WILLMORE SURFACES

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Compact surfaces embedded (or immersed) in $\mathbb{R}^3$ have been studied in differential geometry since its very beginnings. Also various special surfaces have been investigated, such as ellipsoids, tori of revolution,etc. Surprisingly, however, with one exception [2], we do not know any compact surface in $\mathbb{R}^3$ of genus $g > 1$ appearing in the literature because of its special differential geometric properties. The same applies to compact nonorientable surfaces in $\mathbb{R}^3$.

Recently many such surfaces have been obtained as solutions to the following variational problem. The problem is to find compact surfaces,of prescribed topological type,which (on the average) have the least possible curvature, i.e. which are "as smooth as possible". As a measure for the average curvature,one chooses for an immersion $f:M^2 \longrightarrow \mathbb{R}^3$ (M^2 an abstract compact surface) the functional

$$\widetilde{\mathfrak{W}}(f) = \int_{M^2} k_1^1 + k_2^2 \, dA .$$

Here k_1, k_2 are the principal curvatures and dA is the volume element induced from $\mathbb{R}^3$. Instead of $\widetilde{\mathfrak{W}}$ it has become costumary to study the equivalent functional

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$$\mathfrak{W}(f) = \frac{1}{4}\widetilde{\mathfrak{W}}(f) + \pi\chi(M^2) \quad ,$$

where $\chi(M^2)$ denotes the Euler characteristic of M^2. Because of the Gauss-Bonnet theorem, we have

$$\mathfrak{W}(f) = \int_{M^2} H^2 dA \ ,$$

where $H = \frac{1}{2}(k_1 + k_2)$ is the mean curvature of the surface.

For a given abstract surface M^2, we will discuss three problems:

1) Determine $\mathfrak{W}(M^2) := \inf \mathfrak{W}(f)$ over all immersions $f: M^2 \longrightarrow \mathbb{R}^3$.

2) Classify all f for which $\mathfrak{W}(f)$ equals the minimal value $\mathfrak{W}(M^2)$.

3) Determine all critical points f of $\mathfrak{W}$ and the corresponding values $\mathfrak{W}(f)$.

Critical points of $\mathfrak{W}$ are called Willmore surfaces and are characterized by the Euler equation.

$$\Delta H + 2H(H^2 - K) = 0,$$

where $K = k_1 k_2$ is the Gaussian curvature. Willmore surfaces were first studied by Blaschke and Thomsen in 1923 [3,16]. They also established the most important property of $\mathbb{W}$: The functional $\mathbb{W}$ is invariant under conformal mappings $g: \mathbb{R}^3 \cup \{\infty\} \to \mathbb{R}^3 \cup \{\infty\}$. For example, if $f: M^2 \longrightarrow \mathbb{R}^3$ is an immersion such that $0 \notin f(M^2)$, then $\mathbb{W}\left(\frac{f}{|f|^2}\right) = \mathbb{W}(f)$.

For $M^2 \approx S^2$ and $M^2 \approx \mathbb{R}P^2$, the above problems 1), 2), and 3) are completely solved. We have $\mathbb{W}(S^2) = 4\pi$, and the minimum is attained only for round spheres (Willmore 1965 [18]). Recently Bryant [5] classified all Willmore-immersions $f: S^2 \longrightarrow \mathbb{R}^3$. The possible values of $\int_{S^2} H^2 dA$ are $4\pi n$, with n a natural number, where either $n = 1$ or $n \geqq 4$ and n even or $n \geqq 9$ and n odd. Figure 1 shows a Willmore-sphere with $n = 4$.

Any immersed projective plane in $\mathbb{R}^3$ must have a triple point, so, by a result of Li and Yau [12], $\mathbb{W}(\mathbb{R}P^2) \geqq 12\pi$. Recently R. Bryant [6] and independently R. Kusner [9] found explicit immersions of $\mathbb{R}P^2$ for which the minimal value 12π is attained. Indeed, Bryant classified all minimizing $\mathbb{R}P^2$'s in $\mathbb{R}^3$ and found (modulo conformal transformations) a two-parameter family of such surfaces. Figure 2 shows a Willmore-$\mathbb{R}P^2$ with three-fold symmetry. thus providing an "optimal" version of a surface first described qualitatively by W. Boy in 1903 [4].

