TREE-LIKE CURVES AND THEIR NUMBER OF INFLECTION POINTS

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MPI 96-87

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ABSTRACT. In this short note we give a criterion of nonflattening for planar tree-like curves and some upper and lower bounds for the minimal number of inflection points on such curves unremovable by diffeomorphisms of $\mathbf{R}^2$, and finally, calculate the number of tree-like curves with a given Gauss diagram.

§1. INTRODUCTION

This paper provides a partial answer to the following question posed to the author by V.Arnold in June 95. Given a generic immersion $c : S^1 \to \mathbb{R}^2$ (i.e. with double points only) let $\sharp_{inf}(c)$ denote the number of inflection points on c (assumed finite) and let $[c]$ denote the class of c, i.e. the connected component in the space of generic immersions of S^1 to $\mathbb{R}^2$ containing c. Finally, let $\sharp_{inf}[c] = \min_{c' \in [c]} \sharp_{inf}(c')$.

PROBLEM. Estimate $\sharp_{inf}[c]$ in terms of combinatorics of c.

The problem itself is appearently motivated by the following classical result due to Möbius.

THEOREM. Any embedded noncontractible curve on $\mathbb{RP}^2$ has at least 3 inflection points.

The present paper contains some answers for the case when c is a tree-like curve, i.e. satisfies the condition that if p is any double point of c then $c \setminus p$ has 2 connected components. Classes of tree-like curves are naturally enumerated by partially directed trees with a simple additional restriction on directed edges, see §2. It was a pleasant surprise that for the classes of tree-like curves there exists a (relatively) simple combinatorial criterion characterizing when $[c]$ contains a nonflattening curve, i.e. $\sharp_{inf}[c] = 0$ in terms of its tree. On the other hand, all attempts to find a closed formula for $\sharp_{inf}[c]$ in terms of partially directed trees failed. Appearently such a formula does not exist, see Concluding Remarks.

The paper is organized as follows. §2 contains some general information on tree-like curves. §3 contains a criterion of noflattening. §4 presents some upper and lower bounds for $\sharp_{inf}[c]$. Finally, in §5 we solve a natural combinatorial question about tree-like curves, namely, how many classes of tree-like curves have the same Gauss diagram.

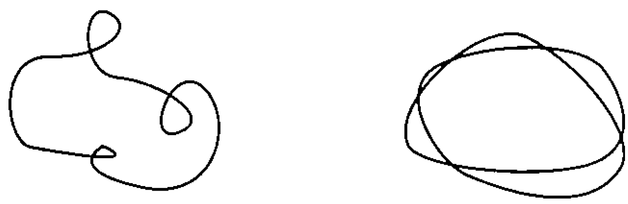
1991 *Mathematics Subject Classification.* Primary 53A04.

Key words and phrases. tree-like curves, Gauss diagram, inflection points.

Acknoledgements. The author is sincerely grateful to V.Arnold for the formulation of the problem and his support and encouragement during many years of our contact. It is difficult to overestimate the role of F.Aicardi for this project. (It was impossible to persuade her to become a coauthor.) Finally, the author wants to acknowledge the hospitality and support of the Max-Planck-Institut during the preparation of this paper.

§2. Some generalities on planar tree-like curves

Recall that a generic immersion $c : S^1 \to \mathbb{R}^2$ is called *a tree-like curve* if removing any of its double points p we get that $c \setminus p$ has 2 connected components, see Fig.1. Some of the results below were first proved in [Ai] and later independently found by the author.



a) tree-like curve b) nontree-like curve

Fig.1. Tree-like and nontree-like immersions.

2.1. Statement, (see Proposition 2.1. in [Ai]). A generic immersion $c : S^1 \to \mathbb{R}^2$ is a tree-like curve iff its Gauss diagram is planar, i.e. can be drawn on $\mathbb{R}^2$ without selfintersections, see Fig.2.

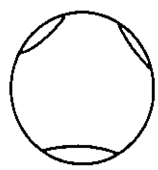


Fig.2. Planar Gauss diagram for ex. 1 a).

2.2. Remark. There is an obvious isomorphism between the set of all planar Gauss diagrams and the set of all planar connected trees. Namely, each planar Gauss diagram GD corresponds to the following planar tree. Let us place a vertex in each connected component of $D^2 \setminus GD$ where D^2 is the disc bounded by the basic circle of GD and connect by edges all vertices lying in the neighboring connected components. The resulting planar tree is denoted by $Tr(GD)$. Leaves of $Tr(GD)$

correspond to the connected components with one neighbor. On GD these connected components have the natural cyclic order according to their position along the basic circle of GD. This cyclic order coincides with the natural cyclic order on the set Lv of leaves of its planar tree $Tr(GD)$.

Decomposition of a tree-like curve. Given a tree-like curve $c : S^1 \to \mathbb{R}^2$ we decompose its image into the union of curvilinear polygons bounding contractable domains as follows. Take the planar Gauss diagram $GD(c)$ of c and consider the connected components of $D^2 \setminus GD(c)$. Each such component has the part of its boundary lying on S^1.

2.3. DEFINITION. The image of the part of the boundary of a connected component in $D \setminus GD(c)$ lying on S^1 forms a closed nonselfintersecting piecewise smooth curve (a curvilinear polygon) called *the building block* of c, see Fig.3. (We call vertices and edges of building blocks *corners* and *sides* to distinguish them from vertices and edges of planar trees used throughout the paper.)

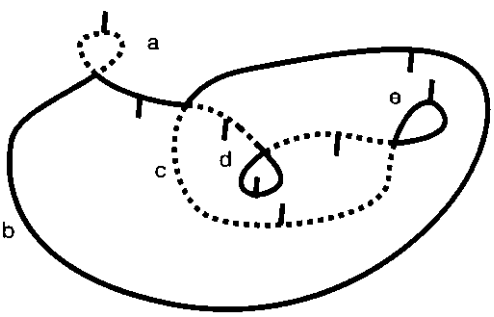


Fig. 3. Splitting of a tree-like curve into building blocks and their coorientation

The union of all building blocks constitutes the whole tree-like curve. Two building blocks have at most one common corner. If they have a common corner then they are called *neighboring*.

