The Nicolas and Robin inequalities with sums of two squares

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Abstract

In 1984, G. Robin proved that the Riemann hypothesis is true if and only if the *Robin inequality* $\sigma(n) < e^\gamma n \log\log n$ holds for every integer $n > 5040$, where $\sigma(n)$ is the sum of divisors function, and γ is the Euler-Mascheroni constant. We exhibit a broad class of subsets $\mathcal{S}$ of the natural numbers such that the Robin inequality holds for all but finitely many $n \in \mathcal{S}$. As a special case, we determine the finitely many numbers of the form $n = a^2 + b^2$ that do not satisfy the Robin inequality. In fact, we prove our assertions with the *Nicolas inequality* $n/\varphi(n) < e^\gamma \log\log n$; since $\sigma(n)/n < n/\varphi(n)$ for $n > 1$ our results for the Robin inequality follow at once.

1 Introduction

Let $\varphi(n)$ denote the *Euler function.* In 1903 Landau (see [4, pp. 217–219]) showed that

$$\varlimsup_{n\to\infty} \frac{n}{\varphi(n)\log\log n} = e^\gamma, \tag{1}$$

where γ is the *Euler-Macheroni constant.* Eighty years later, in a highly interesting work, Nicolas [5] proved that the inequality

$$\frac{n}{\varphi(n)} > e^\gamma \log\log n$$

holds for infinitely many natural numbers n. Moreover, if N_k denotes the product of the first k primes, he proved that

$$\frac{N_k}{\varphi(N_k)} > e^\gamma \log\log N_k$$

holds for every $k \geqslant 1$ on the *Riemann hypothesis* (RH). Assuming RH is false, he also showed there are both infinitely many k for which this inequality holds and infinitely many k for which it does not hold. To acknowledge the many contributions of Nicolas to this subject, we denote by $\mathcal{N}$ the set of numbers $n \in \mathbb{N}$ that satisfy the *Nicolas inequality*:

$$\frac{n}{\varphi(n)} < e^\gamma \log\log n. \tag{2}$$

The principle aim of this paper is to exhibit a broad class of infinite subsets $\mathcal{S} \subset \mathbb{N}$ such that this inequality holds for all but finitely many $n \in \mathcal{S}$. This class includes a set that contains all natural numbers which can be expressed as a sum of two squares.

Let $\sigma(n)$ be the *sum of divisors function*. The analogue of (1) for this function was obtained by Gronwall [2], who proved that

$$\overline{\lim_{n\to\infty}} \frac{\sigma(n)}{n\log\log n} = e^\gamma.$$

Robin [7] showed that if RH is true, then the *Robin inequality*:

$$\frac{\sigma(n)}{n} < e^\gamma \log\log n \tag{3}$$

holds for every integer $n > 5040$, whereas if RH is false, then this inequality fails for infinitely many n. We denote by $\mathcal{R}$ the set of numbers $n \in \mathbb{N}$ that satisfy (3). In view of the elementary inequality

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} \qquad (n > 1),$$

it is clear that $\mathcal{N} \subset \mathcal{R}$. Thus, for the class of subsets $\mathcal{S} \subset \mathbb{N}$ considered in the present paper, the Robin inequality holds for all but finitely many $n \in \mathcal{S}$.

Our work was originally inspired by a recent paper of Choie *et al* [1], which establishes the inclusion in $\mathcal{R}$ of various infinite subsets of the natural numbers $\mathbb{N}$. In particular, in [1] it is shown that $\mathcal{R}$ contains every square-free number $n > 30$, every odd integer $n > 9$, every powerful number $n > 36$, and every integer $n > 1$ not divisible by the fifth power of some prime. As a consequence it follows that the RH holds iff the Robin inequality holds for all natural numbers n divisible by the fifth power of some prime. Note that this criterion does not have the restriction $n \geqslant 5041$. Another "5041-free" criterion was given earlier by Lagarias [3], who showed that RH is true iff

$$\sigma(n) \leqslant H_n + e^{H_n}\log H_n,$$

where

$$H_n = \sum_{j\leqslant n} \frac{1}{j} \qquad (n \geqslant 1).$$

To state our results more precisely, let $\mathbb{P}$ denote the set of prime numbers, and for any subset $\mathcal{A} \subset \mathbb{P}$, put

$$\pi_{\mathcal{A}}(x) = \#\{p \leqslant x \ : \ p \in \mathcal{A}\}$$

Let $\mathcal{P}$ be an arbitrary (fixed) subset of $\mathbb{P}$ such that

$$\overline{\delta} = \varlimsup_{x\to\infty} \frac{\pi_{\mathcal{P}}(x)}{\pi(x)} < 1 \qquad \text{and} \qquad \underline{\delta} = \varliminf_{x\to\infty} \frac{\pi_{\mathcal{P}}(x)}{\pi(x)} > 0, \tag{4}$$

where $\pi(x) = \#\{p \leqslant x\}$ as usual. Let $\mathcal{Q}$ denote the complementary set of primes (i.e., $\mathcal{Q} = \mathbb{P} \setminus \mathcal{P}$), and note that

$$\varlimsup_{x\to\infty} \frac{\pi_{\mathcal{Q}}(x)}{\pi(x)} = 1 - \underline{\delta} < 1 \qquad \text{and} \qquad \varliminf_{x\to\infty} \frac{\pi_{\mathcal{Q}}(x)}{\pi(x)} = 1 - \overline{\delta} > 0. \tag{5}$$

In this paper, we work with the set $\mathcal{S} = \mathcal{S}(\mathcal{P})$ defined by

$$\mathcal{S} = \big\{n \in \mathbb{N} \ : \ \text{if } p \in \mathcal{Q} \text{ and } p \mid n, \ \text{then } p^2 \mid n\big\}. \tag{6}$$

Our main result is the following:

Theorem 1. *The set $\mathcal{N}$ contains all but finitely many of the numbers in $\mathcal{S}$.*

Corollary 1. *Of the numbers n which do not satisfy the Nicolas inequality, all but finitely many are divisible by a prime $q \in \mathcal{Q}$ such that $q^2 \nmid n$.*

In particular, for any fixed $a, m \in \mathbb{N}$ with $\gcd(a, m) = 1$, one can put

$$\mathcal{P} = \big\{p \in \mathbb{P} \ : \ p \not\equiv a \pmod m\big\}$$

and apply Corollary 1 to deduce the following:

Corollary 2. *Of the numbers n which do not satisfy the Nicolas inequality, all but finitely many are divisible by a prime $q \equiv a \pmod m$ such that $q^2 \nmid n$.*

In Section 3 we examine more closely the special case that

$$\mathcal{P} = \big\{p \in \mathbb{P} \ : \ p \equiv 1 \pmod 4\big\} \cup \{2\}.$$

Note that the corresponding set $\mathcal{S}$ contains all natural numbers of the form $n = a^2 + b^2$ (since, by a theorem of Fermat, every prime $q \equiv 3 \pmod 4$ appears with even multiplicity in the prime factorization of n if and only if n can be written as a sum of two squares). Using effective bounds from [6] on the number of primes in arithmetic progressions modulo 4, we are able to determine the set $\mathcal{S} \setminus \mathcal{N}$ completely, leading to:

Theorem 2. *The set $\mathcal{S} \setminus \mathcal{N}$ contains precisely* 347 *natural numbers. In particular, there are precisely* 246 *numbers which can be expressed as a sum of two squares and such that the Nicolas inequality* (2) *does not hold, the largest of which is the number* 5250958134422281281O.

