Method of projections of Drinfeld currents

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Abstract

We study intersections of Borel subalgebras of different types in quantum affine algebra and develop a technique of projections of Drinfeld currents to these intersections. We determine in this way the universal weight function, which is used for the construction of off-shell Bethe vectors. As application, we derive integral expressions for the weight function and for the factors of the universal R-matrix of $U_q(\widehat{sl}_2)$.

1 Introduction

Quantum affine algebra admits two different descriptions: as a quintized Kac-Moody algebra and the so called current realization due to Drinfeld. In particular, it has quite different comultiplication structures and Borel subalgebras, related to these descriptions.

It turns out, that in representation theory and its applications it is convenient to use the current description of the algebra, while coalgebraic structure is usually canonical and comes from Kac-Moody description. The relation between two comultiplication structure was realized by many authors, see, e.g. [5]. In [6] this connection was used for the investigation of properties of the universal R-matrix of quantum affine algebra. In particular, in $U_q(\widehat{\mathfrak{s}l}_2)$ case a differential equation for its factors was derived.

In these investigation the two pairs of Borel subalgebras of quantum affine algebra played the crutial role. One pair consists of opposite Borel subalgebras, generated by positive and negative root vectors in a Kac-Moody description. Another pair is related to current realization. The latter Borel subalgebras could be viewed as a q-deformations of currents into Borel subalgebras of underlying finite-dimensional Lie algebra. In the paper [3] projections of current Borel subalgeras to their intersections with canonical Borel subalgebras were constructed. It was shown how they determine a twist between two comultiplications. Next, it was proved in [7], that the weight function, which is used for the construction of off-shell Bethe vectors and for description of solutions of Knizhnik-Zamolodchikov equation [12], can be defined as an application of the above projector of a product of Drinfeld currents to a highest weight vector of finite-dimensional representation of quantum affine algebra.

In this survey note we determine projection operators to intersections of Borel subalgebras, and study their properties, including these projectors into a general scheme of orthogonal decompositions of Hopf algebras. Such a scheme gives a simple proof of the relation between two comultiplication, based on a study of orthogonal decompositions, related to Lusztig automorphisms. We present such a proof here. Next, we formulate and prove the relation between weight function and projections of products of Drinfeld currents. As an application, we present integral expressions for the weight function and for the factors of the universal R-matrix of $U_q(\widehat{sl}_2)$. The expression for the weight function is taken from [8], the formula for the universal R-matrix, presented here, was not written before.

2 Quantum affine algebra $U_q(\widehat{\mathfrak{g}})$

2.1 $U_q(\widehat{\mathfrak{g}})$ in Chevalley generators

Let $\mathfrak{g}$ be a simple Lie algebra of rank r with Cartan matrix $b_{i,j}$, $i,j=1,...,r$. Denote by $a_{i,j}$, $i,j=0,...,r$ the Cartan matrix of the affine algebra $\widehat{\mathfrak{g}}$ and by (α_i,α_j) the symmetrized matrix $a_{i,j}$, such that $(\alpha_i,\alpha_j)=d_ia_{i,j}=d_ja_{j,i}$, and by δ the minimal positive root $\delta=\sum_{i=0}^r n_i\alpha_i$, $n_i\in\mathbb{Z}_{\geq 0}$, $n_0=1$. Let $\left[{n \atop k}\right]_q=\frac{[n]_q!}{[k]_q![n-k]_q!}$, $[n]_q!=[1]_q[2]_q\cdots[n]_q$, $[n]_q=\frac{q^n-q^{-n}}{q-q^{-1}}$, $q_\alpha=q^{\frac{(\alpha,\alpha)}{2}}$, $q_i=q_{\alpha_i}=q^{d_i}$. We use the notation $\Pi=\{\alpha_1,...,\alpha_r\}$ for the set of simple positive roots of $\mathfrak{g}$ and $\Pi_0=\{\alpha_0,\alpha_1,...,\alpha_r\}$ for the set of simple positive roots of affine Lie algebra $\widehat{\mathfrak{g}}$.

Quantum affine Lie algebra $U_q(\widehat{\mathfrak{g}})$ is generated by central elements $\gamma^{\pm1}$, Chevalley generators $e_{\pm\alpha_i}$, $k_{\alpha_i}^{\pm1}$, where $i=0,1,...,r$ and grading elements $D^{\pm1}$, such that $\gamma^2=k_\delta=\prod_{i=0}^r k_{\alpha_i}^{n_i}$, subject to the relations

$$De_{\pm\alpha_i}D^{-1} = q^{\pm\delta_{i,0}}e_{\pm\alpha_i}, \qquad k_{\alpha_i}e_{\pm\alpha_j}k_{\alpha_i}^{-1} = q_i^{\pm a_{ij}}e_{\pm\alpha_j}\,, \tag{2.1}$$

$$[e_{\alpha_i},e_{-\alpha_j}] = \delta_{ij}\frac{k_{\alpha_i}-k_{\alpha_i}^{-1}}{q_i-q_i^{-1}}\,, \tag{2.2}$$

$$\sum_{r=0}^{n_{ij}}(-1)^r\begin{bmatrix} n_{ij} \\ r\end{bmatrix}_{q_i} e_{\pm\alpha_i}\cdots e_{\pm\alpha_i}e_{\pm\alpha_j}e_{\pm\alpha_i}\cdots e_{\pm\alpha_i}=0\,, \tag{2.3}$$

where $n_{i,j}=1-a_{i,j}$.

One of the possible Hopf structures (which we will call a standard Hopf structure) is given by the formulas:

$$\begin{array}{c}\Delta(D)=D\otimes D, \qquad \Delta(\gamma)=\gamma\otimes\gamma, \qquad \Delta(k_{\alpha_i}) = k_{\alpha_i}\otimes k_{\alpha_i}\,, \\ \Delta(e_{\alpha_i})=e_{\alpha_i}\otimes 1+k_{\alpha_i}\otimes e_{\alpha_i}\,, \quad \Delta(e_{-\alpha_i})=1\otimes e_{-\alpha_i}+e_{-\alpha_i}\otimes k_{\alpha_i}^{-1}\,, \\ \varepsilon(D)=1, \qquad \varepsilon(e_{\pm\alpha_i})=0\,, \quad \varepsilon(k_{\alpha_i}^{\pm1})=1\,, \quad \varepsilon(\gamma)=0, \quad a(\gamma)=\gamma^{-1}, \\ a(e_{\alpha_i})=-k_{\alpha_i}^{-1}e_{\alpha_i}\,, \quad a(e_{-\alpha_i})=-e_{-\alpha_i}k_{\alpha_i}\,, \quad a(k_{\alpha_i}^{\pm1})=k_{\alpha_i}^{\mp1}\,, \quad a(D)=D^{-1}\end{array} \tag{2.4}$$

where Δ, ε and a are comultiplication, counit and antipode maps respectively.

In the following we use sometimes the shortened notations $U=U_q(\widehat{\mathfrak{g}})$ for quantum affine algebra $U_q(\widehat{\mathfrak{g}})$.

2.2 The current realization of $U_q(\widehat{\mathfrak{g}})$

The quantum affine algebra, $U_q(\widehat{\mathfrak{g}})$ admits current realization [2]. In this description $U_q(\widehat{\mathfrak{g}})$ is generated by central element γ, grading element D, and by the elements $e_\alpha[n]$,

$f_\alpha[n]$, where $\alpha \in \Pi$, $n \in \mathbb{Z}$; $k_\alpha^{\pm 1}$, $h_\alpha[n]$, where $n \in \mathbb{Z}\backslash\{0\}$, $\alpha \in \Pi$. They are gathered into generating functions

$$e_\alpha(z) \;=\; \sum_{n\in\mathbb{Z}} e_\alpha[n]z^{-n}\;, \quad f_\alpha(z) \;=\; \sum_{n\in\mathbb{Z}} f_\alpha[n]z^{-n}\;, \tag{2.5}$$

$$\psi_\alpha^\pm(z) \;=\sum_{n>0}\psi_\alpha^\pm[n]z^{\mp n} = k_\alpha^{\pm 1}\exp\left(\pm(q_\alpha - q_\alpha^{-1})\sum_{n>0}h_\alpha[\pm n]z^{\mp n}\right)\;, \tag{2.6}$$

