Compactification of moduli spaces of Einstein-Hermitian connections for null-correlation bundles

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§0. Introduction

In 1970's by an effective use of twistor theory originated from Penrose [P], gauge-theoretic studies of anti-self-dual connections over 4-manifolds were inaugurated by Atiyah, Hitchin and Singer (see for instance [A-H-S], [A-J], [A-W]). Almost at the same time, Hartshorne determined the moduli spaces of anti-self-dual connections for SU(2)-bundles over S^4 through a purely algebraic study of the null-correlation bundles over $\mathbf{P}^3(\mathbf{C})$. A little later, Kobayashi [K] introduced the concept of Einstein-Hermitian vector bundles over Kähler manifolds, which is in some sense a higher dimensional analogue of anti-self-dual connections over 4-manifolds (see for instance Kobayashi [K] for a general theory of Einstein-Hermitian connections).

The purpose of this paper is to construct a compactified family of Einstein-Hermitian connections on null-correlation bundles over odd-dimensional complex projective spaces $\mathbf{P}^{2m+1}(\mathbf{C})$. Let $\mathbf{P}^m(\mathbf{H}) = \mathrm{Sp}(m+1)/\mathrm{Sp}(m)\times\mathrm{Sp}(1)$ be the m-dimensional quaternionic projective space, and $p : \mathbf{P}^{2m+1}(\mathbf{C}) \to \mathbf{P}^m(\mathbf{H})$ the corresponding twistor space. The homogeneous space $\mathrm{Sp}(m+1)/\mathrm{id}\times\mathrm{Sp}(1)$ is a principal fibre bundle over $\mathbf{P}^m(\mathbf{H})$ with typical fibre $\mathrm{Sp}(m)$. Let τ be the standard representation of $\mathrm{Sp}(m)$ in $\mathbf{C}^{2m}$. Then $V := (\mathrm{Sp}(m+1)/\mathrm{id}\times\mathrm{Sp}(1))\times_\tau \mathbf{C}^{2m}$ is a complex vector bundle over $\mathbf{P}^m(\mathbf{H})$. Since $\mathrm{Sp}(m)$ is contained on $\mathrm{U}(2m)$, the vector bundle V carries a natural Hermitian metric h_V. Salamon introduced in [S] a certain type of connections (which we call B_2-connections) on vector bundles over quaternionic Kähler manifolds, and such connections are later studied by Berard-Bergery and Ochiai [B-O] in a more general setting. We showed that B_2-connections are Yang-Mills connections and studied them in [Ni1], which is also obtained by Capria and

Salamon independently. They constructed an interesting family of Yang-Mills connections for the vector bundle (V, h_V) parametrized roughly by $\mathrm{SL}(m+1, \mathbf{H})/\mathrm{Sp}(m+1)$. By generalizing the Penrose twistor correspondence to higher dimensional quaternionic Kähler manifolds, we obtained the following :

Theorem ([Ni2]). *The moduli space of B_2-connections on (V, h_V) is imbedded as a totally real submanifold of the moduli space of Einstein-Hermitian connections on (p^*V, p^*h_V).*

This theorem allows us to construct a family of Einstein-Hermitian connections on (p^*V, p^*h_V) parametrized by $\mathrm{PGL}(2m+2, \mathbf{C})/\mathrm{PSp}(m+1, \mathbf{C})$ (cf. Section 1). Thus, we obtained a mapping ψ of $\mathrm{PGL}(2m+2, \mathbf{C})/\mathrm{PSp}(m+1, \mathbf{C})$ to the moduli space of Einstein-Hermitian connectoins for (p^*V, p^*h_V). This mapping ψ is regarded as a complexification of the one constructed by Capria and Salamon, and moreover we obtain (cf. Section 2) :

Theorem. *The mapping ψ is injective.*

On the other hand, $\mathrm{PGL}(2m+2, \mathbf{C})/\mathrm{PSp}(m+1, \mathbf{C})$ can be embedded as an open dense subset of $\mathbf{P}^l(\mathbf{C})$ (where $l = m(2m+3)$). Let $\mathcal{L}(p^*V, p^*h_V)$ be the set of Einstein-Hermitian connections for (p^*V, p^*h_V) possibly with singularities, and consider the unitary gauge transformatioan group $\mathcal{G}(p^*V, p^*h_V)$ consisting of all bundle automorphisms on p^*V preserving p^*h_V. Then we define an equivalence relation on $\mathcal{L}(p^*V, p^*h_V)$ as follows : for $\nabla_1, \nabla_2 \in \mathcal{L}(p^*V, p^*h_V)$, we say that ∇_1 is equivalent to ∇_2 if there is a gauge transformation $s \in \mathcal{G}(p^*V, p^*h_V)$ such that $s^*\nabla_1 = \nabla_2$ off the singular sets. We denote the resulting set of equivalence class by $\mathcal{L}(p^*V, p^*h_V)/\mathcal{G}(p^*V, p^*h_V)$. In section 4, we extend ψ to a mapping $\tilde{\psi}$ from $\mathbf{P}^l(\mathbf{C})$ to $\mathcal{L}(p^*V, p^*h_V)/\mathcal{G}(p^*V, p^*h_V)$, which gives us a compactification of the image $\psi(\mathrm{PGL}(2m+2, \mathbf{C})/\mathrm{PSp}(m+1, \mathbf{C}))$. Furthermore, we have:

Theorem. *The family of Yang-Mills connections constructed by Capria and Salamon is realized as a connected component of the moduli space of B_2-connections on (V, h_V).*

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§1. Notation, conventions and preliminaries

For this section, we refer to [C-S], [Ni1] and [Ni2].

(1.1.1). The quaternionic projective space $\mathbf{P}^m(\mathbf{H})$ is the set of all quaternionic lines through 0 sitting in the right $\mathbf{H}$-module $\mathbf{H}^{m+1}$. In this paper we make use of column vectors in order to describe elements in vector space over $\mathbf{C}$ or $\mathbf{H}$. Thus $\mathbf{P}^m(\mathbf{H}) = \{(u) | u = {}^t(u^0, \cdots, u^m) \in \mathbf{H}^{m+1} - \{0\}\}$, where (u) means the quaternionic line including a vector $u (\in \mathbf{H}^{m+1})$. Recall that $\mathbf{P}^m(\mathbf{H})$ has a natural quaternionic Kähler structure. The right $\mathbf{H}$-module $\mathbf{H}^{m+1}$ has a standard quaternionic Hermitian inner product $h_{\mathbf{H}^{m+1}}(u, v) = {}^t\bar{u}v$ $(u, v \in \mathbf{H}^{m+1})$, which induces the quaternionic Hermitian metric h_0 on the trivial vector bundle F $:= \mathbf{P}^m(\mathbf{H}) \times \mathbf{H}^{m+1}$. Let V be the quaternionic vector subbundle of $\mathbf{P}^m(\mathbf{H}) \times \mathbf{H}^{m+1}$ such that each fibre $V_{(u)}$ over $(u) (\in \mathbf{P}^m(\mathbf{H}))$ is the orthogonal complement of the quaternionic line (u) with respect to h_0. The restriction of h_0 on V is denoted by h_V.

(1.1.2). When $\mathbf{H}^{m+1}$ is identified with $\mathbf{C}^{2m+2}$ by the isomorphism which sends each $u_1 + ju_2 \in \mathbf{H}^{m+1}$ to $(u_1, u_2) \in \mathbf{C}^{2m+2}$, we regard V and h_V as a complex vector bundle and a (complex) Hermitian metric respectively. The vector bundle $\wedge^2 \mathrm{T}^*(\mathbf{P}^m(\mathbf{H}))$ of covectors of degree 2 is expressed as a direct sum of three holonomy invariant vector subbundles A_2', A_2'' and B_2 (cf.[Ni1]). A Hermitian connection ∇ on (V, h_V) is called a B_2-connection, if the curvature R^∇ of ∇ is an $\mathrm{End}(V)$-valued B_2-form. Let ∇ be a B_2-connection on (V, h_V). Then ∇ induces elliptic complexes $C_\nabla = \{(A^i, d_i)\}$ and $\tilde{C}_\nabla = \{(\tilde{A}^i, \tilde{d}_i)\}$ (see [Ni2;(2.1)] for definition of C_∇ and $\tilde{C}_\nabla$).

