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A SPECIAL CONFIGURATION OF 12 CONICS AND GENERALIZED KUMMER SURFACES

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Abstract. The generalized Kummer surface X associated to an abelian surface possessing an order 3 symplectic automorphism contains a $9\mathbf{A}_2$ configuration of (-2)-curves. Such a configuration plays the role of the $16\mathbf{A}_1$ configurations for usual Kummer surfaces. In this paper we construct 9 other such $9\mathbf{A}_2$ configurations on the generalized Kummer surface associated to the double cover of the plane branched over the sextic dual curve of a cubic curve. The new $9\mathbf{A}_2$ configurations are obtained by taking the pull-back of a certain configuration of 12 conics which are in special position with respect to the branch curve, plus some singular quartic curves. We also give various models of X and of the generic fiber of its natural elliptic pencil.

1. Introduction

A Kummer surface $\mathrm{Km}(A)$ is the minimal desingularization of the quotient of an abelian surface A by the involution $[-1]$. It is a K3 surface that contains 16 disjoint (-2)-curves (over the 16 singularities of $A/[-1]$), such set of curves is also called a $16\mathbf{A}_1$ configuration. A well known result of Nikulin gives the converse: if a K3 surface X contains a $16\mathbf{A}_1$ configuration, then it is a Kummer surface, which means that there exists an abelian surface A such that $X = \mathrm{Km}(A)$ and the 16 (-2)-curves are the resolution of singularities of $A/[-1]$.

Shioda then asked the following question: if two complex tori A, B are such that $\mathrm{Km}(A) \simeq \mathrm{Km}(B)$, is it true that $A \simeq B$? Gritsenko and Hulek gave a negative answer to that question in general. In [11], [12], we studied and constructed examples of two $16\mathbf{A}_1$ configurations on the same K3 surface such that their associated complex torus are not isomorphic.

Kummer surfaces have natural generalizations. By example if the group $\mathbb{Z}/3\mathbb{Z}$ acts symplectically on an abelian surface A, then the quotient surface $A/(\mathbb{Z}/3\mathbb{Z})$ has 9 singularities $\mathbf{A}_2$ (cups) and its minimal desingularization, denoted by $\mathrm{Km}_3(A)$ is a K3 surface which contains 9 disjoint $\mathbf{A}_2$-configurations i.e. 9 pairs of two (-2)-curves C, C' such that $CC' = 1$. It is then natural to ask if an isomorphism $\mathrm{Km}_3(A) \simeq \mathrm{Km}_3(B)$ between to generalized Kummer surfaces implies that A and B are isomorphic.

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With this question in mind, in the present paper we construct geometrically several $9\mathbf{A}_2$ configurations on some generalized Kummer surfaces previously studied in [4] by Birkenhake and Lange. Their construction is as follows:
The dual of a cubic curve $E_\lambda = \{x^3 + y^3 + z^3 - 3\lambda xyz = 0\}$, is a sextic curve C_λ with a set $\mathcal{P}_9$ of $9\mathbf{A}_2$ singularities corresponding to the nine inflection points on E. The minimal desingularization X_λ of the double cover of $\mathbb{P}^2$ branched over C_λ is a generalized Kummer surface with a natural $9\mathbf{A}_2$ configuration. The surface X_λ has a natural elliptic fibration for which the 18 (-2)-curves in the $9\mathbf{A}_2$ configuration are sections, and the reduced strict transform of C_λ is a fiber.

In order to find other (-2)-curves on X_λ we study the set $\mathcal{C}_{12}$ of conics that contain at least 6 points in $\mathcal{P}_9$. One has

Theorem 1. *The set $\mathcal{C}_{12}$ has order 12. Each conic in $\mathcal{C}_{12}$ contains exactly 6 points in $\mathcal{P}_9$ and through each point in $\mathcal{P}_9$ there are 8 conics. The sets $(\mathcal{P}_9, \mathcal{C}_{12})$ form therefore a*

$$(9_8, 12_6)$$

point-conic configuration.

That configuration has interesting symmetries e.g. the 8 conics that goes through one fixed point q in $\mathcal{P}_9$ and the 8 points in $\mathcal{P}_9 \setminus \{q\}$ form a 8_5 point-conic configuration (the freeness of the arrangement of curves $\mathcal{C}_{12}$ is studied in [9], where we learned that this configuration has been also independently discovered in [6]).

The irreducible components of the curves in X_λ above the 12 conics are 24 (-2)-curve on the K3 surface X_λ. That set of 24 (-2)-curves possesses nine $8\mathbf{A}_2$ sub-configurations $\mathcal{A}_1, \dots, \mathcal{A}_9$ (coming from the nine 8_5 sub-configurations). Using the pull-back to X_λ of some 9 special (singular) quartics curves, we are able to complete each of these $8\mathbf{A}_2$ configurations into a $9\mathbf{A}_2$ configuration.

We then continue our study of the surface X_λ by obtaining various models in projective space, and a model of the generic fiber E_{K3} of the natural elliptic fibration $X_\lambda \to \mathbb{P}^1$. We obtain in particular:

Theorem 2. *A Hessian model of the generic fiber of the fibration $X_\lambda \to \mathbb{P}^1$ is*

$$E_{K3} \quad x^3 + y^3 + z^3 + \frac{\lambda^3(t^2+3) - 4t^2}{\lambda^2(t^2-1)} xyz = 0.$$

We also obtain a Weierstrass model of E_{K3}. It turns out that the Mordell-Weil group of the elliptic fibration $X_\lambda \to \mathbb{P}^1$ has rank 1. Using the translation maps obtained from the model E_{K3}, we can construct other $9\mathbf{A}_2$ configurations from the previously known once.

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2. Preliminaries

2.1. **Notations and conventions.** Let $\eta : Y \to Z$ be a dominant map between two surfaces and let $C \hookrightarrow Z$ be a curve. In this paper, the reduced pull-back of C minus the irreducible components contracted by η is called the strict transform of C on Y.

Definition 3. Let $n \in \mathbb{N}^*$ be an integer. We say that (-2)-curves $E_1, \dots, E_{2n}$ on a K3 surface form a $n\mathbf{A}_2$-configuration if their intersection matrix is the diagonal matrix with n blocs of type:

$$\left(\begin{array}{cc} -2 & 1 \\ 1 & -2 \end{array}\right).$$

2.2. **The Hesse $(9_4, 12_3)$, the dual Hesse, and the 8_3 configurations.** Because we will meet a configuration which has properties analogous to the Hesse configuration, let us recall some properties of that configuration:

A smooth elliptic curve $E \hookrightarrow \mathbb{P}^2$ possesses 9 inflection points. The Hesse configuration is the set of 12 lines through the 9 inflection points: each line contains 3 inflection points, each inflection points is contained in 4 lines, so that it is a $(9_4, 12_3)$-configuration.

If 9 points are the inflection points of an elliptic curve, then these points are the inflection points of a pencil of elliptic curves through these points.

Removing one point of the Hesse configuration and its 4 incident lines, one get a 8_3 configuration of 8 points and 8 lines.

By taking the 12 points in the dual space corresponding to the 12 lines of the Hesse configuration, and by considering the lines trough these points, one get the dual configuration of 12 points and 9 lines, called the dual Hesse configuration. As an abstract configuration, it can be realized in the plane $\mathbb{P}^2(\mathbb{F}_3)$ by taking the 13 lines in $\mathbb{P}^2(\mathbb{F}_3)$ and removing from this set the 4 lines passing through a fixed point (the sets of lines and points in $\mathbb{P}^2(\mathbb{F}_3)$ form a 13_4 configuration).

