THE ORNSTEIN-WEISS LEMMA FOR DISCRETE AMENABLE GROUPS.

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Abstract. In this note we prove a convergence theorem for invariant subadditive functions defined on the finite subsets of a discrete amenable group. The theorem can be proved using a quasi-tiling result due to D.S. Ornstein and B. Weiss but the proof given here follows ideas of M. Gromov.

Let us point out that we generalize here to discrete amenable groups the version of the convergence theorem given for countable amenable groups in a preceding paper by the same author.

1. Introduction

Let G be a group. We denote by $\mathcal{F}(G)$ the set of all finite subsets of G. Let K and A be subsets of G. The K-*boundary* of A denoted by $\partial_K(A)$ is the set of all elements g in G such that $Kg = \{kg \colon k \in K\}$ intersects both A and $G \setminus A$.

There are several equivalent definitions of amenable groups in the literature. The following is a characterization due to Følner [Føl]. For a more complete description of this class of groups see for example [Gre] or [Pat].

A (discrete) group G is said to be *amenable* if for all $\epsilon > 0$ and for all $K \in \mathcal{F}(G)$, there exists $F \in \mathcal{F}(G)$ satisfying

$$|\partial_K(F)| \leq \epsilon |F|,$$

where $|F|$ denotes the cardinality of the set F. Such a set F is called an (ϵ, K)-*invariant set* of G, or an (ϵ, K)-*invariant Følner set* of G .

It can be shown that a group G is amenable if and only if there exists a net $(F_i)_{i \in I}$ of elements of $\mathcal{F}(G)$ such that

$$\lim_i \frac{|\partial_K(F_i)|}{|F_i|} = 0$$

for every $K \in \mathcal{F}(G)$. Such a net $(F_i)_{i \in I}$ is called a *Følner net* of G.

The class of amenable groups includes finite groups, abelian groups, it is closed under the operations of taking subgroups, taking factors, taking extensions and taking increasing unions. Typical examples of non-amenable groups are the groups containing a subgroup isomorphic to a non-abelian free group. But this property of containing a non-abelian free group don't characterize at all the non-amenable groups (see for example [OlS]).

The aim of this paper is to prove the following convergence theorem:

Theorem 1.1 (Ornstein-Weiss)**.** *Let G be an amenable group and $h \colon \mathcal{F}(G) \to \mathbf{R}$ a function satisfying the following conditions:*

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(a) *h is subadditive, i.e.*

$$h(A \cup B) \leq h(A) + h(B) \textit{ for all } A, B \in \mathcal{F}(G);$$

(b) *h is right invariant, i.e.*

$$h(Ag) = h(A) \textit{ for all } g \in G \textit{ and } A \in \mathcal{F}(G).$$

Then, for every Følner net $(F_i)_{i \in I}$ *for* G*, the limit*

$$\lambda = \lambda(G, h) = \lim_i \frac{h(F_i)}{|F_i|}$$

exists and is finite. Moreover, this limit does not depend on the choice of the Følner net for G*.*

This theorem is a generalized version of the one containing in [Kri]. For countable amenable groups and with more stronger conditions of h, this convergence theorem was proved using the Ornstein-Weiss quasi-tiling result [OrW, Section I.2, Th. 6] in [LiW, Th. 6.1]. In [Gro, Section 1.3], Gromov gives a sketch of the proof of 1.1 by using tools introduced in [OrW]. The proof given here follows the ideas of Gromov.

Theorem 1.1 is used to define topological invariants of amenable group actions as metric entropy, mean topological dimension (see [Gro], [LiW], [OrW], [CoK]). One can find in [Mou] such a convergence theorem for invariant functions satisfying a more stronger assumption than subadditivity. The result in [Mou] is sufficient for defining the metric entropy of amenable groups.

The paper is organized as follows. In Section 2, we recall the notion of K-boundary of a subset of a group G and the definition of Følner subsets. Then we establish the Følner characterization of amenability in terms of Følner nets for (non-necessary countable) discrete groups. We prove in Section 3 the Filling-lemma (Lemma 3.5) used in the induction step of the proof of Theorem 1.1. In Section 4, we prove the Theorem 1.1. The idea of the proof is the following. We construct by induction a process needed to ϵ-cover every large Følner subset D of the group by translates of some fixed small Følner subsets. The property of this particular covering will make easy the estimation of $\frac{h(D)}{|D|}$ needed in the proof of the convergence theorem.