As an application, we obtain the unconditional result that

$$\{1, 2, 4, 5, 8, 9, 10, 16, 18, 20, 36, 72, 180, 360, 720\}$$

is a complete list of those natural numbers which can be expressed as a sum of two squares and such that the Robin inequality (3) does not hold; this result is consistent with the truth of the Riemann Hypothesis.

Results like those of Theorem 2 can be established for certain quadratic forms other than a^2+b^2. For example, using similar techniques one finds that there are precisely 261 numbers that can be expressed in the form $n = a^2+3b^2$ and for which the Nicolas inequality (2) does not hold, the largest of which is the number 397999936131188090700.

Throughout the paper, any implied constants in the symbols O, $\ll$, $\gg$ and $\asymp$ depend (at most) on the set $\mathcal{P}$ and are absolute otherwise. We recall that for positive functions f, g the notations $f = O(g)$, $f \ll g$ and $g \gg f$ are all equivalent to the assertion that $f \leqslant cg$ for some constant $c > 0$, and the notation $f \asymp g$ means that $f \ll g$ and $g \ll f$.

2 Proof of Theorem 1

For every natural number n we put

$$F(n) = \frac{n}{\varphi(n)} = \prod_{p \mid n} \frac{p}{p-1}\,.$$

Note that

$$F(n) = F(\kappa(n)) \qquad \text{and} \qquad \omega(n) = \omega(\kappa(n)), \tag{7}$$

where $\omega(n)$ is the number of distinct prime divisors of n, and $\kappa(n)$ is the square-free kernel of n:

$$\kappa(n) = \prod_{p \mid n} p.$$

Let

$$\mathcal{N}^\circ = \mathbb{N} \setminus \mathcal{N} = \big\{n \in \mathbb{N} \ : \ F(n) \geqslant e^\gamma \log\log n\big\},$$

and for every integer $k \geqslant 0$, let

$$\mathcal{V}_k = \{n \in \mathbb{N} \ : \ \omega(n) \geqslant k\} \qquad \text{and} \qquad \mathcal{W}_k = \mathcal{S} \cap \mathcal{N}^\circ \cap \mathcal{V}_k.$$

Since $\mathcal{V}_0 = \mathbb{N}$, Theorem 1 is the assertion that $\mathcal{W}_0 = \mathcal{S} \cap \mathcal{N}^\circ$ is a finite set. In view of the next lemma, it suffices to show that $\mathcal{W}_k = \varnothing$ for some k.

Lemma 1. *For every $k \geqslant 0$, $\mathcal{W}_0 \setminus \mathcal{W}_k$ is a finite set.*

Since $\omega(n) < k$ and $F(n) \geqslant e^\gamma \log\log n$ for all $n \in \mathcal{W}_0 \setminus \mathcal{W}_k$, Lemma 1 is an immediate consequence of the following:

Lemma 2. *For every constant $K > 0$, there are at most finitely many natural numbers n such that $\omega(n) \leqslant K$ and $F(n) \geqslant e^\gamma \log\log n$.*

Proof. If $\overline{p}_1, \overline{p}_2, \ldots$ is the sequence of consecutive prime numbers, then for any such number n we have

$$\prod_{j \leqslant K} \frac{\overline{p}_j}{\overline{p}_j - 1} \geqslant \prod_{p \,|\, n} \frac{p}{p-1} = F(n) \geqslant e^\gamma \log\log n;$$

this shows that n is bounded by a constant which depends only on K. $\square$

For every natural number n, let

$$s(n) = \Bigg(\prod_{\substack{p \,|\, n \\ p \in \mathcal{P}}} p\Bigg)\Bigg(\prod_{\substack{q \,|\, n \\ q \in \mathcal{Q}}} q^2\Bigg),$$

and put

$$\mathcal{Y} = \{n \in \mathbb{N} \ : \ n = s(n)\}.$$

Note that $\mathcal{Y} \subset \mathcal{S}$. The following statements are elementary:

($\mathscr{C}_1$) if $n = pm$ with $p \in \mathcal{P}$ and $p \nmid m$, then $n \in \mathcal{Y}$ if and only if $m \in \mathcal{Y}$;

($\mathscr{C}_2$) if $n = q^2 m$ with $q \in \mathcal{Q}$ and $q \nmid m$, then $n \in \mathcal{Y}$ if and only if $m \in \mathcal{Y}$;

($\mathscr{C}_3$) $s(n) \in \mathcal{S}$ for all n;

($\mathscr{C}_4$) $\kappa(s(n)) = \kappa(n)$ for all n;

($\mathscr{C}_5$) $s(n) \mid n$ for all $n \in \mathcal{S}$; in particular, $s(n) \leqslant n$.

Lemma 3. *If $\mathcal{W}_k \neq \varnothing$ and m_k is the least integer in $\mathcal{W}_k$, then $m_k \in \mathcal{Y}$.*

Proof. Clearly, $s(m_k) \in \mathcal{S}$ by $(\mathscr{C}_3)$. Combining $(\mathscr{C}_4)$ with (7) one sees that

$$F(s(n)) = F(n) \qquad \text{and} \qquad \omega(s(n)) = \omega(n) \qquad (n \in \mathbb{N}).$$

Then, using $(\mathscr{C}_5)$ it follows that

$$F(s(m_k)) = F(m_k) \geqslant e^\gamma \log\log m_k \geqslant e^\gamma \log\log s(m_k),$$

which shows that $s(m_k) \in \mathcal{N}^\circ$. Finally, $s(m_k) \in \mathcal{V}_k$ since

$$\omega(s(m_k)) = \omega(m_k) \geqslant k.$$

Thus, we have shown that $s(m_k) \in \mathcal{S} \cap \mathcal{N}^\circ \cap \mathcal{V}_k = \mathcal{W}_k$. Since m_k is the *least* integer in $\mathcal{W}_k$, the equality $m_k = s(m_k)$ follows from $(\mathscr{C}_5)$, hence $m_k \in \mathcal{Y}$. $\square$

Next, for every integer $k \geqslant 0$ let

$$\mathcal{Z}_k = \big\{n \in \mathbb{N} \ : \ \Omega(n) = k\big\} \qquad \text{and} \qquad \mathcal{T}_k = \mathcal{N}^\circ \cap \mathcal{Y} \cap \mathcal{Z}_k.$$

Here, $\Omega(n)$ is the number of prime divisors of n, counted with multiplicity. Using Lemma 3 one sees that if $\mathcal{W}_\ell \neq \varnothing$ and m_ℓ is the least integer in $\mathcal{W}_\ell$, then $m_\ell \in \mathcal{T}_k$ for some $k \geqslant \ell$; in particular,

$$\bigcup_{k \geqslant \ell} \mathcal{T}_k = \varnothing \quad \Longrightarrow \quad \mathcal{W}_\ell = \varnothing.$$

As we mentioned earlier, in order to prove Theorem 1 it suffices to show that $\mathcal{W}_\ell = \varnothing$ for some ℓ, hence it is enough to show that $\mathcal{T}_k \neq \varnothing$ for at most finitely many integers $k \geqslant 0$.

