THE CLASSIFICATION OF THE SIMPLE LIE SUPERALGEBRAS OF VECTOR FIELDS

DIMITRY LEITES[1], IRINA SHCHEPOCHKINA[2]

Abstract. Infinite dimensional Lie superalgebras of vector fields with polynomial or formal coefficients *together with particular gradings*, called Weisfeiler gradings, will be briefly called W-graded vectorial Lie superalgebras. The importance of W-gradings and the name of these algebras are due to the fact that they are determined by a maximal subalgebra $\mathcal{L}_0 \subset \mathcal{L}$ of finite codimension, so every vectorial $\mathcal{L}$ is realized by means of vector fields on $(\mathcal{L}/\mathcal{L}_0)^*$. We classify simple complex W-graded vectorial Lie superalgebras, and discuss their deformations (the classification will be given elsewhere).

We show that a straightforward extension of the original É. Cartan's problem to superalgebras, i.e., "classification of *primitive* Lie superalgebras", is a wild problem and distill a tame subproblem solved here.

Observe that (1) unlike Lie algebras, the Lie *super*algebra of vector fields may have several (finitely many) non-isomorphic W-gradings all of which we list; (2) the parameters of deformations of Lie superalgebras can run over a *super*variety.

To the memory of A. N. Tyurin.

Introduction

This is a continuation of [Sh14]. The term "exceptional", first used in [Sh3], implies that a classification is ready or "almost". In [LSh1], a preliminary result was described together with main steps of classification. In 1996-97 we have explicitly announced the classification of W-graded simple complex vectorial Lie superalgebras at various conferences, e.g., [LSh2]. Here is the proof and examples of deformations (the classification will be given elsewhere).

Summary: *there are 4 series of simple W-graded Lie algebras (Cartan); passing to Lie superalgebras we find 34 series and 15 exceptional algebras.*

Several of these superalgebra admit deformations. These deformations may have odd parameters; e.g., the quantization of the Poisson bracket is a deformation with even parameter, the quantization of the antibracket is a deformation with odd parameter.

As abstract ones, W-gradings disregarded, these algebras can be collected into 13 series (one of which is exceptional) and 5 exceptional families. We would like to draw the attention of the reader to the fact that our economical way to describe vectorial Lie superalgebras as *regradings* of several particular ones, chosen as "points of reference", hides somewhat the peculiarity of Lie *super*algebras of having several totally distinct models as vectorial algebras, i.e., as regards to the structures they preserve or, in inner terms, as (generalized) Cartan

1991 *Mathematics Subject Classification.* 17A70 (Primary) 17B01, 17B70 (Secondary).

Key words and phrases. Lie superalgebras, vector fields, deformations of Lie superalgebras.

We are thankful to D. Alekseevsky, J. Bernstein, P. Grozman, I. Kantor, Yu. Kochetkov, G. Post and A. Rudakov for help. Discussions with I. Kantor, D. Alekseevsky and A. Rudakov in 1978 were especially crucial; Kantor was teh first to observe several of the regradings. Financial support of D.L. by NFR throughout the work, and of I.Shch. by INTAS grant 96-0538 at the last stage is thankfully acknowledged. D.L. also thanks Mittag-Leffler Institute, Harvard University; IHES; Hebrew University of Jerusalem, Tel-Aviv University and IAS, Princeton, for hospitality at the earlier stages of this research and where main results and ideas were delivered (1986 – 1991) and especially MPIM, Bonn, where the first draft ([LSh1]) was typed and where the current Introduction was finally moulded.

prolongs. Passage to real forms makes it manifest how vital it is to distinguish the filtration (realization): real forms do not, as a rule, survive regradings.

I.0. On setting of the problem. Selection of Lie algebras with reasonably nice properties is a matter of taste and is influenced by the initial "big" problem considered. One of the usual choices is the class of *simple* algebras: they have a richer structure and, therefore, are easier to study than other types of Lie algebras; they also illuminate important symmetries. Even representatives of the "complementary" types — nilpotent and solvable algebras — are often of real interest only when subalgebras of simple algebras.

Of simple Lie algebras, finite dimensional ones are naturally the first to study. Over $\mathbb{C}$, they constitute three "classical" series ($\mathfrak{sl}$, $\mathfrak{o}$ and $\mathfrak{sp}$) and five exceptional algebras; all are neatly encoded by very simple graphs — Dynkin diagrams — or Cartan matrices.

Next on the agenda are *$\mathbb{Z}$-graded Lie algebras of polynomial growth* (let us call them ZGLAPGs for short). Some of these algebras have Cartan matrix; they resemble finite dimensional simple Lie algebras more than others, and have proved very useful in various branches of mathematics and theoretical physics.

Let us recall the meaning of the terms entering ZGLAPG. A $\mathbb{Z}$-graded algebra $A = \oplus A_i$ is said to be *simple graded* if it has no *homogeneous* or better say *graded* ideals (i.e., no ideals $I = \oplus I_i$ such that $I_i = I \cap A_i$; a simple graded algebra can be nonsimple as an abstract algebra: e.g., loop algebras with values in a simple Lie algebra are simple graded but nonsimple). If $\dim A_i < \infty$ for all i, one can define the *growth* of A to be $\mathrm{gth}(A) = \overline{\lim\limits_{n\to\infty}} \frac{\ln \dim \oplus_{|i|\leq n} A_i}{\ln n}$. We say that A is of *polynomial growth* if $\mathrm{gth}(A) < \infty$.

Around 1966 V. Kac and Weisfeiler started to study simple *filtered* Lie algebras of polynomial growth, i.e., Lie algebras such that the associated graded algebra is of polynomial growth. The first step in the study was, of course, the *simple graded* algebras. Kac classified simple ZGLAPGs **provided they are generated by elements of degree** ± 1. Among simple ZGLAPGs only $\mathfrak{vect}(1) = \mathfrak{der}\ \mathbb{C}[x]$ and $\mathfrak{vect}^L(1) = \mathfrak{der}\ \mathbb{C}[x^{-1}, x]$ are not generated by elements of degree ± 1 with respect to any grading. Kac conjectured that these examples plus his list exhaust all the simple ZGLAPGs. Twenty years after O. Mathieu proved this conjecture in a string of rather complicated papers culminating in [M].

It goes without saying that we look at the *simple* Lie superalgebras as a preliminary (though vital) material. The central extensions, deformations and algebras of outer derivations of simple Lie superalgebras are often more interesting in applications (like affine Kac–Moody algebras are "more useful" than loop algebras; likewise, Poisson algebras and their deformations are often no less appealing to the heart of the physicist than even the Lie algebras of Hamiltonian vector fields, vital in the classical mechanics). So the real object of interest is the answer to the following **Problem 0**:

0. Problem. Starting with a simple Lie algebra of class ZGLAPG, list the results of iterated combinations of the following operations:

a) $\mathfrak{g} \longrightarrow \mathfrak{e}(\mathfrak{g})$, the nontrivial central extension of $\mathfrak{g}$ (example: central extensions of loop algebras; the Poisson algebra);

b) $\mathfrak{g} \longrightarrow \mathfrak{der}(\mathfrak{g})$, the derivation algebra of $\mathfrak{g}$ (example: affine Kac–Moody algebras);

c) deformations of $\mathfrak{g}$ (though they sometimes lead out of the class ZGLAPG, we should not disregard them since such deformed Lie algebras are often no less important in applications than the original algebras; examples: the result of the *quantization* of either the Poisson algebra or of the algebra of Hamiltonian vector fields; P.Golod's and Krichever-Novikov's examples);

d) all filtered completions of $\mathfrak{g}$ and various $\mathbb{Z}$-gradings;

e) forms of these algebras over non-closed fields.

We will loosely refer to the results of the iterated procedures described in Problem 0 as *classical* Lie algebras and call the algebras obtained from a simple one, $\mathfrak{g}$, by the above procedures the *relatives* of $\mathfrak{g}$ and each other. To list them, their counterparts in the supersetting and/or in prime characteristic, their various (not only $\mathbb{Z}$-) gradings is the main strategic goal of our study.

Remark . The reader should not think, as was customary 30 years ago, that "bigger" algebras are useless or too difficult to study. For example, various versions of $\mathfrak{gl}(\infty)$ and algebras studied by Borcherds and Gritsenko-Nikulin are both, though huge, of huge interest and their representations describable.

I.0.1. Types of the classical Lie algebras. Let us qualitatively describe the simple Lie algebras of polynomial growth to better visualize them. They break into the disjoint union of the following types:

1) *finite dimensional* algebras (growth 0).

2) *loop algebras*, perhaps, twisted (growth 1); more important in applications are their non-simple "relatives" called *affine Kac–Moody* algebras, cf. [K3].

3) *vectorial algebras*, i.e., Lie algebras of vector fields[1] with polynomial coefficients (growth is equal to the number of indeterminates) or their completions with formal power series as coefficients.

4) *the stringy*[2] *algebra* $\mathfrak{vect}^L(1) = \mathfrak{der}\ \mathbb{C}[t^{-1}, t]$ (the superscript stands for Laurent). This class consists of one algebra. It is often called *Witt algebra* and denoted $\mathfrak{witt}$ in honor of Witt who considered its analog over fields of prime characteristic, and by physicists *the centerless Virasoro algebra* because its nontrivial central extension (discovered by Fuchs and Gelfand), $\mathfrak{vir}$, is called the *Virasoro algebra* in honor of Virasoro who rediscovered the corresponding central extension and indicated its importance in physical models.

Strictly speaking, stringy algebras are also vectorial, but we retain the generic term *vectorial* for algebras with *polynomial* or *formal* coefficients.

The above are graded algebras. Filtered algebras are not classified (see examples to Problem 0). Among the known examples of filtered algebras, we distinguish various deformations of the above and

5) *the Lie algebra of matrices of complex size and its generalizations*, cf. [GL2, LSe, DGS]. These algebras are simple filtered Lie algebras of polynomial growth whose associated graded algebras are not simple. (Though they are much more numerous than simple *graded* ones, we still hope that it is possible to distill a tame subproblem: up to some twists, the examples known to us (modulo current algebras and algebras of vector fields on affine varieties, cf. [L7]) are multiparameter deformations of the Poisson algebras of functions on the orbits in the coadjoint representation of simple finite dimensional or Lie algebras of matrices over the rings of differential operators. Super version of this class is even wider due to S. Montgomery's and Konstein's constructions ([GL2, KV]).)

[1] These algebras are sometimes known under the recently introduced clumsy name "algebras of Cartan type": just imagine "the Cartan subalgebra in the Lie superalgebra of Cartan type with Cartan matrix".

[2] The term induced by the lingo of imaginative physicists who now play (and share with us sensation of beauty of stringy superalgebras [GSW] and urge to investigate them, or take a calmer attitude [WL]) with the idea that an elementary particle is not a point but rather a slinky springy string; the term "stringy algebra" means "pertaining to string theory" but also mirrors their structure as a collection of several strings — the $\mathfrak{witt}$-modules. In our papers "stringy" means "simple vectorial superalgebra on a supermanifold whose base is a circle or a relative of such algebra (see Problem 0)".

The algebras of the above classes 1) – 5) can be characterized by other properties that are sometimes taken for their definition. For example, algebras $\mathfrak{g}$ of classes 1), 2) and 4) are "symmetric", i.e. with every root space $\mathfrak{g}_\alpha$ they also have $\mathfrak{g}_{-\alpha}$ of the same dimension as $\mathfrak{g}_\alpha$. Additionally, algebras of classes 1) and 2) have Cartan matrix. This matrix is always symmetrizable and, moreover, not only integer but of so simple form that it can be encoded by a simple graph (Dynkin diagram). Algebras with a symmetric Cartan matrix and (most of the) algebras of class 5) possess an invariant symmetric nondegenerate bilinear form.

I.0.2. Superization. After Wess and Zumino made importance of supersymmetries manifest, cf. [D, WZ], it was natural to classify simple "classical" Lie superalgebras. It immediately turned out that the above types of simple Lie algebras and their properties are intermixed under the passage to superalgebras.

— *Finite dimensional superalgebras.* They split into two types:

"symmetric" ones, subdivided into

sy^{CM} the ones with (symmetrizable) Cartan matrix (since the matrix is symmetrizable, the algebras possess a nondegenerate even supersymmetric bilinear form) and

sy^{WCM} the ones without Cartan matrix in the conventional sense but with a nondegenerate supersymmetric bilinear form that can be either even or odd;

"skew-symmetric" or vectorial types: with more positive roots than it has negative ones.

— *(Twisted) loop superalgebras.* It is obvious that loop superalgebras with values in simple Lie superalgebras are also simple (as graded algebras). Unlike Lie algebras, some of the loop superalgebras have no Cartan matrix, some have a non-symmetrizable ones, etc., so classification of (twisted) loop superalgebras splits into several very different cases. The very first examples showed that even if a finite dimensional $\mathfrak{g}$ possesses a Cartan matrix, the twisted loop superalgebra $\mathfrak{g}_\varphi^{(k)}$ (for notations and theory in non-supercase see [K3]) may possess no Cartan matrix and *vice versa.*

An intrinsic characterization of loop superalgebras without apeal to Cartan matrix (deduced from Mathieu [M]) is given in [GLS1] together with same for stringy superalgebras: both types of algebras are $\mathbb{Z}$-graded $\mathfrak{g} = \mathop{\oplus}\limits_{i=-d}^{\infty} \mathfrak{g}_i$ of infinite depth d but in the adjoint representation they act differently:

for the loop algebras every root vector corresponding to any real root acts locally nilpotently in the adjoint representation, $(*)$
for the stringy algebras this is not so.

In the absence of the above intrinsic definition, one of us (DL) tried to approach the classification "from the end" and, in order to get a supply of examples, conjectured that, as for Lie algebras, simple twisted loop superalgebras correspond to outer automorphisms of $\varphi \in \mathrm{Aut}\mathfrak{g}/\mathrm{Int}\mathfrak{g}$. Serganova listed these automorphisms and amended the conjecture: to get distinct algebras, one should factorize Aut $\mathfrak{g}$ modulo the connected component of the unity (which is somewhat greater than Int $\mathfrak{g}$, the group of inner automorphisms of $\mathfrak{g}$), see [Se3], [FLS]. An *a priori* classification of Lie superalgebras of polynomial growth with symmetrizable Cartan matrix is due to J. van de Leur [vdL]; his classification and Serganova's unpublished classification of Lie superalgebras of polynomial growth with non-symmetrizable Cartan matrix cited in [LS1], [GLS1] support Leites-Serganova's conjecture on the completeness of the list from [FLS].

— *Stringy superalgebras.* For the conjectural list of simple ones see [GLS1]. Partly (with several extra conditions) this conjecture is proved in [KvdL], [K8]. Observe that some of the stringy superalgebras possess Cartan matrix, though non-symmetrizable ones [GLS1], and have deformations whereas their vectorial namesakes are rigid.

Remark . In the literature there are several definitions of what we call stringy superalgebras, mostly self-contradicting ones. For example, in all physical papers and books stringy superalgebras are called "superconformal" ones (meaning conformal superalgebras). In reality, only $\mathfrak{witt}$, $\mathfrak{k}^L(1|1)$ and $\mathfrak{k}^M(1|1)$ are conformal in the original sense of the term ([GLS1]); but even if other superalgebras had been conformal, their central extensions are not, though the term "superconformal" is magnanimously applied to all. In some papers the "superconformal" superalgebras are defined as "simple ... such that considered as $\mathfrak{witt}$-module ...", whereas in physical papers everybody calls Virasoro, Neveu-Schwarz and Ramond superalgebras "superconformal" ones, though they are neither simple nor conformal. Therefore, we suggest the term *stringy* for the general class of algebras inside of which some are conformal, some are simple, etc.

Some of the simple stringy superalgebras ([GLS1]) are *distinguished*: they admit a nontrivial central extensions. In [GLS1], there are also indicated exceptional stringy superalgebras and occasional isomorphisms unnoticed in previous papers and ignored, so far, in later ones.

In this paper we consider the remaining type of simple graded Lie superalgebras of polynomial growth:

Vectorial Lie superalgebras. Main examples to keep in the mind's eye are the Lie algebra $L = \mathfrak{der}\ \mathbb{C}[x]$ of polynomial vector fields with grading and filtration given by setting $\deg x_i = 1$ for all i, and its (x)-adic completion, the filtered Lie algebra $\mathcal{L} = \mathfrak{der}\ \mathbb{C}[[x]]$ of formal vector fields.

Cartan came to these algebras from geometrical problems in which "primitive" Lie algebras naturally arise. Infinite dimensional primitive Lie algebras turned out to be isomorphic to the algebras of derivations of simple ones (one of the items in Problem 0), and since $\dim \mathfrak{der}\ \mathfrak{g}/\mathfrak{g} \leq 1$, to classify infinite dimensional *primitive* algebras is practically the same as to classify the simple vectorial ones (the list of finite dimensional primitive algebras is much longer than that of simple ones but still reasonable [O]). Contrariwise, the classification problem of primitive Lie *super*algebras is wild, as shown in [ALSh] and Appendix. So, having distinguished a reasonable problem, we will study simple Lie superalgebras.

I.1. Classificational problems and Main Theorem. Consider infinite dimensional complex filtered Lie superalgebras $\tilde{\mathcal{L}}$ with decreasing filtration of the form

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_{-\tilde{d}} \supset \tilde{\mathcal{L}}_{-\tilde{d}+1} \supset \dots \supset \tilde{\mathcal{L}}_0 \supset \tilde{\mathcal{L}}_1 \supset \dots \qquad (F)$$

where *depth* $\tilde{d}$ is finite and where

1) $\tilde{\mathcal{L}}_0$ is a maximal subalgebra of finite codimension;
2) $\tilde{\mathcal{L}}_0$ does not contain ideals of the whole $\tilde{\mathcal{L}}$.

The algebra $\tilde{\mathcal{L}}$ with such a subalgebra $\tilde{\mathcal{L}}_0$ is called a *primitive Lie superalgebra.*

We assume that these Lie superalgebras $\tilde{\mathcal{L}}$ are complete with respect to a natural topology whose basis of neighborhoods of zero is formed by the spaces of finite codimension, e.g., the $\tilde{\mathcal{L}}_i$. (In the absence of odd indeterminates this topology is the most natural one: we consider two vector fields k*-close* if their coefficients coincide up to terms of degree $> k$.) This topology is naturally (see. §1) called, "briefly", *projective limit topology* but even more unfortunate term "linearly compact topology" is also used. No other topology will be encountered, so all topological terms (complete, open, closed) refer to this topology.

Observe that the very term "filtered algebra" implies that $[\tilde{\mathcal{L}}_i, \tilde{\mathcal{L}}_j] \subset \tilde{\mathcal{L}}_{i+j}$. Set $\tilde{L}_i = \tilde{\mathcal{L}}_i/\tilde{\mathcal{L}}_{i+1}$. Conditions 1) and 2) manifestly imply that $\dim \tilde{L}_i < \infty$ for all k and the $\mathbb{Z}$-graded Lie superalgebra $\tilde{L} = \mathop{\oplus}\limits_{k\geq -\tilde{d}} \tilde{L}_k$ associated with $\tilde{\mathcal{L}}$ grows *polynomially*, i.e., $\dim \mathop{\oplus}\limits_{k\leq n} \tilde{L}_k$ grows as a polynomial in n.

Weisfeiler endowed every such filtered Lie algebra $\tilde{\mathcal{L}}$ with another, refined, filtration $\tilde{\mathcal{L}} = \mathcal{L}$:

$$\mathcal{L} = \mathcal{L}_{-d} \supset \mathcal{L}_{-d+1} \supset \dots \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \dots \qquad (WF1)$$

by setting $\mathcal{L}_0 = \tilde{\mathcal{L}}_0$ and letting $\mathcal{L}_{-1}$ to be a minimal $\mathcal{L}_0$-invariant subspace strictly containing $\mathcal{L}_0$, different from $\mathcal{L}_0$ and $\mathcal{L}_0$-invariant, the other terms being defined by the formula (for $i \geq 1$):

$$\mathcal{L}_{-i-1} = [\mathcal{L}_{-1}, \mathcal{L}_{-i}] + \mathcal{L}_{-i} \ \text{ and } \ \mathcal{L}_i = \{D \in \mathcal{L}_{i-1} \mid [D, \mathcal{L}_{-1}] \subset \mathcal{L}_{i-1}\}. \qquad (WF2)$$

The d in (WF1) is called the *depth* of $\mathcal{L}$ and of the associated with $\tilde{\mathcal{L}}$ graded Lie superalgebra L.

An advantage of the *Weisfeiler filtrations* is that for the corresponding regraded Lie superalgebra $\mathcal{L}$ the L_0-action on L_{-1} is irreducible (see sec. I.0.2). These refined filtrations are called, after [W], *Weisfeiler filtrations* even for Lie superalgebras, where Weisfeiler's construction is literally applied; we will shortly write *W-filtrations* and call the gradings associated with W-filtrations *W-gradings*.

Now, observe that condition 2) on filtrations considered is equivalent to the following condition which is sometimes more convenient to use:

2′) for any non-zero $x \in \tilde{L}_k$ for $k \geq 0$, where $\tilde{L}_k = \tilde{\mathcal{L}}_k / \tilde{\mathcal{L}}_{k+1}$, there exists $y \in \tilde{L}_{-1}$ such that $[x, y] \neq 0$.

When the L_0-module L_{-1} is faithful, as is always the case for simple Lie superalgebras $\mathcal{L}$, such filtered Lie superalgebras $\mathcal{L}$ (and the associated with them graded ones, L) can be realized by *vector fields* on the supermanifold corresponding to the linear superspace $(\mathcal{L}/\mathcal{L}_0)^*$ with formal (resp. polynomial) coefficients. So, being primarily interested in simple Lie superalgebras, we assume that L_0-module L_{-1} is faithful. We refer to the Lie superalgebras of vector fields as *vectorial* ones for short.

The following problems arise:

A′) Classify simple W-graded vectorial Lie superalgebras as abstract ones, i.e., distinction in grading disregarded.

Immediately observe that this problem is very unnatural: quite distinct algebras become equivalent. Indeed, as early as in [ALSh], we observed that $\mathfrak{vect}(1|1)$, the Lie algebra of all vector fields on $(1|1)$-dimensional superspace, is isomorphic as an abstract algebra to $\mathfrak{k}(1|2)$, the Lie algebra of *contact* vector fields on $(1|2)$-dimensional superspace (and even to $\mathfrak{m}(1)$, another, "odd" type of contact Lie superalgebra). A more natural formulation is, therefore, the following one:

A) Classify simple W-graded vectorial Lie superalgebras.

In some applications it suffices to confine ourselves to graded algebras ([L8]), in other applications, e.g., in the study of representations ([GLS4]), *complete* algebras are sometimes more important and natural than the associated with them graded ones. So the following problem might be a reasonable goal:

B) Classify simple W-filtered complete vectorial Lie superalgebras.

Observe natural ("trivial") examples of complete algebras: algebras $\bar{L}$ obtained from the answers L to Problem A by taking formal series, rather than polynomials, as coefficients. In addition to these examples there might occur nontrivial *filtered deforms* (as Gerstenhaber calls the result of a deformation), i.e., complete algebras $\tilde{L}$ such that the associated graded of $\bar{L}$ and $\tilde{L}$ is the same, L. Examples will be given later.

On the road to solution of Problem B the following problem *seems* to be a natural step:

B′) Classify simple W-filtered complete vectorial Lie superalgebras as abstract ones, i.e., distinction in filtrations disregarded.

Observe that whereas the difference between Problems A′ and B′ is as large as that between Problems A and B, the difference between Problems B and B′ is negligible. Actually, it is impossible to solve Problem B′ and not solve Problem B. Indeed, various filtered deformations are not necessarily isomorphic as abstract algebras, so in order to solve Problem B′ we have to know first all the W-filtrations or the associated W-gradings and then describe filtered deformations for *every* W-grading.

However, even Problem B is not the most natural one: there exist deforms of algebras from class B which do not lie in class B: e.g., the result of the factorization of the quantized Poisson algebra modulo center. So the true problem one should solve is

C) Describe all the deformations (not only filtered ones) of the simple W-filtered vectorial Lie superalgebras.

I.1.1. Notations. Comments. We will describe all the Lie superalgebras from the tables in our Main Theorem in detail in due course, but to simplify grasping the general picture from the displayed formulas of the following theorem, let us immediately inform the reader that the prime example, the general vectorial algebra, $\mathfrak{vect}(m|n;r)$, is the Lie superalgebra of vector fields whose coefficients are polynomials (or formal power series, depending on the actual problem) in m commuting and n anticommuting indeterminates with the filtration (and grading) determined by equating the degrees of r $(0 \leq r \leq m)$ of odd indeterminates to 0, the degrees of the remaining indeterminates being equal to 1. The regradings of other series are determined similarly, but in a more complicated way, see I.2.2. Usually, we omit indicating parameter r if $r = 0$.

On vectorial superalgebras, there are two analogs of trace. The divergence is one of them, the linear functional that vanishes on the brackets is another one. Accordingly, the special (divergence free) subalgebra of $\mathfrak{g}$ is denoted by $\mathfrak{sg}$, e.g., $\mathfrak{svect}(n|m)$; the superscript ^{o} singles out the traceless ideal of codimension 1 (or $0|1$). Observe that the usual notation is a $'$, rather that ^{o}, but we are not considering the derived series and ^{o} is more graphic.

$\mathfrak{k}$ and $\mathfrak{h}$ are the straightforward analogs of the contact and hamiltonian series, respectively; $\mathfrak{po}(2n|m)$, the central extension of $\mathfrak{h}(2n|m)$, is the Poisson Lie superalgebra.

$\mathfrak{m}$ and $\mathfrak{le}$ are the "odd" analogs of the contact and hamiltonian series, respectively. The bracket in $\mathfrak{b}$, the central extension of $\mathfrak{le}$, is the classical Schouten bracket, more popular now under the name antibracket. We say "odd" in quotation marks: the series $\mathfrak{m}$ preserves the Pfaff equation with an *even* contact form, and all these "odd" analogs have both even and odd parts.

The antibracket can be deformed, the deformed algebra is $\mathfrak{b}_\lambda$.

The algebras $\mathfrak{sm}$, $\mathfrak{sle}$, $\mathfrak{h}^o$ and $\mathfrak{sb}$ are divergence free subalgebras in respective algebras.

$\widetilde{\mathfrak{sb}}$ is a simple algebra, it is a deformation of a non-simple $\mathfrak{sb}(n|n+1;n)$.

Observe an outstanding property of the bracket in $\mathfrak{h}_\lambda(2|2) \simeq \mathfrak{b}_\lambda(2|2)$: it can be considered as a deformation of an even bracket as well as a deformation of an odd one.

In the sequel for any $\mathfrak{g}$ we write $\mathfrak{cg} = \mathfrak{g} \oplus \mathbb{C} \cdot z$ or $\mathfrak{c}(\mathfrak{g})$ to denote the trivial central extension with the 1-dimensional even center generated by z. Any nontrivial central extension is shorthanded as $\mathfrak{e}(\mathfrak{g})$.

Let $\mathfrak{a} \in\!\!\!\!\subset \mathfrak{b}$ or $\mathfrak{b} \ni\!\!\!\!\supset \mathfrak{a}$ denote the semidirect sum in which $\mathfrak{a}$ is the ideal.

In proofs we use notations of roots in the simple finite dimensional Lie algebras from [OV], Table 5. The standard (identity) representation of a matrix Lie superalgebra $\mathfrak{g}$, i.e., a subalgebra of $\mathfrak{gl}(V)$, in V is denoted by id or, for clarity, $\mathrm{id}_{\mathfrak{g}}$. The trivial representation of $\mathfrak{g}_0$ is denoted by $\mathbb{C}$ (over a simple algebra) whereas $\mathbb{C}[k]$ denotes a representation of $\mathfrak{g}_0$ trivial on its semi-simple part and such that k is the value of the central element z from $\mathfrak{g}_0$ chosen so that $z|_{\mathfrak{g}_i} = i \cdot \mathrm{id}_{\mathfrak{g}_i}$.

Further elucidations. In formulas (S) and (E) below: in parentheses after the "family name" of the algebra there stands the superdimension of the superspace of indeterminates and — after semicolon — a shorthand description of the regrading $r \neq 0$.

Passing from one regrading to another one, we take a "minimal" realization (i.e., with a minimal $\dim \mathcal{L}/\mathcal{L}_0$) as the point of reference. For the exceptional Lie superalgebras another point of reference is often more convenient, the consistent grading, K.

The regradings of the series are governed by a parameter r described in sec. I.2. All regradings are given in (hopefully) suggestive notations, e.g., $\mathfrak{kas}^\xi(1|6; 3\eta)$ means that having taken $\mathfrak{kas}^\xi$ as the point of reference we set $\deg \eta = 0$ for each of the three η's (certainly, this imposes some conditions on the degrees of the other indeterminates).

The exceptional grading E_H of $\mathfrak{h}_\lambda(2|2)$ is described in passing in the list of occasional isomorphisms (OI); it is described in detail for another incarnation of this algebra (I.2.1.1).

The finite dimensional Lie superalgebra $\mathfrak{h}(0|n)$ of hamiltonian vector fields is not simple: it contains a simple ideal $\mathfrak{h}^o(0|n)$.

On drop-outs. Several algebras are "drop-outs" from the series. Examples: $\mathfrak{svect}^\circ(1|n; r)$ are "drop-outs" from the series $\mathfrak{svect}(m|n; r)$ since the latter are not simple for $m = 1$ but contain the simple ideal $\mathfrak{svect}^\circ(1|n; r)$. Similarly, $\mathfrak{le}(n|n; r)$, $\mathfrak{b}_1{}^\circ(n|n+1; r)$ and $\mathfrak{b}_\infty{}^\circ(n|n+1; r)$ are "drop-outs" from the series $\mathfrak{b}_\lambda(n|n+1; r)$ corresponding to $\lambda = 0$, 1 and ∞, respectively, having either a simple ideal of codimension 1 or a center the quotient modulo which is simple. Though $\mathfrak{sm}$ is not a drop-out due to the above reason, it is singled out by its divergence free property, hence, deserves a separate line.

One should not treat the drop-outs lightly, be it the case of characteristic p, or the super-case. People justly consider the Lie algebras of Hamiltonian vector fields and its central extension, Poisson algebra (or loop algebra and its affine Kac-Moody relative) as totally distinct algebras. The difference between $\mathfrak{sl}$ and $\mathfrak{psl}$ in characteristic p and in the super-case is enourmous: e.g., the irreducible representation of least dimension of $\mathfrak{psl}$ is the adjoint one.

Thus, all drop-outs, without exception, are quite separate items on the list.

Theorem . (Solution of Problem A) *The simple W-graded vectorial Lie superalgebras* $\mathfrak{g}$ *constitute the following series* (S) *and five exceptional families of fifteen individual algebras* (E)*. They are pair-wise non-isomorphic, as graded and filtered superalgebras, bar for occasional isomorphisms* (OI)*.*

For completeness, our tables include finite dimensional "degenerations" labelled by "FD" instead of the number.

All these algebras are the results of the (generalized) Cartan prolongation (described in sec 0.5) and, therefore, are determined by the terms $\mathfrak{g}_i$ *with* $i \leq 0$ *(or* $i \leq 1$ *in some cases). These terms are listed in sec. I.2* (and the form of these terms might drastically vary with n and r though they can be united in "families").

N	the family and conditions for its simplicity
1	$\mathfrak{vect}(m|n; r)$ for $m \geq 1$ and $0 \leq r \leq n$
FD	$\mathfrak{vect}(0|n; r)$ for $n > 1$ and $0 \leq r \leq n$
2	$\mathfrak{svect}(m|n; r)$ for $m > 1$, $0 \leq r \leq n$
FD	$\mathfrak{svect}(0|n; r)$ for $n > 2$ and $0 \leq r \leq n$
3	$\mathfrak{svect}^\circ(1|n; r)$ for $n > 1$, $0 \leq r \leq n$
FD	$\widetilde{\mathfrak{svect}}(0|n)$ for $n > 2$

(S_{vect})

4	$\mathfrak{k}(2m+1\vert n;r)$ for $0\leq r\leq [\frac{n}{2}]$ unless $(m\vert n)=(0\vert 2k)$ $\mathfrak{k}(1\vert 2k;r)$ for $0\leq r\leq k$ except $r=k-1$
5	$\mathfrak{h}(2m\vert n;r)$ for $m>0$ and $0\leq r\leq [\frac{n}{2}]$ $\mathfrak{h}_\lambda(2\vert 2;r)$ for $\lambda\neq 0,1,\infty$, and $r=0$, 1 and E_H (see (OI) and sec. I.2.1.1)
FD	$\mathfrak{h}^o(0\vert n)$ for $n>3$

(S_k)

6	$\mathfrak{m}(n\vert n+1;r)$ for $0\leq r\leq n$ except $r=n-1$
7	$\mathfrak{sm}(n\vert n+1;r)$ for $n>1$ but $n\neq 3$ and $0\leq r\leq n$ except $r=n-1$
8	$\mathfrak{b}_\lambda(n\vert n+1;r)$ for $n>1$, where $\lambda\neq 0,1,\infty$ and $0\leq r\leq n$ except $r=n-1$ $\mathfrak{b}_\lambda(2\vert 3;r)$, where $\lambda\neq 0,1,\infty$ for $r=0$, 2 and E (see sec. I.2.1.1)
9	$\mathfrak{b}_1{}^\circ(n\vert n+1;r)$ for $n>1$ and $0\leq r\leq n$ except $r=n-1$
10	$\mathfrak{b}_\infty{}^\circ(n\vert n+1;r)$ for $n>1$ and $0\leq r\leq n$ except $r=n-1$
11	$\mathfrak{le}(n\vert n;r)$ for $n>1$ and $0\leq r\leq n$ except $r=n-1$
12	$\mathfrak{sle}^\circ(n\vert n;r)$ for $n>2$ and $0\leq r\leq n$ except $r=n-1$
13	$\widetilde{\mathfrak{sb}}_\mu(2^{n-1}-1\vert 2^{n-1})$ for $\mu\neq 0$ and $n>2$

(S_m)

Hereafter we inconsistently abbreviate $\mathfrak{m}(n|n+1;r)$, $\mathfrak{b}_\lambda(n|n+1;r)$, $\mathfrak{le}(n|n;r)$, etc., i.e., algebras from (S_m), to $\mathfrak{m}(n;r)$, $\mathfrak{b}_\lambda(n;r)$, $\mathfrak{le}(n;r)$, etc., respectively.

For clarity, instead of $\mathfrak{b}_\lambda(n;r)$, where $\lambda=\frac{2a}{n(a-b)}\in\mathbb{C}\cup\{\infty\}$, we sometimes write $\mathfrak{b}_{a,b}(n;r)$. The parameters a,b, even more natural than λ, are described in sec. 0.7.

Similarly looking notations $\mathfrak{svect}_{a,b}(0|n)$ and $\mathfrak{spe}_{a,b}(n)$ are shorthand for $\mathfrak{g}\Subset\mathbb{C}(az+bd)$, where $\mathfrak{g}$ is $\mathfrak{svect}(0|n)$ or $\mathfrak{spe}(n)$, d is the operator that determines the standard $\mathbb{Z}$-grading of $\mathfrak{g}$, and z the trivial center.

(OI): Occasional isomorphisms:

$$\begin{aligned}
&\mathfrak{vect}(1|1)\cong\mathfrak{vect}(1|1;1);\\
&\mathfrak{svect}(2|1)\cong\mathfrak{le}(2;2);\ \mathfrak{svect}(2|1;1)\cong\mathfrak{le}(2)\\
&\mathfrak{sm}(n)\cong\mathfrak{b}_{2/(n-1)}(n);\ \text{in particular, } \mathfrak{sm}(2)\cong\mathfrak{b}_2(2), \text{ and}\\
&\mathfrak{sm}(3)\cong\mathfrak{b}_1(3), \text{ hence, } \mathfrak{sm}(3) \text{ is not simple;}\\
&\mathfrak{sle}^\circ(3)\cong\mathfrak{sle}^\circ(3;3);\\
&\mathfrak{b}_{1/2}(2;2)\cong\mathfrak{h}_{1/2}(2|2)=\mathfrak{h}(2|2);\ \ \mathfrak{h}_\lambda(2|2)\cong\mathfrak{b}_\lambda(2;2);\ \ \mathfrak{h}_\lambda(2|2;1)\cong\mathfrak{b}_\lambda(2);\\
&\mathfrak{b}_{a,b}(2;E)\cong\mathfrak{b}_{-b,-a}(2)\cong\mathfrak{b}_{b,a}(2) \text{ for } a\neq b \text{ and } b\neq 0;\\
&\mathfrak{b}_{a,0}(2;E)\cong\mathfrak{le}(2) \text{ and } \mathfrak{b}_{a,a}(2;E)\cong\mathfrak{b}^\circ_\infty(2);\\
&\mathfrak{h}_\lambda(2|2)\cong\mathfrak{h}_{-1-\lambda}(2|2),\ \text{in particular, } \mathfrak{b}_0(n)\simeq\mathfrak{b}_{-1}(n) \text{ and } \mathfrak{b}_1(n)\simeq\mathfrak{b}_{-2}(n),\\
&\text{hence, the fundamental domain is } \mathrm{Re}\lambda\geq-\tfrac12 \text{ or } \mathrm{Re}\lambda\leq-\tfrac12;\\
&\widetilde{\mathfrak{sb}}_\mu(2^{n-1}-1|2^{n-1})\cong\widetilde{\mathfrak{sb}}_\nu(2^{n-1}-1|2^{n-1}) \text{ for } \mu\nu\neq 0.
\end{aligned}\tag{OI}$$

Observe that the exceptional regrading E of $\mathfrak{b}_\lambda(2)$ and isomorphisms $\mathfrak{h}_\lambda(2|2)\cong\mathfrak{b}_\lambda(2;2)$ determine the exceptional grading E_H of $\mathfrak{h}_\lambda(2|2)$. These regradings ($E$ and E_H) are not, however, of any interest from the classificational point of view, thanks to the isomorphisms (OI). On the other hand, they contribute to the automorphisms of $\mathfrak{h}_\lambda(2|2)\cong\mathfrak{b}_\lambda(2;2)$.

