Ideals of linear type and some variants

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Introduction

Here we consider the relationships that hold between the arithmetical properties of the Rees algebra $Re(I)$ and the symmetric algebra Sym (I) of an ideal I in a noetherian ring A. As is well known, $Re(I)$ is of paramount importance since it describes the blow up of A along I. It is, however, difficult to handle because its equations are given by the syzygies of all powers I^n. Sym (I), on the contrary, has a simpler structure; its equations are just given by the syzygies of I and so are linear. This makes it clear that one is interested in situations where Sym (I) and $Re(I)$ do not differ much. The simplest case is, of course, where they are just isomorphic, in which case I is said to be of *linear type* (l.t.).

In this report we want to emphasize more the numerical approach to linear type ideals and the geometric aspects of this property of an ideal. The geometric interpreation of linear type means that the normal cone and the normal bundle of a closed subscheme of a given scheme coincide. In the course of our investigations it showed up that a weaker notion which we call *geometric linear type* (g.l.t.), turned out to be not only useful and interesting on its own, but also to unify various aspects in the papers [HSiV1 – 3] and [Hu3] with respect to linear type property and in [SV2] with respect to the dimension of the symmetric algebra (Huneke-Rossi-formula) . The condition to be of geometric linear type refers only to the reduced structures of the normal cone and the normal bundle. To be specific, this means that the fibres of the normal cone projection should be linear. This already shows the closed connection of the g.l.t.-property with the l.t.-property. The main advantage of g.l.t. being that it can be checked by numerical conditions. This provides the general strategy of proving l.t. by first proving g.l.t. and then looking for additional conditions which give "g.l.t.$\Rightarrow$ l.t.". Manifestations of this principle appear in the proof of theorem 2.4 in [Hu3], and in [HSiV3], s. remark, page 80. (One should note that this last result does not depend on the machinery of approximation complexes.) At this point we would like to remark that in general it is very difficult to check the linear type-property for a given ideal. Known are particular examples (for determinantal ideals s. [Hu3]), and, as a general class of ideals of linear type, those which are generated by d-sequences (s. [Hu1], [Va1]). Note, however, that there are linear type ideals (in particular determinantal ideals) which are not generated by d-sequences, s. $[Hu3]$, footnote **.

The material of this report can be divided roughly into the following two parts. The first one is theoretical and concerned with the general theory of l.t., g.l.t. and weaker conditions and their mutual interrelations; a particular weaker condition is contained in Valla's conjecture (§2). To this first part also belong §1, A,C and §§3.4. In §2 we explain Valla's conjecture and prove a particular case of it. In §4, theorem 4.8 we investigate the relationship of linear type, geometric linear type and the almost complete intersection-property (a.c.i.) in Cohen-Macaulay rings, in particular almost complete intersections are of linear type in this situation. This last result is already mentioned in [SV1] and in [Hu4]. Here we give an elementary proof based on a general Primbasissatz, which seems to be new in this form.

The second part of the report is concerend with numerical aspects and various examples and counterexamples illustrating the theory. To it belong §2 , §5 and §1 B. Section 2 appears here again because in Valla's conjecture the l.t.- property is characterized by an effective numerical condition. It is also pointed out that the full validity of this conjecture is doubtful. In §5 we sketch a purely numerical approach to the linear type-property. The results so obtained are only a first step in the direction towards conditions which should be more effective, in particular from a practical point of view. §1 B is devoted to examples. Various other examples appear throughout the text.

The concept of linear type ideals seems to have been introduced in [Mi], although not under this name (which is due to Robbiano and Valla). Some main contributions were given in [S], [MSS] and [Va1], [Va2]. Since then there has been constant interest in this topic. In particular we mention the papers [HSiV1 − 3], [SV2 − 3] of Herzog, Vasconcelos and Simis and [Hu4] of Huneke. This work, however, goes into a direction rather different from our line of thinking. Since we will not return to this point of view in the main text we describe briefly their approach:

An essential purpose of this work is to relate the Cohen-Macaulayness and normality of the Rees algebra and the associated graded algebra of I to the Koszul homology $H_i(I, A)$ of I. This is done via various approximation complexes. The fundamental idea of Vasconcelos and Simis is to construct a graded comples $\underline{M}(I) = \underline{M}$, with the nth graded piece $\underline{M}_m$ of $\underline{M}$

$$\to H_j(I; A) \otimes \text{ Sym }_{m-j}(A^n/IA^n) \to \ldots \to \text{ Sym }_m(A^n/IA^n) \to 0 \quad ,$$

where n is the number of generators of I (which are fixed) . It then follows for the homology of M that $H_0(\underline{M}) \cong \text{ Sym }(I/I^2)$, and $H_i(\underline{M})$ is independent of the generating set of I. Now, if $\underline{M}$ is acyclic, I is of linear type. To get the acyclicity of $\underline{M}$ they ask for conditions on the depth of the $H_i(I, A)$, in particular they ask for the Cohen-Macaulayness of the Koszul homology to obtain in addition the Cohen-Macaulayness of Sym (I) and Sym (I/I^2). These properties are of course difficult to check and moreover in general far too strong if one only wants to have linear type. All these papers, mentioned above, contain various variations of this main theme.

We use the following notations throughout the report:

$$Re(I) = A[IT] = \sum I^n T^n = \text{ Rees algebra of } I$$

$$\begin{aligned}
gr_I(A) &= \sum I^n/I^{n+1} = \text{ graded algebra associated to } I \\
\text{Sym }(M) &= \text{ symmetric algebra of an } A- \text{ module } M \\
\text{ht}(I) &= \text{ height of } I \\
s(I) &= \text{ analytic spread of } I \\
\mu(M) &= \text{ ``least '' number of generators of } M \\
\ell(M) &= \text{ length of } M \quad .
\end{aligned}$$

The Cohen-Macaulay property is sometimes denoted with CM.

The computations were done with the computer algebra system MACAULAY.

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§1. On linear type and geometric linear type

Part A: Generalities

Given an A-module M, we shall be concerned, in this section, with the structural morphism $A \to \text{Sym}\,(M)$. The following will be of use.

Observation 1: Let $\tilde{\pi} : A \to C$ be a morphism of rings inducing π : $\text{Spec}C \to \text{Spec}(A)$, and assume that $\pi^{-1}(P)$ (the fibre over $P \in \text{Spec}(A)$) is irreducible $\forall P \in \text{Spec}(A)$. Then π admits a natural set theoretical section $s : \text{Spec}(A) \to \text{Spec}(C)$, where $s(P)$ is the generic point of $\pi^{-1}(P)$.

For any prime $Q \in \text{Spec}(C)$ we have $Q \supseteq s(\pi(Q))$, in particular equality will hold if Q is a minimal prime of C.

In what follows I is an ideal of a noetherian ring A. Consider the natural morphism

$$0 \to K \to \text{Sym}\,(I) \xrightarrow{\alpha} Re(I) \to 0 \quad .$$

Definition 1: i) I is of linear type (l.t.) if α is an isomorphism.
ii) I is of geometric linear type (g.l.t.) if K is nilpotent.

Observation 2: The following are equivalent:

i) I is of geometric linear type.

ii) $\text{Spec}(\,\text{Sym}\,(I)) \overset{\tilde{\alpha}}{\leftarrow} \text{Spec}(Re(I))$, ($\tilde{\alpha}$ defined via α settheoretically).

Observation 3: For any prime ideal $P \in \text{Spec}(A)$, $k(P) \underset{A}{\otimes} \text{Sym}_A(I) = \text{Sym}_{k(P)}(I \underset{A}{\otimes} k(P)) \simeq k(P)[X_1, \ldots, X_n]$ (a polynomial ring in $n = \mu(I_P)$ variables over the field $k(P) := A_P/PA_P$).

Definition 2: Consider $\pi : \text{Spec}(C) \to \text{Spec}(A)$, where C is an $\mathbb{N}$-graded A-algebra and $C_0 = A$. C is said to be linear at $P \in \text{Spec}(A)$ if $k(P) \underset{A}{\otimes} C$ is isomorphic, as graded ring, to a polynomial ring over the field $k(P)$.

Now we fix the diagram

$$\begin{tikzcd} \text{Spec}(Re(I)) \arrow[rr, "\tilde{\alpha}"] \arrow[dr, "\pi_1"'] & & \text{Spec}(\,\text{Sym}\,(I)) \arrow[dl, "\pi_2"] \\ & \text{Spec}(A) & \end{tikzcd}$$

Proposition 1.1: The following are equivalent:

1) All fibers of π_1 are linear.
2) $\widetilde{\alpha}$ induces an isomorphism on each fibre.
3) $\mu(I_P) = s(I_P) \quad \forall P \in \operatorname{Spec}(A)$.
4) I is of geometric linear type.

Proof: Consider:

$$0 \to K \to \operatorname{Sym}(I) \to Re(I) \to 0 \quad ,$$

where $K_0 = K_1 = 0$. So on each fibre we have:

$$K \underset{A}{\otimes} k(P) \xrightarrow{\beta} \operatorname{Sym}_{k(P)}(I \underset{A}{\otimes} k(P)) \xrightarrow{\gamma} Re(I) \underset{A}{\otimes} k(P) \to 0 \quad .$$

Since the middle ring is a polynomial ring in $\mu(I_P)$ variables over the field $k(P)$, it follows that the ring at the right is a polynomial ring (isomorphic as graded algebra) if and only if $im(\beta) = 0$. This proves the equivalence of 1) and 2), and that 2) implies 3).

Also the implication 3)$\Rightarrow$ 2) follows from this remark and the surjectivity of γ.

Now we prove 4)$\Rightarrow$ 2): since K is nilpotent , the same holds for $im(\beta)$. Therefore $im(\beta) = 0$.