(1.1.3). Let $C_B'(V, h_V)$ be the set of all irreducible B_2-connections ∇ on (V, h_V) where ∇ is said to be irreducible if $\mathrm{H}^0(\mathbf{P}^m(\mathbf{H}), \tilde{C}_\nabla) = \{0\}$. We denote by $\mathcal{B}'(V, h_V)$ the quotient space of $C_B'(V, h_V)$ by the unitary gauge transformation group $\mathcal{G}(V, h_V)$, and $\mathcal{B}'(V, h_V)$ is often called the moduli space of irreducible Hermitian B_2-connections on (V, h_V). Furthermore, let $C_B''(V, h_V)$ be the set of all irreducible Hermitian B_2-connections ∇ on (V, h_V) such that $\mathrm{H}^2(\mathbf{P}^m(\mathbf{H}), \tilde{C}_\nabla) = \{0\}$. We then put $\mathcal{B}''(V, h_V) := C_B''(V, h_V)/\mathcal{G}(V, h_V)$. It is known that $\mathcal{B}''(V, h_V)$ has a natural structure of Riemannian manifold. For examples of Hermitian B_2-connections, see Capria and Salamon [C-S]. Let $M(l, k; \mathbf{H})$ be the set of all quaternionic valued (l, k) matrices. We now set:

$$\mathcal{H} := \{H \in M(l, k; \mathbf{H}) - \{0\} | {}^t\bar{H} = H\},$$

$$\mathcal{H}_0 := \mathcal{H} \cap \mathrm{GL}(m+1, \mathbf{H}).$$

We say that $H_1, H_2 (\in \mathcal{H})$ are equivalent if there exists an element $a (\in \mathbf{R}^*)$ such that $H_1 = aH_2$. We write the equivalence class of $H (\in \mathcal{H})$ as $\widetilde{H}$ and the set of all $\widetilde{H}$ $(H \in \mathcal{H}_0)$ as $\widetilde{\mathcal{H}}_0$. Now the Lie group $\mathrm{SL}(m+1, \mathbf{H})$ transitively acts on $\widetilde{\mathcal{H}}_0$, which is just $\mathrm{SL}(m+1, \mathbf{H}) / \mathrm{Sp}(m+1)$.

(1.1.4). To each $H \in \mathcal{H}_0$, we associate a quaternionic vector subbundle $W(H)$ of the trivial bundle $F = \mathbf{P}^m(\mathbf{H}) \times \mathbf{H}^{m+1}$ by

$$W(H)_{(u)} = \{v \in \mathbf{H}^{m+1} | {}^t\bar{v}Hu = 0\}, \ (u) \in \mathbf{P}^m(\mathbf{H}),$$

where $W(H)_{(u)}$ denotes the fibre of $W(H)$ over (u). Then given $\widetilde{H} \in \widetilde{\mathcal{H}}_0$, one sees that $W(H)$ is independent of the choice of representations H for $\widetilde{H}$. Let $h(H)$ be the quaternionic Hermitian metric on $W(H)$ induced from the standard quaternionic Hermitian metric on the trivial bundle F. The flat connection d of the vector bundle F over $\mathbf{P}^m(\mathbf{H})$ naturally induces a connection $\nabla(H)$ on $W(H)$

$$\nabla(H) = P(H) \circ d,$$

where $P(H) : F \to W(H)$ denotes the fibrewise orthogonal projection of the vector bundle F onto $W(H)$ over $\mathbf{P}^m(\mathbf{H})$. Then the connection $\nabla(H)$ is compatible with the quaternionic Hermitian metric $h(H)$ on $W(H)$, and the corresponding holonomy group is $\mathrm{Sp}(m)$. Especially, $\nabla(H)$ is irreducible.

(1.1.5). Since $\widetilde{\mathcal{H}}_0$ is connected, the vector bundle $W(H)$ $(H \in \mathcal{H}_0)$ is isomorphic to V $(= W(id_{\mathbf{H}^{m+1}}))$ as quaternionic vector bundles. We now note that $\mathrm{Sp}(m)$ is a maximal compact subgroup of $\mathrm{GL}(m, \mathbf{H})$. Hence, for each $H \in \mathcal{H}_0$ there exists a quaternionic isomorphism

$$t_0(H) : (W(id), h(id)) \tilde{\to} (W(H), h(H))$$

preserving the Hermitian structure. The resulting pull-back connection

$$D(H) = t_0(H)^* \nabla(H) := t_0(H) \circ \nabla(H) \circ t_0(H)^{-1}$$

is a quaternionic connection on (V, h_V). By identifying $\mathbf{H}^{m+1}$ with $\mathbf{C}^{2m+2}$ we regard $D(H)$ as a Hermitian connection on the complex Hermitian vector bundle (V, h_V). Recall the following result of Capria and Salamon:

Theorem ([C-S]). *For each $H \in \mathcal{H}_0$ the Hermitian connection $D(H)$ is an irreducible B_2-connection on the complex vector bundle (V, h_V).*

(1.1.6). The equivalence class $[D(H)]$ of $D(H)$ modulo the unitary gauge transformation group $\mathcal{G}(V, h_V)$ depends only on $\tilde{H} \in \tilde{\mathcal{H}}_0$ and is independent of the choice of vector bundle isomorphism $t_0(H)$ as above. We then have the mapping

$$\varphi : \tilde{\mathcal{H}}_0 \ni \tilde{H} \mapsto [D(H)] \in \mathcal{B}''(V, h_V).$$

(1.2.1). The twistor space corresponding to $\mathbf{P}^m(\mathbf{H})$ is

$$p : \mathbf{P}^{2m+1}(\mathbf{C}) \ni [z] \to (z) \in \mathbf{P}^m(\mathbf{H}),$$

where $[z]$ denotes the complex line including a vector z ($z \in \mathbf{C}^{2m+2} \simeq \mathbf{H}^{m+1}$). The pull-back (p^*V, p^*h_V) over $\mathbf{P}^{2m+1}(\mathbf{C})$ is a Hermitian vector bundle with vanishing first Chern class. A Hermitian connection ∇ on (p^*V, p^*h_V) is an Einstein-Hermitian connection if and only if the corresponding Ricci-curvature is a constant multiple of identity. Since the first Chern class of p^*V is zero, the constant is equal to zero.

(1.2.2). Take an Einstein-Hermitian connection ∇ on (p^*V, p^*h_V). Then ∇ induces elliptic complexes A_∇ and $\tilde{B}_\nabla$ defined by Itoh and Kim (see [Ni2;(2.1)] for definition of A_∇ and $\tilde{B}_\nabla$). Let $\mathcal{C}_E(p^*V, p^*h_V)$ be the set of all Einstein-Hermitian connections on (p^*V, p^*h_V). Moreover, let $\mathcal{C}'_E(p^*V, p^*h_V)$ be the set of all irreducible Einstein-Hermitian connections ∇ on (p^*V, p^*h_V) where ∇ is said to be irreducible if $\mathrm{H}^0(\mathbf{P}^{2m+1}(\mathbf{C}), \tilde{A}_\nabla) = \{0\}$. We denote by $\mathcal{E}(p^*V, p^*h_V)$ and $\mathcal{E}'(p^*V, p^*h_V)$ the quotient space of $\mathcal{C}_E(p^*V, p^*h_V)$ and $\mathcal{C}'_E(p^*V, p^*h_V)$ by the unitary gauge transformation group $\mathcal{G}(p^*V, p^*h_V)$. The quotient space $\mathcal{E}'(p^*V, p^*h_V)$ is often called the moduli space of irreducible Einstein-Hermitian connections on (p^*V, p^*h_V). Furthermore, let $\mathcal{C}''_E(p^*V, p^*h_V)$ be the set of irreducible Einstein-Hermitian connections ∇ on (p^*V, p^*h_V) such that $\mathrm{H}^2(\mathbf{P}^{2m+1}(\mathbf{C}), \tilde{B}_\nabla) = \{0\}$. We then put

$$\mathcal{E}''(p^*V, p^*h_V) := \mathcal{C}''_B(p^*V, p^*h_V)/\mathcal{G}(p^*V, p^*h_V).$$

It is known that $\mathcal{E}''(V, h_V)$ has a natural structure of Kähler manifold (cf. [I], [K]).