such that $Da(z)D^{-1} = a(qz)$ for any of these generating functions, and

$$\begin{gathered}
(z - q^{(\alpha,\beta)}w)e_\alpha(z)e_\beta(w) = e_\beta(w)e_\alpha(z)(q^{(\alpha,\beta)}z - w)\;,\\
(z - q^{-(\alpha,\beta)}w)f_\alpha(z)f_\beta(w) = f_\beta(w)f_\alpha(z)(q^{-(\alpha,\beta)}z - w)\;,\\
\psi_\alpha^\pm(z)e_\beta(w)\left(\psi_\alpha^\pm(z)\right)^{-1} = \frac{(q^{(\alpha,\beta)}\gamma^{\pm 1}z - w)}{(\gamma^{\pm 1}z - q^{(\alpha,\beta)}w)}e_\beta(w)\;,\\
\psi_\alpha^\pm(z)f_\beta(w)\left(\psi_\alpha^\pm(z)\right)^{-1} = \frac{(q^{-(\alpha,\beta)}\gamma^{\mp 1}z - w)}{(\gamma^{\mp 1}z - q^{-(\alpha,\beta)}w)}f_\beta(w)\;,\\
\psi_\alpha^\pm(z)\psi_\beta^\pm(w) = \psi_\beta^\pm(w)\psi_\alpha^\pm(z)\;,\\
\frac{(q^{(\alpha,\beta)}z - \gamma^2 w)}{(z - q^{(\alpha,\beta)}\gamma^2 w)}\psi_\alpha^+(z)\psi_\beta^-(w) = \frac{(q^{(\alpha,\beta)}\gamma^2 z - w)}{(\gamma^2 z - q^{(\alpha,\beta)}w)}\psi_\beta^-(w)\psi_\alpha^+(z)\;,\\
[e_\alpha(z), f_\beta(w)] = \frac{\delta_{\alpha,\beta}}{q_i - q_i^{-1}}\left(\delta(z/\gamma^2 w)\;\psi_\alpha^+(\gamma^{-1}z) - \delta(\gamma^2 z/w)\psi_\alpha^-(\gamma^{-1}w)\right)\;,
\end{gathered} \tag{2.7}$$

where $\alpha, \beta \in \Pi$, $\delta(z) = \sum_{k\in\mathbb{Z}} z^k$, and

$$\sum_{r=0}^{n_{ij}}(-1)^r \begin{bmatrix} n_{ij}\\ r\end{bmatrix}_{q_i} \mathrm{Sym}_{z_1,\ldots,z_{n_{ij}}} e_{\pm\alpha_i}(z_1)\cdots e_{\pm\alpha_i}(z_r)e_{\pm\alpha_j}(w)e_{\pm\alpha_i}(z_{r+1})\cdots e_{\pm\alpha_i}(z_{n_{ij}}) = 0 \tag{2.8}$$

where $\alpha_i, \alpha_j \in \Pi$, $\alpha_i \neq \alpha_j$.

Let $\theta = \delta - \alpha_0 = \sum_{i=1}^r n_r\alpha_r$ denotes the longest root of Lie algebra $\mathfrak{g}$. Suppose that the positive root vector $e_\theta \in \mathfrak{g}$ is presented as a multiple commutator $e_\theta = \lambda[e_{\alpha_{i_1}}, [e_{\alpha_{i_2}}, \cdots [e_{\alpha_{i_n}}, e_{\alpha_j}]\cdots]]$ for some $\lambda \in \mathbb{C}$.

Then the assignment [2, 4]

$$\begin{aligned}
&k_{\alpha_i} \mapsto k_{\alpha_i}\,, \qquad e_{\alpha_i} \mapsto e_{\alpha_i}[0]\,, \qquad e_{-\alpha_i} \mapsto f_{\alpha_i}[0]\,, \qquad i = 1, ..., r,\\
&k_{\alpha_0} \to \gamma^2 k_\theta^{-1} = \gamma^2\prod_{i=1}^r k_{\alpha_i}^{-n_i}, \qquad D \mapsto D,\\
&e_{\alpha_0} \mapsto \mu S_{i_1}^- S_{i_2}^- \cdots S_{i_n}^-(f_{\alpha_j}[1])\,, \qquad e_{-\alpha_0} \mapsto \lambda S_{i_1}^+ S_{i_2}^+ \cdots S_{i_n}^+(e_{\alpha_j}[-1])
\end{aligned} \tag{2.9}$$

establishes the isomorphism of two realizations. Here $S_i^\pm : U \to U$ are the following operators of adjoint action:

$$S_i^+(x) = e_{\alpha_i}[0]x - k_{\alpha_i}xk_{\alpha_i}^{-1}e_{\alpha_i}[0], \qquad S_i^-(x) = xf_{\alpha_i}[0] - f_{\alpha_i}[0]k_{\alpha_i}xk_{\alpha_i}^{-1}\;,$$

and the constant μ is chosen in such a way, that the relation (2.2) for $i = j = 0$ will remain valid in the image.

Another Hopf structure $\Delta^{(D)}$ in $U_q(\widehat{\mathfrak{g}})$ is naturally related to the current realization. In terms of currents it looks as follows (here γ_1 and γ_2 means $\gamma\otimes 1$ and $1\otimes\gamma$ respectively):

$$\Delta^{(D)}(D) = D\otimes D, \qquad \Delta^{(D)}(\gamma) = \gamma\otimes\gamma,$$
$$\Delta^{(D)}e_\alpha(z) = e_\alpha(z)\otimes 1 + \psi^-_\alpha(\gamma_1 z)\otimes e_\alpha(\gamma_1^2 z)\,,$$
$$\Delta^{(D)}f_\alpha(z) = 1\otimes f_\alpha(z) + f_\alpha(\gamma_2^2 z)\otimes \psi^+_\alpha(\gamma_2 z)\,,$$
$$\Delta^{(D)}\psi^\pm_\alpha(z) = \psi^\pm_\alpha(\gamma_2^{\pm 1} z)\otimes \psi^\pm_\alpha(\gamma_1^{\mp 1} z)\,,$$
$$a^{(D)}\left(e_\alpha(z)\right) = -\left(\psi^-_\alpha\left(\gamma^{-1}z\right)\right)^{-1} e_\alpha(\gamma^{-2}z)\,, \quad a^{(D)}\left(f_\alpha(z)\right) = -f_\alpha(\gamma^{-2}z)\,\left(\psi^+_\alpha\left(\gamma^{-1}z\right)\right)^{-1},$$
$$a^{(D)}\left(\psi^\pm_\alpha(z)\right) = \left(\psi^\pm_\alpha(z)\right)^{-1}, \quad \varepsilon(e_\alpha(z)) = \varepsilon(f_\alpha(z)) = 0\,, \quad \varepsilon(\psi^\pm_\alpha(z)) = 1\,.$$

The comultiplications Δ of Section 2.1 and $\Delta^{(D)}$ are related by the twist, which can be described explicitely. See Proposition 3.5.

2.3 Borel subalgebras of $U_q(\widehat{\mathfrak{g}})$

Denote by $U_q(\mathfrak{b}_+)$ the subalgebra of $U_q(\widehat{\mathfrak{g}})$, generated by the elements e_{α_i} and $k^{\pm1}_{\alpha_i}$, $i\in\Pi_0$ in Chevalley description of $U_q(\widehat{\mathfrak{g}})$. Denote also by $U_q(\mathfrak{b}_-)$ the subalgebra of $U_q(\widehat{\mathfrak{g}})$, generated by the elements $e_{-\alpha_i}$ and $k^{\pm1}_{\alpha_i}$, $i\in\Pi_0$.

The algebras $U_q(\mathfrak{b}_\pm)$ are Hopf subalgebras of $U_q(\widehat{\mathfrak{g}})$ with respect to standard comultiplication Δ and serve as q-deformations of the enveloping algebras of opposite Borel subalgebras of Lie algebra $\widehat{\mathfrak{sl}}_3$. We call them standard Borel subalgebras. They contain subalgebras $U_q(\mathfrak{n}_\pm)$, which are generated by the elements $e_{\pm\alpha_i}$, $i\in\Pi_0$.

The subalgebra $U_q(\mathfrak{n}_+)$ is a left coideal of $U_q(\mathfrak{b}_+)$ with respect to standard comultiplication and the subalgebra $U_q(\mathfrak{n}_-)$ is a right coideal of $U_q(\mathfrak{b}_-)$ with respect to the same comultiplication, that is

$$\Delta(U_q(\mathfrak{n}_+) \subset U_q(\mathfrak{b}_+)\otimes U_q(\mathfrak{n}_+)\,, \qquad \Delta(U_q(\mathfrak{n}_-)) \subset U_q(\mathfrak{n}_-))\otimes U_q(\mathfrak{b}_-)\,.$$

The algebras $U_q(\mathfrak{n}_\pm)$ serve as q-deformed enveloping algebras of standard nilpotent subalgebras of Lie algebra $\widehat{\mathfrak{g}}$.

Borel subalgebras of another type are related to current realization of $U_q(\widehat{\mathfrak{g}})$.