Finally 2)$\Rightarrow$ 4) is an application of Observation 1 to $C =$ Sym (I); in fact 2) asserts that K is included in all minimal primes of C , i.e. K is nilpotent.

Remark 1: If $I \subseteq A$ is of geometric linear type and $A \to B$ is a flat morphism of rings, then IB is of geometric linear type. In particular $I_P \subseteq A_P$ is of geometric linear type $\forall P \in \operatorname{Spec}(A)$.

Recall that for an ideal I of a local ring $(A, \mathfrak{m})$

$$\mu(I) \geq s(I) \geq \operatorname{ht}(I) \quad ,$$

where $s(I)$ is the analytic spread of I. Indeed $s(I)$ is defined in terms of the dimension of a closed fibre of a proper (moreover projective) morphism, the last inequality being Grothendiecks " upper semicontinuity" of fibre dimensions; s. [H01].

Proposition 1.2: For an ideal I of a local ring $(A, \mathfrak{m})$ the following are equivalent:

i) I is of geometric linear type and equimultiple (i.e. $s(I) = \mathrm{ht}(I)$).

ii) I is an ideal of the principal class, i.e. $\mathrm{ht}(I) = \mu(I)$.

Proof: i)$\Rightarrow$ ii): Equimultiplicity means $\mathrm{ht}(I) = s(I)$ and g.l.t. asserts that $s(I) = \mu(I)$.

ii)$\Rightarrow$ i): For any $P \in \mathrm{Spec}(A)$ we have $\mathrm{ht}(I) \leq \mathrm{ht}(I_P)$ and $\mu(I) \geq \mu(I_P)$. So

$$\mathrm{ht}(I) \leq \mathrm{ht}(I_P) \leq s(I_P) \leq \mu(I_P) \leq \mu(I) \quad .$$

Now ii) implies that $\mu(I_P) = s(I_P) \quad \forall P \in \mathrm{Spec}(A)$, so ii)$\Rightarrow$ i) by Proposition 1.1.

Corollary 1.3: If in addition A is quasi-unmixed and $I = P$ is prime, then A is a domain and P is a complete intersection.

This is well known (s.e.g. [HSV], 3.16.7). We sketch here a proof using the g.l.t.-property.

Proof: Since $\mu(P) = \mathrm{ht}(P)$, there is a surjective morphism

$$A/P[X_1, \ldots, X_r] \xrightarrow{\theta} \mathrm{gr}_{\,P}(A) \to 0$$

where $r = \mu(P)$. Since both rings have the same dimension ($= \dim A$), and the first is a domain, θ must be an isomorphism. So P is a complete intersection. Also $gr_P(A)$ being a domain implies that A is a domain.

Remark 2: An algebraic "motivation" for the condition $\mu(I) = s(I)$ in a local ring $(A, \mathfrak{m})$ is the following observation:

If there exists a minimal prime P of I such that $\mathrm{ht}(P) = \mu(I)$, then $\mu(I) = s(I)$, since $\sup \{\mathrm{ht}(P) | P \in Min(I)\} \leq s(I) \leq \mu(I)$.

Huneke [Hu3] used the condition g.l.t. to prove that if $X = (x_{ij})$ is a generic n×n matrix and $I = I_{n-1}(X)$ is the ideal of $(n-1)$ size minors of X in $A = \mathbb{Z}[x_{ij}]$, then I is of linear type. One important observation (see Proposition 2.2 in [Hu3]) in Huneke's proof is the fact that the Rees algebra of an ideal of geometric linear type is a "generic point" for the symmetric algebra Sym (I) in the following sense:

Proposition 1.4 ([Hu3]): Let A be a reduced (noetherian) ring and let I be an ideal of grade $I \geq 1$ of A. Then the following are equivalent:

i) Sym $(I)^{red} \simeq Re(I)$.

ii) $\mu(I_P) = s(I_P) \quad \forall P \in \mathrm{Spec}(A)$.

Proof: Apply Proposition 1.1.

Remark 3: Huneke's proof is more complicated (see [Hu3] ,p. 325—326). Moreover the assumption grade $I \geq 1$ is unessential.

Only for this section we make the following

Definition 3: I is said to be of reduced linear type if i) of Proposition 1.4 holds.

Part B: Examples

E0) $$A = k[[X,Y]]/(X^2, XY^n) =: k[[x,y]], \quad n \geq 1 \text{ fixed} \quad .$$

$$I := (y)A \subset (x,y)A = \text{ minimal prime of } I \quad .$$

Then: $\dim A = \mathrm{ht}(I) = 1$, grade $(I) = 0$;

$$\mathrm{ht}(I_P) = s(I_P) = \mu(I_P) = 1 \quad , \quad \forall P \supseteq I \quad ,$$

i.e. I is geometric linear type.

But $0 : y \neq 0 : y^2$, hence [S]: Sym $(I) \not\cong Re(I)$.

Note: Sym $(I)^{red} \not\cong Re(I)$, since A and so $Re(I)$ are not reduced.

E1) $$A = k[[X,Y]]/(X \cdot Y) =: k[[x,y]] \quad , \quad \text{reduced} .$$
$$P = (x)A \quad \text{is a prime of height} \quad 0 \quad .$$

Here: $0 : x = 0 : x^2$,hence Sym $(P) \cong Re(P)$.

E2)[Ha] $$A = k[[X,Y,Z]]$$
$$P = (Y^2 - XZ; X^3 - YZ; Z^2 - X^2Y) \quad ,$$

defining the rational curve (t^3, t^4, t^5) in A^3.

Then:
$$\mu(P) = 3 = s(P) \underset{\neq}{>} \mathrm{ht}(P) = 2$$

$$\mu(P_P) = s(P_P) = 2 \quad , \quad \text{since } A_P \text{ regular} \quad .$$

Hence P is of geometric linear type. But P is also of linear type since it is generated by a d-sequence.

[Note that P is generated by the 2 size minors of $\begin{bmatrix} Y, & Z, & X^2 \\ X, & Y, & Z \end{bmatrix}$, and it is an almost complete intersection.].

Another argument for the linear type property comes from a theorem of Valla [Va1] and [Va2]: Since $\mu(P) = \text{grade}\,(P) + 1$ and $\mu(P_P) = \text{grade}\,(P) = 2$ we get: $\text{Sym}\,(P) \cong Re(P)$.

In this example we have:

$$Re(P) \cong k[[X,Y,Z]][u,v,w]/\mathfrak{a} \quad \text{, where}$$
$$\mathfrak{a} = (X^2u + Zv + Yw; Zu + Yv + Xw) \quad ,$$

i.e. in particular that $Re(P)$ is Cohen—Macaulay and normal.

E3) $A = k[[s^2, s^3, st, t]]$. A is Buchsbaum, but not Cohen—Macaulay .
$P = (st, t)$

Then:

$\text{ht}(P) = s(P) = 1 \underset{\neq}{<} \mu(P)$, hence not of geometric linear type and not of linear type,

A_P regular,

$\mu(P) \leq \text{grade}\,(P) + 1$.

Note that Valla's conditions in [Va2], thm. 3.4b) are fulfilled except that A is not Cohen—Macaulay. Here we have $Re(P) = \text{Sym}\,(P)/(u^2 - av^2)$, where

$$Re(P) = k[[a,b,c,d]][u,v]/\mathfrak{a}_1 \qquad \text{and}$$
$$\text{Sym}\,(P) = k[[a,b,c,d]][u,v]/\mathfrak{a}_2 \qquad \text{with}$$

$$\mathfrak{a}_2 = (ac-bd; du-cv; au-bv; c^2-ad^2; bc-a^2d; a^3-b^2; bu-a^2v; cu-adv)$$

and

$$\mathfrak{a}_1 = (\ \mathfrak{a}_2, u^2 - av^2) \quad .$$

Remark to E3): Note that P is an almost complete intersection since $\mu(P) = 2 = \text{ht}(P) + 1$ and A_P is regular. Hence by §4, thm. 4.7 there exists an ideal J and an element $x \notin J$ such that:

$$P = J + xA = tR + (st)R \quad , \quad J = \text{ complete intersection} \quad .$$

Note that $(t \underset{A}{:} st) \neq (t \underset{A}{:} (st)^2) = A$, which comes from the fact that J is not height—unmixed since $\text{Ass}(tA) = \{P, \mathfrak{m}\}$.

E4) $A := k[[s^2, st, t]] \supset P = (st, t)$.
Now A is Cohen—Macaulay (even a hypersurface), but A_P is not regular. Again we see that P is not of geometric linear type, since $\mathrm{ht}(P) = s(P) = 1 < \mu(P) = 2$.
Note that

$$Re(P) = k[[a, c, d]][u, v]/(c^2 - ad^2; du - cv; cu - adv; u^2 - av^2) \quad .$$

E5) (see [Val], Remark 3.7). We consider the coordinate ring of the rational curve in $\mathbb{P}^3_k$:

$$\begin{aligned} A &= k[[s^4, s^3t, st^3, t^4]] \\ &= k[[X_0, X_1, X_2, X_3]]/P =: k[[x_0, x_1, x_2, x_3]] \quad , \end{aligned}$$

where:

$$P = (X_0^2X_2 - X_1^3; X_0X_3 - X_1X_2; X_0X_2^2 - X_1^2X_3; X_1X_3^2 - X_2^3) \quad .$$

Here we get $\mu(P) = 4 > s(P) = 3$. Hence P is not of geometric linear type.