3. Nine new $9\mathbf{A}_2$ configurations

3.1. $(9_8, 12_6)$ **and** 8_5 **configurations of conics.** Let us fix $\lambda \notin \{1, \omega, \omega^2\}$, for ω such that $\omega^2 + \omega + 1 = 0$. The dual C_λ of the elliptic curve

$$E_\lambda = \{x^3 + y^3 + z^3 - 3\lambda xyz = 0\},$$

is a 9-cuspidal sextic curve, (i.e. a sextic curve with 9 cusps) and conversely any 9-cuspidal sextic curve is obtained in that way. The images by the dual map of the 9 inflection points of E_λ are the 9 cusps of C_λ and the curve C_λ has equation:

$$\begin{aligned} C_\lambda = \{&(x^6 + y^6 + z^6) + 2(2\lambda^3 - 1)(x^3y^3 + x^3z^3 + y^3z^3) \\ &-6\lambda^2 xyz(x^3 + y^3 + z^3) - 3\lambda(\lambda^3 - 4)x^2y^2z^2 = 0\}. \end{aligned}$$

The set $\mathcal{P}_9$ of the 9 cusps $p_1, \dots, p_9$ is

$$\begin{array}{lll} p_1 = (\lambda : 1 : 1), & p_4 = (\lambda : \omega : \omega^2), & p_7 = (\lambda : \omega^2 : \omega) \\ p_2 = (1 : \lambda : 1), & p_5 = (\omega^2 : \lambda : \omega), & p_8 = (\omega : \lambda : \omega^2) \\ p_3 = (1 : 1 : \lambda), & p_6 = (\omega : \omega^2 : \lambda), & p_9 = (\omega^2 : \omega : \lambda). \end{array}$$

When λ varies, the closure of the set of points p_j is a line, denoted by L_j; we obtain in that way a set $\mathcal{L}_9$ of 9 lines. Dually, the points on L_j correspond to the pencil of lines meeting in the inflection point (corresponding to p_j) of the elliptic curve E_λ. One can check moreover that the line L_j is the tangent line to the cusp $p_j \in C_\lambda$.

Theorem 4. *The set $\mathcal{C}_{12}$ of conics that contain 6 points in $\mathcal{P}_9$ has order 12; each conic of $\mathcal{C}_{12}$ is smooth. Each point of $\mathcal{P}_9$ is on 8 conics, thus the sets $\mathcal{P}_9$, $\mathcal{C}_{12}$ of points and conics form a $(9_8, 12_6)$-configuration.*
The set of intersection points of the 12 conics in $\mathcal{C}_{12}$ is the union of $\mathcal{P}_9$ and a set $\mathcal{P}_{12}$ of 12 points, these 12 points have multiplicity 2 for the curve $\sum_{C \in \mathcal{C}_{12}} C$. The intersections between the conics in $\mathcal{C}_{12}$ are transverse. A conic C in $\mathcal{C}_{12}$ meet 9 conics in $\mathcal{C}_{12}$ in 4 points contained in $\mathcal{P}_9$ and each of the two remaining conics in 3 points contained in $\mathcal{P}_9$ and one point in $\mathcal{P}_{12}$. The set $\mathcal{P}_{12}$ is also the set of intersection points of the 9 lines in $\mathcal{L}_9$, and the sets $(\mathcal{P}_{12}, \mathcal{L}_9)$ form a $(12_3, 9_4)$-configuration which is the dual Hesse Configuration.

Proof. By a computer search, the 12 conics are:

$$\begin{array}{l}
C_{1,2,3,4,5,6} = \{x^2 + (\lambda+1)(\omega xy + \omega^2 xz + yz) + \omega^2 y^2 + \omega z^2 = 0\}, \\
C_{1,2,3,7,8,9} = \{x^2 + (\lambda+1)(\omega^2 xy + \omega xz + yz) + \omega y^2 + \omega^2 z^2 = 0\}, \\
C_{1,2,4,5,7,8} = \{xy - \lambda z^2 = 0\}, \\
C_{1,2,4,6,8,9} = \{x^2 + (\omega\lambda+1)(xy + \omega xz + \omega yz) + y^2 + \omega^2 z^2 = 0\}, \\
C_{1,2,5,6,7,9} = \{x^2 + (\omega^2\lambda+1)(xy + \omega^2 xz + \omega^2 yz) + y^2 + \omega z^2 = 0\}, \\
C_{1,3,4,5,8,9} = \{x^2 + (\omega\lambda+\omega^2)(xy + yz + \omega xz) + \omega y^2 + z^2 = 0\}, \\
C_{1,3,4,6,7,9} = \{-\lambda y^2 + xz = 0\}, \\
C_{1,3,5,6,7,8} = \{x^2 + (\omega^2\lambda+\omega)(xy + yz + \omega^2 xz) + \omega^2 y^2 + z^2 = 0\}, \\
C_{2,3,4,5,7,9} = \{x^2 + (\lambda+\omega^2)(xy + xz + \omega^2 yz) + \omega y^2 + \omega z^2 = 0\}, \\
C_{2,3,4,6,7,8} = \{x^2 + (\lambda+\omega)(xy + xz + \omega yz) + \omega^2(y^2 + z^2) = 0\}, \\
C_{2,3,5,6,8,9} = \{\lambda x^2 - yz = 0\}, \\
C_{4,5,6,7,8,9} = \{x^2 + (\lambda+1)(xy + xz + yz) + y^2 + z^2 = 0\},
\end{array}$$

where the index $i, j, \dots, n$ of the conic $C_{i,j,\dots,n}$ means that this conic contains the 6 points p_s, $s \in \{i, j, \dots, n\}$. It is easy to see that the points in $\mathcal{P}_9$ are in general position: no line contains 3 cusps, thus the conics are smooth. From the data of the conics and the knowledge of the points in $\mathcal{P}_9$ they contain, one can check the assertions about the configuration of the 12 conics and the 9 points. If one renumbers the 12 conics by their order $C_1, \dots, C_{12}$ from the top to bottom of the above list, one obtains that the pair of (indexes of) conics which have an intersection point not in $\mathcal{P}_9$ are

$$(1,2),\ (1,12),\ (2,12),\ (3,7),\ (3,11),\ (4,8),$$
$$(4,9),\ (5,6),\ (5,10),\ (6,10),\ (7,11),\ (8,9),$$

and correspondingly, the 12 points are

$$(1:1:1),\ (\omega:\omega^2:1),\ (\omega^2:\omega:1),\ (1:0:0),\ (0:1:0),\ (\omega^2:1:1),$$
$$(1:\omega^2:1),\ (\omega:1:1),\ (1:\omega:1),\ (\omega^2:\omega^2:1),\ (0:0:1),\ (\omega:\omega:1).$$

respectively. One can check easily that these 12 points in $\mathcal{P}_{12}$ are the intersection points of the lines in $\mathcal{L}_9$, which lines form the dual Hesse arrangement (see e.g. [2, Section 1]). By Bézout Theorem the intersections between the conics are transverse. □

Let $q \in \mathcal{P}_9$ and define $\mathcal{P}_q = \mathcal{P}_9 \setminus \{q\}$.

Theorem 5. *The subset $\mathcal{C}_q$ of conics containing the point q has order 8. The set of points $\mathcal{P}_q$ and the set of conics $\mathcal{C}_q$ form a 8_5 configuration: each point is on 5 conics and each conic contains 5 of the points in $\mathcal{P}_q$. For each conic C in $\mathcal{C}_q$ there exists a unique conic $C' \in \mathcal{C}_q$ such that there is a unique point in the intersection of C and C' which is not in $\mathcal{P}_q$.*

Proof. That can be checked directly from the datas in the proof of Proposition 4. □

Let X_λ be the minimal desingularization of the double cover branched over the sextic curve C_λ with 9 cusps. We denote by $\eta : X_\lambda \to \mathbb{P}^2$ the natural map and we denote by A_j, A'_j the two (-2)-curves in X above the point p_j in $\mathcal{P}_9$ (so that the curves A_j, A'_j, $j \in \{1, \dots, 9\}$ form a $9\mathbf{A}_2$-configuration). We have

Lemma 6. *The strict transform by the map $\eta : X_\lambda \to \mathbb{P}^2$ of a conic $C \in \mathcal{C}_{12}$ is the union of two (-2)-curves θ_C, θ'_C. Let C, D be two conics in $\mathcal{C}_{12}$. Suppose that C and D meet in 4 points in $\mathcal{P}_9$. Then the (-2)-curves $\theta_C, \theta'_C, \theta_D, \theta'_D$ are disjoint. Suppose that C and D meet in 3 points in $\mathcal{P}_9$. Then, up to exchanging θ_D and θ'_D, the curves $\theta_C, \theta_D, \theta'_C, \theta'_D$ form a $2\mathbf{A}_2$-configuration.*