When $\mathcal{T}_k \neq \varnothing$ we shall use the following notation. Let n_k denote the least integer in $\mathcal{T}_k$. Let $\widehat{p}_k$ be the largest prime $p \in \mathcal{P}$ that divides n_k, and put $\widehat{p}_k = 1$ if no such prime exists. Similarly, let $\widehat{q}_k$ be the largest prime $q \in \mathcal{Q}$ that divides n_k, and set $\widehat{q}_k = 1$ if no such prime exists. Finally, let

$$P_k^+ = \max\{\widehat{p}_k, \widehat{q}_k\} \qquad \text{and} \qquad P_k^- = \min\{\widehat{p}_k, \widehat{q}_k\}. \tag{8}$$

Note that P_k^+ is the largest prime factor of n_k.

Lemma 4. *Suppose $\mathcal{T}_k \neq \varnothing$:*

(*i*) *if $p \in \mathcal{P}$ with $p < \widehat{p}_k$, then $p \mid n_k$;*

(*ii*) *if $q \in \mathcal{Q}$ with $q < \widehat{q}_k$, then $q \mid n_k$.*

Proof. Suppose on the contrary that $p \in \mathcal{P}$ with $p < \widehat{p}_k$ and $p \nmid n_k$. Since $n_k = s(n_k)$ we can write $n_k = \widehat{p}_k m$ with $\widehat{p}_k \nmid m$. Put $n^* = pm$. Since $n_k \in \mathcal{N}^\circ$, $F(p) > F(\widehat{p}_k)$, and $n^* < n_k$, it follows that

$$F(n^*) = F(p)\,F(m) > F(\widehat{p}_k)\,F(m) = F(n_k) \geqslant e^\gamma \log\log n_k > e^\gamma \log\log n^*,$$

where we have used the fact that F is multiplicative; this shows that $n^* \in \mathcal{N}^\circ$. As $n_k \in \mathcal{Y}$, ($\mathscr{C}_1$) implies that $n^* \in \mathcal{Y}$. Finally, since Ω is (completely) additive, we see that

$$\Omega(n^*) = \Omega(m) + 1 = \Omega(n_k) = k,$$

which shows that $n^* \in \mathcal{Z}_k$, and thus $n^* \in \mathcal{N}^\circ \cap \mathcal{Y} \cap \mathcal{Z}_k = \mathcal{T}_k$. But this is impossible since $n^* < n_k$ (the least number in $\mathcal{T}_k$), and this contradiction completes our proof of (*i*). Using ($\mathscr{C}_2$), the proof of (*ii*) is similar; we omit the details. □

Lemma 5. *Suppose that $\mathcal{T}_k \neq \varnothing$ and $\widehat{p}_k < \widehat{q}_k$. Then there is at most one prime $p \in \mathcal{P}$ such that $\widehat{p}_k < p < \widehat{q}_k$.*

Proof. Suppose on the contrary that there are two primes $p_1, p_2 \in \mathcal{P}$ such that $\widehat{p}_k < p_1 < p_2 < \widehat{q}_k$. Since $n_k = s(n_k)$ we can write $n_k = \widehat{q}_k^{\,2} m$, and it is clear that $\gcd(m, p_1 p_2 \widehat{q}_k) = 1$. Put $n^* = p_1 p_2 m$. Since $n_k \in \mathcal{N}^\circ$, $F(p_1 p_2) > F(\widehat{q}_k^{\,2})$, and $n^* < n_k$, we have

$$F(n^*) = F(p_1 p_2)\,F(m) > F(\widehat{q}_k^{\,2})\,F(m) = F(n_k) \geqslant e^\gamma \log\log n_k > e^\gamma \log\log n^*,$$

which shows that $n^* \in \mathcal{N}^\circ$. As $n_k \in \mathcal{Y}$, ($\mathscr{C}_1$) implies that $n^* \in \mathcal{Y}$. Finally, since

$$\Omega(n^*) = \Omega(m) + 2 = \Omega(n_k) = k,$$

we see that $n^* \in \mathcal{Z}_k$, and thus $n^* \in \mathcal{N}^\circ \cap \mathcal{Y} \cap \mathcal{Z}_k = \mathcal{T}_k$. But this is impossible since $n^* < n_k$, and this contradiction implies the result. □

Lemma 6. *Suppose that $\mathcal{T}_k \neq \varnothing$ and $\widehat{p}_k > \widehat{q}_k$. Let p be the largest prime in $\mathcal{P}$ that is less than $\widehat{p}_k$, and let q be the smallest prime in $\mathcal{Q}$ that is greater than $\widehat{q}_k$. Then $q > p/2$.*

Proof. Suppose on the contrary that $q \leqslant p/2$. Since $n_k = s(n_k)$ and $p \mid n_k$ (by Lemma 4) but $q \nmid n_k$ (since $q > \widehat{q}_k$), we can write $n_k = p\widehat{p}_k m$, where $\gcd(m, p\widehat{p}_k q) = 1$. Put $n^* = q^2 m$. As in the proofs of Lemmas 4 and 5, we see that $n^* \in \mathcal{Y} \cap \mathcal{Z}_k$. Since $p < \widehat{p}_k$ and $q \leqslant p/2$, we have

$$F(p\widehat{p}_k) = \frac{p\widehat{p}_k}{(p-1)(\widehat{p}_k - 1)} < \frac{p^2}{(p-1)^2} < \frac{q}{q-1} = F(q^2);$$

therefore,

$$F(n^*) = F(q^2)\,F(m) > F(p\widehat{p}_k)\,F(m) = F(n_k) \geqslant e^\gamma \log\log n_k > e^\gamma \log\log n^*,$$

which shows that $n^* \in \mathcal{N}^\circ$. Thus, $n^* \in \mathcal{N}^\circ \cap \mathcal{Y} \cap \mathcal{Z}_k = \mathcal{T}_k$. But this is impossible since $n^* < n_k$, and this contradiction implies the result. $\square$

As mentioned above, in order to prove Theorem 1 it suffices to show that $\mathcal{T}_k \neq \varnothing$ for at most finitely many integers $k \geqslant 0$. Arguing by contradiction, we shall assume that the set

$$\mathcal{K} = \{k \geqslant 0 \ : \ \mathcal{T}_k \neq \varnothing\}$$

has infinitely many elements.

Since $\Omega(n_k) = k$, we see that $n_k \to \infty$ as $k \to \infty$ with $k \in \mathcal{K}$; using Lemma 2 it follows that $\omega(n_k) \to \infty$ as well, and therefore $P_k^+ \to \infty$.

We claim that

$$\widehat{p}_k \asymp \widehat{q}_k \qquad (k \in \mathcal{K}), \tag{9}$$

which by (8) is equivalent to

$$P_k^+ \asymp P_k^- \qquad (k \in \mathcal{K}). \tag{10}$$

To see this, we express $\mathcal{K}$ as a disjoint union $\mathcal{A} \cup \mathcal{B}$, where $\mathcal{A}$ [resp. $\mathcal{B}$] is the set of numbers $k \in \mathcal{K}$ for which $\widehat{p}_k < \widehat{q}_k$ [resp. $\widehat{p}_k > \widehat{q}_k$]. To prove (9) it suffices to show:

($\mathscr{D}_1$) $\quad \widehat{p}_k \gg \widehat{q}_k$ for all $k \in \mathcal{A}$;

($\mathscr{D}_2$) $\quad \widehat{p}_k \ll \widehat{q}_k$ for all $k \in \mathcal{B}$.