E6) Same A as in E5. Take $I = (s^4, t^4)$. Since s^4, t^4 in a s.o.p. in the Buchsbaum ring A, they form a d-sequence, hence I is of linear type. The Rees ring is

$$Re(I) = A[u, v]/(x_3u - x_0v; x_2^2u - x_1^2v) \quad .$$

E7) Same A as in E5. Take $I = (st^3, t^4)$ Then:

a) $\mathrm{ht}(I) = 1 < s(I) = 2 = \mu(I)$.

b) $I \subset P := (st^3, s^3t, t^4)$ and $A_P = k((s^4))(\frac{t}{s})$.

c) $I_P = (\frac{t}{s})^3 A_P$ and $\sqrt{I} = P$.

d) $\mu(I_P) = s(I_P) = 1$.

The Rees ring of this ideal is $R(I) = A[u, v]/\mathfrak{a}$, where

$$\mathfrak{a} = (x_3u - x_2v; x_1u - x_0v; x_2^2u - x_1x_3v; x_0x_2u - x_1^2v; x_2u^2 - x_1v^2) \quad .$$

[Note that this is a minimal base by MACAULAY.]

Hence I is of geometric linear type - and also of reduced linear type, but not of linear type.

We come back to this example in section 2 (see "Valla's conjecture").

E8)
$$A = k[[s^2, st, t^2, sz, tz, z]]$$
$$P = (sz, tz, z) \quad .$$
Then A and A/P are CM and A is normally CM along P, hence $\mathrm{ht}(P) = s(P) = 1$. We have
$$\mu(P) = 3 > s(P) = 1 \quad ;$$
i.e. P is not of geometric linear type.

Part C: On linear type in complex analytic geometry

One introduces stronger algebraic structures on complex spaces over a given complex space $\underline{S}$, (see [Fi],1.4-1.7). This allows a geometric intepretation of linear and geometric linear type.

Throughout, we use the notation
$$\underline{X} = (X, \mathcal{O}_X)$$
for a complex space, and
$$\underline{f} : \underline{X} \to \underline{Y}$$
for a morphism
$$(f, f^0) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \quad .$$
Further, $\underline{cpl/S}$ denotes the category of comples spaces on the given complex space $\underline{S}$.

Definition 1: Let $\underline{S}$ be a complex analytic space. A relative complex analytic space $\underline{L} \xrightarrow{\pi} \underline{S}$ is called a linear space over $\underline{S}$ (or simply a linear fibre space) if there are morphisms
$$\begin{array}{rclcll} \alpha & : & \underline{L} & \times_{\underline{S}} & \underline{L} & \longrightarrow & \underline{L} \\ \mu & : & \underline{\mathbb{C}}_S & \times_{\underline{S}} & \underline{L} & \longrightarrow & \underline{L} \end{array} \quad ,$$
in $\underline{cpl/S}$, where $\underline{\mathbb{C}}_S := \underline{S} \times \underline{\mathbb{C}}$, such that the module axioms hold.

Remark 1: If $\mathcal{F}$ is a coherent sheaf on $\underline{S}$, $\underline{\mathrm{Specan}}(\mathrm{Sym}\,(\check{\mathcal{F}})) \to \underline{S}$, where $\mathrm{Sym}\,(\check{\mathcal{F}})$ is the symmetric algebra on $\check{\mathcal{F}}$, is a linear space. It can be shown that, conversely, any linear space $\underline{L} \xrightarrow{\pi} \underline{S}$ arises this way by putting
$$\mathcal{F} := \text{ sheaf of germs of linear forms on } \underline{L} \quad .$$
This gives an anti-equivalence
$$\underline{lin/S} \overset{\simeq}{\leftrightarrows} \underline{coh/S}$$
of the category of linear spaces and linear morphism over $\underline{S}$ with the category of coherent sheaves on $\underline{S}$ (Duality Theorem).

Remark 2: If $\underline{L} \xrightarrow{\underline{\pi}} \underline{S}$ is a linear space then, for all $s \in S$, the fibre $\underline{\pi}^{-1}(s)$ is a finite dimensional vectorspace (if $\underline{L} = \underline{\text{Specan}}\ \text{Sym}\ (\check{\mathcal{F}})$, $\underline{\pi}^{-1}(s) = \underline{\text{Specan}}(\ \text{Sym}$ $((\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x)^\vee))$. The converse need not be true.

Let $\underline{X} \in cpl$, $\underline{Y} \hookrightarrow \underline{X}$ be a closed complex subspace defined by a coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$.

Definition 2: $\underline{C}(\underline{X}, \underline{Y}) := \underline{\text{Specan}}(\bigoplus_{k \geq 0} \mathcal{I}^k/\mathcal{I}^{k+1})$, the normal cone of $\underline{Y}$ in $\underline{X}$

$\underline{N}(\underline{X}, \underline{Y}) := \underline{\text{Specan}}(\ \text{Sym}\ (\mathcal{I}/\mathcal{I}^2))$, the normal bundle of $\underline{Y}$ in $\underline{X}$.

We have natural maps

$$\begin{tikzcd} \bigoplus_{k \geq 0} \mathcal{I}^k/\mathcal{I}^{k+1} & & \text{Sym}\ (\mathcal{I}/\mathcal{I}^2) \arrow[ll] \\ & \mathcal{O}_X/\mathcal{I} \arrow[ul] \arrow[ur] & \end{tikzcd},$$

giving the commutative diagram of spaces

$$\begin{tikzcd} \underline{C}(\underline{X}, \underline{Y}) \arrow[rr, hook, "\underline{j}"] \arrow[dr, "\underline{\nu}"'] & & \underline{N}(\underline{X}, \underline{Y}) \arrow[dl, "\underline{\pi}"] \\ & \underline{Y} & \end{tikzcd}.$$

$\underline{N}(\underline{X}, \underline{Y}) \xrightarrow{\underline{\pi}} \underline{Y}$ is a linear space, and $\underline{C}(\underline{X}, \underline{Y}) \xrightarrow{\underline{\nu}} \underline{Y}$ a so-called conebundle, the fibres of which are cones sitting in the linear fibres of $\underline{\pi}$.

Definition 3: (i) $\underline{Y} \overset{\underline{i}}{\hookrightarrow} \underline{X}$ is of linear type: $\iff$ $\underline{j}$ is an isomorphism of complex spaces over $\underline{Y}$.

(ii) $\underline{Y} \overset{\underline{i}}{\hookrightarrow} \underline{X}$ is of geometric linear type: $\iff$ $\underline{j}_{red}$ is an isomorphism of reduced complex spaces, i.e. the fibres of $\underline{\nu}$ are linear.

Definition 4: $\underline{R}(\underline{X}, \underline{Y}) := \underline{\text{Specan}}(\bigoplus_{k \geq 0} \mathcal{I}^k)$, the "Rees space" of $\underline{Y}$ in $\underline{X}$;

$\underline{S}(\underline{X}, \underline{Y}) := \underline{\text{Specan}}(\ \text{Sym}\ (\mathcal{I}))$, the normal space of $\underline{Y}$ in $\underline{X}$.

The natural homomorphisms

$$\begin{tikzcd}
\bigoplus_{k\geq 0} \mathcal{I}^k & & \mathrm{Sym}\,(\mathcal{I}) \arrow[ll] \\
& \mathcal{O}_X \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

gives the commutative diagram of spaces

$$\begin{tikzcd}
\underline{R}(\underline{X},\underline{Y}) \arrow[rr, hook, "\underline{k}"] \arrow[dr, "\underline{r}"'] & & \underline{S}(\underline{X},\underline{Y}) \arrow[dl, "\underline{s}"] \\
& \underline{X} = \mathrm{Specan}(\mathcal{O}_X) &
\end{tikzcd} \tag{2}$$

Over $\underline{Y} \hookrightarrow \underline{X}$, this diagramm pulls back to the diagram (1). In particular, there are natural inclusions

$$\begin{tikzcd}
\underline{C}(\underline{X},\underline{Y}) \arrow[r, hook] \arrow[d] & \underline{R}(\underline{X},\underline{Y}) \arrow[d] \\
\underline{N}(\underline{X},\underline{Y}) \arrow[r, hook] & \underline{S}(\underline{X},\underline{Y})
\end{tikzcd}$$

corresponding to

$$\begin{tikzcd}
\bigoplus_{k\geq 0} \mathcal{I}^k/\mathcal{I}^{k+1} & \bigoplus_{k\geq 0} \mathcal{I}^k \arrow[l] \\
\mathrm{Sym}\,(\mathcal{I}/\mathcal{I}^2) \arrow[u] & \mathrm{Sym}\,(\mathcal{I}) \arrow[l] \arrow[u]
\end{tikzcd}$$

We now come to the analytic ("schematic") closure, (see [Fi], 0.44):

1) Let $\underline{Y} \hookrightarrow \underline{X}$ be a locally closed subspace of the complex space $\underline{X}$. The smallest closed complex subspace $\underline{Z} \hookrightarrow \underline{X}$ containing $\underline{Y}$ as a locally closed subspace is called, if it exists, the <u>analytic closure of $\underline{Y}$ in $\underline{X}$</u> and denoted $\overline{\underline{Y}}$.

2) Let $\underline{Y} \hookrightarrow \underline{X}$ be a closed subspace, $A \subseteq X$ an analytic set, and let $\mathcal{A} \subseteq \mathcal{O}_X$ be any coherent ideal with $A = \mathrm{supp}(\mathcal{O}_X/\mathcal{A})$. Let $\underline{Y}$ be defined by the

coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$, and define the gap sheaf $\mathcal{I}[A]$ of $\mathcal{I}$ with respect to A: If $U \subseteq X$ is open, then

$$\mathcal{I}[A](U) := \{f \in \mathcal{O}_X(U) | f \mid U - A \in \mathcal{I}(U - A)\} \quad .$$

Then, using Rückert's Nullstellensatz, one shows

$$\mathcal{I}[A] = \bigcup_{k \geq 0} (\mathcal{I} \ : \ \mathcal{A}^k) \quad ,$$

and so is coherent. Thus the analytic closure $\overline{\underline{Y} - A}$ in X exists and is given by $\mathcal{I}[A]$.