2.4. LEMMA. Given a coorientation of a tree-like curve c one gets that all sides of any building block are either inward or outward cooriented w.r.t. the interior of the block.

PROOF. Simple induction on the number of building blocks. □

Since every building block bounds a contractible domain the outward and inward coorientation have the clear meaning. We denote the outward coorientation of a building block by '$+$' and the inward by '$-$' placed near the corresponding vertex of $Tr(c)$.

2.5. DEFINITION. Given a tree-like curve c we associate to it the following planar partially directed tree $Tr(c)$. At first we take the undirected tree $Tr(GD(c))$ where $GD(c)$ is the Gauss diagram of c, see Remark 2.2. (Vertices of $Tr(GD(c))$ are in 1-1-correspondence with building blocks of c. Neighboring blocks correspond to adjacent vertices of $Tr(GD(c))$.) For each pair of neighboring building blocks b_1 and b_2 we do the following. If a building block b_1 contains a neighboring building block b_2 then we direct the corresponding edge (b_1, b_2) of $Tr(GD(c))$ from b_1 to b_2. The resulting partially directed planar tree is denoted by $Tr(c)$.

Since $Tr(c)$ depends only on the class $[c]$ we will also use the notation $Tr[c]$.

2.6. DEFINITION. Consider a partially directed tree Tr (i.e. some of its edges are directed). Tr is called *a noncolliding partially directed tree or ncpd-tree* if any path of Tr does not contain edges pointing at each other. The usual tree Tr' obtained by forgetting directions of all edges of Tr is called *underlying*.

2.7. LEMMA. a) For any tree-like curve c its $Tr(c)$ is noncolliding;

b) the set of classes of tree-like curves is in 1-1-correspondence with the set of ncpd-trees.

PROOF. A connected component of tree-like curves with a given Gauss diagram is uniquely determined by the enclosure of neighboring building blocks. The obvious restriction that if two building blocks contain the third one then one of them is contained in the other is equivalent to the noncolliding property. (See an example on Fig. 4.) □

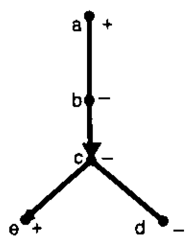


Fig. 4. Ncpd-tree Tr[c] for the example on Fig.3 with the coorientation of its vertices

2.8. REMARK. In terms of the above ncpd-tree one can easily describe the Whitney index (or the total rotation) of a given tree-like curve c as well as the coorientation of its building blocks. Namely, fixing the inward or outward coorientation of some building block we determine the coorientation of any other building block as follows. Take the (only) path connecting the vertex corresponding to the fixed block with the vertex corresponding to the other block. If the number of undirected edges in this path is odd then the coorientation chances and if this number is even then it is preserved. (In other words, $Coor(b_1) = (-1)^{q(b_1,b_2)}Coor(b_2)$ where $q(b_1, b_2)$ is the number of undirected edges on the above path.)

2.9. LEMMA, (see theorem 3.1 of [Ai]).

$$ind(c) = \sum_{b_i \in Tr(c)} Coor(b_i).$$

PROOF. Obvious. □

§3. NONFLATTENING OF TREE-LIKE CURVES

In this section we give a criterion for nonflattening of a tree-like curve in terms of its ncpd-tree. (The author is aware of the fact that some of the proofs below are rather sloppy since they are based on very simple explicit geometric constructions on $\mathbb{R}^2$ which are not so easy to describe with complete rigorousness.)

3.1. DEFINITION. A tree-like curve c (or its class $[c]$) is called *nonflattening* if $[c]$ contains a generic immersion without inflection points.

3.2. DEFINITION. *The convex coorientation* of a nonflattening tree-like curve $c : S^1 \to \mathbb{R}^2$ is defined as follows. The tangent line at any $p \in c$ belongs locally to one component of $\mathbb{R}^2 \setminus c$ and we choose at p a vector transversal to c and pointing at that connected component and extend it by continuity on the whole curve, see Fig.5.

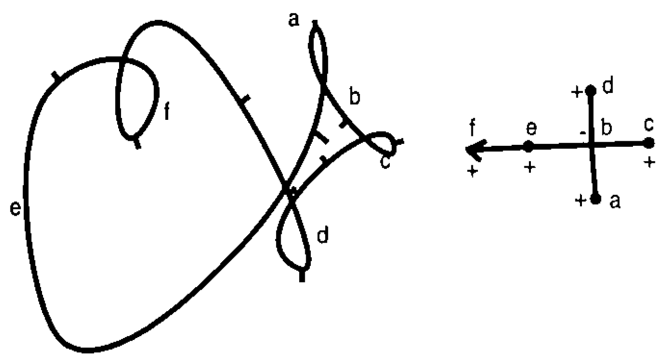


Fig. 5. Nonflattening curve with the convex coorientation and its ncpd-tree.

3.3. DEFINITION. Given a building block b of a tree-like curve c we call a corner v of b is called *of* $\vee$*-type (of* $\wedge$*-type resp.)* if the interior angle between the tangents to its sides at v is bigger (smaller resp.) than 180^0, see Fig.6. (The interior angle is the one contained in the interior of b.)

3.4. REMARK. If v is a $\vee$-type corner then the neighboring block b' sharing the corner v with b lies inside b, i.e. $\vee$-type corners are in 1-1-correspondence with edges of the ncpd-tree of c directed from the vertex corresponding to b.

3.5. CRITERION OF NONFLATTENING. A tree-like curve c is nonflattening iff the following 3 conditions hold for one of two possible coorientations of its ncpd-tree, (see lemma 2.4).

a) all vertices of degree 1 are outward cooriented;

b) all vertices of degree 2 are outward cooriented;

c) any inward cooriented vertex has degree $k \geq 3$ and at most $k - 3$ leaving edges (i.e. edges directed from this vertex).

