Density of rational points on Enriques surfaces

F. A. Bogomolov
Courant Institute of Mathematical Sciences, N.Y.U.
251 Mercer str.
New York, (NY) 10012, U.S.A.
e-mail: bogomolo@cims.nyu.edu
and
Yu. Tschinkel
Dept. of Mathematics, U.I.C.
851 South Morgan str.
Chicago, (IL) 60607-7045, U.S.A.
e-mail: yuri@math.uic.edu

1 Introduction

Let X be a smooth projective algebraic variety defined over a number field K. We will say that rational points on X are potentially dense if there exists a finite extension K' of K such that the set of K'-rational points is Zariski–dense in X. It seems reasonable to ask whether or not rational points are potentially dense if neither X nor its unramified coverings admit a morphism onto a variety of general type. In dimension 1 the answer is affirmative. In [3] and [2] it is proved that rational points are potentially dense on all Fano threefolds with the possible exception of double covers of $\mathbb{P}^3$ ramified in smooth surfaces of degree 6.

In this paper we will study the question of density under the additional assumption that X admits the structure of an elliptic fibration $\varphi : X \to B$. over some base B of dimension ≥ 1. As one of the applications we prove that rational points on Enriques surfaces are potentially dense.

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2 Elliptic fibrations - Generalities

Definition 2.1 Let X be a smooth projective algebraic variety. We say that X admits a structure of an elliptic fibration if there exists a morphism $\varphi : X \to B$ onto a smooth projective variety B whose fibers are connected curves such that the generic fiber is a smooth curve of genus 1. We say that X admits a structure of a Jacobian elliptic fibration if there exists a section $e : B \to X$. We denote by X_b the fiber over $b \in B$.

It is well known (cf. [1] or [4]) that to every elliptic fibration $\varphi : \mathcal{E} \to B$ one can associate a Jacobian elliptic fibration $\varphi_{\mathcal{J}} : \mathcal{J} = \mathcal{J}(\mathcal{E}) \to B$ (over the same groundfield), which over the generic point of the base B is given by classes of divisors of degree zero in the fiber. The zero section corresponds to the trivial class. There is a fiberwise action of $\mathcal{J}$ on $\mathcal{E}$, more precisely, a rational map

$$\psi : \mathcal{J} \times_B \mathcal{E} \to \mathcal{E}$$

which is regular in non-singular points of the fibers of $\mathcal{J}$ and $\mathcal{E}$ and which induces a transitive action of $\mathcal{J}_b$ on $\mathcal{E}_b$ (for smooth fibers).

2.1 Multisections

Definition 2.2 Let $\varphi : \mathcal{E} \to B$ be an elliptic fibration. We say that a subvariety $\mathcal{M} \subset \mathcal{E}$ is a *multisection* of degree d if $\mathcal{M}$ is irreducible and if the degree d of the projection $\varphi : \mathcal{M} \to B$ is non-zero.

Definition 2.3 Let $\varphi : \mathcal{E} \to B$ be an elliptic fibration. A multisection $\mathcal{M}$ is said to be of *of order* m if m is the smallest integer such that for any $b \in B$ and any pair of points $p_b, p'_b \in \mathcal{M} \cap \mathcal{E}_b$ the image of the zero-cycle $p_b - p'_b$ in $\mathcal{J}_b$ is torsion of order m.

Let $\Phi_m \subset \mathcal{J}$ be the set of all points $p \in \mathcal{J}$ which are of order m in the group of rational points of the fiber $\mathcal{J}_{\varphi_{\mathcal{J}}(p)}$ and which are not contained in the zero section $e(B)$.

Lemma 2.4 *Either the intersection of $\mathcal{M}$ with $\psi(\Phi_m \times_B \mathcal{M})$ is a proper subvariety in $\mathcal{M}$ or $\mathcal{M}$ is a multisection of order m.*

Proof. Tautology.