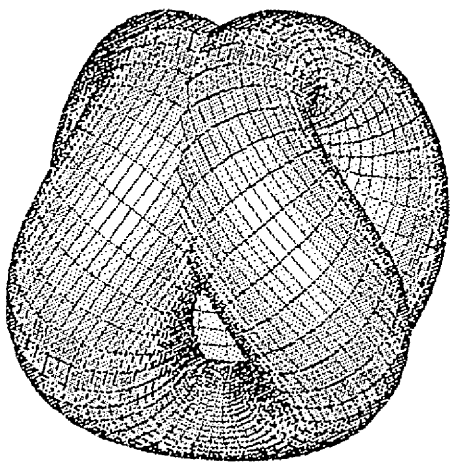

Figure 1

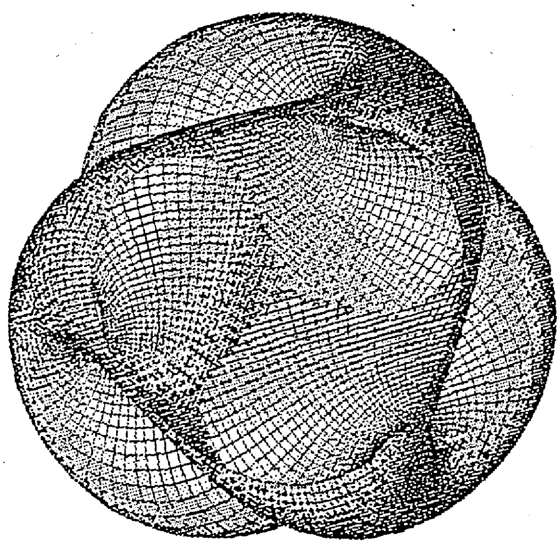

Figure 2

For the torus T^2, there is the long-standing "Willmore Conjecture": $\mathfrak{W}(T^2) = 2\pi^2$. The value $2\pi^2$ is actually attained for a certain torus of revolution whose generating circle has radius 1 and distance $\sqrt{2} - 1$ from the axis of revolution (See Fig. 3).

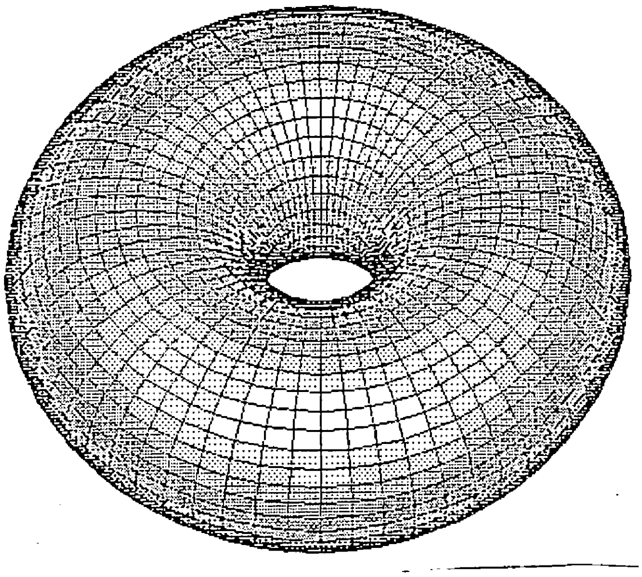

Figure 3

The Willmore inequality $\int H^2 dA \geq 2\pi^2$ has been proved for various special classes of immersed tori (such as tori of revolution [10]), but, in general, it is only known that for any immersion $f: T^2 \longrightarrow \mathbb{R}^3$ we have $\mathbb{W}(f) > 4\pi$ [18]. Recently L. Simon [15] proved the existence of a minimizing immersion f of T^2 with $\mathbb{W}(f) = \mathbb{W}(T^2)$. This implies then $\mathbb{W}(T^2) > 4\pi$.