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2. Amenability

In this section we introduce the notions of K-interior, K-exterior and K-boundary of a subset of a group G. These tools are very convenient for proving the Filling-lemma (Lemma 3.5). We present the Følner characterization of amenability using K-boundaries. We introduce Følner nets which are generalizations of Følner sequences and are very convenient for handling with non-countable amenable groups.

2.1. **Relative amenability.** Let K and A subsets of a group G. The *K-interior* (resp. *K-exterior*) of A is the subset $\mathrm{Int}_K(A)$ (resp. $\mathrm{Ext}_K(A)$) of the elements g in G such that $Kg = \{kg \colon k \in K\}$ is contained in A (resp. in $G \setminus A$). We define the *K-boundary* of A as follow:

$$\partial_K(A) = G \setminus \big(\mathrm{Int}_K(A) \cup \mathrm{Ext}_K(A)\big).$$

Thus, the K-boundary of A is the subset of all elements g in G such that Kg intersects both A and $G \setminus A$.

An immediate consequence of the definition of K-boundary is the following:

Proposition 2.1. *Let K, A, B be subsets of a group G and g an element of G. We have:*

(i) $\partial_K(A) = \partial_K(G \setminus A)$;
(ii) $\partial_K(A \cup B) \subset \partial_K(A) \cup \partial_K(B)$;
(iii) $\partial_K(A \setminus B) \subset \partial_K(A) \cup \partial_K(B)$;
(iv) $\partial_K(A) \subset \partial_{K'}(A)$ *si* $K \subset K' \subset G$;
(v) $\partial_{Kg}(A) = g^{-1}\partial_K(A)$;
(vi) $\partial_K(Ag) = \partial_K(A)g$.

□

Suppose K and A be finite subsets of G. Then $\partial_K(A)$ is finite. Suppose $A \neq \varnothing$. We define the *relative amenability constant* of A with respect to K denoted by $\alpha(A, K)$ by:

$$\alpha(A, K) = \frac{|\partial_K(A)|}{|A|}.$$

Equalities (v) et (vi) of Proposition 2.1 imply

$$\alpha(A, Kg) = \alpha(Ag, K) = \alpha(A, K) \text{ for all } g \in G. \tag{2.1}$$

If $\alpha = \alpha(A, K)$, the set A is called an (α, K)-*invariant* Følner subset of G or simply an (α, K)-*invariant* subset of G.

Recall that a discrete group G is said to be *amenable* if for all $\epsilon > 0$ and $K \in \mathcal{F}(G)$ there exists an (ϵ, K)-invariant subset of G.

If the group G is countable then the amenability of G is equivalent to the existence of a *Følner sequence* (F_n) of G, i.e. a sequence of elements of $\mathcal{F}(G)$ satisfying the following:

$$\lim_{n\to\infty} \frac{|\partial_K(F_n)|}{|F_n|} = 0 \quad \text{ for all } K \in \mathcal{F}(G).$$

If the group is not countable, there is an analogue of Følner sequences in terms of nets (see Proposition 2.2 below).

Nets are very useful tools since much results about sequences in topological spaces extend to nets. Let us recall some results about nets needed in this paper.

2.2. **Basic facts about nets.** We give here some basic definitions and results about nets (also called *generalized sequences*). For more details, see for example [Ke] or [DuS].

Recall that a partially ordered set $(I, \geq)$ is said to be *directed* if I is not empty and if every finite subset of I has an upper bound.

A map f from a directed set I to a set X is called a *net* in X. We will use the notation x_i instead of $f(i)$ (for $i \in I$) and also (x_i) instead of f.

A net (x_i) in a topological space X is said to *converge* to $x \in X$ if for every open neighborhood V of x, there exists $i_0 \in I$ such that $x_i \in V$ for all $i \geq i_0$. If (x_i) converges to x, we note $\lim_i x_i = x$.

Let (x_i) and (y_j) be nets in a topological space X. The net (y_j) is called a *subnet* of (x_i) if there exists a function $\varphi\colon J \to I$ satisfying the two conditions:

(1) $y_j = x_{\varphi(j)}$ for all $j \in J$;
(2) for all $i \in I$ there is $m \in J$ with the property that, if $j \geq m$ then $\varphi(j) \geq i$.