Definition 2.5 Let $\varphi : \mathcal{E} \to B$ be an elliptic fibration. A *non-torsion multisection* of $\mathcal{E}$ is, by definition, a multisection $\mathcal{M}$ which intersects a smooth fiber $\mathcal{E}_b$ at some point p_b with local intersection multiplicity ≥ 2.

Proposition 2.6 *Let $\mathcal{M}$ be a non-torsion multisection of $\mathcal{E}$. Then $\mathcal{M}$ is not of order m for any $m > 0$.*

Proof. By assumption, nearby fibers of $\mathcal{E}_b$ are smooth. We can find a sequence (b_i) of points in the base converging to b and pairs of distinct points p_i, p'_i in the fibers $\mathcal{E}_{b_i}$ which converge to p. If we assume that the cycle $p_i - p'_i$ is torsion of order m (for some $m \geq 2$) in the Jacobian we obtain a contradiction since it converges to $0 = e(b)$.

Definition 2.7 Let $\mathcal{M}$ be a multisection of degree d of the elliptic fibration $\varphi : \mathcal{E} \to B$ and $b \in B$. We denote by $\mathrm{Tr}_{\mathcal{M}}(b)$ be the zero-cycle $\mathcal{M} \cap \mathcal{E}_b$. Define the class map

$$\tau_{\mathcal{M}} : \mathcal{E}_B \to \mathcal{J}_B$$

by the following rule:

$$\tau_{\mathcal{M}}(p) = [d \cdot p - \mathrm{Tr}_{\mathcal{M}}(\varphi(p))]$$

for $p \in \mathcal{E}$.

Lemma 2.8 *Suppose that the multisection $\mathcal{M}$ is of degree d and not of order d' with $d'|d$. Then the map*

$$\tau_{\mathcal{M}} : \mathcal{M} \to \tau_{\mathcal{M}}(\mathcal{M})$$

is a birational isomorphism.

Proof. Consider all points $p, p' \in \mathcal{E}$ such that $\tau_{\mathcal{M}}(p) = \tau_{\mathcal{M}}(p')$. Then the cycle $p - p'$ is of order $d'|d$ in $\mathcal{J}_b$. On the other hand, for a given d' pairs of distinct points differing by torsion of order d' (more precisely, by a translation by $\Phi_{d'}$) constitute a Weil divisor in $\mathcal{M}$. Therefore, the map $\tau_{\mathcal{M}}$ is a birational isomorphism.

Proposition 2.9 *The map $\tau_{\mathcal{M}}$ is regular outside of singular fibers of $\mathcal{E} \to B$.*

Proof. Evident.

Corollary 2.10 *Let $\mathcal{M}$ be a non-torsion multisection of $\mathcal{E}$. Then $\tau_{\mathcal{M}}(\mathcal{M})$ is a non-torsion multisection of $\mathcal{J}$.*

Proof. The map $\tau_{\mathcal{M}}$ is unramified on the set of non-singular fibers. By lemma 2.8, it is a birational isomorphism.

Remark 2.11 Let $\mathcal{M} \subset \mathcal{E}$ be a multisection which is not of order m for any m. Then $\tau_{\mathcal{M}}(\mathcal{M})$ is not of order m for any m.

From now on we will work over a number field K and we restict to the case of the base $B = \mathbb{P}^1$. Merel's theorem implies: Content-Length: 10595 X-Lines: 284 Status: RO

Proposition 2.12 *[2] Let $\varphi_{\mathcal{J}} : \mathcal{J} \to \mathbb{P}^1$ be a Jacobian elliptic fibration defined over K with a non-torsion multisection $\mathcal{M}$. Then for all but finitely many $b \in \varphi_{\mathcal{J}}(\mathcal{M}(K)) \subset \mathbb{P}^1(K)$ the fibers J_b have infinitely many rational points.*

Theorem 2.13 *Let $\varphi : \mathcal{E} \to \mathbb{P}^1$ be an elliptic fibration defined over K. Assume that there exists a rational or elliptic non-torsion multisection $\mathcal{M}$ defined over K. Then rational points on $\mathcal{E}$ are potentially dense.*