Li and Yau [12] proved that for any immersed surface with self-intersections one has $\int H^2 dA \geq 8\pi$. Moreover, for any genus g, there are compact orientable surfaces in $\mathbb{R}^3$

with $\int H^2 dA < 8\pi$ (see below). Therefore, all surfaces of genus g which are absolute minima of $\mathbb{W}$ (if they exist) are necessarily embedded and, of course, are Willmore surfaces. Most of the known examples of embedded Willmore surfaces come from compact minimal surfaces in the unit sphere $S^3 \subset \mathbb{R}^4$. Already Balschke and Thomsen had proved that stereographic projections $\sigma(M)$ of compact minimal surfaces M in S^3 are always Willmore surfaces. The area of $M^2 \subset S^3$ equals $\int H^2 dA$ for the stereographic projection of M^2 in $\mathbb{R}^3$.

In 1970, Lawson [11] discovered many such Willmore surfaces. He found that every compact surface but the projective plane (which is prohibited) can be minimally immersed into S^3. Moreover, every compact, orientable surface can be minimally embedded in S^3. These surfaces are obtained by first solving the Plateau problem for certain geodesic quadrilaterals in S^3 and then extending this solution surface by reflection across its geodesic boundary arcs.

For example, one such family of compact surfaces, $\{M_g\}$ (g = genus), is obtained by starting with such quadrilaterals having edge lengths $\pi/2$ and angles $\pi/2$, $\pi/g+1$, $\pi/2$, $\pi/g+1$. For these examples, $4\pi < \int_{\sigma(M_g)} H^2 dA < 8\pi$ and $\lim_{g\to\infty} \int_{\sigma(M_g)} H^2 dA = 8\pi$. (See Figs. 4-5).

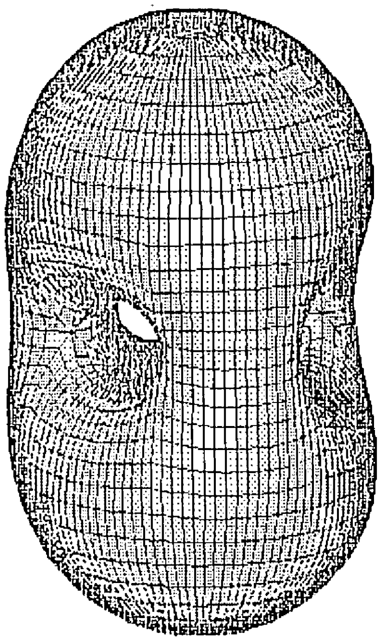

Figure 4

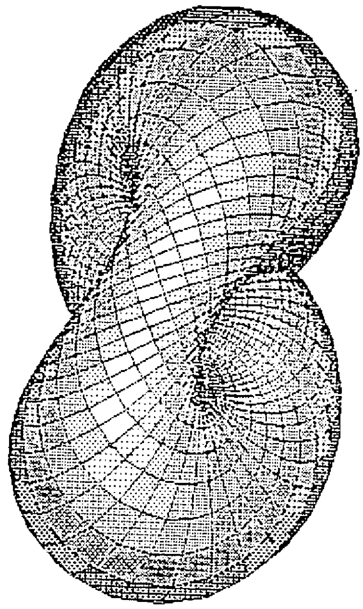

Figure 5

New examples of compact embedded minimal surfaces in S^3, and hence embedded Willmore surfaces, were recently discovered (Karcher-Pinkall-Sterling 1986 [8]). These examples are based on the tessellations of S^2 into cells having the symmetry of a Platonic solid. For example, S^3 is naturally tessellated by five tetrahedra (as S^3 is by four triangles) or S^3 is tessellated by two cubes (as S^2 is by two hexagons), etc. Dividing a cell by its planes of symmetry one obtains as a fundamental region for the group of symmetries a tetrahedron T.

To construct a minimal surface in S^3, one first finds a minimal surface with boundary, called a "patch"(Fig. 6), within T, which intersects orthogonally all the plane-faces of T in planar geodesics. From the patch, one obtains a certain piece of the whole surface, called a "bone" (Fig. 7), by repeatedly reflecting patches through those plane-faces of T which are not contained in faces of a cell. Finally, one builds the complete surface using reflections through faces of the cells (See Fig. 8-13).