Denote by U_F the subalgebra of $U_q(\widehat{\mathfrak{g}})$, generated by the elements $k^{\pm1}_\alpha$, $f_\alpha[n]$, where $\alpha\in\Pi$, $n\in\mathbb{Z}$; $h_\alpha[n]$, $\alpha\in\Pi$, $n>0$ and by U_E the subalgebra of $U_q(\widehat{\mathfrak{g}})$, generated by the elements $k^{\pm1}_\alpha$, $e_\alpha[n]$, where $\alpha\in\Pi$, $n\in\mathbb{Z}$; $h_\alpha[n]$, $\alpha\in\Pi$, $n<0$. They are Hopf subalgebras of $U_q(\widehat{\mathfrak{g}})$ with respect to comultiplication $\Delta^{(D)}$. We call them current Borel subalgebras. Current Borel subalgebra U_F contains the subalgebra U_f, generated by the elements $f_\alpha[n]$, where $\alpha\in\Pi$, $n\in\mathbb{Z}$. Current Borel subalgebra U_E contains the subalgebra U_e, generated by the elements $e_\alpha[n]$, where $\alpha\in\Pi$, $n\in\mathbb{Z}$.

The algebra U_f is a right coideal of U_F with respect to $\Delta^{(D)}$, the algebra U_e is a left coideal of U_E with respect to $\Delta^{(D)}$:

$$\Delta^{(D)}(U_f)\subset U_f\otimes U_F, \qquad \Delta^{(D)}(U_e)\subset U_E\otimes U_e\ .$$

They serve as q-deformed enveloping algebras of algebra of currents into nilpotent subalgebras $\mathfrak{n}_\pm$.

In the following we are interested in intersections of Borel subalgebras of different type. Denote by U_F^+, U_f^-, U_e^+ and U_E^- the following intersections of current Borel algebras:

$$U_f^- = U_F \cap U_q(\mathfrak{b}_-) = U_F \cap U_q(\mathfrak{n}_-)\,, \qquad U_F^+ = U_F \cap U_q(\mathfrak{b}_+)\,, \tag{2.10}$$
$$U_e^+ = U_E \cap U_q(\mathfrak{b}_+) = U_E \cap U_q(\mathfrak{n}_+)\,, \qquad U_E^- = U_E \cap U_q(\mathfrak{b}_-)\,. \tag{2.11}$$

The notataions are given in such a way, that an upper sign says which Borel subalgebra $U_q(\mathfrak{b}_\pm)$ contains the given algebra, the lower letter says to which current Borel subalgebra U_F or U_E, it is included. These letter are capital, if the subalgebra contains imaginary root generators $h_i[n]$ and is small otherwise. These intersections have coideal properties with respect to both comultiplications:

$$\begin{aligned}
\Delta^{(D)}(U_F^+) &\subset U \otimes U_F^+, & \Delta^{(D)}(U_f^-) &\subset U_f^- \otimes U,\\
\Delta^{(D)}(U_E^-) &\subset U_E^- \otimes U, & \Delta^{(D)}(U_e^+) &\subset U \otimes U_e^+,\\
\Delta(U_F^+) &\subset U \otimes U_F^+, & \Delta(U_f^-) &\subset U_f^- \otimes U,\\
\Delta(U_E^-) &\subset U_E^- \otimes U, & \Delta(U_e^+) &\subset U \otimes U_e^+.
\end{aligned}$$

3 Orthogonal decompositions and twists

3.1 Orthogonal decompositions of Hopf algebras

In these section we review the theory of orthogonal decompositions of Hopf algebras, following [3, 6].

Let $\mathcal{A}$ be a bialgebra with unit 1 and counit ε. We say that its subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ determine an orthogonal decomposition of $\mathcal{A}$, if

(i) Algebra $\mathcal{A}$ admits a decomposition $\mathcal{A} = \mathcal{A}_1\mathcal{A}_2$, that is the multiplication map $\mu : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{A}$ establishes an isomorphism of linear spaces;

(ii) $\mathcal{A}_1$ is left coideal, $\mathcal{A}_2$ is right coideal:

$$\Delta(\mathcal{A}_1) \subset \mathcal{A} \otimes \mathcal{A}_1\ , \qquad \Delta(\mathcal{A}_2) \subset \mathcal{A}_2 \otimes \mathcal{A}\ . \tag{3.1}$$

In this case operators

$$P_1 : P_1(a_1a_2) = a_1\varepsilon(a_2)\ , \quad P_2 : P_2(a_1a_2) = \varepsilon(a_1)a_2\ , \quad a_1 \in \mathcal{A}_1\ , \quad a_2 \in \mathcal{A}_2$$

are well defined projection operators from $\mathcal{A}$ to $\mathcal{A}_i$, such that for any $a \in \mathcal{A}$ with $\Delta(a) = \sum_i a_i' \otimes a_i''$

$$\sum_i P_1(a_i')P_2(a_i'') = a\ . \tag{3.2}$$

For the proof of (3.2) denote by $\phi : \mathcal{A} \to \mathcal{A}$ the linear map $\phi(a) = \sum_i P_1(a_i')P_2(a_i'')$ We claim that ϕ is the map of left $\mathcal{A}_1$-modules and of right $\mathcal{A}_2$-modules, that is, $\phi(a_1a) = a_1\phi(a)$, $\phi(aa_2) = \phi(a)a_2$ for any $a \in \mathcal{A}$, $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$. Indeed, in Sweedler notations we can write

$$\phi(a_1a) = P_1(a_1'a')P_2(a_1''a'').$$

From (3.1) we know that $a_1'' \in A_1$, so $P_2(a_1''a'') = \varepsilon(a_1'')P_2(a'')$ and

$$\phi(a_1 a) = P_1(a_1'\varepsilon(a_1'')a')P_2(a'') = P_1(a_1 a')P_2(a'') = a_1 P_1(a')P_2(a'') = a_1\phi(a) \ .$$

Analogously, $\phi(aa_2) = \phi(a)a_2$ for $a_2 \in \mathcal{A}_2$. Since $\phi(1) = 1$, we have $\phi(a) = a$ for any $a \in \mathcal{A}$, which proves the relation (3.2).

Let now $\mathcal{B}$ be a bialgebra dual to $\mathcal{A}$ with opposite comultiplication, that is there exists nondegenerate Hopf pairing $\langle\, , \rangle : \mathcal{A} \otimes \mathcal{B} \to \mathbb{C}$, satisfying the conditions

$$\langle a, b_1 b_2\rangle = \langle \Delta(a), b_1 \otimes b_2\rangle \ , \qquad \langle a_1 a_2, b\rangle = \langle a_2 \otimes a_1, \Delta(b)\rangle \ .$$

Let $R = \sum a^\alpha \otimes b_\alpha$ be the tensor of the pairing. Set $R_i = (P_i \otimes \mathrm{id})R$. The addition identity (3.2) yields the factorization

$$R = R_1 \cdot R_2 \ . \tag{3.3}$$

Indeed, the tensor R is uniquely characterized by one of the properties

$$\langle R, b \otimes 1\rangle = b, \quad \text{for any}\, b \in \mathcal{B}, \qquad \langle R, 1 \otimes a\rangle = a \quad \text{for any}\, a \in \mathcal{A} \ .$$

Let us calculate $\langle R_1 R_2, 1 \otimes a\rangle$. We have

$$\langle R_i, 1 \otimes a\rangle = \langle (P_i \otimes \mathrm{id})R, 1 \otimes a\rangle = P_i\langle R, 1 \otimes a\rangle = P_i(a) \ .$$

Then

$$\langle R_1 R_2, 1 \otimes a\rangle = \langle R_1, 1 \otimes a'\rangle\langle R_2, 1 \otimes a''\rangle = P_1(a')P_2(a'') = a$$

due to (3.2). It proves (3.3).

Denote by $\mathcal{B}^{op}$ the bialgebra $\mathcal{B}$ with opposite comultiplication. Suppose that subalgebras $\mathcal{B}_1 \subset \mathcal{B}$ and $\mathcal{B}_2 \subset \mathcal{B}$ determine an orthogonal decomposition of the bialgebra $\mathcal{B}^{op}$, that is, the subalgebra $\mathcal{B}_1$ is a right coideal of $\mathcal{B}$, $\Delta(\mathcal{B}_1) \subset \mathcal{B}_1 \otimes \mathcal{B}$, the subalgebra $\mathcal{B}_2$ is a left coideal of $\mathcal{B}$, $\Delta(\mathcal{B}_2) \subset \mathcal{B} \otimes \mathcal{B}_2$ and the multiplication rule in $\mathcal{B}$ establishes an isomorphism of vector spaces $\mathcal{B}$ and $\mathcal{B}_1 \otimes \mathcal{B}_2$.