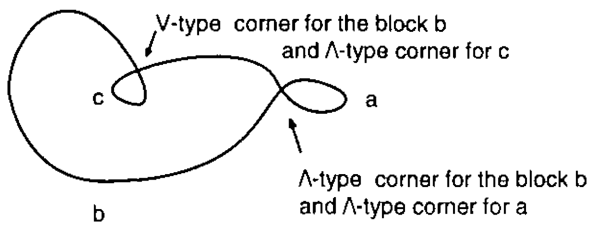


Fig. 6. V- and Λ-type corners for a building block.

PROOF. The necessity of a) - c) is rather obvious. Indeed, in the cases a) and b) a vertex of degree ≤ 2 corresponds to the building block with at most 2 corners. If such a building block belongs to a nonflattening tree-like curve then it must be globally convex and therefore outward cooriented w.r.t. the above convex coorientation. For c) consider an inward cooriented (w.r.t. convex coorientation) building block b of a nonflattening curve. Such b is a curvilinear polygon with locally concave edges. Assuming that b has k corners one gets that the sum of its interior angles is less than $\pi(k-3)$. Therefore the number of V-type corners (or leaving edges at the corresponding vertex) is less than $k-3$. (See Fig.7 for violations of conditions a)-c).)

Sufficiency of a)-c) is proved by a relatively explicit construction. Given a ncpd-tree satisfying a) - c) let us construct a nonflattening curve with this tree using induction of the number of vertices. While constructing this curve inductively we provide additionally that every building block is star-shaped with respect to some interior point, i.e. the segment connecting this point with a point on the boundary of the block always lies in its convex hull.

Case 1. A ncpd-tree contains an outward cooriented leaf connected to an outward cooriented vertex (and therefore the connecting edge is directed). Obviously, the tree obtained by removal of this leaf is also an ncpd-tree. By the inductive hypothesis we can construct a nonflattening curve corresponding to the reduced tree and then depending on orientation of the removed edge either glue inside the appropriate locally convex building block a small convex loop (which is obviously possible) or glue a big locally convex loop containing the whole curve. The possibility to glue a big locally convex loop containing the whole curve is proved in lemma 3.9.

Case 2. All leaves are connected to inward cooriented vertices. (By conditions a) and b) these vertices are of degree ≥ 3.) Using the ncpd-tree we can find at least 1 inward cooriented vertex b which is not smaller than any other vertex, i.e. the corresponding building block contains at least 1 exterior side. Let k be the degree of b and $e_1, ..., e_k$ be its edges in the cyclic order. (Each e_i is either undirected or leaving.) By assumption c) the number of leaving edges is at most $k-3$. If we remove b with all its edges then the remaining forest consists of k trees. Each of the trees connected to b by an undirected edge is an ncpd-tree. We make every tree connected to b by a leaving edge into an ncpd-tree by gluing the undirected edge instead of the removed directed and we mark the extra vertex we get. By induction, we can construct k nonflattening curves corresponding to each of k obtained ncpd-trees. Finally, we have to glue them to the corners of a locally concave k-gon with the sequence of $\vee$- and $\wedge$-type corners prescribed by $e_1, ..., e_k$. The possibility of such a gluing is proved in lemmas 3.11 and 3.12. $\square$

SUPPORTING LEMMAS FOR CASE 1.

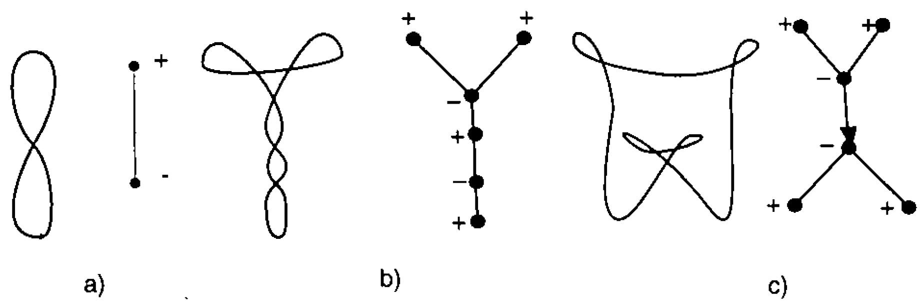


Fig.7. Curves and their ncpd-trees violating each of the 3 conditions of proposition 3.5. separately

3.6. Important construction. The following operation called *contracting homothety* will be extensively used below. It does not change the class of a tree-like curve and the number of inflection points.

Taking a tree-like curve c and its double point p we split $c \setminus p$ into 2 parts c^+ and c^- intersecting only at p. Let Ω^+ and Ω^- denote the union of convex hulls of building blocks contained in c^+ and c^- resp. There are 2 options a) one of domains contains the other, say, $\Omega^- \subset \Omega^+$; or b) $\overline{\Omega}^- \cap \overline{\Omega}^+ = \{p\}$.

In case a) the result of contracting homothety (the usual homothety applied to c^- and then smoothening of the 2nd and higher derivatives at p) is a tree-like curve c_1 isotopic to c and such that $c^+ = c_1^+$ while c_1^- lies in an arbitrary small neighborhood of p.

In case b) we can apply a contracting homothety to either of 2 parts and get 2 nonflattening tree-like curves c_1 and c_2 isotopic to c and such that either $c^+ = c_1^+$ while c_1^- lies in an arbitrary small neighborhood of p or $c^- = c_2^-$ while c_2^+ lies in an arbitrary small neighborhood of p. See Fig.8 for the illustration of contracting homothety.

3.7. DEFINITION. Consider a locally convex domain Ω in $\mathbb{R}^2$ with a piecewise C^2-smooth boundary $\partial\Omega$. Ω is called *rosette-shaped* if for any side e of $\partial\Omega$ there exists a point $p(e) \in e$ such that Ω lies in one of the closed halfspaces $\mathbb{R}^2 \setminus l_p(e)$ w.r.t the tangent line $l_p(e)$ to $\partial\Omega$ at $p(e)$.

3.8. REMARK. For a rosette-shaped Ω there exists a smooth convex curve $\gamma(e)$ containing Ω in its interior and tangent to $\partial\Omega$ at exactly one point lying on a given side e of $\partial\Omega$.