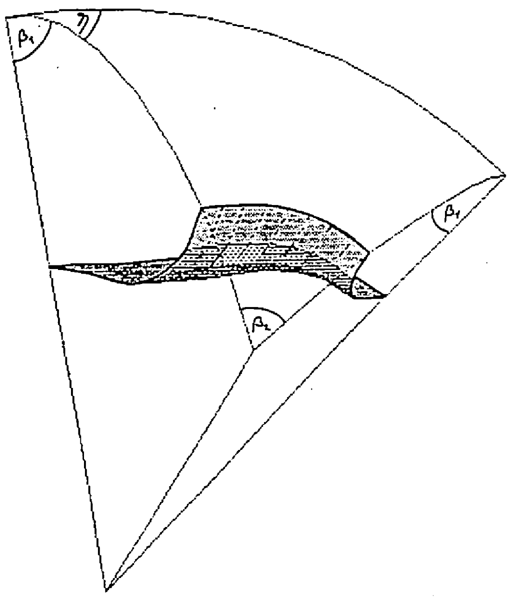

Figure 6

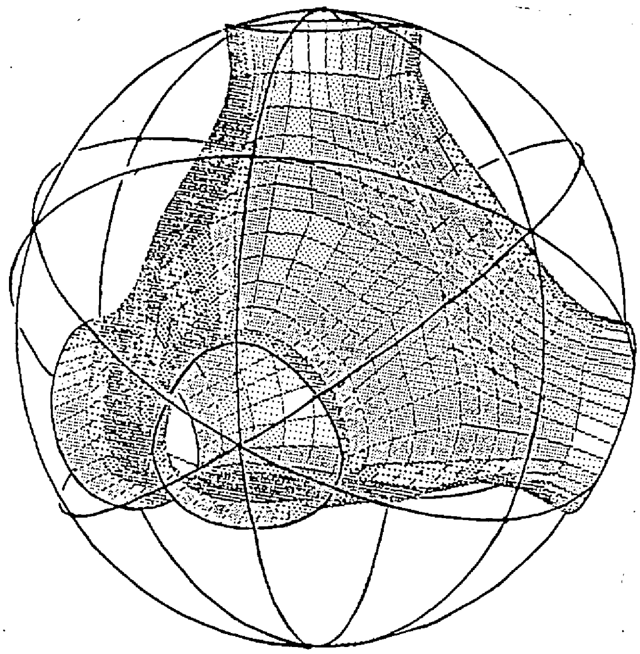

Figure 7

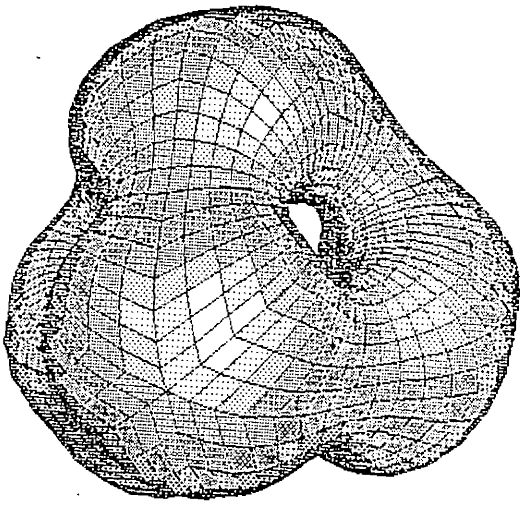

Figure 8

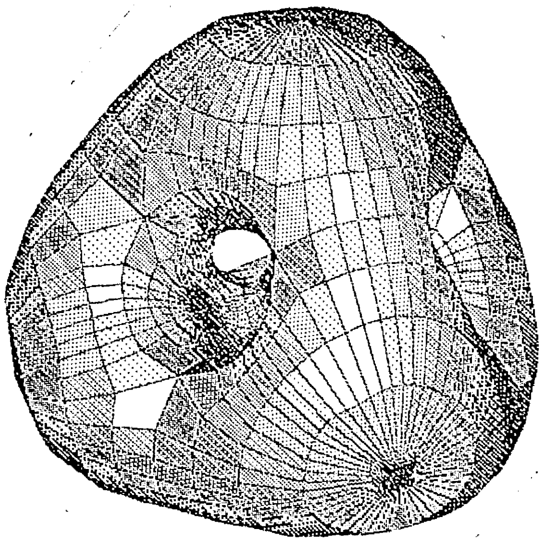

Figure 9

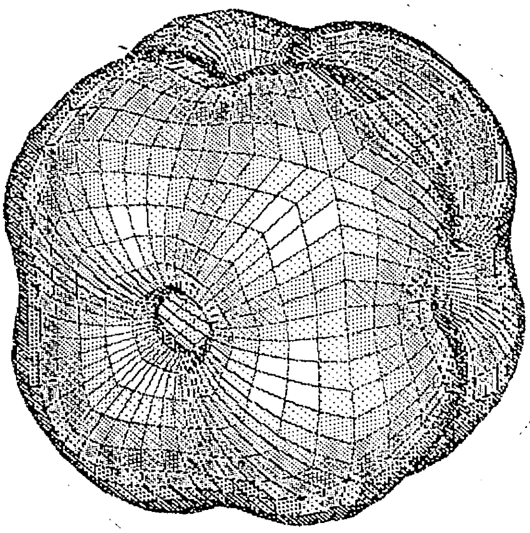

Figure 10

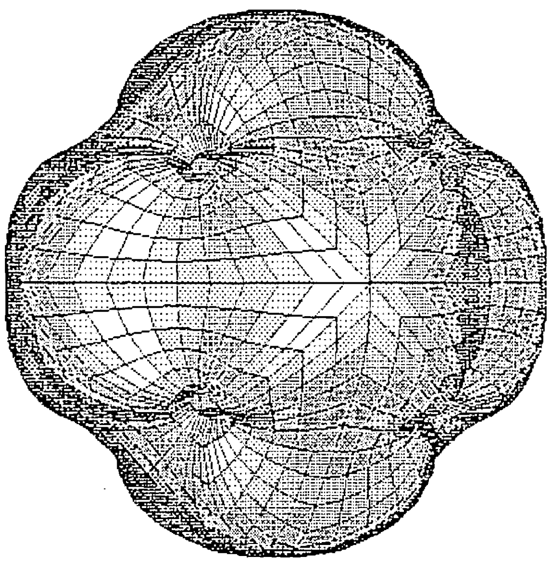

Figure 11

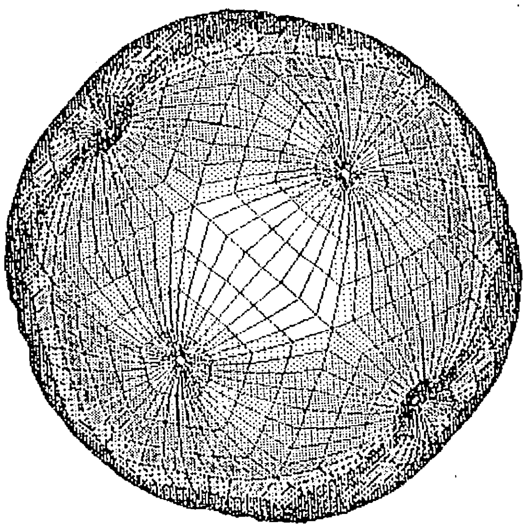

Figure 12

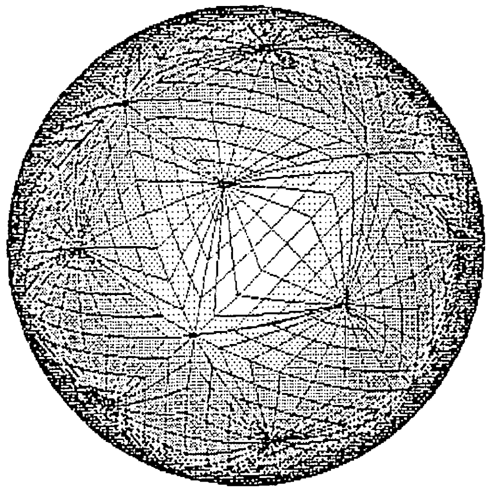

Figure 13

Remarks:

1) The computer plays a crucial role in the investigation of minimal surfaces in S^3. For example, computer estimates indicate the area of Lawson's three-holed torus M_3 is less than that of the genus three surface in Figure 8. This lends evidence to the conjecture that stereographic projections of Lawson's n-holed tori M_n are "optimal", in the sense that they are absolute minima of the Willmore integral among all genus n surfaces.