Proof. Rather than performing a double cover and taking the resolution of surface singularities, we perform three blow-ups at each cusp $q \in \mathcal{P}_9$ of C_λ, so that the branch locus is smooth and near q it is the union of the strict transform of C_λ and a (-2)-curve E_2. The three exceptional curves are E_2 and E_1, E_3 where $E_j^2 = -j$ and $E_1E_2 = E_1E_3 = 1$, $E_2E_3 = 0$. On the double cover, the reduced image inverse of the curves E_1, E_2, E_3 are respectively a (-2)-curve, a (-1)-curve and two disjoint (-3)-curves. Contracting the (-1)-curve and then the image of the (-2)-curve, we get the K3 surface X_λ. By that local computation, we see that for $C \in \mathcal{C}_{12}$, the curves $\theta_C, \theta_{C'}$ are disjoint (the strict transform $\bar{C}$ of C under the blow-up map do not meet the branch locus, and the two curves above $\bar{C}$ remains disjoint after contracting the 9 (-1)-curves on the double cover, and then contracting the images of the 9 (-2)-curves).
Suppose C and D meet in 4 points in $\mathcal{P}_9$. The intersection being transverse, the strict transform $\bar{C}, \bar{D}$ of the curves C, D under the 3 blow-ups at each

cusps are two disjoint curves not meeting the branch curve. As above the 4 curves above the remain disjoint in X_λ after contracting the (-1)-curves.
If C and D meet in 3 points in $\mathcal{P}_9$ then they meet transversely at a unique point not in $\mathcal{P}_9$. Then taking the above notations, we have this time $\bar{C}\bar{D}=1$, so that the last assertion holds. □

Let $\mathcal{P}_q$ and $\mathcal{C}_q$ as above. Using Theorem 5 and Lemma 6, we get:

Corollary 7. *The* 16 (-2)-*curves that are strict transform of the* 8 *conics in* $\mathcal{C}_q$ *form a* $8\mathbf{A}_2$-*configuration.*

For each point $q=p_j$, $j\in\{1,\dots,9\}$, we denote by $\mathcal{A}_j$ the corresponding $8\mathbf{A}_2$-configuration on X_λ. In order to obtain new generalized Nikulin configurations, one needs to find other $\mathbf{A}_2$-configurations, this will be done in the next section by using singular quartics instead of conics.

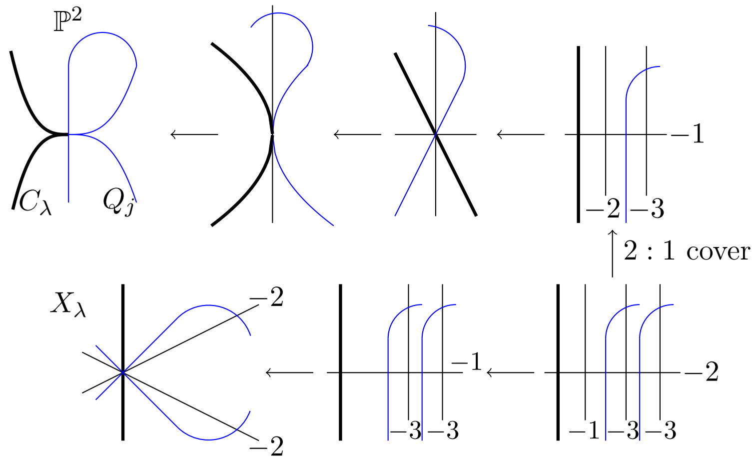
Remark 8. Using a computer, we found eight $8A_2$-configurations in the set of 32 (-2)-curves which is the union of the two $8\mathbf{A}_2$-configurations $\mathcal{A}_j$ and $\{A_1,A_1',\dots,A_9,A_9'\}\setminus\{A_j,A_j'\}$. However one can compute that the orthogonal complement of 6 of them are lattices with no (-2)-classes, thus one cannot complete these 6 configurations into $9\mathbf{A}_2$-configurations. The 32 (-2)-curves can be realized as lines in a projective model of X_λ, see Proposition 11.

3.2. 9 **new** $9\mathbf{A}_2$**-configurations.** Let $p_j\in\mathcal{P}_9$ be one of the 9 cusp singularity of the sextic C_λ.

Theorem 9. *There exists a quartic curve* Q_j *that contains all points in* $\mathcal{P}_9$, *such that* Q_j *has a unique singularity, which is at the point* p_j *and is of multiplicity* 3. *That singularity has two tangents, one branch is smooth while the other branch is a cusp singularity. The tangent to the cusp singularity of* Q_j *is also the tangent to the cusp singularity of the sextic* C_λ *at* p_j.
The curve Q_j *has geometric genus* 0. *Its strict transform on* X_λ *is the union of two* (-2)-*curves* θ_j,θ_j' *which form a* $\mathbf{A}_2$-*configuration. The curves* θ_j,θ_j' *and the* 16 *curves in* $\mathcal{A}_j$ *form a* $9\mathbf{A}_2$-*configuration.*

Proof. We give in the Appendix the equations of the 9 curves Q_j, $j\in\{1,\dots,9\}$. These curves have been constructed using the `LinSys` program by C. Rito which enables to find curves of given degree with prescribed singularities and given tangencies at a set of points in the plane. Conversely, one can check that the singularity of Q_j at p_j has multiplicity 3, is resolved by one blow-up, with the exceptional divisor meeting the strict transform in two points, one of multiplicity 2.
The curve Q_j has genus 0, (see e.g. [7, Chapter 4, Section 2]). By Bézout's Theorem, the intersections of the quartic Q_j with the 8 conics in $\mathcal{C}_{12}$ that contain p_j are transverse, so that the curves in $\mathcal{A}_j$ are disjoint from θ_j,θ_j' and we thus get a $9\mathbf{A}_2$ configuration. □

FIGURE 3.1. Behavior of the quartic Q_j under the double cover



The horizontal arrows are blow-up maps

4. Projective, Hessian and Weierstrass models

4.1. **A degree 8 model and the fibration associated to the double cover.** Let L be the big and nef divisor on X_λ which is the pull back of a line in $\mathbb{P}^2$. For λ generic, the divisors $L, A_1, A_1' \dots, A_9, A_9'$ form a $\mathbb{Q}$-base $\mathcal{B}$ of $\mathrm{NS}(X_\lambda)_{/\mathbb{Q}}$, they generate an index 3^6 lattice of $\mathrm{NS}(X_\lambda)$.

Let $\mu : Y_\lambda \to \mathbb{P}^2$ be the blow-up of the plane at the 9 cusps of the sextic curve C_λ and let $E_1, \dots, E_9$ be the exceptional curves. The strict transform by μ of the curve C_λ is the smooth genus 1 curve

$$\bar{C_\lambda} = \mu^* C_\lambda - 2\sum_{i=1}^{9} E_i, \text{ such that } \bar{C_\lambda}^2 = 0,$$

where $E_1, \dots, E_9$ are the exceptional curves over $p_1, \dots, p_9$. The surface X_λ is the double cover of Y_λ branched over $\bar{C_\lambda}$; we denote by

$$\eta : X_\lambda \to Y_\lambda$$

the double cover morphism (so that $\eta^* E_j = A_j + A_j'$) and by F the ramification locus, so that $2F = \eta^* \bar{C}_\lambda$. Since $2F = \eta^* \bar{C}_\lambda \equiv 6L - 2\sum_{i=1}^{9} A_j + A_j'$, we get

$$F \equiv 3L - \left(\sum_{j=1}^{9} A_j + A_j' \right).$$

Let $D \hookrightarrow \mathbb{P}^2$ be a line; the curve C_λ belong to linear system

$$\delta = |6D - 2\sum_{j=1}^{9} p_j|$$

of sextic curves with a double point at points in $\mathcal{P}_9$. One computes that this linear system is 1 dimensional. Moreover there exists a unique cubic curve $\mathrm{C_a}(\lambda)$ (called the Cayleyan curve, see [1]) that contains the 9 points in $\mathcal{P}_9$, which is

$$\mathrm{C_a}(\lambda) = \{x^3 + y^3 + z^3 - \frac{1}{\lambda}(\lambda^3 + 2)xyz = 0\},$$

so that $2\mathrm{C_a}(\lambda) \in \delta$. The linear system δ lifts to a base point free linear system δ' on Y_λ with $\bar{C}_\lambda \in \delta'$. The linear system δ' defines a morphism $\varphi' : Y_\lambda \to \mathbb{P}^1$ and induces an elliptic fibration

$$\varphi : X_\lambda \to \mathbb{P}^1$$

for which F is a fiber. Let p, q be the images of the strict transforms of $\mathrm{C_a}(\lambda)$ and C_λ by φ'. In fact the surface X_λ is the fiber product of the fibration φ' and the quadratic transformation $\mathbb{P}^1 \to \mathbb{P}^1$ branched at p, q. Indeed both maps $X_\lambda \to Y_\lambda$ and $Y_\lambda \times_{\mathbb{P}^1} \mathbb{P}^1$ has the same branch locus in the rational surface Y_λ.
The curves $A_1, A'_1, \dots, A_9, A'_9$ are sections of φ, and one can check that the curves $\theta_1, \theta'_1, \dots, \theta_9, \theta'_9$ are also sections.