Let $\widetilde{P}_1 : \mathcal{B} \to \mathcal{B}_1$ and $\widetilde{P}_2 : \mathcal{B} \to \mathcal{B}_2$ be the corresponding projection operators, $\widetilde{P}_1(b_1 b_2) = b_1\varepsilon(b_2)$, $\widetilde{P}_2(b_1 b_2) = b_1\varepsilon(b_2)$, and

$$R = \widetilde{R}_1 \cdot \widetilde{R}_2 \ . \tag{3.4}$$

where $\widetilde{R}_i = (\mathrm{id} \otimes \widetilde{P}_i)R$ be the related decomposition of the tensor of the pairing.

Proposition 3.1 *Decompositions (3.3) and (3.4) coincide,*

$$R_i = (P_i \otimes \mathrm{id})R = (\mathrm{id} \otimes \widetilde{P}_i)R, \qquad i = 1, 2 \tag{3.5}$$

if and only if subalgebras $\mathcal{A}_i$ and $\mathcal{B}_j$ are mutually orthogonal, that is

$$\langle a_i, b_j\rangle = \varepsilon(a_i)\varepsilon(b_j), \qquad \text{for any} \quad a_i \in \mathcal{A}_i, b_j \in \mathcal{B}_j, \quad i \neq j \ . \tag{3.6}$$

We call further the decompositions $\mathcal{A} = \mathcal{A}_1\mathcal{A}_2$, $\mathcal{B} = \mathcal{B}_1\mathcal{B}_2$, satisfying the conditions of proposition 3.1, as biorthogonal decompositions of $\mathcal{A}$ and $\mathcal{B} \equiv (\mathcal{A}^*)^{op}$.

Proof. Compute $\langle R_1, b\otimes a\rangle$ and $\langle \tilde{R}_1, b\otimes a\rangle$ for any $a \in \mathcal{A}$, $b \in \mathcal{B}$. We have

$$\begin{aligned}\langle R_1, b_1b_2\otimes a_1a_2\rangle &= \langle a_1\varepsilon(a_2), b_1b_2\rangle = \varepsilon(a_2)\langle a_1', b_1\rangle\langle a_1'', b_2\rangle = \\ &= \varepsilon(a_2)\varepsilon(b_2)\langle a_1'\varepsilon(a_1''), b_1\rangle = \varepsilon(a_2)\varepsilon(b_2)\langle a_1, b_1\rangle \ ,\end{aligned} \tag{3.7}$$

since $a_1'' \in \mathcal{A}_1$. Analogously,

$$\begin{aligned}\langle \tilde{R}_1, b_1b_2\otimes a_1a_2\rangle &= \langle a_1a_2, b_1\varepsilon(b_2)\rangle = \varepsilon(b_2)\langle a_1, b_1''\rangle\langle a_2, b_1'\rangle = \\ &= \varepsilon(a_2)\varepsilon(b_2)\langle a_1, b_1''\varepsilon(b_1')\rangle = \varepsilon(a_2)\varepsilon(b_2)\langle a_1, b_1\rangle \ ,\end{aligned} \tag{3.8}$$

since $b_1' \in \mathcal{B}_1$. We see that $R_1 = \tilde{R}_1$. The same story takes place for other pair.

In other direction. Suppose that decompositions (3.3) and (3.4) coincide. The substitution $b_1 = a_2 = 1$ into the first line of (3.7) gives

$$\langle R_1, b_2\otimes a_1\rangle = \langle a_1', 1\rangle\langle a_1'', b_2\rangle = \varepsilon(a_1')\langle a_1'', b_2\rangle = \langle a_1, b_2\rangle \ . \tag{3.9}$$

On the other hand, the first line of equality (3.8) shows that the dependence of (3.9) on b_2 is precisely the factor $\varepsilon(b_2)$. Presenting $b_2 = \varepsilon(b_2)1 + \tilde{b}_2$, where $\varepsilon(\tilde{b}_2) = 0$ we deduce that $\langle a_1, b_2\rangle = \varepsilon(a_1)\varepsilon(b_2)$. □

Suppose now that $\mathcal{A} = \mathcal{A}_1\mathcal{A}_2$, $\mathcal{B} = \mathcal{B}_1\mathcal{B}_2$ are biorthogonal decompositions of *Hopf* algebras $\mathcal{A}$ and $\mathcal{B} \equiv (\mathcal{A}^*)^{op}$. Let $R = R_1R_2$ be the corresponding factorization of the tensor of the pairing, equal to the universal R-matrix of the double $D(\mathcal{A})$.

Denote by $D^{R_1}(\mathcal{A})$ the double $D(\mathcal{A})$ with twisted comultiplication

$$\Delta^{R_1}(x) = \left(R_1^{21}\right)^{-1}\Delta(x)R_1^{21} = R_2^{21}\Delta^{op}(x)\left(R_1^{21}\right)^{-1} \ .$$

Proposition 3.2

(i) *The tensors* $(R_1^{21})^{-1}$ *and* R_2 *are two-cocycles in the double* $D(\mathcal{A})$*:*

$$R_2^{12}\cdot(\Delta\otimes 1)\,R_2 = R_2^{23}\cdot(1\otimes\Delta)\,R_2 \ , \tag{3.10}$$

$$(\Delta^{op}\otimes 1)\,R_1\cdot R_1^{12} = (1\otimes\Delta^{op})\,R_1\cdot R_1^{23} \ . \tag{3.11}$$

so that $D^{R_1}(\mathcal{A})$ *is a Hopf algebra;*

(ii) *Let* $\widetilde{\mathcal{A}}$ *be the subalgebra of* $D^{R_1}(\mathcal{A})$*, generated by* $\mathcal{A}_2$ *and* $\mathcal{B}_1$*,* $\widetilde{\mathcal{B}}$ *be the subalgebra of* $D^{R_1}(\mathcal{A})$*, generated by* $\mathcal{A}_1$ *and* $\mathcal{B}_2$*. They are Hopf subalgebras of* $D^{R_1}(\mathcal{A})$*;*

(iii) *The restriction of the Hopf pairing* $\langle\mathcal{A},\mathcal{B}\rangle \to \mathbb{C}$ *to subalgebras* $\mathcal{A}_i$*,*$\mathcal{B}_j$ *extends to a nondegenerate Hopf (with respect to* Δ^{R_1}*) pairing* $\langle\widetilde{\mathcal{A}},\widetilde{\mathcal{B}}\rangle \to \mathbb{C}$*;*

(iv) *We have biorthogonal decompositions* $\widetilde{\mathcal{A}} = \mathcal{A}_2\mathcal{B}_1$*,* $\widetilde{\mathcal{B}} = \mathcal{B}_2\mathcal{A}_1$*.*

We call biorthogonal decompositions $\widetilde{\mathcal{A}} = \mathcal{A}_2\mathcal{B}_1$, $\widetilde{\mathcal{B}} = \mathcal{B}_2\mathcal{A}_1$ the *mutation* of biorthogonal decompositions $\mathcal{A} = \mathcal{A}_1\mathcal{A}_2$, $\mathcal{B} = \mathcal{B}_1\mathcal{B}_2$.

Remark 3.3 Formal algebraical manipulations show that $D^{R_1}(\mathcal{A})$ is a quasitriangular Hopf algebra with universal R-matrix $\overline{R} = R_2R_1^{21}$ and is isomorphic as a Hopf algebra to the double $D(\widetilde{\mathcal{A}})$. Nevertheless, if we do not take special care of topology and convergence, products $R_1^{21}R_2^{23}$ and $R_1^{32}R_2^{12}$ can be not well defined as well as the Yang–Baxter equation for $\overline{R}$.

We present the proof of (i). The rest is left to the reader. One can see that both sides of (3.10) belong to $\mathcal{A}_2 \otimes \mathcal{D}(\mathcal{A}) \otimes \mathcal{B}_2$. On the other hand, we have

$$(\Delta^{op} \otimes 1)\, R_1 \cdot R_1^{12} \cdot R_2^{12} \cdot (\Delta \otimes 1)\, R_2 = (1 \otimes \Delta^{op})\, R_1 \cdot R_1^{23} \cdot R_2^{23} \cdot (1 \otimes \Delta)\, R_2$$

due to the properties of universal R-matrix, so the coassociator

$$\Phi = R_2^{12} \cdot (\Delta \otimes 1)\, R_2 \cdot \left(R_2^{23} \cdot (1 \otimes \Delta)\, R_2\right)^{-1}$$

can be presented also as

$$\Phi = \left((\Delta^{op} \otimes 1)\, R_1 \cdot R_1^{12}\right)^{-1} \cdot (1 \otimes \Delta^{op})\, R_1 \cdot R_1^{23}$$

and thus belongs to the intersection of $\mathcal{A}_2 \otimes \mathcal{D}(\mathcal{A}) \otimes \mathcal{B}_2$ and $\mathcal{A}_1 \otimes \mathcal{D}(\mathcal{A}) \otimes \mathcal{B}_1$, which means that it has a form $1 \otimes d \otimes 1$ for some $d \in \mathcal{D}(\mathcal{A})$. Then the pentagon identity on Φ says that there is no nontrivial coassociator of such a form. $\square$