(1.3). The pull-back $\nabla \mapsto p^*\nabla$ of connections induces an imbedding $p^* : \mathcal{B}'(V, h_V) \to \mathcal{E}'(p^*V, p^*h_V)$ ($p^* : \mathcal{B}''(V, h_V) \to \mathcal{E}''(p^*V, p^*h_V)$). Furthermore we obtained :

Theorem ([Ni2]). *The embedding* $p^*: \mathcal{B}''(V, h_V) \hookrightarrow \mathcal{E}''(p^*V, p^*h_V)$ *is totally real, (i.e.,* $\mathcal{B}''(V, h_V)$ *is embedded in* $\mathcal{E}''(p^*V, p^*h_V)$ *by* p^* *as a totally real submanifold).*

§2. Construction of Einstein-Hermitian connections

In this section, we construct a family of Einstein-Hermitian connections on the Hermitian vector bundle (p^*V, p^*h_V) over $\mathbf{P}^{2m+1}(\mathbf{C})$. It will be shown that connections constructed here are parametrized by symplectic structures on $\mathbf{C}^{2m+2}$ i.e., we shall obtain a mapping of the set of all symplectic structures of $\mathbf{C}^{2m+2}$ onto a family of Einstein-Hermitian connections on (p^*V, p^*h_V).

(2.1.1). Let $M(k; \mathbf{C})$ be the set of complex-valued square matrices of degree k. A complex-valued skew-symmetric matrix $S \in M(2m+2; \mathbf{C})$ induces a skew-symmetric bilinear form on $\mathbf{C}^{2m+2}$ by

$$S(\xi, \eta) = {}^t\xi S \eta, \quad (\xi, \eta \in \mathbf{C}^{2m+2}).$$

Then this bilinear form is non-degenerate if and only if the matrix S is of full rank. We identify each S with the corresponding bilinear form defined as above, when no confusion is likely to occur.

(2.1.2). We put

$$\mathfrak{S} := \{0 \neq S \in M(2m+2; \mathbf{C}) \mid S \text{ is skew-symmetric } \},$$
$$\mathcal{S} := \{S \in \mathfrak{S} \mid S \text{ is non-degenerate } \}.$$

Then $\mathbf{C}^*$ naturally acts on $\mathfrak{S}$ by

$$\mathbf{C}^* \times \mathfrak{S} \ni (c, S) \mapsto cS \in \mathfrak{S}$$

Note that this $\mathbf{C}^*$-action preserves the subset $\mathcal{S}$ of $\mathfrak{S}$. We now define :

$$\widetilde{\mathfrak{S}} := \mathfrak{S}/\mathbf{C}^*,$$
$$\widetilde{\mathcal{S}} := \mathcal{S}/\mathbf{C}^*.$$

For each $S \in \mathfrak{S}$, we denote by $\widetilde{S}$ the corresponding element of $\mathfrak{S}$. Then it is easily seen that $\widetilde{\mathcal{S}}$ is nothing but $\mathrm{PGL}(2m+2, \mathbf{C})/\mathrm{PSp}(m+1, \mathbf{C})$.

(2.2.1). Recall that the vector bundle p^*F is the trivial bundle $\mathbf{P}^{2m+1}(\mathbf{C}) \times \mathbf{C}^{2m+2}$ over $\mathbf{P}^{2m+1}(\mathbf{C})$. For $\widetilde{S} \in \mathcal{S}$, we define a complex subbundle $V(S)$ of p^*F such that the fibre $V(S)_{[z]}$ over $[z] \in \mathbf{P}^{2m+1}(\mathbf{C})$ is the vector subspace $\{y \in \mathbf{C}^{2m+2} \mid {}^t y S z = 0, \ {}^t\overline{y}({}^t\overline{S}S)^{1/2} z = 0\}$ of

$\mathbf{C}^{2m+2}$. Since the two vectors $\overline{Sz}$ and $({}^t\overline{S}S)^{1/2}z$ are orthogonal, $V(S)$ is a complex vector bundle of rank 2m. Note that $V(S) = V(S')$ whenever $\widetilde{S} = \widetilde{S'}$.

(2.2.2). Let $k(S)$ be the Hermitian metric on $V(S)$ induced from the standard Hermitian metric on p^*F. Then the flat connection d on the trivial bundle p^*F induces a Hermitian connection $\nabla(S)$ on $V(S)$ by

$$\nabla(S) = Q(S) \circ d,$$

where $Q(S)$ denotes the orthogonal projection of p^*F onto $V(S)$. We then obtain:

Theorem 2.2.3. *For each S, the Hermitian connection $\nabla(S) = \nabla$ is an Einstein-Hermitian connection on $(V(S), k(S))$.*

Proof. Let $N(S)$ be the vector subbundle of p^*F obtained as the orthogonal complement of $V(S)$ in p^*F. We denote by $\widetilde{Q} = \widetilde{Q(S)}$ the orthogonal projection of p^*F onto $N(S)$. Put $H = ({}^t\overline{S}S)^{1/2}$. For $z \in \mathbf{C}^{2m+2}$, let A be the $(2m+2, 2)$-matrix consisting of two column vectors Hz and $\overline{Sz}$. Then the projection $\widetilde{Q}$ is written as follows

$$\widetilde{Q} = A({}^t\overline{A}A)^{-1}\, {}^t\overline{A}, \tag{1}$$

at $[z] \in \mathbf{P}^{2m+1}(\mathbf{C})$. For a section $f \in \Gamma(\mathbf{P}^{2m+1}(\mathbf{C}), \mathrm{C}^\infty(V(S)))$,

$$\begin{aligned}\nabla f &= (id - \widetilde{Q})(df)\\ &= df + d(\widetilde{Q})f,\end{aligned}$$

since $\widetilde{Q}f = 0$. The curvature $R = R(S)$ for ∇ is given by

$$\begin{aligned}R &= (d + d\widetilde{Q}) \circ (d + d\widetilde{Q})\\ &= d\widetilde{Q} \wedge d\widetilde{Q}.\end{aligned}$$

More precisely, $R = Q(d\widetilde{Q} \wedge d\widetilde{Q})Q$, where we denote $Q(S)$ by Q for simplicity. Since

$$Q(Hz, \overline{Sz}) = 0 \text{ and } {}^t(\overline{Hz}, Sz)Q = 0,$$

we obtain from (1) the expression:

$$R = QdA({}^t\overline{A}A)^{-1}\, {}^t\overline{(dA)}Q, \tag{2}$$

where $dA = (Hdz, \overline{Sdz})$. Moreover,

$$(3) \qquad {}^t\overline{A}A = \begin{pmatrix} |Hz|^2 & 0 \\ 0 & |Sz|^2 \end{pmatrix}.$$

By (2) and (3),

$$(4) \qquad R = (\det({}^t\overline{A}A))^{-1} Q dA \begin{pmatrix} |Sz|^2 & 0 \\ 0 & |Hz|^2 \end{pmatrix} {}^t\overline{(dA)} Q$$
$$= \frac{Q\{|Sz|^2 Hdz \wedge {}^t\overline{dz}{}^t\overline{H} + |Hz|^2 \overline{Sdz} \wedge {}^t dz {}^t S\}Q}{\det({}^t\overline{A}A)}.$$