Proposition 10. *The fibration φ contracts the 24 (-2)-curves θ_C, θ'_C above the 12 conics $C \in \mathcal{C}_{12}$. The singular fibers of φ are 8 fibers of type $\tilde{\mathbf{A}}_{\mathbf{2}}$. For λ generic, the fibration φ has fibers with non-constant moduli and the Mordell-Weil group of the fibration φ is $\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2$.*

Proof. The following four sextic curves

$$C_{123456} + C_{123789} + C_{456789},\ C_{124578} + C_{134679} + C_{235689},$$
$$C_{124689} + C_{135678} + C_{234579},\ C_{125679} + C_{134589} + C_{234678},$$

belong to the linear system δ of sextic curves that have multiplicity 2 at the points in $\mathcal{P}_9$; actually their singularities are nodes. By the results in the proof of Theorem 4, the strict transform to Y_λ of the above 4 sextic form 4 fibers of type $\tilde{\mathbf{A}}_{\mathbf{2}}$, which lies in the étale locus of η. Their strict transform on X_λ is therefore the union of eight fibers of type $\tilde{\mathbf{A}}_{\mathbf{2}}$. A fiber of type $\tilde{\mathbf{A}}_{\mathbf{2}}$ contributes to 3 in the Euler characteristic of X_λ, which is equal to 24. Since there are $8\tilde{\mathbf{A}}_{\mathbf{2}}$ singular fibers, the fibration has no other singular fibers. The 24 curves θ_C, θ'_C above the 12 conics are in the fibers, thus are contracted by φ.

The strict transform $\mathrm{C_a}(\lambda)'$ on X_λ of $\mathrm{C_a}(\lambda)$ is smooth, of genus 1 (see Remark 20 for its link with $\mathrm{C_a}(\lambda)$). Since $\mathrm{C_a}(\lambda)' \cdot F = 0$, we have that $\mathrm{C_a}(\lambda)' \equiv F$. The curve F is isomorphic to E_λ. For generic λ the curves $\mathrm{C_a}(\lambda)$ and E_λ have distinct j-invariants, thus the fibers of φ have a non-constant moduli. Since the fibration is not isotrivial, results of Shioda (see [14, Corollary 1.5]) apply and tell that the Mordell-Weil group of sections of $\varphi : X_\lambda \to \mathbb{P}^1$ has rank $1 = 19 - (2 + 8(3-1))$.

In fact elliptic fibrations of K3 surfaces are classified by Shimada in [16]. A table with the 3278 possible cases is avaliable in [17]. Our fibration is case number 2373 in that table, where one can find moreover that the torsion part of its Mordell-Weil group is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$. □

The divisor

$$D_{14} = 4L - \left(\sum_{j=1}^{9} A_j + A'_j\right)$$

is linearly equivalent to $L + F$ and is effective. Let us define $D_8 = D_{14} - (A_1 + A'_1)$.

Proposition 11. *The divisors D_8 and D_{14} are ample of square $D_8^2 = 8$, $D_{14}^2 = 14$. The linear system $|D_8|$ is base point free, non-hyperelliptic, and defines an embedding*

$$X_\lambda \hookrightarrow \mathbb{P}^5$$

as a degree 8 complete intersection surface. For $d \in \mathbb{N}^*$, let n_d be the number of (-2)-curves of degree d for D_8. The series $\sum n_d T^d$ begins with

$$32T + 20T^2 + 334T^4 + 576T^5 + 880T^6 + 8640T^7 + 17784T^8...,$$

in particular X_λ contains 32 lines and 20 conics.

Proof. Let B be a (-2)-curve such that $D_{14}B \leq 0$. Since L is effective and $L^2 > 0$, one has $LB \geq 0$, moreover since F is a fiber, $FB \geq 0$ and we must have $LB = 0 = FB$. That implies that B is an irreducible component of a singular fiber, ie $B \in \{\theta_C, \theta'_C \,|\, C \in \mathcal{C}_{12}\}$. But since $L\theta_C = L\theta'_C = 2$ for $C \in \mathcal{C}_{12}$, such a curve B cannot exist, thus D_{14} is ample.

Let us prove that D_8 is ample. We have

$$\theta_1 + \theta'_1 \equiv 4L - 2(A_1 + A'_1) - \sum_{j=1}^{9}(A_j + A'_j),$$

thus

$$D_8 \equiv A_1 + A'_1 + \theta_1 + \theta'_1$$

and the divisor D_8 is effective. We check that $D_8A_1 = D_8A'_1 = D_8\theta_1 = D_8\theta'_1 = 1$ and $D_8^2 = 8$, therefore D_8 is nef and big. Suppose that there is a (-2)-curve B on X_λ such that $D_8B = 0$. Then by the above expression of D_8, one has $A_1B = A'_1B = 0$. Let $D \hookrightarrow \mathbb{P}^2$ be a line. For $j \in \{2, \dots, 9\}$, let us consider the linear system

$$\delta_j = |4D - (p_1 + p_j + \sum_{k=1}^{9} p_k)|$$

of the quartic curves that go through the points in $\mathcal{P}_9$ and with multiplicity 2 at p_1 and p_j. Using `LinSys`, one can compute that for each $j > 1$, the linear system δ_j is a pencil of curves and the base points set is $\mathcal{P}_9$. Moreover, the generic element γ_j of δ_j is an irreducible curve of geometric genus 1 which cuts C_λ in $\mathcal{P}_9$ and two more points. Thus we obtain that for each $j > 1$, the strict transform of γ_j is an irreducible curve Γ_j such that

$$D_8 \equiv \Gamma_j + A_j + A'_j \text{ and } \Gamma_j^2 = 2.$$

Since $D_8B = 0$, we obtain $A_jB = A'_jB = 0$ for all $j \in \{1, \dots, 9\}$. Since the orthogonal of the classes A_j, A'_j, $j \in \{1, \dots, 9\}$ (on which B belongs) is generated by L, the class of B must be a multiple of L and have positive square, which is absurd. Therefore D_8 is ample.

Suppose that there is a fiber F' such that $D_8F' \in \{1, 2\}$. Observe that by using the expression for D_8, we get that $F'\Gamma_j = 0, 1, 2$. If $F'\Gamma_j = 0$, then Γ_j is contained in a fiber of the fibration determined by F', but this is not possible since $\Gamma_j^2 = 2$. If $F'\Gamma_j = 1$, then Γ_j is a section of the fibration so is a rational curve, but again this is not possible. If $F'\Gamma_j = 2$ (we can assume that this holds for all j, otherwise we are in a previous case), then F' is in the orthogonal complement of the A_j, A'_j but this is not possible since this is generated by L, which is of square 2. Therefore there are no such fiber F' and using [13], we obtain that the linear system $|D_8|$ is base-point free and gives an embedding of X_λ.