The set C of all *cluster points* (sometimes called *limit points*) of a net (x_i) in a topological space X is the (closed) subset of X defined by

$$C = \bigcap_{i \in I} \overline{\{x_k \colon k \geq i\}}.$$

The point x is a cluster point of the net (x_i) if and only if there is a subnet (y_j) of (x_i) converging to x.

If $C \neq \varnothing$, the *limit inferior* $\liminf_i x_i$ (resp. *limit superior* $\limsup_i x_i$) of a net (x_i) of real numbers is the supremum (resp. the infimum) of its cluster points. If $\liminf_i x_i$ (resp. $\limsup_i x_i$) is finite, then it is the minimum (resp. maximum) of the set of cluster points. Thus, if $\liminf_i x_i = \limsup_i x_i < \infty$ then the net (x_i) will converge to its unique cluster point.

Recall also that every net in a compact space admits at least one cluster point.

2.3. **Amenability and Følner nets.** The next result gives a characterization of amenable discrete groups in terms of Følner nets:

Proposition 2.2. *A (discrete) group G is amenable if and only if there exists a net $(F_i)_{i \in I}$ in $\mathcal{F}(G)$ satisfying*

$$\lim_i \frac{|\partial_K(F_i)|}{|F_i|} = 0,$$

for all $K \in \mathcal{F}(G)$. Such a net (F_i) is called a Følner net *of G.*

Proof. Suppose that G is amenable. Let I be the (non-empty) set defined by

$$I = \{(\epsilon, K) \colon \epsilon > 0 \text{ and } K \in \mathcal{F}(G)\}.$$

Direct I as follow:

$$(\epsilon_2, K_2) \geq (\epsilon_1, K_1) \Leftrightarrow \epsilon_2 \leq \epsilon_1 \text{ and } K_1 \subset K_2.$$

As G is amenable, we can choose for every $i = (\eta, L)$ an (η, L)-invariant subset F_i of G. This define a net (F_i) in $\mathcal{F}(G)$. Let $\epsilon > 0$ and $K \in \mathcal{F}(G)$. Let $i_0 = (\epsilon, K)$. For $i = (\eta, K') \geq i_0$ we have

$$\frac{|\partial_K(F_i)|}{|F_i|} \leq \frac{|\partial_{K'}(F_i)|}{|F_i|} \leq \eta \leq \epsilon,$$

since $K \subset K'$ and since $\eta \leq \epsilon$ (see Proposition 2.1.(iv)). Hence $\lim_i \frac{|\partial_K(F_i)|}{|F_i|} = 0$.

Suppose now that there exists a net (F_i) of finite subsets of G satisfying

$$\lim_i \frac{|\partial_K(F_i)|}{|F_i|} = 0,$$

for all $K \in \mathcal{F}(G)$. We will show that G is amenable. In fact, let $\epsilon > 0$ and $K \in \mathcal{F}(G)$. As $\lim_i \frac{|\partial_K(F_i)|}{|F_i|} = 0$, there is an $i_0 \in I$ such that $\frac{|\partial_K(F_i)|}{|F_i|} \leq \epsilon$ for all $i \geq i_0$. In particular, F_{i_0} is (ϵ, K)-invariant. Hence G is amenable. $\square$

3. The filling lemma

In this section we introduce some tools needed for proving the Filling-lemma (Lemma 3.5). This lemma is the key-result in the induction step of the proof of Theorem 1.1.

Let X be a set and $\epsilon > 0$. A family $(A_i)_{i\in I}$ of finite subsets of X is said to be *ϵ-disjoint* if there is a family $(B_i)_{i\in I}$ of disjoint subsets of X such that $B_i \subset A_i$ and $|B_i| \geq (1-\epsilon)|A_i|$ for all $i \in I$.

Lemma 3.1. *Let X be a set and $(A_1, A_2, \dots, A_n)$ be an ϵ-disjoint family of subsets of X. Then*

$$(1-\epsilon)\sum_{i=1}^{n} |A_i| \leq |\bigcup_{i=1}^{n} A_i|.$$

Proof. Since $(A_1, A_2, \dots, A_n)$ is ϵ-disjoint, there exists a disjoint family $(B_1, B_2, \dots, B_n)$ of subsets of X such that $B_i \subset A_i$ and $|B_i| \geq (1-\epsilon)\ |A_i|$ for all $1 \leq i \leq n$. Thus

$$(1-\epsilon)\sum_{i=1}^{n} |A_i| \leq \sum_{i=1}^{n}\ |B_i| = |\bigcup_{i=1}^{n} B_i| \leq |\bigcup_{i=1}^{n} A_i|.$$

