfied with $a_0 = b_0 = 0$. We shall prove the following Claim.

Claim: If $k \in \mathbb{Z}_+$ and Π_k is satisfied, then either Π_{k+1} is satisfied or

(Σ_{k+1}) $xz^{k+1}x = xz^{k+1}y \,(\mathrm{mod}\, I)$ and $xz^{k+2} = 0 \,(\mathrm{mod}\, I)$.

Proof of Claim. Assume that Π_k holds. Then $xz^k x = a_k xz^{k+1} \,(\mathrm{mod}\, I)$ and $xz^k y = b_k xz^{k+1} \,(\mathrm{mod}\, I)$. Using (4.2), we see that $xz^k yz = xz^{k+1}x - xz^{k+1}y - \alpha xz^{k+2}$ for some $a_k, b_k \in \mathbb{K}$. On the other hand, $xz^k yz = b_k xz^{k+2} \,(\mathrm{mod}\, I)$. These equalities yield

$$-xz^{k+1}x + xz^{k+1}y + (b_k + \alpha)xz^{k+2} = 0 \,(\mathrm{mod}\, I).$$

By (4.2), $xz^k xy = xz^k yy + \alpha xz^{k+1}x - \gamma xz^{k+2}$. Using (Π_k), we then have $xz^k xy = b_k xz^{k+1}y + \alpha xz^{k+1}x - \gamma xz^{k+2} \,(\mathrm{mod}\, I)$. Directly from (Π_k), we have $xz^k xy = a_k xz^{k+1}y \,(\mathrm{mod}\, I)$. These two equalities yield

$$\alpha xz^{k+1}x + (b_k - a_k)xz^{k+1}y - \gamma xz^{k+2} = 0 \,(\mathrm{mod}\, I).$$

Again, by (4.2), $xz^k xx = xz^{k+1}x - xz^{k+1}y - \alpha xz^{k+2}$. By (Π_k), $xz^k xx = a_k xz^{k+1}x \,(\mathrm{mod}\, I)$. By the last two equalities,

$$(a_k - 1)xz^{k+1}x + xz^{k+1}y + \alpha xz^{k+2} = 0 \,(\mathrm{mod}\, I).$$

In the matrix form, the equations in the above three displays have the form

$$M \begin{pmatrix} xz^{k+1}x \\ xz^{k+1}y \\ xz^{k+2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \,(\mathrm{mod}\, I), \quad \text{where } M = \begin{pmatrix} -1 & 1 & b_k + \alpha \\ \alpha & b_k - a_k & -\gamma \\ a_k - 1 & 1 & \alpha \end{pmatrix}. \tag{4.3}$$

If the first two columns of matrix M are linearly independent, Π_{k+1} follows straight away.

It remains to consider the case when the first two columns of matrix M are proportional. In this case $a_k = 0$ and $b_k = -\alpha$ and M has the form

$$M = \begin{pmatrix} -1 & 1 & 0 \\ \alpha & -\alpha & -\gamma \\ -1 & 1 & \alpha \end{pmatrix}.$$

Since $(\alpha,\gamma) \neq (0,0)$, (4.3) is now equivalent to $xz^{k+1}x = xz^{k+1}y \,(\mathrm{mod}\, I)$ and $xz^{k+2} = 0\,(\mathrm{mod}\, I)$, which is Σ_{k+1}. This concludes the proof of the claim. □

According to the above claim we have two options: either Π_k holds for all $k \in \mathbb{Z}_+$ or for some $k \in \mathbb{Z}_+$, $\Pi_0,\dots,\Pi_k$ and Σ_{k+1} are satisfied. Note that the first option is what happens for generic $(\alpha,\gamma) \in \mathbb{K}^2$, while the validity of $\Pi_0,\dots,\Pi_k$ and Σ_{k+1} occurs for (α,γ) from an algebraic curve.