3) We have $\overline{\underline{Y} - A} \hookrightarrow \underline{Y}$, and $\overline{\underline{Y} - A} = \underline{Y}$ if and only if the vanishing ideal sheaf $\mathcal{I}_{A \cap Y} \subseteq \mathcal{O}_Y$ of $A \cap Y$ in Y contains a nonzerodivisor in every stalk, in which case $A \cap Y$ is called analytically rare in Y. Geometrically this means that at any point $y \in A \cap Y$ there is an $f \in \mathcal{O}_{Y,y}$ which vanishes on $A \cap Y$ near y, but not on any of the local components (including the embedded ones) of Y at y.

As a corollary we obtain a geometric version of a theorem proved algebraically by [HSiV2], thm. 3.2.

Theorem 1.5: Let $\underline{X}$ be a complex space, $\underline{Y} \hookrightarrow \underline{X}$ a closed complex subspace. The following statements are equivalent:

(i) $\underline{R}(\underline{X}, \underline{Y})_{red} \overset{k_{red}}{\hookrightarrow} \underline{S}(\underline{X}, \underline{Y})_{red}$ is the identity

(ii) $\underline{C}(\underline{X}, \underline{Y})_{red} \overset{i_{red}}{\hookrightarrow} \underline{N}(\underline{X}, \underline{Y})_{red}$ is the identity.

Proof (i)$\Rightarrow$(ii): is clear.

(ii)$\Rightarrow$(i): By [Fi], p. 163, we have that the canonical immersion

$$\underline{X} - Y \hookrightarrow \mathbb{P}\underline{S}(\underline{X}, \underline{Y})$$

gives, upon closure, the blowup of $\underline{X}$ along $\underline{Y}$:

$$\mathbb{P}\underline{R}(\underline{X}, \underline{Y}) = \overline{\underline{X} - Y} \quad .$$

Now $\mathbb{P}\underline{C}(\underline{X}, \underline{Y})$ is analytically rare in $\mathbb{P}\underline{R}(\underline{X}, \underline{Y})$, because it is a divisor, and so in $\mathbb{P}\underline{S}(\underline{X}, \underline{Y})$, being contained in $\mathbb{P}\underline{R}(\underline{X}, \underline{Y})$. Hence

$$\mathbb{P}\underline{S}(\underline{X}, \underline{Y}) = \overline{\mathbb{P}\underline{S}(\underline{X}, \underline{Y}) - \mathbb{P}C(\underline{X}, \underline{Y})} \quad .$$

Now, if we have (ii), we have $\mathbb{P}C(\underline{X},\underline{Y}) = \mathbb{P}N(\underline{X},\underline{Y})$, but $\underline{\mathbb{P}S}(\underline{X},\underline{Y}) - \underline{\mathbb{P}N}(\underline{X},\underline{Y}) = \underline{X} - Y$, hence

$$\underline{\mathbb{P}S}(\underline{X},\underline{Y}) = \overline{\underline{X} - Y} = \underline{\mathbb{P}R}(\underline{X},\underline{Y}) \quad .$$

This implies

$$\underline{S}(\underline{X},\underline{Y}) - X = \underline{R}(\underline{X},\underline{Y}) - X \quad .$$

Since X is nowhere dense in both, $S(\underline{X},\underline{Y})$ and $R(\underline{X},\underline{Y})$, we get (i) upon taking the settheoretic closures. Q.e.d.

§2. On Valla's conjecture

Conjecture 1: Let A be a regular ring an P a prime ideal in A. Then if $\mu(P) = s(P)$, P is of linear type.

If P is not prime Valla showed us in a letter a counterexample to that conjecture:

Let $I = (X^3, X^2Y, Y^2Z) \subseteq k[X,Y,Z] =: A$. Then:

$$\mu(I) = s(I) = 3$$
$$\text{Sym}\,(I) = A[T_1, T_2, T_3]/(YT_1 - XT_2; YZT_2 - X^2T_3] \quad .$$

Now $ZT_2^2 - XT_1T_3$ is vanishing for $(T_1, T_2, T_3) = (X^3, X^2Y, Y^2Z)$, but this element is not in $(YT_1 - XT_2; YZT_2 - X^2T_3)$.

A "weaker" conjecture is the following:

Conjecture 2: Let A be regular and P a prime ideal of geometric linear type. Then P is of linear type.

If A is not regular and if we also accept ideals I being only primary (to some ideal P), then example 7 of §1 is a counterexample.

Also this weaker form of Valla's conjecture is far from being trivial: Note that "geometric linear type" implies $\text{Sym}\,(P)^{red} \simeq Re(P)$ by Proposition 1.4. Moreover since A is a domain, $Re(P)$ and therefore $\text{Sym}\,(P)^{red}$ is a domain. What we want to show for proving the conjecture is that $\text{Sym}\,(P)$ itself is a domain, s. [Mi], [Hu5], theorem 1.1.

For this it is sufficient to show (see [Hu3], Lemma 2.3) that there is an element x of positive degree in $\text{Sym}\,(P)$, not nilpotent, such that $\text{Sym}\,(P)/(x)$ is reduced. Indeed, then we have $N := \text{nilrad}(\text{Sym}\,(P)) \subseteq (x)$ and given an element $y \in N$, say $y = sx$, we know that $s \in N$ (since N is prime in our case). So $N \subseteq xN$, hence $N = 0$.

Within the framework of these conjectures we consider the following special situation (*), where we can give a partial answer in the affirmative.

(*) Let $(A, \mathfrak{m})$ be a regular local ring and P a prime ideal of A contained in $\mathfrak{m}^2$. Assume that A/P is Cohen-Macaulay with maximal embedding dimension, i.e.

$$\mu(A/P) = e(A/P) + \dim(A/P) - 1 \quad .$$

Remark 1: The local ring at a rational surface singularity satisfies these hypotheses on A/P.

Remark 2: (see also [GoShi], p. 71) (*) implies [Sa1] that $\mu(P) = \binom{e}{2}$, where $e = e(A/P)$. Moreover $\mu(A/P) = \dim(\mathfrak{m}/\mathfrak{m}^2)$ since $P \subset \mathfrak{m}^2$, hence $\mu(A/P) = d$, since A is regular. So we get

(1) $$d = e + d^* - 1 \quad ,$$

where $d = \dim A$ and $d^* = \dim(A/P)$, in particular we have $h := \mathrm{ht}(P) = e - 1$.

Theorem 2.1: Under the assumptions (*) we have:

a) P is an almost complete intersection (a.c.i.) iff $e = 3$ (i.e. $h = 2$).

b) If $h = 2$ then the asymptotic depths $\liminf(\text{ depth } (P^n/P^{n+1}))$ and $\liminf(\text{ depth } (R/P^n))$ denoted by $a(P)$ and $b(P)$ are $d - 3$.

c) If $d^* \leq 2$, then: $\mu(P) = s(P)$ iff P is of linear type.

Proof: a) P is a.c.i. iff $\mu(P) = h + 1$, hence iff $\binom{e}{2} = e$, i.e. $e = 3$.

b) $h = 2$ implies that P is a.c.i., hence P is of linear type (since P can be generated by a d-sequence, s.e.g. [GoShi], lemma 2:5). Therefore we have $\mu(P) = s(P)$.

Moreover (Burch's inequality):

(2) $s(P) \leq d - \inf \text{ depth } (P^n/P^{n+1}) \leq d - \inf \text{ depth } (A/P^n)$.

Using (1) we get:

$$\inf(\text{ depth } (P^n/P^{n+1})) \leq d^* - 1 \quad ,$$
$$\inf(\text{ depth } (A/P^n)) \leq d^* - 1 \quad .$$

Since P is generated by a d-sequence, say $P = (x_1, x_2, x_3)$, where (x_1, x_2) is a regular sequence in A, we know that

$$\dim A - \text{ depth } (A/(x_1, \cdots, x_k) \leq k$$

for $1 \leq k \leq 3$. Then [Hu], theorem 4.2:

$$\inf(\text{ depth } (A/P^n)) = d - 3 = d^* - 1 \quad ,$$

hence $\inf(\text{ depth } (P^n/P^{n+1}) = \inf(\text{ depth } (A/P^n) = d-3$. So equality occurs in Burch's inequality (2), therefore $a(P) = b(P) = d - 3$.

c) If $\underline{d^* = 1}$ then $d = e = h + 1$. Assuming $\mu(P) = s(P)$ we get

$$s(P) = \binom{h+1}{2} \leq d - a(P) = (h + 1) - a(P) \quad ,$$

hence $h = 1$ or 2 ($h = 2$ is possible because of b)). If $h = 1$ then P is a complete intersection and so P is of linear type. If $h = 2$ then P is a.c.i., hence of linear type too.

If $\underline{d^* = 2}$ and $\mu(P) = s(P)$ then $d = h + 2$ and

$$s(P) = \binom{h+1}{2} \leq (h+2) - a(P) \quad .$$

On the other hand $a(P) = 0$ or 1, hence again $h \leq 2$, i.e.P is of linear type, q.e.d.

Remark 3: Valla's conjecture would be always true for the situation (*), if $a(P) \geq d^* - 2$.