With respect to the divisor D_8, the degrees of the curves $A_1, A'_1, \theta_1, \theta'_1$ equal 2 and the degrees of curves A_i, A'_i, $i \geq 2$ is 1. For the assertions on the number of rational curves of degree $d \leq 8$ we used the algorithm in [10], which computes the classes of (-2)-curves in $\mathrm{NS}(X_\lambda)$ of given degrees with respect to a fixed ample class. □

Proceeding in a similar way as in the proof of Proposition 11, we obtain:

Proposition 12. *Let $i, j \in \{1, \dots, 9\}$, $i \neq j$. The divisor*

$$D_{i,j} = D_{14} - (A_i + A'_i + A_j + A'_j)$$

is nef of square 2 and the linear system $|D_{i,j}|$ is base point free.

One can compute that the intersection with D_{ij} is 0 for the 10 curves $\theta_{ijklmn}, \theta'_{ijklmn}$ (where $\{k, l, m, n\} \subset \{1, \dots, 9\}$ is a set of 4 elements such that the conic C_{ijklmn} exists), and for the (-2)-curve which is the strict transform on X_λ of the line through cusps p_i, p_j.

4.2. **A Hessian model of the K3 surface X_λ.** Let f_λ be the equation of the 9 cuspidal sextic C_λ which is the dual of E_λ, and let c_λ be the equation of the Cayleyan elliptic curve, the unique cubic curve that goes through the 9 cusps.

We recall that Y_λ is the blow-up of the plane at the 9 points in $\mathcal{P}_9$; it has a natural elliptic fibration. A singular model of Y_λ is obtained as the surface in $\mathbb{P}^1 \times \mathbb{P}^2$ with equation $uf_\lambda - v(c_\lambda)^2 = 0$, where u, v are the coordinates of $\mathbb{P}^1$. The projection onto $\mathbb{P}^1$ induces the fibration $Y_\lambda \to \mathbb{P}^1$. A singular model of the $K3$ surface X_λ is the surface X_λ^{sing} in $\mathbb{P}^1 \times \mathbb{P}^2$ with equation $u^2 f_\lambda - v^2(c_\lambda)^2 = 0$; again the projection onto $\mathbb{P}^1$ induces the natural fibration $X_\lambda^{sing} \to \mathbb{P}^1$.

In order to obtain a smooth model of X_λ, let us consider the linear system $L_4(\mathcal{P}_9)$ of quartics that contain the 9 cusps. The linear system $L_4(\mathcal{P}_9)$ has (projective) dimension 5 and defines a rational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^5$. One computes that the image of X_λ^{sing} by the rational map

$$(i_d, \phi) : \mathbb{P}^1 \times \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^5$$

is a smooth model of X_λ; the image of the cusps being the 18 (-2)-curves on X_λ forming a $9\mathbf{A}_2$ configuration. Taking the generic point over $\mathbb{P}^1$, one get a smooth genus 1 curve in $\mathbb{P}^5_{/\mathbb{Q}(t)}$ (where $t = \frac{v}{u}$). That curve E_{K3} has naturally 18 rational points, corresponding to the 18 (-2)-curves. Using Magma, we computed a Hessian model $E_{K3} \hookrightarrow \mathbb{P}^2_{/\mathbb{Q}(t)}$, which is

Theorem 13. *A model of the generic fiber of the fibration $X_\lambda \to \mathbb{P}^1$ is*

$$E_{K3} \quad x^3 + y^3 + z^3 + \frac{\lambda^3(t^2+3) - 4t^2}{\lambda^2(t^2-1)} xyz.$$

The elliptic curve E_{K3} contains the 9 obvious 3-torsion points

$$\begin{array}{l} Q_1 = (0:-1:1),\ Q_2 = (-1:0:1),\ Q_3 = (-1:1:0), \\ Q_4 = (0:-\omega:1),\ Q_5 = (\omega+1:0:1),\ Q_6 = (-\omega:1:0), \\ Q_7 = (0:\omega+1:1),\ Q_8 = (-\omega:0:1),\ Q_9 = (\omega+1:1:0). \end{array}$$

(where $\omega^2 + \omega + 1 = 0$; we take Q_1 as the neutral element) and the following 9 points

$$\begin{array}{l} P_1 = (-2t : \lambda(t+1) : \lambda t + \lambda), \\ P_2 = (\lambda(t-1) : -2t : \lambda t + \lambda), \\ P_3 = (-\lambda t - \lambda : -\lambda t + \lambda : 2t), \\ P_4 = ((2\omega+2)t : \lambda(\omega t + \omega) : \lambda t + \lambda) \\ P_5 = (\lambda(\omega+1)(-t+1) : -2\omega t : \lambda t + \lambda), \\ P_6 = ((\omega+1)\lambda(t+1) : \omega\lambda(-t+1) : 2t), \\ P_7 = (-2\omega t : -\lambda(\omega+1)(t+1) : \lambda t - \lambda), \\ P_8 = (\lambda\omega(t-1) : (2\omega+2)t : \lambda t + \lambda), \\ P_9 = (-\lambda\omega(t-1) : (\omega+1)\lambda(t-1) : 2t). \end{array}$$

Together, these 18 points are the above-mentioned points corresponding to the 18 sections of the fibration of X_λ.

Remark 14. One can check that the points P_j are all translate of P_1 by the 9 torsion points Q_k.

Once a cubic equation for E_{K3} is known, we get a natural model of the K3 surface X_λ as

$$\lambda^2(u^2 - v^2)(x^3 + y^3 + z^3) + (\lambda^3(u^2 + 3v^2) - 4u^2)xyz = 0.$$

in the space $\mathbb{P}^1 \times \mathbb{P}^2$ (with coordinates u, v, x, y, z). That model is smooth, and the fibers are smooth cubic curves, by contrast with the previous model X_λ^{sing}. Using Magma, it is then possible to obtain the equations of the (-2)-curves (also sections) A_j, resp. A'_j, which are on $X_\lambda \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ corresponding to the point Q_i, resp. P_i.

Lemma 15. *The* 9 *curves* $A_j+A'_j$ *(*$j\in\{1,\dots,9\}$*) form a* $9\mathbf{A}_2$ *configuration.*

Proof. We use the equations of the (-2)-curves A_j, A'_j in the model $X_\lambda \subset \mathbb{P}^1\times\mathbb{P}^2$ to check that $A_jA'_j=1$ and $A_jA'_k=A_jA_k=A'_jA'_k=0$ for $k\neq j$. In fact, one already knows that 3-torsion sections are disjoint by [8, VII, Proposition 3.2] (thus the sections A'_j are also disjoint). □

One can check moreover that the 9 intersection points of A_j with A'_j for $i=1,\dots,9$ are on the same fiber over 0 of the fibration, fiber which is isomorphic to E_λ. Using the addition law on the elliptic curve E_{K3}, one can find other sections. By example the following points

$$\begin{aligned}
R_1 &= (-2t:\lambda(t-1):\lambda t+\lambda),\\
R_2 &= (\lambda(t+1):-2t:\lambda t-\lambda),\\
R_3 &= (-\lambda t+\lambda:-\lambda t-\lambda:2t),\\
R_4 &= ((2\omega+2)t:\lambda\omega(t-1):\lambda t+\lambda),\\
R_5 &= (-\lambda(\omega+1)(t+1):-2\omega t:\lambda t-\lambda),\\
R_6 &= ((\omega+1)\lambda(t-1):-\omega\lambda(t+1):2t),\\
R_7 &= (-2\omega t:\lambda(\omega+1)(-t+1):\lambda t+\lambda),\\
R_8 &= (\lambda\omega(t+1):(2\omega+2)t:\lambda t-\lambda),\\
R_9 &= (-\omega\lambda t+\omega\lambda:(\omega+1)(\lambda t+\lambda):2t),
\end{aligned}$$

are the points $R_i=-P_1+Q_i$, $i\in\{1,\dots,9\}$. Let $\bar{R}_i$ be the section on X_λ corresponding to the point R_i.