We use the following result, which is an easy consequence of the prime number theorem:

Lemma 7. *Let* $c_{\mathcal{P}} = \overline{\delta} \,/\, \underline{\delta}$ *and* $c_{\mathcal{Q}} = (1 - \underline{\delta}\,) \,/\, \big(1 - \overline{\delta}\,\big)$. *For every* $\varepsilon > 0$ *there is a number* $x_0(\varepsilon)$ *such that for all* $x > x_0(\varepsilon)$*:*

(*i*) *if* p *is the smallest prime in* $\mathcal{P}$ *greater than* x*, then* $p \leqslant (c_{\mathcal{P}} + \varepsilon)\, x$*;*

(*ii*) *if* q *is the smallest prime in* $\mathcal{Q}$ *greater than* x*, then* $q \leqslant (c_{\mathcal{Q}} + \varepsilon)\, x$*;*

(*iii*) *if* p *is the largest prime in* $\mathcal{P}$ *less than* x*, then* $p \geqslant \big(c_{\mathcal{P}}^{-1} - \varepsilon\big)\, x$*;*

(*iv*) *if* q *is the largest prime in* $\mathcal{Q}$ *less than* x*, then* $q \geqslant \big(c_{\mathcal{Q}}^{-1} - \varepsilon\big)\, x$*.*

To prove $(\mathscr{D}_1)$ we can assume that $\mathcal{A}$ is an infinite set. Let $k \in \mathcal{A}$, so that $\widehat{p}_k < \widehat{q}_k$. Since $\widehat{q}_k = P_k^+ \to \infty$ as $k \to \infty$ with $k \in \mathcal{A}$, the assertion $(\mathscr{D}_1)$ then follows from Lemmas 5 and 7.

To prove $(\mathscr{D}_2)$ we can assume that $\mathcal{B}$ is an infinite set. Let $k \in \mathcal{B}$, so that $\widehat{p}_k > \widehat{q}_k$. Let p, q be defined as in Lemma 6. Since $\widehat{p}_k = P_k^+ \to \infty$ as $k \to \infty$ with $k \in \mathcal{B}$, on combining Lemmas 6 and 7 it follows that

$$\widehat{p}_k \ll p \ll q \ll \widehat{q}_k,$$

which proves $(\mathscr{D}_2)$ and completes our proof of (9).

Next, for every $n \in \mathbb{N}$ let

$$\omega_{\mathcal{P}}(n) = \#\big\{p \in \mathcal{P} \ : \ p \mid n\big\} \qquad \text{and} \qquad \omega_{\mathcal{Q}}(n) = \#\big\{q \in \mathcal{Q} \ : \ q \mid n\big\}.$$

We claim that

$$\omega_{\mathcal{P}}(n_k) \asymp \omega_{\mathcal{Q}}(n_k) \qquad (k \in \mathcal{K}). \tag{11}$$

Indeed, by Lemma 4 it follows that $\omega_{\mathcal{P}}(n_k) = \pi_{\mathcal{P}}(\widehat{p}_k)$ and $\omega_{\mathcal{Q}}(n_k) = \pi_{\mathcal{Q}}(\widehat{q}_k)$. Therefore, using the prime number theorem together with (4), (5) and (9) we have

$$\omega_{\mathcal{P}}(n_k) = \pi_{\mathcal{P}}(\widehat{p}_k) \asymp \frac{\widehat{p}_k}{\log \widehat{p}_k} \asymp \frac{\widehat{q}_k}{\log \widehat{q}_k} \asymp \pi_{\mathcal{Q}}(\widehat{q}_k) = \omega_{\mathcal{Q}}(n_k),$$

which proves (11).

Finally, we need the following relation:

$$\log \kappa(n_k) \asymp \omega(n_k) \log \omega(n_k) \qquad (k \in \mathcal{K}). \tag{12}$$

To prove this, observe that the definition (8) and Lemma 4 together imply

$$\prod_{p \leqslant P_k^-} p \;\Big|\; \kappa(n_k) \qquad \text{and} \qquad \kappa(n_k) \;\Big|\; \prod_{p \leqslant P_k^+} p.$$

Consequently,
$$\sum_{p\leqslant P_k^-}\log p\leqslant \log\kappa(n_k)\leqslant \sum_{p\leqslant P_k^+}\log p,$$
and also
$$\pi(P_k^-)\leqslant \omega(n_k)\leqslant \pi(P_k^+).$$
By the prime number theorem, for either choice of the sign $\pm$ we have
$$\sum_{p\leqslant P_k^\pm}\log p\sim P_k^\pm \qquad\text{and}\qquad \pi(P_k^\pm)\sim\frac{P_k^\pm}{\log P_k^\pm}\qquad (k\to\infty,\ k\in\mathcal{K}),$$
therefore in view of (10) we see that
$$\log\kappa(n_k)\asymp P_k^+ \qquad\text{and}\qquad \omega(n_k)\asymp\frac{P_k^+}{\log P_k^+}\,,$$
and (12) follows immediately.

Now we come to the heart of the argument. To complete the proof of Theorem 1, we seek a contradiction to our assumption that $\mathcal{K}$ is an infinite set. For this, it is enough to prove both of the following statements with a suitably chosen real number $\varepsilon>0$:

($\mathscr{E}_1$) the inequality $n_k\leqslant\kappa(n_k)^{1+\varepsilon}$ holds for at most finitely many $k\in\mathcal{K}$;

($\mathscr{E}_2$) the inequality $n_k>\kappa(n_k)^{1+\varepsilon}$ holds for at most finitely many $k\in\mathcal{K}$.

In view of (11) and (12), there is a constant $C>1$ such that the inequalities
$$\omega_{\mathcal{P}}(n_k)\leqslant (C-1)\,\omega_{\mathcal{Q}}(n_k) \tag{13}$$
and
$$\log\kappa(n_k)\leqslant C\,\omega(n_k)\log\omega(n_k) \tag{14}$$
both hold if k is sufficiently large. Let C be fixed, and put $\varepsilon=C^{-3}$.