Hence, R is an End($V(S)$)-valued (1,1)-form. Hence ∇ is a Hermitian connection of type (1,0) on $(V(S), k(S))$. Secondly, we shall calculate the Ricci curvature $\gamma(S) = \gamma$ for ∇. Let ω be the Fubini-Study form on $\mathbf{P}^{2m+1}(\mathbf{C})$. Recall that the corresponding Kähler operator

$$L: \{p - forms\} \to \{(p+2) - forms\} \qquad 0 \leqq p \leqq 2(2m+1)$$

is defined by $L(\eta) := \omega \wedge \eta$ for a p-form η on $\mathbf{P}^{2m+1}(\mathbf{C})$. Let Λ be the formal adjoint of L. Then Λ can be naturally extended to the operator $id \otimes \Lambda$ (denoted also by Λ for simplicity) on End($V(S)$) $\otimes \wedge^* \mathrm{T}^* \mathbf{P}^{2m+1}(\mathbf{C})$. Recall that $\gamma = \sqrt{-1}\Lambda R$. Let $\{(U_j, \varphi_j)\}_{0 \leqq j \leqq 2m+1}$ be the standard affine coordinate system for $\mathbf{P}^{2m+1}(\mathbf{C})$, defined by

$$U_j = \{[z] = [{}^t(z^0, \cdots, z^{2m+1})] \in \mathbf{P}^{2m+1}(\mathbf{C}); z^j \neq 0\}$$

and φ_j is the mapping :

$$U_j \ni [{}^t(x^1, \cdots, 1, \cdots, x^{2m+1})] \mapsto {}^t(x^1, \cdots, x^{2m+1}) \in \mathbf{C}^{2m+1}.$$

Let us calculate $\sqrt{-1}\Lambda R$ on U_0. For $z = {}^t(1, x^1, \cdots, x^{2m+1})$, we have:

$$(5) \qquad \sqrt{-1}(1 + |x|^2)(dz \wedge {}^t\overline{dz}) = id + z{}^t\overline{z} - (z, 0) - {}^t(\overline{z}, 0),$$

where $(z, 0)$ denotes the $(2m+2, 2m+2)$-matrix whose first column vector is z and all other entries are 0. Substituting the above expression of R, we now conclude that

$$\gamma = 0.$$

Hence ∇ is an Einstein-Hermitian connection on $(V(S), k(S))$. Q.E.D.

(2.3). Since $\mathcal{S}$ is connected, $(V(S), k(S))$ is isomorphic to (p^*V, p^*h_V) as C^∞-Hermitian vector bundle. We choose such an isomorphism $t(S){:}\,(p^*V, p^*h_V) \simeq (V(S), k(S))$. Let $D(S)$ be the pull-back $t(S)^*\nabla(S) := t(S)^{-1} \circ \nabla(S) \circ t(S)$ of $\nabla(S)$. Then the connection $D(S)$ is also an Einstein-Hermitian connection on (p^*V, p^*h_V). Note that the equivalence class $[D(S)]$ modulo $\mathcal{G}(p^*V, p^*h_V)$ is independent of the choice of the isomorphism $t(S)$. We obtain the mapping $\psi{:}\,\widetilde{\mathcal{S}} \to \mathcal{E}(p^*V, p^*h_V)$ by

$$\psi(\widetilde{S}) = [D(S)] \qquad S \in \mathcal{S}.$$

Since the holonomy group of $D(S)$ is $\mathrm{Sp}(m)$, the connection $D(S)$ is irreducible (for more details see Section 3). Thus ψ is regarded as a mapping : $\widetilde{\mathcal{S}} \to \mathcal{E}'(p^*V, p^*h_V)$.

(2.4.1). Recall that the element j ($\in \mathbf{H}$) induces a real structure j_0 on $\mathbf{C}^{2m+2} (\simeq \mathbf{H}^{m+1})$:

$$j_0{:}\,\mathbf{C}^{2m+2} \ni (a, b) \mapsto (-\overline{b}, \overline{a}) \in \mathbf{C}^{2m+2}.$$

Therefore the subset $\mathcal{S}$ of $M(2m+2; \mathbf{C})$ admits a natural real structure

$$j_{\mathfrak{S}}{:}\,\mathfrak{S} \ni S \mapsto j_0^{-1} S j_0 \in \mathfrak{S}.$$

Since $j_{\mathfrak{S}}(cS)$ ($c \in \mathbf{C}^*, S \in \mathfrak{S}$) is $\overline{c} j_{\mathfrak{S}}(S)$, the real structure $j_{\mathfrak{S}}$ on $\mathfrak{S}$ is pushed down on a real structure (denoted by $j_{\widetilde{\mathfrak{S}}}$) on $\widetilde{\mathfrak{S}}$. Furthermore, $j_{\mathfrak{S}}$ and $j_{\widetilde{\mathfrak{S}}}$ restrict to the real structures $j_{\mathcal{S}}$ and $j_{\widetilde{\mathcal{S}}}$ on $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ respectively.

(2.4.2). Recall that the twistor space $\mathbf{P}^{2m+1}(\mathbf{C})$ has the standard real structure

$$\tau{:}\,[z^1, z^2] \mapsto [-\overline{z^2}, \overline{z^1}] \qquad z^1, z^2 \in \mathbf{C}^{m+1}.$$

Since p^*V is trivial on each fibre of $p : \mathbf{P}^{2m+1}(\mathbf{C}) \to \mathbf{P}^m(\mathbf{H})$, the real structure τ induces a bundle automorphism $\tilde{\tau}$ on p^*V such that the following diagram is commutative:

$$\begin{array}{ccc} p^*V & \xrightarrow{\ \tilde{\tau}\ } & p^*V \\ \downarrow & & \downarrow \\ \mathbf{P}^{2m+1}(\mathbf{C}) & \xrightarrow{\ \tau\ } & \mathbf{P}^{2m+1}(\mathbf{C}). \end{array}$$

By the bundle automorphism $\tilde{\tau}$, we define a mapping τ' of $\mathcal{E}'(p^*V, p^*h_V)$ onto itself as follows:

$$\tau'([D]) = [\tilde{\tau}^* D], \quad ([D] \in \mathcal{E}'(p^*V, p^*h_V))$$

(cf. [Ni2;(3.6)]).

(2.4.3). One can easily check that $\psi \circ j_{\widetilde{S}} = \tau' \circ \psi$. Hence ψ induces the mapping

$$(\psi)_{\mathbf{R}}: \mathcal{S}_{\mathbf{R}} \to \mathcal{E}'(p^*V, p^*h_V)_{\mathbf{R}},$$

where $\mathcal{S}_{\mathbf{R}}$ and $\mathcal{E}'(p^*V,p^*h_V)_{\mathbf{R}}$ are the subsets of all elements of $\widetilde{S}$ and $\mathcal{E}'(p^*V,p^*h_V)$ fixed by the real structures $j_{\widetilde{S}}$ and τ' respectively. Note that $\mathcal{S}_{\mathbf{R}} \simeq \mathcal{H}_0$ and $(\psi)_{\mathbf{R}} = p^* \circ \psi$. By [Ni1;(0.2)], $p^*(\mathcal{B}'(V,h_V))$ is contained in $\mathcal{E}'(p^*V,p^*h_V)_{\mathbf{R}}$. Thus,

$$Image(\psi) \cap p^*(\mathcal{B}'(V,h_V)) = p^*(Image(\phi)).$$

§3. Injectivity of the mapping ψ

In this section we shall prove that the mapping ψ is injective. This injectivity allows us to show that the image of ψ is $\mathrm{PGL}(2m+2,\mathbf{C})/\mathrm{PSp}$ $(m+1,\mathbf{C})$.