Proof. Extending the groundfield, we can assume that $\mathcal{M}$ is defined over K, that K-rational points on $\mathcal{M}$ are Zariski-dense and that it intersects a smooth fiber $\mathcal{E}_{\varphi(p)}$ with local intersection multiplicity ≥ 2 at a K-rational point p. The image $\varphi(\mathcal{M}(K))$ is Zariski-dense in $\mathbb{P}^1$. Every fiber $\mathcal{E}_b$ for $b \in \varphi(\mathcal{M}(K))$ contains at least one point $p_b \in \mathcal{M}(K)$. Therefore, it suffices to show that for almost all $b \in \varphi(\mathcal{M}(K))$ the fiber $\mathcal{J}_b$ has a K-rational point of infinite order. Then we use the action of the $\mathcal{J}$ on $\mathcal{E}$ to translate p_b.

By assumption and 2.10 the curve $\tau_{\mathcal{M}}(\mathcal{M})$ is a non-torsion multisection of $\mathcal{J} \to \mathbb{P}^1$. The point $\tau_{\mathcal{M}}(p_b) \in \tau_{\mathcal{M}}(\mathcal{M})$ is a K-rational point of the fiber $\mathcal{J}_{\varphi(p_b)}$. Moreover, it is contained in the non-torsion multisection $\tau_{\mathcal{M}}(\mathcal{M})$. By theorem 2.12, for all but finitely many fibers $\mathcal{J}_{\varphi(p_b)}$ the point $\tau_{\mathcal{M}}(p_b)$ is a non-torsion point in the group $\mathcal{J}_{\varphi(p_b)}(K)$. This concludes the proof.

Proposition 2.14 *Let X be a smooth algebraic surface defined over a number field K and admitting two different elliptic fibrations $\varphi_1 : \mathcal{E}_1 \to B_1$ and $\varphi_2 : \mathcal{E}_2 \to B_2$ with $B_1 \simeq B_2 \simeq \mathbb{P}^1$. Then rational points on X are potentially dense.*

Proof. If there is at least one $\mathcal{E}_{b_2}$ which is not of order m for any m we are done by 2.11. In view of 2.10, it suffices to consider the case when for all $b_2 \in B_2$ all multiple intersection points of the multisections $\mathcal{E}_{b_2}$ with fibers $\mathcal{E}_{b_1}$ (for $b_1 \in B_1$) are contained in the singular fibers of the fibration $\mathcal{E}_1 \to B_1$. If a generic $\mathcal{E}_{b_2}$ is of some fixed order m_0 we see that (a cover of) X is dominated (birationally) by a product of two elliptic curves.

3 Enriques surfaces

We start with a brief summary of the structure theory of Enriques surfaces (cf. [1], pp. 274-275): Either X has two distinct elliptic fibrations over $\mathbb{P}^1$ (this case is called *non-special* in [1]) or it admits an elliptic fibration with a 2-section which is a (-2)-curve. In the second case (called *special*) the associated K3 double cover $\pi : Y \to X$ admits the structure of an elliptic fibration $Y \to \mathbb{P}^1$ with *two* (non-intersecting) sections (which could both be torsion). The surface Y is the minimal resolution of a double covering of a quadratic cone $Q \subset \mathbb{P}^3$ (given by $z_0 z_1 = z_2^2$ in standard coordinates in $\mathbb{P}^3 = (z_0, z_1, z_2, z_3)$) ramified in an intersection of this cone with a quartic hypersurface in $\mathbb{P}^3$ not passing through the vertex of this cone. The ramification curve R is reduced, of degree 8 and has at most simple singularities. The fibers of the elliptic fibration correspond to the generators of the cone Q and the two sections are mapped to the vertex of Q (cf. [1], p. 278).

Theorem 3.1 *Let X be an Enriques surface over K. Then rational points on X are potentially dense.*

Proof. In the non-special case we apply 2.14 and we are done.