Lemma 16. *The* 9 *curves* $A_i+\bar{R}_i$*,* $i=1,\dots,9$ *form a* $9\mathbf{A}_2$ *configuration.*

Proof. The curves A_i (resp. $\bar{R}_i$) are images of the curves A'_i (resp. A_i) by the translation by $-A'_1$ and we know that the curves A_i, A'_i form a $9\mathbf{A}_2$ configuration. □

Remark 17. One can compute easily the classes in the Néron-Severi group of the curves $\bar{R}_i$; they are given in the Appendix. Using that knowledge, we get the matrix representation on $\mathrm{NS}(X_\lambda)$ of the translation by $-A'_1$ automorphism τ. The characteristic polynomial of τ is

$$(T-1)^3(T^2+T+1)^8.$$

Using the action of τ and its powers, we can obtain more classes in $\mathrm{NS}(X_\lambda)$ of the sections on the K3 surface $X_\lambda\to\mathbb{P}^1$.

Remark 18. We searched among these sections the $9\mathbf{A}_2$ configurations, but we obtained only the expected ones, i.e. the $9\mathbf{A}_2$ configuration that are translate of the configuration $A_i+A'_i$, $i\in\{1,\dots,9\}$. Since these configurations are images of one configuration by an automorphism (the translation by A'_1 and its multiples), these $9\mathbf{A}_2$ configurations gives the same generalized Kummer structures.

For $P\in E_{K3}(\mathbb{Q}(\omega,t))$ let us denote by $(P)\hookrightarrow X_\lambda$ the corresponding section. We have

Proposition 19. *Modulo torsion, the section $A'_1 = (P_1)$ generates the Mordell-Weil lattice $MWL(X_\lambda)$ of sections.*

Proof. Let $O = A_1$ be the zero section and F be a fiber of $X_\lambda \to \mathbb{P}^1$. Using the knowledge of the action automorphism τ^{-1} (the translation by (P_1)) on $\mathrm{NS}(X)$, we get that

$$(6P_1) - 2(3P_1) + O \equiv 6F$$

in $\mathrm{NS}(X_\lambda)$, thus (see e.g. [18, Chapter III, Theorem 9.5]) $\langle P_1, P_1 \rangle = \frac{2}{3}$, where $\langle \cdot, \cdot \rangle$ is the pairing on $\mathrm{MWL}(X_\lambda)$ associated to the canonical height.

Let $\mathrm{Triv}(X_\lambda)$ be the lattice generated the zero section and the fibers components of the fibration. The determinant formula [15, Corollary 6.39] is

$$|\det \mathrm{NS}(X_\lambda)| = |\det \mathrm{Triv}(X_\lambda)| \cdot \det \mathrm{MWL}(X_\lambda)/|\mathrm{MWL}(X_\lambda)|^2.$$

Using lattice theoretic arguments, we obtain a basis of $\mathrm{NS}(X_\lambda)$ and compute that $|\det \mathrm{NS}(X_\lambda)| = 54$. We have moreover $\det \mathrm{Triv}(X_\lambda) = -3^8$ and $|\mathrm{MWL}(X_\lambda)|^2 = 3^4$, thus we obtain that $\det \mathrm{MWL}(X_\lambda) = \frac{2}{3}$.

We know that $\mathrm{MWL}(X_\lambda)$ has rank 1; since $\langle P_1, P_1 \rangle = \frac{2}{3}$, we conclude that P_1 generates $\mathrm{MWL}(X_\lambda)$ modulo torsion. □

We already know that the fiber at 0 of the elliptic K3 surface $X_\lambda \to \mathbb{P}^1$ is (isomorphic to) E_λ.

Remark 20. The fiber at ∞ of $X_\lambda \to \mathbb{P}^1$ is the elliptic curve

$$\mathrm{C_a}(\lambda)' : \quad x^3 + y^3 + z^3 + \frac{(\lambda^3 - 4)}{\lambda^2} xyz = 0.$$

The j-invariants of $\mathrm{C_a}(\lambda)$ and $\mathrm{C_a}(\lambda)'$ are distinct, in particular these curves are not isomorphic. In fact, from the construction, one may expect that there is a degree 2 isogeny between them: this is confirmed by Vélu's formulas (see e.g. [18, Chap. II, Example 6.3.2]).

Since we know a embedding of X_λ in $\mathbb{P}^1 \times \mathbb{P}^2$, we can embed X_λ to $\mathbb{P}^5$ via the Segre embedding. Doing so we obtain that X_λ is a degree 8 $K3$ surface in $\mathbb{P}^5$ which is defined by the following 5 equations:

$$\begin{gathered}
-U_2U_4 + U_1U_5, \quad -U_2U_3 + U_0U_5, \quad -U_1U_3 + U_0U_4, \\
\lambda^2(U_0^2U_3 - U_3^3 + U_1^2U_4 - U_4^3) + (\lambda^3 - 4)U_0U_1U_5 \\
\qquad + \lambda^2(U_2^2U_5 + 3\lambda U_3U_4U_5 - U_5^3), \\
\lambda^2(U_0^3 + U_1^3 + U_2^3 - U_0U_3^2) + (\lambda^3 - 4)U_0U_1U_2 \\
\qquad + \lambda^2(-U_1U_4^2 + 3\lambda U_0U_4U_5 - U_2U_5^2),
\end{gathered}$$

in particular this is not a complete intersection.

4.3. **A Weierstrass equation.** Let ω be such that $\omega^2 + \omega + 1 = 0$. Let us define three polynomials A, B, D in $\mathbb{Q}(\omega)(t)$ as follows:
The polynomial A has degree 8:

$$\begin{aligned}
A = {}& (\lambda^3t^2 + 3\lambda^3 - 4t^2)(\lambda^3t^2 + 3\lambda^3 + (6\omega + 6)\lambda^2t^2 + (-6\omega - 6)\lambda^2 - 4t^2) \\
& \cdot(\lambda^3t^2 + 3\lambda^3 - 6\lambda^2t^2 + 6\lambda^2 - 4t^2)(\lambda^3t^2 + 3\lambda^3 - 6\omega\lambda^2t^2 + 6\omega\lambda^2 - 4t^2),
\end{aligned}$$

the polynomial B has degree 12:

$$\begin{gathered}B=(\lambda^6t^4+6\lambda^6t^2+9\lambda^6+6\lambda^5t^4+12\lambda^5t^2-18\lambda^5-18\lambda^4t^4+36\lambda^4t^2\\ -18\lambda^4-8\lambda^3t^4-24\lambda^3t^2-24\lambda^2t^4+24\lambda^2t^2+16t^4)\\ \cdot(\lambda^6t^4+6\lambda^6t^2+9\lambda^6+(-6\omega-6)\lambda^5t^4+(-12\omega-12)\lambda^5t^2+(18\omega+18)\lambda^5-18\omega\lambda^4t^4\\ +36\omega\lambda^4t^2-18\omega\lambda^4-8\lambda^3t^4-24\lambda^3t^2+(24\omega+24)\lambda^2t^4+(-24\omega-24)\lambda^2t^2+16t^4)\\ \cdot(\lambda^6t^4+6\lambda^6t^2+9\lambda^6+6\omega\lambda^5t^4+12\omega\lambda^5t^2-18\omega\lambda^5+(18\omega+18)\lambda^4t^4\\ +(-36\omega-36)\lambda^4t^2+(18\omega+18)\lambda^4-8\lambda^3t^4-24\lambda^3t^2-24\omega\lambda^2t^4+24\omega\lambda^2t^2+16t^4),\end{gathered}$$

and the polynomial D is the following product of 8 degree 1:

$$\begin{gathered}D=((\lambda+2)t-(2\omega+1)\lambda)((\lambda-2\omega-2)t-(2\omega+1)\lambda)((\lambda+2\omega)t-(2\omega+1)\lambda)\\ \cdot(t^2-1)((\lambda+2)t+(2\omega+1)\lambda)((\lambda-2\omega-2)t+(2\omega+1)\lambda)((\lambda+2\omega)t+(2\omega+1)\lambda).\end{gathered}$$

We have:

Theorem 21. *The following elliptic curve*

$$E_{1/\mathbb{Q}(\omega,t)}:\ y^2=x^3-\frac{1}{48}Ax+\frac{1}{864}B$$

is a minimal Weierstrass model of the elliptic K3 surface X_λ. The 8 singular fibers $\tilde{\mathbf{A}}_2$ of X_λ are over the 8 zeros of D.