(3.1.1). Let $S \in \mathcal{S}$. Then the matrix $H(S)$ in Section 2 induces a Hermitian inner product on $\mathbf{C}^{2m+2}$ by

$$H(S)(\xi,\eta) = {}^t\overline{\xi}H(S)\eta, \quad \xi,\eta \in \mathbf{C}^{2m+2}.$$

This inner product $H(S)(\ ,\ )$ naturally defines a Hermitian metric $k_0(S)$ on the trivial bundle p^*F. Let $k_1(S)$ be the restriction of $k_0(S)$ to the subbundle $V(S)$. The flat connection d on the trivial bundle p^*F induces a Hermitian connection $\nabla_1(S)$ on the Hermitian subbundle $(V(S),k_1(S))$ by

$$\nabla_1(S) := Q_1(S) \circ d,$$

where $Q_1(S)$ denotes the orthogonal projection of p^*F onto $V(S)$. By a calculation similar to Teorem 2.2.3, the Hermitian connection $\nabla_1(S)$ is an Einstein-Hermitian connection on $(V(S),k_1(S))$. By the same argument as in (2.3), there exists an isomorphism $t_1(S) : (p^*V,p^*h_V) \simeq (V(S),k_1(S))$ of C^∞-Hermitian vector bundles. By $D_1(S)$, we denote the pull-back $t_1(S)^*\nabla_1(S)$ of $\nabla_1(S)$ for simplicity. Then $D_1(S)$ is also an Einstein-Hermitian connection on (p^*V,p^*h_V), and its equivalence class $[D_1(S)]$ modulo $\mathcal{G}(p^*V,p^*h_V)$ is independent of the choice of the isomorphism $t_1(S)$. We now define a mapping $\psi_1 : \widetilde{S} \to \mathcal{E}(p^*V,p^*h_V)$ by

$$\psi_1(\widetilde{S}) = [D_1(S)] \qquad S \in \mathcal{S}.$$

(3.1.2). Let $f_1(S)$ be the automorphism of p^*V defined by

$$f_1(S)(\xi) := (H(S)^{-1})^{1/2}\xi, \quad \xi \in p^*F.$$

Then $f_1(S)$ is an isomorphism between C^∞-Hermitian vector bundles $(V(S'), h(S'))$ and $(V(S), k_1(S))$ where $S' := (\overline{H(S)}^{-1})^{1/2}S)$. Obviously,

$$\nabla_1(S) = f_1(S) \circ \nabla(S') \circ f_1(S)^{-1}.$$

Hence $D(S')$ is equivalent to $D_1(S)$ modulo $\mathcal{G}(p^*V, p^*h_V)$. Note that the mapping:

$$\mathcal{S} \ni S \mapsto S' \in \mathcal{S}$$

is bijective. Thus ψ is injective if and only if so is ψ_1.

(3.2). We prepare the following lemma in linear algebra in order to give an explicit expression of the curvature $R_1(S)$ of $D_1(S)$.

Definition 3.2.1. *There exists a* $\mathbb{C}$*-basis* $\{e_1, \cdots, e_{2k}\}$ *for* $\mathbb{C}^{2k}$ *such that the Hermitian inner product* $H(S)$ *and the symplectic form* S *are respectively represented by the matrices* I *and* J *in terms of the basis, where*

$$I := \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & 0 \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = \sum_{i=1}^{2k} \overline{e_1}^* \otimes e_i^*,$$

$$J := \begin{pmatrix} 0 & 1 & & 0 & 0 \\ -1 & 0 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & 0 & 1 \\ 0 & 0 & & -1 & 0 \end{pmatrix} = \sum_{j=1}^{k} (e_{2j-1}^* \otimes e_{2j}^* - e_{2j}^* \otimes e_{2j-1}^*).$$

Such a $\mathbb{C}$*-basis is called a symplectic basis with respect to* S.

(3.2.2). Fix an $S \in \tilde{\mathcal{S}}$. Note that S induces a skew symmetric bilinear form fibrewise on the trivial bundle p^*F. Then $k_1(S)$ and the restriction of the symmetric bilinear form to $V(S)$ allow us to regard $V(S)$ as a vector bundle with $\mathrm{Sp}(m)$-structure. Take a point $[z] \in \mathbb{P}^{2m+1}(\mathbb{C})$. Then we choose a $\mathbb{C}$-basis $\{a_1, a_2, \dots, a_{2m+2}\}$ for $\mathbb{C}^{2m+2}$, which is symplectic with respect to the symplectic structure S, such that the fibre $V(S)_{[z]}$ of $V(S)$ at $[z]$ is generated by the flat sections corresponding to

$a_1, a_2, \ldots, a_{2m}$ over $\mathbf{C}$. Obviously, the connection $\nabla_1(S)$ is $\mathrm{Sp}(m)$-invariant. We shall now show that $\nabla_1(S)$ is irreducible. The curvature $R_1(S)$ of $\nabla_1(S)$ is written in the form

$$({}^t\overline{z}H(S)z)^{-1}UB(U^{-1}dz \wedge {}^td\overline{z}{}^t\overline{U}^{-1} + J\overline{U}^{-1}d\overline{z} \wedge {}^tdz{}^tU^{-1}\,{}^tJ)BU^{-1},$$

at $[z] \in \mathbf{P}^{2m+1}(\mathbf{C})$, where $B = \sum_{i=1}^{2m} e_i \otimes e_i^*$ and U denote the square matrix of degree $2m+2$ whose i-th column vector is a_i for each i. Hence the holonomy group of $\nabla_1(S)$ is exactly $\mathrm{Sp}(m)$. Thus $\nabla_1(S)$ is irreducible.

Theorem 3.2.3. *The mapping $\psi_1 : \mathcal{S} \to \mathcal{E}'(p^*V, p^*h_V)$ is injective, i.e., if $[D_1(S_1)] = [D_1(S_2)]$ for $S_1, S_2 \in \tilde{\mathcal{S}}$, then there exists an element $c \in \mathbf{C}^*$ such that $S_1 = cS_2$.*

Proof. Assume $[D_1(S_1)] = [D_1(S_2)]$. We have an isomorphism $g : (V(S_1), k_1(S_1)) \simeq (V(S_2), k_1(S_2))$ such that $g\,\nabla_1(S_1)\,g^{-1} = \nabla_1(S_2)$. The proof is divided into three steps.

Step 1. Let $[z] \in \mathbf{P}^{2m+1}(\mathbf{C})$ be arbitrary. Then there exists a $\mathbf{C}$-basis $\{e_1, \cdots, e_{2m+2}\}$ for $\mathbf{C}^{2m+2}$, which is symplectic with respect to the symplectic structure S_1, such that $V(S_1)_{[z]}$ is generated by the flat sections $a_1, a_2, \ldots, a_{2m}$ over $\mathbf{C}$. Since the normalizer of $\mathrm{Sp}(m)$ in $\mathrm{U}(2m)$ is $\mathrm{U}(1)\cdot\mathrm{Sp}(m)$, we have an element $c \in \mathbf{C}^*$ such that $\{cg(e_1), \cdots, cg(e_{2m})\}$ is a symplectic $\mathbf{C}$-basis for $V(S_2)_{[z]}$ with respect to the symplectic structure induced by S_2. Hence there exist vectors $f_{2m+1}, f_{2m+2} \in \mathbf{C}^{2m+2}$ such that $\{cg(e_1), \cdots, cg(e_{2m}), f_{2m+1}, f_{2m+2}\}$ is a symplectic $\mathbf{C}$-basis for $\mathbf{C}^{2m+2}$ with respect to S_2. Let $H_i := H(S_i)$, $i = 1, 2$ and let $U_1 = (e_1, \cdots, e_{2m+2})$ be the square matrix, of degree 2m+2, whose i-th column vector is e_i. Moreover, put $U_2 = (cg(e_1), \cdots, cg(e_{2m}), f_{2m+1}, f_{2m+2})$. We then obtain :

$$\text{(a)} \qquad \begin{aligned} &({}^t\overline{z}H_1z)^{-1}U_1BK_1BU_1^{-1} \\ &= ({}^t\overline{z}H_2z)^{-1}g^{-1}U_2BK_2BU_2^{-1}g, \end{aligned}$$

on $V(S_1)_{[z]}$, where

$$K_i = B(U_i^{-1}dz \wedge {}^td\overline{z}{}^t\overline{U_i}^{-1} + J\overline{U_i}^{-1}d\overline{z} \wedge {}^tdz{}^tU_i^{-1}\,{}^tJ)B,$$

for $i = 1, 2$. From our definition of U_1 and U_2, we have :

$$g^{-1}U_2B = c^{-1}U_1B.$$

This together with (a) yields

$$({}^t\overline{z}H_2z)({}^t\overline{z}H_1z)^{-1}K_1 = K_2.$$

Therefore, by setting $T := U_1^{-1}dz \wedge {}^t d\overline{z}({}^t\overline{U_1})^{-1}$ and $C := U_2^{-1}U_1$, we obtain :

$$\text{(b)} \qquad ({}^t\overline{z}H_2z)({}^t\overline{z}H_1z)^{-1}B(T + J\overline{T}J)B = B\{CT{}^t\overline{C} + (\overline{JCJ})J\overline{T}{}^tJ{}^t(JCJ)\}B.$$