2) It was proved by Lawson that a compact minimal embedded surface in S^3 separates S^3 into two diffeomorphic components.

Furthermore, there was the following conjecture ([19], problem #98).

<u>Conjecture.</u> Any compact minimal embedded surface in S^3 separates S^3 into two components of equal volume.

The surface with dodecahedral symmetry in Figure 12 is a counterexample.

It suffices to prove M stays within a distance $\pi/2 - D$ (≈ 23.8) of its equator of reflection, E, where $4D - 2\sin(2D) = \pi$, since this tube around E contains half the volume of S^3. This is obvious from Figure 14 and can be rigorously proved (the actual value is approximately 7°).

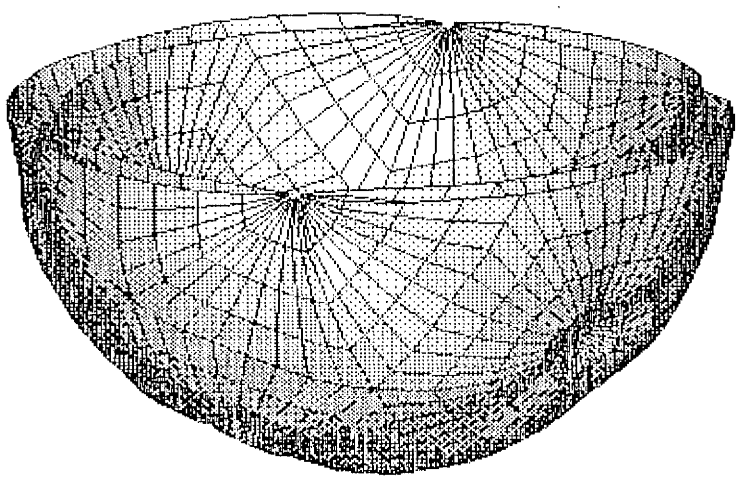

Figure 14

3) In 1985 [13], the first author found the first examples of compact embedded Willmore surfaces which are not stereographic projections of compact embedded minimal surfaces in S^3. Using results of Langer and Singer [10] on elastic curves on S^2 an infinite series of such surfaces is exhibited. (See Fig. 15).

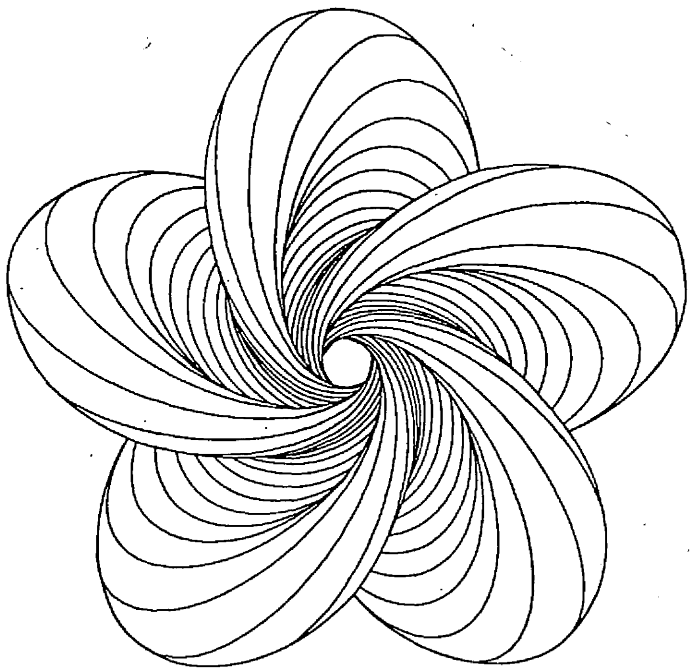

Figure 15

All of these surfaces are however unstable critical points of ω and hence are not candidates for absolute minima.