Case 1: Π_k holds for all $k \in \mathbb{Z}_+$. The only monomials, which do not contain any of yz, xz^kx and xz^ky for $k \in \mathbb{Z}_+$ are z^ky^m and $z^ky^mxz^j$ for $k,m,j \in \mathbb{Z}_+$. Denote this set of monomials N. The number of monomials of degree n in N is exactly $\frac{(n+1)(n+2)}{2}$. Since Π_k is satisfied for each k, B is the linear span of N. It follows that $\dim B_n \leqslant \frac{(n+1)(n+2)}{2}$ for each n and these inequalities turn into equalities precisely when monomials from N are linearly independent in B. On the other hand, by Lemma 3.8, $\dim A_n = \dim B_n \geqslant \frac{(n+1)(n+2)}{2}$ for each n. Hence $\dim B_n = \frac{(n+1)(n+2)}{2}$ for every n, that is, $H_B = (1-t)^{-3}$, and monomials from N are linearly independent in B. Equalities in Π_k can be written as the equalities $xz^kx - a_kxz^{k+1} + f_k$ and $xz^ky - b_kxz^{k+1} + g_k$ in B with f_k, g_k being homogeneous elements of I of degree $k+2$. According to the above observations, the equality $H_B = (1-t)^{-3}$ implies that $yz - zx + zy + \alpha zz$ together with $xz^kx - a_kxz^{k+1} + f_k$ and $xz^ky - b_kxz^{k+1} + g_k$ for $k \in \mathbb{Z}_+$ form a reduced Gröbner basis in the ideal of relations of B and that N is the set of corresponding normal words. Since N is closed under multiplication by z on the left, the map $u \mapsto zu$ from B to itself is injective. Thus B has no non-trivial right annihilators. Since A and B are isomorphic $H_A = (1-t)^{-3}$ and A has non non-trivial right annihilators. By Lemma 3.1 $H_{A^!} = (1+t)^3$ and $wA^!_2 \neq \{0\}$ for every non-zero $w \in A^!_1$. Now Lemma 3.2 implies that A is Koszul.

Case 2: there is $k \in \mathbb{Z}_+$ such that $\Pi_0,\dots,\Pi_k$ and Σ_{k+1} hold. By Σ_{k+1}, $xz^{k+1}x = xz^{k+1}y \,(\mathrm{mod}\, I)$ and $xz^{k+2} = 0\,(\mathrm{mod}\, I)$. Using these equalities and (4.2), we have $xz^{k+1}xx = xz^{k+1}yx \,(\mathrm{mod}\, I)$ and $xz^{k+1}xx = xz^{k+2}x - xz^{k+2}y - \alpha xz^{k+3} = 0\,(\mathrm{mod}\, I)$. Hence

$$xz^{k+1}yx = 0\,(\mathrm{mod}\, I).$$

The only monomials, which do not contain any of yz, xz^jx for $0 \leqslant j \leqslant k+1$, xz^jy for $0 \leqslant j \leqslant k$, xz^{k+2} and $xz^{k+1}yx$ are the words of the form z^jy^mw, where $m,j \in \mathbb{Z}_+$ and w is an initial subword (empty allowed) of the infinite word $xz^{k+1}yyyy\dots$. Denote this set of monomials N. As before, the number of monomials of degree n in N is $\frac{(n+1)(n+2)}{2}$. Now exactly the same argument as in Case 1 yields $H_A = H_B = (1-t)^{-3}$ and shows that A is Koszul. Note that this time the reduced Gröbner basis in the ideal of relations of B turns out to be finite: it consists of $2k+6$ elements. This concludes the proof of Theorem 1.1.

5 Isomorphic Sklyanin algebras

Recall that we still assume that $\mathrm{char}\,\mathbb{K} \neq 3$. Throughout this section we shall also assume that there is $\theta \in \mathbb{K}$ such that $\theta^3 = 1 \neq \theta$ and use θ for this element without further reference. Note that the absence of a non-trivial cubic root of 1 does effect the results below both directly (when θ features in a statement) and indirectly.

By Lemmas 3.5 and 3.6, a non-degenerate Sklyanin algebra $A^{p,q,r}$ is non-isomorphic to any quantum polynomial algebra B^α if and only if $r \neq 0$, $(p+q)^3+r^3 \neq 0$, $(p,q) \neq (0,0)$ and $p^3=q^3=r^3$ fails. That is, by dividing by r, (p,q,r) can be turned into a unique triple $(a,b,1)$ with

$$(a,b) \in M, \quad \text{where} \quad M = \{(a,b) \in \mathbb{K}^2 : (a,b) \neq (0,0),\ (a+b)^3+1 \neq 0,\ (a^3-1,b^3-1) \neq (0,0)\}.$$

Since scaling the triple of parameters does not change the Sklyanin algebra, in order to describe which non-degenerate Sklyanin algebras non-isomorphic to quantum polynomials are isomorphic to each other, it suffices to do so in the case $(p,q,r)=(a,b,1)$ with $(a,b) \in M$.