In the special case we proceed as follows. We will prove that rational points are potentially dense on the K3 cover Y. After blowing up the vertex of the cone we can realize Q birationally as $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(2) \oplus \mathcal{O})$ (see [2]). Taking sections of $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(2))$ (a 3-dimensional linear space) we obtain a family of (conics) $\mathbb{P}^1_s \simeq \mathbb{P}^1 \subset Q$ where each $\mathbb{P}^1_s$ intersects the ramification curve R in 8 points (counted with multiplicities). The double cover D_s of $\mathbb{P}^1_s$ is a multisection of the elliptic fibration $Y \to \mathbb{P}^1$. We want to find an elliptic non-torsion multisection among the D_s. Then we apply 2.13.

Recall that the elliptic fibration $Y \to \mathbb{P}^1$ has two non-intersecting sections e_1, e_2 which are the preimages of the (-2)-curve in $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(2) \oplus \mathcal{O})$. Their difference is torsion in the Picard group of the generic point of $Y \to \mathbb{P}^1$ if there exists an integer $m > 0$ such that $m(e_1 - e_2)$ can be represented as a sum, with integer coefficients, of components of singular fibers of $Y \to \mathbb{P}^1$. If all fibers of $Y \to \mathbb{P}^1$ are irreducible then the difference $e_1 - e_2$ is not torsion and rational points are potentially dense. Otherwise, we have to consider special cases.

Denote by $R^0 \subset R$ the Zariski open subset of points where the curve R is smooth and where it is not tangent to the generators of the cone Q. For every point $P \in R^0$ we consider the affine line L_P of sections $\mathbb{P}^1_s$ which are tangent to R at P. These sections cover the whole cone Q, except the line joining P and the vertex of Q. Consider the subset of sections of $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(2))$ which have at least two distinct points of local intersection multiplicity ≥ 2 with R. If this subset covers birationally the cone Q, then we obtain a 1-dimensional family of elliptic curves which covers Y and which is generically transversal to the elliptic fibration $Y \to \mathbb{P}^1$. Hence we can apply 2.14.

In particular, if the curve R has a singular double point $r \in R$, then for any $P \in R^0$ we can find a tangent section $\mathbb{P}^1_s$ which passes through r.

The family of such sections covers Q (birationally), provided that R has a component which is not a section. Therefore, we have only to consider the case when R consists of 4 distinct components which are sections. Suppose that r lies on the intersection of two components. Then the tangents to some point of a third component passing through r cover Q birationally and the argument above applies.

The only remaining case is the case of a non-singular curve R. For any $P \in R$ there exists a section $s \in L_P$ which has local intersection multiplicity ≥ 2 with R at some other point. Indeed, we have a natural tangent correspondence $\mathrm{TC}_P \subset L_P \times R$ defined by

$$\mathrm{TC}_P = \{(s, r) \mid s \in L_P, r \in \mathbb{P}^1_s \cap R\}.$$

Since R is irreducible this correspondence is irreducible. Since R is not a rational curve this correspondence has ramification points over L_P. Ramification points correspond to sections having at least two intersection points of local multiplicity ≥ 2 with R. Hence, the family of sections with this property (as P moves over R^0) covers (birationally) the cone Q and defines a transversal elliptic fibration on $Y \to \mathbb{P}^1$.

References

[1] W. Barth, C. Peters, and A. van de Ven, *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*, Springer-Verlag, Berlin, 1984.

[2] F. A. Bogomolov and Yu. Tschinkel, On the density of rational points on elliptic fibrations, Preprint 1998.

[3] J. Harris and Yu. Tschinkel, Rational points on quartics, Preprint 1998.

[4] V. A. Iskovskikh and I. R. Shafarevich. Algebraic surfaces, In *Algebraic geometry, II*, volume 35 of *Encyclopaedia Math. Sci.*, pages 127–262. Springer, Berlin, 1996.

[5] L. Merel, Bornes pour la torsion des courbes elliptiques sur les corps de nombres, *Invent. Math.*, 124(1-3):437–449, 1996.