In general this condition is much too strong, as pointed out to us by Shin Ikeda: Let $X = (x_{ij})$ be a generic $n \times (n+1)$-matrix and $P = I_n(X)$ the ideal of n size minors of X in $k[x_{ij}] =: A$. Then

$$d^* = \dim\ A/P = n(n+1) - 2$$

and $a(P) = n^2 - 1$. Hence the condition $a(P) \geq d^* - 2$ would imply $n \leq 3$.

Remark 4: Note that for a.c.i. P in the situation (*) we have $a(P) = 0$ if $d^* = 1$ and $a(P) = 1$ if $d^* = 2$.

Note furthermore that for a.c.i. ideals P in the situation (*) we know [GoShi], Cor. 3.5, theorem 1.2, that $gr_P(A)$ and $Re(P)$ are both Gorenstein. But A/P is of course not Gorenstein [Ku]. Under the assumption of situation (*) we mention a quite elementary proof of the fact, that almost complete intersections are not Gorenstein rings:

P is a.c.i. means $h = \mathrm{ht}(P) = 2$, hence

$$\text{depth } (A/P) = \dim(A/P) = d - 2 \quad .$$

Then we know by [Sa2], theorem 3.2, p. 84 that

$$\mu(P) \leq (n+1)e(A) = n + 1 \quad , \qquad \text{where}$$

$n := \dim_k Ext_A^{d-2}(k, A/P)$, $\quad k = A/\mathfrak{m}$. Since

$$\begin{aligned} Ext_A^{d-2}(k, A/P) &\cong \mathrm{Hom}_A(k, A/(P, x_1, \ldots, x_{d-2})) \\ &\cong \mathrm{Hom}_{A/P}(k, A/(P, x_1, \ldots, x_{d-2})) \\ &\cong Ext_{A/P}^{d-2}(k, A/P) \quad , \end{aligned}$$

where $x_1, \ldots, x_{d-2}$ is a regular sequence on A/P, n is the highest Bass-number of A/P at $\mathfrak{m}/P$. In our case $\mu(P) = h + 1 = 3$, i.e. $n \geq 2$, hence A/P is not Gorenstein.

Examples for the situation (*):

E1) $A = k[[X]]$, where $X = (x_{ij})$ is a generic 2×3 matrix. Let $P = I_2(X)$ the ideal generated by the 2 size minors. Then A/P is Cohen-Macaulay with:

$$e(A/P) = e = 3 \quad ; \quad d = G \quad ; \quad d^* = 4 \quad , \quad \mu(P) = 3 \quad .$$

E2) Example E2) of §1 .

We come back to Valla's conjecture in a more general frame in §4 .

To conclude we mention another situation where Valla's conjecture is true:

For the ideals $I_k(X)$ generated by the k size minors of a generic matrix X, the following statements are equivalent:

(i) $I_k(X)$ is of linear type

(ii) $I_k(X)$ is geometric linear type

(iii) $\mu(I_k(X)) = s(I_k(x))$.

The reason is that all three conditions are equivalent with the fact that there are no non-trivial Plücker relations (s. [Hu3]).

§3. Krull dimension of symmetric algebras

Here we mention the Huneke-Rossi-formula [HuR] linking the dimension of Sym (M) to the Forster number that bounds the number of generators M as expressed in its local data; Simis and Vasconcelos showed in [SiV1] how to derive this result from a theorem expressing $\dim(A)$ in terms of ideals of linear type. Here we put these arguments into a somewhat more general framework and show how Simis' and Vasconcelos'arguments fit into a general dimension formula for arbitrary ideals. We base our approach on the following theorem.

Theorem 3.1: Let A be a noetherian ring, $I \subseteq A$ an ideal contained in the Jacobson radical of A. Then

$$\dim(A) = \sup_{P \supseteq I}\{\dim(A/P) + s(I_P)\} \quad .$$

Remark 1: This is clear if A is catenarian, since we have always

$$\dim(A) \geq \sup_{P \supseteq I}\{\dim(A/P) + s(I_P)\} \quad .$$

If A is catenarian, pick $P_0 \in$ Min (A/I) to be such that $\mathrm{ht}(I) = \mathrm{ht}(P_0)$. Then $\dim(A) = \dim(A/P_0) + s(I_{P_0})$, which proves the claim.

For the general case we need some generalities on the dimension of noetherian rings.

Lemma 3.2: Let B be a domain which is a finitely generated k-algebra over a field k. Then

$$tr.d._k(Quot(B)) = \dim(B) \quad .$$

Proof: By Noether normalization, B is finite over a polynomial subring $k[X_1, \ldots, X_d] \hookrightarrow B$, where $d = tr.d._k(Quot(B))$. Then, by going up

$$\dim(B) = \dim k[X_1, \ldots, X_d] = d \quad .$$

Lemma 3.3: Let B be finitely generated over a subring A, $P \in \mathrm{Spec}(B)$, and $p := P \cap A$. Then

$$\begin{aligned}\dim(B/P) &\leq \dim(A/p) + tr.d._{k(p)}k(P) \\ &\leq \dim(A/p) + \dim(B \otimes k(p)) \quad .\end{aligned}$$

Proof: For the first inequality, replacing B/P by B and A/p by A, we may assume that A and B are integral domains and $P = (0)$, $p = (0)$. By the dimension inequality in [Ma], Theorem 23, we have, for any $Q \in \operatorname{Spec}(B)$ and $q := Q \cap A$:

$$\begin{aligned} \operatorname{ht}(Q) &\leq \operatorname{ht}(q) + tr.d._A B - tr.d._{k(q)} k(Q) \\ &\leq \dim(A) + tr.d._A B \quad . \end{aligned}$$

Choosing Q with $\operatorname{ht}(Q) = \dim(B)$ proves the first inequality.

For the second inequality consider the ring extension $B/P \hookleftarrow A/p$, which establishes $B/P \otimes k(p) = B/P \otimes Quot(A/p)$ as a finitely generated $k(p)$-algebra; hence , by Lemma 3.2:

$$tr.d._{k(p)} k(P) = \dim(B/P \otimes k(p)) \quad ,$$

since $Quot(B/P \otimes k(p)) = Quot(B/P) = k(P)$. But by right exactness of "$\otimes$", $B/P \otimes k(p)$ is a quotient of $B \otimes k(p)$, whence $\dim(B/P \otimes k(p)) \leq \dim(B \otimes k(p))$. Q.e.d.

Remark 2: It might be instructive to look at the geometric significance of this fomula. Let $f : X \to Y$ be a morphism of finite type of noetherian schemes (corresponding to $A \hookrightarrow B$), and let $X' \hookrightarrow X$, $Y' = \overline{f(X')} \hookrightarrow Y$ be closed subschemes (corresponding to P and p). This gives the commutative diagramm

$$\begin{array}{ccccc} & X' & \hookrightarrow & X \\ f' := f|X' & \downarrow & & \downarrow f \\ & Y' & \hookrightarrow & Y \end{array}$$

Then Lemma 3.3 can be interpreted as:

$$\begin{aligned} \dim X' &\leq \dim Y' + \text{“generic fibre dimension of } f' \text{''} \\ &\leq \dim Y' + \text{“generic fibre dimension of } f \text{ along } Y' \text{''}, \end{aligned}$$

the second inequality being geometrically obvious from $(f')^{-1}(y) \subseteq f^{-1}(y)$ for all $y \in Y'$.

Proof of Theorem 3.1: For any $P \supseteq I$, one has (see Remark 1):

$$\dim(A/P) + s(I_P) \leq \dim(A) \quad .$$

It suffices to exhibit one $P \supseteq I$ such that

$$\dim(A/P) + s(I_P) \geq \dim(A) \quad .$$

For this consider $B := gr_I(A)$, $\widetilde{A} := A/I$. For any $\mathcal{P} \in \mathrm{Spec}(B)$, we have by Lemma 3.3:

$$\dim(B/\mathcal{P}) \leq \dim(A/P) + s(I_P)$$

(where we denote by P the inverse image of $\mathcal{P} \cap \widetilde{A}$ under $A \to \widetilde{A} = A/I$). Choosing $\mathcal{P}$ so that $\dim(B/\mathcal{P}) = \dim B$ proves the claim, since $\dim B = \dim(gr_I(A)) = \dim A$. Q.e.d.

Remark 3: Geometrically one looks at the normal cone map

$$\nu : C(X, Y) \to Y$$

with $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(A/I)$, and restricts it to an irreducible component of $C(X,Y)$ having the maximal dimension.

Before proving the theorem of Huneke and Rossi we make a last remark.

Remark 4: Using the natural $\mathrm{Sym}\,(M)$- homomorphism

$$M \underset{A}{\otimes} \mathrm{Sym}\,(M) \xrightarrow{\varphi} \mathrm{Sym}\,(M)_+ \to 0$$

and taking the n-th symmetric power of φ we get

$$\mathrm{Sym}\,_n(M) \otimes \mathrm{Sym}\,(M) \longrightarrow \mathrm{Sym}\,_n(\mathrm{Sym}\,(M)_+) \to 0 \quad ;$$

then the universal property of such powers with respect to the base change $A \to \mathrm{Sym}\,(M)$ shows immediately that

$$(\mathrm{Sym}\,(M)_+)^n \cong \mathrm{Sym}\,_n(\mathrm{Sym}\,(M)_+) \quad ,$$

i.e $I := \mathrm{Sym}\,(M)_+$ is of linear type; s. [HSiV1], Expl. 2.3, p. 87, or for another argument [Val], Prop. 3.11.