To prove ($\mathscr{E}_1$), we suppose on the contrary that $n_k\leqslant\kappa(n_k)^{1+\varepsilon}$ holds for infinitely many $k\in\mathcal{K}$. Let k be large, and put
$$r=\omega_{\mathcal{P}}(n_k)=\pi_{\mathcal{P}}(\widehat{p}_k) \qquad\text{and}\qquad s=\omega_{\mathcal{Q}}(n_k)=\pi_{\mathcal{Q}}(\widehat{q}_k)$$
By what we have already seen it is clear that $\min\{r,s\}\to\infty$ as $k\to\infty$ with $k\in\mathcal{K}$, thus by (13) we have
$$r\leqslant (C-1)s \tag{15}$$

if k is large enough. By Lemma 4 and the fact that $n_k \in \mathcal{Y}$, it follows that

$$n_k = \Bigg(\prod_{\substack{p\leqslant \widehat{p}_k\\ p\in\mathcal{P}}} p\Bigg)\Bigg(\prod_{\substack{q\leqslant \widehat{q}_k\\ q\in\mathcal{Q}}} q^2\Bigg) \qquad \text{and} \qquad \kappa(n_k) = \Bigg(\prod_{\substack{p\leqslant \widehat{p}_k\\ p\in\mathcal{P}}} p\Bigg)\Bigg(\prod_{\substack{q\leqslant \widehat{q}_k\\ q\in\mathcal{Q}}} q\Bigg).$$

Hence, our assumption that $n_k \leqslant \kappa(n_k)^{1+\varepsilon}$ implies that

$$\kappa(n_k) \geqslant \left(\frac{n_k}{\kappa(n_k)}\right)^{1/\varepsilon} = \Bigg(\prod_{\substack{q\leqslant \widehat{q}_k\\ q\in\mathcal{Q}}} q\Bigg)^{1/\varepsilon}. \tag{16}$$

If $\overline{p}_1, \overline{p}_2, \ldots$ is the sequence of consecutive prime numbers, then by the prime number theorem (and recalling our choice of ε) we derive that

$$\log\kappa(n_k) \geqslant C^3 \sum_{\substack{q\leqslant \widehat{q}_k\\ q\in\mathcal{Q}}} \log q \geqslant C^3 \sum_{p\leqslant \overline{p}_s} \log p \sim C^3\overline{p}_s \sim C^3 s\log s$$

as $k \to \infty$ with $k \in \mathcal{K}$. On the other hand, using (14), (15) and the fact that $\omega(n_k) = r + s$, it follows that

$$\log\kappa(n_k) \leqslant C(r+s)\log(r+s) \leqslant C^2 s\log(Cs) \sim C^2 s\log s.$$

Since $C^3 > C^2$, these two inequalities for $\log\kappa(n_k)$ lead to a contradiction once k is sufficiently large, and this completes the proof of $(\mathscr{E}_1)$.

To prove $(\mathscr{E}_2)$ we use some ideas from Choie *et al* [1]. Suppose that $n_k > \kappa(n_k)^{1+\varepsilon}$, and put $t = \omega(n_k)$. We claim that either

$$\sum_{p\leqslant \overline{p}_t} \log p < (1+\varepsilon)^{-1/2}\,\overline{p}_t, \tag{17}$$

or

$$\overline{p}_t \leqslant \exp\big(2/\log(1+\varepsilon)\big). \tag{18}$$

Assuming the claim, it is easy to see that $\omega(n_k)$ is bounded above by a constant K that depends only on ε. By Lemma 2, n_k can take only finitely many distinct values, which implies $(\mathscr{E}_2)$.

To prove the claim, assume that (17) fails:

$$\log(\overline{p}_1\cdots\overline{p}_t) = \sum_{p\leqslant \overline{p}_t} \log p \geqslant (1+\varepsilon)^{-1/2}\,\overline{p}_t.$$

Thanks to Rosser and Schoenfeld [8] it is known that

$$\prod_{p\leqslant x}\frac{p}{p-1}\leqslant e^\gamma\left(\log x+\frac{1}{\log x}\right)\qquad (x>1).$$

Therefore, taking $x=\overline{p}_t$ and noting that $\kappa(n_k)\geqslant\overline{p}_1\cdots\overline{p}_t$, we derive that

$$\begin{aligned} e^\gamma\left(\log\overline{p}_t+\frac{1}{\log\overline{p}_t}\right)&\geqslant\prod_{j=1}^t\frac{\overline{p}_j}{\overline{p}_j-1}\geqslant\frac{n_k}{\varphi(n_k)}\geqslant e^\gamma\log\log n_k\\ &> e^\gamma\log\left((1+\varepsilon)\log\kappa(n_k)\right)\\ &\geqslant e^\gamma\log\left((1+\varepsilon)\log(\overline{p}_1\cdots\overline{p}_t)\right)\\ &\geqslant e^\gamma\log\left((1+\varepsilon)^{1/2}\,\overline{p}_t\right)=e^\gamma\left(\log\overline{p}_t+0.5\log(1+\varepsilon)\right); \end{aligned}$$

that is,

$$\frac{1}{\log\overline{p}_t}\geqslant 0.5\log(1+\varepsilon),$$

which is equivalent to (18). This proves the claim and completes our proof of Theorem 1.

3 Proof of Theorem 2

We continue to use the notation of the previous section, but we focus on the special case that

$$\begin{aligned}\mathcal{P}&=\big\{p\in\mathbb{P}\ :\ p\equiv 1\pmod 4\big\}\cup\{2\},\\ \mathcal{Q}&=\big\{q\in\mathbb{P}\ :\ q\equiv 3\pmod 4\big\}.\end{aligned}$$

Note that the corresponding set $\mathcal{S}$ contains every natural number that can be expressed as a sum of two squares. As before, we write

$$\mathcal{T}_k=\big\{n\in\mathbb{N}\ :\ F(n)\geqslant e^\gamma\log\log n,\ n=s(n),\ \text{and}\ \Omega(n)=k\big\}$$

and put

$$\mathcal{K}=\{k\geqslant 0\ :\ \mathcal{T}_k\neq\varnothing\}.$$

Lemma 8. *If $k\in\mathcal{K}$, then $P_k^-<50000$.*

Proof. For every real number $x\geqslant 10$, let

- $g_{\mathcal{P}}(x) =$ the smallest prime in $\mathcal{P}$ greater than x;
- $g_{\mathcal{Q}}(x) =$ the smallest prime in $\mathcal{Q}$ greater than x;
- $\ell_{\mathcal{P}}(x) =$ the largest prime in $\mathcal{P}$ less than x;
- $\ell_{\mathcal{Q}}(x) =$ the largest prime in $\mathcal{Q}$ less than x.

Also, put

$$\vartheta_{\mathcal{P}}(x) = \sum_{\substack{p \leqslant x \\ p \in \mathcal{P}}} \log p \qquad \text{and} \qquad \vartheta_{\mathcal{Q}}(x) = \sum_{\substack{q \leqslant x \\ q \in \mathcal{Q}}} \log q.$$

Using the explicit bounds of Theorems 1 and 2 of Ramaré and Rumely [6], we see that the inequalities

$$0.49\,x < \vartheta_{\mathcal{P}}(x) < 0.51\,x \qquad \text{and} \qquad 0.49\,x < \vartheta_{\mathcal{Q}}(x) < 0.51\,x. \tag{19}$$

hold for all $x \geqslant 45000$ (note that $\vartheta_{\mathcal{P}}(x) = \log 2 + \theta(x;4,1)$ and $\vartheta_{\mathcal{Q}}(x) = \theta(x;4,3)$ in the notation of [6]). Consequently, for any $x \geqslant 50000$ we have

$$\tfrac{49}{51}\,x < \ell_{\mathcal{P}}(x) < x < g_{\mathcal{P}}(x) < \tfrac{51}{49}\,x$$

and

$$\tfrac{49}{51}\,x < \ell_{\mathcal{Q}}(x) < x < g_{\mathcal{Q}}(x) < \tfrac{51}{49}\,x.$$

Now suppose that $P_k^- \geqslant 50000$. Using Lemma 5 and the preceding bounds we have

$$\widehat{q}_k < g_{\mathcal{P}}(g_{\mathcal{P}}(\widehat{p}_k)) < \left(\tfrac{51}{49}\right)^2 \widehat{p}_k.$$