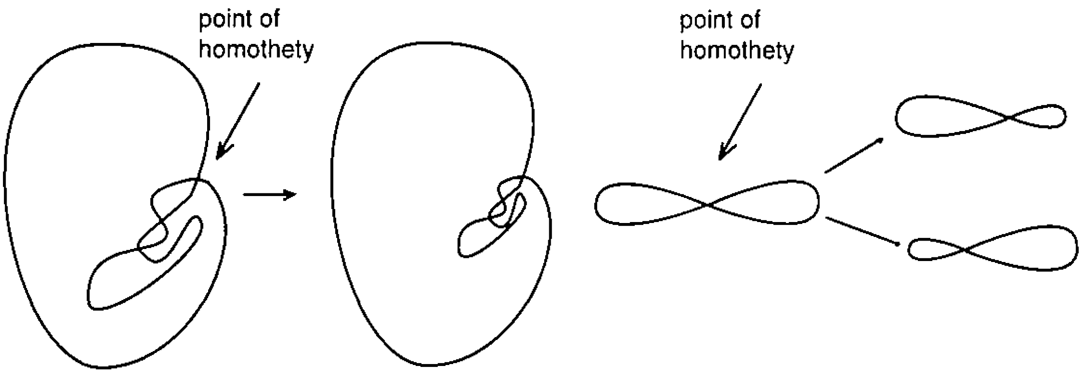


Fig. 8. Contracting homothety for enclosed and not enclosed building blocks.

3.9. LEMMA. Consider a nonflattening tree-like curve c with the convex coorientation and its locally convex building block b containing at least one exterior side, i.e. a side bounding the noncompact exterior domain on $\mathbb{R}^2$. There exists a nonflattening curve c' isotopic to c such that its building block b' corresponding to b bounds a rosette-shaped domain.

PROOF. Step 1. Let k denote the number of corners of b. Consider connected components $c_1, ..., c_k$ of $c \setminus b$. By assumption that b contains an exterior side one has that every c_i lies either inside or outside b (can not contain b). Therefore using contracting homothety we can make every c_i small and lying in the small neighborhood of its corner preserving the nonflattening property.

Step 2. Take the standard unit circle $S^1 \subset \mathbb{R}^2$ and choose k points on S^1. Then deform S^1 slightly into a piecewise smooth locally convex curve $\tilde{S}^1$ with the same sequence of $\vee$- and $\wedge$-type corners as on b. Now glue the small components $c_1, ..., c_k$ (after appropriate linear transformation applied to each c_i) to $\tilde{S}^1$ in the same order as they sit on b. The resulting curve c' is a nonflattening tree-like curve with the same ncpd-tree as c. □

3.10. COROLLARY. Using the remark 3.8. one can glue a big locally convex loop containing the whole c' and tangent to c' at one point on any exterior edge and then deform this point of tangency into a double point and therefore get the nondegenerate tree-like curve required in case 1.

SUPPORTING LEMMAS FOR CASE 2.

Take any polygon Pol with k vertices and with the same sequence of $\vee$- and $\wedge$-type vertices as given by $e_1, ..., e_k$, see notations in the proof of case 2. The existence of such a polygon is exactly guaranteed by condition c), i.e. $k \geq 3$ and the number of interior angles $> \pi$ is less or equal than $k - 3$. Deform it slightly to make it into a locally concave curvilinear polygon which we denote by $\widetilde{Pol}$.

3.11. LEMMA. It is possible to glue a nonflattening curve $\tilde{c}$ (after an appropriate diffeomorphism) through its convex exterior edge to any $\wedge$-type vertex of $\widetilde{Pol}$ placing it outside $\widetilde{Pol}$ and preserving nonflattening of the union.

PROOF. We assume that the building block containing the side e of the curve $\tilde{c}$ to which we have to glue v is rosette-shaped. We choose a point p on e and substitute e by 2 convex sides meeting transversally at p. Then we apply to $\tilde{c}$ a linear transformation putting the origin at p in order to a) make $\tilde{c}$ small; b) make the angle between the new sides equal to the angle at the $\wedge$-type vertex to which we have to glue $\tilde{c}$. After that we glue $\tilde{c}$ and smoothen the higher derivatives. □

3.12. LEMMA. It is possible to glue a nonflattening curve $\tilde{c}$ after cutting away its exterior building block with 1 corner to a $\vee$-type vertex of $\widetilde{Pol}$ and preserving nonflattening of the union. The curve is placed inside $\widetilde{Pol}$.

PROOF. The argument is essentially the same as above. We cut away a convex exterior loop from $\tilde{c}$ and apply to the remaining curve a linear transformation making it small and making the angle between 2 sides at the corner where we have cut away a loop equal to the angle at the $\vee$-type vertex. Then we glue the result to the $\vee$-type vertex and smoothen the higher derivatives. □

§4. UPPER AND LOWER BOUNDS OF $\sharp_{inf}[c]$ FOR TREE-LIKE CURVES

Violation of any of the above 3 conditions of nonflattening leads to the appearance of unremovable by diffeomorphisms inflection points on a tree-like curve. At first we reduce the question about the minimal number $\sharp_{inf}[c]$ of inflection points on the classes of tree-like curves to a purely combinatorial problem and then give some upper and lower bounds for this number. Some of the geometric proofs are only sketched for the same reasons as in the previous section. Since we are interested in inflections which survive under the action of diffeomorphisms of $\mathbb{R}^2$ we will assume from now on that all considered curves have only locally unremovable inflection points. (For example, the germ (t, t^4) is not interesting since its inflection disappears after a small deformation of the germ.)

4.1. DEFINITION. A generic immersion $c : S^1 \to \mathbb{R}^2$ the inflection points of which coincide with some of its double points is called *normalized.*

4.2. PROPOSITION. Every tree-like curve is isotopic to a normalized tree-like curve with at most the same number of inflection points.

PROOF.