Our original notations of exceptional simple vectorial superalgebras, though reflect the way they are constructed and the geometry preserved ($\mathfrak{ksle}$ reflects that it is a subalgebra of $\mathfrak{k}$ related to $\mathfrak{sle}$, and so on), are rather long. But to write just $\mathfrak{e}(\dim)$ is to create confusion: the superdimensions of the superspaces $(\mathcal{L}/\mathcal{L})^*$ on which the algebra $\mathcal{L}$ is realized coincide sometimes for different regradings. So, in accordance with table $(E\subset S)$in sec. I.2, we set (by K we denote *consistent* gradings, CK is an exceptional grading found by Cheng and Kac):

Lie superalgebra	its regradings (shorthand)
$\mathfrak{vle}(4\vert 3;r)$, $r=0,1,K$	$\mathfrak{vle}(4\vert 3)$, $\mathfrak{vle}(5\vert 4)$, and $\mathfrak{vle}(3\vert 6)$
$\mathfrak{vas}(4\vert 4)$	$\mathfrak{vas}(4\vert 4)$
$\mathfrak{kas}(1\vert 6;r)$, $r=0,1\xi,3\xi,3\eta$	$\mathfrak{kas}(1\vert 6)$, $\mathfrak{kas}(5\vert 5)$, $\mathfrak{kas}(4\vert 4)$, and $\mathfrak{kas}(4\vert 3)$
$\mathfrak{mb}(4\vert 5;r)$, $r=0,1,K$	$\mathfrak{mb}(4\vert 5)$, $\mathfrak{mb}(5\vert 6)$, and $\mathfrak{mb}(3\vert 8)$
$\mathfrak{ksle}(9\vert 6;r)$, $r=0,2,K,CK$	$\mathfrak{ksle}(9\vert 6)$, $\mathfrak{ksle}(11\vert 9)$, $\mathfrak{ksle}(5\vert 10)$, and $\mathfrak{ksle}(9\vert 11)$

(E)

Remark . 1) Though $\mathfrak{b}_\lambda(2;r)$ and $\mathfrak{h}_\lambda(2|2;r)$ are isomorphic, we consider them separately because they are deformations of very distinct structures: the odd and the even bracket, respectively. In I.2.4.2 we show wherefrom comes the mysterious at this point isomorphism $\mathfrak{h}_\lambda(2|2)\cong\mathfrak{h}_{-1-\lambda}(2|2)$.

Actually, $\mathfrak{h}_\lambda(2|2)\cong\mathfrak{b}_\lambda(2;2)$ should be considered as an exceptional family.

2) **Warning**. Isomorphic abstract Lie superalgebras might be quite distinct as filtered or graded: e.g., regradings provide us with isomorphisms ([ALSh])

$$\mathfrak{k}(1|2)\cong\mathfrak{vect}(1|1)\cong\mathfrak{m}(1) \text{ as abstract algebras.} \tag{IAA}$$

Observe that of the above three nonisomorphic graded algebras only one is W-graded.

3) The excluded regrading $\mathfrak{k}(1|2)$ is often considered in physical papers, despite the fact that this grading (or filtration) is not a Weisfeiler one: for it the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is reducible. Over $\mathbb{R}$, however, $\mathfrak{k}(1|2k;k-1)$ is a W-grading, at least, for some real forms, and it is these forms that are considered.

4) $\widetilde{\mathfrak{svect}}(0|n)$, as well as $\widetilde{\mathfrak{sb}}_\mu(2^{n-1}-1|2^{n-1})$, depend on an odd parameter if n is odd.

The above Lie superalgebras sometimes admit deformations that do not possess Weisfeiler filtrations. These deformations are considered in detail in [LS5]; far partial results see [L4], [Ko1], [LSh3].

The five families of exceptional Lie superalgebras are given below in their minimal realizations as Cartan's prolongs $(\mathfrak{g}_{-1},\mathfrak{g}_0)_*$ or generalized (see sect. 0.5) Cartan's prolongs $(\mathfrak{g}_{-},\mathfrak{g}_0)_*^{mk}$.

For $\mathfrak{g}_-=\mathop{\oplus}\limits_{-2\leq i\leq -1}\mathfrak{g}_i$, instead of $(\mathfrak{g}_{-},\mathfrak{g}_0)_*^{mk}$ we write $(\mathfrak{g}_{-2},\mathfrak{g}_{-1},\mathfrak{g}_0)_*^{mk}$. Indicated is also one of the Lie superalgebras from the list of series (S) as an ambient which contains the exceptional one as a maximal subalgebra. The superalgebras of depth 3 appear as regradings of the listed ones at certain value of r; the corresponding terms will be given below.

$\mathfrak{vle}(4\|3;r)=(\Pi(\Lambda(3))/\mathbb{C}\cdot 1,\mathfrak{cvect}(0\|3))_*\subset\mathfrak{vect}(4\|3;R),\quad r=0,1,K$
$\mathfrak{vas}(4\|4)=(\mathrm{spin},\mathfrak{as})_*\subset\mathfrak{vect}(4\|4)$
$\mathfrak{kas}^\xi(1\|6;r)\subset\mathfrak{k}(1\|6;r),\quad r=0,1\xi,3\xi$ $\mathfrak{kas}^\xi(1\|6;3\eta)=(Vol_0(0\|3),\mathfrak{c}(\mathfrak{vect}(0\|3)))_*\subset\mathfrak{svect}(4\|3)$
$\mathfrak{mb}(4\|5;r)=(\mathfrak{ab}(4),\mathfrak{cvect}(0\|3))_*^m\subset\mathfrak{m}(4\|5;R),\quad r=0,1,K$
$\mathfrak{ksle}(9\|6;r)=(\mathfrak{hei}(8\|6),\mathfrak{svect}(0\|4)_{3,4})_*^k\subset\mathfrak{k}(9\|6;r),\quad r=0,2,\text{ CK}$ $\mathfrak{ksle}(9\|6;K)=(\mathrm{id}_{\mathfrak{sl}(5)},\Lambda^2(\mathrm{id}^*_{\mathfrak{sl}(5)}),\mathfrak{sl}(5))_*^k\subset\mathfrak{svect}(5\|10;R)$

$(E\subset S)$

In $(E\subset S)$, most of the regradings R of the ambients are the same as that of the embedded algebra, i.e., $R(r)=r$. Certain regradings $R(r)$ of the ambients are so highly nonstandard that even the homogeneous fibers are of infinite dimension. The following table describes these regradings (for $\mathfrak{kas}$ notations $r=0,\ 1\xi,\ 3\xi$ are clear):

$\mathfrak{vle}(4\|3)$	$R(0)=(1111\|111),\ R(1)=(2110\|011),\ R(K)=(2220\|111)$
$\mathfrak{mb}(4\|5)$	$R(0)=(1111\|1111;2),\ R(1)=(0211\|2011;2),\ R(K)=(0222\|3111;3)$
$\mathfrak{ksle}(9\|6)$	$R(K)=(22222\|1\dots 1)$

(R)

The description of the degrees of the generating functions is given in Theorem I.2.1.2.

Summary. 1) In [ALSh] we have singled out a tame subproblem: *classify simple vectorial Lie superalgebras*, from its straightforward but wild classical analog: *classify primitive Lie superalgebras* (cf. problem formulation in [K2], [K7], [K9]).

2) **Solution of Problem A for depth 1**. (The most part of this solution was preprinted in [LSh2].)

3) In several of our earlier papers, we also gave examples of simple vectorial Lie superalgebras of greater depth. The union of these examples ([ALSh], [LSh1], [Sh5]) was our conjectural list solving Problem A announced in [LSh2]. And the list is, indeed, the complete one. We make use of the proof from [K7, CK1, CK2], where the case of consistent grading and what we call *Kac's grading* for depth >1 are considered, to **solve Problem A in full generality**.

Related results. 1) For the list of simple "stringy" Lie superalgebras, their explicit representation by generating functions, and for related structures see [GLS1]. Observe a mysterious difference in presentations for $\mathfrak{kas}$ generated by polynomials and $\mathfrak{kas}^L$ generated by Laurent polynomials, see [GLS4]: we suspect that there are infinitely many relations between generators of $\mathfrak{kas}$ whereas $\mathfrak{kas}^L$ is finitely presented.

2) The description of the irreducible representations of simple vectorial Lie superalgebras was started in [BL1] for the general series $\mathfrak{vect}(m|n)$ in its standard realization following the pioneering works of A. Rudakov. For a review of results on representations for other series of vectorial Lie superalgebras by 1980 see [L2]. For a description of the irreducible representations of two or three exceptional vectorial Lie superalgebras see [KR]; for an updated review and description of irreducible representations for 11 of the 15 exceptions in a wider setting see [GLS3].

3) For the classification of simple vectorial Lie superalgebras over $\mathbb{R}$ see [LSh4].

4) For the classification of deformations of simple vectorial Lie superalgebras see [LS5].

I.2.1. Nonstandard realizations. As follows from Step 7 of our proof (see I.3), the following nonstandard gradings exhaust all the W-gradings of all the simple vectorial Lie superalgebras. In particular, the gradings in the series $\mathfrak{vect}$ induce the gradings in the series $\mathfrak{svect}$, $\mathfrak{svect}^\circ$ and the exceptional families $\mathfrak{vle}(4|3)$ and $\mathfrak{vas}(4|4)$; the gradings in $\mathfrak{m}$ induce the gradings in $\mathfrak{b}_\lambda$, $\mathfrak{le}$, $\mathfrak{sle}$, $\mathfrak{sle}^\circ$, $\mathfrak{b}$, $\mathfrak{sb}$, $\mathfrak{sb}^\circ$ and the exceptional family $\mathfrak{mb}$; the gradings in $\mathfrak{k}$ induce the gradings in $\mathfrak{po}$, $\mathfrak{h}$ and the exceptional families $\mathfrak{kas}$ and $\mathfrak{ksle}$.

The indeterminates are introduced in sec. 0.2 below. In what follows we consider $\mathfrak{k}(2n+1|m)$ as preserving the Pfaff equation $\tilde\alpha = 0$, where (see (0.2.4))

$$\tilde\alpha = dt + \sum_{i\leq n}(p_i dq_i - q_i dp_i) + \sum_{j\leq r}(\xi_j d\eta_j + \eta_j d\xi_j) + \sum_{k\geq m-2r}\theta_k d\theta_k.$$

The standard realizations correspond to $r = 0$, they are marked by an $(*)$. Observe that the codimension of $\mathcal{L}_0$ attains its minimum in the standard realization.

Lie superalgebra	its $\mathbb{Z}$-grading
$\mathfrak{vect}(n\vert m;r)$, $0\leq r\leq m$	$\deg u_i = \deg\xi_j = 1$ for any i,j $(*)$
	$\deg\xi_j = 0$ for $1\leq j\leq r$; $\deg u_i = \deg\xi_{r+s} = 1$ for any i,s
$\mathfrak{m}(n;r)$, $0\leq r\leq n$, $r\neq n-1$	$\deg\tau = 2$, $\deg q_i = \deg\xi_i = 1$ for any i $(*)$
	$\deg\tau = \deg q_i = 2$, $\deg\xi_i = 0$ for $1\leq i\leq r<n$; $\deg q_{r+j} = \deg\xi_{r+j} = 1$ for any j
$\mathfrak{m}(1\vert n;n)$	$\deg\tau = \deg q_i = 1$, $\deg\xi_i = 0$ for $1\leq i\leq n$
$\mathfrak{k}(2n+1\vert m;r)$,	$\deg t = 2$, $(*)$ $\deg p_i = \deg q_i = \deg\xi_j = \deg\eta_j = \deg\theta_k = 1$ for any i,j,k
$0\leq r\leq [\frac{m}{2}]$, $r\neq k-1$ for $m=2k$ and $n=0$	$\deg t = \deg\xi_i = 2$, $\deg\eta_i = 0$ for $1\leq i\leq r\leq[\frac{m}{2}]$; $\deg p_i = \deg q_i = \deg\theta_j = 1$ for $j\geq 1$ and all i
$\mathfrak{k}(1\vert 2m;m)$	$\deg t = \deg\xi_i = 1$, $\deg\eta_i = 0$ for $1\leq i\leq m$

I.2.1.1. The exceptional nonstandard regrading E. This is a regrading of $\mathfrak{b}_{a,b}(2)$ given by the formulas:

$$\deg\tau = 0;\quad \deg\xi_1 = \deg\xi_2 = -1;\quad \deg q_1 = \deg q_2 = 1. \qquad (r = E)$$

Further

$$\begin{aligned}&b = 0 \text{ or } a = b\text{: } \mathfrak{b}_{a,0}(2;E)\cong\mathfrak{le}(2) \text{ and } \mathfrak{b}_{a,a}(2;E)\cong\mathfrak{b}^\circ_\infty(2), \text{ in part. } \mathfrak{g}_{-2}=0;\\ &a \text{ and } b \text{ generic: } \mathfrak{g}_{-2} = Span\{\mathrm{Le}_{\xi_1\xi_2}\} \text{ and } \mathfrak{g}_{-1} = Span\{\mathrm{Le}_{\xi_1}, \mathrm{Le}_{\xi_2}, \mathrm{Le}_{Q_1}, \mathrm{Le}_{Q_2}\},\end{aligned} \qquad (I.2.1.1)$$

where $Q_1 = A\xi_1\xi_2 q_1 + B\tau\xi_2$, $Q_2 = A\xi_1\xi_2 q_2 - B\tau\xi_1$ and where A and B are some coefficients determined by a and b. The bracket on $\mathfrak{g}_{-1}$ is determined by the odd form $\omega = c\sum dQ_i d\xi_i$, so $\mathfrak{g}_0$ must be contained in $\mathfrak{m}(2)_0$. The direct calculations show that $\dim\mathfrak{g}_0 = 4|4$ and

$$\mathfrak{g}_0 = \mathfrak{spe}(2)\oplus\mathbb{C}X,\ \text{ where } X = \mathrm{Le}_{a\tau + b\sum q_i\xi_i}.$$

Indeed,

$$\mathfrak{spe}(2)_0\cong\mathfrak{sl}(2) = Span\{\mathrm{Le}_{q_1\xi_2}, \mathrm{Le}_{q_2\xi_1}, \mathrm{Le}_{q_1\xi_1 - q_2\xi_2}\},\quad \mathfrak{spe}(2)_{-1} = \mathbb{C}\cdot\mathrm{Le}_1,$$

and

$$\mathfrak{spe}(2)_1 = \mathbb{C}\mathrm{Le}_{\alpha\xi_1\xi_2 P(q) + \beta\tau\Delta(\xi_1\xi_2 P(q))},$$

where $P(q)$ is a monomial of degree 2 and α, β are some constants.

The eigenvalues of X on $\mathfrak{g}_{-1}$ are: $-a+b$ on the even part and $a+b$ an the odd part. So

$$\mathfrak{b}_{a,b}(2;E) \cong \mathfrak{b}_{-b,-a}(2) \cong \mathfrak{b}_{b,a}(2).$$

For $n>2$, as well as for $\mathfrak{m}(n)$ with $n>1$, similar regradings are not Weisfeiler ones.

The exceptional grading E of $\mathfrak{b}_\lambda(2)$ induces the exceptional grading E_H of the isomorphic algebra $\mathfrak{h}_\lambda(2|2)$, see (OI).

Thus, the exceptional regradings E or E_H do not prowide with new W-graded vectorial algebras. Still, they are important for description of automorphisms.

I.2.1.2. The W-regradings of exceptional algebras.

Theorem . ([Sh14], [CK2]) *The W-regradings of the exceptional simple vectorial Lie superalgebras are given by the following regradings of their "standard" ambients listed in Introduction* $(E\subset S)$*:*

1) $\mathfrak{vle}(4|3;r) = (\Pi(\Lambda(3)/\mathbb{C}\cdot 1), \mathfrak{cvect}(0|3))_* \subset \mathfrak{vect}(4|3), \quad r=0,1,K;$
 $\underline{r=0}$: $\deg y = \deg u_i = \deg \xi_i = 1;$
 $\underline{r=1}$: $\deg y = \deg \xi_1 = 0$, $\deg u_2 = \deg u_3 = \deg \xi_2 = \deg \xi_3 = 1$, $\deg u_1 = 2;$
 $\underline{r=K}$: $\deg y = 0$, $\deg u_i = 2$; $\deg \xi_i = 1$.

2) $\mathfrak{vas}(4|4) = (\text{spin}, \mathfrak{as})_* \subset \mathfrak{vect}(4|4);$

3) $\mathfrak{kas} \subset \mathfrak{k}(1|6;r)$, $r=0,1,3\xi;$
 $\underline{r=0}$: $\deg t = 2, \quad \deg \eta_i = 1; \quad \deg \xi_i = 1;$
 $\underline{r=1}$: $\deg \xi_1 = 0, \quad \deg \eta_1 = \deg t = 2, \quad \deg \xi_2 = \deg \xi_3 = \deg \eta_2 = \deg \eta_3 = 1;$
 $\underline{r=3\xi}$: $\deg \xi_i = 0, \quad \deg \eta_i = \deg t = 1;$
 $\underline{r=3\eta}$: $\deg \eta_i = 0$, $\deg \xi_i = \deg t = 1 \quad \mathfrak{kas}(1|6;3\eta) \subset \mathfrak{svect}(4|3)$.

4) $\mathfrak{mb}(4|5;r) = (\mathfrak{ab}(4), \mathfrak{cvect}(0|3))_*^m \subset \mathfrak{m}(4), \quad r=0,1,K;$
 $\underline{r=0}$: $\deg \tau = 2, \quad \deg u_i = \deg \xi_i = 1$ for $i=0,1,2,3;$
 $\underline{r=1}$: $\deg \tau = \deg \xi_0 = \deg u_1 = 2, \ \deg u_2 = \deg u_3 = \deg \xi_2 = \deg \xi_3 = 1; \ \deg \xi_1 = \deg u_0 = 0;$
 $\underline{r=K}$: $\deg \tau = \deg \xi_0 = 3, \quad \deg u_0 = 0, \quad \deg u_i = 2; \quad \deg \xi_i = 1$ for $i>0$.

5) $\mathfrak{ksle}(9|6;r) = (\mathfrak{hei}(8|6), \mathfrak{svect}_{3,4}(4))_*^k \subset \mathfrak{k}(9|6), \quad r=0,\ 2,$ CK;
 $\underline{r=0}$: $\deg t = 2, \quad \deg p_i = \deg q_i = \deg \xi_i = \deg \eta_i = 1;$
 $\underline{r=2}$: $\deg t = \deg q_3 = \deg q_4 = \deg \eta_1 = 2, \ \deg q_1 = \deg q_2 = \deg p_1 = \deg p_2 = \deg \eta_2 = \deg \eta_3 = \deg \zeta_2 = \deg \zeta_3 = 1; \quad \deg p_3 = \deg p_4 = \deg \zeta_1 = 0;$
 $\underline{r=K}$: $\deg t = \deg q_i = 2, \quad \deg p_i = 0; \quad \deg \zeta_i = \deg \eta_i = 1$; $\mathfrak{ksle}(9|6;K) \subset \mathfrak{svect}(5|10;R)$;
 $\underline{r=CK}$: $\deg t = \deg q_1 = 3, \quad \deg p_1 = 0; \quad \deg q_2 = \deg q_3 = \deg q_4 = \deg \zeta_1 = \deg \zeta_2 = \deg \zeta_3 = 2; \quad \deg p_2 = \deg p_3 = \deg p_4 = \deg \eta_1 = \deg \eta_2 = \deg \eta_3 = 1$.

Thus, from the point of view of classification of the W-filtered Lie superalgebras, there are five *families* of exceptional algebras consisting of 15 individual algebras.

I.2.2. Several first terms that determine the Cartan and mk-prolongations.

(To recall the notion of Cartan prolongation and for definition of its generalizations see sec. 0.5.) To facilitate the comparison of various vectorial superalgebras, we offer the following Table. The most interesting phenomena occur for extremal values of parameter r and small values of superdimension $m|n$.

The central element $z \in \mathfrak{g}_0$ is supposed to be chosen so that it acts on $\mathfrak{g}_k$ as $k \cdot \mathrm{id}$.

Let $\Lambda(r) = \mathbb{C}[\xi_1, \dots, \xi_r]$ be the Grassmann superalgebra generated by the ξ_i of degree 0 for all i. We set $\Lambda(0) = \mathbb{C}$, the $\mathfrak{vect}(0|n)$-module $\Lambda(n)/\mathbb{C}\cdot 1$ is denoted by $T_0(\vec{0})$; set

$$Vol_0(0|m) = \{v \in Vol(0|m) \mid \int v = 0\};$$

the $\mathfrak{svect}$-module $T_0^0(\vec{0})$ is defined as $Vol_0(0|m)/\mathbb{C}\cdot 1$ (for more elucidations see sec. 0.6).Over $\mathfrak{svect}$ it is convenient to consider Vol_0 as a submodule of Λ.

Recall that the range of the parameter r (see I.2.1 and sec. 0.11) is the set of integer from $[0, m]$, where m is the number of odd indeterminates, for the series $\mathfrak{k}$ and $\mathfrak{h}$ the range is $[0, [\frac{m}{2}]]$, and for $\mathfrak{m}(n)$, $\mathfrak{b}_\lambda(n)$, etc. the range is $[0, n]$ with value $r = n-1$ excluded.

Recall that we exclude certain values of r, namely, $r = k-1$ for $\mathfrak{k}(1|2k)$ and $r = n-1$ for $\mathfrak{m}(n)$ and their subalgebras because for these values of r the corresponding grading is not a W-grading: the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is reducible.

N	$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$
1	$\mathfrak{vect}(n\|m; r)$	$-$	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{gl}(n\|m-r)\otimes\Lambda(r) \Subset \mathfrak{vect}(0\|r)$
2	$\mathfrak{vect}(1\|m; m)$	$-$	$\Lambda(m)$	$\Lambda(m) \Subset \mathfrak{vect}(0\|m)$
3	$\mathfrak{svect}(n\|m; r), n \neq 1$	$-$	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{sl}(n\|m-r)\otimes\Lambda(r) \Subset \mathfrak{vect}(0\|r)$
4	$\mathfrak{svect}^\circ(1\|m; r), r \neq m$	$-$	$\mathrm{id}\otimes Vol_0(0\|r)$	$\mathfrak{sl}(n\|m-r)\otimes\Lambda(r) \Subset \mathfrak{vect}(0\|r)$
5	$\mathfrak{svect}^\circ(1\|m; m)$	$-$	$Vol_0(0\|m)$	$\Lambda(m) \Subset \mathfrak{svect}(0\|m)$
	$\mathfrak{svect}^\circ(1\|2)$	$-$	$T_0(\vec{0})$	$\mathfrak{vect}(0\|2) \cong \mathfrak{sl}(1\|2)$
	$\mathfrak{svect}(2\|1)$	$-$	$\Pi(T_0(\vec{0}))$	$\mathfrak{vect}(0\|2) \cong \mathfrak{sl}(2\|1)$

6	$\mathfrak{h}(2n\|m)$	$-$	id	$\mathfrak{osp}(m\|2n)$
7	$\mathfrak{h}(2n\|m; r)$	$T_0(\vec{0})$	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{osp}(m-2r\|2n)\otimes\Lambda(r) \Subset \mathfrak{vect}(0\|r)$
8	$\mathfrak{h}(2n\|2r; r)$	$-$	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{sp}(2n)\otimes\Lambda(r) \Subset \mathfrak{vect}(0\|r)$
9	$\mathfrak{h}_\lambda(2\|2)$	$-$	$\Pi(Vol^\lambda(0\|2))$	$\mathfrak{osp}(2\|2) \cong \mathfrak{vect}(0\|2)$
10	$\mathfrak{h}_\lambda(2\|2; 1)$	$-$	$\mathrm{id}_{\mathfrak{sp}(2)}\otimes Vol^\lambda(0\|1)$	$\mathfrak{sp}(2)\otimes\Lambda(1) \Subset \mathfrak{vect}(0\|1)$
11	$\mathfrak{k}(2n+1\|m; r)$	$\Lambda(r)$	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{cosp}(m-2r\|2n)\otimes\Lambda(r) \Subset \mathfrak{vect}(0\|r)$
12	$\mathfrak{k}(1\|2m; m)$	$-$	$\Lambda(m)$	$\Lambda(m) \Subset \mathfrak{vect}(0\|m)$
13	$\mathfrak{k}(1\|2m+1; m)$	$\Lambda(m)$	$\Pi(\Lambda(m))$	$\Lambda(m) \Subset \mathfrak{vect}(0\|m)$

We set $p(\mu) \equiv n \pmod 2$, so μ can be odd indeterminate. The Lie superalgebras $\widetilde{\mathfrak{svect}}_\mu(0|n)$ are isomorphic for nonzero μ's; hence, so are the algebras $\widetilde{\mathfrak{sb}}_\mu(2^{n-1}-1|2^{n-1})$. So for n even we can set $\mu = 1$, while if μ is odd we should consider it as an additional indeterminate on which the coefficients depend.

N	$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$
14	$\mathfrak{m}(n; r)$	$\Pi(\Lambda(r))$	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{cpe}(n-r)\otimes\Lambda(r) \Subset \mathfrak{vect}(0\|r)$
15	$\mathfrak{m}(n; n)$	$-$	$\Pi(\Lambda(n))$	$\Lambda(n) \Subset \mathfrak{vect}(0\|n)$
16	$\widetilde{\mathfrak{sb}}_\mu(2^{n-1}-1\|2^{n-1})$	$-$	$\frac{\Pi(Vol(0\|n))}{\mathbb{C}(1+\mu\xi_1\dots\xi_n)vol(\xi)}$	$\widetilde{\mathfrak{svect}}_\mu(0\|n)$

In what follows $\lambda = \frac{2a}{n(a-b)} \neq 0,\ 1,\ \infty$; the three exceptional cases (corresponding to the "drop-outs" $\mathfrak{le}(n)$, $\mathfrak{b}_1{}^\circ(n)$ and $\mathfrak{b}^\circ_\infty(n)$, respectively) are considered separately. The irreducibility condition of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ for $\mathfrak{g} = \mathfrak{b}^\circ_\infty$ excludes $r = n-1$. The case $r = n-2$ is extra exceptional, so in the following tables, unless specified,

$$0 < r < n-2; \quad \text{additionally } a \neq b \text{ and } (a,b) \neq \alpha(n, n-2) \text{ for any } \alpha \in \mathbb{C}.$$

To further clarify the following tables, denote the superspace of the identity $k|k$-dimensional representation of $\mathfrak{spe}(k)$ by V; let $d = \mathrm{diag}(1_k, -1_k) \in \mathfrak{pe}(k)$. Let $W = V \otimes \Lambda(r)$ and $D \in \mathfrak{vect}(0|r)$. Let $\Xi = \xi_1 \cdots \xi_n \in \Lambda(\xi_1, \dots, \xi_n)$. Denote by T^r the representation of $\mathfrak{vect}(0|r)$ in $\mathfrak{spe}(n-r) \otimes \Lambda(r)$ given by the formula

$$T^r(D) = 1 \otimes D + d \otimes \frac{1}{n-r}\mathrm{div} D. \tag{T^r}$$

In $\mathfrak{sle}^\circ(n;r)_0$ for $r \neq n-2$:

$$\begin{aligned} &\mathfrak{vect}(0|r) \text{ acts on the ideal } \mathfrak{spe}(n-r)\otimes\Lambda(r) \text{ via } T^r;\\ &\text{any } X \otimes f \in \mathfrak{spe}(n-r)\otimes\Lambda(r) \text{ acts in } \mathfrak{g}_{-1} \text{ as } \mathrm{id}\otimes f \text{ and in } \mathfrak{g}_{-2} \text{ as } 0;\\ &\text{any } D \in \mathfrak{vect}(0|r) \text{ acts in } \mathfrak{g}_{-1} \text{ via } T^r \text{ and in } \mathfrak{g}_{-2} \text{ as } D. \end{aligned} \tag{$I.2.1.1$}$$

In $\mathfrak{sle}^\circ(n;n-2)_0$, observe that $\mathfrak{spe}(2) \cong \mathbb{C}(\mathrm{Le}_{q_1\xi_1 - q_2\xi_2}) \Subset \mathbb{C}\mathrm{Le}_{\xi_1\xi_2}$, whereas $\mathfrak{g}_{-2}$ and $\mathfrak{g}_{-1}$ are as above, for $r < n-2$. Set $\mathfrak{h} = \mathbb{C}(\mathrm{Le}_{q_1\xi_1 - q_2\xi_2})$. In this case

$$\mathfrak{g}_0 \cong \underline{(\mathfrak{h}\otimes\Lambda(n-2) \Subset \mathbb{C}\mathrm{Le}_{\xi_1\xi_2}\otimes(\Lambda(n-2)\setminus\mathbb{C}\xi_3\cdots\xi_n))} \Subset T^1(\mathfrak{vect}(0|n-2)). \tag{$I.2.1.2$}$$

The action of $\mathfrak{vect}(0|n-2)$, the quotient of $\mathfrak{g}_0$ modulo the underlined ideal, is via (I.2.1.1). In the subspace $\xi_1\xi_2\otimes\Lambda(n-2) \subset \mathfrak{g}_0$ this action is the same as in the space of volume forms. So we can throw away Ξ, or, speaking more correctly, take the irreducible submodule of forms with integral 0.

In the following table the terms "$\mathfrak{g}_{-i}$" denote the superspace isomorphic to the one in quotation marks but with the action given by formulas (I.2.1.1) and (I.2.1.2).

N	$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$
17	$\mathfrak{le}(n)$	–	id	$\mathfrak{pe}(n)$
18	$\mathfrak{le}(n;r)$	$\Pi(T_0(\vec{0}))$	$\mathrm{id}\otimes\Lambda(r)$	$\mathfrak{pe}(n-r)\otimes\Lambda(r) \Subset \mathfrak{vect}(0\vert r)$
19	$\mathfrak{le}(n;n)$	–	$\Pi(T_0(\vec{0}))$	$\mathfrak{vect}(0\vert n)$
20	$\mathfrak{sle}^\circ(n)$	–	id	$\mathfrak{spe}(n)$
21	$\mathfrak{sle}^\circ(n;r)$	"$\Pi(T_0(\vec{0}))$"	"$\mathrm{id}\otimes\Lambda(r)$"	$\mathfrak{spe}(n-r)\otimes$ "$\Lambda(r)$" $\Subset T^1(\mathfrak{vect}(0\vert r))$
22	$\mathfrak{sle}^\circ(n;n-2)$	"$\Pi(T_0(\vec{0}))$"	"$\mathrm{id}\otimes\Lambda(r)$"	see (I.2.1.2)
23	$\mathfrak{sle}^\circ(n;n)$	–	$\Pi(T^0_0(\vec{0}))$	$\mathfrak{svect}(0\vert n)$

Next, we consider $\mathfrak{b}_{a,b}(n;r)$ for $0 < r < n-2$ and $ar - bn \neq 0$; in particular, this excludes $\mathfrak{b}^\circ_\infty(n;n) = \mathfrak{b}^\circ_{a,a}(n;n)$ and $\mathfrak{b}^\circ_1(n;n-2) = \mathfrak{b}^\circ_{n,n-2}(n;n-2)$.

If z is the central element of $\mathfrak{cspe}(n-r)$ that acts on $\mathfrak{g}_{-1}$ as $-\mathrm{id}$, then

$$z\otimes\psi \text{ acts on } \mathfrak{g}_{-1} \text{ as } -\mathrm{id}\otimes\psi, \text{ and on } \mathfrak{g}_{-2} \text{ as } -2\mathrm{id}\otimes\psi. \tag{$I.2.1.3$}$$

Set

$$c = \frac{a}{ar - bn}.$$

Let $a\cdot\mathrm{str}\otimes\mathrm{id}$ be the representation of $\mathfrak{pe}(n-r)=\mathfrak{spe}(n-r)\in\mathbb{C}d$ which is $\mathrm{id}_{\mathfrak{spe}(n-r)}$ on $\mathfrak{spe}(n-r)$ and sends d to $2a\mathrm{id}$. Recall that $\mathbb{C}[k]$ is the 1-dimensional representation of $\mathfrak{g}_0$ where k is the value of the central element z from $\mathfrak{g}_0$ chosen so that $z|_{\mathfrak{g}_i}=i\cdot\mathrm{id}_{\mathfrak{g}_i}$.

N	$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$
24	$\mathfrak{b}_\lambda(n)$	$\Pi(\mathbb{C}[-2])$	id	$\mathfrak{spe}(n)\in\mathbb{C}(az+bd)$
25	$\mathfrak{b}_\lambda(n;r)$	$\Pi((-c)\mathrm{str})\otimes(Vol(0\|r))^{2c}$	$\left((-\frac{c}{2})\mathrm{str})\otimes\mathrm{id}\right)\otimes(Vol(0\|r))^c$	$(\mathfrak{pe}(n-r)\otimes\Lambda(r))\in\mathfrak{vect}(0\|r)$
26	$\mathfrak{b}_\lambda(n;n)$	$-$	$\Pi(Vol^\lambda(0\|n))$	$\mathfrak{vect}(0\|n)$

In the following tables the terms "$\mathfrak{g}_i$" denote the superspace isomorphic to the one in quotation marks but with the action given by formula (I.2.1.3).

In the exceptional case $ar=bn$, i.e., $\lambda=\dfrac{2}{n-r}$, the $\mathfrak{vect}(0|r)$-action on the ideal $\mathfrak{cspe}(n-r)\otimes\Lambda(r)\in\mathfrak{vect}(0|r)$ of $\mathfrak{g}_0$ and on $\mathfrak{g}_-$ is the same as for $\mathfrak{sle}^\circ$, see (I.2.1.1).