Proof. One computes that the j-invariant of the elliptic curve E_{K3} is

$$j=-\frac{A^3}{(\lambda^2(\lambda^3-1)D)^3}.$$

For any $J\notin\{0,1728\}$, the elliptic curve

$$E_0(J)\ \ y^2=x^3-\frac{1}{48}\frac{J}{J-1728}x+\frac{1}{864}\frac{J}{J-1728}$$

has j-invariant equal to J. In our case, we compute that we have

$$\frac{j}{j-1728}=\frac{A^3}{B^2},$$

where A and B are as above. By taking the change of variables

$$x'=u^2x,\ y'=u^3y$$

with $u=(B/A)^{1/2}$ in the equation of $E_0(j)$, we obtain the elliptic curve E_1. The curve E_1 has also its j-invariant equals to j, is also in Weierstrass form, but its coefficients are coprime degree 8 and 12 polynomials in t. The discriminant of the equation of E_1 is

$$\Delta=-(\lambda^2(\lambda^3-1)D)^3,$$

where D is the above product of 8 degree 1 polynomials in t. According to [8, Table IV.3.1], the associated elliptic surface is a K3 surface with 8 singular fibers of type $\tilde{\mathbf{A}}_2$.

Using Magma, we finally obtain an isomorphism defined over $\mathbb{Q}(\omega,t)$ between the Hesse model E_{K3} and the Weierstrass model E_1. □

5. Appendix

Let us define the following classes in the $\mathbb{Q}$-base $\mathcal{B} = (L, A_1, A_1', \dots, A_9, A_9')$:

$$B_1 = 2L - \frac{1}{3}\left(\sum_{j=1}^{9} 2A_j + A_j'\right), \ B_2 = 2L - \frac{1}{3}\left(\sum_{j=1}^{9} A_j + 2A_j'\right).$$

We remark that $B_1^2 = B_2^2 = 2$, $B_1B_2 = 5$ and $B_1 + B_2 = D_{14}$. We have $D_{14}A_j = D_{14}A_j' = 1$, therefore $B_iA_j \in \{0,1\}$, $B_iA_j' \in \{0,1\}$.

Using algorithms described in [10], we find that for $j \in \{1,\dots,9\}$, the classes of the curves θ_j, θ_j' are

$$\theta_j = B_1 - (A_j + A_j'),\ \theta_j' = B_2 - (A_j + A_j').$$

It is easy to check that $\theta_j^2 = \theta_j'^2 = -2$, $\theta_j\theta_j' = 1$, and for $1 \le i \ne j \le 9$, we have $\theta_i\theta_j = \theta_i'\theta_j' = 0$ and $\theta_i\theta_j' = 3$. In fact, using that the image in $\mathbb{P}^2$ of θ_j, θ_j' is a quartic curve that goes through the points in $\mathcal{P}_9$ with a multiplicity 3 at p_j, one gets

$$4L \equiv \theta_j + \theta_j' + 2(A_j + A_j') + \sum_{j=1}^{9}(A_j + A_j').$$

The classes in the $\mathbb{Q}$-base $\mathcal{B} = (L, A_1, A_1', \dots, A_9, A_9')$ of the classes of the 24 (-2)-curves $\theta_{i,\dots,n}, \theta_{i,\dots,n}'$ above the 12 conics $C_{i,\dots,n}$ in $\mathcal{C}_{12}$ are

$$\begin{aligned}
\theta_{123456} &= \tfrac{1}{3}(3,-2,-1,-2,-1,-2,-1,-1,-2,-1,-2,-1,-2,0,0,0,0,0,0),\\
\theta_{123456}' &= \tfrac{1}{3}(3,-1,-2,-1,-2,-1,-2,-2,-1,-2,-1,-2,-1,0,0,0,0,0,0),\\
\theta_{123789} &= \tfrac{1}{3}(3,-2,-1,-2,-1,-2,-1,0,0,0,0,0,0,-1,-2,-1,-2,-1,-2),\\
\theta_{123789}' &= \tfrac{1}{3}(3,-1,-2,-1,-2,-1,-2,0,0,0,0,0,0,-2,-1,-2,-1,-2,-1),\\
\theta_{124578} &= \tfrac{1}{3}(3,-2,-1,-1,-2,0,0,-2,-1,-1,-2,0,0,-2,-1,-1,-2,0,0),\\
\theta_{124578}' &= \tfrac{1}{3}(3,-1,-2,-2,-1,0,0,-1,-2,-2,-1,0,0,-1,-2,-2,-1,0,0),\\
\theta_{124689} &= \tfrac{1}{3}(3,-2,-1,-1,-2,0,0,-1,-2,0,0,-2,-1,0,0,-2,-1,-1,-2),\\
\theta_{124689}' &= \tfrac{1}{3}(3,-1,-2,-2,-1,0,0,-2,-1,0,0,-1,-2,0,0,-1,-2,-2,-1),\\
\theta_{125679} &= \tfrac{1}{3}(3,-2,-1,-1,-2,0,0,0,0,-2,-1,-1,-2,-1,-2,0,0,-2,-1),\\
\theta_{125679}' &= \tfrac{1}{3}(3,-1,-2,-2,-1,0,0,0,0,-1,-2,-2,-1,-2,-1,0,0,-1,-2),\\
\theta_{134589} &= \tfrac{1}{3}(3,-2,-1,0,0,-1,-2,-1,-2,-2,-1,0,0,0,0,-1,-2,-2,-1),\\
\theta_{134589}' &= \tfrac{1}{3}(3,-1,-2,0,0,-2,-1,-2,-1,-1,-2,0,0,0,0,-2,-1,-1,-2),\\
\theta_{134679} &= \tfrac{1}{3}(3,-2,-1,0,0,-1,-2,-2,-1,0,0,-1,-2,-2,-1,0,0,-1,-2),\\
\theta_{134679}' &= \tfrac{1}{3}(3,-1,-2,0,0,-2,-1,-1,-2,0,0,-2,-1,-1,-2,0,0,-2,-1),\\
\theta_{135678} &= \tfrac{1}{3}(3,-2,-1,0,0,-1,-2,0,0,-1,-2,-2,-1,-1,-2,-2,-1,0,0),\\
\theta_{135678}' &= \tfrac{1}{3}(3,-1,-2,0,0,-2,-1,0,0,-2,-1,-1,-2,-2,-1,-1,-2,0,0),\\
\theta_{234579} &= \tfrac{1}{3}(3,0,0,-2,-1,-1,-2,-2,-1,-1,-2,0,0,-1,-2,0,0,-2,-1),\\
\theta_{234579}' &= \tfrac{1}{3}(3,0,0,-1,-2,-2,-1,-1,-2,-2,-1,0,0,-2,-1,0,0,-1,-2),\\
\theta_{234678} &= \tfrac{1}{3}(3,0,0,-2,-1,-1,-2,-1,-2,0,0,-2,-1,-2,-1,-1,-2,0,0),\\
\theta_{234678}' &= \tfrac{1}{3}(3,0,0,-1,-2,-2,-1,-2,-1,0,0,-1,-2,-1,-2,-2,-1,0,0),\\
\theta_{235689} &= \tfrac{1}{3}(3,0,0,-2,-1,-1,-2,0,0,-2,-1,-1,-2,0,0,-2,-1,-1,-2),\\
\theta_{235689}' &= \tfrac{1}{3}(3,0,0,-1,-2,-2,-1,0,0,-1,-2,-2,-1,0,0,-1,-2,-2,-1),
\end{aligned}$$

$$\theta_{456789} = \tfrac{1}{3}(3,0,0,0,0,0,0,-1,-2,-1,-2,-1,-2,-2,-1,-2,-1,-2,-1),$$
$$\theta'_{456789} = \tfrac{1}{3}(3,0,0,0,0,0,0,-2,-1,-2,-1,-2,-1,-1,-2,-1,-2,-1,-2).$$