Finally, we want to mention that also the tori of constant mean curvature in $\mathbb{R}^3$, discovered recently by H. Wente [17], are related to the Willmore functional: A. Garsia [7] and R. Rüedy [14] had proved that any compact Riemann surface (i.e. an oriented surface with a conformal structure) can be conformally immersed into $\mathbb{R}^3$. Again one might ask for an "optimal" model in $\mathbb{R}^3$ (in the sense of the Willmore functional) for a given compact Riemann surface (M,g) (g a Riemannian metric defining the conformal structure). More generally, we are interested in "constrained Willmore surfaces", i.e. critical points of the functional $\mathfrak{W}$ restricted to the space of all <u>conformal</u> immersions $f:(M,g) \longrightarrow \mathbb{R}^3$. It has been observed by J. Langer that any compact surface in $\mathbb{R}^3$ of constant mean curvature is a constrained Willmore surface. This follows from the fact that the Gauss map of a constant mean curvature surface is harmonic, and the functional $\tilde{\mathfrak{W}}$ is essentially just the energy of the Gauss map.

The figures 16-19 show an immersed torus with constant H that was explicitly constructed by U. Abresch [1]. The whole surface (Figure 16) consists of six congruent pieces, which are immersed annuli (Figure 17). The figures 18 and 19 show one half and one third of the torus, respectively.

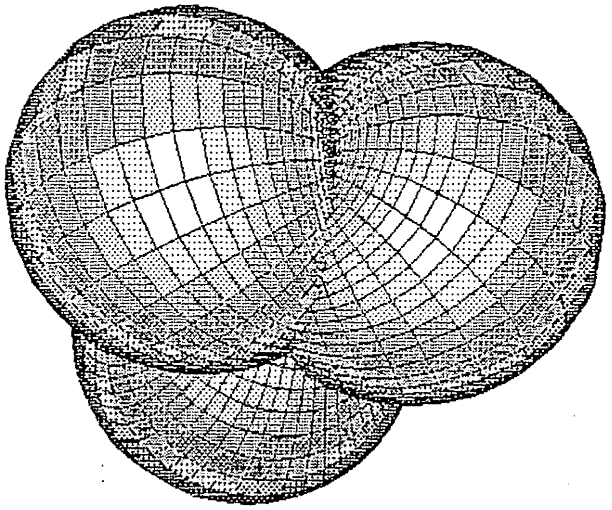

Figure 16

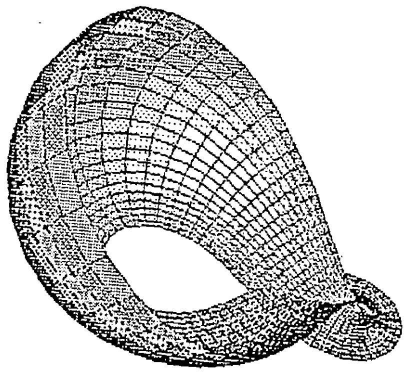

Figure 17

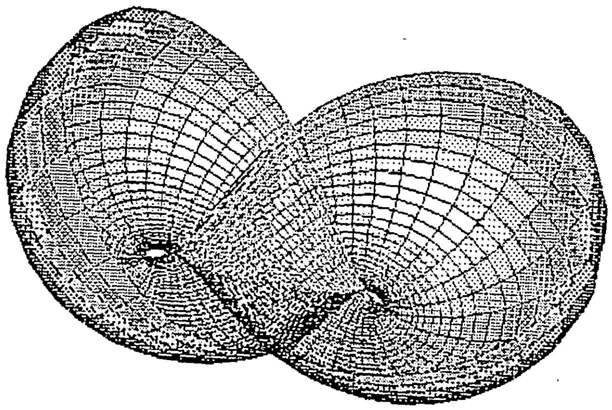

Figure 18

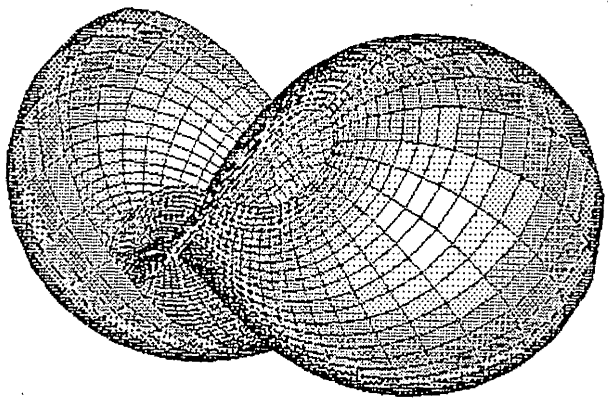

Figure 19

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