N	$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$
27	$\mathfrak{b}_{2/(n-r)}(n;r)$	$\Pi(\mathbb{C})\otimes\Lambda(r)$	$\mathrm{id}\otimes$ "$\Lambda(r)$"	$\mathfrak{cpe}(n-r)\otimes$ "$\Lambda(r)$" $\in T^r(\mathfrak{vect}(0\|r))$
28	$\mathfrak{b}^\circ_\infty(n)$	$\Pi(\mathbb{C})$	id	$\mathfrak{spe}(n)_{a,a}$
29	$\mathfrak{b}^\circ_\infty(n;r)$	$\Pi(\mathbb{C})\otimes\Lambda(r)$	$\mathrm{id}\otimes\Lambda(r)$	$((\mathfrak{spe}(n-r)_{a,a})\otimes\Lambda(r))\in\mathfrak{vect}(0\|r)$
30	$\mathfrak{b}^\circ_\infty(n;n), n>2$	$-$	$\Pi(\Lambda(n))$	$(\Lambda(n)\setminus\mathbb{C}\Xi)\in\mathfrak{svect}(0\|n)$
31	$\mathfrak{b}^\circ_1(n)$	$\Pi(\mathbb{C})$	id	$\mathfrak{spe}(n)_{n,n-2}$
32	$\mathfrak{b}^\circ_1(n;r)$	"$\Pi(Vol_0(0\|r))$"	$\mathrm{id}\otimes$ "$\Lambda(r)$"	$((\mathfrak{spe}(n-r)_{n,n-2})\otimes$ "$\Lambda(r)$"$)\in T^r(\mathfrak{vect}(0\|r))$
33	$\mathfrak{b}^\circ_1(n;n-2)$	"$\Pi(T_0(\vec{0}))$"	$\mathrm{id}\otimes$ "$\Lambda(r)$"	(I.2.1.2) for the above line with $\mathfrak{cspe}(2)$ instead of $\mathfrak{spe}(2)$
34	$\mathfrak{b}^\circ_1(n;n)$	$-$	$\Pi(Vol_0(0\|n))$	$\mathfrak{vect}(0\|n)$

I.2.3. The exceptional Lie subsuperalgebras. Here are the terms $\mathfrak{g}_i$ for $-2\le i\le 0$ of the 15 exceptional W-graded algebras (cf. sec. I.2.3):

$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$	$\dim\mathfrak{g}_-$
$\mathfrak{vle}(4\|3)$	$-$	$\Pi(\Lambda(3)/\mathbb{C}1)$	$\mathfrak{c}(\mathfrak{vect}(0\|3))$	$4\|3$
$\mathfrak{vle}(4\|3;1)$	$\mathbb{C}[-2]$	$\mathrm{id}\otimes\Lambda(2)$	$\mathfrak{c}(\mathfrak{sl}(2)\otimes\Lambda(2)\in T^{1/2}(\mathfrak{vect}(0\|2))$	$5\|4$
$\mathfrak{vle}(4\|3;K)$	$\mathrm{id}_{\mathfrak{sl}(3)}\otimes\mathbb{C}[-2]$	$\mathrm{id}^*_{\mathfrak{sl}(3)}\otimes\mathrm{id}_{\mathfrak{sl}(2)}\otimes\mathbb{C}[-1]$	$\mathfrak{sl}(3)\oplus\mathfrak{sl}(2)\oplus\mathbb{C}z$	$3\|6$
$\mathfrak{vas}(4\|4)$	$-$	spin	$\mathfrak{as}$	$4\|4$
$\mathfrak{kas}$	$\mathbb{C}[-2]$	$\Pi(\mathrm{id})$	$\mathfrak{co}(6)$	$1\|6$
$\mathfrak{kas}(;1\xi)$	$\Lambda(1)$	$\mathrm{id}_{\mathfrak{sl}(2)}\otimes\mathrm{id}_{\mathfrak{gl}(2)}\otimes\Lambda(1)$	$(\mathfrak{sl}(2)\oplus\mathfrak{gl}(2)\otimes\Lambda(1))\in\mathfrak{vect}(0\|1)$	$5\|5$
$\mathfrak{kas}(;3\xi)$	$-$	$\Lambda(3)$	$\Lambda(3)\oplus\mathfrak{sl}(1\|3)$	$4\|4$
$\mathfrak{kas}(;3\eta)$	$-$	$Vol_0(0\|3)$	$\mathfrak{c}(\mathfrak{vect}(0\|3))$	$4\|3$
$\mathfrak{mb}(4\|5)$	$\Pi(\mathbb{C}[-2])$	$Vol(0\|3)$	$\mathfrak{c}(\mathfrak{vect}(0\|3))$	$4\|5$
$\mathfrak{mb}(4\|5;1)$	$\Lambda(2)/\mathbb{C}1$	$\mathrm{id}_{\mathfrak{sl}(2)}\otimes\Lambda(2)$	$\mathfrak{c}(\mathfrak{sl}(2)\otimes\Lambda(2)\in T^{1/2}(\mathfrak{vect}(0\|2))$	$5\|6$
$\mathfrak{mb}(4\|5;K)$	$\mathrm{id}_{\mathfrak{sl}(3)}\otimes\mathbb{C}[-2]$	$\Pi(\mathrm{id}^*_{\mathfrak{sl}(3)}\otimes\mathrm{id}_{\mathfrak{sl}(2)}\otimes\mathbb{C}[-1])$	$\mathfrak{sl}(3)\oplus\mathfrak{sl}(2)\oplus\mathbb{C}z$	$3\|8$
$\mathfrak{ksle}(9\|6)$	$\mathbb{C}[-2]$	$\Pi(T^0_0(\vec{0}))$	$\mathfrak{svect}(0\|4)_{3,4}$	$9\|6$
$\mathfrak{ksle}(9\|6;2)$	$\Pi(\mathrm{id}_{\mathfrak{sl}(1\|3)})$	$\mathrm{id}_{\mathfrak{sl}(2)}\otimes\Lambda(3)$	$(\mathfrak{sl}(2)\otimes\Lambda(3))\in\mathfrak{sl}(1\|3)$	$11\|9$
$\mathfrak{ksle}(9\|6;K)$	id	$\Pi(\Lambda^2(\mathrm{id}^*))$	$\mathfrak{sl}(5)$	$5\|10$
$\mathfrak{ksle}(9\|6;CK)$	$\mathrm{id}^*_{\mathfrak{sl}(3)}\otimes\Lambda(1)$	$\mathrm{id}_{\mathfrak{sl}(2)}\otimes\left(\mathrm{id}_{\mathfrak{sl}(3)}\otimes\Lambda(1)\right)$	$\mathfrak{sl}(2)\oplus(\mathfrak{sl}(3)\otimes\Lambda(1)\in\mathfrak{vect}(0\|1))$	$9\|11$

Observe that none of the simple W-graded vectorial Lie superalgebras is of depth > 3 and only two algebras are of depth 3: $\mathfrak{mb}(4|5;K)$, for which we have

$$\mathfrak{mb}(4|5;K)_{-3} \cong \Pi(\mathrm{id}_{\mathfrak{sl}(2)}),$$

and another one, $\mathfrak{ksle}(9|6;CK) = \mathfrak{ksle}(9|11)$, for which we have :

$$\mathfrak{ksle}(9|11)_{-3} \simeq \Pi(\mathrm{id}_{\mathfrak{sl}(2)} \otimes \mathbb{C}[-3]).$$

I.2.3.1 The exceptional Lie subsuperalgebra $\mathfrak{kas}$ of $\mathfrak{k}(1|6)$. Like $\mathfrak{vect}(1|m;m)$, this Lie superalgebra is not determined by its nonpositive part and requires a closer study. The Lie superalgebra $\mathfrak{g} = \mathfrak{k}(1|2n)$ is generated by the functions from $\mathbb{C}[t, \xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n]$. The standard $\mathbb{Z}$-grading of $\mathfrak{g}$ is induced by the $\mathbb{Z}$-grading of $\mathbb{C}[t,\xi,\eta]$ given by

$$\deg K_f = \deg f - 2, \text{ where } \deg t = 2, \quad \deg \xi_i = \deg \eta_i = 1.$$

Clearly, in this grading $\mathfrak{g}$ is of depth 2. Let us consider the functions that generate several first homogeneous components of $\mathfrak{g} = \underset{i\geq -2}{\oplus} \mathfrak{g}_i$:

component	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$	$\mathfrak{g}_1$
its generators	1	$\Lambda^1(\xi,\eta)$	$\Lambda^2(\xi,\eta) \oplus \mathbb{C}\cdot t$	$\Lambda^3(\xi,\eta) \oplus t\Lambda^1(\xi,\eta)$

As one can prove directly, the component $\mathfrak{g}_1$ generates the whole subalgebra $\mathfrak{g}_+$ of elements of positive degree. The component $\mathfrak{g}_1$ splits into two $\mathfrak{g}_0$-modules $\mathfrak{g}_{11} = \Lambda^3$ and $\mathfrak{g}_{12} = t\Lambda^1$. It is obvious that $\mathfrak{g}_{12}$ is always irreducible and the component $\mathfrak{g}_{11}$ is trivial for $n = 1$.

The partial Cartan prolongs of $\mathfrak{g}_{11}$ and $\mathfrak{g}_{12}$ are well-known:

$$(\mathfrak{g}_- \oplus \mathfrak{g}_0, \mathfrak{g}_{11})^{mk}_* \cong \mathfrak{po}(0|2n) \oplus \mathbb{C}\cdot K_t \cong \mathfrak{d}(\mathfrak{po}(0|2n));$$
$$(\mathfrak{g}_- \oplus \mathfrak{g}_0, \mathfrak{g}_{12})^{mk}_* = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{12} \oplus \mathbb{C}\cdot K_{t^2} \cong \mathfrak{osp}(2n|2).$$

Observe a remarkable property of $\mathfrak{k}(1|6)$: only for it does the component $\mathfrak{g}_{11}$ split into 2 irreducible modules that we will denote $\mathfrak{g}^\xi_{11}$ and $\mathfrak{g}^\eta_{11}$: one is contains $\xi_1\xi_2\xi_3$, the other one $\eta_1\eta_2\eta_3$.

Observe further, that $\mathfrak{g}_0 = \mathfrak{co}(6) \cong \mathfrak{gl}(4)$. As $\mathfrak{gl}(4)$-modules, $\mathfrak{g}^\xi_{11}$ and $\mathfrak{g}^\eta_{11}$ are the symmetric squares $S^2(\mathrm{id})$ and $S^2(\mathrm{id}^*)$ of the identity 4-dimensional representation and its dual, respectively.

Theorem . ([Sh5], [Sh14]) *The partial Cartan prolong* $\mathfrak{kas}^\xi = (\mathfrak{g}_- \oplus \mathfrak{g}_0, \mathfrak{g}^\xi_{11} \oplus \mathfrak{g}_{12})^{mk}_*$ *is infinite dimensional and simple. It is isomorphic to* $\mathfrak{kas}^\eta = (\mathfrak{g}_- \oplus \mathfrak{g}_0, \mathfrak{g}^\eta_{11} \oplus \mathfrak{g}_{12})^{mk}_*$.

When it does not matter which of isomorphic algebras $\mathfrak{kas}^\xi \simeq \mathfrak{kas}^\eta$ to take, we will simply write $\mathfrak{kas}$.

I.2.4. Grozman's theorem and a description of $\mathfrak{g}$ as $\mathfrak{g}_{\bar 0}$ and $\mathfrak{g}_{\bar 1}$. It is convenient to describe the Lie superalgebra $\mathfrak{g}$ of twisted polyvector fields as $\mathfrak{vect}$-module. Similarly, in [CK2] the exceptional algebras are described as $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$. For most of the series such description is of little value because each homogeneous component $\mathfrak{g}_{\bar 0}$ and $\mathfrak{g}_{\bar 1}$ has a complicated structure. For the exceptions the situation is totally different! Observe that apart from being beautiful, such a description is useful for the construction of simple Volichenko algebras, i.e., inhomogeneous with respect to parity subalgebras of simple Lie superalgebras, cf. [LS].

Recall in relation to this a theorem of Grozman. He completely described bilinear differential operators acting in tensor fields and invariant under all changes of coordinates. Miraculously, almost all of the first order operators determine a Lie superalgebra on its domain. Some of these superalgebras turn out to be simple or close to simple. In the constructions below we use some of these invariant operators.

Let ρ be an irreducible representation of the group $GL(n)$ in a finite dimensional vector space V and $\lambda = (\lambda_1, \dots, \lambda_n)$ its lowest weight (see [OV]). The *tensor field of type* ρ or V on an n-dimensional connected manifold M is any section t of the locally trivial vector bundle over M with fiber V such that under the change of coordinates

$$t(y(x)) = \rho(\frac{\partial y}{\partial x})t(x).$$

The space of tensor field of type ρ or V will be denoted by $T(\rho)$ or $T(V)$ or even $T(\lambda)$.

Denote by $T(\mu, \dots, \mu) = Vol^\mu$ the space of μ-densities. Other important examples: Ω^r, the space of differential r-forms, is $T(0, \dots, 0, 1, \dots, 1)$ with r-many 1's; in particular, $T(0) = \Omega^0$ is the space of functions; $\mathfrak{vect}(n) = T(-1, 0, \dots, 0)$ and $L^r = E^r_{\Omega^0}(\mathfrak{vect}(n)) = T(-1, \dots, -1, 0, \dots, 0)$ with r-many -1's is the space of r-vector fields.

The spaces of twisted r-forms and twisted r-vector fields with twist μ are defind to be $\Omega^r_\mu = \Omega^r \otimes_{\Omega^0} Vol^\mu$ and $L^r_\mu = L^r \otimes_{\Omega^0} Vol^\mu$. Obviously, $L^r_\mu \simeq \Omega^{n-r}_{\mu-1}$ and $Vol^1 = \Omega^n$.

The following statements are excerpts from Grozman's difficult result [G]. To describe one of the operators, P_4, the Nijenhuis bracket, originally defined to be

$$\begin{gathered}\omega^k \otimes \xi, \omega^l \otimes \eta \mapsto (\omega^k \otimes \omega^l) \otimes [\xi, \eta] + \\ \big(\omega^k \wedge L_\xi(\omega^l) + (-1)^k d\omega^k \wedge \iota(\xi)(\omega^l)\big) \otimes \eta + \\ \big(-L_\eta(\omega^k) \wedge \omega^l + (-1)^l \iota(\eta)(\omega^k) \wedge d\omega^l)\big) \otimes \xi,\end{gathered}$$

where ι is the inner product. The Nijenhuis bracket has the following interpretation which implies its invariance: the invariant operator $D : (\Omega^k \otimes \mathfrak{vect}(M), \Omega^\bullet) \longrightarrow \Omega^\bullet$ given by the formula

$$\begin{gathered}D(\omega^k \otimes \xi, \omega) = \\ d\left(\omega^k \wedge \iota(\xi)(\omega) + (-1)^k \omega^k \wedge \iota(\xi)(d\omega)\right) = d\omega^k \wedge \iota(\xi)(\omega) + (-1)^k \omega^k \wedge L_\xi(\omega)\end{gathered}$$

is, for a fixed $\omega^k \otimes \xi$, a superdifferentiation of the supercommutative superalgebra $\Omega^\bullet$ and he Nijenhuis bracket is just the supercommutator of these superdifferentiations. So we can identify $\Omega^\bullet \otimes \mathfrak{vect}(M)$, with the Lie subsuperalgebra of $\mathfrak{vect}(\hat M)$ (recall that $\hat M$ is the supermanifold $(M, \Omega^\bullet(M))$ and $\hat x_i = dx_i$):

$$C(d) = \{D \in \mathfrak{vect}(\hat M) \mid [D, d] = 0\}, \quad \text{the centralizer of the exterior differential.}$$

I.2.4.1. Statement . ([G]) *The following first order differential operators are, up to dualization and permutation of arguments, all first order bilinear operators $D : T(\rho_1) \otimes T(\rho_2) \longrightarrow T(\rho_3)$ (hereafer tensoring over the algebra of functions) invariant under arbitrary changes of variables:*

$$\mathbf{P_1} : \Omega^r \otimes T(\rho_2) \longrightarrow T(\rho_3), \qquad (w, t) \mapsto Z(dw, t),$$

where Z is the zero-th order operator, the extension of the projection $\rho_1 \otimes \rho_2 \longrightarrow \rho_3$ onto the irreducible component;

$$\mathbf{P_2} : Vect \otimes T(\rho) \longrightarrow T(\rho), \quad \textit{the Lie derivative};$$

$$\mathbf{P_3} : T(S^p(\mathrm{id}^*)) \otimes T(S^q(\mathrm{id}^*)) \longrightarrow T(S^{p+q-1}(\mathrm{id}^*)), \quad \textit{the Poisson bracket};$$

$\mathbf{P_4}$*: On manifolds, the bracket in $C(d)$ is called the Nijenhuis bracket. This bracket is a linear combination of operators P_1, P_1^{*1}, their composition with the permutation operator $T(V) \otimes T(W) \longrightarrow T(W) \otimes T(V)$, and an "irreducible" operator P_4;*

$$\mathbf{P_5} : \Omega^p \otimes \Omega^q \longrightarrow \Omega^{p+q+1}; \quad \omega_1, \omega_2 \mapsto (-1)^{p(\omega_1)} a(d\omega_1 \omega_2) + b(\omega_1 d\omega_2), \ \textit{where } a, b \in \mathbb{C};$$

$\mathbf{P_6}: \Omega^p_\mu\otimes\Omega^q_\nu\longrightarrow\Omega^{p+q+1}_{\mu+\nu}$, *where* $|\mu|^2+|\nu|^2\neq 0$ *and* $p+q<n$ *is given by the formula*

$$\omega_1 vol^\mu, \omega_2 vol^\nu\mapsto\big(\nu(-1)^{p(\omega_1)}d\omega_1\omega_2-\mu\omega_1 d\omega_2\big)\, vol^{\mu+\nu};$$

$$\mathbf{P_7}: L^p\otimes L^q\longrightarrow L^{p+q-1}\quad \textit{the Schouten bracket};$$

$\mathbf{P_8}: L^p_\mu\otimes L^q_\nu\longrightarrow L^{p+q-1}_{\mu+\nu}$, *a generalization of the Schouten bracket (on manifolds, for* $p+q\leq n$*; on supermanifolds of dimension* $n|1$ *for* $p,q\in\mathbb{C}$*) given by the formula*

$$Xvol^\mu, Yvol^\nu\mapsto\big((\nu-1)(\mu+\nu-1)\mathrm{div}X\cdot Y+(-1)^{p(X)}(\mu-1)(\mu+\nu-1)X\mathrm{div}Y-$$
$$(\mu-1)(\nu-1)\mathrm{div}(XY)\big)\, vol^{\mu+\nu},$$

where the divergence of a polyvector field is best described in local coordinates $(x,\check{x})$ *on the supermanifold* $\check{M}$ *associated to any manifold* M (see Remark after formula (0.3.7)).

I.2.4.2. Statement . *The following natural invariant operators determine Lie superalgebras on their domains. Some of these Lie superalgebras are close to simple ones:*

P_3 *and* P_4 *(by definition). The vectorial Lie superalgebra* $C(d)$ *is not, however, transitive.*

P_5*: for* $ab=0$ *it determines an associative superalgebra structure; for* $a=b$ *determines the structure of a nilpotent Lie superalgebra on* $\Pi(\Omega^\bullet)$*. The operator* $d\omega_1\omega_2-\omega_1 d\omega_2$ *determines a Lie algebra structure on the space* $\Omega^\bullet$*.*

P_6 *multiplied by* $\frac{\mu-\nu}{\mu\nu}$ *determines structures of nilpotent Lie superalgebras on*

$$\Pi(\mathop{\oplus}\limits_{\lambda\in\mathbb{C}}\Omega^\bullet\otimes Vol^\lambda)/d\Omega^\bullet,\quad \Omega^\bullet_+=\Pi(\mathop{\oplus}\limits_{\lambda>0;\ \lambda\in\mathbb{R}}\Omega^\bullet\otimes Vol^\lambda),\quad \textit{and } \Omega^\bullet_-=\Pi(\mathop{\oplus}\limits_{\lambda<0;\ \lambda\in\mathbb{R}}\Omega^\bullet\otimes Vol^\lambda).$$

P_6 *multiplied by* $\frac{\mu-\nu}{\mu+\nu}$ *determines a nilpotent Lie superalgebra structure on*

$$\Pi(d\Omega^\bullet\oplus\mathop{\oplus}\limits_{\lambda\neq 0;\ \lambda\in\mathbb{R}}\Omega^\bullet\otimes Vol^\lambda).$$

P_8 *multiplied by* $\frac{1}{\mu\nu}$ *determines nilpotent Lie superalgebra structures on* $\Omega^\bullet_+$ *and* $\Omega^\bullet_-$*.*

I.2.4.3. In addition to the W-gradings it is sometimes convenient to consider gradings which lead our vectorial Lie superalgebras out of the polynomial growth class but still preserve the finiteness of depth. Namely, we set $\deg x_i=0$ for all even indeterminates x; we denote this grading $\bar{n}$. Then the exceptional Lie superalgebras and $\mathfrak{b}_\lambda(n;\bar{n})$ look as follows (with the unspecified brackets being given by P_6 or P_7):

$\underline{\mathfrak{g}=\mathfrak{e}(5|10)}$: $\mathfrak{g}_{\bar 0}=\mathfrak{svect}(5|0)\simeq d\Omega^4$, $\mathfrak{g}_{\bar 1}=\Pi(d\Omega^1)$ with the natural $\mathfrak{g}_{\bar 0}$-action on $\mathfrak{g}_{\bar 1}$ and the bracketing of odd elements being their product; we identify

$$dx_i\wedge dx_j\wedge dx_k\wedge dx_l\otimes vol^{-1}=\mathrm{sign}(ijklm)\frac{\partial}{\partial x_m}\ \text{for any permutation } (ijklm) \text{ of } (12345).$$

$\underline{\mathfrak{g}=\mathfrak{vas}(4|4)}$: $\mathfrak{g}_{\bar 0}=\mathfrak{vect}(4|0)$, and $\mathfrak{g}_{\bar 1}=\Omega^1\otimes Vol^{-1/2}$ with the natural $\mathfrak{g}_{\bar 0}$-action on $\mathfrak{g}_{\bar 1}$ and the bracketing of odd elements being

$$[\frac{\omega_1}{\sqrt{vol}},\frac{\omega_2}{\sqrt{vol}}]=\frac{d\omega_1\wedge\omega_2+\omega_1\wedge d\omega_2}{vol},$$

where we identify

$$\frac{dx_i\wedge dx_j\wedge dx_k}{vol}=\mathrm{sign}(ijkl)\frac{\partial}{\partial x_l}\ \text{for any permutation } (ijkl) \text{ of } (1234).$$

$\underline{\mathfrak{g}=\mathfrak{vle}(3|6)}$: $\mathfrak{g}_{\bar 0}=\mathfrak{vect}(3|0)\oplus\mathfrak{sl}(2)^{(1)}_{\geq 0}$, where $\mathfrak{g}^{(1)}_{\geq 0}=\mathfrak{g}\otimes\mathbb{C}[x_1,x_2,x_3]$, and

$$\mathfrak{g}_{\bar 1}=\Big(\Omega^1\otimes Vol^{-1/2}\Big)\otimes\mathrm{id}_{\mathfrak{sl}(2)^{(1)}_{\geq 0}}\ \text{with the natural } \mathfrak{g}_{\bar 0}\text{-action on } \mathfrak{g}_{\bar 1}.$$

Recall that $\mathrm{id}_{\mathfrak{sl}(2)}$ is the irreducible $\mathfrak{sl}(2)$-module L^1 with highset weight 1; its tensor square splits into $L^2 \simeq \mathfrak{sl}(2)$ and the trivial module L^0; accordingly, denote by $v_1 \wedge v_2$ and $v_1 \bullet v_2$ the projections of $v_1 \otimes v_2 \in L^1 \otimes L^1$ onto the skew-symmetric and symmetric components, respectively. For $f_1, f_2 \in \Omega^0$, $\omega_1, \omega_2 \in \Omega^1$ and $v_1, v_2 \in L^1$ we set

$$[\frac{\omega_1 \otimes v_1}{\sqrt{vol}}, \frac{\omega_2 \otimes v_2}{\sqrt{vol}}] = \frac{(\omega_1 \wedge \omega_2) \otimes (v_1 \wedge v_2) + (d\omega_1 \wedge \omega_2 + \omega_1 \wedge d\omega_2) \otimes (v_1 \bullet v_2)}{vol},$$

where we identify Ω^0 with $\Omega^3 \otimes_{\Omega^0} Vol^{-1}$ and $\Omega^2 \otimes_{\Omega^0} Vol^{-1}$ with $\mathfrak{vect}(3|0)$ by setting

$$\frac{dx_i \wedge dx_j}{vol} = \mathrm{sign}(ijk)\frac{\partial}{\partial x_k} \text{ for any permutation } (ijk) \text{ of } (123).$$

$\underline{\mathfrak{g} = \mathfrak{mb}(3|8)}$: $\mathfrak{g}_{\bar 0} = \mathfrak{vect}(3|0) \oplus \mathfrak{sl}(2)^{(1)}_{\geq 0}$, as for $\mathfrak{vle}(3|6)$, while $\mathfrak{g}_{\bar 1} = \mathfrak{g}_{11} \oplus \mathfrak{g}_{12}$, where

$$\mathfrak{g}_{11} = \left(Vol^{-1/2}\right) \otimes \mathrm{id}_{\mathfrak{sl}(2)^{(1)}_{\geq 0}} \text{ and } \mathfrak{g}_{12} = \left(\Omega^2 \otimes Vol^{-1/2}\right) \otimes \mathrm{id}_{\mathfrak{sl}(2)^{(1)}_{\geq 0}}.$$

Multiplication is similar to that of $\mathfrak{g} = \mathfrak{vle}(3|6)$. For $f_1, f_2 \in \Omega^0$, $\omega_1, \omega_2 \in \Omega^1$ and $v_1, v_2 \in L^1$ we set

$$\begin{aligned}
&[\frac{\omega_1 \otimes v_1}{\sqrt{vol}}, \frac{\omega_2 \otimes v_2}{\sqrt{vol}}] = 0,\\
&[\frac{f_1 \otimes v_1}{\sqrt{vol}}, \frac{f_2 \otimes v_2}{\sqrt{vol}}] = \frac{(df_1 \wedge df_2) \otimes (v_1 \wedge v_2)}{vol},\\
&[\frac{f_1 \otimes v_1}{\sqrt{vol}}, \frac{\omega_1 \otimes v_2}{\sqrt{vol}}] = \frac{f_1\omega_1 \otimes (v_1 \wedge v_2) + (df_1\omega_1 + f_1 d\omega_1) \otimes (v_1 \bullet v_2)}{vol}.
\end{aligned}$$

$\underline{\mathfrak{g} = \mathfrak{kas}}$: $\mathfrak{g}_{\bar 0} = \mathfrak{vect}(1|0) \oplus \mathfrak{sl}(4)^{(1)}_{\geq 0}$, where $\mathfrak{sl}(4)^{(1)}_{\geq 0} = \mathfrak{sl}(4) \otimes \mathbb{C}[x]$, and $\mathfrak{g}_{\bar 1} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$, where $\mathfrak{g}_{-1} = \Lambda^2(\mathrm{id}_{\mathfrak{sl}(4)^{(1)}_{\geq 0}})$ and $\mathfrak{g}_1 = S^2(\mathrm{id}^*_{\mathfrak{sl}(4)^{(1)}_{\geq 0}})$; clearly, one can interchange $\mathfrak{g}_{\pm 1}$.

The multiplication is natural: the bracket of $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ (or the other way round) is the product of the skew matrix by symmetric one (or vice versa); the action of $\mathfrak{sl}(4)^{(1)}_{\geq 0}$ on $\mathfrak{g}_{\pm 1}$ is the natural action in the space of bilinear forms (or its dual), $\mathfrak{vect}(1|0)$ acts on functions-coefficients.

$\underline{\mathfrak{g} = \mathfrak{b}_\lambda(n; \bar n)}$: here $\bar n$ denotes the non-Weisfeiler grading given by the formulas $\deg q_i = 0$, $\deg \xi_i = 0$ for $i = 1, \dots, n$. Then we have $\mathfrak{g}_i = (\Pi(\Lambda^{i-1}(\mathfrak{vect}(n|0))) \otimes Vol^{-(i-1)\lambda}$ for $i = -1, 0, \dots, n-1$. The multiplication is given by P_8.

Consider $n = 2$ more attentively. Clearly, one can interchange $\mathfrak{g}_{\pm 1}$; this possibility explains the mysterious isomorphism $\mathfrak{b}_\lambda(2; \bar 2) \simeq \mathfrak{b}_{-1-\lambda}(2; \bar 2)$ mentioned in (OI). In particular, we have additional outer automorphisms of $\mathfrak{b}_{-1/2}(2; \bar 2)$, whereas for $\lambda = \frac{1}{2}$ (and $\lambda = -\frac{3}{2}$) there is a nontrivial central extension missed in [Ko1] and studied in [LSh3].

I.3. The steps of our classification. (From [LSh1], 1988, pp.235–289; compare with [K9], §3.)

Step 1. Formulation of the main ideas of our proof. Recapitulations and comparison with the Lie algebra case.

(1) Following Weisfeiler, we make use of the fact that every simple filtered vectorial Lie superalgebra has a W-filtration such that the associated graded Lie superalgebra is simple (see [W]);

(2) We show that any simple $\mathbb{Z}$-graded vectorial Lie superalgebra $\underset{i \geq -1}{\oplus} \mathfrak{g}_i$ is infinite dimensional if and only if $\mathfrak{g}_0$ acts on $\mathfrak{g}_{-1}$ so that the action has a rank 1 operator with an even covector.

We have listed all simple $\mathbb{Z}$-graded vectorial Lie superalgebras of depth 1 in the union of [ALSh], [Sh3], [Sh-M], [Sh14]. This union also contains all the examples of simple vectorial superalgebras of depth $d > 1$ considered as abstract ones and examples of various W-regradings.

(3) The cases when $\mathfrak{g}_{-1}$ is a purely odd superspace (consistent grading) or $d > 1$. We left them to consider at the end. Meanwhile Cheng and Kac investigated them ([K7], [CK2]) and found new (as compared with the known to us at that time) regradings of exceptional superalgebras, but no new abstract examples.

Step 2. Criterion for infinite dimension of a Cartan prolongation: existence of rank 1 operators with an even covector.

Step 3. Main reduction: the three cases. For $\mathfrak{g}_0$ and an irreducible faithful representation ρ of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ with an operator of rank 1 the following cases are possible:

I) ρ is induced (from a representation of a Lie subsuperalgebra of $\mathfrak{g}_0$ of codimension $0|k$).

II) $\mathfrak{g}_0$ is almost simple or $\mathfrak{g}_0$ is a central extension of an almost simple Lie superalgebra;

III) $\dim \mathfrak{g}_{-1} = (2,2)$ and $\mathfrak{g}_0$ is contained in a particular Lie superalgebra.

Step 4. Description of prolongations of induced representations (case I of step 3).

Step 5. Description of prolongations of almost simple Lie superalgebras and their central extensions (case II of step 3).

Step 6. Case III of step 3. (The unique simple infinite dimensional Lie superalgebra arising at this step is a regrading of $\mathfrak{b}^{\circ}_{\infty}(2)$.)

Step 7. Description of W-gradings in the above-found simple superalgebras.

Step 8. The case of consistent grading and depth > 1. See [K7] and [K9].

Step 9. Description of deformations. This is a separate problem, see [LS5].

I.4. On history. Victor Kac was the first to start classification of simple vectorial Lie superalgebras. This was even in a pre-super era: in [K0] three of the four series of simple finite dimensional vectorial Lie superalgebras are described. The realization of supersymmetric nature of the Universe encouraged several researchers to begin classification of simple Lie superalgebras, of finite dimension, to start with. Interesting partial results were obtained by Scheunert, Nahm, Rittenberg [SNR], I. Kaplansky ([FK], [Kap]) and Kac ([K1], [K1C]). The union of these results added up to a complete classification (where various cases required absolutely different approach); Kac summarized the results in [K1.5].

None of these researchers, however, considered deformations with odd parameters; these deformations in the natural category of supermanifolds, i.e., with odd parameters, were first classified by J. Bernstein and Leites (for proof see [LS5]) and simultaneously and independently by M. Duflo. Observe that to throw away odd parameters is contrary to the very idea of supersymmetry.[3]

Scheunert, Nahm, Rittenberg [SNR] classified all of simple finite dimensional Lie superalgebras except vectorial ones and Kaplansky independently classified the ones with Cartan matrix. Kac considered all finite dimensional algebras but missed some of them ([K1], [K1C]). Kaplansky's preprint-1975 (a gist of it appeared with some extensions as [FK], and finally as [Kap]) helped Kac to repair gaps in his independent proof of the classification ([K1], [K1C]). Since various cases have to be treated separately, anyway, it is instructive to compare different techniques of [SNR], and [FK], [Kap], with [K2], where earlier partial results of Pais and Rittenberg and Djoković are cited.

The paper [K2] — a review of the state of the art at that moment — consists of the classification, skillfully rounded up, and a "continuation of the theory" — elements of the

[3]It is interesting why for some people it is easier, psychologically, to accept odd central extensions than odd deformations ... Study of real forms (also a cohomological problem) also causes similar problems.

representation theory, further classification theorems, etc. Though every original major statement (Theorems 7–10, description of systems of simple roots, even the highest weight theorem) in the "continuation" is somewhat wrong (for some corrections see [S], [Se1]–[Se3], [L2], and this paper), the paper [K2] contains not only a classification, it is the first review of Lie superalgebra theory, partly developed by the author only. The paper was and still is impressive, and it became an unjust habit to write instead of "for classification see [K2]" (as the best sourse) "Kac's classification [K2]", thus attributing to Kac the whole classification rather than the summary of (dis)joint efforts.

Next on the agenda were *infinite dimensional* Lie superalgebras: to start with, say, $\mathbb{Z}$-graded of polynomial growth with the aim to obtain their complete classification similar to that of Kac–Mathieu [M] or to go even further: describe their completions and deformations (the latter task is not completed even for loop algebras and though we hope a version of it is tame, it might be wild as stated, cf. [GL2]).

Simple vectorial Lie algebras were classified by É. Cartan who considered them in the framework of the class of *primitive* Lie algebras, almost identical with the class of simple Lie algebras.

Kac's original idea was to follow Cartan and classify primitive Lie superalgebras; he conjectured [K2] (Th. 10 and Conjecture 1) that infinite dimensional primitive Lie superalgebras of vector fields with polynomial or formal coefficients are only straightforward analogs of the four well-known Cartan series of simple algebras $\mathfrak{vect}(n)$, $\mathfrak{svect}(n)$, $\mathfrak{h}(2n)$ and $\mathfrak{k}(2n+1)$ and the Lie superalgebras of their derivations.

Since superdimension is a pair of numbers, Kac's examples of *simple* vectorial Lie superalgebras "double" Cartan's list of simple vectorial Lie algebras. It soon became clear that the actual list of simple vectorial Lie superalgebras "doubles" that of Cartan at least twice, not once: the "odd" counterparts $\mathfrak{m}$ of $\mathfrak{k}$ as well as $\mathfrak{le}$ — the counterpart of $\mathfrak{h}$ — were discovered [L1], in [ALSh] their divergence-free subalgebras, deformations $\mathfrak{b}_\lambda(n;r)$ (misattributed to [Ko1] in [CK2], [K7], [K9]) and $\widetilde{\mathfrak{sb}}(2^n-1|2^n)$ were added.

At about the same time Grozman found $\mathfrak{b}_\lambda(n)$ in a different setting, cf. I.2.4

After [L1] the number of Lie superalgebras serving as the linear part of a simple vectorial Lie superalgebras grew so rapidly and their shapes were so diverse that at one moment we feared that the classification of *simple* vectorial superalgebras was a wild problem. Then we had understood that even the Lie superalgebras from the four well-known series ($\mathfrak{vect}$, $\mathfrak{svect}$, $\mathfrak{h}$ and $\mathfrak{k}$), have, in addition to the dimension of the superspace on which they are usually realized, one more discrete parameter governing their other, *nonstandard*, realizations ([ALSh]).

In other words, one Lie superalgebra of vector fields has several (but finitely many) distinct *nonisomorphic* realizations as a *filtered or graded* Lie superalgebra. This phenomenon is not only totally missed in [K2] but even in Kac' latest papers on the subject [K7], [K9] it is stated that to describe all filtered vectorial superalgebras (not just as abstract ones) is "probably impossible" (see [K7], the penultimate paragraph on p. 5 addresses an equivalent problem). In [K9] this problem (classify all finite dimensional irreducible faithful representations of Lie superalgebras) is unequivocally qualified as impossible. Perhaps it is, but it is not equivalent to our problem. The equivalent problem is, contrarywise, tame and in this paper we solve it.

Furthermore, several of these simple vectorial Lie superalgebras have deformations (see [L2], [Ko1], [L3], [LSh2]). Some of these deformed Lie superalgebras are very interesting due to their applications to physics. Observe, that it is natural not to confine ourselves to the classification problem of simple Lie superalgebras of certain class (say, filtered deformations) but to investigate where the deformations lead us; in this way we get, among other examples, deforms similar to the result of quantization of the Poisson Lie algebra.

Next, three exceptional vectorial algebras were discovered [Sh3], followed by a fourth exception [Sh-M] and, finally, fifth and the last exception [Sh5]. Regrettably, together with any other of our results this somehow escaped even such a detailed featured review as the one S. Sternberg wrote for Math. Reviews (99m:17006) heralding (somewhat prematurely concidering the *proof*, as we will see) solution of Problem B′ in [K7].

According to our method (which is a rather straightforward superization of Guillemin, Quillen and Sternberg's approach [GQS]) — to seek the irreducible modules over finite dimensional Lie superalgebras with a rank 1 even covector — cf. [K7], [K9] — we had to consider various cases and we leisurely passed from case to case. The end became visible after the exceptional cases and a difficult regular case (see §3 and [Sh-M]) were completely done with: *we have completely classified infinite dimensional simple vectorial Lie superalgebras of depth 1.* Having added center to L_0 considered, we got a number of examples of depth > 1 and were sure that there are no more examples though.

So, we have announced the complete classification in [LSh2] (as an aside remark to another story), as well as during several talks, in particular, at a seminar of E. Ivanov, JINR, Dubna (July, 1996) and Voronezh Winter School Jan. 12–18, 1997; conference in honor of D. Buchsbaum, Northeastern Univ., Boston, November 1997, see [LSh2].

Observe that, as early as in [ALSh], we wrote that the notion of "primitive Lie superalgebras" is not as "useful", so to say, as in the Lie algebra setting and resembles that of "semisimple" Lie superalgebras (or even Lie algebras: infinite dimensional or over a field of prime characteristic). Indeed, while for finite dimensional and vectorial complex Lie algebras these notions (primitive and semisimple) can be reduced to a reasonable notion of simple algebras, there are "too many primitive Lie superalgebras". For this simple but important fact, announced in [ALSh], see Appendix. So, having established that the problem of classification of primitive Lie *super*algebras is a wild one ([ALSh]), we only classify *simple* Lie superalgebras.

Kac's talk [K6], where he considered consistent gradings of simple exceptional vectorial Lie superalgebras which proved to be of (WF) type, i.e., gave their new, as compared with known to us, realizations was an unexpected contribution to our proof after we had confirmed that we did had announced the classification. Though brief, the talk left the impression that Kac has a new powerful method which proves our classification of *simple* vectorial Lie superalgebras (Problem A or even B) quicker than we do.

It was, however, unclear why the list of the series of simple vectorial superalgebras Kac gave in [K6] looked shorter than ours, and that of exceptions longer.

While we were eagerly waiting for the details of [K6] we tidied up our proof, and submitted [Sh5], a short version of [Sh14], which by that time was being refereed for almost two years.

When the details of [K6] were published in [K7] we saw, to our amazement, that Kac's method is very similar to ours, the one we lectured and discussed. Actually, Kac applied the same approach (superization of [GQS]) but more ingeniously and considered the cases we left out so far. While we were busy considering various filtrations, he distinguished one particular filtration for each of complete simple vectorial Lie superalgebras.

Since it does not matter which base of neighborhoods of zero one takes to determine the topology, Kac gave examples of complete simple Lie superalgebras *considered as abstract ones and isomorphic to the completions of our graded ones.*

The totally new and vital items in [K7], as compared with our results published by that time, were (1) the study of the case of consistent grading (which, as we conjectured, did not give any new examples to our final list if we consider algebras as abstract ones, not as graded or filtered) and (2) recovery, following Weisfeiler, of algebras of depth > 1 from the depth 1 ones.

In [K7], [CK1], [CK2], Cheng and Kac considered a *part* of our problem only, namely, the classification of complete vectorial Lie superalgebras considered as *abstract* ones (presumably under a wrong impression that there are "too many" W-filtrations). Still, their result doublechecked the completeness of our classification *if we consider our algebras as abstract ones.*

Meanwhile we had listed all the distinct W-gradings of these Lie superalgebras and were finishing the proof of the completeness of our list. We considered the following tasks:

(1) to list the examples of simple algebras of depth 1;

(1′) to prove completeness of the similar list for depth > 1;

(2) for a given Lie superalgebra list its W-filtrations;

(3) classify the deformations.

Task (3) leads us outside the realm of the original problem, but applications of its solution make it worth the trouble.

In [K7] Kac skillfully simplified problems (1) and (1′) and ignored problem (2) (dubbing it as "perhaps, impossible") and in [CK1] together with Cheng he slackened the formulation of problem (3): they only considered *filtered* deformations, moreover, disregarding odd parameters and throwing them away when they popped up.