Lemma 3.2. *Let G be a group, K a finite subset of G, and $0 < \varepsilon < 1$. Let $A_1, A_2, \dots, A_n$ be an ϵ-disjoint family of non empty finite subsets of G and let $\eta > 0$ such that $\alpha(A_i, K) \leq \eta$ for all $1 \leq i \leq n$. Then one has*

$$\alpha(\bigcup_{i=1}^{n} A_i, K) \leq \frac{\eta}{1-\epsilon}.$$

Proof. Using Proposition 2.1.(ii), we obtain

$$\partial_K(\bigcup_{i=1}^{n} A_i) \subset \bigcup_{i=1}^{n} \partial_K(A_i).$$

Thus

$$|\partial_K(\bigcup_{i=1}^{n} A_i)| \leq \sum_{i=1}^{n} |\partial_K(A_i)| = \sum_{i=1}^{n} \alpha(A_i, K)|A_i| \leq \eta \sum_{i=1}^{n} |A_i|.$$

As the family $(A_i)_{1\leq i\leq n}$ is ϵ-disjoint, Lemma 3.1 implies

$$(1-\epsilon)\sum_{i=1}^{n} |A_i| \leq |\bigcup_{i=1}^{n} A_i|.$$

We deduce

$$\alpha(\bigcup_{i=1}^{n} A_i, K) = \frac{|\partial_K(\bigcup_{i=1}^{n} A_i)|}{|\bigcup_{i=1}^{n} A_i|} \leq \frac{\eta}{1-\epsilon}.$$

Lemma 3.3. *Let G be a group and let K, A and Ω be finite subsets of G such that $\varnothing \neq A \subset \Omega$. Suppose that there exists $\epsilon > 0$ such that $|\Omega \setminus A| \geq \epsilon|\Omega|$. Then*

$$\alpha(\Omega \setminus A, K) \leq \frac{\alpha(\Omega, K) + \alpha(A, K)}{\epsilon}.$$

Proof. The Proposition 2.1.(iii) gives

$$\partial_K(\Omega \setminus A) \subset \partial_K(\Omega) \cup \partial_K(A).$$

Thus

$$|\partial_K(\Omega \setminus A)| \leq |\partial_K(\Omega)| + |\partial_K(A)| = \alpha(\Omega, K)|\Omega| + \alpha(A, K)|A|.$$

Since $|\Omega \setminus A| \geq \epsilon|\Omega| \geq \epsilon|A|$, we deduce

$$\alpha(\Omega \setminus A, K) = \frac{|\partial_K(\Omega \setminus A)|}{|\Omega \setminus A|} \leq \frac{\alpha(\Omega, K) + \alpha(A, K)}{\epsilon}.$$

Lemma 3.4. *Let G be a group and let A and B be two finite subsets of G. Then one has*
$$\sum_{g\in G}|Ag\cap B| = |A||B|.$$

Proof. For $E\subset G$, denote by $\chi_E\colon G\to\{0,1\}$ the characteristic fonction of E. We have
$$\sum_{g\in G}|Ag\cap B| = \sum_{g\in G}\sum_{g'\in G}\chi_{Ag\cap B}(g') = \sum_{g\in G}\sum_{g'\in G}\chi_A(g'g^{-1})\chi_B(g').$$
Now, after changing the order of summation and changing the variable, we obtain:
$$\sum_{g\in G}|Ag\cap B| = \sum_{g'\in G}\chi_B(g')\sum_{g\in G}\chi_A(g'g^{-1}) = |B||A|.$$

Let G be a group. Let K and Ω be finite subsets of G and $\epsilon>0$. A subset $R\subset G$ is called an (ϵ,K)-*filling of* Ω if the following conditions are satisfied:

(C1) $R\subset Int_K(\Omega)$;

(C2) the family $(Kg)_{g\in R}$ is ϵ-disjoint.

Remark that an (ϵ,K)-filling is a finite set and that it could be empty.