3.2 Orthogonal decompositions related to Lusztig automorphisms

G.Lusztig [10] defined automorphisms $T_i \in \mathrm{End}(U)$, $i = 0, 1, ..., r$ which are called Lusztig automorphisms. They satisfy braid group relations and are given as (we essentually replace ν in [10] by q^{-1})

$$T_i(e_{\alpha_i}) = -e_{-\alpha_i} k_{\alpha_i}^{-1}, \qquad T_i(e_{\alpha_j}) = \sum_{p+s=-a_{i,j}} (-1)^p q_i^s e_{\alpha_i}^{(p)} e_{\alpha_j} e_{\alpha_i}^{(s)} \qquad \text{if} \quad i \neq j,$$
$$T_i(e_{-\alpha_i}) = -k_{\alpha_i} e_{\alpha_i}, \qquad T_i(e_{-\alpha_j}) = \sum_{p+s=-a_{i,j}} (-1)^p q_i^{-s} e_{-\alpha_i}^{(s)}, e_{-\alpha_j} e_{-\alpha_i}^{(p)} \qquad \text{if} \quad i \neq j$$

where $e_{\pm\alpha_i}^{(p)} = e_{\pm\alpha_i}^p / [p]_{q_i}!$.

The automorphisms T_i are compatible with comultiplication Δ in the following sense. For any automorphism T of the algebra U denote by $\Delta^T : U \to U \otimes U$ the comultiplication $\Delta^T(x) = T \otimes T\, (\Delta(T^{-1}(x)))$. Then

$$\Delta^{T_i}(x) = R_{\alpha_i}^{21} \Delta(x) \left(R_{\alpha_i}^{21}\right)^{-1}, \tag{3.12}$$

where $R_{\alpha_i} = \exp_{q_i}((q_i^{-1} - q_i) e_{\alpha_i} \otimes e_{-\alpha_i})$, $\exp_q(x) = \sum_{n \geq 0} \frac{x^n}{(n)_q!}$ and $(n)_q = \frac{(q^n-1)}{(q-1)}$.

In other words, the semidirect product of U and the braid group, given by the relations $T_i x T_i^{-1} = T_i(x)$ for any $x \in U$, can be equiped with a structure of a Hopf algebra, if we put $\Delta(T_i) = R_{\alpha_i}^{21} T_i \otimes T_i$.

Any element w of the Weyl group W of Lie algebra $\widehat{\mathfrak{g}}$ uniquely determines an automorphism $T_w : U \to U$ by the rule: $T_w = T_{i_1} T_{i_2} \cdots T_{i_n}$ if $w = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_n}}$ is a reduced decomposition of w into a product of simple reflections. Put

$$\mathcal{A}_1^w = U_q(\mathfrak{b}_+) \cap T_w(U_q(\mathfrak{n}_-)), \qquad \mathcal{A}_2^w = U_q(\mathfrak{b}_+) \cap T_w(U_q(\mathfrak{b}_+)), \tag{3.13}$$
$$\mathcal{B}_1^w = U_q(\mathfrak{b}_-) \cap T_w(U_q(\mathfrak{n}_+)), \qquad \mathcal{B}_2^w = U_q(\mathfrak{b}_-) \cap T_w(U_q(\mathfrak{b}_-)). \tag{3.14}$$

Proposition 3.4 *For any $w \in W$ the algebras $\mathcal{A}_1^w$, $\mathcal{A}_2^w$, $\mathcal{B}_1^w$, $\mathcal{B}_2^w$ form biorthogonal decompositions $U_q(\mathfrak{b}_+) = \mathcal{A}_1^w \mathcal{A}_2^w$ and $U_q(\mathfrak{b}_-) = \mathcal{B}_1^w \mathcal{B}_2^w$ of dual Hopf algebras $U_q(\mathfrak{b}_\pm)$*

Proof. Coideal properties and orthogonality conditions follow inductively from (3.12). The decomposition condition (i) of section 3.1 is the consequence of the theory of Cartan-Weyl basis, see [1, 9]. □

Recall that the total ordering $<$ of the system of positive roots Δ_+ of Lie algebra $\widehat{\mathfrak{g}}$ is called *normal* if for any positive roots α, β, $\alpha < \beta$ such that one of them is real and $\alpha + \beta$ is a root we have $\alpha < \alpha + \beta < \beta$. Let δ be a minimal positive imaginary root of Δ_+, α_0 a positive affine root, $\alpha_1, ..., \alpha_r$ be simple positive roots of $\mathfrak{g}$. Choose a normal ordering, satisfying the following conditions:

(i) $\alpha_k < \delta$ for all $k = 1, ..., r$, and $\delta < \alpha_0$,

(ii) all positive real roots, which are less then δ are enumerated successively by integer numbers: $\gamma_1 < \gamma_2 < .... < \delta$.

Such a normal ordering (more precisely, its part, preceeding δ), determines a sequence $\alpha_{i_1}, \alpha_{i_2}, ...\alpha_{i_n}, ...$ of positive simple roots and a sequence $w_0, w_1, w_2, ..., w_n, ...$ of the elements of W by the inductive rule:

$$w_0 = 1, \quad \alpha_{i_1} = \gamma_1, \qquad \text{and} \qquad w_{n+1} = w_n \cdot s_{\alpha_{i_{n+1}}}, \quad \alpha_{i_{n+1}} = w_n^{-1}(\gamma_{n+1}) \ .$$

For the sequence $w_1, w_2, ...$ we have inclusions:

$$\mathcal{A}_1^{w_1} \subset \mathcal{A}_1^{w_2} \subset ... \subset \mathcal{A}_1^{w_n} \subset ..., \quad \mathcal{A}_2^{w_1} \supset \mathcal{A}_2^{w_2} \supset ... \supset \mathcal{A}_2^{w_n} \supset ... \ ,$$
$$\mathcal{B}_1^{w_1} \subset \mathcal{B}_1^{w_2} \subset ... \subset \mathcal{B}_1^{w_n} \subset ..., \quad \mathcal{B}_2^{w_1} \supset \mathcal{B}_2^{w_2} \supset ... \supset \mathcal{B}_2^{w_n} \supset ... \ .$$

Moreover, the algebra $\mathcal{A}_1^{w_n}$ is generated by Cartan-Weyl generators $e_{\gamma_k} = T_{w_{k-1}}(e_{\alpha_{i_k}})$ with $k \leq n$ and the algebra $\mathcal{B}_1^{w_n}$ is generated by Cartan-Weyl generators $e_{-\gamma_k} = T_{w_{k-1}}(e_{-\alpha_{i_k}})$ with $k \leq n$.

We have the equalities

$$U_e^+ = \cup_n \mathcal{A}_1^{w_n}, \quad U_F^+ = \cap_n \mathcal{A}_2^{w_n}, \quad U_f^- = \cup_n \mathcal{B}_1^{w_n}, \quad U_E^- = \cap_n \mathcal{B}_2^{w_n} \ .$$

and biorthogonal decompositions

$$U_q(\mathfrak{b}_+) = U_e^+ U_F^+ \qquad \text{and} \qquad U_q(\mathfrak{b}_-) = U_f^- U_E^- \ . \tag{3.15}$$

Denote by $\mathcal{R} \in U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}^-)$ the universal $\mathcal{R}$ matrix of the algebra U and by

$$\mathcal{R} = \mathcal{R}_1 \mathcal{R}_2 \tag{3.16}$$

its factorization with respect to biorthogonal decompositions (3.15). We have

Proposition 3.5 *Drinfeld comultiplication $\Delta^{(D)}$ is the twist of canonical comultiplication Δ by means of the cocycle $\mathcal{R}_1^{21}$: for any $x \in U$ we have*

$$\Delta^{(D)}(x) \ = \ \Delta^{\mathcal{R}_1}(x) \ = \ \left(\mathcal{R}_1^{21}\right)^{-1} \Delta(x) \mathcal{R}_1^{21} \ . \tag{3.17}$$

Note that here, unlike to proposition 3.4, we are precisely in the setting of remark 3.3. Thus we treate further the product $\overline{\mathcal{R}} = \mathcal{R}_2 \mathcal{R}_1^{21}$ just as a tensor of the pairing and not as a universal R-matrix.