Since [CK1] only considers one particular filtration, the problem of complete classification of filtered deformations is still open.

Comparison of notations. Strangely enough, not all of the results from [ALSh] or [Ko4] are reproduced in [K7]. Cheng and Kac considered the third parameter r for exceptional Lie superalgebras only. Cheng and Kac totally ignored subtleties described in I.2.2; so far we had published only (IAA) and some of the isomorphisms listed in (OI) and only these were mentioned in [K7]. Here is a comparison of series.

our notations	notations from [K7]
$\mathfrak{vect}(m\vert n;r)$	$W(m\vert n)$
$\mathfrak{svect}(m\vert n;r)$	$S(m\vert n)$
$\mathfrak{svect}^\circ(1\vert n;r)$	$S(1\vert n)$
$\widetilde{\mathfrak{svect}}(0\vert n)$	$\tilde{S}(0\vert n)$

(S_{vect})

$\mathfrak{k}(2m+1\vert n;r)$	$K(2m+1\vert n)$
$\mathfrak{h}(2m\vert n;r)$	$H(2m\vert n)$
$\mathfrak{h}_\lambda(2\vert 2;r)$	not singled out
$\mathfrak{h}^o(0\vert n)$	$H(0\vert n)$

(S_k)

$\mathfrak{m}(n\|n+1;r)$	$KO(n\|n+1)$	
$\mathfrak{sm}(n\|n+1;r)$	$SKO(n\|n+1)$	
$\mathfrak{b}_\lambda(n\|n+1;r)$	$SKO(n\|n+1;\beta)$, where $\beta=\frac{b}{a}=1-\frac{2}{n\lambda}\in\mathbb{C}$	
$\mathfrak{b}_1{}^\circ(n\|n+1;r)$	$SKO(n\|n+1;\frac{n-2}{n})$	(S_m)
$\mathfrak{b}_\infty{}^\circ(n\|n+1;r)$	$SKO(n\|n+1;1)$	
$\mathfrak{le}(n\|n;r)$	$HO(n\|n)$	
$\mathfrak{sle}^\circ(n\|n;r)$	$SHO(n\|n)$	
$\widetilde{\mathfrak{sb}}_\mu(2^{n-1}-1\|2^{n-1})$	not recognized as vectorial; appears as a filtered deformation	

The parameter λ of $\mathfrak{b}_\lambda(n|n+1;r)$ is a projective one: $\lambda\in\mathbb{C}P^1$. At $\lambda=0$ the Lie superalgebra $\mathfrak{b}_0(n|n+1)$ is the Buttin superalgebra — a central extension of $\mathfrak{le}(n|n)$; it is an "odd" analog of the Poisson algebra $\mathfrak{po}(2n)$. Therefore, the whole family $\mathfrak{b}_\lambda(n|n+1)$ can be considered as a deformation of the Buttin superalgebra. In Kac's notations the parameter β is an affine one: $\beta\in\mathbb{C}$. However, as is clear from the translation formulas, $\lambda=0$ corresponds to $\beta=\infty$.

Deformations: discussion. Let us forget at the moment that one should not disregard odd deformations, leave alone general deformations, and consider the following result of computations of filtered deformations [CK1] (considered as final, e.g., in [K9]):

There are only three series $\tilde{\mathcal{L}}$ of filtered deformations of Lie superalgebras $\mathcal{L}$ and all of them can be described as follows. Let $\mathcal{L}\subset\mathfrak{vect}(n|m;r)$ be a subalgebra with the same negative part, let Θ be the product of all m odd indeterminates and t a parameter, $p(t)=p(m)$. Then

$$\tilde{\mathcal{L}}=(1+t\Theta)\mathcal{L} \qquad (d.1)$$

is a subalgebra of $\mathfrak{vect}(n|m;r)$. It so happens that usually $\tilde{\mathcal{L}}\simeq\mathcal{L}$ but there are three series of exceptions: $\mathfrak{svect}(0|n)$, $\mathfrak{sle}(n)$ and $\mathfrak{b}_1(n)$.

Now, observe that

(1) all these deformations were known: the first one appeared in [K2], the other two are due to Kochetkov. Kochetkov's description of deformations is partly buried in inaccessible VINITI depositions [Ko4] (reproduced in [L6]) and partly unpublished (a letter to Leites, at that time Kochetkov's Ph.D. advisor).

(2) Kochetkov [Ko4] had also found several odd cocycles, see [L6]. In [CK1] they are rediscovered but deliberately thrown away.

(3) In 1985 Kochetkov further observed (unpublished) that nontrivial deformations of the form $(d.1)$ correspond to the Lie superalgebras with a trace, i.e., with an ideal of codimension 1. Likewise, the existence of an integral on the space of values of the divergence map gives rise to similar deformations.

(4) In [CK1] it is shown that filtered deformations of the graded Lie superalgebra $\mathfrak{g}$ of nonzero depth and with a decreasing filtration correspond to $\mathfrak{g}_0$-invariants of the superspace $H^2(\mathfrak{g}_-;\mathfrak{g})$ (of which Cheng and Kac only take the even part). A problem arises: let $\mathfrak{g}\simeq\mathfrak{h}$ be two $\mathbb{Z}$-gradings of the same abstract algebra. Is the formula

$$H^2(\mathfrak{g}_-;\mathfrak{g})^{\mathfrak{g}_0}\simeq H^2(\mathfrak{h}_-;\mathfrak{h})^{\mathfrak{h}_0} \qquad (d.2)$$

true?

If not, [CK1] contains not the description of filtered deformations of all simple W-graded vectorial Lie superalgebras, as claimed[4], but $\frac{1}{m}$th of all possibilities for each $n|m$-dimensional supermanifold.

If yes, it is remarkable: why should this be true?! A proof is desirable. (For example, the whole spaces $H^2(\mathfrak{g}_-;\mathfrak{g})$ and $H^2(\mathfrak{h}_-;\mathfrak{h})$ are always distinct in the cases considered in [GLS2].) Dzhumadildaev told us that $(d.2)$ is not true for some simple Lie algebras in characteristic p.

For other examples of deformations see [L6] and [LSh3]. For the answer to the above question $(d.2)$ and the complete description of deformations see [LS5], where we give a simplified proof of the results of [Ko4], and cases left, and so close the problem.

§0. Background and basic examples

0.1. Linear algebra in superspaces. Generalities. A *superspace* is a $\mathbb{Z}/2$-graded space; for any superspace $V = V_{\bar 0}\oplus V_{\bar 1}$ denote by $\Pi(V)$ another copy of the same superspace: with the shifted parity, i.e., $(\Pi(V))_{\bar i} = V_{\bar i+\bar 1}$. The *superdimension* of V is $\dim V = p + q\varepsilon$, where $\varepsilon^2 = 1$ and $p = \dim V_{\bar 0}$, $q = \dim V_{\bar 1}$. (Usually, $\dim V$ is expressed as a pair (p, q) or $p|q$; this obscures the fact that $\dim V\otimes W = \dim V\cdot\dim W$; this fact is clear with the use of ε.)

A superspace structure in V induces the superspace structure in the space $\mathrm{End}(V)$. A *basis of a superspace* is always a basis consisting of *homogeneous* vectors; let $Par = (p_1, \dots, p_{\dim V})$ be an ordered collection of their parities. We call Par the *format* of (the basis of) V. A square *supermatrix* of format (size) Par is a $\dim V\times\dim V$ matrix whose ith row and ith column are of the same parity $p_i\in Par$.

One usually considers one of the simplest formats Par, e.g., the format Par_{st} of the form $(\bar 0,\dots,\bar 0;\bar 1,\dots,\bar 1)$ is called *standard*. In this paper we can do without nonstandard formats. But they are vital in the classification of systems of simple roots of simple finite dimensional simple Lie superalgebras. In particular, the reader might be interested in them in connection with applications to q-quantization or integrable systems. Systems of simple roots corresponding to distinct nonstandard formats are related by so-called odd reflections — analogs of our nonstandard regradings.

The matrix unit E_{ij} is supposed to be of parity $p_i + p_j$ and the bracket of supermatrices (of the same format) is defined via **Sign Rule**:

if something of parity p moves past something of parity q the sign $(-1)^{pq}$ accrues; the formulas defined on homogeneous elements are extended to arbitrary ones via linearity.

Examples of application of the Sign Rule: setting $[X,Y] = XY - (-1)^{p(X)p(Y)}YX$ we get the notion of the supercommutator and the ensuing notions of the supercommutative superalgebra; Lie superalgebra is the one which in addition to superskew-commutativity satisfies the super Jacobi identity, i.e., the Jacobi identity amended with the Sign Rule); the superderivation of a superalgebra A is a linear map $D : A\longrightarrow A$ that satisfies the super Leibniz rule

$$D(ab) = D(a)b + (-1)^{p(D)p(a)}aD(b).$$

In particular, let $A = \mathbb{C}[x]$ be the free supercommutative polynomial superalgebra in $x = (x_1,\dots,x_n)$, where the superstructure is determined by the parities of the indeterminates: $p(x_i) = p_i$. Partial derivatives are defined (with the help of super Leibniz Rule) by the formulas $\frac{\partial x_i}{\partial x_j} = \delta_{i,j}$. Clearly, the collection $\mathfrak{der}\,\mathbb{C}[x]$ of all superdifferentiations of A is a Lie superalgebra whose elements are of the form $\sum f_i(x)\frac{\partial}{\partial x_i}$ with $f_i(x)\in\mathbb{C}[x]$ for all i.

Observe that sometimes applying Sign Rule requires some dexterity (we have to distinguish between two versions both of which turn in non-supercase into one, called either skew- or

[4]Actually, the claim of [CK1] is more vague and subject to interpretations: "we determine all simple filtered deformations of certain $\mathbb{Z}$-graded Lie superalgebras classified in [K3]", and since in [K3] the algebras are only classified as abstract ones, one might argue that $(d.2)$ is not claimed there.

anti-commutativity):

$$\begin{array}{ll} ba = (-1)^{p(b)p(a)}ab & \text{(symmetry)} \\ ba = -(-1)^{p(b)p(a)}ab & \text{(anti-symmetry)} \\ ba = (-1)^{(p(b)+1)(p(a)+1)}ab & \text{(skew-symmetry)} \\ ba = -(-1)^{(p(b)+1)(p(a)+1)}ab & \text{(antiskew-symmetry)} \end{array}$$

In what follows the superskew-symmetric bilinear forms are named according to the above definitions.

Given the supercommutative superalgebra $\mathcal{F}$ of "functions" in indeterminates x, define the supercommutative superalgebra Ω of differential forms as polynomial algebra over $\mathcal{F}$ in dx, where $p(d) = \bar{1}$. Since dx is even for an odd x, we can consider not only polynomials in dx. Smooth or analytic functions in the differentials of the x are called *pseudodifferential forms* on the supermanifold with coordinates x, see [BL]. We will need them to interpret $\mathfrak{b}_\lambda(n)$. The exterior differential is defined on the space of (pseudo) differential forms by the formulas (mind super Leibniz rule):

$$d(x_i) = dx_i \text{ and } d^2 = 0.$$

The Lie derivative is defined (minding same Rules) by the formula

$$L_D(df) = (-1)^{p(D)} d(D(f)).$$

In particular,

$$L_D\left((df)^\lambda\right) = \lambda(-1)^{p(D)} d(D(f))(df)^{\lambda-1} \text{ for any } \lambda \in \mathbb{C} \text{ and odd } f.$$

The *general linear* Lie superalgebra of all supermatrices of size *Par* is denoted by $\mathfrak{gl}(Par)$, where $Par = (p_1, \dots)$ is the ordered collection of parities of the rows identical to that of the columns; usually, for the *standard* (simplest) format $\mathfrak{gl}(\bar{0}, \dots, \bar{0}, \bar{1}, \dots, \bar{1})$ is abbreviated to $\mathfrak{gl}(\dim V_{\bar{0}} | \dim V_{\bar{1}})$. Any nonzero supermatrix from $\mathfrak{gl}(Par)$ can be uniquely expressed as the sum of its even and odd parts; in the standard format this is the following block expression:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad p\left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right) = \bar{0},\ p\left(\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right) = \bar{1}.$$

The *supertrace* is the map $\mathfrak{gl}(Par) \longrightarrow \mathbb{C}$, $(A_{ij}) \mapsto \sum (-1)^{p_i} A_{ii}$, where $Par = (p_1, \dots)$. Since $\mathrm{str}[x, y] = 0$, the subsuperspace of supertraceless matrices constitutes the *special linear* Lie subsuperalgebra $\mathfrak{sl}(Par)$.

There are, however, at least two super versions of $\mathfrak{gl}(n)$, not one. The other version is called the *queer* Lie superalgebra and is defined as the one that preserves the complex structure given by an *odd* operator J, i.e., is the centralizer $C(J)$ of J:

$$\mathfrak{q}(n) = C(J) = \{X \in \mathfrak{gl}(n|n) : [X, J] = 0\}, \text{ where } J^2 = -\mathrm{id}.$$

It is clear that by a change of basis we can reduce J to the form $J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1 & 0 \end{pmatrix}$ in the standard format and then $\mathfrak{q}(n)$ takes the form

$$\mathfrak{q}(n) = \left\{\begin{pmatrix} A & B \\ B & A \end{pmatrix}\right\}.$$

On $\mathfrak{q}(n)$, the *queertrace* is defined: $\mathrm{qtr} : \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mapsto \mathrm{tr} B$. Denote by $\mathfrak{sq}(n)$ the Lie superalgebra of *queertraceless* matrices.

Observe that the identity representations of $\mathfrak{q}(V)$ and $\mathfrak{sq}(V)$ in V, though irreducible in supersetting, are not irreducible in the nongraded sense: take homogeneous (with respect to parity) and linearly independent vectors $v_1, \dots, v_n$ from V; then $Span(v_1 + J(v_1), \dots, v_n + J(v_n))$ is an invariant subspace of V which is not a subsuperspace.

A representation is *irreducible* of *general type* or just of *G-type* if there is no invariant subspace, otherwise it is called *irreducible of Q-type* (Q is after the general queer Lie superalgebra — a specifically "superish" analog of $\mathfrak{gl}$); an irreducible representation of Q-type has no invariant sub*super*space but *has* an invariant subspace.

Lie superalgebras that preserve bilinear forms: two types. To the linear map $F: V \longrightarrow W$ of superspaces there corresponds the dual map $F^*: W^* \longrightarrow V^*$ between the dual superspaces. In a basis consisting of the vectors v_i of format Par, the formula $F(v_j) = \sum\limits_i v_i A_{ij}$ assigns to F the supermatrix A. In the dual bases, to F^* the *supertransposed* matrix A^{st} corresponds:

$$(A^{st})_{ij} = (-1)^{(p_i+p_j)(p_i+p(A))} A_{ji}.$$

The supermatrices $X \in \mathfrak{gl}(Par)$ such that

$$X^{st}B + (-1)^{p(X)p(B)}BX = 0 \quad \text{for an homogeneous matrix } B \in \mathfrak{gl}(Par)$$

constitute the Lie superalgebra $\mathfrak{aut}(B)$ that preserves the bilinear form B^f on V whose matrix B is given by the formula $B_{ij} = (-1)^{p(B^f)p(v_i)} B^f(v_i, v_j)$ for the basis vectors v_i.

Recall that the *supersymmetry* of the homogeneous form B^f means that its matrix B satisfies the condition $B^u = B$, where $B^u = \begin{pmatrix} R^t & (-1)^{p(B)}T^t \\ (-1)^{p(B)}S^t & -U^t \end{pmatrix}$ for the matrix $B = \begin{pmatrix} R & S \\ T & U \end{pmatrix}$. Similarly, *skew-supersymmetry* of B means that $B^u = -B$. Thus, we see that the *upsetting* of bilinear forms $u: \mathrm{Bil}(V, W) \longrightarrow \mathrm{Bil}(W, V)$, which for the *spaces* and when $V = W$ is expressed on matrices in terms of the transposition, is a new operation.

Most popular canonical forms of the nondegenerate supersymmetric form are the ones whose supermatrices in the standard format are the following canonical ones, B_{ev} or B'_{ev}:

$$B'_{ev}(m|2n) = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad \text{where } J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

or

$$B_{ev}(m|2n) = \begin{pmatrix} \mathrm{antidiag}(1, \dots, 1) & 0 \\ 0 & J_{2n} \end{pmatrix}.$$

The usual notation for $\mathfrak{aut}(B_{ev}(m|2n))$ is $\mathfrak{osp}(m|2n)$ or, more precisely, $\mathfrak{osp}^{sy}(m|2n)$. Observe that the passage from V to $\Pi(V)$ sends the supersymmetric forms to superskew-symmetric ones, preserved by the "symplectico-orthogonal" Lie superalgebra, $\mathfrak{spo}(2n|m)$ or, more prudently, $\mathfrak{osp}^{sk}(m|2n)$, which is isomorphic to $\mathfrak{osp}^{sy}(m|2n)$ but has a different matrix realization.

In the standard format the matrix realizations of these algebras are:

$$\mathfrak{osp}(m|2n) = \left\{ \begin{pmatrix} E & Y & X^t \\ X & A & B \\ -Y^t & C & -A^t \end{pmatrix} \right\}; \quad \mathfrak{osp}^{sk}(m|2n) = \left\{ \begin{pmatrix} A & B & X \\ C & -A^t & Y^t \\ Y & -X^t & E \end{pmatrix} \right\},$$
$$\text{where } \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{sp}(2n), \quad E \in \mathfrak{o}(m).$$

A nondegenerate supersymmetric odd bilinear form $B_{odd}(n|n)$ can be reduced to a canonical form whose matrix in the standard format is J_{2n}. A canonical form of the superskew odd nondegenerate form in the standard format is $\Pi_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. The usual notation for $\mathfrak{aut}(B_{odd}(Par))$ is $\mathfrak{pe}(Par)$. The passage from V to $\Pi(V)$ establishes an isomorphism $\mathfrak{pe}^{sy}(Par) \cong \mathfrak{pe}^{sk}(Par)$. This Lie superalgebra is called, as A. Weil suggested, *periplectic*. The matrix realizations in the standard format of these superalgebras is shorthanded to:

$$\begin{aligned}\mathfrak{pe}^{sy}\ (n) &= \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix},\ \text{where } B = -B^t,\ C = C^t \right\};\\ \mathfrak{pe}^{sk}(n) &= \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix},\ \text{where } B = B^t,\ C = -C^t \right\}.\end{aligned}$$

Observe that though the Lie superalgebras $\mathfrak{osp}^{sy}(m|2n)$ and $\mathfrak{osp}^{sk}(m|2n)$, as well as $\mathfrak{pe}^{sy}(n)$ and $\mathfrak{pe}^{sk}(n)$, are isomorphic, the difference between them is sometimes crucial, see [Sh5].

The *special periplectic* superalgebra is

$$\mathfrak{spe}(n) = \{X \in \mathfrak{pe}(n) \mid \mathrm{str} X = 0\}.$$

Of particular interest to us will be also

$$\mathfrak{spe}(n)_{a,b} = \mathfrak{spe}(n) \Subset \mathbb{C}(az + bd),\ \text{where } z = 1_{2n},\ d = \mathrm{diag}(1_n, -1_n).$$

0.1.1. What a Lie superalgebra is. Dealing with superalgebras it sometimes becomes useful to know their definition. Lie superalgebras were distinguished in topology in 1930's or earlier. So when somebody offers a "better than usual" definition of a notion which seemed to have been established about 70 year ago this might look strange, to say the least. Nevertheless, the answer to the question "what is a Lie superalgebra?" is still not a common knowledge.

So far we operated as if we know. However, the naive definition suggested above ("apply the Sign Rule to the definition of the Lie algebra") is manifestly inadequate for considering the (singular) supervarieties of deformations and to apply representation theory to mathematical physics, for example, in the study of the coadjoint representation of the Lie supergroup which can act on a supermanifold but never on a superspace (an object from another category). So, to deform Lie superalgebras and apply group-theoretical methods in "super" setting, we must be able to recover a supermanifold from a superspace, and vice versa.

A proper definition of Lie superalgebras is as follows, cf. [L3]–[L5]. The *Lie superalgebra* in the category of supermanifolds corresponding to the "naive" Lie superalgebra $L = L_{\bar 0} \oplus L_{\bar 1}$ is a linear supermanifold $\mathcal{L} = (L_{\bar 0}, \mathcal{O})$, where the sheaf of functions $\mathcal{O}$ consists of functions on $L_{\bar 0}$ with values in the Grassmann superalgebra on $L_{\bar 1}^*$; this supermanifold should be such that for "any" (say, finitely generated, or from some other appropriate category) supercommutative superalgebra C, the space $\mathcal{L}(C) = \mathrm{Hom}(\mathrm{Spec} C, \mathcal{L})$, called *the space of C-points of $\mathcal{L}$*, is a Lie algebra and the correspondence $C \longrightarrow \mathcal{L}(C)$ is a functor in C. (A. Weil introduced this approach in algebraic geometry in 1953; in super setting it is called *the language of points* or *families*, see [L4], [D].) This definition might look terribly complicated, but fortunately one can show that the correspondence $\mathcal{L} \longleftrightarrow L$ is one-to-one and the Lie algebra $\mathcal{L}(C)$, also denoted $L(C)$, admits a very simple description: $L(C) = (L \otimes C)_{\bar 0}$.

A *Lie superalgebra homomorphism* $\rho : L_1 \longrightarrow L_2$ in these terms is a functor morphism, i.e., a collection of Lie algebra homomorphisms $\rho_C : L_1(C) \longrightarrow L_2(C)$ compatible with morphisms of supercommutative superalgebras $C \longrightarrow C'$. In particular, a *representation* of a Lie superalgebra L in a superspace V is a homomorphism $\rho : L \longrightarrow \mathfrak{gl}(V)$, i.e., a collection of Lie algebra homomorphisms $\rho_C : L(C) \longrightarrow (\mathfrak{gl}(V) \otimes C)_{\bar 0}$.

Example . Consider a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. The tangent space of the moduli superspace of deformations of ρ is isomorphic to $H^1(\mathfrak{g}; V \otimes V^*)$. For example, if $\mathfrak{g}$ is the $0|n$-dimensional (i.e., purely odd) Lie superalgebra (with the only bracket possible: identically equal to zero), its only irreducible representations are the trivial one, $\mathbf{1}$, and $\Pi(\mathbf{1})$. Clearly, $\mathbf{1} \otimes \mathbf{1}^* \simeq \Pi(\mathbf{1}) \otimes \Pi(\mathbf{1})^* \simeq \mathbf{1}$, and because the Lie superalgebra $\mathfrak{g}$ is commutative, the differential in the cochain complex is trivial. Therefore, $H^1(\mathfrak{g}; \mathbf{1}) = E^1(\mathfrak{g}^*) \simeq \mathfrak{g}^*$, so there are $\dim \mathfrak{g}$ odd parameters of deformations of the trivial representation. If we consider $\mathfrak{g}$ "naively" all of the odd parameters will be lost.

Which of these infinitesimal deformations can be extended to a global one is a separate much tougher question, usually solved *ad hoc*.

Examples that lucidly illustrate why one should always remember that a Lie superalgebra is not a mere linear superspace but a linear supermanifold are, e.g., deforms $\widetilde{\mathfrak{svect}}(0|2n+1)$ and $\widetilde{\mathfrak{sb}}_\mu(2^{2n}-1|2^{2n})$ with odd parameters. In the category of supermanifolds these are simple Lie superalgebras.

0.1.2. Projectivization. If $\mathfrak{s}$ is a Lie algebra of scalar matrices, and $\mathfrak{g} \subset \mathfrak{gl}(n|n)$ is a Lie subsuperalgebra containing $\mathfrak{s}$, then the *projective* Lie superalgebra of type $\mathfrak{g}$ is $\mathfrak{pg} = \mathfrak{g}/\mathfrak{s}$. Lie superalgebras $\mathfrak{g}_1 \bigodot \mathfrak{g}_2$ described in sect. 3.1 are also projective.

Projectivization sometimes leads to new Lie superalgebras, for example: $\mathfrak{pgl}(n|n)$, $\mathfrak{psl}(n|n)$, $\mathfrak{pq}(n)$, $\mathfrak{psq}(n)$; whereas $\mathfrak{pgl}(p|q) \cong \mathfrak{sl}(p|q)$ if $p \neq q$.

0.1.3. What a semisimple Lie superalgebra is. Recall that the Lie superalgebra $\mathfrak{g}$ without proper ideals and of dimension > 1 is called *simple*. Examples: $\mathfrak{sl}(m|n)$ for $m > n \geq 1$, $\mathfrak{psl}(n|n)$ for $n > 1$, $\mathfrak{psq}(n)$ for $n > 2$, $\mathfrak{osp}(m|2n)$ for $mn \neq 0$ and $\mathfrak{spe}(n)$ for $n > 2$.

In §5 we will need the remaining simple finite dimensional Lie superalgebras of non-vectorial type. These superalgebras, discovered by I. Kaplansky (an unpublished preprint, see [FK]) are $\mathfrak{osp}_\alpha(4|2)$, the deforms of $\mathfrak{osp}(4|2)$, and the two exceptions that we denote by $\mathfrak{ag}_2$ and $\mathfrak{ab}_3$. These Lie superalgebras do not contribute to our classification; for their description we refer to [FK], [K2] completed in [vdL], [Sud] and [GL1].

We say that $\mathfrak{h}$ is *almost simple* if it can be sandwiched (non-strictly) between a simple Lie superalgebra $\mathfrak{s}$ and the Lie superalgebra $\mathfrak{Der}\ \mathfrak{s}$ of derivations of $\mathfrak{s}$, i.e., $\mathfrak{s} \subset \mathfrak{h} \subset \mathfrak{Der}\ \mathfrak{s}$.

By definition, $\mathfrak{g}$ is *semisimple* if its radical is zero. Literally following the description of semisimple Lie algebras over the fields of prime characteristic, V. Kac [K2] gave the following description of semisimple Lie superalgebras. Let $\mathfrak{s}_1, \dots, \mathfrak{s}_k$ be simple Lie superalgebras, let $n_1, \dots, n_k$ be *pairs* of non-negative integers $n_j = (n_j^{\bar 0}, n_j^{\bar 1})$, let $\mathcal{F}(n_j)$ be the supercommutative superalgebra of polynomials in $n_j^{\bar 0}$ even and $n_j^{\bar 1}$ odd indeterminates, and $\mathfrak{s} = \bigoplus\limits_j (\mathfrak{s}_j \otimes \mathcal{F}(n_j))$. Then

$$\mathfrak{Der}\ \mathfrak{s} = \bigoplus_j \left((\mathfrak{Der}\ \mathfrak{s}_j) \otimes \mathcal{F}(n_j) \Subset \mathrm{id}_{\mathfrak{s}_j} \otimes \mathfrak{vect}(n_j)\right). \qquad (0.1.3)$$

Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{Ders}$ containing $\mathfrak{s}$. If the projection of $\mathfrak{g}$ on $\mathrm{id}_{\mathfrak{s}_j} \otimes \mathfrak{vect}(n_j)_{-1}$ is **onto** *for each j, then $\mathfrak{g}$ is semisimple and all semisimple Lie superalgebras arise in the manner indicated*, i.e., as sums of *almost simple superalgebras* corresponding to the summands of (0.1.3).

0.1.4. A. Sergeev's central extension. In 70's A. Sergeev proved that there is just one nontrivial central extension of $\mathfrak{spe}(n)$ for $n > 2$. It exists only for $n = 4$ and we denote it by $\mathfrak{as}$. Let us represent an arbitrary element $A \in \mathfrak{as}$ as a pair $A = x + d \cdot z$, where $x \in \mathfrak{spe}(4)$, $d \in \mathbb{C}$ and z is the central element. The bracket in $\mathfrak{as}$ is

$$\left[\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} + d \cdot z, \begin{pmatrix} a' & b' \\ c' & -a'^t \end{pmatrix} + d' \cdot z\right] = \left[\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & -a'^t \end{pmatrix}\right] + \mathrm{tr}\ c\tilde{c}' \cdot z, \quad (0.1.4.1)$$

where $\tilde{}$ is extended via linearity from matrices $c_{ij} = E_{ij} - E_{ji}$ on which $\tilde{c}_{ij} = c_{kl}$ for any even permutation $(1234) \mapsto (ijkl)$.

The Lie superalgebra $\mathfrak{as}$ can also be described with the help of the spinor representation. For this we need several vectorial superalgebras defined in sect. 0.3. Consider $\mathfrak{po}(0|6)$, the Lie superalgebra whose superspace is the Grassmann superalgebra $\Lambda(\xi, \eta)$ generated by $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$ and the bracket is the Poisson bracket (0.3.6). Recall that $\mathfrak{h}(0|6) = Span(H_f : f \in \Lambda(\xi, \eta))$.

Now, observe that $\mathfrak{spe}(4)$ can be embedded into $\mathfrak{h}(0|6)$. Indeed, setting $\deg \xi_i = \deg \eta_i = 1$ for all i we introduce a $\mathbb{Z}$-grading on $\Lambda(\xi, \eta)$ which, in turn, induces a $\mathbb{Z}$-grading on $\mathfrak{h}(0|6)$ of the form $\mathfrak{h}(0|6) = \mathop{\oplus}\limits_{i \geq -1} \mathfrak{h}(0|6)_i$. Since $\mathfrak{sl}(4) \cong \mathfrak{o}(6)$, we can identify $\mathfrak{spe}(4)_0$ with $\mathfrak{h}(0|6)_0$.

It is not difficult to see that the elements of degree -1 in the standard gradings of $\mathfrak{spe}(4)$ and $\mathfrak{h}(0|6)$ constitute isomorphic $\mathfrak{sl}(4) \cong \mathfrak{o}(6)$-modules. It is subject to a direct verification that it is possible to embed $\mathfrak{spe}(4)_1$ into $\mathfrak{h}(0|6)_1$.

Sergeev's extension $\mathfrak{as}$ is the result of the restriction to $\mathfrak{spe}(4) \subset \mathfrak{h}(0|6)$ of the cocycle that turns $\mathfrak{h}(0|6)$ into $\mathfrak{po}(0|6)$. The quantization deforms $\mathfrak{po}(0|6)$ into $\mathfrak{gl}(\Lambda(\xi))$; the through maps $T_\lambda : \mathfrak{as} \longrightarrow \mathfrak{po}(0|6) \longrightarrow \mathfrak{gl}(\Lambda(\xi))$ are representations of $\mathfrak{as}$ in the $4|4$-dimensional modules spin_λ isomorphic to each other for all $\lambda \neq 0$. The explicit form of T_λ is as follows:

$$T_\lambda : \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} + d \cdot z \mapsto \begin{pmatrix} a & b - \lambda\tilde{c} \\ c & -a^t \end{pmatrix} + \lambda d \cdot 1_{4|4}, \tag{0.1.4.2}$$

where $1_{4|4}$ is the unit matrix and $\tilde{c}$ is defined in formula (0.1.4.1). Clearly, T_λ is an irreducible representation for any λ.

0.2. Vectorial Lie superalgebras. The standard realization. The elements of the Lie algebra $\mathcal{L} = \mathfrak{der}\, \mathbb{C}[[u]]$ are considered as vector fields. The Lie algebra $\mathcal{L}$ has only one maximal subalgebra $\mathcal{L}_0$ of finite codimension (consisting of the fields that vanish at the origin). The subalgebra $\mathcal{L}_0$ determines a filtration of $\mathcal{L}$: set

$$\mathcal{L}_{-1} = \mathcal{L} \ \text{ and } \ \mathcal{L}_i = \{D \in \mathcal{L}_{i-1} \mid [D, \mathcal{L}] \subset \mathcal{L}_{i-1}\} \text{ for } i \geq 1. \tag{0.2.1}$$

The associated graded Lie algebra $L = \mathop{\oplus}\limits_{i \geq -1} L_i$, where $L_i = \mathcal{L}_i / \mathcal{L}_{i+1}$, consists of the vector fields with *polynomial* coefficients.

Superization. For a simple Lie superalgebra $\mathcal{L}$ (for example, take $\mathcal{L} = \mathfrak{der}\, \mathbb{C}[u, \xi]$), suppose $\mathcal{L}_0 \subset \mathcal{L}$ is a maximal subalgebra of finite codimension. Let $\mathcal{L}_{-1}$ be a minimal subspace of $\mathcal{L}$ containing $\mathcal{L}_0$, different from $\mathcal{L}_0$ and $\mathcal{L}_0$-invariant. A *Weisfeiler filtration* of $\mathcal{L}$ is determined by setting for $i \geq 1$:

$$\mathcal{L}_{-i-1} = [\mathcal{L}_{-1}, \mathcal{L}_{-i}] + \mathcal{L}_{-i} \ \text{ and } \ \mathcal{L}_i = \{D \in \mathcal{L}_{i-1} \mid [D, \mathcal{L}_{-1}] \subset \mathcal{L}_{i-1}\}. \tag{0.2.2}$$

Since the codimension of $\mathcal{L}_0$ is finite, the filtration takes the form

$$\mathcal{L} = \mathcal{L}_{-d} \supset \dots \supset \mathcal{L}_0 \supset \dots \tag{0.2.3}$$

for some d. This d is called the *depth* of $\mathcal{L}$ and of the associated graded Lie superalgebra L.

Considering the subspaces (0.2.1) as the basis of a topology, we can complete the graded or filtered Lie superalgebras L or $\mathcal{L}$; the elements of the completion are the vector fields with formal power series as coefficients. Though the structure of the graded algebras is easier to describe, in applications the completed Lie superalgebras are usually needed.

Remarks . 1) Not all filtered or graded Lie superalgebras of finite depth are *vectorial*, i.e., realizable with vector fields on a supermanifold of the same dimension as that of $\mathcal{L}/\mathcal{L}_0$; only those with faithful L_0-action on L_- are.

2) Unlike Lie algebras, simple vectorial *super*algebras possess *several* nonisomorphic maximal subalgebras of finite codimension, see I.2.2.

1) General algebras. Let $x = (u_1, \dots, u_n, \theta_1, \dots, \theta_m)$, where the u_i are even indeterminates and the θ_j are odd ones. Set $\mathfrak{vect}(n|m) = \mathfrak{der}\ \mathbb{C}[x]$; it is called *the general vectorial Lie superalgebra.*

2) Special algebras. The *divergence* of the field $D = \sum\limits_i f_i \frac{\partial}{\partial u_i} + \sum\limits_j g_j \frac{\partial}{\partial \theta_j}$ is the function (in our case: a polynomial, or a series)

$$\mathrm{div} D = \sum_i \frac{\partial f_i}{\partial u_i} + \sum_j (-1)^{p(g_j)} \frac{\partial g_i}{\partial \theta_j}. \tag{0.2.4}$$

• The Lie superalgebra $\mathfrak{svect}(n|m) = \{D \in \mathfrak{vect}(n|m) : \mathrm{div} D = 0\}$ is called the *special* (or *divergence-free*) *vectorial superalgebra.*

It is clear that it is also possible to describe $\mathfrak{svect}(n|m)$ as $\{D \in \mathfrak{vect}(n|m) \mid L_D vol_x = 0\}$, where vol_x is the volume form with constant coefficients in coordinates x (see 0.6) and L_D the Lie derivative with respect to D.

• The Lie superalgebra $\mathfrak{svect}_\lambda(0|m) = \{D \in \mathfrak{vect}(0|m) \mid \mathrm{div}(1 + \lambda\theta_1 \cdot \dots \cdot \theta_m)D = 0\}$, where $p(\lambda) \equiv m \pmod 2$, — the deform of $\mathfrak{svect}(0|m)$ — is called the *deformed special* (or *divergence-free*) *vectorial superalgebra.* Clearly, $\mathfrak{svect}_\lambda(0|m) \cong \mathfrak{svect}_\mu(0|m)$ for $\lambda\mu \neq 0$. So we briefly denote these deforms by $\widetilde{\mathfrak{svect}}(0|m)$.

Observe that for odd m the parameter of deformation, λ, is odd.

Remark . As is customary in differential geometry, where meaningful notations prevail, we sometimes write $\mathfrak{vect}(x)$ or $\mathfrak{vect}(V)$ if $V = Span(x)$ and use similar notations for the subalgebras of $\mathfrak{vect}$ introduced below. Some algebraists sometimes abbreviate $\mathfrak{vect}(n)$ and $\mathfrak{svect}(n)$ to W_n (in honor of Witt) and S_n, respectively.

3) The algebras that preserve Pfaff equations and differential 2-forms.

• Set $u = (t, p_1, \dots, p_n, q_1, \dots, q_n)$; let

$$\tilde{\alpha}_1 = dt + \sum_{1\leq i\leq n} (p_i dq_i - q_i dp_i) \ + \sum_{1\leq j\leq m} \theta_j d\theta_j \quad \text{and} \quad \tilde{\omega}_0 = d\tilde{\alpha}_1 \ . \tag{0.2.5}$$

The form $\tilde{\alpha}_1$ is called *contact*, the form $\tilde{\omega}_0$ is called *symplectic*. Sometimes it is more convenient to redenote the θ's and set $\Theta = (\xi, \eta)$, or, perhaps, $\Theta = (\xi, \eta, \theta)$, where

$$\xi_j = \frac{1}{\sqrt{2}}(\theta_j - i\theta_{r+j}); \quad \eta_j = \frac{1}{\sqrt{2}}(\theta_j + i\theta_{r+j}) \ \text{ for } j \leq r = [m/2] \text{ (here } i^2 = -1), \quad \theta = \theta_{2r+1}$$

and in place of $\tilde{\omega}_0$ or $\tilde{\alpha}_1$ take α_1 and $\omega_0 = d\alpha_1$, respectively, where

$$\alpha_1 = dt + \sum_{1\leq i\leq n} (p_i dq_i - q_i dp_i) + \sum_{1\leq j\leq r} (\xi_j d\eta_j + \eta_j d\xi_j) \begin{cases} & \text{if } \ m = 2r \\ +\theta d\theta & \text{if } \ m = 2r+1. \end{cases} \tag{0.2.6}$$

The Lie superalgebra that preserves the *Pfaff equation* $\alpha_1 = 0$, i.e., the superalgebra

$$\mathfrak{k}(2n+1|m) = \{D \in \mathfrak{vect}(2n+1|m) \mid L_D\alpha_1 = f_D\alpha_1 \text{ for some } f_D \in \mathbb{C}[t, p, q, \theta]\},$$

is called the *contact superalgebra.* The Lie superalgebra

$$\mathfrak{po}(2n|m) = \{D \in \mathfrak{k}(2n+1|m) \mid L_D\alpha_1 = 0\}$$

is called the *Poisson* superalgebra. (A geometric interpretation of the Poisson superalgebra: it is the Lie superalgebra that preserves the connection with form α in the line bundle over a symplectic supermanifold with the symplectic form $d\alpha$, cf. [Kos].)