Lemma 3.5 (Filling-lemma)**.** *Let Ω and K be non-empty finite subsets of a group G. For all $\epsilon\in]0;1]$, there exists a finite subset $R\subset G$ such that:*

(a) *R is an (ϵ,K)-filling of Ω;*

(b) *$\left|\bigcup_{g\in R}Kg\right|\geq\epsilon(1-\alpha_0)|\Omega|$, where $\alpha_0=\alpha(\Omega,K)$ is the relative amenability constant of Ω with respect to K.*

Proof. Since $K\neq\varnothing$, we can suppose $1_G\in K$ (otherwise choose $k_0\in K$, replace K with Kk_0^{-1} and remark that $\alpha(\Omega,K)=\alpha(\Omega,Kk_0^{-1})$ according to Equalities (2.1)). As $1_G\in K$, we have $\mathrm{Int}_K(\Omega)\subset\Omega$ and $\mathrm{Ext}_K(\Omega)\subset G\setminus\Omega$. We deduce
$$\Omega\setminus\partial_K(\Omega)=\mathrm{Int}_K(\Omega)$$
thus
$$(1-\alpha_0)|\Omega|\leq|\Omega\setminus\partial_K(\Omega)|=|\,\mathrm{Int}_K(\Omega)|. \tag{3.1}$$
Since $\mathrm{Int}_K(\Omega)\subset\Omega$, every (ϵ,K)-filling of Ω is contained in Ω and has a bounded cardinality. Thus we can choose an (ϵ,K)-filling $R\subset G$ of Ω with maximal cardinality. Define $A=\bigcup_{g\in R}Kg$. We will prove that $|A|\geq\epsilon(1-\alpha_0)|\Omega|$, which is exactly condition (b). Lemma 3.4 implies
$$\sum_{g\in\mathrm{Int}_K(\Omega)}|Kg\cap A|\leq|K||A|. \tag{3.2}$$
Let us prove that
$$\epsilon|K|\leq|Kg\cap A|\quad\text{for all } g\in\mathrm{Int}_K(\Omega). \tag{3.3}$$
If $g\in R$, then $Kg\cap A=Kg$ and (3.3) is true since $\epsilon\leq1$. Suppose now $g\in\mathrm{Int}_K(\Omega)\setminus R$ and $|Kg\cap A|<\epsilon|K|$. Then
$$|Kg\setminus A|>(1-\epsilon)|Kg|,$$

which implies that $R \cup \{g\}$ is an (ϵ, K)-filling of Ω. This contradicts the maximality of the cardinality of R. Thus Inequality (3.3) is true. We deduce

$$\epsilon |K| |\operatorname{Int}_K(\Omega)| \leq \sum_{g \in \operatorname{Int}_K(\Omega)} |Kg \cap A|. \tag{3.4}$$

Inequalities (3.1), (3.2) and (3.4) imply

$$|A| \geq \epsilon(1 - \alpha_0)|\Omega|.$$

□

4. Proof of the Theorem 1.1

Let us first give some remarks:

(1) If one choose $A = B$ in condition (a) of Theorem 1.1, we get $h(A) \leq 2h(A)$ for all $A \in \mathcal{F}(G)$. This shows that $h \geq 0$.

(2) To prove Theorem 1.1 it is sufficient to prove the existence of the limit of $(h(F_i)/|F_i|)$ for all Følner net (F_i) of G. In fact, the limit will be independent of the choice of the Følner net. To see this fact, let (A_i) and (B_j) be two Følner nets of G. Let $\hat{I} = \{\hat{i} \colon i \in I\}$ (resp. $\hat{J} = \{\hat{j} \colon j \in J\}$) a copy of the directed set I (resp. J). Direct the set $K = (I \times J) \bigsqcup (\hat{I} \times \hat{J})$ with the binary relation $\geq$ defined in the natural way by:
$$\begin{array}{lcl} (i,j) \;\geq\; (i',j') & \Leftrightarrow & i \geq i' \text{ and } j \geq j' \text{ if } (i,j) \in I \times J \text{ and } i', j' \in I \times J, \\ (i,j) \;\geq\; (\hat{i}',\hat{j}') & \Leftrightarrow & i \geq i' \text{ and } j \geq j' \text{ if } (i,j) \in I \times J \text{ and } \hat{i}', \hat{j}' \in \hat{I} \times \hat{J}, \\ (\hat{i},\hat{j}) \;\geq\; (i',j') & \Leftrightarrow & i \geq i' \text{ and } j \geq j' \text{ if } (\hat{i},\hat{j}) \in \hat{I} \times \hat{J} \text{ and } (i', j') \in I \times J, \\ (\hat{i},\hat{j}) \;\geq\; (\hat{i}',\hat{j}') & \Leftrightarrow & i \geq i' \text{ and } j \geq j' \text{ if } (\hat{i},\hat{j}) \in \hat{I} \times \hat{J} \text{ and } (\hat{i}', \hat{j}') \in \hat{I} \times \hat{J}. \end{array}$$
Now define the net (F_k) as follows: if $k = (i,j)$ then let $F_k = A_i$ and if $k = (\hat{i},\hat{j})$ then let $F_k = B_j$. Defined in this way, the net (F_k) is a Følner net of G. Moreover, if $\big(h(F_k)/|F_k|\big)$ converges to λ with respect to the directed set K, then both nets $\big(h(A_i)/|A_i|\big)$ and $\big(h(B_j)/|B_j|\big)$ converge to λ with respect to their directed set.