Step 2. Put $E_{ij} = e_i \otimes e_j^* - e_j \otimes e_i^* \quad (i \neq j)$ and

$$E_{ii} = e_i \otimes e_i^* \quad .$$

Write the matrix C as (c_{ij}). Then there exist a (1,0)-vector $v_1 \in T_{[z]}\mathbf{P}^{2m+1}(\mathbf{C})$ such that

$$T(v_1, \overline{v_1}) = E_{11}.$$

Hence the identity (b) implies

$$|c_{11}|^2 + |c_{21}|^2 = 1, \quad c_{i1} = 0 \ \ (3 \leqq i \leqq 2m).$$

Similarly, we have $v_2 \in T_{[z]}\mathbf{P}^{2m+1}(\mathbf{C})$ such that $T(v_2, \overline{v_2}) = E_{22}$. It then follows that

$$|c_{12}|^2 + |c_{22}|^2 = 1, \quad c_{i2} = 0 \ \ (3 \leqq i \leqq 2m).$$

Inductively, we obtain

$$\begin{aligned} &|c_{2s-1,2s-1}|^2 + |c_{2s,2s-1}|^2 = 1, \\ &|c_{2s-1,2s}|^2 + |c_{2s,2s}|^2 = 1, \\ &c_{2s-1,j} = c_{2s,j} = 0 \qquad (j \neq 2s-1, 2s) \ , \end{aligned}$$

for all s with $1 \leqq s \leqq m$. For suitable v', $v'' \in \mathrm{T}_{[z]}(\mathbf{P}^{2m+1}(\mathbf{C})$ corresponding to the following four values of $T(v', v'')$,

$$T(v', v'') = E_{12}, \sqrt{-1}E_{12}, \sqrt{-1}E_{11}, \sqrt{-1}E_{22}$$

we contract the equality (b) by $v \wedge v$, We then have

$$a_{21} = a_{12} = 0$$

and there is a $\theta \in \mathbf{R}$ such that

$$a_{11} = a_{22} = e^{i\theta}.$$

Similarly, taking $T(v', v'')$ to be either $E_{2j-1,2j}$,$\sqrt{-1}E_{2j-1,2j}$,$\sqrt{-1}E_{2j,2j}$ or$\sqrt{-1}E_{2j-1,2j-1}$ we have

$$a_{2j-1,2j} = a_{2j,2j-1} = 0 \ \ (2 \leqq j \leqq m)$$

and $\theta_j \in \mathbf{R}$ $(2 \leqq j \leqq m)$ such that

$$a_{2j-1,2j-1} = a_{2j,2j} = e^{i\theta_j}.$$

Furthermore, let $T(v', v'')$ be either $E_{2i,2j-1}$ (i $\neq$ j) or $E_{k,2m+1}$ ($1 \leqq k \leqq 2m-1$). Then the identities

$$\theta_1 = \cdots = \theta_m \quad .$$

and

$$a_{i,2m+1} = a_{i,2m+2} = 0 \quad (1 \leqq i \leqq 2m).$$

follow. Hence we obtain :

$$C = \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & \vdots & \vdots \\ 0 & 0 & e^{i\theta} & 0 & 0 \\ a_{2m+1,1} & \cdots & a_{2m+1,2m} & a_{2m+1,2m+1} & a_{2m+1,2m+2} \\ a_{2m+2,1} & \cdots & a_{2m+2,2m} & a_{2m+2,2m+1} & a_{2m+2,2m+2} \end{pmatrix}.$$

Step 3. Since ${}^t\overline{U_2}H_2U_2 = I$, the matrix ${}^t\overline{C}$ is just ${}^t\overline{U_1}H_2U_2$. Thus,

(c) $$H_2(f_1, \cdots, f_{2m}) = e^{\sqrt{-1}\theta} H_1(e_1, \cdots, e_{2m}) \quad (1 \leqq j \leqq 2m).$$

Since $\{e_1, \cdots, e_{2m}\}$ is a unitary basis for $\mathbf{C}^{2m}$ with respect to the Hermitian inner product H_1, the (i,j)-entry $(H_1)_{ij}$ is given by

$$(H_1)_{ij} = ({}^t\overline{H_1^{-1}H_2 f_i}) H_1 (H_1^{-1} H_2 f_j) = \delta_{ij},$$

i.e., when restricted to the subspace $\sum_{i=1}^{2m} f_i$, the Hermitian inner products associated with $H_2 H_1^{-1} H_2$ and H_2 coincide on the space. Changing $[z] \in \mathbf{P}^{2m+1}(\mathbf{C})$ is arbitrarily, we have $H_2 H_1^{-1} H_2 = H_2$ on $\mathbf{C}^{2m+2}$. Hence $H_2 = H_1$. Now by (c),

$$f_j = e^{-i\theta} e_j \ (1 \leqq j \leqq 2m),$$

and we have $V(S_2)_{[z]} = V(S_1)_{[z]}$. Now, since $(V(S_2)_{[z]})^{\perp} = \mathbf{C}z + \mathbf{C}\overline{S_2 z}$, and $(V(S_2)_{[z]})^{\perp} = \mathbf{C}z + \mathbf{C}\overline{S_1 z}$, there exists a holomorphic functions $c(z)$ on $\mathbf{C}^{2m+2}$ - $\{0\}$ such that $S_1 = c(z)S_2 z$ for all $z \in \mathbf{C}^{2m+2}$. By Hartogs' Theorem, we can extend $c(z)$ to a holomorphic function on $\mathbf{C}^{2m+2}$. Using the Taylor expansion of $c(z)$ at $z = 0$, we see that $c(z)$ is constant on $\mathbf{C}^{2m+2}$. Thus we obtain the composite c such that $S = cJ$ for constant c, as required. Q.E.D.

(3.3). In (2.4.3), we have $p^* \circ \varphi$ coincides with $(\psi)_{\mathbb{R}}$. Hence in view of (3.2.2), we have

Corollary 3.3.1. *The mapping φ is injective, and so the image of φ is* $\mathrm{SL}(m+1,\mathbf{H})/\mathrm{Sp}(m+1)$.

§4. The moduli space of B_2-connections on (V, h_V)

The moduli space $\mathcal{B}''(V, h_V)$ is written as a union of connected components $\mathcal{B}_i(V, h_V)$:

$$\mathcal{B}''(V, h_V) = \bigcup_{i \in I} \mathcal{B}_i(V, h_V).$$

By $\mathcal{B}_1(V, h_V)$, we denote the component containing the image of ϕ. Using the same method as in [A-H-S] and [F], we shall examine $\mathcal{B}_1(V, h_V)$.

Theorem 4.1.1. *$\mathcal{B}_1(V, h_V)$ is nothing but the image of ϕ, i.e., $\mathcal{B}_1(V, h_V)$ is diffeomorphic to* $\mathrm{SL}(m+1,\mathbf{H})/\mathrm{Sp}(m+1)$.