• Similarly, set $u = q = (q_1, \dots, q_n)$, let $\theta = (\xi_1, \dots, \xi_n; \tau)$ be odd. Set

$$\alpha_0 = d\tau + \sum_i (\xi_i dq_i + q_i d\xi_i), \qquad \omega_1 = d\alpha_0$$

and call these forms the *odd-contact* (actually it is even, though "odd") and *periplectic*, respectively. The periplectic form is odd.

The Lie superalgebra that preserves the Pfaff equation $\alpha_0 = 0$, i.e., the superalgebra

$$\mathfrak{m}(n) = \{D \in \mathfrak{vect}(n|n+1) \mid L_D\alpha_0 = f_D \cdot \alpha_0 \text{ for some } f_D \in \mathbb{C}[q, \xi, \tau]\}$$

is called the *odd-contact superalgebra.*[5]

The Lie superalgebra

$$\mathfrak{b}(n) = \{D \in \mathfrak{m}(n) \mid L_D\alpha_0 = 0\}$$

is called the *Buttin* superalgebra ([L4]). (A geometric interpretation of the Buttin superalgebra: it is the Lie superalgebra that preserves the connection with form α_1 in the line bundle of rank $\varepsilon = (0|1)$ over a *periplectic* supermanifold, i.e., over a supermanifold with the periplectic form $d\alpha_0$.)

The Lie superalgebras

$$\mathfrak{sm}(n) = \{D \in \mathfrak{m}(n) \mid \operatorname{div} D = 0\}\ ,\ \mathfrak{sb}(n) = \{D \in \mathfrak{b}(n) \mid \operatorname{div} D = 0\}$$

are called the *divergence-free* (or *special*) *odd-contact* and *special Buttin* superalgebras, respectively.

Remark . A relation with finite dimensional geometry is as follows. Clearly, $\ker \alpha_1 = \ker \tilde{\alpha}_1$. The restriction of $\tilde{\omega}_0$ to $\ker \alpha_1$ is the orthosymplectic form $B_{ev}(m|2n)$; the restriction of $\tilde{\omega}_0$ to $\ker \tilde{\alpha}_1$ is $B'_{ev}(m|2n)$. Similarly, the restriction of ω_1 to $\ker \alpha_0$ is $B_{odd}(n|n)$.

0.3. Generating functions. A laconic way to describe $\mathfrak{k}$, $\mathfrak{m}$ and their subalgebras is via generating functions.

• Odd form α_1. For any $f \in \mathbb{C}[t, p, q, \theta]$ set :

$$K_f = (2 - E)(f)\frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t}E, \tag{0.3.1}$$

where $E = \sum\limits_i y_i \frac{\partial}{\partial y_i}$ (here the y_i are all the coordinates except t) is the *Euler operator* (which counts the degree with respect to the y_i), and H_f is the hamiltonian field with Hamiltonian f that preserves $d\tilde{\alpha}_1$:

$$H_f = \sum_{i \leq n} \left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right) - (-1)^{p(f)} \left(\sum_{j \leq m} \frac{\partial f}{\partial \theta_j}\frac{\partial}{\partial \theta_j}\right). \tag{0.3.2}$$

The choice of the form α_1 instead of $\tilde{\alpha}_1$ only affects the shape of H_f that we give for $m = 2k + 1$:

$$H_f = \sum_{i \leq n} \left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right) - (-1)^{p(f)} \sum_{j \leq k} \left(\frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j}\frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}\right).$$

• Even form α_0. For any $f \in \mathbb{C}[q, \xi, \tau]$ set:

$$M_f = (2 - E)(f)\frac{\partial}{\partial \tau} - Le_f - (-1)^{p(f)}\frac{\partial f}{\partial \tau}E, \tag{0.3.3}$$

[5] To call $\mathfrak{m}$ "odd contact superalgebra" is misleading since the even part of $\mathfrak{m}$ is nonzero. A really good name is desired.

where $E=\sum\limits_i y_i\frac{\partial}{\partial y_i}$ (here the y_i are all the coordinates except τ), and

$$Le_f=\sum_{i\leq n}(\frac{\partial f}{\partial q_i}\,\frac{\partial}{\partial \xi_i}+(-1)^{p(f)}\frac{\partial f}{\partial \xi_i}\,\frac{\partial}{\partial q_i}). \tag{0.3.4}$$

Since

$$\begin{aligned} L_{K_f}(\alpha_1)&=2\tfrac{\partial f}{\partial t}\alpha_1=K_1(f)\alpha_1,\\ L_{M_f}(\alpha_0)&=-(-1)^{p(f)}2\tfrac{\partial f}{\partial \tau}\alpha_0=-(-1)^{p(f)}M_1(f)\alpha_0,\end{aligned} \tag{0.3.5}$$

it follows that $K_f\in\mathfrak{k}(2n+1|m)$ and $M_f\in\mathfrak{m}(n)$. Observe that

$$p(Le_f)=p(M_f)=p(f)+\bar 1.$$

• To the (super)commutators $[K_f,K_g]$ or $[M_f,M_g]$ there correspond *contact brackets* of the generating functions:

$$[K_f,K_g]=K_{\{f,\,g\}_{k.b.}};\qquad [M_f,M_g]=M_{\{f,\,g\}_{m.b.}}$$

The explicit formulas for the contact brackets are as follows. Let us first define the brackets on functions that do not depend on t (resp. τ).

The *Poisson bracket* $\{\cdot,\cdot\}_{P.b.}$ (in the realization with the form $\widetilde{\omega}_0$) is given by the formula

$$\{f,g\}_{P.b.}=\sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\,\frac{\partial g}{\partial q_i}-\,\frac{\partial f}{\partial q_i}\,\frac{\partial g}{\partial p_i}\right)-(-1)^{p(f)}\sum_{j\leq m}\,\frac{\partial f}{\partial \theta_j}\,\frac{\partial g}{\partial \theta_j}\ \text{for any } f,g\in\mathbb{C}[p,q,\theta] \tag{0.3.6}$$

and in the realization with the form ω_0 for $m=2k+1$ it is given by the formula

$$\begin{aligned}\{f,g\}_{P.b.}=&\sum_{i\leq n}\left(\tfrac{\partial f}{\partial p_i}\,\tfrac{\partial g}{\partial q_i}-\,\tfrac{\partial f}{\partial q_i}\,\tfrac{\partial g}{\partial p_i}\right)-\\ &(-1)^{p(f)}\left[\sum_{j\leq m}(\tfrac{\partial f}{\partial \xi_j}\,\tfrac{\partial g}{\partial \eta_j}+\tfrac{\partial f}{\partial \eta_j}\,\tfrac{\partial g}{\partial \xi_j})+\tfrac{\partial f}{\partial \theta}\,\tfrac{\partial g}{\partial \theta}\right]\ \text{for } f,g\in\mathbb{C}[p,q,\xi,\eta,\theta].\end{aligned}$$

The *Buttin bracket* $\{\cdot,\cdot\}_{B.b.}$ is given by the formula

$$\{f,g\}_{B.b.}=\sum_{i\leq n}\left(\frac{\partial f}{\partial q_i}\,\frac{\partial g}{\partial \xi_i}+(-1)^{p(f)}\,\frac{\partial f}{\partial \xi_i}\,\frac{\partial g}{\partial q_i}\right)\ \text{for any } f,g\in\mathbb{C}[q,\xi]. \tag{0.3.7}$$

Remark . The what we call here "Buttin bracket" was discovered in pre-super era by Schouten; Buttin was the first to prove that this bracket establishes a Lie superalgebra structure. The interpretations of the Buttin superalgebra similar to that of the Poisson algebra and of the elements of $\mathfrak{le}$ as analogs of Hamiltonian vector fields was given in [L1]. The Buttin bracket and "odd mechanics" introduced in [L1] was rediscovered by Batalin with Vilkovisky (and, even earlier, by Zinn-Justin, but his papers went mainly unnoticed, see [FLSf]); it gained a great deal of currency under the name *antibracket*, cf. [GPS]. The *Schouten bracket* was originally defined on the superspace of polyvector fields on a manifold, i.e., on the superspace of sections of the exterior algebra (over the algebra $\mathcal{F}$ of functions) of the tangent bundle, $\Gamma(\Lambda^{\boldsymbol{\cdot}}(T(M)))\cong\Lambda^{\boldsymbol{\cdot}}_{\mathcal{F}}(Vect(M))$. The explicit formula of the Schouten bracket (in which the hatted slot should be ignored, as usual) is

$$\begin{gathered}[X_1\wedge\dots\wedge\dots\wedge X_k,Y_1\wedge\dots\wedge Y_l]=\\ \textstyle\sum_{i,j}(-1)^{i+j}[X_i,Y_j]\wedge X_1\wedge\dots\wedge\hat X_i\wedge\dots\wedge X_k\wedge Y_1\wedge\dots\wedge\hat Y_j\wedge\dots\wedge Y_l.\end{gathered} \tag{$*$}$$

With the help of Sign Rule we easily superize formula $(*)$, i.e., replace M with a supermanifold $\mathcal{M}$. Let x and ξ be the even and odd coordinates on $\mathcal{M}$. By setting

$$\theta_i=\Pi(\frac{\partial}{\partial x_i})=\check x_i,\quad q_j=\Pi(\frac{\partial}{\partial \xi_j})=\check\xi_j \tag{$**$}$$

we get an identification of the Schouten bracket of polyvector fields on $\mathcal{M}$ with the Buttin bracket of functions on the supermanifold $\check{\mathcal{M}}$ the coordinates on it being x, ξ and $\check{x}$, $\check{\xi}$, where the transformation rule of the checked variables is induced by that of unchecked ones via $(**)$.

In terms of the Poisson and Buttin brackets, respectively, the contact brackets are

$$\{f, g\}_{k.b.} = (2-E)(f)\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}(2-E)(g) - \{f, g\}_{P.b.} \tag{0.3.8}$$

and

$$\{f, g\}_{m.b.} = (2-E)(f)\frac{\partial g}{\partial \tau} + (-1)^{p(f)}\frac{\partial f}{\partial \tau}(2-E)(g) - \{f, g\}_{B.b.}. \tag{0.3.9}$$

The Lie superalgebras of *Hamiltonian fields* (or *Hamiltonian superalgebra*) and its special subalgebra (defined only if $n = 0$) are

$$\mathfrak{h}(2n|m) = \{D \in \mathfrak{vect}(2n|m) \mid L_D\omega_0 = 0\} \text{ and } \mathfrak{h}^o(m) = \{H_f \in \mathfrak{h}(0|m) \mid \int f vol_\theta = 0\}.$$

The "odd" analogues of the Lie superalgebra of Hamiltonian fields are the Lie superalgebra of vector fields Le_f introduced in [L1] and its special subalgebra:

$$\mathfrak{le}(n) = \{D \in \mathfrak{vect}(n|n) \mid L_D\omega_1 = 0\} \text{ and } \mathfrak{sle}(n) = \{D \in \mathfrak{le}(n) \mid \mathrm{div} D = 0\}.$$

It is not difficult to prove the following isomorphisms (as superspaces):

$$\begin{array}{rclrcl} \mathfrak{k}(2n+1|m) & \cong & Span(K_f \mid f \in \mathbb{C}[t, p, q, \xi]); & \mathfrak{le}(n) & \cong & Span(Le_f \mid f \in \mathbb{C}[q, \xi]); \\ \mathfrak{m}(n) & \cong & Span(M_f \mid f \in \mathbb{C}[\tau, q, \xi]); & \mathfrak{h}(2n|m) & \cong & Span(H_f \mid f \in \mathbb{C}[p, q, \xi]). \end{array}$$

Set $\mathfrak{po}^o(m) = \{K_f \in \mathfrak{po}(0|m) \mid \int f vol_\xi = 0\}$; then $\mathfrak{h}^o(m) = \mathfrak{po}^o(m)/\mathbb{C}\cdot K_1$.

0.3.1. Deduction of the explicit form of the odd-contact bracket. In what follows F_x is a shorthand for $\frac{\partial}{\partial x}F$.

On the one hand, since $M_F\alpha_0 = (-1)^{p(F)+1}2F_\tau\alpha_0$, it follows that

$$\begin{aligned} &(M_FM_G - (-1)^{(p(F)+1)(p(G)+1)}M_GM_F)\alpha_0 = \\ &\{(-1)^{p(G)+1}2[(\xi F_\tau + F_q)G_{\xi\tau} + (-1)^{p(F)}(F_\xi - qF_\tau)G_{q\tau} + (-1)^{p(G)(p(F)+1)}2G_\tau F_\tau] \\ &-(-1)^{(p(F)+1)p(G)}2[(\xi G_\tau + G_q)F_{\xi\tau} + (-1)^{p(G)}(G_\xi - qG_\tau)F_{q\tau} + (-1)^{p(F)(p(G)+1)}2F_\tau G_\tau]\}\alpha_0. \end{aligned} \tag{0.3.10}$$

On the other hand, we have

$$((2-E)(F)G_\tau)_\tau = 2F_\tau G_\tau + \xi F_{\tau\xi}G_\tau - qF_{\tau q}G_\tau; \tag{0.3.11}$$

$$(F_\tau(2-E)(G))_\tau = (-1)^{p(F)+1}(2F_\tau G_\tau + F_\tau\xi G_{\tau\xi} - F_\tau qG_{\tau q}); \tag{0.3.12}$$

$$(F_qG_\xi + (-1)^{p(F)}F_\xi G_q)_\tau = F_{\tau q}G_q + (-1)^{p(F)}F_qG_{\tau\xi} - F_\xi G_{\tau q}. \tag{0.3.13}$$

Clearly,

$$(-1)^{p(F)p(G)}(0.3.11) + (-1)^{p(G)}(0.3.12) + (-1)^{p(F)+p(G)}(0.3.13) = \{\dots\},$$

where $\{\dots\}$ is the coefficient of α_0 in (0.3.10). Therefore,

$$\{F, G\}_{m.b.} = (-1)^{p(F)+p(G)}[(2-E)(F)G_\tau + (-1)^{p(F)}F_\tau(2-E)(G) + \{F, G\}_{\mathrm{B.b.}}].$$

The arguments for the even contact bracket are similar with α_0 replaced with α_1 or $\tilde{\alpha}_1$ and fields M_* with K_*, respectively.

0.4. Divergence-free subalgebras. Since, as is easy to calculate,

$$\operatorname{div} K_f = (2n+2-m)K_1(f),$$

it follows that the divergence-free subalgebra of the contact Lie superalgebra either coincides with it (for $m = 2n+2$) or is the Poisson superalgebra. For the "odd" contact series the situation is more interesting: the divergence free subalgebra is simple and new (as compared with the above list).

Since

$$\operatorname{div} M_f = (-1)^{p(f)} 2\left((1-E)\frac{\partial f}{\partial \tau} - \sum_{i\leq n}\frac{\partial^2 f}{\partial q_i \partial \xi_i}\right), \tag{0.4.1}$$

it follows that the divergence-free subalgebra of the odd-contact superalgebra is

$$\mathfrak{sm}(n) = Span\left(M_f \in \mathfrak{m}(n) \mid (1-E)\frac{\partial f}{\partial \tau} = \sum_{i\leq n}\frac{\partial^2 f}{\partial q_i \partial \xi_i}\right).$$

In particular,

$$\operatorname{div} Le_f = (-1)^{p(f)} 2\sum_{i\leq n}\frac{\partial^2 f}{\partial q_i \partial \xi_i}. \tag{0.4.2}$$

The odd analog of the Laplacian, namely, the operator

$$\Delta = \sum_{i\leq n}\frac{\partial^2}{\partial q_i \partial \xi_i} \tag{0.4.3}$$

on a periplectic supermanifold appeared in physics under the name of *BRST operator*, cf. [GPS], or Batalin-Vilkovysky operator. Observe that it is just the Fourier transform (with respect to the ghost indeterminates (odd ones, if considered on a manifold) of the exterior oerator.

The divergence-free vector fields from $\mathfrak{sle}(n)$ are generated by *harmonic* functions, i.e., such that $\Delta(f) = 0$.

Lie superalgebras $\mathfrak{sle}(n)$, $\mathfrak{sb}(n)$ and $\mathfrak{svect}(1|n)$ have traceless ideals $\mathfrak{sle}^\circ(n)$, $\mathfrak{sb}^\circ(n)$ and $\mathfrak{svect}^\circ(n)$ of codimension 1 defined from the exact sequences

$$\begin{gathered}
0 \longrightarrow \mathfrak{sle}^\circ(n) \longrightarrow \mathfrak{sle}(n) \longrightarrow \mathbb{C}\cdot Le_{\xi_1\dots\xi_n} \longrightarrow 0,\\
0 \longrightarrow \mathfrak{sb}^\circ(n) \longrightarrow \mathfrak{sb}(n) \longrightarrow \mathbb{C}\cdot M_{\xi_1\dots\xi_n} \longrightarrow 0,\\
0 \longrightarrow \mathfrak{svect}^\circ(n) \longrightarrow \mathfrak{svect}(1|n) \longrightarrow \mathbb{C}\cdot \xi_1\dots\xi_n\frac{\partial}{\partial t} \longrightarrow 0.
\end{gathered}$$

0.5. The Cartan prolongs. We will repeatedly use the Cartan prolongation. So let us recall the definition and generalize it somewhat. Let $\mathfrak{g}$ be a Lie algebra, V a $\mathfrak{g}$-module, S^i the operator of the i-th symmetric power. Set $\mathfrak{g}_{-1} = V$, $\mathfrak{g}_0 = \mathfrak{g}$ and for $k > 0$ define the k-th *Cartan prolong* (the result of Cartan's *prolongation*) of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ as

$$\begin{aligned}
\mathfrak{g}_k = \{X \in \operatorname{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{k-1}) \mid X(v_0)(v_1, v_2, \dots v_k) = X(v_1)(v_0, v_2, \dots, v_k)\\
\text{for any } \ v_0, v_1, \dots v_k \in \mathfrak{g}_{-1}\}.
\end{aligned}$$

Equivalently, let

$$i : S^{k+1}(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1} \longrightarrow S^k(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1} \tag{0.5.1}$$

be the natural embedding and

$$j : S^k(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_0 \longrightarrow S^k(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1} \tag{0.5.2}$$

the natural map. Then $\mathfrak{g}_k = i(S^{k+1}(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1}) \cap j(S^k(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_0)$.

The *Cartan prolong* of the pair $(V, \mathfrak{g})$ is $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathop{\oplus}\limits_{k\geq -1} \mathfrak{g}_k$.

(In what follows $\cdot$ in superscript denotes, as is now customary, the collection of all degrees, while $*$ is reserved for dualization; in the subscripts we retain the old-fashioned $*$ instead of $\cdot$ to avoid too close a contact with the punctuation marks.)

Suppose that the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is *faithful*. Then, clearly,

$$(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \subset \mathfrak{vect}(n) = \mathfrak{der}\ \mathbb{C}[x_1, ..., x_n], \quad \text{where} \quad n = \dim\ \mathfrak{g}_{-1} \quad \text{and}$$
$$\mathfrak{g}_i = \{D \in \mathfrak{vect}(n) \mid \deg D = i, [D, X] \in \mathfrak{g}_{i-1} \text{ for any } X \in \mathfrak{g}_{-1}\}.$$

It is subject to an easy verification that the Lie algebra structure on $\mathfrak{vect}(n)$ induces same on $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$.

Of the four simple vectorial Lie algebras, three are Cartan prolongs:

$$\mathfrak{vect}(n) = (\mathrm{id}, \mathfrak{gl}(n))_*, \quad \mathfrak{svect}(n) = (\mathrm{id}, \mathfrak{sl}(n))_*, \quad \mathfrak{h}(2n) = (\mathrm{id}, \mathfrak{sp}(n))_*.$$

The fourth one — $\mathfrak{k}(2n+1)$ — is the result of a trifle more general construction described as follows.

0.5.1. A generalization of the Cartan prolong. Let $\mathfrak{g}_- = \mathop{\oplus}\limits_{-d\leq i\leq -1} \mathfrak{g}_i$ be a nilpotent $\mathbb{Z}$-graded Lie algebra and $\mathfrak{g}_0 \subset \mathfrak{der}_0\mathfrak{g}$ a Lie subalgebra of the $\mathbb{Z}$-grading-preserving derivations. Let

$$i : S^{k+1}(\mathfrak{g}_-)^* \otimes \mathfrak{g}_- \longrightarrow S^k(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}_- \tag{0.5.1$'$}$$

and

$$j : S^k(\mathfrak{g}_-)^* \otimes \mathfrak{g}_0 \longrightarrow S^k(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}_- \tag{0.5.2$'$}$$

be the natural maps similar to (0.5.1) and (0.5.2), respectively. For $k > 0$ define the k-th prolong of the pair $(\mathfrak{g}_-, \mathfrak{g}_0)$ to be:

$$\mathfrak{g}_k = (j(S^\cdot(\mathfrak{g}_-)^* \otimes \mathfrak{g}_0) \cap i(S^\cdot(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-))_k\,,$$

where the subscript k in the right hand side singles out the component of degree k.

Set $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \mathop{\oplus}\limits_{i\geq -d} \mathfrak{g}_i$; then, as is easy to verify, $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ is a Lie algebra.

What is the Lie algebra of contact vector fields in these terms? Denote by $\mathfrak{hei}(2n)$ the Heisenberg Lie algebra: its space is $W \oplus \mathbb{C} \cdot z$, where W is a $2n$-dimensional space endowed with a nondegenerate skew-symmetric bilinear form B and the bracket in $\mathfrak{hei}(2n)$ is given by the following relations:

$$z \text{ is in the center and } [v, w] = B(v, w) \cdot z \text{ for any } v, w \in W. \tag{0.5.3}$$

Clearly, $\mathfrak{k}(2n+1) \cong (\mathfrak{hei}(2n), \mathfrak{csp}(2n))_*$.

0.5.2. Lie superalgebras of vector fields as Cartan's prolongs. The superization of the constructions from sec. 0.5 are straightforward: via Sign Rule. We thus get infinite dimensional Lie superalgebras

$$\mathfrak{vect}(m|n) = (\mathrm{id}, \mathfrak{gl}(m|n))_*;\ \mathfrak{svect}(m|n) = (\mathrm{id}, \mathfrak{sl}(m|n))_*;$$
$$\mathfrak{h}(2m|n) = (\mathrm{id}, \mathfrak{osp}^{sk}(m|2n))_*;$$
$$\mathfrak{le}(n) = (\mathrm{id}, \mathfrak{pe}^{sk}(n))_*;\ \mathfrak{sle}(n) = (\mathrm{id}, \mathfrak{spe}^{sk}(n))_*.$$

Remark. Observe that the Cartan prolongs $(\mathrm{id}, \mathfrak{osp}^{sy}(m|2n))_*$ and $(\mathrm{id}, \mathfrak{pe}^{sy}(n))_*$ are finite dimensional.

The generalization of Cartan's prolongations described in sec. 0.5.1 was first defined in [ALSh] (and repeatedly used later, e.g., in [CK1]). Observe that after superization it has **two** analogs associated with the contact series $\mathfrak{k}$ and $\mathfrak{m}$, respectively.

• Define the Lie superalgebra $\mathfrak{hei}(2n|m)$ on the direct sum of a $(2n,m)$-dimensional superspace W endowed with a nondegenerate skew-symmetric bilinear form B and a $(1,0)$-dimensional space spanned by z by (0.5.3).

Clearly, we have

$$\mathfrak{k}(2n+1|m) = (\mathfrak{hei}(2n|m), \mathfrak{cosp}^{sk}(m|2n))_*$$

and, given $\mathfrak{hei}(2n|m)$ and a subalgebra $\mathfrak{g}$ of $\mathfrak{cosp}^{sk}(m|2n)$, we call $(\mathfrak{hei}(2n|m), \mathfrak{g})_*$ the *k-prolong* of $(W, \mathfrak{g})$, where W is the identity $\mathfrak{osp}^{sk}(m|2n)$-module.

• The "odd" analog of $\mathfrak{k}$ is associated with the following "odd" analog of $\mathfrak{hei}(2n|m)$. Denote by $\mathfrak{ab}(n)$ the *antibracket* Lie superalgebra: its space is $W \oplus \mathbb{C}\cdot z$, where W is an $n|n$-dimensional superspace endowed with a nondegenerate skew-symmetric odd bilinear form B; the bracket in $\mathfrak{ab}(n)$ is given by the following relations:

$$z \text{ is odd and lies in the center; } [v,w] = B(v,w)\cdot z \text{ for any } v, w \in W. \tag{0.5.4}$$

Clearly,

$$\mathfrak{m}(n) = (\mathfrak{ab}(n), \mathfrak{cpe}^{sk}(n))_*$$

and, given $\mathfrak{ab}(n)$ and a subalgebra $\mathfrak{g}$ of $\mathfrak{cpe}^{sk}(n)$, we call $(\mathfrak{ab}(n), \mathfrak{g})_*$ the *m-prolong* of $(W, \mathfrak{g})$, where W is the identity $\mathfrak{pe}^{sk}(n)$-module.

Generally, given a nondegenerate form B on a superspace W and a superalgebra $\mathfrak{g}$ that preserves B, we refer to the above generalized prolongations as to *mk-prolongation* of the pair $(W, \mathfrak{g})$.

0.5.3. A partial Cartan prolong or the prolong of a positive part. Take a $\mathfrak{g}_0$-submodule $\mathfrak{h}_1$ in $\mathfrak{g}_1$ such that $[\mathfrak{g}_{-1}, \mathfrak{h}_1] = \mathfrak{g}_0$. If such $\mathfrak{h}_1$ exists (usually, $[\mathfrak{g}_{-1}, \mathfrak{h}_1] \subset \mathfrak{g}_0$), define the 2nd prolongation of $(\underset{i\leq 0}{\oplus} \mathfrak{g}_i, \mathfrak{h}_1)$ to be

$$\mathfrak{h}_2 = \{D \in \mathfrak{g}_2 \mid [D, \mathfrak{g}_{-1}] \in \mathfrak{h}_1\}.$$

The terms $\mathfrak{h}_i$, $i > 2$, are similarly defined. Set $\mathfrak{h}_i = \mathfrak{g}_i$ for $i < 0$ and $\mathfrak{h}_* = \oplus \mathfrak{h}_i$.

Examples: $\mathfrak{vect}(1|n;n)$ is a subalgebra of $\mathfrak{k}(1|2n;n)$. The former is obtained as Cartan's prolong of the same nonpositive part as $\mathfrak{k}(1|2n;n)$ and a submodule of $\mathfrak{k}(1|2n;n)_1$, cf. Table I.2.2. The simple exceptional superalgebra $\mathfrak{kas}$ introduced in I.2.3 ([Sh5, Sh14]) is another example.

Remark . In nonsuper setting generalized Cartan prolongation was first introduced by Tanaka [T], see a review [Y]. Different from ours emphasis of these papers delayed our recognition of similarity of constructions. The partial prolong seems to be first discovered in [ALSh].

0.6. The modules of tensor fields. To advance further, we have to recall the definition of the modules of tensor fields over $\mathfrak{vect}(m|n)$ and its subalgebras, see [BL2], [L2]. For any other $\mathbb{Z}$-graded vectorial Lie superalgebra the construction is identical.

Let $\mathfrak{g} = \mathfrak{vect}(m|n)$ and $\mathfrak{g}_{\geq} = \underset{i\geq 0}{\oplus} \mathfrak{g}_i$. Clearly, $\mathfrak{vect}_0(m|n) \cong \mathfrak{gl}(m|n)$. Let V be the $\mathfrak{gl}(m|n)$-module with the *lowest* weight $\lambda = \mathrm{lwt}(V)$. Make V into a $\mathfrak{g}_{\geq}$-module setting $\mathfrak{g}_+ \cdot V = 0$ for $\mathfrak{g}_+ = \underset{i>0}{\oplus} \mathfrak{g}_i$. Let us realize $\mathfrak{g}$ by vector fields on the $m|n$-dimensional linear supermanifold $\mathcal{C}^{m|n}$ with coordinates $x = (u, \xi)$. The superspace $T(V) = \mathrm{Hom}_{U(\mathfrak{g}_{\geq})}(U(\mathfrak{g}), V)$ is isomorphic, due to the Poincaré–Birkhoff–Witt theorem, to $\mathbb{C}[[x]] \otimes V$. Its elements have a natural interpretation as formal *tensor fields of type* V. When $\lambda = (a, \dots, a)$ we will simply write $T(\vec{a})$ instead of $T(\lambda)$. We will usually consider $\mathfrak{g}$-modules induced from irreducible $\mathfrak{g}_0$-modules.

Examples: $\mathfrak{vect}(m|n)$ as $\mathfrak{vect}(m|n)$- and $\mathfrak{svect}(m|n)$-modules is $T(\mathrm{id})$. More examples: $T(\vec{0})$ is the superspace of functions; $Vol(m|n) = T(1,\dots,1;-1,\dots,-1)$ (the semicolon separates the first m ("even") coordinates of the weight with respect to the matrix units E_{ii} of $\mathfrak{gl}(m|n)$) is the superspace of *densities* or *volume forms*. We denote the generator of $Vol(m|n)$ corresponding to the ordered set of coordinates x by $vol(x)$. The space of λ-densities is $Vol^\lambda(m|n) = T(\lambda,\dots,\lambda;-\lambda,\dots,-\lambda)$. In particular, $Vol^\lambda(m|0) = T(\vec{\lambda})$ but $Vol^\lambda(0|n) = T(\overrightarrow{-\lambda})$. We set:

$$Vol_0(0|m) = \{v \in Vol \mid \int v = 0\} \text{ and } T_0(\vec{0}) = \Lambda(m)/\mathbb{C}\cdot 1.$$

If the generator vol of Vol is fixed, then $Vol \cong T(\vec{0})$, as $\mathfrak{svect}(m|n)$-modules. We set $T_0^0(\vec{0})$ to denote the $\mathfrak{svect}(m|n)$-module $Vol_0(0|m)/\mathbb{C}\ vol(\xi)$.

Remark. To view the volume element as "$d^m u d^n \xi$" is totally wrong: the superdeterminant can never appear as a factor under the changes of variables. We can try to use the usual notations of differentials provided *all* the differentials anticommute. Then the linear transformations that do not intermix the even u's with the odd ξ's multiply the volume element $vol(x)$, viewed as the fraction $\frac{du_1\cdot\ldots\cdot du_m}{d\xi_1\cdot\ldots\cdot d\xi_n}$, by the Berezinian of the transformation. But how could we justify this? Let $X = (x,\xi)$. If we consider the usual, exterior, differential forms, then the dX_i's super anti-commute, hence, the $d\xi_i$ commute; whereas if we consider the *symmetric* product of the differentials, as in the metrics, then the dX_i's supercommute, hence, the dx_i commute. However, the $\frac{\partial}{\partial\xi_i}$ anticommute and, from transformations' point of view, $\frac{\partial}{\partial\xi_i} = \frac{1}{d\xi_i}$. The notation, $du_1\cdot\ldots\cdot du_m\cdot\frac{\partial}{\partial\xi_1}\cdot\ldots\cdot\frac{\partial}{\partial\xi_n}$, is, nevertheless, still wrong: almost any transformation $A:(u,\xi)\mapsto(v,\eta)$ sends $du_1\cdot\ldots\cdot du_m\cdot\frac{\partial}{\partial\xi_1}\cdot\ldots\cdot\frac{\partial}{\partial\xi_n}$ to the correct element, $\mathrm{ber}(A)(du^m\cdot\frac{\partial}{\partial\xi_1}\cdot\ldots\cdot\frac{\partial}{\partial\xi_n})$, plus extra terms. Indeed, the fraction $du_1\cdot\ldots\cdot du_m\cdot\frac{\partial}{\partial\xi_1}\cdot\ldots\cdot\frac{\partial}{\partial\xi_n}$ is the highest weight vector of an *indecomposable* $\mathfrak{gl}(m|n)$-module and $vol(x)$ is the notation of the image of this vector in the 1-dimensional quotient module modulo the invariant submodule that consists precisely of all the extra terms.

0.7. Deformations of the Buttin superalgebra. (After [ALSh].) As is clear from the definition of the Buttin bracket, there is a regrading (namely, $\mathfrak{b}(n;n)$ given by $\deg\xi_i = 0, \deg q_i = 1$ for all i) under which $\mathfrak{b}(n)$, initially of depth 2, takes the form $\mathfrak{g} = \mathop{\oplus}\limits_{i\geq -1}\mathfrak{g}_i$ with $\mathfrak{g}_0 = \mathfrak{vect}(0|n)$ and $\mathfrak{g}_{-1}\cong\Pi(\mathbb{C}[\xi])$. Replace now the $\mathfrak{vect}(0|n)$-module $\mathfrak{g}_{-1}$ of functions (with inverted parity) with the module of λ-densities, i.e., set $\mathfrak{g}_{-1}\cong\Pi(Vol(0|n)^\lambda)$, where

$$L_D(vol_\xi)^\lambda = \lambda\mathrm{div}D\cdot vol_\xi^\lambda \text{ and } p(vol_\xi)^\lambda = \bar{1}.$$

0.7.1. Define $\mathfrak{b}_\lambda(n;n)$ as the Cartan's prolong

$$(\mathfrak{g}_{-1},\mathfrak{g}_0)_* = (\Pi(Vol(0|n)^\lambda), \mathfrak{vect}(0|n))_*.$$

Clearly, it is a deform of $\mathfrak{b}(n;n)$. The collection of these $\mathfrak{b}_\lambda(n;n)$ for all λ's is called the *main deformation*. (Though main, this deformation is not the quantization of the Buttin bracket, cf. [Ko1], [L4], [LSh3].)

The deform $\mathfrak{b}_\lambda(n)$ of $\mathfrak{b}(n)$ is a regrading of $\mathfrak{b}_\lambda(n;n)$ described as follows. Set

$$\mathfrak{b}_\lambda(n) = \{M_f\in\mathfrak{m}(n) \mid a\ \mathrm{div}M_f = (-1)^{p(f)}2(a-bn)\frac{\partial f}{\partial\tau}\}. \tag{0.7}$$

For future use, we will denote the operator that singles out $\mathfrak{b}_\lambda(n)$ in $\mathfrak{m}(n)$ as follows:

$$\mathrm{div}_\lambda = (bn - aE)\frac{\partial}{\partial\tau} - a\Delta. \tag{0.8}$$

Taking into account the explicit form (0.4.1) of the divergence of M_f we get

$$\mathfrak{b}_\lambda(n) = \{M_f\in\mathfrak{m}(n) \mid (bn-aE)\frac{\partial f}{\partial\tau} = a\Delta f\} = \{D\in\mathfrak{vect}(n|n+1) \mid L_D(vol^a_{q,\xi,\tau}\alpha_0^{a-bn}) = 0\}.$$

It is subject to a direct check that $\mathfrak{b}_{a,b}(n)$ is isomorphic to $\mathfrak{b}_\lambda(n)$ for $\lambda = \frac{2a}{n(a-b)}$. This isomorphism shows that λ actually runs over $\mathbb{C}P^1$, not $\mathbb{C}$.

Observe that $\mathfrak{b}_{nb,b}(n) \cong \mathfrak{sm}(n)$. Observe also that $\mathfrak{b}_{a,-a}(2;2) \cong \mathfrak{h}_{1/2}(2|2)$, see (OI).

As follows from the description of $\mathfrak{vect}(m|n)$-modules ([BL2]) and the criteria for simplicity of $\mathbb{Z}$-graded Lie superalgebras ([K2]), the Lie superalgebras $\mathfrak{b}_\lambda(n)$ are simple for $n > 1$ and $\lambda \neq 0, 1, \infty$. It is also clear that the $\mathfrak{b}_\lambda(n)$ are nonisomorphic for distinct λ's.

The Lie superalgebra $\mathfrak{b}(n) = \mathfrak{b}_0(n)$ is not simple: it has an ε-dimensional, i.e., $(0|1)$-dimensional, center. At $\lambda = 1$ and ∞ the Lie superalgebras $\mathfrak{b}_\lambda(n)$ are not simple either: they has an ideal of codimension ε^n and ε^{n+1}, respectively. The corresponding exact sequences are

$$0 \longrightarrow \mathbb{C}M_1 \longrightarrow \mathfrak{b}(n) \longrightarrow \mathfrak{le}(n) \longrightarrow 0,$$

$$0 \longrightarrow \mathfrak{b}_1{}^\circ(n) \longrightarrow \mathfrak{b}_1(n) \longrightarrow \mathbb{C}\cdot M_{\xi_1\dots\xi_n} \longrightarrow 0,$$

$$0 \longrightarrow \mathfrak{b}_\infty{}^\circ(n) \longrightarrow \mathfrak{b}_\infty(n) \longrightarrow \mathbb{C}\cdot M_{\tau\xi_1\dots\xi_n} \longrightarrow 0.$$

0.8. The exceptional simple vectorial Lie superalgebras as Cartan's prolongs. For the description of the exceptions as Cartan prolongs see Introduction. For a more detailed description of the "standard" realizations of the exceptions see [Sh14], [ShP].