Step 1. The idea of the proof is to separate building blocks as much as possible and then substitute every block by a curvilinear polygon with nonflattening sides. Namely, given a tree-like c let us partially order the vertices of its ncpd-tree $Tr[c]$ by choosing one vertex as the root (vertex of level 1). Then we assign to all its adjacent vertices level 2, etc. The only requirement for the choice of the root is that all the directed edges point from the lower level to the higher. One can immediately see that noncolliding property garantees the existence of at least one root. Given such a partial order we apply consecutively a series of contracting homotheties to all double points as follows. We start with double points which are the corners of the building block b corresponding to the root. Then we apply contracting homothety to all connected components of $c \setminus b$. Then we apply contracting homothety to all connected components of $c \setminus (\cup b_i)$ where b_i has level less or equal 2 etc. (See an example on Fig.9.) Note that every building block except for the root has its father to which it is attached through a $\wedge$-type corner since the root contains an exterior edge. The resulting curve $\tilde{c}$ has the same type and number of inflection points as

c and widely separated building blocks. Every building block lies in a very small neighborhood of the corresponding corner of its father and the radius of smallness increases from level to level.

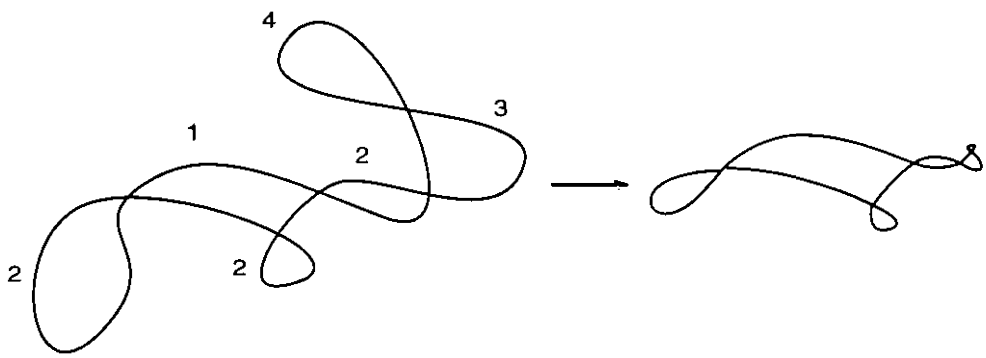


Fig.9. Separation of building blocks of different levels by contracting homothety. (Numbers show different levels of building blocks.)

Step 2. Now we substitute every side of every building block by a nonflattening arc not increasing the number of inflection points. Fixing some orientation of $\tilde{c}$ we assign at every double point 2 oriented tangent elements to 2 branches of $\tilde{c}$ in the obvious way. Note that we can assume that any 2 of these tangent elements not sharing the same vertex are in general position, i.e. the line connecting the footpoints of the tangent elements is different from both tangent lines.

Initial change. At first we will substitute every building block of the highest level by a convex loop. There exists a smooth (except for the corner) convex loop gluing which instead of the building block will make the whole new curve C^1-smooth and isotopic to $\tilde{c}$ in the class of C^1-smooth curves. This convex loop lies on the definite side w.r.t. both tangent lines at the double point. Note that if the original removed building block lies wrongly w.r.t. one (both resp.) tangent lines then it has at least 1 (2 resp.) inflection points. After constructing a C^1-smooth curve we change it slightly in a small neighborhood of the double point in order to provide for each branch a) if the branch of $\tilde{c}$ changes convexity at the double point then we produce a smooth inflection at the double point; b) if the branch does not change the convexity then we make it smooth. The above remark garantees that the total number of inflection points does not increase.

Typical change. Assume that all blocks of level $> i$ already have nonflattening sides. Take any block b of level i. By the choice of the root it has a unique $\wedge$-type corner with its father. The block b has a definite sequence of its $\vee$- and $\wedge$-type corners starting with the attachment corner and going around b clockwise. We cut away all connected components of $c \setminus b$ which have level $> i$ then substitute b by a curvilinear polygon with nonflattening sides and then glue back the blocks we cut of. Let us draw the usual polygon Pol with the same sequence of $\vee$- and $\wedge$-type vertices as for b. Now we will deform its sides into convex and concave arcs depending on the sides of the initial b. The tangent elements to the ends of some side of b can be in one of 2 typical normal or 2 typical abnormal positions (up to orientation-preserving affine transformations of $\mathbb{R}^2$), see Fig.10.

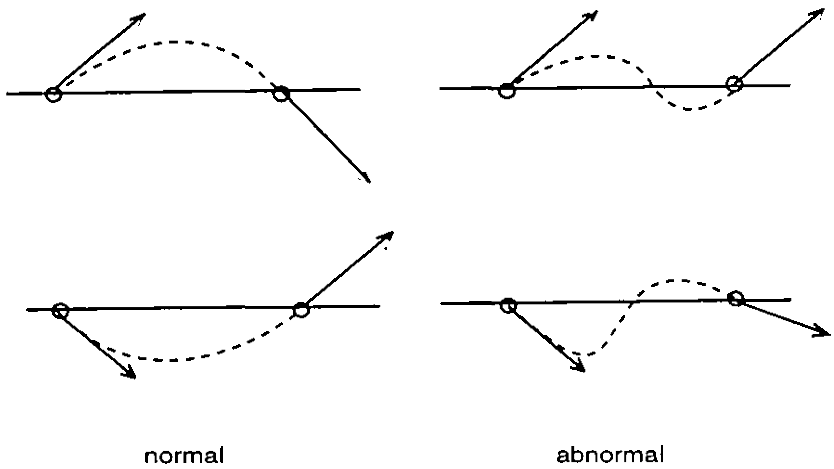


Fig.10. Normal and abnormal positions of the tangent elements to a side.

If the position of the tangent elements is normal then we deform the corresponding side of Pol to get a nonflattening arc with the same position of the tangent elements as for the initial side of b. If the position is abnormal then we deform the side of Pol to get a nonflattening arc which has the same position w.r.t. the tangent element at the beginning as the original side of b. Analogous considerations as before show that after gluing everything back and smoothening the total number of inflections will not increase. □

4.3. DEFINITION. Given a planar tree Tr we denote by its *fat ncpd-tree* FTr the planar graph obtained from Tr by 'blowing up' each vertex into a polygon with the number of vertices equal to the degree of the vertex, see Fig. 11. These polygons are called *fat vertices*. Again we call by *sides* edges of the fat vertices to distinguish them from the edges of the original tree Tr. A fat tree with signs '+' or '-' on every side of each fat vertex is called *labelled*.