Proof of Theorem 1.1. Let (F_i) be a Følner net of G and fix $\epsilon \in]0, \frac{1}{2}]$. Remark that the properties of h imply that $h(A) \leq h(\{1_G\})|A|$ for all $A \in \mathcal{F}(G)$ which shows that the net defined by $x_i = \frac{h(F_i)}{|F_i|}$ is bounded. More precisely, the numbers x_i are contained in $[0, h(\{1_G\})]$. As every net in a compact space admits at least one cluster point, we can define the real number

$$\lambda = \liminf_i x_i,$$

which is in fact the least cluster point.

Fix an integer $n \geq 2$. Then there exists a finite sequence $K_1, K_2, \ldots, K_n$ extracted from (F_i) and satisfying the following conditions:

(C1) $h(K_j)/|K_j| \leq \lambda + \epsilon$ for all $1 \leq j \leq n$,

(C2) $\alpha(K_j, K_i) \leq \epsilon^{2n}$ for all $1 \leq i < j \leq n$.

In fact, as λ is the least cluster point of x_i we can find a subnet $(x_{\varphi(k)})_{k \in K}$ and $k_0 \in K$ satisfying

$$x_{\varphi(k)} \leq \lambda + \epsilon,$$

for all $k \geq k_0$. Remark that $(F_{\varphi(k)})$ is also a Følner net of G, i.e. for all $K \in \mathcal{F}(G)$ we have $\lim_k \frac{|\partial_K(F_{\varphi(k)})|}{|F_{\varphi(k)}|} = 0$. Thus, it is possible to extract a finite sequence

$K_1, K_2, \dots, K_n$ from $(F_{\varphi(k)})$ satisfying condition (C2).

Let D be a non-empty finite subset of G such that

$$\alpha(D, K_j) \leq \epsilon^{2n} \quad \text{for all } 1 \leq j \leq n. \tag{4.1}$$

We will show that for a large enough integer n, there is an ϵ-disjoint family in D composed by certain translates of the type $K_j g$ (with $1 \leq j \leq n$ and $g \in G$) which partially cover D, i.e. such that the proportion of D covered be these sets is at least $1-\epsilon$. After that, we will use this partial cover and the properties of h to prove $\limsup_{i\to\infty} h(F_i)/|F_i| \leq \lambda$, ending the proof of the Theorem 1.1.

Let us define by induction a process to ϵ-cover D in at most n steps:

Step 1. Recall that $\alpha(D, K_j) \leq \epsilon^{2n}$ for all $1 \leq j \leq n$. Using Lemma 3.5 with $\Omega = D$ and $K = K_n$, there is $R_n \subset G$ an (ϵ, K_n)-filling of D such that

$$\frac{|\bigcup_{g\in R_n} K_n g|}{|D|} \geq \epsilon\big(1 - \alpha(D, K_n)\big) \geq \epsilon(1 - \epsilon^{2n}).$$

Put $D_1 = D \setminus \bigcup_{g \in R_n} K_n g$. The previous inequality implies:

$$|D_1| \leq |D|\big(1 - \epsilon(1 - \epsilon^{2n})\big). \tag{4.2}$$

We continue this covering process by induction as follows. Put $D_0 = D$. Suppose that the covering process applies k times, with $1 \leq k \leq n-1$.
The induction hypothesis at step k is:

(H1) $\alpha(D_{k-1}, K_j) \leq (2(k-1)+1)\epsilon^{2n-k+1}$ for all $1 \leq j \leq n-k+1$;
(H2) $R_{n-k+1} \subset G$ is an (ϵ, K_{n-k+1})-filling of D_{k-1};
(H3) If we write

$$D_k = D_{k-1} \setminus \bigcup_{g \in R_{n-k+1}} K_{n-k+1} g,$$

then

$$|D_k| \leq |D| \prod_{i=0}^{k-1} \left(1 - \epsilon\big(1 - (2i+1)\epsilon^{2n-i}\big)\right).$$

Remark that this hypothesis is satisfied for $k = 1$. Let us construct step $k+1$:

Step $\boldsymbol{k+1}$. If $|D_k| \leq \epsilon|D_{k-1}|$ then $|D_k| \leq \epsilon|D|$ and we stop the covering process. Otherwise, we have $|D_k| > \epsilon|D_{k-1}|$. Let $1 \leq j \leq n-k$. Lemma 3.3 implies

$$\alpha(D_k, K_j) \leq \frac{\alpha(\bigcup_{g\in R_{n-k+1}} K_{n-k+1} g, K_j)}{\epsilon} + \frac{\alpha(D_{k-1}, K_j)}{\epsilon}. \tag{4.3}$$

Equalities (2.1) and condition (C2) imply

$$\alpha(K_{n-k+1} g, K_j) = \alpha(K_{n-k+1}, K_j) \leq \epsilon^{2n}.$$

Since the family $(K_{n-k+1} g)_{g \in R_{n-k+1}}$ is ϵ-disjoint, Lemma 3.2 gives

$$\alpha(\bigcup_{g \in R_{n-k+1}} K_{n-k+1} g, K_j) \leq \frac{\epsilon^{2n}}{1-\epsilon}.$$

Using Inequality (4.3) and the induction hypothesis (H1), we deduce

$$\alpha(D_k, K_j) \leq \frac{\epsilon^{2n}}{(1-\epsilon)\ \epsilon} + \frac{\big(2(k-1)+1\big)\epsilon^{2n-k+1}}{\epsilon} \leq (2k+1)\epsilon^{2n-k}$$

for all $1 \leq j \leq n-k$. The latter inequality is (H1) for $k+1$. Using Lemma 3.5 with $\Omega = D_k$ and $K = K_{n-k}$, we get the existence of $R_{n-k} \subset G$ an (ϵ, K_{n-k})-filling of D_k satisfying

$$\frac{|\bigcup_{g\in R_{n-k}} K_{n-k}g|}{|D_k|} \geq \epsilon\ \big(1-\alpha(D_k, K_{n-k})\big) \geq \epsilon\ \big(1-(2k+1)\epsilon^{2n-k}\big).$$

In particular, hypothesis (H2) is satisfied for $k+1$. Define

$$D_{k+1} = D_k \setminus \bigcup_{g\in R_{n-k}} K_{n-k}g.$$

Then we have

$$|D_{k+1}| \leq |D_k|\big(1-\epsilon\big(1-(2k+1)\epsilon^{2n-k}\big)\big).$$

Using the induction hypothesis (H3) and the latter inequality, we obtain

$$|D_{k+1}| \leq |D| \prod_{i=0}^{k} \big(1-\epsilon\big(1-(2i+1)\epsilon^{2n-i}\big)\big)$$

which is exactly (H3) for $k+1$. This finishes the construction of step $k+1$ and proves the induction step.

Now, suppose that this covering process continues until step n, and that we have $|D_{n-1}| > \epsilon|D_{n-2}|$. Using (H3) for $k = n$, we obtain

$$|D_n| \leq |D| \prod_{i=0}^{n-1} \big(1-\epsilon\big(1-(2i+1)\epsilon^{2n-i}\big)\big)\ . \tag{4.4}$$

The next step is to show that for n large enough (only depending on ϵ) we get $|D_n| \leq \epsilon|D|$. From Inequality (4.4), we deduce:

$$|D_n| \leq |D|\big(1-\epsilon(1-(2n-1)\epsilon^{n+1})\big)^n. \tag{4.5}$$

Since $\lim_{i\to\infty}(2i-1)\epsilon^{i+1} = 0$ and $\lim_{i\to\infty}(1-\frac{\epsilon}{2})^i = 0$, there is an integer n_0 such that for all $i \geq n_0$, we have $(2i-1)\epsilon^{i+1} \leq \frac{1}{2}$ and $(1-\frac{\epsilon}{2})^i \leq \epsilon$. If $n \geq n_0$, Inequality (4.5) implies

$$|D_n| \leq |D|(1-\frac{\epsilon}{2})^n \leq \epsilon|D|.$$

From now, we suppose that the integer n fixed at the beginning of the proof is greater than this n_0.