The proof of proposition 3.5 follows from calculation of [9], where the limit in a proper completion of the comultiplication Δ twisted by power of affine shift in the Weyl group of $U_q(\widehat{\mathfrak{sl}}_2)$ was calculated. It is sufficient for the calculation of the comultiplication $\Delta^{\mathcal{R}_1}$ of any Drinfeld current. □

3.3 Decompositions of current Borel subalgebras

There are two pairs of biorthogonal decompositions of current Borel algebras, equiped with comultiplication $\Delta^{(D)}$. The first pair consists of decompositions $U_F = U_F^+ U_f^-$, and $U_E = U_E^- U_e^+$; the second consists of decompositions $U_E = U_e^+ U_E^-$ and $U_F = U_f^- U_F^+$.

Denote the projection operators in Borel subalgebra U_F, correesponding to the first decomposition, by $\check{P}^\pm$, and projection operators in the same subalgebra with respect to second decomposition by $P^\pm$, so that for any $f_+ \in U_F^+$ and any $f_- \in U_f^-$

$$\check{P}^+(f_+f_-) = f_+\varepsilon(f_-), \qquad \check{P}^-(f_+f_-) = \varepsilon(f_+)f_-, \tag{3.18}$$
$$P^+(f_-f_+) = \varepsilon(f_-)f_+, \qquad P^-(f_-f_+) = f_-\varepsilon(f_+)\ . \tag{3.19}$$

The operator P^+ will be also denoted as P with all indices suppressed.

Denote by $\overline{\mathcal{R}}$ the tensor of the Hopf pairing $U_F \otimes U_E \to \mathbb{C}$ with respect to comultiplication $\Delta^{(D)}$ and let

$$\overline{\mathcal{R}} = \overline{\mathcal{R}}_1\overline{\mathcal{R}}_2, \qquad \overline{\mathcal{R}}_1 = (\check{P}^+ \otimes \mathrm{id})\overline{\mathcal{R}}, \quad \overline{\mathcal{R}}_2 = (\check{P}^- \otimes \mathrm{id})\overline{\mathcal{R}} \tag{3.20}$$

be the factorization of $\overline{\mathcal{R}}$ related to biorthogonal decompositions $U_F = U_F^+ U_f^-$, and $U_E = U_E^- U_e^+$. Since this decompositions are obtained by a mutation of biorthogonal decompositions (3.15), we have the equality

$$\overline{\mathcal{R}}_1 = \mathcal{R}_2, \qquad \overline{\mathcal{R}}_2 = \mathcal{R}_1^{21} \tag{3.21}$$

where $\mathcal{R}_i$ are the factors of the decomposition (3.16) of the universal R matrix for U with respect to comultiplication Δ.

The equality (3.21) enables us to compute the universal R-matrix for quantum affine algebra U by means of applications of projections $\check{P}^\pm$ to the tensor $\overline{\mathcal{R}}$. In section 5.2 we apply these ideas to derive an integral formula for the universal R-matrix for $U_q(\widehat{\mathfrak{sl}}_2)$.

4 Universal weight function

4.1 The definition

Let V be a representation of $U_q(\widehat{\mathfrak{g}})$ and v be a vector in V. We call v a highest weight vector with respect to current Borel subalgebra U_E, if

$$\begin{aligned} e_\alpha(z)v &= 0\,, & \alpha \in \Pi\,,\\ \psi_\alpha^\pm(z)v &= \lambda_\alpha(z)v\,, & \alpha \in \Pi, \end{aligned} \tag{4.1}$$

where $\lambda_\alpha(z)$ is a meromorphic function, decomposed in a series over z^{-1} for $\psi_i^+(z)$ and into a series over z for $\psi_i^-(z)$. Representation V is called a representation with highest weight vector $v \in V$ with respect to U_E, if it is generated by v over $U_q(\widehat{\mathfrak{g}})$.

An ordered set $I = i_1, ..., i_{|I|}$ together with a map $\iota : I \to \Pi$, is called ordered Π-multiset.

Suppose that for any ordered Π-multiset I, $|I| = n$, it is chosen an element $W(t_{i_1}, ..., t_{i_n})$, which is a formal power series over variables $t_{i_2}/t_{i_1}, t_{i_3}/t_{i_2}, ..., t_{i_n}/t_{i_{n-1}}$, $1/t_{i_n}$ with coefficients in polynomials $U_q(\widehat{\mathfrak{g}})[t_{i_1}, t_{i_1}^{-1}, ..., t_{i_n}, t_{i_n}^{-1}]$ such that

(1) for any highest weight with respect to U_E representation V with highest weight v the function

$$w_V(t_{i_1},...,t_{i_n}) = W(t_{i_1},...,t_{i_n})v$$

converges in a region $|t_{i_1}| \gg \cdots \gg |t_{i_n}|$ to a meromorphic V-valued function;

(2) if $I=\emptyset$ then $W=1$ and $w_V=v$;

(3) let $V=V_1\otimes V_2$ be a tensor product of highest weight representations with highest vectors v_1, v_2 and highest weight series $\{\lambda^{(1)}_\alpha(z)\}$ and $\{\lambda^{(2)}_\alpha(z)\}$, $\alpha\in\Pi$. Then for any ordered Π-multiset I we have

$$\begin{gathered} w_V(\{t_a|_{a\in\ I}\}) = \sum_{I=I_1\coprod I_2} w_{V_1}(\{t_a|_{a\in\ I_1}\})\otimes w_{V_2}(\{t_a|_{a\in\ I_2}\})\times \\ \prod_{a\in I_1}\lambda^{(2)}_{\iota(a)}(t_a)\times\prod_{a<b,\ a\in I_1,\ b\in I_2}\frac{q^{-(\iota(a),\iota(b))}t_a-t_b}{t_a-q^{-(\iota(a),\iota(b))}t_b}. \end{gathered} \tag{4.2}$$

A collection $W(t_{i_1},...,t_{i_n})$ is called a *universal weight function.* A collection $w(t_{i_1},...,t_{i_n})$ is called a *weight function.*

The weight function is closely related to off-shell Bethe vectors and is systematically used in investigations of solutions of q-difference Knizhnick-Zamolodchikov equations [7, 11, 12].

4.2 Projections and the weight function

Let $I=\{i_1,...,i_n\}$ be an ordered Π-multiset. Put

$$W(t_{i_1},...,t_{i_n}) = P\left(f_{\iota(i_1)}(t_{i_1})\cdots f_{\iota(i_n)}(t_{i_n})\right). \tag{4.3}$$

Theorem 1 *[7] The collection $W(t_{i_1},...,t_{i_n})$, defined in (4.3) is a universal weight function*

Theorem 1 follows from a bit more general statement, which is given and proved below.

Denote by J the left ideal of $U=U_q(\widehat{\mathfrak{g}})$, generated by the elements $e_\alpha[n]$, where $\alpha\in\Pi$, $n\in\mathbb{Z}$,

$$J = U\cdot\langle e_\alpha[n],\ \alpha\in\Pi, n\in\mathbb{Z}\rangle\ .$$

Theorem 2 *For any element $f\in U_F$*

$$\Delta\big(P(f)\big) = (P\otimes P)\left(\Delta^{(D)}(f)\right) \quad \mathrm{mod} \quad U\otimes J. \tag{4.4}$$

The proof of theorem 2 includes two statements, which we formulate and prove separately.

Proposition 4.1 *For any element $x\in U^-_f$ we have*

$$\Delta^{(D)}x = 1\otimes x + \sum_i a_i\otimes b_i, \quad \textit{such that} \quad a_i\in U^-_f \quad \textit{and} \quad \varepsilon(a_i)=0\,. \tag{4.5}$$

The proof of Proposition 4.1 consists of two observations. First, note that the statement (4.5) is multiplicative over x and sufficient to prove it for generators of the algebra U_f^-. Next, the algebra U_f^- is generated by the elements $f_\alpha[n]$, where $\alpha\in\Pi$, $n\le 0$. For this elements the property (4.5) follow directly from the precise form of comultiplication $\Delta^{(D)}$. $\square$

Proposition 4.2 *For any element $x\in U_F$ we have*

$$(P\otimes P)\big(\Delta^{(D)}(f)\big)=(P\otimes 1)\big(\Delta^{(D)}(f)\big) \tag{4.6}$$

For the proof of Proposition 4.2 we note, that any element $f\in U_F$ can be presented as

$$f=f_1+\sum_i x_iy_i,\qquad f_1,y_i\in U_F^+,\quad x_i\in U_f^-,\qquad \varepsilon(x_i)=0\ . \tag{4.7}$$