4.4. DEFINITION. Given a tree-like normalized curve c we associate to it the following labelled fat ncpd-tree. We take the fat tree $FTr[c]$ obtained from $Tr[c]$. (Pay attention to the fact that if we contract the edges of the original tree $Tr[c]$ then the fat tree $FTr[c]$ is homeomorphic to the original curve c and the sides of c are in 1-1-correspondence with the sides of $FTrc$.) Then we put on each side of each fat vertex of $FTr[c]$ '+' or '-' depending on whether the corresponding side of the corresponding building block is convex or concave w.r.t. the interior of the block, see Fig. 11. (Note that this labelling depends on a particular choice of c and not only on $[c]$.)

The following proposition is closely related to the criterion of nonflattening from §2.

4.5. PROPOSITION (CRITERION OF REALIZABILITY OF A LABELLED FAT TREE). There exists a tree-like normalized curve with a given labelled fat ncpd-tree if and only if the following 3 conditions hold

a) the side of every 1-sided fat vertex is marked by '+';

b) 2 sides of every 2-sided fat vertex are marked either by '++' or by '+-';

c) if all sides of some fat vertex with $k \geq 3$ sides are marked by '-' then there exists at most $k-3$ directed edges of the initial tree leaving this fat vertex.

SKETCH OF PROOF. The necessity of a)-c) is obvious. These conditions garantee the existence of all building blocks with nonflattening sides. It is easy to see that they are, in fact, sufficient. Realizing each building block by some curvilinear polygon with nonflattening sides we can glue them together in a global normalized tree-like curve. Namely, we start from some building block which contains an exterior edge. Then we glue all its neighbors to its corners. (In order to be able to glue them we make them small and adjust the gluing angles by appropriate linear transformations.) Finally, we smoothen higher derivatives at all corners and then proceed in the same way for all new corners. □

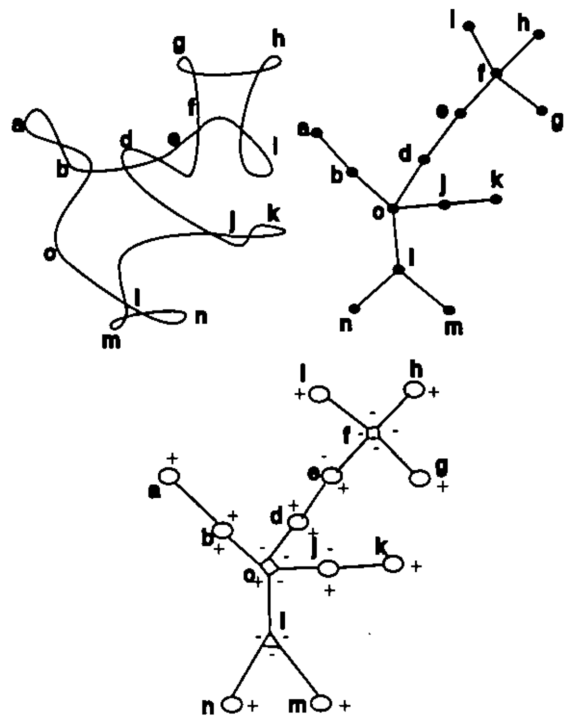


Fig.11. Ncpd-tree and the fat labeled ncpd-tree obtained from a given normalized tree-like curve.

Combinatorial setup. The above proposition 4.5. allows us to reformulate the question about the minimal number of inflection points $\sharp_{inf}[c]$ for tree-like curves combinatorially.

4.6. DEFINITION. A labelling of a fat ncpd-tree is called *admissible* if it satisfies the conditions a)-c) of proposition 4.5.

Note that sides of all fat vertices of a given planar fat tree have the natural cyclic order.

4.7. DEFINITION. Given a labelled fat ncpd-tree $LFTr$ consider 2 consecutive (in the natural cyclic order) sides belonging to 2 different fat vertices (i.e. these sides are connected by an edge of the original tree). We say that these sides *create an inflection point* if either a) their signs coincide and the connecting edge is directed; or b) their signs are different and the connecting edge is undirected. For a given labelled fat tree $LFTr$ let $\sharp_{inf}(LFTr)$ denote the total number of created inflection points.

4.8. PROPOSITION (combinatorial reformulation). For a given tree-like curve c one has

$$\sharp_{inf}[c] = \min \sharp_{inf}(LFTr)$$

where the minimum is taken over the set of all admissible labellings of $FTr[c]$ and $FTr[c]$ is the fat ncpd-tree of c.

PROOF. This is the direct corollary of propositions 4.2 and 4.5. Namely, for every tree-like curve $\tilde{c}$ isotopic to c there exists a normalized curve $\tilde{c}'$ with at most the same number of inflection points. The number of inflection points of $\tilde{c}'$ coincides with that of its labelled fat ncpd-tree. On the other side, for every admissible labelling of $FTr[c]$ there exists a normalized curve c' with such a labelled fat ncpd-tree. □

Lower bound. A natural lower bound for $\sharp_{inf}[c]$ can be obtained in terms of the ncpd-tree $Tr[c]$ (without any use of $FTr[c]$). Choose any coorientation of c and the corresponding coorientation of $Tr[c]$, see §2. All 1-sided building blocks of c (corresponding to the leaves of $Tr[c]$) have the natural cyclic order. (This order coincides with the natural cyclic order on all leaves of $Tr[c]$ according to their position on the plane.)

4.9. DEFINITION. A neighboring pair of 1-sided building blocks (or of leaves on $Tr[c]$) is called *reversing* if the coorientations of these blocks are different. Let $\sharp_{rev}[c]$ denote the total number of reversing neighboring pairs of building blocks.

Note that $\sharp_{rev}[c]$ is even and independent on the choice of coorientation of c. Moreover, $\sharp_{rev}[c]$ depends only on the class $[c]$ and therefore we can use the above notation instead of $\sharp_{rev}(c)$.