To prove Theorem 4.1.1, we compute the dimension of $\mathcal{B}_1(V, h_V)$. By Borel-Weil-Kostant-Bott's theorem (cf. [Mu]) we shall show the following:

Lemma 4.1.2. *The real dimension of $\mathcal{B}_1(V, h_V)$ is $m(2m+3)$* ($= \dim_{\mathbb{R}} \mathrm{SL}(m+1,\mathbf{H})/\mathrm{Sp}(m+1)$).

Proof. By [Ni2], $\mathcal{B}_1(V, h_V)$ is $\dim_{\mathbb{C}} \mathrm{H}^1(\mathbb{P}^{2m+1}(\mathbb{C}), A_D)$, where D denotes the Einstein-Hermitian connection $\nabla(I)$ on (p^*V, p^*h_V). Since the vector bundle p^*V is homogeneous, and since $\mathbb{P}^{2m+1}(\mathbb{C}) = \mathrm{Sp}(m+1)/\mathrm{Sp}(m) \times \mathrm{U}(1)$, we can write the vector bundle $\mathrm{End}(p^*V)$ as $\mathrm{Sp}(m+1) \times_{(\rho\otimes\rho^*)} \mathfrak{gl}(2m, \mathbb{C})$, where ρ is the unitary representation of $\mathrm{Sp}(m) \times \mathrm{U}(1)$ on $\mathbb{C}^{2m}$ defined by

$$\rho : \mathrm{Sp}(m) \times \mathrm{U}(1) \ni (a, b) \mapsto \rho(a, b) := a \in \mathrm{Sp}(m) \subset \mathrm{U}(2m).$$

The representation $\rho \otimes \rho^*$ is equivalent to $\rho^* \otimes \rho^*$ and is expressible as a direct sum $\mathbb{C}\omega_{\mathbb{C}^{2m}} \oplus \wedge_0^2 \rho^* \oplus \mathrm{S}^2 \rho^*$ of irreducible representations $\mathbb{C}\omega_{\mathbb{C}^{2m}}$, $\wedge_0^2 \rho^*$ and $\mathrm{S}^2 \rho^*$, where $\omega_{\mathbb{C}^{2m}}$ is such that

$$\omega_{\mathbb{C}^{2m}}(a, b)(\xi, \zeta) := {}^t\xi J \zeta, \quad \xi, \zeta \in \mathbb{C}^{2m}.$$

Recall that $\wedge_0^2 \rho^* := (\mathbb{C}\omega_{\mathbb{C}^{2m}})^\perp \cap \wedge^2 \rho^*$ and that $\mathrm{S}^2 \rho^*$ is the symmetric part of $\rho^* \otimes \rho^*$. Now, the vector bundle is written as a direct sum $L_1 \oplus L_2$

$\oplus$ L_3 of homogeneous vector bundles L_1, L_2, L_3 corresponding to representations $\mathbf{C}\omega_{\mathbf{C}^{2m}}$, $\wedge_0^2\rho^*$, $S^2\rho^*$, respectively. Hence the complex A_{D_0} is decomposed into three components $A(L_1)$, $A(L_2)$, $A(L_3)$. Applying Borel-Weil-Kostant-Bott's theorem to $A(L_i)$ (i=1,2,3), we obtain

$$\dim_{\mathbf{C}} \mathrm{H}^1(A(L_i)) = 0 \qquad (i = 1, 3),$$
$$\dim_{\mathbf{C}} \mathrm{H}^1(A(L_2)) = (2m + 3)m.$$

Summing these up, we have $\dim_{\mathbf{C}} \mathrm{H}^1(A_{D_0}) = $ (2m+3)m, as required.
Q.E.D.

(4.1.3). By using Lemma 4.1.2, we prove Theorem 4.1.1. Consider the frame bundle P of unitary bases. Let $M(2m+2, 2m; \mathbf{C})$ be the set of $(2m + 2, 2m)$-matrices. Then P is naturally regarded as a submanifold of $M(2m + 2, 2m, \mathbf{C})$ as folows:

Let $(u) \in \mathbf{P}^m(\mathbf{H})$ and let $(f_1, \cdots, f_{2m})$ be a unitary basis for $(V_{(u)}, (h_V)_{(u)})$. Now, the Lie group $\mathrm{SL}(m + 1, \mathbf{H})$ acts on P by

$$\nu : \mathrm{SL}(m + 1, \mathbf{H}) \times P \ni (g, B) \mapsto gB({}^t\overline{gB}gB)^{-1/2} \in P.$$

Let η be the action of $\mathrm{SL}(m + 1, \mathbf{H})$ on $\mathbf{P}^m(\mathbf{H})$ such that

$$\eta : \mathrm{SL}(m + 1, \mathbf{H}) \times \mathbf{P}^m(\mathbf{H}) \ni (g, (u)) \mapsto ({}^t\overline{g}^{-1}u) \in \mathbf{P}^m(\mathbf{H}).$$

In terms of these actions, the natural projection of P onto $\mathbf{P}^m(\mathbf{H})$ is equivalent. The vector bundle $\wedge^i \mathrm{T}^* \mathbf{P}^m(\mathbf{H})$ splits into a direct sum $A_i \oplus B_i$ in such a way that A_i and B_i are holonomy invariant vector subbundles (cf. [Ni1;(3.1)]). Since the decomposition $\wedge^i \mathrm{T}^* \mathbf{P}^m(\mathbf{H}) = A_i \oplus B_i$ $(1 \leqq i \leqq 2m)$ depends only on the $\mathrm{GL}(m, \mathbf{H}) \cdot \mathrm{GL}(1, \mathbf{H})$-structure of the tangent bundle of $\mathbf{P}^m(\mathbf{H})$, the action ν induces the one of $\mathrm{SL}(m+1, \mathbf{H})$ on $\mathcal{B}_1(V, h_V)$. By an argument similar to [A-H-S;Section 9] and [F; Section 2], the isotropy subgroup of $\mathrm{SL}(m + 1, \mathbf{H})$ is compact. Since $\mathrm{Sp}(m + 1)$ is a maximal compact subgroup of $\mathrm{SL}(m+1, \mathbf{H})$ and $\dim_{\mathbf{R}}(\mathcal{B}_1(V, h_V)) =$ (2m+1)m (Lemma (4.1)), the isotropy subgroup is equal to $\mathrm{Sp}(m + 1)$. Hence $\mathcal{B}_1(V, h_V) = \mathrm{SL}(m + 1, \mathbf{H})/\,\mathrm{Sp}(m + 1)$ and it coincides with the image of ϕ, as required.

(4.2). Let N be a holomorphic vector bundle of rank $2m$ over $\mathbf{P}^{2m+1}(\mathbf{C})$. Recall that N is a null-correlation bundle if there exists a following exact sequence :

$$0 \to N \to T \otimes H^{-1} \to H \to 0,$$

where T, H are respectively the holomorphic tangent bundle and the hyperplane bundle over $\mathbf{P}^{2m+1}(\mathbf{C})$. By $\mathcal{N}$ we denote the set of null-correlation bundles over $\mathbf{P}^{2m+1}(\mathbf{C})$. Then we obtain :

Proposition 4.2.1. *We have a natural bijection of $\mathcal{N}$ onto the image of ψ.*

Proof. Given $S \in \mathfrak{S}$, we denote by σ_S the holomorphic section to $H^2 \otimes T^*$ defined by

$$\sigma_S([z]) = {}^t zSz, \quad [z] \in \mathbf{P}^{2m+1}(\mathbf{C}).$$

Then the mapping $\mathfrak{S} \ni S \mapsto \sigma_S \in \mathrm{H}^0(\mathbf{P}^{2m+1}(\mathbf{C}), H^2 \otimes T^*)$ is bijective. Restricting to $\mathcal{S}$, we have the parametrization of $\mathcal{N} = \{N_{[S]}; [S] \in \mathfrak{S}\}$ by $\mathcal{S}$. Endow the tangent bundle T of $\mathbf{P}^{2m+1}(\mathbf{C})$ with the Fubini-Study metric. Since the natural (1,0)-connection on the holomorphic subbundle $N_{[S]}$ of $T \otimes H^{-1}$ is obtained from the dual bundle $(V(S), \nabla(S))^*$, we obtain the bijections

$$\mathcal{N} \approx \mathcal{S} \approx Image\psi, \quad N_{[S]} \leftrightarrow [S] \leftrightarrow (V(S), \nabla(S))$$

as required. Q.E.D.