Theorem 3.4 ([HuR]:) Let A be a noetherian ring and M a finitely generated A- module. Then

$$\dim(\mathrm{Sym}\,(M)) = \sup_{P \in \mathrm{Spec}(A)} \{\dim(A/P) + \mu(M_P)\} \quad .$$

Proof: Localizing $S := \mathrm{Sym}\,(M)$ at the multiplicative set $1 + I$ (where $I = \mathrm{Sym}\,(M)_+$) we get $IS_{(1+I)}$ in the Jacobson radical of $S_{(1+I)}$; and all primes p over I have the form $P + I$ with $P \in \mathrm{Spec}A$. Since I is in particular of geometric linear type, we know that

$$s(I_p) = \mu(I_p) = \mu(M_P)$$

by the universal property of $\mathrm{Sym}\,(M)$, which proves Theorem 3.4.

§4. Almost complete intersections

The main result in this section is the characterization of a generalized notion of an almost complete intersection (a.c.i.), in arbitrary local rings. As an application we get a characterization of an a.c.i. in Cohen-Macaulay rings. In particular those a.c.i.'s are of linear type. This was mentioned by Huneke [Hu4]; somewhat earlier Vasconcelos and Simis [SV1] indicated a proof for that fact using ideas related to our Primbasissatz (s. Theorem 4.2). This Primbasissatz for general ideals in local rings seems to have gone unmentioned, therefore we sketch the proof. In this way we analyse finally the interplay between properties of an ideal to be an a.c.i., to be of linear type or to be of geometric linear type.

Part A: A general Primbasissatz for local rings

In this section $(A, \mathfrak{m})$ will be a local ring, noetherian, with infinite residue field $k = A/\mathfrak{m}$ [the reason for this being that, for a finite dimensional vectorspace over k, a finite union of proper subvarieties cannot exhaust the whole space, and this allows for genericity arguments.] We denote the n-fold product $I \times \ldots \times I$ of an ideal $I \subseteq A$ by $I^{[n]}$, to prevent confusion with the n-fold idealtheoretic product $I \cdot \ldots \cdot I = I^n$. Elements $(x_1, \ldots, x_n) \in I^{[n]}$ are also denoted by $\underline{x}$. We endow $I^{[n]}$ with the topology induced on it by the projection $\pi : I^{[n]} \to (I/\mathfrak{m}I)^{\oplus n}$ from the Zariski-topology on the finite dimensional k-vectorspace $(I/\mathfrak{m}I)^{\oplus n}$.

Definition 1: A subset $U \subseteq I^{[n]}$ is called generic if U contains a nonempty open subset.

In the following proposition, we collect various properties of generic sets.

Proposition 4.1:

(i) A generic set is dense.

(ii) The intersection of finitely many generic sets is generic.

(iii) Let $m \leq n$ and $\pi : I^{[n]} \to I^{[m]}$ be the projection. If $U \subseteq I^{[n]}$ is generic, so is $\pi(U) \subseteq I^{[m]}$. If $V \subseteq I^{[m]}$ is generic, so is $\pi^{-1}(V)$.

(iv) Let $\varphi : A \to S$ be a surjective homomorphism of local rings, $J \subseteq S$ an ideal, and let $I := \varphi^{-1}(J)$. If $V \subseteq J^{[n]}$ is generic so is $\varphi^{-1}(V) \subseteq I^{[n]}$.

(v) Let $(a_1, \ldots, a_m)$ be an idealbasis of the ideal I, $U \subseteq I^{[n]}$. Then U is generic if and only if the following holds:

There are polynomials $F_j(X) \in A[X]$, $\quad j = 1, \ldots, \ell$, where X is an n×m-matrix of indeterminates, say

$$X := \begin{pmatrix} X_1^1, \ldots, X_1^m \\ \ldots\ldots\ldots \\ X_n^1, \ldots, X_n^m \end{pmatrix} \quad ,$$

such that, if $(x_1, \ldots, x_n) \in I^{[n]}$, i.e.

$$x_i = \sum_{j=1}^{m} \xi_i^j a_j \quad ; \quad \xi_j^i \in A \quad , \quad 1 \le i \le n \quad ,$$

the fact that some $F_k(\xi)$ is a unit in A implies $(x_1, \ldots, x_n) \in U$.

Before formulating the Primbasissatz, we need two more concepts.

Definition 2: Let A be any ring, I an ideal. A height sequence for I (of length h) is a sequence $(a_1, \ldots, a_h) \in I^{[h]}$ such that $(a_1, \ldots, a_h)A$ is an ideal of height h.

Definition 3: Let A be a ring, $a \in A$ a nonunit. Then a is called active

$:\Leftrightarrow$ a is nonzerodivisor in A_{red}

$\Leftrightarrow$ $\forall P \in \mathrm{Min}(A) : a \notin P$

$\Leftrightarrow$ $\mathrm{ht}(aA) = 1$.

We can now formulate the Primbasissatz; s. [Mu,] , [Na].

Theorem 4.2: (Primbasissatz) Let $(A, \mathfrak{m})$ be a noetherian local ring with infinite residue field $k = A/\mathfrak{m}$. Let $I \subseteq A$ be an ideal of A. Then there exists a generic set $\mathcal{P}r(I) \subseteq I^{[\mu(I)]}$ ("Primbasen") such that for all $(a_1, \ldots, a_{\mu(I)}) \in \mathcal{P}r(I)$ the following statements hold:

(i) $(a_1, \ldots, a_{\mu(I)})$ is a minimal system of generators for I.

(ii) If $\mathrm{ht}(I) > 0$, all a_j are active or if grade $(I) > 0$, all a_j are nonzerodivisors.

(iii) $\forall P \in \mathrm{Ass}(A/I) : (a_1, \ldots, a_{\mu(IA_P)}) \cdot A_P = IA_P$.

(iv) $(a_1, \ldots, a_{\mathrm{ht}(I)})$ is a maximal height sequence for I; in particular

$$\mathrm{ht}((a_1, \ldots, a_{\mathrm{ht}(I)}) \cdot A) = \mathrm{ht}(I) \quad .$$

Addendum Because of Proposition 4.1(ii) one may constrain $\mathcal{Pr}(I)$ further by any finitely many additional generic conditions. So one may, in addition, require

$$\text{grade }((a_1,\ldots,a_{\text{grade }(I)})\cdot A) = \text{grade }(I)$$
$$s((a_1,\ldots,a_{s(I)})\cdot A) = s(I) \quad .$$

The proof of Theorem 4.2 proceeds by showing that any of the conditions (i)-(iv) imposed is generic and applying Proposition 4.1(ii). In fact, from Proposition 4.1, one deduces the following series of statements 4.3-4.6. Note for this that a subset of a vectorspace V is called maximally independent if any finite subset of $k \leq \dim V$ elements is linearly independent.

Proposition 4.3. Let I be an ideal, $m \in \mathbb{N}$. The set

$$\mathcal{B}(I,m) := \{(a_1,\ldots,a_m) \in I^{[m]} \mid \{a_1,\ldots,a_m\} \text{ is maximally independent modulo } \mathfrak{m}I\}$$

is generic.

Proof: Fix a basis of I. The independency conditions, then, are expressed in terms of nonvanishing of maximal minors modulo $\mathfrak{m}$ of the matrix expressing $a_1,\ldots,a_m$ in terms of the given basis. By Proposition 4.1(v), $\mathcal{B}(I,m)$ is generic.

Proposition 4.4. Let $\mathcal{P}$ be a finite set of primes of A, $I \subseteq A$ an idal with $I \not\subseteq \bigcup_{P\in\mathcal{P}} P$, $m \in \mathbb{N}$. Then the set

$$\mathcal{A}(I,m;\mathcal{P}) := \{(a_1,\ldots,a_m) \in I^{[m]} \mid \forall i : a_i \notin \bigcup_{P\in\mathcal{P}} P\}$$

is generic.

Proof: In $I/\mathfrak{m}I$, $\bigcup_{P\in\mathcal{P}} P$ is contained in a finite union of hyperplanes.

Proposition 4.5. Let $P \in \operatorname{Spec}(A)$ be a prime,

$$\lambda_P : A \longrightarrow A_P$$

the localization map, $I \subseteq A$ an ideal, $m \in \mathbb{N}$. Then, if $V \subseteq IA_P^{[m]}$ is generic, $\lambda_P^{-1}(V) \subseteq I^{[m]}$ is generic.

Proof: Choose, according to Proposition 4.1, polynomials defining V. Multiplying these by a common denominator lifts these to polynomials defining $\lambda_P^{-1}(V)$.

Proposition 4.6. Let $k \in \mathbb{N}$, $h \leq \mathrm{ht}(I)$, and

$$\mathcal{H}(I,h) := \{(a_1,\ldots,a_h) \in I^{[k]} \mid \mathrm{ht}((a_1,\ldots,a_h)A) = h\} \quad .$$

Then $\mathcal{H}(I,h) \subseteq I^{[h]}$ is generic.

Proof. If $\mathrm{ht}((a_1,\ldots,a_h)\cdot A) = h$, $\mathrm{ht}((a_1,\ldots,a_j)\cdot A) = j$ for $1 \leq j \leq h$. Hence, by Proposition 4.1(iii), it suffices to consider the case $h = \mathrm{ht}(I)$. Let $P \in \mathrm{Min}(A/I)$ with $\mathrm{ht}(I) = \mathrm{ht}(P)$. Then IA_P is P-primary, and s.o.p.'s, being minimal reductions, are generic by [NR], p. 153. Then apply Proposition 4.5.