4.10. PROPOSITION. $\sharp_{rev}[c] \leq \sharp_{inf}[c]$.

PROOF. Pick a point p_i in each of 1-sided building blocks b_i such that the side is locally convex near p_i w.r.t. the interior of b_i. (Such a choice is obviously possible since b_i has just 1 side.) The proof is accomplished by the following simple observation.

Take an immersed segment $\gamma : [0,1] \to \mathbb{R}^2$ such that $\gamma(0)$ and $\gamma(1)$ are not inflection points and the total number of inflection points on γ is finite. At each nonflattening point p of γ we can choose the convex coorientation, see §3, i.e. since the tangent line to γ at p belongs locally to 1 connected component of $\mathbb{R}^2 \setminus \gamma$ we can choose a transversal vector pointing at that halfspace. Let us denote the convex coorientation at p by $n(p)$.

4.11. LEMMA. Assume that we have fixed a global coorientation $Coor$ of γ. If $Coor(0) = n(0)$ and $Coor(1) = n(1)$ then γ contains an even number of locally unremovable inflections. If $Coor(0) = n(0)$ and $Coor(1)$ is opposite to $n(1)$ then γ contains an odd number of locally unremovable inflections.

PROOF. Recall that we have assumed that all our inflection points are unremovable by local deformations of the curve. Therefore passing through such an inflection point the convex coorientation changes to the opposite. □

Upper bound.

4.12. DEFINITION. Each pair of neighboring 1-sided blocks of c (leaves of $FTr[c]$ resp.) is joined by the unique segment of c (path in $FTr[c]$ resp.) called *connecting.* If a connecting path joins a pair of neighboring 1-blocks (leaves resp.) with the opposite coorientations then it is called *an reversing connecting path*, compare with 4.9.

4.13. DEFINITION. Let us call by *a joint* of a tree-like curve a nonextendable sequence of 2-sided building blocks not contained in each other. (On the level of its ncpd-tree one gets a sequence of degree 2 vertices connected by undirected egdes.)

Every joint consists of 2 smooth intersecting segments of c called *threads* belonging to 2 different connecting paths.

4.14. DEFINITION. For every nonreversing connecting path ρ in $FTr[c]$ we determine *the standard sign distribution* of this path as follows. First we put '+' ... previous side on one end of ρ. If the next side of ρ is connected to the ~~end~~ by an undirected edge of $Tr[c]$ then we change the sign and if the edge is directed then we keep the sign. etc. (By definition both ends of ρ will be labelled by '+'.)

4.15. DEFINITION. A joint is called *suspicious* if either

a) both its threads lie on nonreversing paths and both sides of some 2-sided block from this joint are labelled by '-' w.r.t. the standard sign distribution of nonreversing paths; or

b) one thread lies on nonreversing path and there exists a block from this joint the side of which lying on the nonreversing path is labelled by '-' (w.r.t. the standard sign distribution); or

c) both threads lie on reversing paths.

Let $\sharp_{jt}$ denote the total number of suspicious joints.

4.16. DEFINITION. A building block with k sides is called *suspicious* if

a) if contains at least $k-3$ other blocks, i.e. at least $k-3$ edges are leaving the corresponding vertex of the tree;

b) all sides lying on nonreversing paths are labelled with '-' w.r.t. the standard sign distributions of these nonreversing paths.

Let $\sharp_{bl}$ denote the total number of suspicious blocks.

4.17. PROPOSITION.

$$\sharp_{inf}[c] \leq \sharp_{rev}[c] + 2(\sharp_{jt} + \sharp_{bl}).$$

PROOF. According to the statement 4.8 for any tree-like curve c one has $\sharp_{inf}[c] \leq \sharp_{inf}(LFTr)$ where $LFTr$ is some admissible labelling of the fat ncpd-tree $FTr[c]$ of c. Let us show that there exists an admissible labelling of $FTr[c]$ with at most $\sharp_{rev}[c] + 2(\sharp_{jt} + \sharp_{bl})$ inflection points, see Def 4.7. First we fix the standard sign distribution of all nonreversing paths. Then for each reversing path we choose any sign distribution requiring exactly 1 inflection point to get the necessary signs of all leaves. Now the labelling of the whole fat ncpd-tree is fixed but it is not admissible, in general. In order to make it admissible we have to provide conditions b) and c) of Proposition 4.5 for at most $\sharp_{jt}$ suspicious joints and at most $\sharp_{bl}$ suspicious blocks. To make each such suspicious joint or block admissible we need to introduce at most 2 inflection point. Proposition follows. $\square$

§5. Enumeration of tree-like curves with a given Gauss diagram.

In this section we calculate the number of different classes of tree-like curves which have the same Gauss diagram.

5.1. Proposition. There exists a 1-1-correspondence between classes of oriented tree-like curves on nonoriented $\mathbb{R}^2$ and the set of all planar ncpd-trees on oriented $\mathbb{R}^2$.

Proof. Obvious.

5.2. Definition. For a given planar tree Tr consider the subgroup $Diff(Tr)$ of all orientation-preserving diffeomorphisms of $\mathbb{R}^2$ sending Tr homeomorphically onto itself as an embedded 1-complex. The subgroup $PAut(Tr)$ of the group $Aut(Tr)$ of automorphisms of Tr as an abstract tree induced by $Diff(Tr)$ is called *the group of planar automorphisms of* Tr.

The following simple proposition gives a complete description of different possible groups $PAut(Tr)$. (Unfortunately, the author was unable to find the corresponding reference.)

5.3. Statement.

(1) The group $PAut(Tr)$ of planar automorphisms of a given planar tree Tr is isomorphic to $\mathbf{Z}/\mathbf{Z}_p$ and is conjugate by an appropriate diffeomorphism to the rotation about some centre by multiples of $2\pi/p$.
(2) If $PAut(Tr) = \mathbf{Z}/\mathbf{Z}_p$ for $p > 2$ then the above centre of rotation is a vertex of Tr.
(3) For $p = 2$ the centre of rotation is either a vertex of Tr or the middle of its edge.
(4) If the centre of rotation is a vertex of Tr then the action $PAut(Tr)$ on Tr is free except of the centre and the quotient can be identified with a connected subtree $STr \subset Tr$ containing the centre.
(5) For $p = 2$ if the centre is the middle of an edge then the action of $PAut(Tr)$ on Tr is free except for this edge.