§5. Compactification of $\psi(\mathcal{S})$

In this section, we give a certain type of compactification of $\tilde{\mathcal{S}}$, by which we study the ends of the family of Einstein-Hermitian connections constructed in Section 2.

(5.1.1). Let $\mathfrak{S}_k$ be the subset of $\mathfrak{S}$ defined by

$$\mathfrak{S}_k := \{S \in \mathfrak{S}; \mathrm{rank}_{\mathbf{C}}\, S = 2k\}.$$

Then $\mathfrak{S}_{m+1}$ is nothing but $\mathcal{S}$ and $\mathfrak{S}$ is represented as a union of $\mathfrak{S}_k$'s, $1 \leqq k \leqq m+1$. Each $\mathfrak{S}_k$ is isomorphic to the complex homogeneous manifold $\mathrm{GL}(2m+2, \mathbf{C}) / \mathrm{G_k}$ where

$$\mathrm{G_k} = \{ \begin{pmatrix} C & 0 \\ D & E \end{pmatrix} \in \mathrm{GL}(2(m+1), \mathbf{C}); C \in \mathrm{Sp}(k, \mathbf{C}) \}.$$

(5.1.2). Note that $\tilde{\mathfrak{S}}$ is a complex projective space of complex dimension m(2m+3). Since $\mathfrak{S}$ is a union of $\mathfrak{S}_k$s,

$$\tilde{\mathfrak{S}} = \bigcup_{1 \leqq k \leqq m+1} \tilde{\mathfrak{S}}_k,$$

by setting $\widetilde{\mathfrak{S}}_k = \mathfrak{S}_k/\mathbb{C}^*$. Obviously, we have $\widetilde{\mathfrak{S}}_k \cong \mathrm{PGL}(2m+2;\mathbb{C})/\widetilde{\mathrm{G}}_k$, where

$$\widetilde{\mathrm{G}}_k = \{\begin{pmatrix} \widetilde{C} & 0 \\ \widetilde{D} & \widetilde{E} \end{pmatrix} \in \mathrm{PGL}(2(m+1),\mathbb{C}); \widetilde{C} \in \mathrm{PSp}(k,\mathbb{C})\}.$$

Since $\widetilde{\mathfrak{S}}_{m+1}$ is just $\widetilde{\mathfrak{S}}$, the boundary of $\widetilde{S}$ in $\widetilde{\mathfrak{S}}$ is a union $\bigcup_{1\leqq k\leqq m} \widetilde{\mathfrak{S}}_k$.

(5.1.3). Let $\mathcal{L}(p^*V, p^*h_V)$ be the set of all Einstein-Hermitian connections on (p^*V, p^*h_V) possibly with singularities. Then we have an equivalence relation on $\mathcal{L}(p^*V, p^*h_V)$ as follows. For ∇_1, $\nabla_2 \in \mathcal{L}(p^*V, p^*h_V)$, we say that ∇_1 is equivalent to ∇_2 if (1) the singular sets for ∇_1 and ∇_2 coincide, and (2) there exists a unitary gauge transformation $t \in \mathcal{G}(p^*V, p^*h_V)$ such that $t\ \nabla_1\ t^{-1} = \nabla_2$ outside the singularities. We denote the equivalence class of ∇ by $[\nabla]$ and the set of all equivalence classes

$$\{[\nabla] : \nabla \in \mathcal{L}(p^*V, p^*h_V)\} = \mathcal{L}(p^*V, p^*h_V)/\mathcal{G}(p^*V, p^*h_V)$$

by $\widetilde{\mathcal{E}}(p^*V, p^*h_V)$. We shall now study the Einstein-Hermitian connections corresponding to the boundary of $\widetilde{S}$ in $\widetilde{\mathfrak{S}}$. Let $\widetilde{S} \in \widetilde{\mathfrak{S}} - \widetilde{S}$. Then, we can define $V(S)$, $h(S)$ and $\nabla(S)$ for $\widetilde{S} \in \widetilde{\mathfrak{S}}$ by the method similar to (2.2). Moreover, we put

$$F(S) = \{[z] \in P^{2m+1}(\mathbb{C}); Sz = 0\}.$$

Then, outside $F(S)$, the vector bundle $V(S)$ has a natural holomorphic structure such that $\nabla(S)$ is an Einstein-Hermitian connection on $(V(S), h(S))$. Since $\widetilde{S}$ is open-dense in $\widetilde{\mathfrak{S}}$, there exists a sequence $\{\widetilde{S}_i\}$ in $\widetilde{S}$ converging to $\widetilde{S}$. For the corresponding sequence $\{D(S_i)\}$, we have unitary gauge transformations g_i such that $\{g_i D(S_i) g_i^{-1}\}$ converges to $D(S) \in \mathcal{L}(p^*V, p^*h_V)$ with respect to C^∞-topology on every compact subset of $P^{2m+1}(\mathbb{C})$ - $F(S)$.

(5.1.4). We now have C^∞ bundle isomorphism $t : (p^*V, p^*h_V) \to (V(S), h(S))$ outside $F(S)$, such that

$$tD(S)t^{-1} = \nabla(S).$$

The gauge equivalence class $[D(S)]$ depends only on $\widetilde{S}$. Furthermore, there is an element $K \in \mathrm{PGL}(2m+2, \mathbb{C})$ such that $\widetilde{S}$ is written as

${}^tK\tilde{J}_jK$. Hence the set $F(S)$ is $K^{-1}F(J_j)$, which is a space of complex dimension $2m+1-2j$. Hence we obtain the mapping

$$\widetilde{\psi} : \widetilde{\mathfrak{S}} \ni \widetilde{S} \to [D(S)] \in \widetilde{\mathcal{E}}(p^*V, p^*h_V).$$

Obviously, $\widetilde{\mathfrak{S}}$ is compact and therefore the image of $\widetilde{\psi}$ is a compactification of $\psi(\mathcal{S}) \approx \mathcal{N}$.

(5.2). The space $\tilde{\mathcal{E}}(p^*V, p^*h_V)$ carries the real structure

$$\widetilde{\tau} : \tilde{\mathcal{E}}(p^*V, p^*h_V) \ni [D] \mapsto \widetilde{\tau}([D]) := [\tau^{\sharp} \circ D \circ \tau^{\sharp}] \in \tilde{\mathcal{E}}(p^*V, p^*h_V),$$

which is a natural extension of the real structure τ' on $\mathcal{E}'(p^*V, p^*h_V)$. By calculation, $\tilde{\psi}$ is compatible with the real structures $j_{\mathfrak{S}}$ (cf. (2.4.1)) and $\tilde{\tau}$. Hence $\tilde{\psi}$ restricts to the real points

$$(\widetilde{\psi})_{\mathbf{R}} : \widetilde{\mathfrak{S}}_{\mathbf{R}} \to \tilde{\mathcal{E}}(p^*V, p^*h_V)_{\mathbf{R}}.$$

Since we have a natural identification of $\widetilde{\mathfrak{S}}_{\mathbf{R}}$ with

$$\{\text{positive semi-definite quaternionic Hermitian matrices}\}/\mathbf{R}^*,$$

the image of $(\tilde{\psi})_{\mathbf{R}}$ gives us a compactification of ψ.

Added in Proof. After the completion of this paper, the auther received a preprint by H.Doi and T.Okai entitled "1-instantons on $\mathbf{H}P^n$", which gives a result slightly stronger that Theorem 4.1.1 .

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