Proposition 4.1 implies that for any $x\in U_f^-$, such that $\varepsilon(x)=0$, we have $\Delta^{(D)}(x)=\sum_i a_i\otimes b_i$, where $a_i\in U_f^-$ and $\varepsilon(a_i)=0$. So by definition of the operator P and Proposition4.1 we have

$$(P\otimes 1)\big(\Delta^{(D)}(\sum_i x_iy_i)\big)\ =\ 0,\qquad P(\sum_i x_iy_i)\big)\ =\ 0\ , \tag{4.8}$$

such that in the notations of (4.7) we have

$$(P\otimes 1)\big(\Delta^{(D)}(f)\big)\ =\ (P\otimes 1)\big(\Delta^{(D)}(f_1)\big),\qquad P(f)=P(f_1)\ , \tag{4.9}$$

where $f_1\in U_F^+$. The equality (4.9) also implies the equality

$$(P\otimes P)\big(\Delta^{(D)}(f)\big)\ =\ (P\otimes P)\big(\Delta^{(D)}(f_1)\big)\ . \tag{4.10}$$

On the other hand, we know, that the algebra U_F^+ is the left coideal of U_F with respect to comultiplication $\Delta^{(D)}$, $\Delta^{(D)}(U_F^+)\subset U\otimes U_F^+$, which implies the equality

$$(P\otimes P)\big(\Delta^{(D)}(f_1)\big)=(P\otimes 1)\big(\Delta^{(D)}(f_1)\big)\ . \tag{4.11}$$

Combining (4.9), (4.10) and (4.11) we get the proof of Propostion 4.2. $\square$

Proof of theorem 2. Note first that due to (4.9) and (4.10) both sides of (4.4) do not change if we replace f by $f_1\in U_F^+$, according to the notations of (4.7). Taking in mind proposition 4.2, we see that it is sufficient to prove an equality

$$\Delta(f)=(P\otimes 1)\left(\Delta^{(D)}(f)\right)\mod U\otimes J\ . \tag{4.12}$$

for any $f\in U_F^+$.

Remind the relation between coproducts Δ and $\Delta^{(D)}$. By Proposition 3.5 and the equality (3.21) we have for any $x\in U$

$$\Delta^{(D)}(x)=\mathcal{R}_2^{-1}\Delta(x)\mathcal{R}_2 \tag{4.13}$$

By (4.13) and (3.20) we have $\mathcal{R}_2\in U_f^-\otimes U_E$, so

$$(P\otimes 1)\left(\Delta^{(D)}(f)\right)=(P\otimes 1)\left(\mathcal{R}_2^{-1}\Delta(f)\mathcal{R}_2\right)=(P\otimes 1)\left(\Delta(f)\right)\mod U\otimes J\ . \tag{4.14}$$

Here we drop the factor $\mathcal{R}_2$, since $\mathcal{R}_2 = 1 \bmod U \otimes J$ and drop the factor $\mathcal{R}_2^{-1}$, since its first tensor component belongs to U_f^- and $(\varepsilon\otimes 1)(\mathcal{R}_2^{-1}) = 1$. Note also that due to (4.13) $\Delta(f) \in U_F \otimes U$, so the expression $(P\otimes 1)(\Delta(f))$ is well defined. Moreover, since $U_q(\mathfrak{b})$ is a Hopf subalgebra of U with respect to comultiplication Δ and $U_F^+ = U_q(\mathfrak{b}) \cap U_F$, we have an inclusion

$$\Delta(f) \in U_F^+ \otimes U \qquad \text{for any} \quad f \in U_F^+ \ ,$$

which implies

$$(P\otimes 1)(\Delta(f)) = \Delta(f) \qquad \text{for any} \quad f \in U_F^+ \ . \tag{4.15}$$

The substitution of (4.15) into (4.14) gives (4.12) and the statement of Theorem 2. □

5 Calculations for $U_q(\widehat{\mathfrak{sl}}_2)$

In this section we present two applications of projections to intersection of Borel subalgebras. The calculation of the universal weight function for $U_q(\widehat{\mathfrak{sl}}_2)$ is taken from in [8], the integral presentation of the universal R-matrix for the same algebra is obtained by the same calculations with projection operators $\check{P}^{\pm}$.

For the sets of variables $\bar{z} = \{z_1, ..., z_k\}$ and $\bar{w} = \{w_1, ..., w_k\}$ we define the following formal power series:

$$\begin{aligned} X(\bar{z}) &= \prod_{1\le i<j\le n} \frac{1 - z_i/z_j}{q - q^{-1}z_i/z_j}, & X'(\bar{z}) &= \prod_{1\le i<j\le n} \frac{1 - z_j/z_i}{q - q^{-1}z_j/z_i}, \\ Y(\bar{z};\bar{w}) &= \prod_{i=1}^{k} \frac{1}{1 - w_i/z_i} \prod_{j=1}^{i-1} \frac{q^{-1} - q w_j/z_i}{1 - w_j/z_i}\ , & Z(\bar{z};\bar{w}) &= Y(\bar{z};\bar{w}) \prod_{i=1}^{k} \frac{w_i}{z_i} \end{aligned}$$

The notation ${}^{\hat\omega}\overline{z}$ for a set $\bar{z} = \{z_1, ..., z_k\}$ means the set $\overline{z}$ with reversed order: ${}^{\hat\omega}\overline{z} = \{z_n, \dots, z_1\}$.

5.1 The universal weight function

Theorem 3 *[8] The universal weight function for $U_q(\widehat{\mathfrak{sl}}_2)$ can be written as a following formal integral:*

$$\begin{aligned} W(t_1, \dots, t_a) &= P\left(f_\alpha(t_1)\cdots f_\alpha(t_n)\right) = \\ &= X'(\overline{t}) \oint Z(\overline{t};\, {}^{\hat\omega}\overline{u})\, f_\alpha(u_1)\frac{du_1}{u_1}\cdots f_\alpha(u_n)\frac{du_n}{u_n}\,. \end{aligned} \tag{5.1}$$

We have analogous expressions for the projections $\check{P}^{\pm}\left(f_\alpha(t_1)\cdots f_\alpha(t_n)\right)$, related to the factorization (3.20) of the tensor $\overline{\mathcal{R}}$:

$$\begin{aligned} \check{P}^{+}\left(f_\alpha(t_1)\cdots f_\alpha(t_n)\right) &= X(\overline{t}) \oint Z(\overline{t};\, \overline{u})\, f_\alpha(u_1)\frac{du_1}{u_1}\cdots f_\alpha(u_n)\frac{du_n}{u_n}\,, \\ \check{P}^{-}\left(f_\alpha(t_1)\cdots f_\alpha(t_n)\right) &= X(\overline{t}) \oint Y(\overline{u};\, \overline{t})\, f_\alpha(u_1)\frac{du_1}{u_1}\cdots f_\alpha(u_n)\frac{du_n}{u_n}\,. \end{aligned} \tag{5.2}$$

Proof. Define a set of rational functions of the variable t, depending on parameters $t_1, \ldots, t_b$:

$$\varphi_{t_j}(t; t_1, \ldots, t_b) = \prod_{i=1,\ i\neq j}^{b} \frac{t - t_i}{t_j - t_i} \prod_{i=1}^{b} \frac{q^{-1}t_j - qt_i}{q^{-1}t - qt_i}\, . \tag{5.3}$$

As functions of the variable t, they have simple poles at the points $t = q^2 t_i$, $i = 1, \ldots, b$, tend to zero when $t \to \infty$ and have properties: $\varphi_{t_j}(t_i; t_1, \ldots, t_b) = \delta_{ij}$. Set

$$f_\alpha(t; t_1, \ldots, t_b) = f_\alpha(t) - \sum_{m=1}^{b} \varphi_{t_m}(t; t_1, \ldots, t_b) f_\alpha(t_m)\, . \tag{5.4}$$

Proposition 5.1 *Projections $P\left(f_\alpha(t_1)\cdots f_\alpha(t_n)\right)$ can be presented in the factorized form*

$$P\left(f_\alpha(t_1)\cdots f_\alpha(t_n)\right) = P\left(f_\alpha(t_1)\right) P\left(f_\alpha(t_2; t_1)\right) \cdots P\left(f_\alpha(t_n; t_1, \ldots, t_{n-1})\right)\, . \tag{5.5}$$

The Proposition 5.1 implies the Theorem 3 due to the following calculation:

$$\begin{aligned} P\left(f_\alpha(t; t_1, \ldots, t_b)\right) &= \oint \frac{du}{u} f_\alpha(u) \left(\frac{1}{1 - u/t} - \sum_{m=1}^{b} \varphi_{t_m}(t; t_1, \ldots, t_b)\ \frac{1}{1 - u/t_m} \right) \\ &= \oint \frac{du}{u} f_\alpha(u)\ \frac{1}{1 - u/t} \prod_{i=1}^{b} \frac{1 - t/t_i}{1 - u/t_i}\ \frac{q - q^{-1}u/t_i}{q - q^{-1}t/t_i}\, . \end{aligned}$$

□

Proof of the Proposition 5.1. We claim first that the projection $P\left(f_\alpha(t_1)\cdots f_\alpha(t_n)\right)$ admits the following presentation:

$$P\left(f_\alpha(t_1)\cdots f_\alpha(t_n)\right) = P\left(f_\alpha(t_1)\cdots f_\alpha(t_{n-1})\right) P(f_\alpha(t_n)) + \sum_{j=1}^{n-1} \frac{G_j(t_1, ..., t_{n-1})}{t_n - q^2 t_j}\, , \tag{5.6}$$

where $G_j(t_1, ..., t_{n-1})$ are some operator-valued functions.