On the other hand, by Lemma 6 we have

$$\tfrac{51}{49}\,\widehat{q}_k > g_{\mathcal{Q}}(\widehat{q}_k) > \tfrac{1}{2}\,\ell_{\mathcal{P}}(\widehat{p}_k) > \tfrac{49}{102}\,\widehat{p}_k.$$

Hence, it follows that

$$0.92\,\widehat{q}_k < \widehat{p}_k < 2.2\,\widehat{q}_k. \tag{20}$$

By Lemma 4 it is clear that

$$\log \kappa(n_k) = \sum_{\substack{p \leqslant \widehat{p}_k \\ p \in \mathcal{P}}} \log p + \sum_{\substack{q \leqslant \widehat{q}_k \\ q \in \mathcal{Q}}} \log q = \vartheta_{\mathcal{P}}(\widehat{p}_k) + \vartheta_{\mathcal{Q}}(\widehat{q}_k).$$

On the other hand, arguing as in the proof of Theorem 1, it follows from (16) that

$$\log \kappa(n_k) \geqslant \varepsilon^{-1}\vartheta_{\mathcal{Q}}(\widehat{q}_k)$$

if $\varepsilon > 0$ is fixed and $n_k \leqslant \kappa(n_k)^{1+\varepsilon}$. Combining the two preceding results with (19), we see that

$$0.51\,(\widehat{p}_k + \widehat{q}_k) \geqslant \vartheta_{\mathcal{P}}(\widehat{p}_k) + \vartheta_{\mathcal{Q}}(\widehat{q}_k) \geqslant \varepsilon^{-1}\vartheta_{\mathcal{Q}}(\widehat{q}_k) \geqslant 0.49\,\varepsilon^{-1}\widehat{q}_k$$

since $P_k^- \geqslant 50000$; taking into account (20), we further have

$$0.51\,(1 + 2.2)\,\widehat{q}_k \geqslant 0.51\,(\widehat{p}_k + \widehat{q}_k) \geqslant 0.49\,\varepsilon^{-1}\widehat{q}_k,$$

which implies that $\varepsilon \geqslant 0.3002$. Thus, for the smaller value $\varepsilon = 0.3$, we see that the condition $n_k \leqslant \kappa(n_k)^{1.3}$ implies $P_k^- < 50000$.

On the other hand, if $n_k > \kappa(n_k)^{1.3}$, we put $t = \omega(n_k)$ as in the proof of Theorem 1. Since $\varepsilon = 0.3$, we derive from (17) and (18) that either

$$\vartheta(\overline{p}_t) = \sum_{p \leqslant \overline{p}_t} \log p < (1.3)^{-1/2}\,\overline{p}_t < 0.88\,\overline{p}_t, \tag{21}$$

or

$$\overline{p}_t \leqslant \exp(2/\log 1.3) < 2045.$$

Using again Theorems 1 and 2 of Ramaré and Rumely [6] (see also [8]), it is easy to see that the inequality (21) implies $\overline{p}_t < 300$, hence the inequality $\overline{p}_t < 2045$ holds in both cases. It follows that $t < 310$, and therefore,

$$\min\{\pi_{\mathcal{P}}(\widehat{p}_k), \pi_{\mathcal{Q}}(\widehat{q}_k)\} = \min\{\omega_{\mathcal{P}}(n_k), \omega_{\mathcal{Q}}(n_k)\} \leqslant \omega(n_k) = t < 310,$$

which implies that $P_k^- < 5000$. This completes the proof. $\square$

Corollary 3. *If $k \in \mathcal{K}$, then $k < 10000$.*

Proof. For any $k \in \mathcal{K}$ we have

$$k = \Omega(n_k) = \omega_{\mathcal{P}}(n_k) + 2\,\omega_{\mathcal{Q}}(n_k) = \pi_{\mathcal{P}}(\widehat{p}_k) + 2\,\pi_{\mathcal{Q}}(\widehat{q}_k).$$

If $P_k^- = \widehat{p}_k$ (i.e., $\widehat{p}_k < \widehat{q}_k$), then by Lemmas 5 and 8 it follows that

$$\begin{aligned} k &\leqslant \max_{p < 50000} \bigl\{\pi_{\mathcal{P}}(p) + 2\,\pi_{\mathcal{Q}}\bigl(g_{\mathcal{P}}(g_{\mathcal{P}}(p))\bigr)\bigr\} \\ &\leqslant \pi_{\mathcal{P}}(50000) + 2\,\pi_{\mathcal{Q}}\bigl(g_{\mathcal{P}}(g_{\mathcal{P}}(50000))\bigr) = 7718. \end{aligned}$$

If $P_k^- = \widehat{q}_k$ (i.e., $\widehat{q}_k < \widehat{p}_k$), then by Lemmas 6 and 8 it follows that

$$\begin{aligned} k &\leqslant \max_{q<50000} \max_{\substack{p\in\mathbb{P} \\ \ell_{\mathcal{P}}(p)<2g_{\mathcal{Q}}(q)}} \{\pi_{\mathcal{P}}(p) + 2\,\pi_{\mathcal{Q}}(q)\} \\ &= \max_{q<50000} \max_{\substack{p\in\mathbb{P} \\ \ell_{\mathcal{P}}(p)<2g_{\mathcal{Q}}(q)}} \{1 + \pi_{\mathcal{P}}(\ell_{\mathcal{P}}(p)) + 2\,\pi_{\mathcal{Q}}(q)\} \\ &\leqslant \max_{q<50000} \{1 + \pi_{\mathcal{P}}(2\,g_{\mathcal{Q}}(q)) + 2\,\pi_{\mathcal{Q}}(q)\} \\ &\leqslant 1 + \pi_{\mathcal{P}}(2\,g_{\mathcal{Q}}(50000)) + 2\,\pi_{\mathcal{Q}}(50000) = 9951. \end{aligned}$$

The result follows. $\square$

Now let $\overline{\overline{p}}_1, \overline{\overline{p}}_2, \dots$ be the sequence of consecutive primes in $\mathcal{P}$, and let $\overline{\overline{q}}_1, \overline{\overline{q}}_2, \dots$ be the consecutive primes in $\mathcal{Q}$. For any integers $r, s \geqslant 0$, let

$$N_{r,s} = \Bigg(\prod_{i=1}^{r} \overline{\overline{p}}_i\Bigg)\Bigg(\prod_{j=1}^{s} \overline{\overline{q}}_j^{\,2}\Bigg).$$

It is easy to see that $N_{r,s} \in \mathcal{Y}$ for all $r, s \geqslant 0$, and for every $k \in \mathcal{K}$ one has

$$n_k = N_{r,s}, \qquad \widehat{p}_k = \overline{\overline{p}}_r, \qquad \widehat{q}_k = \overline{\overline{q}}_s \qquad \text{and} \qquad k = r + 2s,$$

where $r = \omega_{\mathcal{P}}(n_k)$ and $s = \omega_{\mathcal{Q}}(n_k)$. By a straightforward computation, one verifies the following:

Lemma 9. *If $r, s \geqslant 0$, then $N_{r,s} \in \mathcal{N}^\circ$ if and only if the pair (r, s) lies in the set*

$$\begin{aligned} \mathcal{X} = \big\{&(0,0), (1,0), (0,1), (2,0), (1,1), (2,1), (1,2), (3,1), (2,2), (4,1), \\ &(3,2), (2,3), (4,2), (3,3), (5,2), (4,3), (3,4), (5,3), (4,4), (6,3), \\ &(5,4), (4,5), (7,3), (6,4), (5,5), (7,4), (6,5), (7,5), (8,5)\big\}. \end{aligned}$$

We remark that, in view of Corollary 3, it suffices to check the condition $N_{r,s} \in \mathcal{N}^\circ$ only for those pairs (r, s) with $r + 2s < 10000$.