0.9. The structures preserved. It is always desirable to find the structure preserved by the Lie superalgebra under the study. To see what do the vectorial superalgebras in nonstandard realizations preserve, we have to say, first of all, what is the structure that $\mathfrak{g}_0 = \mathfrak{vect}(0|n)$ preserves on $\mathfrak{g}_{-1} = \Lambda(n)$.

Let $\mathfrak{g} = \mathfrak{vect}(0|n)$, set further

$$W = \Lambda(n),\ V = \Lambda(n)/\mathbb{C}\cdot 1,\ V_0 = \{\varphi \in \Lambda(n) \mid \varphi(0) = 0\}.$$

The projection $p : W \longrightarrow V$ establishes a natural isomorphism between V and V_0. Let $i : V_0 \longrightarrow W$ be the "inverse" embedding.

Denote by $\mathrm{mult} : W \otimes W \longrightarrow W$ the multiplication on W given by the tensor mult of valency $(2,1)$. Since V_0 is an ideal in the associative supercommutative superalgebra W, the image $\mathrm{mult}|_{V_0\otimes V_0}$ is contained in V_0. Denote by mult° the tensor which coincides with mult on $V_0 \otimes V_0$ and vanishes on $\mathbb{C}\cdot 1 \otimes W \oplus W \otimes \mathbb{C}\cdot 1$. By means of the projection p and the embedding i we can $\mathfrak{g}$-invariantly transport mult° to V. The tensor obtained will be also denoted by mult°.

For any monomial $\varphi \in W$ denote by φ^* the dual functional (let $B(W)$ denote the monomial basis). Then

$$\mathrm{mult} = \mathrm{mult}^\circ + \sum_{\varphi\in B(W)} 1^*\otimes\varphi^*\otimes\varphi = \mathrm{mult}^\circ + 1^*\otimes \sum_{\varphi\in B(W)} \varphi^*\otimes\varphi.$$

By definition of $\mathfrak{g}$, it preserves mult, i.e., $L_D(\mathrm{mult}) = 0$ for any $D \in \mathfrak{g}$. Hence,

$$L_D(\mathrm{mult}^\circ) = -L_D(1^*)\otimes \sum_{\varphi\in B(W)} \varphi^*\otimes\varphi - (1^*)\otimes L_D\left(\sum_{\varphi\in B(W)} \varphi^*\otimes\varphi\right).$$

Under the restriction onto $V_0 \otimes V_0$ the second summand vanishes. Observe that

$$\sum_{\varphi\in B(W)} \varphi^*\otimes\varphi = \mathrm{id}|_W.$$

Thus,

$$L_D(\mathrm{mult}^\circ|_{V_0\otimes V_0}) = -L_D(1^*)\otimes \mathrm{id}|_{V_0}.$$

The lift of this identity operator to W reads as follows:

$$L_D(\mathrm{mult}) = \alpha(D) \otimes \mathrm{id}|_W \text{ for a 1-form } \alpha \text{ on } W.$$

Thus, all the structures preserved by $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ are clear, except for those preserved by several of the exceptional algebras. Namely, these structures are:

(1) the tensor products $B \otimes \mathrm{mult}$ of a bilinear or a volume form B preserved (perhaps, conformally, up to multiplication by a scalar) in the fiber of a vector bundle over a $0|r$-dimensional supermanifold on which the structure governed by mult is preserved,

(2) mult°, or mult twisted by divergence with factor λ. Observe that the volume element B may be not just $vol(\xi)$ but $(1 + \alpha\xi_1 \dots \xi_n)vol(\xi)$ as well.

The structures of another type, namely, certain pseudodifferential forms preserved by $\mathfrak{b}_\lambda(n)$, are already described.

0.9.1. On Shmelev's interpretations of $\mathfrak{h}_\lambda(2|2) \cong \mathfrak{b}_\lambda(2;2)$. When $n = 2$ the algebras $\mathfrak{b}_\lambda(n)$ seemed to have one more realization. G. Shmelev interpreted $\mathfrak{h}_\lambda(2|2) \cong \mathfrak{b}_\lambda(2;2)$ as a deformation $\mathfrak{h}_\lambda(2|2)$ of the Lie superalgebra $\mathfrak{h}(2|2)$ of Hamiltonian vector fields as preserving either the pseudodifferential form

$$d\eta^{\frac{2\lambda-1}{1-\lambda}}\left(\lambda dqdp + (1-\lambda)d\xi d\eta\right)$$

or, equivalently, as preserving the pseudodifferential form

$$d\eta^{\frac{1}{\lambda}-2}\left(dqdp + d\xi d\eta\right).$$

We trustingly reproduced this in [ALSh] and other places, but verification shows that Shmelev made a mistake calculating Lie derivatives and these interpretations are wrong.

§1. Step 1. Main ideas of our proof

For Lie algebras the following facts and ideas (sec. 1.0) are known, though, perhaps, not as well as they deserve. For completeness sake we recall them without (or only with a sketch of a) proof. We learned them from a preprint by A. Joseph [J] that we saw in Moscow ca. 1974, where Cartan's list of primitive Lie algebras is obtained as in [GQS].

1.0. Lie algebras. Given a primitive Lie algebra $\mathcal{L}$, consider its Weisfeiler filtration (WF), see Introduction, and the associated graded Lie algebra $\mathfrak{g} := \underset{i\geq -d}{\oplus} \mathfrak{g}_i$, where $\mathfrak{g}_i = \mathcal{L}_i/\mathcal{L}_{i+1}$.

1.0.1. Proposition . *If $\mathfrak{g}$ is any primitive graded Lie algebra, then*

1) *$\mathfrak{g}_{-1}$ generates $\mathfrak{g}_- := \underset{i<0}{\oplus} \mathfrak{g}_i$;*

2) *Let $\mathfrak{g}^0 := \underset{i\geq 0}{\oplus} \mathfrak{g}_i$. If $a \in \mathfrak{g}^0$ and $[\mathfrak{g}_{-1}, a] = 0$, then $a = 0$ (transitivity condition);*

3) *the closure $\overline{\mathfrak{g}} = \prod \mathfrak{g}_i$ is an algebra with projective limit topology, $\overline{\mathfrak{g}}^0$ is its maximal subalgebra, and* $\mathrm{gr}\overline{\mathfrak{g}} \cong \mathfrak{g}$.

The notion of projective limit topology is useful in theoretical questions. Recall that the vector space endowed with one of the following topologies is isomorphic and homeomorphic to a space endowed with any of the other two topologies:

a) the dual to a space with discrete topology;

b) the direct product of finite dimensional spaces;

c) the projective limit of finite dimensional spaces.

The topological vector space whose topology is given by any of the above ways will be referred to as a space with a *projective limit* topology. Notice that this topology is highly disconnected: every subspace W containing one space V from the base of neighborhoods of zero is open (since W is the union of shifts of V) and closed (since the complement to W is also the union of shifted base spaces). The "topology" here is a means to impose finiteness type conditions on subspaces and maps.

The algebra with a *projective limit* topology is a one whose space is endowed with such a topology and the multiplication is a continuous map with respect to it. In applications, one usually considers continuous representations of algebras with a *projective limit* topology in either projective limit or discrete spaces, cf. [R1], [BL2].

Example. Consider $O_n = \mathbb{C}[[x_1, \dots, x_n]]$ with the (x)-adic topology; the (x)-adic topology is equivalent to the one determined (for a given $c \in \mathbb{N}$, $c > 1$) by the norm $|f| = c^{-\deg f}$ for any $f \in O_n$.

Then $O_n \simeq \mathbb{C}^{\mathbb{N}}$ with the product topology (the right hand side is the set of coefficients of monomials). The continuity of the multiplication $O_n \times O_n \longrightarrow O_n$ follows from the fact that $O_n \cdot (x)^p \simeq (x)^p \cdot O_n \subset (x)^p$ for any p.

1.0.2. Lemma . *Let* $\mathfrak{g} = \mathop{\oplus}\limits_{i \geq -d} \mathfrak{g}_i$ *be a primitive graded algebra. Then*

a) $[a, \mathfrak{g}_s] = \mathfrak{g}_{r+s}$ *for any nonzero* $a \in \mathfrak{g}_r$*, where* $r, s < 0$ *and* $r + s \geq -d$*;*

b) *for any nonzero* $a \in \mathfrak{g}_r$*,* $-d \leq r < 0$ *we have either* $\mathfrak{g}_r = Span(a)$ *or* $\mathfrak{g}_r = [a, \mathfrak{g}_0]$.

An action of a Lie algebra $\mathfrak{g}$ in a space V is called *transitive* if $\mathfrak{g}v = V$ for any nonzero $v \in V$.

Thus, Lemma 1.0.2 shows that $\mathfrak{g}_0 : \mathfrak{g}_{-1}$ is a faithful transitive representation for every primitive Lie algebra $\mathfrak{g}$. There are very few such pairs:

1.0.3. Proposition . *Let* $\mathfrak{g} : V$ *be a faithful transitive representation. Then the following cases are possible:*

1) $\mathfrak{g} = \mathfrak{s}$*, where* $\mathfrak{s}$ *is simple and, moreover,* $\mathfrak{s}$ *is either* $\mathfrak{sl}(n)$ *or* $\mathfrak{sp}(2n)$ *and* V *is either the identity representation* id *of* $\mathfrak{s}$ *or its dual,* id**;*

2) $\mathfrak{g} = \mathbb{C}z$*, the* 1*-dimensional commutative Lie algebra,*

3) $\mathfrak{g} = \mathfrak{s} \oplus \mathbb{C}z$*, the trivial central extension of the case* 1) *and the* $\mathfrak{g}$*-action on* V *is the natural extension of the* $\mathfrak{s}$*-action:* z *acts as multiplication by a scalar.*

Proof. 1) Let $\mathfrak{r} \subset \mathfrak{g}$ be a radical. By Lie's theorem, there is an eigenspace

$$V_\chi = \{v \in V \mid rv = \chi(r)v \text{ for any } r \in \mathfrak{r} \text{ and some character } \chi : \mathfrak{r} \longrightarrow \mathbb{C}\}.$$

To prove that $V_\chi = V$, it suffices to show that V_χ is a submodule. Indeed,

$$r(gv) = [r, g]v + g(rv) = \chi([r, g])v + \chi(r)gv \text{ for any } \; r \in \mathfrak{r}, \; g \in \mathfrak{g}, \; v \in V_\chi.$$

Let us show that $\chi([r, g]) = 0$. Indeed, in the space W spanned by $v, gv, g^2v, \dots$ (W is an eigenspace of r and g) the elements of the radical $\mathfrak{r}$ are represented in the above-indicated basis by matrices with $\chi(r)$ on the diagonal, i.e., $\chi(r) \dim W = \mathrm{tr}_W r$. But

$$0 = \mathrm{tr}_W [r, g] = \chi([r, g]) \dim W,$$

hence, V_χ is a submodule. So we deduce:

$$\textbf{Transitivity implies irreducibility}. \qquad (1.0.3.1)$$

Hence, $V_\chi = V$, and the radical is at most the center of $\mathfrak{g}$ due to the faithfulness of the $\mathfrak{g}$-action on V.

2) If $\mathfrak{g} = \mathbb{C}z$, then $\dim V = 1$.

3) Let $\mathfrak{s} \neq 0$. In $\mathfrak{s}$, select the maximal torus $\mathfrak{h}$; let R^+ be the set of positive roots of $\mathfrak{s}$ (with respect to an ordering) and $\mathfrak{b}$ the Borel subalgebra corresponding to R^+. Let ν^+ and ν^- be the highest and the lowest weights of V, $w^\pm$ the corresponding weight vectors. By transitivity, $gw^- = w^+$ for some $g \in \mathfrak{g}$. Let $g = \sum\limits_{\alpha \in R \cup \{0\}} c_\alpha X_\alpha + \lambda z$. Then

$$w^+ = \sum c_\alpha X_\alpha w^- + \lambda w^-.$$

Since the vectors of distinct weights are linearly independent, only one summand in the right-hand side is nonzero for $\alpha = \nu^+ - \nu^-$. Since $-\nu^-$ is the highest weight of V^*, we have a representation of α as the sum of two positive (even dominant) weights.

This implies simplicity of $\mathfrak{s}$. Indeed, if $\mathfrak{s}=\mathfrak{s}_1\oplus\dots\oplus\mathfrak{s}_k$, where the summands are simple, then $V=V_1\otimes\dots\otimes V_k$, where V_i is an irreducible $\mathfrak{s}_i$-module. Now,

$$\nu^+=\nu_1^++\dots+\nu_k^+,\quad -\nu^-=-\nu_1^--\dots-\nu_k^-$$

with nonnegative summands $\nu_i^+,-\nu_i^-\in\mathfrak{h}_i^*$. But since α is a root, $\nu^+-\nu^-=\alpha\in\mathfrak{h}_i^*$ for some i and the other summands are all zero. Hence, $k=1$ by faithfulness of the module V and, moreover,

$$\textbf{A root of } \mathfrak{g}_0 \textbf{ is the sum of dominant weights}. \tag{1.0.3.2}$$

From representation theory (see, e.g., [OV]) we know that this can only happen if $\mathfrak{s}=\mathfrak{sl}(n)$ or $\mathfrak{sp}(2n)$ and these dominant weights are:

for $\mathfrak{sl}(n)$: $\omega_1+\omega_{n-1}=\varepsilon_1-\varepsilon_n$ is a root for the fundamental weights $\omega_1=\varepsilon_1$ and $\omega_{n-1}=\varepsilon_1+\dots+\varepsilon_{n-1}=-\varepsilon_n$.

for $\mathfrak{sp}(2n)$: $\omega_1+\omega_1=2\varepsilon_1$ is a root.

The representation corresponding to ω_1 is the standard one; the case of $\mathfrak{sl}(n)$-module id* with highest weight ω_{n-1} reduces to the standard module id with the help of the outer automorphism given by $X\mapsto -X^t$. □

1.0.4. Proposition . *Let $\mathfrak{g}$ be an infinite dimensional primitive graded Lie algebra. Then*

1) $\mathfrak{g}_0$ *is* $\mathfrak{s}$ *or* $\mathfrak{s}\oplus\mathbb{C}z$ *with* $\mathfrak{s}=\mathfrak{sl}(n)$ *or* $\mathfrak{sp}(2n)$ *and* $\mathfrak{g}_{-1}$ *is the standard* $\mathfrak{g}_0$*-module;*

2) $\dim\mathfrak{g}_{-2}=0$ *or* 1*;*

3) $\mathfrak{g}_{-3}=0$.

Proof. 1) is already proved.

2) Since $\mathfrak{g}_{-2}=[a,\mathfrak{g}_{-1}]$ for any nonzero $a\in\mathfrak{g}_{-1}$ and $[a,a]=0$, we have $\dim\mathfrak{g}_{-2}\le\dim\mathfrak{g}_{-1}-1$. The action of $\mathfrak{s}$ on $\mathfrak{g}_{-2}$ is transitive and, since $\dim\mathfrak{g}_{-2}\neq\dim\mathfrak{g}_{-1}$, the description of transitive $\mathfrak{s}$-actions implies that the $\mathfrak{s}$-action on $\mathfrak{g}_{-2}$ is zero. Hence, $\dim\mathfrak{g}_{-2}\le 1$.

3) By the same arguments as in 2) $\dim\mathfrak{g}_{-3}\le 1$. If $\mathfrak{g}_{-3}\neq 0$, then there exists a unique $a\in\mathfrak{g}_{-2}$ which can organize an epimorphism $\mathrm{ad}a:\mathfrak{g}_{-1}\longrightarrow\mathfrak{g}_{-3}$ and $\ker(\mathrm{ad}a)|_{\mathfrak{g}_{-1}}\neq 0$ (otherwise $\dim\mathfrak{g}_{-1}=1$ and $\mathfrak{g}_{-2}=[\mathfrak{g}_{-1},\mathfrak{g}_{-1}]=0$).

By Lemma 1.0.2 $[g,\mathfrak{g}_{-2}]=\mathfrak{g}_{-3}$ for any $g\in\mathfrak{g}_{-1}$; now, take $g\in\ker\mathrm{ad}a$ to get $\mathfrak{g}_{-3}=0$. □

1.0.5. Theorem . *The following table lists all primitive infinite dimensional Lie algebras* $\mathfrak{g}$.

$\mathfrak{g}_0$	$\mathfrak{sl}(n)$	$\mathfrak{sl}(n)\oplus\mathbb{C}z$	$\mathfrak{sp}(2n)$	$\mathfrak{sp}(2n)\oplus\mathbb{C}z$
$\mathfrak{g}$ with $\mathfrak{g}_{-2}=0$	$\mathfrak{svect}(n)$	$\mathfrak{vect}(n)$ $\mathfrak{der}(\mathfrak{svect}(n))$	$\mathfrak{h}(2n)$	$\mathfrak{der}\ \mathfrak{h}(2n)$
$\mathfrak{g}$ with $\mathfrak{g}_{-2}\neq 0$	–	–	–	$\mathfrak{k}(2n+1)$

Proof is based on the following facts:

1) if $\mathfrak{g}_{-2}=0$, then $\mathfrak{g}\subset\mathfrak{vect}(n)$, where $n=\dim\mathfrak{g}_{-1}$, by the uniquness of Cartan's prolongation;

2) the semisimple part of $\mathfrak{g}_0$ is either $\mathfrak{sl}$ or $\mathfrak{sp}$;

3) every $\mathfrak{g}_i$ is a completely reducible $\mathfrak{g}_0$-module;

4) $\mathfrak{g}_{i+1}\subset\mathfrak{g}_i^{(1)}$, the Cartan prolong of $\mathfrak{g}_i$;

5) the first Cartan prolong of a $\mathfrak{g}_0$-submodule is a $\mathfrak{g}_0$-submodule (indeed, since $[v,[g,h]]=[[v,g],h]+[g,[v,h]]$ for any $v\in V, g\in\mathfrak{g}_0, h\in\mathfrak{g}_i^{(1)}$, it follows that $[g,h]\in\mathfrak{g}_i^{(1)}$).

It remains to decompose $\mathfrak{g}_i$ into $\mathfrak{g}_0$-submodules and collect from them the Lie algebras from the table. Let us decompose $S^i(V^*)\otimes V$ into irreducible $\mathfrak{s}$-modules:

$$S^i(V^*)\otimes V=\begin{cases}L^{(i-1)\omega_1}\oplus L^{i\omega_1+\omega_{n-1}} & \text{for } \mathfrak{s}=\mathfrak{sl}(n)\\ L^{(i-1)\omega_1}\oplus L^{(i+1)\omega_1}\oplus L^{(i-1)\omega_1+\omega_2} & \text{for } \mathfrak{s}=\mathfrak{sp}(2n).\end{cases}$$

Here are all the possibilities ($\mathfrak{g}_1\not\in\mathfrak{g}_0^{(1)}$ means that this case is impossible):

$\mathfrak{g}_0=\mathfrak{sl}(n)$	$\mathfrak{g}_1=L^{\omega_1}$	$\Longrightarrow$	$\mathfrak{g}_1\not\in\mathfrak{g}_0^{(1)}$
	$\mathfrak{g}_1=L^{2\omega_1+\omega_{n-1}}$	$\Longrightarrow$	$\mathfrak{g}=\mathfrak{svect}(n)$
$\mathfrak{g}_0=\mathfrak{gl}(n)$	$\mathfrak{g}_1=L^{\omega_1}$	$\Longrightarrow$	$\mathfrak{g}=\mathfrak{sl}(n+1)$
	$\mathfrak{g}_1=L^{2\omega_1+\omega_{n-1}}$	$\Longrightarrow$	$\mathfrak{g}=\mathfrak{der}(\mathfrak{svect}(n))$
$\mathfrak{g}_0=\mathfrak{sp}(2n)$,	$\mathfrak{g}_1=L^{2\omega_1}$	$\Longrightarrow$	$\mathfrak{g}=\mathfrak{h}(2n)$
	$\mathfrak{g}_1=L^{\omega_1}$	$\Longrightarrow$	$\mathfrak{g}_1\not\in\mathfrak{g}_0^{(1)}$
	$\mathfrak{g}_1=L^{\omega_2+\omega_1}$	$\Longrightarrow$	$\mathfrak{g}_1\not\in\mathfrak{g}_0^{(1)}$
$\mathfrak{g}_0=\mathfrak{sp}(2n)\oplus\mathbb{C}z$,	$\mathfrak{g}_1=L^{2\omega_1}$	$\Longrightarrow$	$\mathfrak{g}=\mathfrak{der}\ \mathfrak{h}(2n)$
	$\mathfrak{g}_1=L^{\omega_1}$	$\Longrightarrow$	$\mathfrak{g}_1\not\in\mathfrak{g}_0^{(1)}$
	$\mathfrak{g}_1=L^{\omega_2+\omega_1}$	$\Longrightarrow$	$\mathfrak{g}_1\not\in\mathfrak{g}_0^{(1)}$

1.1. Lie superalgebras. Given a Lie superalgebra $\mathcal{L}$ with Weisfeiler filtration (WF), see Introduction, consider the associated graded Lie superalgebra $\mathfrak{g}:=\mathop{\oplus}\limits_{i\geq -d}\mathfrak{g}_i$, where $\mathfrak{g}_i=\mathcal{L}_i/\mathcal{L}_{i+1}$. There are two distinct cases: that of a consistent $\mathbb{Z}$-grading, i.e., when $\mathfrak{g}_{\bar 0}=\mathop{\oplus}\limits_{i\equiv 0 \pmod 2}\mathfrak{g}_i$ and $\mathfrak{g}_{\bar 1}=\mathop{\oplus}\limits_{i\equiv 1 \pmod 2}\mathfrak{g}_i$, and when the grading is inconsistent.

To the inconsistent grading certain arguments of sec. 1.0 are applicable: in particular, the proof of the facts that

1) $\mathfrak{g}_{-1}$ generates $\mathfrak{g}_-:=\mathop{\oplus}\limits_{i<0}\mathfrak{g}_i$;

2) If $a\in\mathfrak{g}^0$ and $[\mathfrak{g}_{-1},a]=0$, then $a=0$ (transitivity condition);

3) the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is transitive, moreover, it is irreducible.

The proof of the fact that

$$\text{a root of } \mathfrak{g}_0 \text{ is the sum of positive weights of } \mathfrak{g}_{-1} \tag{1.1}$$

is given in §2, where (1.1) reads as follows: *the Cartan prolong of the pair $\mathfrak{g}_0:\mathfrak{g}_{-1}$ is of infinite dimension if and only if the action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ has a rank 1 operator (the coroot corresponding to* (1.1)*) with the even covector.*

After that we "only" have to find the pairs $\mathfrak{g}_0:\mathfrak{g}_{-1}$ for which (1.1) holds. The search of such pairs and the proof of (mutatis mutandis) the analog of Theorem 1.0.5 occupy §§3–6.

Observe that analogs of headings 2) and 3) of Prop. 1.0.4 fail to remain true *as stated* for Lie superalgebras but the fact that $\mathfrak{g}_0$ has only two or three simple subquotients and possibility to apply Prop. 1.0.4 to a nonzero *even* $a\in\mathfrak{g}_{-1}$ demonstrates that the only cases of depth >1 correspond, same as for Lie algebras, to a nondegenerate bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}_{-1}$, perhaps, with values in a multidimensional vector superspace in contradistinction from the one dimensional target space for Lie algebras.

Finally, it is only *a posteriori* that we see the restrictions on depth of W-filtered simple vectorial Lie superalgebras given in Introduction.

1.2. Lie superalgebras: consistent gradings.

Theorem . ([K7]) *The only simple vectorial Lie superalgebras with consistent Weisfeiler filtration are:* $\mathfrak{k}(1|m)$ *and the exceptions* $\mathfrak{kas}$, $\mathfrak{vle}(4|3;K)$, $\mathfrak{ksle}(9|6;K)$ *and* $\mathfrak{mb}(4|5;K)$, *see sec. I.2.2.*

§2. Step 2. Criterion for infinite dimension of the Cartan prolong

Here we generalize to Lie superalgebras the following Guillemin–Quillen–Sternberg's theorem proved à la Wilson.

2.1. Theorem . *Let* $\mathfrak{g}$ *be an infinite dimensional transitive Lie algebra of the form* $\mathfrak{g} = \underset{i\geq -1}{\oplus} \mathfrak{g}_i$. *Then there exists* $g \in \mathfrak{g}_0$ *such that* $\mathrm{rk}(\mathrm{ad} g)|_{\mathfrak{g}_{-1}} = 1$.

The converse statement is obvious: if such g exists, then, clearly, Cartan's prolong of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ is infinite dimensional. Indeed, in some basis, g takes the form $x\frac{\partial}{\partial y}$ and all the elements $x^n\frac{\partial}{\partial y}$ belong to the Cartan prolong.

2.2. A reformulation of Theorem. This theorem is, in fact, a special case of a more general statement. Let $\mathcal{A} = \underset{i\leq 0}{\oplus} \mathcal{A}_i$ be the commutative $\mathbb{Z}$-graded algebra with unity generated by $\mathcal{A}_{-1} = A$ and $M = \underset{i\geq k}{\oplus} M_i$ a $\mathbb{Z}$-graded $\mathcal{A}$-module. (In the case we are interested in $\mathcal{A} = S^{\bullet}(\mathfrak{g}_{-1})$ and $M = \mathfrak{g}$.) We will say that M is *transitive* if

$$Am \neq 0 \text{ for any nonzero } m \in M_i \text{ and } i \geq k+1.$$

Notice that each element $m \in M$ determines a linear map

$$m : A \longrightarrow M, \quad a \mapsto am.$$

Clearly, M is transitive if and only if there exists $i \geq k+1$ and a nonzero $m \in M_i$ such that $\ker m \neq A$.

Now it is clear that Theorem 2.1 can be reformulated in the following equivalent form which is more convenient for us.

Theorem . *Let* $M = \underset{i\geq k}{\oplus} M_i$ *be a* $\mathbb{Z}$*-graded infinite dimensional transitive* $\mathcal{A}$*-module. Then there exists* $m \in M_{k+1}$ *such that* $\operatorname{codim}\ker m = 1$*; hence,* $\mathrm{rk}\ m = 1$.

Proof. Without loss of generality we may assume that $k = -1$, i.e., $M = \underset{i\geq -1}{\oplus} M_i$ and $m \in M_0$.

1°. Due to the commutativity of $\mathcal{A}$, $\ker am \supset \ker m$ for any $a \in A$, $m \in M$.

2°. Set $r_i = \max\limits_{m\in M_i}\{\dim\ker m\}$. By the above, $r_{i+1} \leq r_i$; hence, the sequence r_i stabilizes: $r_N = r_{N+1} = \dots := r$ for some N. Due to transitivity, $r < \dim A$, so it remains to prove that $r = \dim A - 1$.

3°. We will say that m is *maximally degenerate* if $\dim\ker m = r$.

By item 1°, if $a \in A$ and $m \in M_n$ for $n \geq N+1$ is maximally degenerate, so am is maximally degenerate too and $\ker am = \ker m$.

4°. Let $m_0 \in M_{N+l}$ for $l \geq 0$ be a maximally degenerate element. For any $n \geq N$ set

$$V_n(m_0) = \{m \in M_n \mid \ker m \supset \ker m_0\}.$$

It is clear that any nonzero element $m \in V_n(m_0)$ is maximally degenerate and $\ker m = \ker m_0$.

5°. The following Lemma is crucial in the next item. Actually, this Lemma is the key point of the whole proof of Theorem 2.2.

Lemma . *Let U, V, W be finite dimensional vector spaces, $U \subset \mathrm{Hom}(V,W)$ a subspace such that any nonzero homomorphism $u \in U$ is a monomorphism. Then* $\dim U \leq \dim W - \dim V + 1$.

Proof (This elegant proof due to J. Bernstein replaces an original elementary but ugly proof.) Let $P(X)$ be the projective space corresponding to the vector space X. Since $U \otimes V \to W$ is a monomorphism for any nonzero $u \in U$, there exists a map $i : P(U) \times P(V) \longrightarrow P(W)$ which induces embeddings $i : a \times P(V) \longrightarrow P(W)$ and $i : P(U) \times b \longrightarrow P(W)$ for each fixed $a \in P(U)$ and $b \in P(V)$.

Let $h \in H^2(P(W))$, $h_1 \in H^2(P(U))$ and $h_2 \in H^2(P(V))$ be the generators of the respective cohomology rings. Since $i : a \times P(V) \longrightarrow P(W)$ and $i : P(U) \times b \longrightarrow P(W)$ are embeddings, it follows that $i^*(h) = h_1 + h_2$.

On the other hand, the only relations h, h_1 and h_2 satisfy are

$$h^{\dim W+1} = 0, \quad h_1^{\dim U+1} = 0, \quad h_2^{\dim V+1} = 0.$$

Hence, replacing h with $i^*(h) = h_1 + h_2$ we see that the first identity is only possible if

$$\dim U + \dim V \leq \dim W + 1. \quad \square$$

6°. Each $m \in V_n(m_0)$ determines a monomorphism $A/\ker m_0 \longrightarrow V_{n-1}(m_0)$. By Lemma 5° we have

$$\dim V_n(m_0) \leq \dim V_{n-1}(m_0) - (\dim A - r) + 1 \Longrightarrow$$
$$\dim V_{n-1}(m_0) \geq \dim V_n(m_0) + (\dim A - r) - 1.$$

7°. Suppose $r < \dim A - 1$. Then by item 6°, $\dim V_{n-1}(m_0) \geq \dim V_n(m_0) + 1$. Let us take $m_0 \in M_{N+s}$, where $s > 0$ is arbitrary. Then $V_{N+s}(m_0) \neq 0$; hence,

$$\dim M_n \geq \dim V_N(m_0) \geq V_{N+1}(m_0) + 1 \geq \dots \geq \dim V_{N+s}(m_0) + s \geq 1 + s.$$

Since $\dim M_N < \infty$, we get a contradiction. Thus, $r = \dim A - 1$.

8°. So for some $s \geq N$ we get an element $m_0 \in M_s$ with $\operatorname{codim} \ker m_0 = 1$. By transitivity there are $a_1, \cdots, a_s \in A$ such that $0 \neq m = a_1 \cdots a_s m_0 \in M_0$. Due to item 1°, $\ker m \supset \ker m_0$ and by transitivity $\ker m \neq A$. Then $\ker m = \ker m_0$ and m is the element we need. $\square$

Remark . The proof of the reformulated theorem is outlined in this way by D. Alekseevsky and elaborated by A. Rudakov (1979). An advantage of this proof is that it is short and superizable as shown in sec 2.3.

2.3. Let $\mathcal{A} = \oplus_{i \leq -1} \mathcal{A}_i$ be the supercommutative superalgebra generated by $\mathcal{A}_{-1} = A = A_{\bar 0} \oplus A_{\bar 1}$. Let $M \oplus_{i \geq -1} M_i$ be an $\mathcal{A}$-module. As above, we will say that M is *transitive* if $\ker m \neq 0$ for any nonzero $m \in M_i$, where $i \geq 0$.

Theorem . *In any infinite dimensional transitive graded $\mathcal{A}$-module M there exists an element $m \in M_0$ such that* $\operatorname{codim} \ker m = 1|0$.

Proof. Let $\dim A_{\bar 1} = 0|n$.

1°. Due to supercommutativity of $\mathcal{A}$ we have

$$a^2 m = 0 \text{ for any } a \in A_{\bar 1}, m \in M; \tag{2.3.1}$$

$$a_1 \cdots a_i m = 0 \text{ for any } m \in M \text{ and } a_1, \cdots, a_i \in A_{\bar 1} \text{ if } i > n. \tag{2.3.2}$$

2°. Due to (2.3.2) and transitivity of M we see that $A_{\bar 0} m \neq 0$ for any $m \in M_i$ with $i \geq n$.

3°. Let $\mathcal{A}_0 \subset \mathcal{A}$ be the subalgebra generated by $A_{\bar 0}$. Let $\tilde M = M/\oplus_{i=-1}^{n-1} M_i$. Let us consider $\tilde M$ as an $\mathcal{A}_0$-module and apply Theorem 2.2 to $\tilde M$. We get an element $m \in M_{n+1}$ such that $\dim \ker m \cap A_{\bar 0} = \dim A_{\bar 0} - 1$. If $\ker m \cap A_{\bar 1} \neq A_{\bar 1}$, there exists $a_1 \in A_{\bar 1}$ such that $a_1 \notin \ker m$. Then $a_1 m \neq 0$ and by (2.3.1), $\ker a_1 m \supset \ker m \oplus \mathbb{C} a_1$.

Similarly, for $i \leq n$, there exist $a_1, \cdots, a_i$ such that $m_1 = a_1 \cdots a_i m \neq 0$ and $\ker m_1 \supset \ker m + A_{\bar{1}}$.

Notice that $m_1 \in M_s$, where $s = n + 1 - i > 0$. Hence, $\ker m_1 \neq A$; thus, $\operatorname{codim} \ker m_1 = 1|0$.

Now we can complete the proof in the same way as in item 8° of the proof of Theorem 2.2. □

Corollary . *Let $\mathfrak{g} = \oplus_{i \geq -1} \mathfrak{g}_i$ be a transitive infinite dimensional $\mathbb{Z}$-graded Lie superalgebra with $\dim \mathfrak{g}_i < \infty$ for all i. Then there exists $v^* \otimes w \in \mathfrak{g}_0$, where $v \in (\mathfrak{g}_{-1})_{\bar{0}}$ and $w \in \mathfrak{g}_{-1}$.*

§3. Step 3. Main reduction: the three cases

3.0. Lemma . *If $\mathfrak{g}_* = \underset{i \geq -1}{\oplus} \mathfrak{g}_i$ is simple, then*

1) $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$;

2) *the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_{-1}$ is irreducible, i.e., $\mathfrak{g}_{\geq} = \underset{i \geq 0}{\oplus} \mathfrak{g}_i$ is a maximal Lie subsuperalgebra;*

3) *the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_{-1}$ is faithful.*

Proof. When one of the conditions 1) − 3) is violated it is easy to find an ideal in $\mathfrak{g}_*$.

If 1) fails, take $\mathfrak{i} = \mathfrak{g}_{-1} \oplus [\mathfrak{g}_{-1}, \mathfrak{g}_1] \oplus (\oplus_{i>0} \mathfrak{g}_i)$.

In the other cases take $\mathfrak{i} = \underset{n \geq 0}{\oplus} \mathfrak{i}_n$, where $\mathfrak{i}_n$ are recursively defined (here ρ is the $\mathfrak{g}_0$-action in $\mathfrak{g}_{-1}$):

$$\mathfrak{i}_{-1} = \begin{cases} \text{a } \mathfrak{g}_0\text{-invariant subsuperspace} & \text{in case 2)} \\ 0 & \text{in case 3)} \end{cases}$$

$$\mathfrak{i}_0 = \ker \rho \quad \text{in case 3)}$$

$$\mathfrak{i}_n = \{g \in \mathfrak{g}_n \mid [\mathfrak{g}_{-1}, g] \subset \mathfrak{i}_{n-1}\} \quad \text{for } n \geq \begin{cases} 0 & \text{in case 2)} \\ 1 & \text{in case 3);} \end{cases}$$

The Jacobi identity provides the inclusion

$$[\mathfrak{g}_{-1}, [\mathfrak{g}_k, \mathfrak{i}_n]] \subset [\mathfrak{g}_{k-1}, \mathfrak{i}_n] + [\mathfrak{g}_k, [\mathfrak{g}_{-1}, \mathfrak{i}_n]];$$

the double induction on k and n shows that $\mathfrak{i}$ is an ideal in $\mathfrak{g}_*$. □

The conditions 2) and 3) of Lemma allow us to think of $\mathfrak{g}_0$ as of an irreducible linear Lie superalgebra, i.e., as of a subalgebra of $\mathfrak{gl}(V)$, where $V = \mathfrak{g}_{-1}$, and the identity representaiton in V is irreducible.

3.1. Some notations. The four cases.

1) Let V_1 and V_2 be linear superspaces, $V = V_1 \otimes V_2$. The tensor product of $\mathfrak{gl}(V_i)$-actions in $V_1 \otimes V_2$ gives us a representation

$$\rho(A, B) = A \otimes 1_{V_2} + 1_{V_1} \otimes B \quad \text{for any} \quad A \in \mathfrak{gl}(V_1), B \in \mathfrak{gl}(V_2).$$

It is clear that ρ has a one-dimensional kernel $\mathbb{C}(1_{V_1} - 1_{V_2})$. Denote the image $\rho(\mathfrak{gl}(V_1) \oplus \mathfrak{gl}(V_2))$ by

$$\mathfrak{gl}(V_1) \bigodot \mathfrak{gl}(V_2).$$

2) The space of all operators in a superspace V may be endowed with two structures: that of the Lie superalgebra $\mathfrak{gl}(V)$ and of the associative superalgebra $\operatorname{Mat}(V)$. We will carefully distinguish them. (Recall that we endow with index $_L$ the Lie superalgebra constructed on the space of an associative superalgebra with the dot product replaced with the bracket. In particular, $\operatorname{Mat}(V)_L = \mathfrak{gl}(V)$.)

Let V_0 be a $(1,1)$-dimensional superspace and $\mathcal{A}\subset \mathrm{Mat}(1|1)=\mathrm{Mat}(V_0)$ the associative subsuperalgebra of supermatrices of the form $\begin{pmatrix} a & b\\ b & a\end{pmatrix}$, where $a,b\in\mathbb{C}$ and $C(\mathcal{A})\subset \mathrm{Mat}(V_0)$ is the associative subsuperalgebra of supermatrices of the form $\begin{pmatrix} c & d\\ -d & c\end{pmatrix}$, where $c,d\in\mathbb{C}$.

Set $J=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$ and $\Pi=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}$; clearly 1 and J generate $\mathcal{A}$ while 1 and Π generate $C(\mathcal{A})$ as associative superalgebras.