Lemma 5.1. *If both (a,b) and (a',b') belong to M, then the Sklyanin algebras $Q^{a,b,1}$ and $Q^{a',b',1}$ are isomorphic if and only if (a,b) and (a',b') are in the same orbit of the group action on M generated by two maps $(a,b) \mapsto (\theta a,\theta b)$ and $(a,b) \mapsto \left(\frac{\theta a+\theta^2 b+1}{a+b+1}, \frac{\theta^2 a+\theta b+1}{a+b+1}\right)$. This group is finite, consists of 24 elements (thus if $\mathbb{K}$ is infinite, for generic $(a,b) \in M$, there are exactly 23 other elements of M giving rise to an isomorphic Sklyanin algebra) and is isomorphic to $SL_2(\mathbb{Z}_3)$.*

The complete list of pairs $(a',b') \in M$ such that for a given $(a,b) \in M$, $Q^{a,b,1}$ and $Q^{a',b',1}$ are isomorphic is as follows:

- $(\theta^j a, \theta^j b)$ *and* $(\theta^j b, \theta^j a)$ *with* $j \in \{0,1,2\}$;
- $\left(\frac{\theta^j a+\theta^k b+\theta^m}{a+b+\theta^n}, \frac{\theta^k a+\theta^j b+\theta^m}{a+b+\theta^n}\right)$ *with* $n \in \{0,1,2\}$ *and* $\{j,k,m\}=\{0,1,2\}$.

The first line line in the above list provides 6 pairs, while the second line yields 18.

Proof. Assume that (a,b) and (a',b') belong to M, then the Sklyanin algebras $A=Q^{a,b,1}=A(V,R)$ and $B=Q^{a',b',1}=A(V,R')$ are isomorphic. By Theorem 1.1, $\dim A_3=\dim B_3=10$. Since $\dim V^3=27$, we have $\dim(VR+RV)=\dim(VR'+R'V)=27-10=17$. Since $\dim VR=\dim RV=\dim VR'=\dim R'V=9$, it follows that $\dim(RV\cap VR)=\dim(R'V\cap V'R)=1$. On the other hand, obviously, the potentials $F=x^3+y^3+z^3+axyz^{\circlearrowleft}+bxzy^{\circlearrowleft}$ and $F'=x^3+y^3+z^3+a'xyz^{\circlearrowleft}+b'xzy^{\circlearrowleft}$ for A and B respectively satisfy $F \in RV\cap VR$ and $F' \in R'V\cap VR'$. Since a linear substitution $T \in GL_3(\mathbb{K})$ facilitating the graded isomorphism of A and B must send $RV\cap VR$ to $R'V\cap VR'$, it transforms F to F' up to a scalar multiple. Hence T must transform the abelianization of F $G=x^3+y^3+z^3+3(a+b)xyz \in \mathbb{K}[x,y,z]$ (the image of F under the canonical map from $\mathbb{K}\langle x,y,z\rangle$ to $\mathbb{K}[x,y,z]$) into the abelianization $G'=x^3+y^3+z^3+3(a'+b')xyz \in \mathbb{K}[x,y,z]$ of F' up to a scalar multiple. Hence, T provides an isomorphism between the elliptic (projective) curves C given by $G=0$ and C' given by $G'=0$.

Since (a,b) and (a',b') belong to M, we have $(a+b)^3+1\neq 0$ and $(a'+b')^3+1\neq 0$ and therefore the curves C and C' are regular. Note that the each of the curves C and C' have the same exactly collection of nine inflection points which are the nine lines L_k for $1\leqslant k\leqslant 9$ spanned by $(1,-1,0)$, $(1,-\theta,0)$, $(1,-\theta^2,0)$, $(1,0,-1)$, $(1,0,-\theta)$, $(1,0,-\theta^2)$, $(0,1,-1)$, $(0,1,-\theta)$ and $(0,1,-\theta^2)$ respectively. Since T is an isomorphism between $C^{p,q}$ and $C^{p',q'}$, T must leave the union of L_j invariant. It is a routine exercise to verify that the subgroup G of $GL_3(\mathbb{K})$ leaving the union of L_j invariant is generated by

$$\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}\ (\lambda\in\mathbb{K}^*),\quad \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\quad \begin{pmatrix}1&0&0\\0&1&0\\0&0&\theta\end{pmatrix}\quad\text{and}\quad \begin{pmatrix}\theta&\theta^2&1\\\theta^2&\theta&1\\1&1&1\end{pmatrix}. \tag{5.1}$$

One way to see it is to notice that, first of all, each linear map in the above display leaves the union of L_j invariant. Next, one easily checks that the group K generated by matrices in the above display acts transitively on the set of triples of L_j that span $\mathbb{K}^3$. This fact together with an (easily verifiable) observation that K contains all permutation matrices and all scalar matrices yields that K contains all elements of G and therefore $K=G$. Thus $T\in G$.