Sketch of proof. The action of $PAut(Tr)$ on the set $Lv(Tr)$ of leaves of Tr preserves the natural cyclic order on $Lv(Tr)$ and thus reduces to the $\mathbf{Z}/\mathbf{Z}_p$-action for some p. Now each element $g \in PAut(Tr)$ is determined by its action on $Lv(Tr)$ and thus the whole $PAut(Tr)$ is isomorphic to $\mathbf{Z}/\mathbf{Z}_p$. Indeed consider some $\mathbf{Z}/\mathbf{Z}_p$-orbit O on $Lv(Tr)$ and all vertices of Tr adjacent to O. They are all pairwise different or all coincide since otherwise they can not form an orbit of the action of diffeomorphisms on Tr. $\square$

Proof. Obvious. $\square$

5.4. Proposition. The number $\natural(GD)$ of all classes of oriented tree-like curves on nonoriented $\mathbb{R}^2$ with a given Gauss diagram GD on n vertices is equal

$$\natural(GD) = \begin{cases} a)\ 2^{n-1} + (n-1)2^{n-2}, \text{ if } PAut(GD) \text{ is trivial;} \\ b)\ 2^{2k-2} + (2k-1)2^{2k-3} + 2^{k-1} \text{ where } n = 2k, \text{if } PAut(GD) = \mathbf{Z}/2\mathbf{Z} \\ \quad \text{and the rotation centre is the middle of the side;} \\ c)\ 2^k + (2^{n-1} + (n-1)2^{n-2} - 2^k)/p, \text{ where } n = kp+1 \text{ and} \\ PAut(GD) = \mathbf{Z}/p\mathbf{Z} \text{ for some prime p (including } PAut(GD) = \mathbf{Z}/2\mathbf{Z} \\ \quad \text{with a central vertex);} \\ d)\ \text{ for the general case see Proposition 8 below.} \end{cases}$$

PROOF. By Proposition 5.1 we enumerate ncpd-trees with a given underlying planar tree $Tr(DG)$.

Case a). Let us first calculate only ncpd-trees all edges of which are directed. The number of such ncpd-trees equals the number n of vertices of $Tr(DG)$ since for any such tree there exists such a source-vertex (all edges are directed from this vertex). Now let us calculate the number of ncpd-trees with l undirected edges. Since $Aut(GD)$ is trivial we can assume that all vertices of $Tr(GD)$ are enumerated. There exist $\binom{n-1}{l}$ subgraphs in $Tr(GD)$ containing l edges and for each of these subgraphs there exist $(n-l)$ ncpd-trees with such a subgraph of undirected edges. Thus the total number $\natural(GD) = \sum_{l=0}^{n-1}\binom{n-1}{l}(n-l) = 2^{n-1} + (n-1)2^{n-2}$.

Case b). The $\mathbf{Z}/2\mathbf{Z}$-action on the set of all ncpd-trees splits them into 2 groups according to the cardinality of orbits. The number of $\mathbf{Z}/2\mathbf{Z}$-invariant ncpd-trees equals the number of all subtrees in a tree on k vertices where $n = 2k$ (since the source-vertex of such a tree necessarily lies in the centre). The last number equals 2^{k-1}. This gives $\natural(GD) = (2^{n-1} + (n-1)2^{n-2} - 2^{k-1})/2 + 2^{k-1} = 2^{2k-2} + (2k-1)2^{2k-3} + 2^{k-1}$.

Case c). The $\mathbf{Z}/p\mathbf{Z}$-action on the set of all ncpd-trees splits them into 2 groups according to the cardinality of orbits. The number of $\mathbf{Z}/p\mathbf{Z}$-invariant ncpd-trees equals to the number of all subtrees in a tree on $k+1$ vertices where $n = pk+1$ (since the source-vertex of such a tree lies in the centre). The last number equals 2^k. This gives $\natural(GD) = 2^k + (2^{n-1} + (n-1)2^{n-2} - 2^k)/p$.

5.5. PROPOSITION. Consider a Gauss diagram GD with the tree $Tr(GD)$ on n vertices which has $Aut(Tr) = \mathbf{Z}/p\mathbf{Z}$ where p is not a prime. Then for each nontrivial factor d of p the number of ncpd-trees with the $\mathbf{Z}/d\mathbf{Z}$-group of symmetry equals

$$\sum_{d'|d} \mu(d')2^{\frac{kd}{d'}}$$

where $\mu(d')$ is the Möbius function. (This gives a rather unpleasant expression for the number of all tree-like curves with a given GD if p is an arbitrary positive integer.)

PROOF. Consider for each d such that $d|p$ the subtree STr_d on $km+1$ vertices $m = \frac{n-1}{d}$ 'spanning' Tr with respect to the $\mathbf{Z}/d\mathbf{Z}$-action. The number of ncpd-trees invariant at least w.r.t $\mathbf{Z}/d\mathbf{Z}$ equals 2^{km} where $p = dm$ and $n = kp+1$. Thus by inclusion-exclusion formula one gets that the number of ncpd-trees invariant exactly w.r.t. $\mathbf{Z}/d\mathbf{Z}$ equals $\sum_{d'|d}\mu(d')2^{\frac{kd}{d'}}$. $\square$

PROBLEM. Calculate the number of ncpd-trees with a given underlying tree and of a given index.

§6. Concluding remarks.

In spite of the fact that there exists a reasonable criterion of nonflattening in the class of tree-like curves in terms of their ncpd-trees the author is convinced that there is no closed formula for $\sharp_{inf}[c]$. Combinatorial reformulation 4.7. reduces calculation of $\sharp_{inf}[c]$ to a rather complicated discrete optimization problem which can hardly have any closed answer. (One can even make speculations about the computational complexity of the above optimization problem.)

The lower and upper bounds presented in §4 can be improved by using much more complicated charactcrics of an ncpd-tree. On the other side, both of them are exact on some ncpd-trees. Since the closed formula is unavailable the author was not trying to get the best possible estimations.

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