Observe that

$$J^2=-1,\quad \Pi^2=1;\quad J\Pi=\Pi J=0$$

and, therefore, $\mathcal{A}\cong C(\mathcal{A})$ over $\mathbb{C}$ and $[\mathcal{A}, C(\mathcal{A})]=0$. (This justifies our notations: $C(\mathcal{A})$ is the centralizer of $\mathcal{A}$ in $\mathrm{Mat}(1|1)$.)

If V_1 and V_2 are linear superspaces of dimension $m\varepsilon$ and n, respectively, then the Lie superalgebras $(\mathrm{Mat}(V_1)\otimes\mathcal{A})_L$ and $(C(\mathcal{A})\otimes\mathrm{Mat}(V_2))_L$ of the operators in $V_1\otimes V_0$ and $V_0\otimes V_2$, respectively, can be identified with $\mathfrak{q}(m)$ and $\mathfrak{q}(n)$, respectively.

Now, let $V=V_1\otimes V_0\otimes V_2$. Then a natural $\mathfrak{q}(n_1)\oplus\mathfrak{q}(n_2)$-action ρ in V is determined:

$$\text{for any } \begin{pmatrix} A & B\\ B & A\end{pmatrix}\in\mathfrak{q}(m), \text{ and } \begin{pmatrix} C & D\\ D & C\end{pmatrix}\in\mathfrak{q}(n) \text{ set } \rho(\begin{pmatrix} A & B\\ B & A\end{pmatrix}, \begin{pmatrix} C & D\\ D & C\end{pmatrix}) = (A\otimes 1_{V_0}+B\otimes J)\otimes 1_{V_2}+1_{V_1}\otimes(1_{V_0}\otimes C+\Pi\otimes D).$$

This representation has a one-dimensional kernel. Denote by $\mathfrak{q}(n_1)\bigodot\mathfrak{q}(n_2)$ the image of the Lie superalgebra $\mathfrak{q}(n_1)\oplus\mathfrak{q}(n_2)$ under this representation.

3) Let V_1 be a linear superspace of dimension (r,s), $\mathfrak{n}=\mathfrak{gl}(V_1)\otimes\Lambda(n)$, $\mathfrak{k}=\mathfrak{vect}(0|n)$ and $\mathfrak{g}=\mathfrak{n}\subplus\mathfrak{k}$ with the natural action of $\mathfrak{k}$ on the ideal $\mathfrak{n}$. The Lie superalgebra $\mathfrak{g}$ has a natural faithful representation ρ in the linear superspace $V=V_1\otimes\Lambda(n)$ defined by the formulas

$$\begin{array}{lcl}\rho(A\otimes\varphi)(v\otimes\psi) &=& (-1)^{p(\varphi)p(v)}Av\otimes\varphi\psi\\ \rho(D)(v\otimes\psi) &=& -(-1)^{p(D)p(v)}v\otimes D(\psi)\end{array}\quad \text{for any }\ A\otimes\varphi\in\mathfrak{n}, D\in\mathfrak{k}, v\otimes\psi\in V.$$

In the sequel, we will always identify the elements of $\mathfrak{g}$ with their images under ρ. Therefore, we will consider $\mathfrak{g}$ embedded in $\mathfrak{gl}(V)$.

4) Let $\mathfrak{hei}(0|2n)=Span(\xi_1,\dots,\xi_n;\eta_1,\dots,\eta_n;z)$ be the Heisenberg superalgebra, see 0.5.2, with the nonzero relations $[\xi_i,\eta_j]=\delta_{ij}z$. The nonzero irreducible finite dimensional representation of $\mathfrak{hei}(0|2n)$ that maps z to the operator of multiplication by 1 is realized in $\Lambda(\xi)$ via $\xi_i\mapsto\xi_i, \eta_i\mapsto\partial_i$. The normalizer of $\mathfrak{hei}(0|2n)$ in $\mathfrak{gl}(\Lambda(n))$ is $\mathfrak{g}=\mathfrak{hei}(0|2n)\subplus\mathfrak{o}(2n)$ or, in terms of differential operators:

$$\mathfrak{hei}(0|2n)\subplus\mathfrak{o}(2n)=Span(1,\xi_i,\partial_i,\xi_i\partial_j,\partial_i\partial_j,\xi_i\xi_j)_{i,j=1}^n.$$

3.2. Theorem . *If $\mathfrak{g}\subset\mathfrak{gl}(V)$ is an irreducible linear Lie superalgebra neither almost simple nor a central extension of an almost simple, then $\mathfrak{g}$ is contained in one of the following four Lie superalgebras:*

1) $\mathfrak{g}\subset\mathfrak{gl}(V_1)\bigodot\mathfrak{gl}(V_2)\subset\mathfrak{gl}(V)$ *for any V_1, V_2 such that $V=V_1\otimes V_2$ and* $\dim V_i>1$ *for* $i=1,2$*;*

2) $\mathfrak{g}\subset\mathfrak{q}(m)\bigodot\mathfrak{q}(n)\subset\mathfrak{gl}(V)$*, where $V=V_1\otimes V_0\otimes V_2$ for any V_0, V_1, V_2 such that* $\dim V_0=(1,1)$*,* $\dim V_1=m\varepsilon$*,* $\dim V_2=n$ *and $mn>1$;*

3) $\mathfrak{g}\subset\mathfrak{gl}(V_1)\otimes\Lambda(n)\subplus\mathfrak{vect}(0|n)\subset\mathfrak{gl}(V)$ *for any V_1 and n such that $V=V_1\otimes\Lambda(n)$; the natural representation of $\mathfrak{g}$ in V is induced;*

4) $\mathfrak{g}\subset\mathfrak{hei}(0|2n)\subplus\mathfrak{o}(2n)\subset\mathfrak{gl}(V)$ *for $n>1$ if $V\cong\Lambda(n)$ or $\Pi(\Lambda(n))$.*

For proof see [Sh-M] or sec. 3.3.

Corollary . *If $\mathfrak{gl}(V)$ is an irreducible linear superalgebra with a rank 1 operator in V, then the following 3 cases are possible:*

1) *the natural representation of $\mathfrak{g}$ in V is induced;*
2) *$\mathfrak{g}$ is almost simple or a central extension of an almost simple Lie superalgebra;*
3) $\dim V = (2,2)$ *and* $\mathfrak{g} \subset \mathfrak{hei}(0|4) \Subset \mathfrak{o}(4)$.

Indeed, neither the matrix Lie superalgebras $\mathfrak{gl}(V_1) \bigodot \mathfrak{gl}(V_2)$ with $\dim V_i > 1$ nor $\mathfrak{q}(n_1) \bigodot \mathfrak{q}(n_2)$ with $n_1 n_2 > 1$ have operators of rank 1; the only possibility for the Lie superalgebra $\mathfrak{g} = \mathfrak{hei}(0|2n) \Subset \mathfrak{o}(2n)$ to have an operator of rank 1 occures for $n = 2$.

3.3. Proof of Theorem 3.2. In this section $\mathfrak{g}$ is an irreducible linear Lie superalgebra, ρ its standard representation in superspace V. We will assume that $\dim V > 1$ (because the case $\dim V = 1$ is trivial). By hypothesis $\mathfrak{g}$ contains an ideal $\mathfrak{k}$.

Our proof of Theorem 3.2 is largely based on the following Theorem 3.3.1 — the superization of a statement well-known to the reader from Dixmier's book [Di], sec.5.4.1.

3.3.1. Theorem . *Let $\mathfrak{g}$ be an irreducible linear Lie superalgebra, ρ its standard representation (in particular, ρ is faithful). Let $\mathfrak{k}$ be an ideal of $\mathfrak{g}$ and* $\dim \mathfrak{k} > 1$. *Then only one of the following 3 cases is possible:*

A) *$\rho|\mathfrak{k}$ is irreducible;*

B) *$\rho|\mathfrak{k}$ is a multiple of an irreducible $\mathfrak{k}$-module τ and the multiplicity of τ is > 1;*

C) *there exists a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{k} \subset \mathfrak{h}$ and $\rho \equiv \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}} \sigma$ for an irreducible $\mathfrak{k}$-module σ.*

Proof largely follows the lines of [Di] with one novel case: irreducible modules of Q-type might occure. Fortunately, their treatment is rather straightforward and we would rather save paper by ommiting the verification.

We will say that the *representation ρ is of type A, B, or C with respect to the ideal* $\mathfrak{k}$ if the corresponding case holds. Lemmas 3.3.5, 3.3.2 and 3.3.3 deal with types A, B and C, respectively.

3.3.2. Lemma . B) *If ρ is of type B with respect to the ideal $\mathfrak{k}$, then either $\mathfrak{g} \subset \mathfrak{gl}(V_1) \bigodot \mathfrak{gl}(V_2)$ or $\mathfrak{g} \subset \mathfrak{q}(V_1) \bigodot \mathfrak{q}(V_2)$ for some V_1 and V_2.*

Proof. Let V be the superspace of representation τ. Then $V = V_1 \otimes U$ for some U and

$$\rho(h)(v \otimes u) = \tau(h)v \otimes u \quad \text{for any} \quad h \in \mathfrak{k},\ v \in V_1,\ u \in U$$

i.e.,

$$\text{the operators } \rho(h) \text{ are of the form } \tau(h) \otimes 1. \tag{3.3.2.1}$$

On the other hand, $\mathfrak{g}$ is the stabilizer of τ. So for any $g \in \mathfrak{g}$ there exists $s(g) \in \operatorname{End} V_1$ such that $\tau([g,h]) = [s(g), \tau(h)]$ for any $h \in \mathfrak{k}$.

Let us consider the operator $T(g) = \rho(g) - s(g) \otimes 1$ in V. For $h \in \mathfrak{k}$ we have

$$[T(g), \rho(h)] = [\rho(g) - s(g) \otimes 1, \rho(h)] = \rho([g,h]) - [s(g), \tau(h)] \otimes 1 = 0\ ,$$

i.e., $T(g)$ commutes with all the operators of the form $\rho(h)$, $h \in \mathfrak{h}$, and, since τ is irreducible, $T(g)$ is of the form

$$T(g) = \begin{cases} 1 \otimes A(g) & \text{if } \tau \text{ is of } G\text{-type} \\ 1 \otimes A(g) + J \otimes B(g) & \text{if } \tau \text{ is of } Q\text{-type}\ , \end{cases}$$

where J is an odd generator of $C(\tau)$, the centralizer of τ.

Therefore, $\rho(g)$ is either of the form

$$\rho(g) = s(g) \otimes 1 + 1 \otimes A(g), \text{ if } \tau \text{ is of } G\text{-type} \tag{3.3.2.2}$$

or

$$\rho(g) = s(g) \otimes 1 + 1 \otimes A(g) + J \otimes B(g) \text{ if } \tau \text{ is of } Q\text{-type} . \qquad (3.3.2.3)$$

Formulas (3.3.2.1)–(3.3.2.3) show that if τ is of G-type, then $\mathfrak{g}$ is contained in $\mathfrak{gl}(V_1) \bigodot \mathfrak{gl}(V_2) \subset \mathfrak{gl}(V)$, where $V_2 = U$, and if τ is of Q-type, then $\mathfrak{g}$ is contained in $\mathfrak{q}(V_1) \bigodot \mathfrak{q}(V_2) \subset \mathfrak{gl}(V)$, where $V_2 = Span(1, J) \otimes U$. □

3.3.3. Lemma . C) *If ρ is of type C with respect to the ideal $\mathfrak{k}$, then $\mathfrak{g} \subset \mathfrak{gl}(V) \otimes \Lambda(n) \Subset \mathfrak{vect}(0|n)$ for some V and n.*

Lemma C follows from the definition of the induced representaiton.

3.34. To complete the proof of Theorem 3.2 it suffices to consider the case when ρ possesses the following property: for any ideal $\mathfrak{k} \subset \mathfrak{g}$ either ρ is of type A with respect to $\mathfrak{k}$ or $\rho|_{\mathfrak{k}}$ is the multiple of a character.

Let $\mathfrak{r}$ be the radical of the linear Lie superalgebras $\mathfrak{g}$.

Lemma . *If $\mathfrak{r}$ is commutative and ρ belongs to type A or B with respect to $\mathfrak{r}$, then* $\dim \mathfrak{r} = 0$ *or* 1.

Proof follows from the fact that any irreducible representation of a commutative Lie superalgebra is 1-dimensional (up to the change of parity) and the faithfulness of ρ. □

3.3.5. In the case when $\mathfrak{r}$ is not commutative, consider the derived series of $\mathfrak{r}$:

$$\mathfrak{r} \supset \mathfrak{r}_1 \supset \cdots \supset \mathfrak{r}_k \supset \mathfrak{r}_{k+1} = 0,$$

where $\mathfrak{r}_{i+1} = [\mathfrak{r}_i, \mathfrak{r}_i]$. Clearly, each $\mathfrak{r}_i$ is an ideal in $\mathfrak{g}$ and the last nonzero ideal, $\mathfrak{r}_k$, is commutative.

Lemma . A) *If ρ is of type A with respect to $\mathfrak{r}_{k-1}$ and $\rho|_{\mathfrak{r}_k}$ is scalar, then either $\mathfrak{r}_{k-1} \cong \mathfrak{hei}(0|2n)$ or $\mathfrak{r}_{k-1} \cong \mathfrak{hei}(0|2n-1)$ and $V \cong \Lambda(n)$ or $\Pi(\Lambda(n))$ and $\mathfrak{g} \subset \mathfrak{hei}(0|2n) \Subset \mathfrak{o}(2n)$.*

Proof. We see that if the restrictions $\rho|_{\mathfrak{r}_i}$ are irreducible for all i, then

1) $\dim \mathfrak{r}_k = 1$ because ρ is faithful;

2) $\mathfrak{r}_{k-1} = \mathfrak{hei}(0|m)$ for some m;

3) $\rho|_{\mathfrak{r}_{k-1}}$ is irreducible and faithful, so it can be realized in the superspace of functions, $\Lambda(2n)$, or in $\Pi(\Lambda(2n))$, where $n = [\frac{m}{2}] + 1$;

4) $\mathfrak{g}$ is contained in the normalizer of $\mathfrak{hei}(0|2n)$ in its spinor representation, i.e., $\mathfrak{g} \subset \mathfrak{hei}(0|2n) \Subset \mathfrak{o}(2n)$. □

3.3.6. Lemma . *Let* $\dim \mathfrak{r} \leq 1$*, i.e., $\mathfrak{g}$ is either semisimple or a central extension of a semisimple Lie superalgebra but NOT an almost simple or a central extension of an almost simple Lie superalgebra. Then we can always choose an ideal $\mathfrak{k}$ such that ρ is of type B or C with respect to $\mathfrak{k}$.*

Proof. Let $\dim \mathfrak{r} = 0$, i.e., let $\mathfrak{g}$ be semisimple. As $\mathfrak{g}$ is not almost simple, then due to [K2] the alternative arises: either

$$\mathfrak{g} \text{ contains an ideal } \mathfrak{k} \text{ of the form } \mathfrak{k} = \mathfrak{h} \otimes \Lambda(n) \text{ with simple } \mathfrak{h} \text{ and } n > 0 \qquad (3.3.6)$$

or

$$\mathfrak{g} = \bigoplus_{j \leq k} \mathfrak{s}_j, \text{ where each } \mathfrak{s}_j \text{ is almost simple and } k > 1.$$

3.3.6.1. Lemma . *If $\mathfrak{g} = \bigoplus\limits_{j \leq k} \mathfrak{s}_j$ and $k \geq 2$, then any irreducible faithful representation of $\mathfrak{g}$ is of type B with respect to any its ideal $\mathfrak{s}_j$.*

Proof. Since the stabilizer of any irreducible representation of $\mathfrak{s}_j$ is the whole $\mathfrak{g}$, the type of any irreducible representation of $\mathfrak{g}$ with respect to $\mathfrak{s}_j$ can be either A or B. Due to faithfulness case A is ruled out. □

3.3.6.2. Lemma . *Let $\mathfrak{h}$ be a simple Lie superalgebra and $\mathfrak{k} = \mathfrak{h} \otimes \Lambda(n)$. Then $\mathfrak{k}$ has no faithful irreducible finite dimensional representations.*

For proof see 3.3.6.6.

3.3.6.3. Corollary . *If $\mathfrak{g}$ contains an ideal $\mathfrak{k}$ of the form* (3.3.6)*, then $\mathfrak{g}$ can not have any faithful irreducible finite dimensional representation of type A with respect to the ideal $\mathfrak{k}$.*

3.3.6.4. Corollary . *Lemma* 3.3.6.1 *and Corollary* 3.3.6.2 *prove Lemma* 3.3.6 *for semisimple Lie superalgebras.*

3.3.6.5. Lemma . *If $\mathfrak{h}$ is a simple Lie superalgebra, then*

$$[\mathfrak{h}_{\bar 1}, \mathfrak{h}_{\bar 1}] = \mathfrak{h}_{\bar 0} \quad \textit{and} \quad [\mathfrak{h}_{\bar 0}, \mathfrak{h}_{\bar 1}] = \mathfrak{h}_{\bar 1} \ .$$

Proof is left to the reader. (*Hint*: see [K2].) □

3.3.6.6. Proof of Lemma 3.3.6.2. For $n > 0$ the Lie superalgerba $\mathfrak{k}$ contains supercommutative ideal

$$\mathfrak{n}_1 = \mathfrak{h} \otimes \xi_1 \dots \xi_n.$$

Moreover, if $n > 2$, then $\mathfrak{g}$ contains a supercommutative ideal

$$\mathfrak{n}_2 = \mathfrak{h} \otimes \Lambda^{n-1}(\xi) \oplus \mathfrak{n}_1.$$

Thanks to Theorem 3.3.1, any irreducible representation ρ of $\mathfrak{k}$ is obtained as follows: take an irreducible representation (character χ) of an ideal $\mathfrak{n}_i$, the subsuperalgebra $\mathfrak{st}_{\mathfrak{k}}(\chi)$, and an irreducible representation $\tilde\rho$ of $\mathfrak{st}_{\mathfrak{k}}(\chi)$ whose restriction onto $\mathfrak{n}_i$ is a multiple of χ. Then $\rho = \mathrm{ind}^{\mathfrak{k}}_{\mathfrak{st}_{\mathfrak{k}}(\chi)}(\tilde\rho)$ for some $\tilde\rho$.

Let us show that if $\dim \rho < \infty$, then $\chi|_{\mathfrak{h}\otimes\xi_1\cdots\xi_n} = 0$. Since $\mathfrak{h} \otimes \xi_1 \cdots \xi_n$ is an ideal in $\mathfrak{k}$, this would contradict the faithfulness of ρ. Indeed, $\dim \rho < \infty$ if and only if $\mathfrak{st}_{\mathfrak{k}}(\chi) \supset \mathfrak{k}_{\bar 0}$.

1) $n = 2k + 1$. For any $f \in \mathfrak{h}$ and $g = h \otimes 1 \in \mathfrak{h}_0 \otimes 1$ we have $0 = \chi([g, f \otimes \xi_1 \cdots \xi_n]) = \chi([h, f] \otimes \xi_1 \cdots \xi_n)$ implying that

$$\chi|_{[\mathfrak{h}_{\bar 0},\mathfrak{h}_{\bar 1}]\otimes\xi_1\cdots\xi_n} \overset{(\text{Lemma 3.3.6.5})}{=} \chi|_{(\mathfrak{h}_{\bar 1}\otimes\xi_1\cdots\xi_n)} = 0.$$

Since χ is a character, then $\chi|_{\mathfrak{h}_{\bar 0}\otimes\xi_1\cdots\xi_n} = 0$, i.e., $\chi|_{\mathfrak{h}\otimes\xi_1\cdots\xi_n} = 0$.

2) $n = 2k > 2$. For any

$$\begin{array}{l} g = h \otimes l_1(\xi) \in \mathfrak{h}_{\bar 1} \otimes \Lambda^1(\xi) \subset \mathfrak{k}_{\bar 0} \\ g_1 = h_1 \otimes l_{n-1}(\xi) \in \mathfrak{h}_{\bar 1} \otimes \Lambda^{n-1}(\xi) \subset (\mathfrak{n}_2)_{\bar 0} \end{array}$$

we have

$$0 = \chi([g, g_1]) = \pm\chi([h, h_1]) \otimes l_1(\xi) l_{n-1}(\xi),$$

i.e.,

$$\chi|_{[\mathfrak{h}_{\bar 1},\mathfrak{h}_{\bar 1}]\otimes\xi_1\cdots\xi_n} = \chi|_{\mathfrak{h}_{\bar 0}\otimes\xi_1\cdots\xi_n} = 0 \ .$$

But χ is a character; therefore,

$$\chi|_{\mathfrak{h}_{\bar 1}\otimes\xi_1\cdots\xi_n} = 0$$

and, finally,

$$\chi|_{\mathfrak{h}\otimes\xi_1\cdots\xi_n = 0} \ .$$

3) $n = 2$. Then $\mathfrak{k}$ contains a commutative ideal $\mathfrak{n} = \mathfrak{h} \otimes \xi_1\xi_2$; let χ be a character of $\mathfrak{n}$.

If χ generates a finite dimensional representation of $\mathfrak{k}$, then $\mathfrak{st}_{\mathfrak{k}}(\chi) \supset \mathfrak{k}_{\bar{0}}$, i.e.,

$$\chi([f, g \otimes \xi_1\xi_2]) = 0 \text{ for any } f \in \mathfrak{h}_{\bar{0}},\ g \in \mathfrak{h}.$$

If $g \in \mathfrak{h}_{\bar{1}}$, then formula (3.3.6) holds automatically by the parity considerations. Let $g \in \mathfrak{h}_{\bar{0}}$. Then, nevertheless,

$$\chi([f, g \otimes \xi_1\xi_2]) = \chi([f, g] \otimes \xi_1\xi_2) = 0$$

and, therefore, $\chi|_{[\mathfrak{h}_{\bar{0}},\mathfrak{h}_{\bar{0}}]\otimes\xi_1\xi_2} = 0$.

Set

$$\mathfrak{m} = \mathfrak{h} \otimes (\Lambda^1(\xi) \oplus \Lambda^2(\xi)).$$

Clearly, $\mathfrak{m}$ is an ideal of $\mathfrak{k}$ and $\mathfrak{st}_{\mathfrak{n}}(\chi) = \mathfrak{m}$. Therefore, the irreducible representation ρ of $\mathfrak{m}$ given by χ is such that $\rho|_{\mathfrak{n}}$ is a multiple of χ. Let

$$\mathfrak{n}_1 = \mathfrak{h}_{\bar{0}} \otimes \Lambda^1(\xi) \oplus ([\mathfrak{h}_{\bar{0}}, \mathfrak{h}_{\bar{0}}] \oplus \mathfrak{h}_{\bar{1}}) \otimes \Lambda^2(\xi).$$

Clearly, $\mathfrak{n}_1$ is an ideal in $\mathfrak{m}$ and

$$[\mathfrak{n}_1, \mathfrak{m}] \subset ([\mathfrak{h}_{\bar{0}}, \mathfrak{h}_{\bar{0}}] \oplus \mathfrak{h}_{\bar{1}}) \otimes \Lambda^2(\xi) \subset \ker\ \chi.$$

Thus, $\mathfrak{n}_1 \subset \ker\ \rho$ since the subspace $V^{\rho(\mathfrak{n}_1)} = \{v \in V \mid \rho(\mathfrak{n}_1)v = 0\}$ is $\rho(\mathfrak{m}_1)$-invariant and non-zero. But $\mathfrak{m}/\mathfrak{n}_1$ is a solvable Lie algebra and by the Lie theorem any its irreducible finite dimensional representation is of the same form as χ. Let it be χ, for definiteness sake.

This means that $\chi|_{[\mathfrak{h}_{\bar{1}},\mathfrak{h}_{\bar{1}}]\otimes\Lambda^2(\xi)} = 0$, i.e., $\chi|_{\mathfrak{h}\otimes\Lambda^2(\xi)} = 0$ and since $\mathfrak{h} \otimes \xi_1\xi_2$ is an ideal in $\mathfrak{k}$, then the representation of $\mathfrak{k}$ given by χ cannot be faithful. Lemma 3.3.6.2 is proved. □

3.3.6.7. Central extensions of semisimple Lie superalgebras. Let $\mathfrak{g}$ be a Lie superalgebra, $\mathfrak{r}$ its radical of dimension 1. Then $\mathfrak{r}$ is the center of $\mathfrak{g}$ and $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{r}$ is semisimple.

First, consider the case when $\tilde{\mathfrak{g}} = \underset{i\leq k}{\oplus} \mathfrak{s}_i$, where the $\mathfrak{s}_i$ are almost simple and $k > 1$. Let $\pi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ be the natural projection. The Lie superalgebra $\mathfrak{k} := \pi^{-1}(\mathfrak{s}_1)$ is an ideal in $\mathfrak{g}$ and $\dim \mathfrak{k} > 1$.

3.3.6.8. Lemma . *If $k > 1$, then ρ can not be irreducible of type A with respect to $\mathfrak{k}$.*

Proof. Set $\mathfrak{g}_+ = \underset{i\geq 1}{\oplus} (\mathfrak{s}_i)_{\bar{0}}$. Clearly, $[\mathfrak{s}_1, \mathfrak{g}_+] = 0$. Since $\rho(\mathfrak{r})$ acts by scalar operators and ρ is a finite dimensional representation, it follows that $[\pi^{-1}(\mathfrak{g}_+), \mathfrak{k}] = 0$, i.e., $\rho(\pi^{-1}(\mathfrak{g}_+))$ consists of intertwining operators of $\rho|_{\mathfrak{n}}$, i.e., $\rho(\pi^{-1}(\mathfrak{g}_+)) = \mathbb{C}\cdot 1$ and, since ρ is faithful, then $\pi^{-1}(\mathfrak{g}_+) = \mathfrak{r}$ implying $\mathfrak{g}_+ = 0$. □

3.3.6.9. Now, let $\tilde{\mathfrak{g}}$ contain an ideal $\mathfrak{k}$ of the form (3.3.6). The central extension is defined by a cocycle $c : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \longrightarrow \mathbb{C}$. The cocycle condition is

$$c(f, [g, h]) = c([f, g], h) + (-1)^{p(f)p(g)}c(g, [f, h]) \quad \text{for any} f, g, h \in \tilde{\mathfrak{g}}.$$

As earlier, we assume that $\mathfrak{g}$ has a faithful finite dimensional representation; so the restriction of c to $\tilde{\mathfrak{g}}_{\bar{0}} \times \tilde{\mathfrak{g}}_{\bar{0}}$ is trivial. Besides, $c|_{\tilde{\mathfrak{g}}_{\bar{0}}\times\tilde{\mathfrak{g}}_{\bar{1}}} = 0$ by parity considerations. Therefore,

$$c|_{\tilde{\mathfrak{g}}_{\bar{0}}\times\tilde{\mathfrak{g}}} = 0. \tag{3.5}$$

Lemma . *Set $\mathfrak{L}^m = \mathfrak{h} \otimes \Lambda^m(\xi_1, \dots, \xi_n)$. We have $c|_{\mathfrak{L}^n\times\mathfrak{L}^n} = 0$.*

Proof. If $n = 2k + 1$ the condition (3.5) means that

$$c|_{\mathfrak{h}_{\bar{1}}\otimes\Lambda^n(\xi)\times\mathfrak{h}_{\bar{1}}\otimes\Lambda^n(\xi)} = 0.$$

Let $g_{\bar{1}}, h_{\bar{1}} \in \mathfrak{h}_{\bar{1}}$ and $h_{\bar{0}} \in \mathfrak{h}_{\bar{0}}$. Then

$$\begin{gathered} c(f_{\bar{0}} \otimes \xi_1 \cdot\dots\cdot \xi_n, [g_{\bar{1}}, h_{\bar{1}}] \otimes \xi_1 \cdot\dots\cdot \xi_n) = -c(f_{\bar{0}} \otimes \xi_1 \cdot\dots\cdot \xi_n, [g_{\bar{1}} \otimes \xi_1 \cdot\dots\cdot \xi_n, h_{\bar{1}}]) = \\ -c([f_{\bar{0}} \otimes \xi_1 \cdot\dots\cdot \xi_n, g_{\bar{1}} \otimes \xi_1 \cdot\dots\cdot \xi_n], h_{\bar{1}}) - c(g_{\bar{1}} \otimes \xi_1 \cdot\dots\cdot \xi_n, [f_{\bar{0}} \otimes \xi_1 \cdot\dots\cdot \xi_n, h_{\bar{1}}]) = 0. \end{gathered}$$

Since $\mathfrak{h}_{\bar 0} = [\mathfrak{h}_{\bar 1}, \mathfrak{h}_{\bar 1}]$ (by Lemma 3.3.6.5), it follows that $c|_{\mathfrak{L}^n\times\mathfrak{L}^n} = 0$.

If $n = 2k$ the condition (3.5) means that

$$c|_{\mathfrak{h}_{\bar 0}\otimes\Lambda^n(\xi)\times\mathfrak{h}_{\bar 0}\otimes\Lambda^n(\xi)} = 0.$$

Let $f_{\bar 1}, h_{\bar 1} \in \mathfrak{h}_{\bar 1}$ and $g_{\bar 0} \in \mathfrak{h}_{\bar 0}$. Then

$$c(f_{\bar 1}\otimes\xi_1\cdot\dots\cdot\xi_n, [g_{\bar 0}, h_{\bar 1}]\otimes\xi_1\cdot\dots\cdot\xi_n) = -c(f_{\bar 1}\otimes\xi_1\cdot\dots\cdot\xi_n, [g_{\bar 0}\otimes\xi_1\cdot\dots\cdot\xi_n, h_{\bar 1}]) =$$
$$-c([f_{\bar 1}\otimes\xi_1\cdot\dots\cdot\xi_n, g_{\bar 0}\otimes\xi_1\cdot\dots\cdot\xi_n], h_{\bar 1}) - c(g_{\bar 0}\otimes\xi_1\cdot\dots\cdot\xi_n, [f_{\bar 1}\otimes\xi_1\cdot\dots\cdot\xi_n, h_{\bar 1}]) = 0.$$

Since $\mathfrak{h}_{\bar 1} = [\mathfrak{h}_{\bar 0}, \mathfrak{h}_{\bar 1}]$ (by Lemma 3.3.6.5), it follows that $c|_{\mathfrak{L}^n\times\mathfrak{L}^n} = 0$. □

3.3.6.10. Lemma . *$c|_{\mathfrak{L}^k\times\mathfrak{L}^m} = 0$ for any $m = 0, 1, \dots, n$ and $k = n-m,\ n-m+1,\ \dots,\ n$.*

Proof. We will perform the inverse double induction on k and m. For $k = m = n$ Lemma 3.3.6.9 will do.

Let the statement be true for all $k > k_0$ and $k_0 + m \geq n$. Let us show that it is true for $k = k_0$ as well. Observe that due to [K2], $\tilde{\mathfrak{g}}$ contains n elements η_i such that $\mathrm{ad}\eta_i|_{\mathfrak{h}} = \partial_{\xi_i} + D_i$, where $D_i \in \mathfrak{vect}(0|n)$ and $\deg D_i > 0$ in the standard grading of $\mathfrak{vect}(0|n)$.

Let $\varphi \in \Lambda^{k_0+1}(\xi_1, \dots, \xi_n), \psi \in \Lambda^m(\xi_1, \dots, \xi_n)$; let $g, h \in \mathfrak{h}$ be such that $p(g\otimes\psi) = \bar 0$; $p(h\otimes\psi) = \bar 1$.

Then we have

$$\begin{aligned} c([\eta_i, g\otimes\varphi], h\otimes\psi) &= (-1)^{p(g)}c(g\otimes\eta_i(\varphi), h\otimes\psi) \\ &= (-1)^{p(g)}\left(c(g\otimes\tfrac{\partial\varphi}{\partial\xi_i}, h\otimes\psi) + c(g\otimes D_i\varphi, h\otimes\psi)\right). \end{aligned}$$

As to $D_i\varphi \in \left(\bigoplus\limits_{s\geq k_0+1}\Lambda^s(\xi)\right)$ from the last summand, it vanishes by the inductive hypothesis.

On the other hand,

$$c([\eta_i, g\otimes\varphi], h\otimes\psi) = c(\eta_i, [g\otimes\varphi, h\otimes\psi]) + c([\eta_i, h\otimes\psi], g\otimes\varphi).$$

As $\deg\varphi + \deg\psi = k_0 + 1 + m > n$, the bracket in the first summand above is equal to 0. The second summand is equal to 0 by (3.5). So $c(g\otimes\frac{\partial\varphi}{\partial\xi_i}, h\otimes\psi) = 0$, for arbitrary $g\otimes\frac{\partial\varphi}{\partial\xi_i} \in \mathfrak{L}^{k_0}_{\bar 1}$ and $h\otimes\psi \in \mathfrak{L}^m_{\bar 1}$. □

3.3.6.11. Lemma . *$\mathfrak{k}$ has no faithful irreducible finite dimensional representations.*

Proof. Word-for-word reproduction of the proof of Lemma 3.3.6.2 with the help of Lemmas 3.3.6.9 and 3.3.6.10. □

Corollary . *ρ can not be of type A with respect to the ideal $\mathfrak{k}$.*

□

Lemmas 3.3.2 (Lemma B), 3.3.3 (Lemma C), 3.3.4, 3.3.5 (Lemma A) and Lemma 3.3.6 put together prove Theorem 3.2. □

§4. Step 4. Prolongation of the induced representations

In this step we describe all the simple infinite dimensional $\mathbb{Z}$-graded Lie superalgebras of depth 1 for which the irreducible representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is induced from a representation of a subsuperalgebra $\mathfrak{h} \subset \mathfrak{g}_0$.

But, first of all, we describe certain universal Lie superalgebras.

4.1. Universality of $\mathfrak{vect}$, $\mathfrak{k}$ and $\mathfrak{m}$. Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded transitive Lie superalgebra.

1. If $\mathfrak{g}=\mathop{\oplus}\limits_{i=-1}^{\infty}\mathfrak{g}_i$ and $\dim\mathfrak{g}_{-1}=(m|n)$, then $\mathfrak{g}\subset\mathfrak{vect}(m|n)$.
2. If $\mathfrak{g}=\mathop{\oplus}\limits_{i=-2}^{\infty}\mathfrak{g}_i$ and $\mathfrak{g}_{<0}=\mathfrak{hei}(2m|n)$, then $\mathfrak{g}\subset\mathfrak{k}(2m+1|n)$.
3. If $\mathfrak{g}=\mathop{\oplus}\limits_{i=-2}^{\infty}\mathfrak{g}_i$ and $\mathfrak{g}_{<0}=\mathfrak{ab}(n)$ (see 0.5.2), then $\mathfrak{g}\subset\mathfrak{m}(n)$.

The following are all the Lie superalgebras with "big" nonpositive part:

$1'$. Let $\mathfrak{g}=\mathop{\oplus}\limits_{i=-1}^{\infty}\mathfrak{g}_i$ and $\dim\mathfrak{g}_{-1}=(m|n)$. Let, besides, $\mathfrak{g}_0\supset\mathfrak{sl}(m|n)$ and $\dim\mathfrak{g}=\infty$. If $m>1$, then, $\mathfrak{g}=\mathfrak{vect}(m|n)$, or $\mathfrak{svect}(m|n)$, or $\mathfrak{dsvect}(m|n)$. If $m=1$, then there are two more possibilities: $\mathfrak{g}=\mathfrak{svect}^\circ(1|n)$ and $\mathfrak{dsvect}^\circ(1|n)$.

$2'$. Let $\mathfrak{g}=\mathop{\oplus}\limits_{i=-2}^{\infty}\mathfrak{g}_i\subset\mathfrak{k}(2m+1|n)$ be such that $\mathfrak{g}_{\leq 0}=\mathfrak{k}(2m+1|n)_{\leq 0}$ and $\dim\mathfrak{g}=\infty$. Then $\mathfrak{g}=\mathfrak{k}(2m+1|n)$ or $\mathfrak{g}=\mathfrak{dpo}(2m|n)$ or, if $m=0$ and $n=6$, then $\mathfrak{g}=\mathfrak{kas}$.

$3'$. If $\mathfrak{g}=\mathop{\oplus}\limits_{i=-2}^{\infty}\mathfrak{g}_i\subset\mathfrak{m}(n)$ is such that $\mathfrak{g}_{<0}=\mathfrak{ab}(n)$, $\dim\mathfrak{g}=\infty$ and $\mathfrak{g}$ is simple, then $\mathfrak{g}=\mathfrak{m}(n)$ or $\mathfrak{g}=\mathfrak{b}_\infty(n)$ or $\mathfrak{g}=\mathfrak{b}_\lambda(n)$ for $\lambda\neq 0$.

The described universal properties are immediate corollaries of the properties of Cartan prolongs.

4.2. Lemma . *Let $\mathfrak{g}=\mathop{\oplus}\limits_{i=-1}^{\infty}\mathfrak{g}_i$ be a $\mathbb{Z}$-graded transitive Lie superalgebra and $W\subset\mathfrak{g}_{-1}$ a linear subsuperspace. Let $F\in\mathfrak{g}_k$, $k>0$, be such that $[w,F]=0$ for any $w\in W$. Then for any $u\in\mathfrak{g}_{-1}$ the commutator $[u,F]$ also commutes with W.*

Proof. By the Jacoby identity

$$[w,[u,F]]=(-1)^{p(w)p(u)}[u,[w,F]]+[[w,u],F]=0.\quad\square$$

Actually, Lemma 4.2 says that if the expression of F does not involve certain indeterminates, then $[\partial_i,F]$ does not involve these indeterminates for any i, either.