Proof of the relations (5.6) is based on the inductive use of the following Lemma, which shows, that during the move of the current $P^-\left(f_\alpha(t_n)\right)$ to the left calculating the projection P the only the simple poles at the points $t_n = q^2 t_j$, $j = 1, \ldots, n-1$ will appear, such that the corresponding operator valued coefficient G_j at $(t_n - q^2 t_j)^{-1}$ does not depend on t_n.

Lemma 5.2 *The following relation is valid:*

$$\begin{aligned} f_\alpha(t_1) P^-\left(f_\alpha(t_2)\right) &= \frac{q^{-1}t_1 - qt_2}{qt_1 - q^{-1}t_2} P^-\left(f_\alpha(t_2)\right) f_\alpha(t_1) \\ &\quad + \frac{qt_1(q^{-2} - q^2)}{qt_1 - q^{-1}t_2} P\left(f_\alpha(q^2 t_1)\right) f_\alpha(t_1)\, . \end{aligned} \tag{5.7}$$

Proof. The relation (5.7) follows from the application of the integral transform $\oint \frac{du}{u}\frac{1}{1-t_2/u}$ to the relation

$$f_\alpha(t_1)f_\alpha(u) = \frac{q^{-1}t_1 - qu}{qt_1 - q^{-1}u} f_\alpha(u)f_\alpha(t_1) + (q^{-2}-q^2)\delta(q^2t_1/u)f_\alpha(q^2t_1)f_\alpha(t_1)\,.$$

□

Let us come back to the proof of the Proposition 5.1. The commutation relations (2.7) imply that the product $f_\alpha(t_1)\cdots f_\alpha(t_n)$ has simple zeros at hyperplanes $t_n = t_i$, $i = 1,\dots,n-1$. Substitution of these conditions to the equality (5.6) gives the systems of $n-1$ linear equations over the field of rational functions $\mathbb{C}(t_1,...,t_{n-1})$ for the operators $G_j(t_1,\dots,t_{n-1})$:

$$\sum_{j=1}^{n-1} \frac{G_j(t_1,\dots,t_{n-1})}{t_i - q^2t_j} = G\cdot P\left(f_\alpha(t_i)\right)\,, \quad i = 1,...,n-1\,, \tag{5.8}$$

where $G = P\left(f_\alpha(t_1)\cdots f_\alpha(t_{n-1})\right)$. The matrix $B_{i,j} = (t_i - q^2t_j)^{-1}$ of this system has nonzero in $\mathbb{C}(t_1,...,t_{n-1})$ determinant,

$$\det(B) = (-q^2)^{\frac{n(n-1)}{2}} \frac{\prod_{i\neq j}(t_i-t_j)^2}{\prod_{i,j}(t_i-q^2t_j)}\,,$$

so the system has unique solution over $\mathbb{C}(t_1,...,t_{n-1})$. This implies that operators G_j are linear combinations over $\mathbb{C}(t_1,...,t_{n-1})$ of operators $G\cdot P^+\left(f_\alpha(t_j)\right)$, $j = 1,...,n-1$, so the projection of the product can be presented as

$$P\left(f_\alpha(t_1)\cdots f_\alpha(t_n)\right) = G\cdot\left(P\left(f_\alpha(t_n)\right) - \sum_{j=1}^{n-1}\tilde\varphi_{t_j}(t_n;t_1,\dots,t_{n-1})P\left(f_\alpha(t_j)\right)\right)\,,$$

where $\tilde\varphi_{t_j}(t_n;t_1,\dots,t_{n-1}) = A_j(t_n;t_1,\dots,t_{n-1})/\prod_{m=1}^{n-1}(t_n-q^2t_m)$ are rational functions which nominators $A_j(t_n;t_1,\dots,t_{n-1})$ are polynomials over t_n of degree less then $n-1$. The system (5.8) is satisfied if rational functions $\tilde\varphi_{t_j}(t_n;t_1,\dots,t_{n-1})$ enjoy the property

$$\tilde\varphi_{t_j}(t_i;t_1,\dots,t_{n-1}) = \delta_{i,j}\,, \qquad i,j = 1,...,n-1\,.$$

This interpolation problem has unique solution given by the formula (5.3). □

5.2 The universal R-matrix

According to (3.16), the universal R-matrix $\mathcal{R}$ for the algebra $U_q(\widehat{\mathfrak{sl}}_2)$ with comultiplication Δ admits a factorization $\mathcal{R} = \mathcal{R}_1\mathcal{R}_2$, where

$$\mathcal{R}_1^{21} = (\check P^-\otimes \mathrm{id})\overline{\mathcal{R}}, \qquad \mathcal{R}_2 = (\check P^+\otimes\mathrm{id})\overline{\mathcal{R}}\,, \tag{5.9}$$

and $\overline{\mathcal{R}}$ is the tensor of the Hopf pairing $U_F\otimes U_E\to\mathbb{C}$ with respect to comultiplication $\Delta^{(D)}$. It can be presented as

$$\overline{\mathcal{R}} = \mathcal{K}\cdot\widetilde{\mathcal{R}}, \quad \widetilde{\mathcal{R}} = \sum_{n\geq 0}\frac{(q^{-1}-q)^n}{n!}\widetilde{\mathcal{R}}^{(n)}\,,$$

where

$$\widetilde{\mathcal{R}}^{(n)} = \oint \cdots \oint \prod_{i=1}^{n} \frac{dz_i}{z_i} f_\alpha(z_1)\cdots f_\alpha(z_n) \otimes e_\alpha(z_1)\cdots e_\alpha(z_n) \ ,$$

and in the notations $k_\alpha = q^{h_\alpha}$, $\gamma = q^{c/2}$, $D = q^d$

$$\mathcal{K} = q^{-\frac{h_\alpha \otimes h_\alpha}{2}} q^{\frac{-c\otimes d - d\otimes c}{2}} \exp\left((q^{-1}-q) \sum_{n>0} \frac{n}{[2n]_q}\ h_\alpha[n] \otimes h_\alpha[-n] \right) q^{\frac{-c\otimes d - d\otimes c}{2}} \ . \qquad (5.10)$$

Set

$$\begin{aligned} \mathcal{R}_1^{(n)} &= \oint \prod_{i=1}^{n} \frac{dz_i}{z_i}\frac{du_i}{u_i} X(\bar z)\ Y(\bar u;\bar z)\ e(z_1)\cdots e(z_n) \otimes f(u_1)\cdots f(u_n) = \\ &= \oint \prod_{i=1}^{n} \frac{dz_i}{z_i}\frac{du_i}{u_i} X({}^{\hat\omega}\bar u) Y({}^{\hat\omega}\bar u;\ {}^{\hat\omega}\bar z)\ e(z_1)\cdots e(z_n) \otimes f(u_1)\cdots f(u_n) \ , \end{aligned} \qquad (5.11)$$

$$\begin{aligned} \mathcal{R}_2^{(n)} &= \oint \prod_{i=1}^{n} \frac{dz_i}{z_i}\frac{du_i}{u_i} X(\bar z)\ Z(\bar z;\bar u)\ f(u_1)\cdots f(u_n) \otimes e(z_1)\cdots e(z_n) = \\ &= \oint \prod_{i=1}^{n} \frac{dz_i}{z_i}\frac{du_i}{u_i} X({}^{\hat\omega}\bar u) Z({}^{\hat\omega}\bar z;\ {}^{\hat\omega}\bar u)\ f(u_1)\cdots f(u_n) \otimes e(z_1)\cdots e(z_n) \ . \end{aligned} \qquad (5.12)$$

The substitution of (5.2) into (5.9) gives the following equalities:

$$\mathcal{R}_1 = \sum_{n\geq 0} \frac{(q^{-1}-q)^n}{n!} \mathcal{R}_1^{(n)}, \qquad \mathcal{R}_2 = \mathcal{K}\cdot \left(\sum_{n\geq 0} \frac{(q^{-1}-q)^n}{n!} \mathcal{R}_2^{(n)} \right) \qquad (5.13)$$

where $\mathcal{K}$ is given by the expression (5.10). We have finally

Theorem 4 *The universal R-matrix for $U_q(\widehat{\mathfrak{sl}}_2)$ can be written as a product of series of formal integrals, $\mathcal{R} = \mathcal{R}_1\mathcal{R}_2$, with the factors $\mathcal{R}_1$ and $\mathcal{R}_2$, defined by (5.13).*

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