Now the linear transformations given by the first two matrices in the above display provide automorphisms of each $Q^{p,q,1}$, while the linear transformation given by the third matrix in the above display facilitates an isomorphism between $Q^{p,q,1}$ and $Q^{\theta p,\theta q,1}$ for each $(p,q)\in M$. Finally, a direct computation shows that the linear transformations given by the last matrix in the above display provides an isomorphism of $Q^{p,q,1}$ and $Q^{p',q',1}$ for every $(p,q)\in M$, where $(p',q')=\left(\frac{\theta p+\theta^2 q+1}{p+q+1},\frac{\theta^2 p+\theta q+1}{p+q+1}\right)$. This completes the proof of the isomorphism statement. Thus the existence of an isomorphism between A and B is equivalent to A and B being in the same orbit of the group action on M generated by two maps $(a,b)\mapsto(\theta a,\theta b)$ and $(a,b)\mapsto\left(\frac{\theta a+\theta^2 b+1}{a+b+1},\frac{\theta^2 a+\theta b+1}{a+b+1}\right)$.

A direct computation shows that this group consists of the maps $(a,b)\mapsto(\theta^j a,\theta^j b)$ with $j\in\{0,1,2\}$ $(a,b)\mapsto(\theta^j b,\theta^j a)$ with $j\in\{0,1,2\}$ and $(a,b)\mapsto\left(\frac{\theta^j a+\theta^k b+\theta^m}{a+b+\theta^n},\frac{\theta^k a+\theta^j b+\theta^m}{a+b+\theta^n}\right)$ with $n\in\{0,1,2\}$ and $\{j,k,m\}=\{0,1,2\}$. Hence the group has 24 elements. As for this group being isomorphic to $SL_2(\mathbb{Z}_3)$, this can be done by computing enough features of this group (for instance, it has a two-element center, maximal order of an element in it is 6 etc.) to be able to identify it in the well-known list of 24-element groups. □

Theorem 1.2 is just an amalgamation of Lemmas 3.5, 3.6 and 5.1. That is, it is already proven. Note that analyzing the conclusion of Theorem 1.2, it is easy to see that it has the following neat corollary, where we do not distinguish between different types of Sklyanin algebras, like degenerate and non-degenerate, quantum polynomials or not. The argument for those classes of algebras may differ (one of the reasons to treat quantum polynomials separately is that the elliptic curve from the proof of Lemma 5.1 becomes degenerate in this case) but the results admit a 'uniform' description.

Corollary 5.2. *Two Sklyanin algebras $Q^{p,q,r}$ and $Q^{p',q',r'}$ are isomorphic if and only if (p,q,r) and (p',q',r') lie in the same orbit of the natural action of the subgroup H of $GL_3(\mathbb{K})$ generated by*

$$\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}\ (\lambda\in\mathbb{K}^*),\quad \begin{pmatrix}1&0&0\\0&1&0\\0&0&\theta\end{pmatrix}\quad \textit{and}\quad \begin{pmatrix}\theta&\theta^2&1\\\theta^2&\theta&1\\1&1&1\end{pmatrix}. \tag{5.2}$$

Curiously enough, the above group H is a subgroup of the group G from the proof of Lemma 5.1.

6 Some remarks

1. Observe that our results on Sklyanin algebras collapse in characteristic 3. Indeed, a cubic root of 1 plays an essential role in the substitution we construct as well as in the description of isomorphic Sklyanin algebras.