Corollary 4. *If $k \in \mathcal{K}$, then $k \leqslant 18$.*

Corollary 5. *If $n \in \mathcal{S} \cap \mathcal{N}^\circ$, $r = \omega_{\mathcal{P}}(n)$ and $s = \omega_{\mathcal{Q}}(n)$, then $(r, s) \in \mathcal{X}$. In particular, $\omega(n) \leqslant 13$.*

Proof. Since

$$F(N_{r,s}) = \left(\prod_{i=1}^{r} \frac{\overline{\overline{p}}_i}{\overline{\overline{p}}_i - 1}\right)\left(\prod_{j=1}^{s} \frac{\overline{\overline{q}}_j}{\overline{\overline{q}}_j - 1}\right) \geqslant \left(\prod_{\substack{p\,|\,n\\ p\in\mathcal{P}}} \frac{p}{p-1}\right)\left(\prod_{\substack{q\,|\,n\\ q\in\mathcal{Q}}} \frac{q}{q-1}\right) = F(n)$$

and

$$n \geqslant s(n) = \left(\prod_{\substack{p\,|\,n\\ p\in\mathcal{P}}} p\right)\left(\prod_{\substack{q\,|\,n\\ q\in\mathcal{Q}}} q^2\right) \geqslant \left(\prod_{i=1}^{r} \overline{\overline{p}}_i\right)\left(\prod_{j=1}^{s} \overline{\overline{q}}_j^2\right) = N_{r,s},$$

we have

$$F(N_{r,s}) \geqslant F(n) \geqslant e^\gamma \log\log n \geqslant e^\gamma \log\log N_{r,s},$$

which shows that $N_{r,s} \in \mathcal{N}^\circ$. $\square$

We now turn to a description of our method for generating the elements of $\mathcal{S}\setminus\mathcal{N} = \mathcal{S}\cap\mathcal{N}^\circ$. For any given $n \in \mathcal{S}\cap\mathcal{N}^\circ$ with $r = \omega_{\mathcal{P}}(n)$ and $s = \omega_{\mathcal{Q}}(n)$, we can write

$$s(n) = p_1 \cdots p_r\, q_1^2 \cdots q_s^2,$$

where $p_1 < \cdots < p_r$ are primes in $\mathcal{P}$ and $q_1 < \cdots < q_s$ are primes in $\mathcal{Q}$. For fixed $i = 1, \ldots, r$, let γ_i be the largest non-negative integer such that the number

$$\left(\prod_{\ell=1}^{i-1} \overline{\overline{p}}_\ell\right)\left(\prod_{\ell=i}^{r} \overline{\overline{p}}_{\ell+\gamma_i}\right)\left(\prod_{j=1}^{s} \overline{\overline{q}}_j^2\right)$$

lies in $\mathcal{N}^\circ$, which exist by Lemma 2. Using an argument similar to that in the proof of Lemma 4, one can deduce that

$$\overline{\overline{p}}_i \leqslant p_i \leqslant \overline{\overline{p}}_{i+\gamma_i} \qquad (i = 1, \ldots, r). \tag{22}$$

Similarly, for fixed $j = 1, \ldots, s$, let δ_j be the largest non-negative integer such that the number

$$\left(\prod_{i=1}^{r} \overline{\overline{p}}_i\right)\left(\prod_{\ell=1}^{j-1} \overline{\overline{q}}_\ell^2\right)\left(\prod_{\ell=j}^{s} \overline{\overline{q}}_{\ell+\delta_j}^2\right)$$

lies in $\mathcal{N}^\circ$. Then,

$$\overline{\overline{q}}_j \leqslant q_j \leqslant \overline{\overline{q}}_{j+\gamma_j} \qquad (j = 1, \ldots, s). \tag{23}$$

Therefore, for fixed $(r,s) \in \mathcal{X}$, if $n \in \mathcal{S} \cap \mathcal{N}^\circ$ with $r = \omega_{\mathcal{P}}(n)$ and $s = \omega_{\mathcal{Q}}(n)$, then the number $s(n)$ must lie in the finite set $\mathcal{A}_{r,s}$ of integers of the form

$$m = p_1 \cdots p_r \, q_1^2 \cdots q_s^2, \tag{24}$$

where $p_1 < \cdots < p_r$ are primes in $\mathcal{P}$, $q_1 < \cdots < q_s$ are primes in $\mathcal{Q}$, the primes p_i and q_j satisfy the bounds (22) and (23), and $m \in \mathcal{N}^\circ$. The set $\mathcal{A}_{r,s}$ can be explicitly determined by a numerical computation, and we obtain a finite list of "admissible" values for the quantity $s(n)$.

To determine explicitly all of the numbers $n \in \mathcal{S} \cap \mathcal{N}^\circ$ with $r = \omega_{\mathcal{P}}(n)$ and $s = \omega_{\mathcal{Q}}(n)$, for every $m \in \mathcal{A}_{r,s}$ we need to find all such numbers for which $s(n) = m$. To do this, factor m as in (24). For fixed $i = 1, \ldots, r$, let α_i be the largest integer such that the number $mp_i^{\alpha_i - 1}$ lies in $\mathcal{N}^\circ$. Similarly, for fixed $j = 1, \ldots, s$, let β_j be the largest integer such that the number $mq_j^{\beta_j - 1}$ lies in $\mathcal{N}^\circ$. Put

$$M = m \cdot p_1^{\alpha_1 - 1} \cdots p_r^{\alpha_r - 1} q_1^{\beta_1 - 1} \cdots q_s^{\beta_s - 1}.$$

Then, it is easy to see that $m \mid n$ and $n \mid M$ for any $n \in \mathcal{S} \cap \mathcal{N}^\circ$ such that $s(n) = m$. Hence, n can take only finitely many values which can be determined explicitly for each $m \in \mathcal{A}_{r,s}$.

For example, taking $r = s = 2$ we find that

$$\{4410, 8820, 10890, 13230, 17640, 21780, 22050, 26460, 30870, 35280, 39690,$$
$$44100, 52920, 61740, 66150, 70560, 79380, 88200, 92610, 105840, 110250\}$$

is a complete list of the numbers $n \in \mathcal{S} \setminus \mathcal{N}$ with $\omega_{\mathcal{P}}(n) = \omega_{\mathcal{Q}}(n) = 2$. Examining the lists generated as (r,s) varies over the pairs in $\mathcal{X}$, we are lead to the statement of Theorem 2.