Proof of Theorem 4.2. Let $\mathcal{P} \subseteq \mathrm{Ass}(A/I)$ be either empty if $\mathrm{ht}(I) = 0$ or $\mathrm{Min}(A/I)$ if $\mathrm{ht}(I) > 0$ or $\mathrm{Ass}(A/I)$ if grade $(I) > 0$. Let $\mu := \mu(I)$, $h = \mathrm{ht}(I)$, and

$$\pi : I^{[\mu]} \longrightarrow I^{[h]}$$

be the projection. Then put

$$\begin{aligned}\mathcal{P}r(I) :=& \mathcal{B}(I,\mu) \cap \mathcal{A}(I,\mu;\mathcal{P}) \cap \\ & \bigcap_{P \in \mathrm{Ass}(A/I)} \lambda_P^{-1}(\mathcal{B}(IA_P,\mu)) \cap \pi^{-1}(\mathcal{H}(I,h)) \quad .\end{aligned}$$

Then Theorem 4.2 follows from Prop. 4.3, Prop. 4.4, Prop. 4.5, Prop. 4.6, and Prop. 4.1(ii). Q.e.d.

Part B: Characterization of almost complete intersections

Definition 1: Let A be a ring, $I \subseteq A$ an ideal.

(i) I is called a complete intersection (c.i.) if $\mathrm{ht}(I) = \mu(I)$.

(ii) I is called a local complete intersection (l.c.i.) if

$$\forall P \in \mathrm{Min}(A/I) : IA_P \quad \text{is c.i. in} \quad A_P \quad .$$

(iii) I is called an almost complete intersection (ac.i.) if I is l.c.i. and $\mathrm{ht}(I) + 1 = \mu(I)$.

Theorem 4.7. Let $(A, \mathfrak{m})$ be a local ring, $I \subseteq A$ an ideal.

(i) Let A have an infinite residue field. If I is an a.c.i., there is an ideal $J \subseteq I$ and $a \in I$, $a \notin J$, such that $I = J + aR$, and J is a complete intersection. If J is height unmixed (i.e. $\mathrm{ht}(J) = \mathrm{ht}(P)$ for all $P \in \mathrm{Ass}(A/J)$), $(J : a) = (J : a^2)$.

(ii) Conversely, if $I = J + aR$, with J a complete intersection and $(J : a) = (J : a^2)$, then I is either c.i. or a.c.i.

Proof. Let $h = \mathrm{ht}(I)$ and $\mu = \mu(I)$.

Ad (i): Choose $(a_1, \ldots, a_\mu)$ to be a Primbasis of I according to Theorem 4.2. Put $J := (a_1, \ldots, a_h) \cdot A$ and $a := a_{h+1} = a_\mu$. Then $a \notin J$, $I = J + aA$, and J is c.i.

Now let J be height unmixed. By standard arguments it suffices to show that $(J : I) \cap I = J$, and for this it suffices to show: $(JA_P : IA_P) \cap IA_P \subseteq JA_P$ for all $P \in \mathrm{Ass}(A/J) = \mathrm{Min}(A/J)$; and it clearly suffices to assume $P \in \mathrm{Min}(A/I)$. But then $\mathrm{ht}\,(P) = \mathrm{ht}\,(IA_P) = \mathrm{ht}(JA_P) = h$, so $JA_P = IA_P$ by Theorem 4.2(iii).

Ad(ii). One always has

$$\begin{aligned}\mathrm{ht}(J) = \mu(J) \leq \mathrm{ht}(I) \leq \mathrm{ht}(IA_P) \leq \mu(IA_P) \leq \mu(I) \\ \leq \mu(J) + 1 = \mathrm{ht}(J) + 1\end{aligned}$$

for all $P \supseteq I$. The only case which is not completely trivial is $\mathrm{ht}(I) = \mu(I) - 1$. Let $P \in \mathrm{Min}(A/I)$; then $\mathrm{ht}(IA_P) = \mathrm{ht}(P)$ is either h or $h+1$. If it is $h+1$, it follows that $\mu(IA_P) = h+1$, so IA_P is c.i.

If it is h, P must be a minimal prime of J. This implies $a^n \in JA_P$ for some n, and so $IA_P = JA_P$, since $(J : a) = (J : a^2)$. This proves thm. 4.7.

Remark 1. For (ii), both cases do occur.

Let us abbreviate the property of an ideal I to be of the form $I = J + aA$ with $J =$ c.i. , and $(J : a) = (J : a^2)$ by saying I is a generalized almost complete intersection (g.a.c.i.), see also [Va2].

We can now characterize almost complete intersections in a Cohen-Macaulay ring.

Theorem 4.8. Let A be a Cohen-Macaulay ring such that all localizations A_P, $P \in \operatorname{Spec}(A)$, have infinite residue field, and let $I \subseteq A$ be an ideal with

$\operatorname{ht}(I) + 1 = \mu(I)$. The following statements are equivalent:

(i) I is an a.c.i.

(ii) I is a g.a.c.i.

(iii) I is of linear type.

(iv) I is of geometric linear type.

The equivalence of (i), (iii), and (iv) holds without the assumption on the localizations.

Proof. We may assume A is local.

(i)⇒(ii) :	This is Theorem 4.7(i).
(ii)⇒(iii):	This is a standard implication, see e.g.[Va1].
(iii)⇒(iv) :	This is obvious.
(iv)⇒(i):	Let $P \in \operatorname{Min}(A/I)$. Then also IA_P is of geometric linear type. So $s(IA_P) = \mu(IA_P)$, i.e. IA_P is a parameter ideal and hence c.i. Q.e.d.

Remark 1. One cannot strengthen the condiditons for a l.c.i. to "IA_P c.i. for all $P \in \operatorname{Ass}(A/I)$" without destroying the equivalences of Theorem 4.8, as easy examples show. However, for a CM ring, one looses in some cases nothing, as the following generalization of the result [HO1], Satz 1, shows, which is in [HI], following a suggestion of Hochster.

Theorem 4.9 ([HI]): Let A be a CM local ring, $I \subseteq A$ a l.c.i. .Then the following are equivalent:

(i) $\operatorname{ht}(I) = s(I)$

(ii) I is c.i. .

Addendum. I is l.c.i. iff $\forall P \in \operatorname{Min}(A/I)$: $\ell(A_P/IA_P) = e(IA_P)$.

Remark 2. The important implication (i) $\Rightarrow$ (iii) is formulated in [Hu4], proposition 2.4, (1). But the reference for proof only refers to the prime ideal case. This case was also obtained independently by Valla [Va2]

§5. Numerical conditions for a.c.i

In the following we mention briefly some simple numerical conditions for an ideal in a local ring $(A, \mathfrak{m})$ to be an a.c.i. and of linear type. The philosophy is to complete the given ideal to an $\mathfrak{m}$-primary ideal by adding a part of a s.o.p. and to impose numerical conditions on this primary ideal. The results so obtained surely are but a first tentative step in the direction towards conditions which should be more effective both from a theoretical and practical point of view.

First we demonstrate our approach by looking for some examples.

1) Example E2 of §1:

$$A = k[[X,Y,Z]] \quad ,$$
$$P = (Y^2 - XZ, X^3 - YZ, Z^2 - X^2Y) \quad \text{is a.c.i.} \quad .$$

Choose the parameter X mod P and consider the $\mathfrak{m}$-primary ideal

$$I := P + XA = (Y^2, YZ, Z^2, X) \quad .$$

Then $q := (Y^2, Z^2, X)$ is a minimal reduction of I, namely $\quad I^2 = q \cdot I$, $I \not\subset q$, and

$$r(I) := \max\{r \mid I^{r-1} \not\subset q\} = 2 \quad .$$

Therefore we have:

$$e(I) = e(q) = \ell(A/q) = 4 \quad \text{and} \quad \ell(A/I) = 3 \quad ,$$

hence:

$$e(I) = \ell(A/I) + r(I) - 1 \quad . \tag{1}$$

Note, if we choose Z instead of X, we get:

$$J := P + ZA$$
$$e(J) = 5 \quad , \quad \ell(A/J) = 4 \quad , \quad r(J) = 2 \quad ,$$

hence the same relation (1) as before.

Moreover, in the example:

$$\mu(I) = \mu(P) + \dim(A/P) \quad . \tag{2}$$

2) Example E5 of §1:

$$A = k[[X_0, X_1, X_2, X_3]]$$
$$P = (X_0^2X_2 - X_1^3; X_0X_3 - X_1X_2; X_0X_2^2 - X_1^2X_3; X_1X_3^2 - X_2^3) \quad .$$

We take X_0, X_3 as a s.o.p. mod P and consider

$$I := P + (X_0, X_3)A = (X_0, X_1^3; X_1X_2; X_2^3; X_3) \quad ,$$

i.e. $\mu(I) = 5 \neq \mu(P) + \dim(A/P)$;
so (2) is not fulfilled.

A minimal reduction of I is $q = (X_0, X_3, X_1^3, X_2^3)$, namely: $I^3 = qI^2$ and $I^2 \not\subset q$. Then

$$e(I) = \ell(A/q) = \ell(A/I) + \ell(I/q + I^2) + \ell(I^2 + q/q) \quad .$$

One can check that: $e(I) = 9$ and $\ell(A/I) = 5$, hence also (1) is not fulfilled. [Note that P is not a.c.i.].

3) (s. [Sa2], p. 84):

$$A = k[[X, Y, Z, W]]$$
$$P = (X^3 - Z^2; XY^2 - W^2; XW - YZ; X^2Y - ZW) \quad .$$

Take the parameters X, Y mod P and consider

$$I := P + (X, Y)A = (Z^2; W^2; ZW; X; Y) \quad ,$$

i.e. (2) is not fulfilled. But relation (1) is fulfilled: take the maximal reduction $q = (X; Y; Z^2; W^2)$ of I, note that $I^2 = qI$ and $r(I) = 2$, hence

$$\begin{array}{ccc} e(I) = & \ell(A/I) + & (r(I) - 1) \\ \| & \| & \| \\ 4 & 3 & 1 \end{array} \quad .$$

Note, that P is not a.c.i. (and not of linear type).