2. Concerning the potential

$$P_{0,0}=x^3-xyz^{\circlearrowleft}+yyz^{\circlearrowleft},$$

which does not correspond to a Sklyanin algebra, the algebra W it does generate is quite peculiar. The defining relations of W are $xx-zx+zy=0$, $xy-yy=0$ and $yz-zx+zy=0$. A direct computation shows that the reduced Gröbner basis in the ideal of relations of W is finite. It comprises $xx-zx+zy$, $xy-yy$, $yz-zx+zy=0$, yyy, $xzx-xzy+zyx-zzx+zzy$ and $xzyx$, which allows us to compute the Hilbert series of $W: H_W=\frac{(1+t)(1+t^2)(1+t+t^2)}{1-t-t^3-2t^4}$. On the other hand, the dual algebra $W^!$ is given by the relations $xx+yz+zx$, $xy+yy$, xz, yx, $zx+zy$ and zz. Those together with yyy, zyz, $yyz-zyy$ and $yzy-zyy$ form the reduced Gröbner basis in the ideal of relations of $W^!$. The corresponding normal words are 1, x, y, z, yy, yz, zy and zyy, yielding $H_{W^!}=(1+t)^3$. Clearly, the the duality relation $H_W(t)H_{W^!}(-t)=1$ fails and therefore W is non-Koszul. Thus W provides an example of a *non-Koszul quadratic potential algebra on three generators.*

3. We would like to make a comment on the groups G and H defined in (5.1) and (5.2), respectively. Both are subgroups of $GL_3(\mathbb{K})$ containing the group $S = \mathbb{K}^*\mathrm{Id}$ of scalar matrices as a normal subgroup. Consider the groups $G_0 = G/S$ and $H_0 = H/S$. Both H_0 and G_0 turn out to be finite (H_0 is a subgroup of G_0). Naturally, H_0 is isomorphic to the group from Lemma 5.1. Thus it has 24 elements and is isomorphic to $SL_2(\mathbb{Z}_3)$. As for G_0, it is rather easy to find its order: G_0 contains 216 elements.

4. For the results on isomorphisms we supposed that the filed is algebraically closed, or at least there is nontrivial cubic root of unity. Analogous results could be obtained if we omit this condition. In fact, the similar methods, we have used above, allow to deal with the case when $\mathrm{char}\,\mathbb{K} \neq 3$ but $\mathbb{K}$ possesses no nontrivial cubic roots of 1. Equivalently, the quadratic equation $t^2 + t + 1 = 0$ has no solutions in $\mathbb{K}$. In this case, there are much fewer isomorphic Sklyanin algebras. Namely, $Q^{p,q,r}$ and $Q^{p',q',r'}$ are isomorphic if and only if either (p', q', r) is proportional to (p, q, r) (in this case the algebras $Q^{p,q,r}$ and $Q^{p',q',r'}$ coincide) or (p', q', r) is proportional to (q, p, r). In the latter case swapping of x and y, while leaving z as it is, provides a required isomorphism. Note that now we have four pairwise non-isomorphic degenerate Sklyanin algebras. The extra one is $Q^{1,1,1}$ and it is not isomorphic to any monomial algebra. Similarly, in this case non-degenerate $Q^{p,q,r}$ satisfying $(p + q)^3 = -r^3 \neq 0$ are no longer isomorphic to quantum polynomials. To give a taste of how all this can be verified, we sketch the proof of the analog of Lemma 5.1. Let $\mathbb{K}_1$ be the extension of $\mathbb{K}$ via the polynomial $t^2 + t + 1$ ($\mathbb{K}_1$ is the quotient of $\mathbb{K}[t]$ by the ideal generated by $t^2 + t + 1$). Let $\theta \in \mathbb{K}_1$ be one of the two solutions of the quadratic equation $t^2 + t + 1 = 0$. Then $\theta^3 = 1 \neq \theta$. In the proof of Lemma 5.1 it is shown that $T \in GL_3(\mathbb{K}_1)$ provides an isomorphism between some $Q^{a,b,1}$ and $Q^{a',b',1}$ for $(a, b), (a', b') \in M$ if and only if T belongs to the group G defined in (5.1) (in which case T preserves the whole class of algebras $Q^{a,b,1}$ with $(a, b) \in M$). Now the substitutions we are interested in (those which work in the case when the ground field is $\mathbb{K}$) are precisely $T \in G$ with all entries from $\mathbb{K}$. These are easily seen to be only the scalar multiples of permutation matrices. If the permutation in question is even, T provides an automorphism of each $Q^{a,b,1}$, while if it is odd, T provides an isomorphism between $Q^{a,b,1}$ and $Q^{b,a,1}$.

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