4 Evaluation of $\overline{\lim\limits_{n\in S}} \dfrac{n}{\varphi(n)\log\log n}$ and $\overline{\lim\limits_{n\in S}} \dfrac{\sigma(n)}{n\log\log n}$

We conclude the paper by giving two propositions and two corollaries that yield the analogue of the work of Landau [4] and Gronwall [2] for any set $\mathcal{S}$ of the form (6) and for the set of natural numbers equal to a sum of two squares. In fact, Corollary 6 shows that Theorem 1 is nontrivial in the sense that $F(n)/\log\log n$ cannot be bounded away from e^γ by any positive constant for all large $n \in \mathcal{S}$. We will use the notation $f(n) = o(g(n))$ to mean that $\lim\limits_{n\to\infty} f(n)/g(n) = 0$.

Proposition 1. *Let $\{a_n\}$ be an infinite sequence of positive integers such that if we write $a_n = \prod_p p^{v(p,n)}$ we have:*

(i) $\kappa(a_n) = \prod_{p \leqslant n} p$ *(i.e.,* $v(p,n) = 0 \iff p > n$*);*

(ii) $a_n = \exp(n^{1+o(1)})$*;*

(iii) $\lim_{n\to\infty} v(p,n) = \infty$ *for each* p*.*

Then,

$$\lim_{n\to\infty} \frac{\sigma(a_n)}{a_n \log\log a_n} = e^{\gamma}.$$

Proof. For all $n \geqslant 1$, let

$$b_n = \prod_{p \leqslant n} p \qquad \text{and} \qquad c_n = \frac{\sigma(a_n)}{a_n} \frac{\varphi(b_n)}{b_n},$$

and observe that (i) implies

$$c_n = \left(\prod_{p \leqslant n} \frac{p^{v(p,n)+1} - 1}{p^{v(p,n)}(p-1)} \right) \left(\prod_{p \leqslant n} \frac{p-1}{p} \right) = \prod_{p \leqslant n} \left(1 - \frac{1}{p^{v(p,n)+1}} \right).$$

Since $v(p,n) + 1 \geqslant 2$ for every prime $p \leqslant n$, we have for any $m \leqslant n$:

$$1 \geqslant c_n > \prod_{p \leqslant m} \left(1 - \frac{1}{p^{v(p,n)+1}} \right) \prod_{p > m} \left(1 - \frac{1}{p^2} \right).$$

Using (iii) we have for every fixed integer m:

$$1 \geqslant \overline{\lim_{n\to\infty}}\, c_n \geqslant \underline{\lim_{n\to\infty}}\, c_n \geqslant \prod_{p > m} \left(1 - \frac{1}{p^2} \right).$$

The product on the right tends to one as $m \to \infty$, hence $\lim_{n\to\infty} c_n = 1$; therefore,

$$\lim_{n\to\infty} \frac{\sigma(a_n)}{a_n \log n} = \lim_{n\to\infty} \frac{b_n}{\varphi(b_n) \log n}.$$

Our assumption (ii) implies that $\log\log a_n = (1 + o(1)) \log n$, and using Mertens' theorem (see, for example, [8]) we have

$$\frac{\varphi(b_n)}{b_n} = \prod_{p \leqslant n} \left(1 - \frac{1}{p} \right) = (1 + o(1)) \frac{e^{-\gamma}}{\log n},$$

and the result follows. $\square$

Using similar ideas (and an easier argument) one can obtain the following analogue of Proposition 1 for the Euler totient function:

Proposition 2. *Let $\{a_n\}$ be an infinite sequence of positive integers such that:*

(*i*) $\kappa(a_n) = \prod_{p\leqslant n} p$;

(*ii*) $a_n = \exp(n^{1+o(1)})$.

Then,

$$\lim_{n\to\infty} \frac{a_n}{\varphi(a_n)\log\log a_n} = e^\gamma.$$

Corollary 6. *For any set $\mathcal{S}$ defined by* (6), *we have*

$$\overline{\lim_{n\in\mathcal{S}}} \frac{\sigma(n)}{n\log\log n} = \overline{\lim_{n\in\mathcal{S}}} \frac{n}{\varphi(n)\log\log n} = e^\gamma.$$

Proof. Since

$$\overline{\lim_{n\to\infty}} \frac{\sigma(n)}{n\log\log n} = \overline{\lim_{n\to\infty}} \frac{n}{\varphi(n)\log\log n} = e^\gamma$$

by [2] and [4], respectively, it suffices to show that there is a sequence $\{a_n\}$ in $\mathcal{S}$ such that

$$\lim_{n\to\infty} \frac{\sigma(a_n)}{a_n\log\log a_n} = \lim_{n\to\infty} \frac{a_n}{\varphi(a_n)\log\log a_n} = e^\gamma.$$

Let $a_1 = 1$, and for every integer $n \geqslant 2$, let

$$b_n = \prod_{p\leqslant n} p, \qquad d_n = \left\lfloor n^{(\log n)^{-1/2}} \right\rfloor \qquad \text{and} \qquad a_n = b_n^{d_n}.$$

It is easy to see that $d_n \geqslant 2$ for $n \geqslant 2$, $d_n = n^{o(1)}$, and d_n tends to infinity with n. Clearly, $a_n \in \mathcal{S}$ for all $n \geqslant 1$, and by the Prime Number Theorem in the form $\sum_{p\leqslant x} \log p = x(1+o(1))$ as $x \to \infty$ we see that

$$\log a_n = d_n \log b_n = n^{o(1)} \sum_{p\leqslant n} \log p = n^{1+o(1)} \qquad (n\to\infty).$$

The sequence $\{a_n\}$ therefore satisfies the hypotheses of Propositions 1 and 2, and the result follows. $\square$

Corollary 7. *We have*

$$\varlimsup_{n=a^2+b^2} \frac{\sigma(n)}{n\log\log n} = \varlimsup_{n=a^2+b^2} \frac{n}{\varphi(n)\log\log n} = e^\gamma.$$

Proof. Defining a_n for all $n \geqslant 1$ as in the proof of Corollary 6, it is easy to see that the sequence $\{a_n^2\}$ satisfies the hypotheses of Propositions 1 and 2; it follows that

$$\varlimsup_{n=a^2} \frac{\sigma(n)}{n\log\log n} = \varlimsup_{n=a^2} \frac{n}{\varphi(n)\log\log n} = e^\gamma,$$

and this implies the stated result. □

References

[1] Y.-J. Choie, N. Lichiardopol, P. Moree and P Solé, 'On Robin's criterion for the Riemann hypothesis,' *J. Théor. Nombres Bordeaux* **19** (2007), 351–366.

[2] T. H. Gronwall, 'Some asymptotic expressions in the theory of numbers,' *Trans. Amer. Math. Soc.* **14** (1913), no. 1, 113–122.

[3] J. C. Lagarias, 'An elementary problem equivalent to the Riemann hypothesis,' *Amer. Math. Monthly* **109** (2002), no. 6, 534–543.

[4] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Leipzig, 1909.

[5] J. L. Nicolas, 'Petites valeurs de la fonction d'Euler,' *J. Number Theory* **17** (1983), no. 3, 375–388.

[6] O. Ramaré and R. Rumely, 'Primes in arithmetic progressions,' *Math. Comp.* **65** (1996), no. 213, 397–425.

[7] G. Robin, 'Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann,' *J. Math. Pures Appl. (9)* **63** (1984), no. 2, 187–213.

[8] J. B. Rosser and L. Schoenfeld, 'Approximate formulas for some functions of prime numbers,' *Illinois J. Math.* **6** (1962), 64–94.