$\underline{4)}$
$$A = k[[s^2, s^3, st, t, s^2u, tu]]$$
$$P = (s^2u, tu) \quad .$$

Then A is CM and A/P is Buchsbaum, but not CM. In [GoShi] it was shown that P is a.c.i. . Corresponding to this, the relations (1) and (2) are fulfilled if $\mathrm{char}(k) \neq 2$, which D. Rees pointed out to us:

Choose parameters s^2, t and consider:
$$I := (s^2u, tu, s^2, t) \quad .$$
Then $J := (t - s^2u, tu, s^2)$ is a reduction of I (note that also $t - \lambda s^2u$ any $\lambda \neq 0$, would be a possible generator of a reduction of I).

Since $s^2u \cdot t = s^2 \cdot tu$ we get
$$(t + s^2u)^2 = (t - s^2u)^2 + 4s^2 \cdot tu \in J^2 \quad ,$$
so that
$$I = (J, t + s^2u) \quad \text{and} \quad I^2 = J \cdot I \quad .$$
Moreover we have
$$\mathfrak{m}^2 = (I + (s^3, st)A)^2 = I\mathfrak{m} + (s^6, s^4t, s^2t^2)A \quad ,$$
hence $\mathfrak{m}^2 = I\mathfrak{m}$; i.e. I is a reduction of $\mathfrak{m}$.

Furthermore
$$I = (J, s^2u); \quad J : I = J : s^2u; \quad J : I \supseteq I \quad .$$
So we get:
$$4 = e(\mathfrak{m}) = e(I) = e(J) = \ell(A/J) = \ell(A/I) + \ell(I/J) \quad ,$$
since $\ell(A/I) = 3$ and $\ell(I/J) = \ell(A/J : I) = 1$. Therefore the relations (1) and (2) are fulfilled; this implies again that P is a.c.i. as the following proposition shows.

Proposition 5.0: Let $(A, \mathfrak{m})$ be CM and let P be a prime ideal in A such that

(i) $\mu(P) \geq \mathrm{ht}(P) + 1$

(ii) A_P is regular.

Assume that there exists a s.o.p. $\underline{x}$ mod P such that

(1) $e(I) = \ell(A/I) + r(I) - 1$

(2) $\mu(I) = \mu(P) + \dim(A/P)$, with $I = P + \underline{x}A$.

Then P is of linear type.

Proof: Let q be a minimal reduction of I such that $I^{r-1} \not\subset q$, where $r = r(I)$. Then

$$e(I) = \ell(A/q) = \ell(A/I) + \ell(I/q + I^2) + \ldots + \ell(I^{r-1} + q/I^r + q) \quad .$$

From (1) we see that $\ell(I/q + I^2) = 1$, hence $I = (a_1, \ldots, a_d, a_{d+1})$, where $q = (a_1, \ldots, a_d)$, $d = \dim A$. Hence by (2)

$$\mu(P) = \mathrm{ht}(P) + 1 \quad ,$$

i.e. P is a.c.i. in a CM ring A, therefore it can be generated by a d-sequence [GoShi]. Q.e.d.

Of course the conditions (1) and (2) are too strong. Therefore we make the following definition.

Definition. Let $(A, \mathfrak{m})$ be a local ring, $I \subseteq A$ an $\mathfrak{m}$-primary ideal, $q \subseteq I$ a minimal reduction. Put

$$N(I,q) := \bigoplus_{k \geq 0} (I^k + q/\mathfrak{m}I^k + q)$$

which is artinian;

$$\begin{aligned} n(I,q) &:= \ell(N(I,q)) \\ r(I,q) &:= \min\{k \mid I^k \subseteq q\} \quad , \qquad \text{and} \\ n(I) &:= \min\{n(I,q) \mid q \subseteq I \quad \text{a minimal reduction }\} \\ r(I) &:= \min\{r(I,q) \mid q \subseteq I \quad \text{a minimal reduction }\} \quad . \end{aligned}$$

Lemma 5.1. (i) $n(I,q) \geq r(I,q)$
(ii) $n(I,q) = r(I,q)$ and $r(I,q) > 1$
iff $\mu(I) = \dim(A) + 1$.

Proof: Use the observation

$$n(I,q) = \sum_{k=0}^{r(I,q)-1} \ell(I^k + q \ / \ \mathfrak{m} \cdot I^k + q) \quad ,$$

which implies (i) and the direction "$\Rightarrow$" of (ii). For the converse of (ii), note that any minimal base of q can be extended to a minimal base of I by [NR], Lemma 3, p. 147; so we have $\ell(I \ /\mathfrak{m} I + q) = 1$, and hence the claim.

Corollary 5.2. $n(I) \geq r(I)$.

Theorem 5.3. Let (A, m) be a quasi-unmixed local ring. Let $I \subseteq A$ be a l.c.i. ideal. Let $\underline{x}$ denote a s.o.p. with respect to I, and put $I(\underline{x}) := \underline{x}A + I$. Finally, let $q \subseteq I(\underline{x})$ be a minimal reduction of $I(\underline{x})$. Define properties $A_q), A)$ and $B)$:

$A_q)$ $\quad n(I(\underline{x}), q) = r(I(\underline{x}), q) > 1$

$A)$ $\quad n(I(\underline{x})) = r(I(\underline{x})) > 1$

$B)$ $\quad \mu(I(\underline{x})) = \mu(I) + \dim(A/I)$

Consider the following statements:

(i) $A_q)$ and $B)$ hold for all $\underline{x}$ and q

(ii) $A_q)$ and $B)$ hold for some $\underline{x}$ and some q

(iii) $A)$ and $B)$ hold for some $\underline{x}$

(iv) I is an a.c.i.

Then the following holds:

(I) (i)$\Rightarrow$ (iii)$\Rightarrow$ (ii)$\Rightarrow$ (iv)

(II) If A and A/I are CM, (iv)$\Rightarrow$(i).

Proof: We only show (II):

Since I is a.c.i., clearly $\dim(A) \leq \mu(I(\underline{x})) \leq \dim(A) + 1$, and we have to rule out the case $\dim(A) = \mu(I(\underline{x}))$. So assume $\dim(A) = \mu(I(\underline{x}))$. Then

$$\begin{aligned} e(I(\underline{x})) &= e(\underline{x}A + I) \geq e(\underline{x}, I, A) \\ &= \sum_{P \in \mathrm{Assh}(A/I)} e(IA_P)e(\underline{x}, A/P) \\ &= \sum_{P \in \mathrm{Assh}(A/I)} \ell(A_P/IA_P)e(\underline{x}, A/P) \\ &= e(\underline{x}, A/I) \\ &= \ell(A/I(\underline{x})) \quad , \quad \text{since} \quad A/I \quad \text{is CM} \quad . \end{aligned}$$

But since A is also CM, we must have equality everywhere. By Theorem 4.9 in §4, then, I is c.i., contrary to the assumption. Q.e.d.

Remark 1: In a CM ring, the conditions $A_q)$, $A)$ can be expressed in terms of the multiplicity $e(I(\underline{x}))$ as follows:

Introduce, for I an $\mathfrak{m}$-primary ideal in a local ring $(A, \mathfrak{m})$, and $q \subseteq I$ a minimal reduction,

$$\begin{aligned} D(I) &:= \bigoplus_{k \geq 0} \mathfrak{m} I^k + q / I^{k+1} + q \\ d(I,q) &:= \ell(D(I)) \\ d(I) &:= \max\{d(I,q) \mid q \subseteq I \quad \text{a minimal reduction}\} \quad . \end{aligned}$$

Then we get:

<u>Corollary 5.4.</u> In Theorem 5.3 the condition $A_q)$ and $A)$ can be replaced by

$A'_q)$ $\qquad e(I(\underline{x})) \leq d(I(\underline{x}), q) + r(I(\underline{x}), q) \quad$ and $\quad r(I(\underline{x}), q) > 1$

$A')$ $\qquad e(I(\underline{x})) \leq d(I(\underline{x})) + r(I(\underline{x})) \quad$ and $\quad r(I(\underline{x})) > 1$

in case A is CM.

The following is a simple sufficient condition for a.c.i., compare Proposition 5.0.

<u>Corollary 5.5.</u> Let the assumption be as in Corollary 5.4. Suppose the conditions:

$A''_q)$ $$e(I(\underline{x})) \leq \ell(A/I(\underline{x})) + r(I(\underline{x}), q) - 1$$

and

$B)$ $$\mu(I(\underline{x})) = \mu(I) + \dim(A/I)$$

hold. Then I is an a.c.i., and so of linear type in the CM ring A.

In many simple examples, $A''_q)$ holds for a.c.i.'s, and one may ask to which extent $A'_q)$ and $A''_q)$ are equivalent. A partial answer is given as follows:

<u>Proposition 5.6.</u> Let $(A, \mathfrak{m})$ be a CM ring, $I \subseteq A$ a l.c.i. such that A/I is CM.

(i) If I is an a.c.i. and $A''_q)$ holds, then

$(*)$ $$\mathfrak{m} I(\underline{x}) \subseteq I(\underline{x})^2 + q \quad .$$

(ii) Conversely, if $(*)$ holds, $A''_q)$ is true and so I is an a.c.i.

Remark 2: Using Herzog's work on ideals of monomial curves [H] , and Lejeune-Jalabert's and Teissier's work [T], Chap. I on integral closures of monomial ideals, one may systematically construct monomial space curves which are a.c.i.'s and CM which do not satisfy $(*)$ and so not satisfy $A_q'')$. This remark is due to O. Villamayor.

Questions: 1) Are there numerical conditions (hopefully practical) which imply $B)$?

2) Is $B)$ true for any ideal and a generic choice of $\underline{x}$?

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