The equations of the quartic curves $Q_1,\dots,Q_9$ that have a singular point at $p_1,\dots,p_9$ are respectively

$$\begin{aligned}
Q_1 = {} & x^4 - 2\lambda x^3y + 3\lambda^2x^2y^2 - (\lambda^3+1)xy^3 + \lambda y^4 - 2\lambda x^3z + (-\lambda^3+1)xy^2z - 2\lambda y^3z \\
& +3\lambda^2x^2z^2 + (-\lambda^3+1)xyz^2 + (\lambda^4+2\lambda)y^2z^2 - (\lambda^3+1)xz^3 - 2\lambda yz^3 + \lambda z^4, \\
Q_2 = {} & x^4 - (\lambda^3+1)/\lambda x^3y + 3\lambda x^2y^2 - 2xy^3 + 1/\lambda y^4 - 2x^3z + (-\lambda^3+1)/\lambda x^2yz - 2y^3z \\
& +(\lambda^3+2)x^2z^2 + (1-\lambda^3)/\lambda xyz^2 + 3\lambda y^2z^2 - 2xz^3 - (\lambda^3+1)/\lambda yz^3 + z^4, \\
Q_3 = {} & x^4 - 2x^3y + (\lambda^3+2)x^2y^2 - 2xy^3 + y^4 - (\lambda^3+1)/\lambda x^3z + (1-\lambda^3)/\lambda x^2yz \\
& +(1-\lambda^3)/\lambda xy^2z - (\lambda^3+1)/\lambda y^3z + 3\lambda x^2z^2 + 3\lambda y^2z^2 - 2xz^3 - 2yz^3 + 1/\lambda z^4, \\
Q_4 = {} & x^4 + (2\omega+2)\lambda x^3y + 3\omega\lambda^2x^2y^2 - (\lambda^3+1)xy^3 - (\omega+1)\lambda y^4 - 2\omega\lambda x^3z \\
& -(\omega^2\lambda^3+\omega+1)xy^2z - 2\omega\lambda y^3z - (3\omega+3)\lambda^2x^2z^2 + (\omega-\omega\lambda^3)xyz^2 \\
& +(\lambda^4+2\lambda)y^2z^2 - (\lambda^3+1)xz^3 + (2\omega+2)\lambda yz^3 + \omega\lambda z^4, \\
Q_5 = {} & x^4 - (\omega^2\lambda^3+\omega^2)/\lambda x^3y + 3\omega\lambda x^2y^2 - 2xy^3 - (\omega+1)/\lambda y^4 - 2\omega x^3z \\
& +(1-\lambda^3)/\lambda x^2yz - 2\omega y^3z - ((\omega+1)\lambda^3+2\omega+2)x^2z^2 + (\omega-\omega\lambda^3)/\lambda xyz^2 \\
& +3\lambda y^2z^2 - 2xz^3 - \omega^2(\lambda^3+1)/\lambda yz^3 + \omega z^4, \\
Q_6 = {} & x^4 + (2\omega+2)x^3y + (\omega\lambda^3+2\omega)x^2y^2 - 2xy^3 + \omega^2y^4 - (\omega\lambda^3+\omega)/\lambda x^3z \\
& +(1-\lambda^3)/\lambda x^2yz - (\omega^2\lambda^3-\omega^2)/\lambda xy^2z - (\omega\lambda^3+\omega)/\lambda y^3z \\
& -(3\omega+3)\lambda x^2z^2 + 3\lambda y^2z^2 - 2xz^3 + (2\omega+2)yz^3 + \omega/\lambda z^4 \\
Q_7 = {} & x^4 - 2\omega\lambda x^3y - (3\omega+3)\lambda^2x^2y^2 - (\lambda^3+1)xy^3 + \omega\lambda y^4 + (2\omega+2)\lambda x^3z \\
& +(\omega-\omega\lambda^3)xy^2z + (2\omega+2)\lambda y^3z + 3\omega\lambda^2x^2z^2 - (\omega^2\lambda^3-\omega^2)xyz^2 \\
& +(\lambda^4+2\lambda)y^2z^2 - (\lambda^3+1)xz^3 - 2\omega\lambda yz^3 + \omega^2\lambda z^4, \\
Q_8 = {} & x^4 - (\omega\lambda^3+\omega)/\lambda x^3y - (3\omega+3)\lambda x^2y^2 - 2xy^3 + \omega/\lambda y^4 + (2\omega+2)x^3z \\
& +(1-\lambda^3)/\lambda x^2yz + (2\omega+2)y^3z + (\omega\lambda^3+2\omega)x^2z^2 - (\omega^2\lambda^3-\omega^2)/\lambda xyz^2 \\
& +3\lambda y^2z^2 - 2xz^3 - (\omega\lambda^3+\omega)/\lambda yz^3 + \omega^2z^4, \\
Q_9 = {} & x^4 - 2\omega x^3y + (\omega^2\lambda^3-2\omega-2)x^2y^2 - 2xy^3 + \omega y^4 - (\omega^2\lambda^3+\omega^2)/\lambda x^3z \\
& +(1-\lambda^3)/\lambda x^2yz + (-\omega\lambda^3+\omega)/\lambda xy^2z - (\omega^2\lambda^3+\omega^2)/\lambda y^3z \\
& +3\omega\lambda x^2z^2 + 3\lambda y^2z^2 - 2xz^3 - 2\omega yz^3 + \omega^2/\lambda z^4,
\end{aligned}$$

where $\omega^2+\omega+1=0$.

Let us define

$$S = \sum_{i=1}^{9} A_i,\ S' = \sum_{i=1}^{9} A'_i$$

The classes of the (-2)-curves $\bar{R}_i$ defined in Section 4.2 are

$$\bar{R}_i = 2L - \frac{1}{3}(S + 2S' + 3A_i + 3A'_i),$$

where L is the pull-back of a line by the double cover map $X_\lambda \to \mathbb{P}^2$. The translation automorphism τ defined in Section 4.2 sends L to the class

$$L' = 7L - \frac{4}{3}(S + 2S').$$

References

[1] Artebani M., Dolgachev I., The Hesse pencil of plane cubic curves, Enseign. Math. (2) 55 (2009), no. 3-4, 235–273.
[2] Bauer T., Harbourne B., Roé J., Szemberg T., The Halphen cubics of order two, Collect. Math. 68 (2017), no. 3, 339–357.
[3] Th. Bauer, S. Di Rocco, B. Harbourne, J. Huizenga, A. Lundman, P. Pokora, T. Szemberg, Bounded Negativity and Arrangements of Lines, Int. Math. Res. Not. IMRN 2015, no. 19, 9456–9471.
[4] Birkenhake C., Lange H., A family of Abelian surfaces and curves of genus four, Manus. Math. 85, (1994), 393–407
[5] Dolgachev I., Abstract configurations in algebraic geometry, The Fano Conference, 423–462, Univ. Torino, Turin, 2004.
[6] Dolgachev I., Laface A., Persson U., Urzúa G, Chilean configuration of conics, lines and points, preprint
[7] Griffiths P., Harris J., Principles of Algebraic Geometry, Wiley-Interscience, New York, 1978. xii+813 pp
[8] Miranda R., The Basic Theory of Elliptic Surfaces, Dottorato di Ricerca in Matematica, ETS Editrice, Pisa, 1989. vi+108 pp.
[9] Pokora P., Szemberg T., Conic-line arrangements in the complex projective plane, preprint, 2002.01760
[10] Roulleau X., On the geometry of K3 surfaces with finite automorphism group: the compact case, preprint, ArXiv 1909.01909
[11] Roulleau X., Sarti A., Construction of Nikulin configurations on some Kummer surfaces and applications, Math. Annalen 373 (2019), no 11, 7651–7668
[12] Roulleau X., Sarti A., Explicit Nikulin configurations on Kummer surfaces, preprint, ArXiv 1907.12215
[13] Saint Donat B., Projective models of K-3 surfaces. Amer. J. Math. 96 (1974), 602–639
[14] Shioda T., On elliptic modular surfaces. J. Math. Soc. Japan 24 (1972), 20–59
[15] Schütt M., Shioda T., Mordell-Weil lattices, Erg. der Math. 3. Folge. A Series of Modern Surveys in Mathematics 70. Springer, 2019. xvi+431 pp
[16] Shimada I., On elliptic K3 surfaces, Michigan Math. J. 47 (2000), no. 3, 423–446
[17] Shimada I., On elliptic K3 surfaces, arXiv:math/0505140
[18] Silverman J., Advanced topics in the arithmetic of elliptic curves, Graduate Texts in Mathematics, 151. Springer-Verlag, New York, 1994. xiv+525 pp.

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