Let us recall what we proved: for all subset D of G satisfying $\alpha(D, K_j) \leq \epsilon^{2n}$ for all $1 \leq j \leq n$, there is an integer k_0 (with $1 \leq k_0 \leq n$) such that $|D_{k_0}| \leq \epsilon|D|$. More precisely, the proportion of D covered by the sets of the following ϵ-disjoint families

$$(K_n g)_{g\in R_n}, (K_{n-1}g)_{g\in R_{n-1}}, \dots, (K_{n-k_0+1}g)_{g\in R_{n-k_0+1}},$$

is at least $1-\epsilon$.

Using this cover, we want to obtain a good upper bound for $h(D)/|D|$. To simplify the notations, let $J = \{n-k_0+1, \dots, n\}$ and write K_jR_j for $\bigcup_{g\in R_j} K_j g$

for all $j \in J$. From now, we also will use subadditivity and right-invariance of the function h. Since

$$D = \bigcup_{j \in J} K_j R_j \cup D_{k_0}$$

with

$$|D_{k_0}| \leq \epsilon |D|,$$

we deduce

$$\frac{h(D)}{|D|} \leq \frac{h(\bigcup_{j\in J} K_j R_j)}{|D|} + \frac{h(D_{k_0})}{|D|} \leq \frac{h(\bigcup_{j\in J} K_j R_j)}{|D|} + \epsilon h(1_G). \tag{4.6}$$

We obtain

$$\frac{h(\bigcup_{j\in J} K_j R_j)}{|D|} \leq \sum_{j\in J} \sum_{g \in R_j} \frac{h(K_j g)}{|D|} = \sum_{j\in J} \sum_{g \in R_j} \frac{h(K_j)}{|K_j|} \frac{|K_j g|}{|D|}.$$

Using condition (C1), we deduce

$$\frac{h(\bigcup_{j\in J} K_j R_j)}{|D|} \leq (\lambda + \epsilon) \sum_{j\in J} \sum_{g \in R_j} \frac{|K_j g|}{|D|}. \tag{4.7}$$

Remark that the family containing the sets $K_j g$, with $j \in J$ and $g \in R_j$, is an ϵ-disjoint family of D. According to Lemma 3.1 we get

$$\sum_{j\in J} \sum_{g \in R_j} |K_j g| \leq \frac{|D|}{1-\epsilon}. \tag{4.8}$$

Thus, inequalities (4.7) and (4.8) imply

$$\frac{h(\bigcup_{j\in J} K_j R_j)}{|D|} \leq \frac{\lambda + \epsilon}{1 - \epsilon}. \tag{4.9}$$

Now, using inequalities (4.6) and (4.9) we have

$$\frac{h(D)}{|D|} \leq \frac{\lambda + \epsilon}{1-\epsilon} + \epsilon h(1_G). \tag{4.10}$$

Since (F_i) is a Følner net, there exists $i_0 \in I$ such that

$$(i \geq i_0) \Rightarrow \alpha(F_i, K_j) \leq \epsilon^{2n} \quad \text{for all } 1 \leq j \leq n.$$

Note that the limit superior μ of the bounded net $\left(\frac{h(F_i)}{|F_i|}\right)$ exists and is the biggest cluster point of this net. In particular, there exists a subnet $\left(\frac{h(F_{\varphi(j)})}{|F_{\varphi(j)}|}\right)_{j\in J}$ converging to μ. Let $j_0 \in J$ such that $\varphi(j) \geq i_0$ for all $j \geq j_0$. Using inequality (4.10) with $D = F_{\varphi(j)}$ and for $j \geq j_0$, we deduce

$$\mu = \lim_j \frac{h(F_{\varphi(j)})}{|F_{\varphi(j)}|} \leq \frac{\lambda + \epsilon}{1-\epsilon} + \epsilon h(1_G).$$

Since the latter inequality is satisfied for all $\epsilon \in]0, \frac{1}{2}]$, we can take the limit when ϵ tends to 0 and we get

$$\limsup_i \frac{h(F_i)}{|F_i|} = \mu \leq \lambda = \liminf_i \frac{h(F_i)}{|F_i|},$$

ending the proof of the theorem. $\square$

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