4.3. Theorem . *Let $\xi=(\xi_1,\dots,\xi_k)$ and V a linear superspace of dimension $m|n$, where $(m|n)\neq(1|0)$ or $(0|1)$.*

a) $(\Lambda(\xi),\Lambda(\xi)\Subset\mathfrak{vect}(\xi))_*=\mathfrak{k}(1|2k;k)$.
b) $(\Pi(\Lambda(\xi)),\Lambda(\xi)\Subset\mathfrak{vect}(\xi))_*=\mathfrak{m}(k;k)$.
c) $(V\otimes\Lambda(\xi),\mathfrak{gl}(V)\otimes\Lambda(\xi)\Subset\mathfrak{vect}(\xi))_*=\mathfrak{vect}(m|n+k;k)$.

Proof. It is clear that the Lie superalgebras in the rhs of a) – c) are subalgebras of the respective Lie superalgebras in the lhs. Let $\mathfrak{g}=\mathop{\oplus}\limits_{l=-1}^{\infty}\mathfrak{g}_l$ be a superalgebra from the lhs.

Let $y=-\mathrm{id}_V\otimes 1\in\mathfrak{gl}(V)\otimes\Lambda(\xi)$. Then $\mathrm{ad}y$ determines a natural $\mathbb{Z}$-grading in $\mathfrak{g}$.
Let $x=\Sigma\xi_i\partial_{\xi_i}\in\mathfrak{g}_0$. Then $\mathrm{ad}x$ determines a $\mathbb{Z}$-grading in each component $\mathfrak{g}_l$.
Let $\mathfrak{g}_{l,i}=\{g\in\mathfrak{g}_l\mid \mathrm{ad}x(g)=i\cdot g\}$. Then $\mathfrak{g}=\mathop{\oplus}\limits_{l,i}\mathfrak{g}_{l,i}$.

Observe that $\mathfrak{g}_{-1}=\mathop{\oplus}\limits_{i\geq 0}\mathfrak{g}_{-1,i}$ and $\mathfrak{g}_0=\mathop{\oplus}\limits_{i\geq -1}\mathfrak{g}_{0,i}$. Set

$$W=\mathfrak{g}_{-1,0}\oplus\mathfrak{g}_{-1,1}=(V\otimes 1)\oplus(V\otimes\Lambda^1(\xi)).$$

We see that $\mathfrak{g}_0$ does not contain nonzero elements that commute with W. Hence, by Lemma 4.2 none of the $\mathfrak{g}_l$ for $l>0$ contain nonzero elements that commute with W, either.

But if $\mathfrak{g}_l=\mathop{\oplus}\limits_{i\geq -s}\mathfrak{g}_{1,i}$, then the components $\mathfrak{g}_{l+1,t}$ with $t<-s-1$ should commute with W. Hence, $\mathfrak{g}_{l+1}=\mathop{\oplus}\limits_{i\geq -s-1}\mathfrak{g}_{1+1,i}$ and $\mathfrak{g}_l=\mathop{\oplus}\limits_{i\geq -l-1}\mathfrak{g}_{1,i}$.

Cases a) and b). Consider the operator $T = \mathrm{ad}(2y + x)$. Its eigenvalues are integers and in each $\mathfrak{g}_l$ they are not less then $l-1$ of linearly independent eigenvectors. Set $\mathfrak{g}^T_s = \{g \in \mathfrak{g} \mid T(g) = s \cdot g\}$. Clearly, $\mathfrak{g}^T_s \subset \mathop{\oplus}\limits_{l=-1}^{s+1} \mathfrak{g}_l$ and $\dim \mathfrak{g}^T_s < \infty$. Hence, in $\mathfrak{g}$, we obtain a new $\mathbb{Z}$-grading: $\mathfrak{g}^T = \mathop{\oplus}\limits_{s=-2}^{\infty} \mathfrak{g}^T_s$. The negative part of this $\mathbb{Z}$-grading is as follows:

$$\begin{aligned}
&\mathfrak{g}^T_{-2} = \mathfrak{g}_{-1,0} = \mathbb{C}\cdot 1 \text{ (case a), or } \mathbb{C}\cdot\pi(1) \text{ (case b) ;}\\
&\mathfrak{g}^T_{-1} = \mathfrak{g}_{-1,1} \oplus \mathfrak{g}_{0,-1} = Span\{\xi_1, \dots, \xi_k, \partial_{\xi_1}, \dots, \partial_{\xi_k}\} \text{ (case a), or}\\
&\qquad\qquad Span\{\pi(\xi_1), \dots, \pi(\xi_k), \partial_{\xi_1}, \dots, \partial_{\xi_k}\} \text{ (case b).}
\end{aligned}$$

Let us check that $\mathfrak{g}^T$ is transitive, i.e., if $g \in \mathfrak{g}^T_l$ with $l \geq 0$ and $[\mathfrak{g}^T_{-1}, g] = 0$, then $g = 0$. Indeed, for such $g \in \mathfrak{g}$ we have $[W, g] = 0$; hence, $g \in \mathfrak{g}_{-1}$. But only the subspace $\mathfrak{g}^T_{-2} = \mathfrak{g}_{-1,0} \subset \mathfrak{g}_{-1}$ commutes with the ∂_{ξ_i} for all i.

Due to the universality of the Lie superalgebras of series $\mathfrak{k}$ and $\mathfrak{m}$ (see sec. 4.1) this means that

in case a) $\mathfrak{g}^T \subset \mathfrak{k}(1|k)$ and thus $\mathfrak{g}^T = \mathfrak{k}(1|k)$;

in case b) $\mathfrak{g}^T \subset \mathfrak{m}(k)$ and thus $\mathfrak{g}^T = \mathfrak{m}(k)$.

Case c). Set $W_0 = \mathfrak{g}_{-1,0} = V \otimes 1$.

4.3.1. Lemma . *If* $\dim V \neq (1|0)$ *or* $(0|1)$ *and* $F \in \mathfrak{g}_1$ *is such that* $[F, W_0] = 0$, *then* $F = 0$. *In particular,* $\mathfrak{g}_{1,-2} = 0$.

Proof. Due to Lemma 4.2 the element $[F, v \otimes \varphi] \in \mathfrak{g}_0$ commutes with W_0 for any $v \in V$ and $\varphi \in \Lambda(\xi)$. Hence, $[F, v \otimes \varphi]$ belongs to $\mathfrak{vect}(0|k) \subset \mathfrak{g}_0$.

Denote $[F, v \otimes \varphi]$ by $D_{v,\varphi}$. By transitivity, $D_{v,\varphi}\psi \neq 0$ for some $v \in V$ and $\varphi, \psi \in \Lambda(\xi)$. Take $u \in V$ linearly independent with v. Then

$$\begin{aligned}
&[[F, v \otimes \varphi], u \otimes \psi] \pm [v \otimes \varphi, [F, u \otimes \psi]] =\\
&\pm u \otimes D_{v,\varphi}\psi \pm v \otimes D_{u,\psi}\varphi \neq 0.
\end{aligned} \tag{4.1}$$

(The symbol $\pm$ stands for a sign; this sign is not essential to us.) By the Jacobi identity (4.1) should vanish. This contradiction completes the proof. □

Lemmas 4.2 and 4.3.1 together mean that none of the $\mathfrak{g}_l$ with $l > 0$ contains elements that commute with W_0. Hence, the least degree of $\mathbb{Z}$-grading wrt ξ in $\mathfrak{g}_{l+1}$ cannot be less than same in $\mathfrak{g}_l$ for $l \geq 0$. Thus, $\mathfrak{g}_l = \mathop{\oplus}\limits_{i\geq -1} \mathfrak{g}_{l,i}$.

Consider operator $T = \mathrm{ad}(y + x)$. Its eigenvalues are integers and in each $\mathfrak{g}_l$ they are not less then $l-1$ of linearly independent eigenvectors. So, as in cases a) and b), $\dim \mathfrak{g}^T_s = \dim\{g \in \mathfrak{g} \mid T(g) = s \cdot g\} < \infty$. Hence, in $\mathfrak{g}$, we get a new $\mathbb{Z}$-grading:

$$\mathfrak{g}^T = \mathop{\oplus}\limits_{s=-1}^{\infty} \mathfrak{g}^T_s.$$

The negative part of this $\mathbb{Z}$-grading is

$$\mathfrak{g}^T_{-1} = \mathfrak{g}_{-1,0} \oplus \mathfrak{g}_{0,-1} = V \otimes 1 \oplus Span(\partial_{\xi_1}, \dots, \partial_{\xi_k}).$$

Moreover, by Lemma 4.3.1 the elements of $\mathfrak{g}$ that commute with $W_0 = V \otimes 1 \subset \mathfrak{g}^T_{-1}$ form the subspace $\mathfrak{g}_{-1} \oplus \mathfrak{vect}(0|k)$. We can directly see that the elements of this subspace that commute with all ∂_{ξ_i} span $\mathfrak{g}^T_{-1}$. Hence, $\mathfrak{g}^T$ is transitive. Due to the universal property 4.1 of the Lie superalgebra $\mathfrak{vect}(m|n+k)$ this means that $\mathfrak{g}^T \subset \mathfrak{vect}(m|n+k)$; thus, $\mathfrak{g}^T = \mathfrak{vect}(m|n+k)$. □

4.3.2. Corollary . 1) *If* $\mathfrak{g} \subset \mathfrak{k}(1|2k;k)$ *with* $\mathfrak{g}_{-1} = \mathfrak{k}(1|2k;k)_{-1} = \Lambda(\xi_1, \cdots, \xi_k)$ *is such that* $\mathfrak{g}_1$ *does not contain nonzero elements that commute with the subspace* $W_0 = \mathbb{C} \cdot 1 \subset \mathfrak{g}_{-1}$*, then* $\mathfrak{g} \subset \mathfrak{vect}(1|k;k)$.

2) *If* $\mathfrak{g} \subset \mathfrak{m}(k;k)$ *with* $\mathfrak{g}_{-1} = \mathfrak{m}(k;k)_{-1} = \Pi(\Lambda(\xi_1, \cdots, \xi_k))$ *is such that* $\mathfrak{g}_1$ *does not contain nonzero elements that commute with the subspace* $W_0 = \mathbb{C} \cdot \Pi(1) \subset \mathfrak{g}_{-1}$*, then* $\mathfrak{g} \subset \mathfrak{vect}(0|k+1;k)$*; hence,* $\dim \mathfrak{g} < \infty$.

The proof is the same as for case c) of Theorem 4.3. □

Now we are able to return to the classification.

4.4. Let, as above, $\xi = (\xi_1, \dots, \xi_k)$ and V an $(m|n)$-dimensional linear superspace. We denote the coordinates in V by $u = (u_1, \dots, u_{n+m})$.

Theorem . *Let* $\mathfrak{g} = \mathop{\oplus}\limits_{i=-1}^{\infty} \mathfrak{g}_i$ *be a simple infinite dimensional transitive* $\mathbb{Z}$*-graded Lie superalgebra with* $\mathfrak{g}_{-1} = V \otimes \Lambda(\xi)$ *and* $\mathfrak{g}_0 \subset \mathfrak{gl}(V) \otimes \Lambda(\xi) \Subset \mathfrak{vect}(\xi)$.

1) *If* $\dim V > 1$*, then* $\mathfrak{g}$ *is either* $\mathfrak{vect}(1|n+k;k)$ *or* $\mathfrak{svect}(m|n+k;k)$ *(for* $m > 1$*) or* $\mathfrak{svect}^\circ(m|n+k;k)$*;*

2) *if* $\dim V = (1,0)$ *then* $\mathfrak{g}$ *is either* $\mathfrak{vect}(1|k;k)$ *or* $\mathfrak{svect}^\circ(1|k;k)$ *or* $\mathfrak{k}(1|2k;k)$ *or* $\mathfrak{kas}^\xi(;\xi)$.

3) *if* $\dim V = (0,1)$*, then* $\mathfrak{g}$ *is either* $\mathfrak{m}(k;k)$ *or* $\mathfrak{b}_\lambda(k;k)$ *for* $\lambda \in \mathbb{C}P^1 \setminus \{0\}$ *.*

Proof. Case 1). By Theorem 4.3 $\mathfrak{g} \subset \mathfrak{vect}(m|n+k;k)$. Thus, we can use the realization of elements of $\mathfrak{g}$ by vector fields in u, ξ. The $\mathbb{Z}$-grading wrt ξ in each component $\mathfrak{vect}(m|n+k;k)_l$ determines a filtration in $\mathfrak{g}_l \subset \mathfrak{g}$.

Let $\mathrm{gr}(\mathfrak{g}) = \mathop{\oplus}\limits_{l=-1}^{\infty} \mathop{\oplus}\limits_{i} \mathrm{gr}(\mathfrak{g})_{l,i}$ be the corresponding bigraded Lie superalgebra, i.e.,

$$\mathrm{gr}(\mathfrak{g})_{l,i} = \frac{\mathfrak{g}_l \cap \mathop{\oplus}\limits_{j\geq i} \mathfrak{vect}(m|n+k;k)_{l,j}}{\mathfrak{g}_l \cap \mathop{\oplus}\limits_{j>i} \mathfrak{vect}(m|n+k;k)_{l,j}}.$$

By Lemma 3.1 the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible. This gives us the following conditions on $\mathrm{gr}(\mathfrak{g})_{0,*} := \mathop{\oplus}\limits_{i} \mathrm{gr}(\mathfrak{g})_{0,i}$:

$$\mathrm{gr}(\mathfrak{g})_{0,-1} = Span\{\partial_{\xi_1}, \dots, \partial_{\xi_k}\} = \mathfrak{vect}(m|n+k;k)_{0,-1}; \tag{4.2}$$

$$\begin{gathered}\mathrm{gr}(\mathfrak{g})_{0,0} \subset \mathfrak{vect}(m|n+k;k)_{0,0} = \mathfrak{gl}(V) \oplus \mathfrak{gl}(\xi) = \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(0|k) \\ \text{acts irreducibly in } V \otimes 1 \subset \mathfrak{g}_{-1}.\end{gathered} \tag{4.3}$$

4.4.1. Without loss of generality we may assume that the $\mathrm{gr}(\mathfrak{g})_{0,0}$-module $\mathrm{gr}(\mathfrak{g})_{0,-1} = Span(\partial_{\xi_i} \mid i = 1, \dots, k)$ is irreducible. Indeed, if it is reducible and, e.g., $Span(\partial_{\xi_1}, \dots, \partial_{\xi_s})$ is a submodule, then

$$\mathfrak{g} \subset \Big((\mathfrak{gl}(V) \otimes \Lambda(\xi_1, \dots, \xi_s) \Subset \mathfrak{vect}(\xi_1, \dots, \xi_s)) \otimes \Lambda(\xi_{s+1}, \dots, \xi_k)\Big) \Subset \mathfrak{vect}(\xi_{s+1}, \dots, \xi_k).$$

So we set $\tilde{V} = V \otimes \Lambda(\xi_1, \dots, \xi_s)$ and $\tilde{\xi} = (\xi_{s+1}, \cdots, \xi_k)$. Now let us use the condition $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$. As we have seen in the proof of Theorem 4.3,

$$\mathfrak{vect}(m|n+k;k)_1 = \mathop{\oplus}\limits_{i\geq -1} \mathfrak{vect}(m|n+k;k)_{1,i}$$

and the component $\mathrm{gr}(\mathfrak{g})_{0,-1}$ can be only obtained by commuting $\mathrm{gr}(\mathfrak{g})_{-1,0} = V \otimes 1 = Span(\partial_{u_1}, \dots, \partial_{u_{m+n}})$ with $\mathfrak{vect}(m|n+k;k)_{1,-1} = Span(u_i\partial_{\xi_j} \mid \text{all possible } i,j)$.

By (4.2), for each $j = 1, \cdots, k$, there exists $u_\alpha\partial_{\xi_j} \in \mathrm{gr}(\mathfrak{g})_{1,-1}$. But, due to (4.3), $\mathrm{gr}(\mathfrak{g})_{0,0}$ acts irreducibly on V^*. Thus, $\mathrm{gr}(\mathfrak{g})_{1,-1}$ contains the elements of the form $u_\alpha\partial_{\xi_j}$ for any $\alpha = 1, \dots, m+n$ and $j = 1, \dots, k$. Hence, $\mathrm{gr}(\mathfrak{g})_{1,-1} = \mathfrak{vect}(m|n+k;k)_{1,-1}$.

Thus, for any $\alpha, \beta = 1, \dots, m+n$ and $i, j = 1, \dots, k$ we have:

$$[\xi_i \partial_{u_\beta}, u_\alpha \partial_{\xi_j}] = \delta_{\alpha\beta}\xi_i\partial_{\xi_j} - (-1)^{(p(u_\alpha)+1)(p(u_\beta)+1)}\delta_{ij}u_\alpha\partial_{u_\beta} \in \mathrm{gr}(\mathfrak{g})_{0,0}.$$

Consequently, $\mathrm{gr}(\mathfrak{g})_{0,0} \supset (\mathfrak{gl}(V) \oplus \mathfrak{gl}(\xi)) \cap \mathfrak{svect}(m|n+k)$.

Now, consider the regrading $\mathfrak{g}^T$ of $\mathrm{gr}(\mathfrak{g})$ into the standard $\mathbb{Z}$-grading of $\mathfrak{vect}(m|n+k)$. Then

$$\mathfrak{g}^T_{-1} = \mathrm{gr}(\mathfrak{g})_{-1,0} \oplus \mathrm{gr}(\mathfrak{g})_{0,-1} = \mathfrak{vect}(m|n+k;k)_{-1};$$
$$\mathfrak{g}^T_0 = \mathrm{gr}(\mathfrak{g})_{-1,1} \oplus \mathrm{gr}(\mathfrak{g})_{0,0} \oplus \mathrm{gr}(\mathfrak{g})_{1,-1} \supset \mathfrak{sl}(m|n+k).$$

Next, by property $1'$, sect. 4.1, we see that $\mathrm{gr}(\mathfrak{g}) = \mathfrak{vect}(m|n)$, or $\mathfrak{svect}(m|n)$, or $\mathfrak{dsvect}(m|n)$ for $m > 1$.

If $m = 1$, then there are two more possibilities: $\mathrm{gr}(\mathfrak{g}) = \mathfrak{svect}^\circ(1|n)$ and $\mathrm{gr}(\mathfrak{g}) = \mathfrak{dsvect}^\circ(1|n)$.

Case 2). If $\dim V = (1|0)$, then $\mathfrak{g}_{-1} = \Lambda(\xi)$ and $\mathfrak{g}_0 \subset \Lambda(\xi) \in \mathfrak{vect}(0|k)$.

By Corollary 4.3.2, if the component $\mathfrak{g}_1$ does not contain nonzero elements that commute with $\mathfrak{g}_{-1,0} = \mathbb{C}\cdot 1$, then $\mathfrak{g} \subset \mathfrak{vect}(1|k;k)$ and by above arguments $\mathfrak{g} = \mathfrak{vect}(1|k;k)$ or $\mathfrak{svect}^\circ(1|k;k)$.

Let the component $\mathfrak{g}_1$ contain nonzero elements that commute with $\mathfrak{g}_{-1,0} = \mathbb{C}\cdot 1$. Then $\mathfrak{g} \subset \mathfrak{k} := \mathfrak{k}(1|2k;k)$.

In terms of the standard indeterminates $t, \xi_1, \dots, \xi_k, \eta_1, \dots, \eta_k$ for $\mathfrak{k}$ we have

$$\begin{aligned}
&\mathfrak{k}_{-1} = \mathfrak{g}_{-1} = \Lambda(\xi);\\
&\Lambda(\xi) \otimes (t - \textstyle\sum \xi_i\eta_i) \cong \Lambda(k) \subset \mathfrak{k}_0;\\
&\Lambda(\xi) \otimes \Lambda^1(\eta) \cong \mathfrak{vect}(0|k) \subset \mathfrak{k}_0;\\
&\mathfrak{g}_1 = t^1 \otimes \Lambda(\xi) \bigoplus t \otimes \Lambda(\xi) \otimes \Lambda^1(\eta) \bigoplus \Lambda(\xi) \otimes \Lambda^2(\eta).
\end{aligned}$$

The subspace $W_1 \subset \mathfrak{k}_1$ of elements that commute with $\mathfrak{g}_{-1,0}$ coincides with $\Lambda(\xi) \otimes \Lambda^2(\eta)$.

Consider $\mathrm{gr}(\mathfrak{g})$. By (4.2), $\mathrm{gr}(\mathfrak{g})_{0,-1} = Span(\eta_1, \dots, \eta_k)$ and η_i acts in W_1 as ∂_{ξ_i}.

Since $\mathfrak{g}_1 \cap W_1 \neq 0$, then $\mathfrak{g}_1 \cap \mathfrak{k}_{1,-2} \neq 0$, where $\mathfrak{k}_{1,-2} = \Lambda^2(\eta) \otimes 1$. Hence, $\mathrm{gr}(\mathfrak{g})_{1,-2} \neq 0$. Set

$$\mathfrak{h} = \mathop{\oplus}_{i=-1}^{k-1} \mathfrak{h}_i = \mathrm{gr}(\mathfrak{g})_{0,*} \cap \Lambda^1(\eta) \otimes \Lambda(\xi).$$

Clearly, $\mathfrak{h} \supset [\mathfrak{g}_{-1}, \mathrm{gr}(\mathfrak{g})_{1,-2}]$.

4.4.2. Lemma . *$\mathfrak{h}_{k-2}$ is not contained in $\mathfrak{svect}(0|k)_{k-2}$. In particular, $\mathfrak{h}_{k-2} \neq 0$.*

Proof. Let $X = \xi_1 \cdots \xi_k \in \mathfrak{g}_{-1}$. Then

$$\begin{aligned}
&[X, \sum_{i<j} \alpha_{ij}\eta_i\eta_j] = (-1)^{k+1} \sum_{i<j} \alpha_{ij}(\tfrac{\partial X}{\partial \xi_i}\eta_j - \tfrac{\partial X}{\partial \xi_j}\eta_i) \mapsto\\
&(-1)^{k+1} \sum_{i<j} \alpha_{ij}(\tfrac{\partial X}{\partial \xi_i}\partial_{\xi_j} - \tfrac{\partial X}{\partial \xi_j}\partial_{\xi_i}) = D.
\end{aligned}$$

But $\mathrm{div} D = 2(-1)^{k+1} \sum_{i<j} \alpha_{ij} \frac{\partial^2 X}{\partial \xi_i \partial \xi_j}$. So $\mathrm{div} D = 0$ if and only if $\alpha_{ij} = 0$ for all i, j. Since $\mathrm{gr}(\mathfrak{g})_{1,-2} \neq 0$, this proves Lemma. □

We have established that:

1) $\mathfrak{h} \subset \mathfrak{vect}(0|k)$;

2) $\mathfrak{h}_{-1} = \mathfrak{vect}(0|k)_{-1}$ and $\mathfrak{h}_{-1}$ is an irreducible $\mathfrak{h}_0$-module;

3) $\mathfrak{h}_{k-2}$ is not contained in $\mathfrak{svect}(0|k)_{k-2}$.

Comparing these conditions with the list in [K2] (p.71, Th. 4) we conclude that $\mathfrak{h} = \mathfrak{vect}(0|k)$ for $n > 3$. If $n = 3$, then, additionally, $\mathfrak{h}$ can be equal to $\mathfrak{sl}(1|3) \subset \mathfrak{vect}(0|3)$. Anyway, $\mathrm{gr}(\mathfrak{g})_{0,*} \supset \Lambda^1(\eta) \otimes \Lambda^1(\xi) \cong \mathfrak{gl}(k)$. Thus, by the irreducibility of the exterior square of the standard $\mathfrak{gl}(k)$-module, $\mathrm{gr}(\mathfrak{g})_1 \supset \Lambda^2(\eta)$.

The irreducibility of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ means that $\mathrm{gr}(\mathfrak{g})_{0,*}$ should contain at least some elements from $\Lambda(\xi)\otimes(t-\sum\xi_i\eta_i)$. Hence, $t\in\mathrm{gr}(\mathfrak{g})_{0,*}$.

Now we are able to consider the regrading $\mathfrak{g}^T$ of $\mathrm{gr}(\mathfrak{g})$ into the standard $\mathbb{Z}$-grading of $\mathfrak{k}(1|2k)$. By the above, $\mathfrak{g}^T_{\leq 0}=\mathfrak{k}(1|2k)_{\leq 0}$. Hence, $\mathfrak{g}^T$ is either $\mathfrak{k}(1|2k)$ or $\mathfrak{kas}^\xi\subset\mathfrak{k}(1|6)$ and $\mathfrak{g}=\mathfrak{k}(1|2k;k)$ or $\mathfrak{kas}^\xi(;\xi)$, respectively.

Case 3). If $\dim V=(0|1)$, then $\mathfrak{g}_{-1}=\Pi(\Lambda(\xi))$, $\mathfrak{g}_0\subset\Lambda(\xi)\Subset\mathfrak{vect}(\xi)$ and $\mathfrak{g}\subset\mathfrak{m}(k;k)$. The operators of rank 1 with even covector in $\Lambda(\xi)\Subset\mathfrak{vect}(\xi)$ are

$$g_i=\xi_1\cdots\xi_n\partial_i,\ \text{ and }\ h_i=\xi_1\dots\widehat{\xi_i}\dots\xi_n\partial_i\ \text{ for }\ i=1,\dots,n.$$

By Theorem 2.1 $\mathfrak{g}_0$ contains at least one of them. Sect. 4.4.1 implies that $\mathfrak{g}_0$ contains either all the g_i or all the h_i.

a) Let $\mathfrak{g}_0\supset Span(g_1,\dots,g_n)$. Then from (4.2) we deduce that $\mathfrak{g}_0\supset\mathfrak{vect}(\xi_1,\dots\xi_n)$. Since the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible, $\mathfrak{g}_0=\Lambda(\xi)\Subset\mathfrak{vect}(\xi)$.

b) For the same reason, if $\mathfrak{g}_0\supset Span(h_1,\dots,h_n)$, then $\mathfrak{g}_0\supset\mathfrak{svect}(\xi)$ and the following cases are possible:

1) $\mathfrak{g}_0\cong\mathfrak{g}_0^\lambda=\{v+\lambda\cdot\mathrm{div}\,(v)\mid v\in\mathfrak{vect}(\xi)\}$, where $\lambda\neq 0$;
2) $\mathfrak{g}_0\cong Vol_0(0|n)\Subset\mathfrak{svect}(\xi)$;
3) $\mathfrak{g}_0\cong\Lambda(\xi)\Subset\mathfrak{svect}(\xi)$.

In all the cases $\mathfrak{g}_0$ is $\mathbb{Z}$-graded by powers of ξ and $\mathfrak{g}_{0,0}\supset\mathfrak{sl}(0|k)$. Due to Corollary 4.3.1 $\mathfrak{g}_{1,-2}\neq 0$; hence, $\mathfrak{g}_{1,-2}=S^2(\mathrm{id})$.

Consider the regrading $\mathfrak{g}^T$ of $\mathfrak{g}$ into the standard $\mathbb{Z}$-grading of $\mathfrak{m}(k)$. We get $\mathfrak{g}^T_{<0}=\mathfrak{m}_{<0}$ and $\mathfrak{g}_0\supset\mathfrak{spe}(k)$. Then by universal property $3'$ of sec. 4.1 $\mathfrak{g}=\mathfrak{m}(k)$ or $\mathfrak{b}_\lambda(k)$, where $\lambda\in\mathbb{C}P^1\setminus\{0\}$. □

4.5. Corollary . *For the simple vectorial Lie superalgebra $\mathfrak{g}$ with W-filtration the Lie superalgebra $\mathfrak{g}_0$ is solvable in the following cases only: $\mathfrak{g}=\mathfrak{vect}(1|1)$ or $\mathfrak{svect}^\circ(1|2;1)$ or $\mathfrak{vect}(1|2;1)$ and the corresponding $\mathfrak{g}_0$ is isomorphic to $\mathfrak{gl}(1|1)$ or $\mathfrak{sl}(1|1)\otimes\Lambda(1)\Subset\mathfrak{vect}(0|1)$ or $\mathfrak{gl}(1|1)\otimes\Lambda(1)\Subset\mathfrak{vect}(0|1)$, respectively.*

§5. Step 5. Prolongations of almost simple Lie superalgebras and their central extensions

In this step there are two essentially different cases: vectorial and non-vectorial Lie superalgebras (Steps 5B and 5A, respectively).

Let $\mathfrak{g}$ be an almost simple Lie superalgebra or a central extension of an almost simple classical Lie superalgebra (all such Lie superalgebras are listed in Tables 8.1 and 8.2). Let T be a faithful irreducible representation of $\mathfrak{g}$ in the finite dimensional superspace V. Our goal is to describe all the collections $(\mathfrak{g},T,V)$ for which there exists a rank 1 operator $T(g)$, $g\in\mathfrak{g}$, with an even covector.

Irreducible representations of simple vectorial Lie superalgebras and their "relatives" — central extensions of almost simple Lie superalgebras related to them — are completely described, and to make exposition selfcontained we give an account in Appendix. From this description we easily deduce the list of admissible collections $(\mathfrak{g},T,V)$.

For non-vectorial Lie superalgebras, there is a complete description of the typical (i.e., generic) representations (Berezin [B1], [B2] and Kac [K5]). But (bar a couple of exceptions) the rank 1 operators occure, if ever, in atypical representations whose description is only conjectural.

The established results concerning atypical representations are, so far, only partial (for reviews covering various aspects of the problem see [L2], [KW], [Se1]).

Though we share the belief in the validity of the remarkable general formula of Penkov and Serganova, we should better rely on something else: it is conjectural at the moment. Therefore, our proof in Step 5A does not appeal to the description of irreducible representations. Inevitably then, the proof is rather lengthy.

Our method of description and of search after such collections $(\mathfrak{g}, T, V)$ is based on pretty straightforward considerations which, nevertheless, provide with essential restrictions on the weight diagram of T.

Step 5A (sec. 5.0–5.6). Non-vectorial Lie superalgebras

To help the reader, we have gathered main results of Step 5A in Proposition 5.0.

5.0. Proposition . *For the almost simple non-vectorial Lie superalgebra $\mathfrak{g}_0$ or any central extension of such, the representations (T, V) with a rank 1 operator $T(g)$, $g \in \mathfrak{g}_0$, with an even covector are only the following ones (we have normed the action of the generator of the center of $\mathfrak{g}_0$ to be equal to* id*):*

$V = \mathfrak{g}_{-1}$	$\mathfrak{g}_0$	$(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$
id or id* $\Pi(\text{id})$ or $\Pi(\text{id}^*)$	$\mathfrak{sl}(m\|n)$ and $\mathfrak{gl}(m\|n)$ $\mathfrak{sl}(n\|m)$ and $\mathfrak{gl}(n\|m)$	$\mathfrak{svect}(m\|n)$ and $\mathfrak{vect}(m\|n)$ $\mathfrak{svect}(n\|m)$ and $\mathfrak{vect}(n\|m)$
id	$\mathfrak{osp}^{sk}(m\|2n)$ $\mathfrak{cosp}^{sk}(m\|2n)$	$\mathfrak{h}(2n\|m)$ $\mathfrak{d}(\mathfrak{h}(2n\|m))$
id	$\mathfrak{spe}^{sk}(n)$ and $\mathfrak{pe}^{sk}(n)$	$\mathfrak{sle}(n)$ and $\mathfrak{le}(n)$
id	$\mathfrak{c}(\mathfrak{spe}^{sk}(n))$ and $\mathfrak{c}(\mathfrak{pe}^{sk}(n))$	$\mathfrak{d}(\mathfrak{sle}(n))$ and $\mathfrak{d}(\mathfrak{le}(n))$
spin_λ, $\lambda \neq 0$	$\mathfrak{as}$	$\mathfrak{vle}(4\|4)$

Now we only have to test the simplicity of the prolongs $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$.

5.1. Proposition . *If $T(\mathfrak{g})$ contains an odd operator of rank 1, then either $T(\mathfrak{g})$ contains even operators of rank $(1, 0)$ and $(0, 1)$ or $T(\mathfrak{g})$ contains an even operator of rank $(1, 1)$.*

Proof. Let $\xi \in \mathfrak{g}_{\bar{1}}$ be such that $\text{rk}T(\xi) = 1$, i.e., $T(\xi)v = \alpha(v) \cdot v_\xi$, where $\alpha \in V^*$ and v_ξ is a fixed nonzero vector. (In other words, $T(\xi) = \alpha \otimes v_\xi$.) Let $\eta \in \mathfrak{g}_{\bar{1}}$. Then

$$T([\eta, \xi]) = T(\eta)T(\xi) + T(\xi)T(\eta).$$

Consider the two operators:

$$A_\eta = \alpha \otimes T(\eta)v_\xi \text{ and } B_\eta = -T(\eta)^* \cdot \alpha \otimes v_\xi$$

i.e., explicitly

$$\begin{aligned} A_\eta(v) &= T(\eta)T(\xi)v = \alpha(v) \cdot T(\eta)T(\xi), \\ B_\eta(v) &= T(\xi)T(\eta)v = \alpha(T(\eta)v_\xi. \end{aligned}$$

Clearly, for each fixed η both A_η and B_η are of rank not greater than 1; besides, they act in the spaces of opposite parity.

5.2. Lemma . *Let $W = \bigcap\limits_{\eta \in \mathfrak{g}_{\bar{1}}} \ker T(\eta)$ and $U = T(\mathfrak{g}_{\bar{1}})V$. Then the subspaces U and W are $\mathfrak{g}$-invariant.*

Since T is irreducible, $U = V$ and $W = 0$.

Proof. $\mathfrak{g}_{\bar{1}}$-invariance is obvious. Let $g \in \mathfrak{g}_{\bar{0}}$, $\xi \in \mathfrak{g}_{\bar{1}}$, $v \in V$ and $u = \xi v \in U$. Let us show that $gu \in U$. Indeed,

$$gu = g\xi v = \xi gv + [g, \xi]v \in T(\mathfrak{g}_{\bar{1}})V = V$$

because $gv \in V$ and $[g, \xi] \in \mathfrak{g}_{\bar{1}}$.

Let $w \in W$, $g \in \mathfrak{g}_{\bar{0}}$. Let us show that $gw \in W$. Indeed,

$$\eta(gw) = g(\eta w) + [\eta, g]w = 0$$

because $\eta w = 0$ and $[\eta, g] \in \mathfrak{g}_{\bar 1}$. Consequently, $gw \in W$. □

Corollary . *There exist* $\eta_1, \eta_2 \in \mathfrak{g}_{\bar 1}$ *such that* $\mathrm{rk} A_{\eta_1} = 1$ *and* $\mathrm{rk} B_{\eta_2} = 1$.

To complete the proof of Proposition, it suffices to observe that if $\mathrm{rk} B_{\eta_1} = 1$ (or $\mathrm{rk} A_{\eta_2} = 1$), then

$$\mathrm{rk} T([\eta_1, \xi]) = \mathrm{rk}(A_{\eta_1} + B_{\eta_1}) = (1,1), \text{ resp. } \mathrm{rk} T([\eta_2, \xi]) = (1,1),$$

whereas if $\mathrm{rk} B_{\eta_1} = \mathrm{rk} A_{\eta_2} = 0$, then the operators $T([\eta_1, \xi])$ and $T([\eta_2, \xi])$ are of rank 1 and act in the spaces of opposite parity. □

Conclusion. If $\dim \mathfrak{g} = \infty$, then there exists an element $g_0 \in \mathfrak{g}_{\bar 0}$ such that $\mathrm{rk}(T(g_0))|_{V_{\bar 0}} = 1$ and $\mathrm{rk}(T(g_0)|_{V_{\bar 1}} \leq 1$.

5.3. Proposition . *Let* $\mathfrak{g}_j$ *for* $j = 1, \dots, k$ *be simple Lie algebras (*not super*algebras),* $\mathfrak{r}$ *a solvable Lie algebra and* $\mathfrak{g}$ *their* direct *sum. Let* T *be a (perhaps, reducible) representation of* $\mathfrak{g}$ *in space* V *with a rank* 1 *operator. Then either*

1) *there exists a one-dimensional* $\mathfrak{g}$*-invariant subspace of* V*, or*

2) *up to renumbering,* $\mathfrak{g}_1 = \mathfrak{sl}(n)$ *or* $\mathfrak{sp}(2n)$ *and* $V = W \oplus U$*, where* $U = \ker T(\mathfrak{g}_1)$ *and* W *is isomorphic, as* $\mathfrak{g}$*-module, to either* $\mathrm{id} \otimes \chi_0 \otimes \lambda$ *or* $\mathrm{id}^* \otimes \chi_0 \otimes \lambda$*, where* id *is the identity representation of* $\mathfrak{g}_1$ *and* χ_0 *is the zero (one-dimensional) representation of* $\mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_k$*, and* λ *is a character of* $\mathfrak{r}$.

Proof. Since every finite dimensional representation of the semisimple Lie algebra $\mathfrak{g}_{ss} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$ is completely reducible and the sum of $\mathfrak{g}_{ss}$ with $\mathfrak{r}$ is the direct one, then V can be represented in the form $V = \oplus_i V_i$, where $T|_{V_i} = \tau_1^i \otimes \dots \otimes \tau_k^i \otimes \tau_{\mathfrak{r}}^i$ and τ_j^i is an irreducible representation of $\mathfrak{g}_j$ and $\tau_{\mathfrak{r}}^i$ is an irreducible representation of $\mathfrak{r}$.

Since T contains a rank 1 operator, so does the restriction $T|_{V_{i_0}}$ for some i_0. Therefore, not more than one of the spaces of the representations $\tau_1^{i_0}$, $\dots \tau_k^{i_0}$, $\tau_{\mathfrak{r}}^{i_0}$ can be of dimension > 1.

If $\dim \tau_1^{i_0} = \dots = \dim \tau_k^{i_0} = \dim \tau_{\mathfrak{r}}^{i_0} = 1$, then $\dim V_{i_0} = 1$ and possibility 1) of Proposition 5.3 is realized.

If $\dim \tau_1^{i_0} = \dots = \dim \tau_k^{i_0} = 1$, but $\dim \tau_{\mathfrak{r}}^{i_0} > 1$, then, due to the fact that $\mathfrak{r}$ is solvable, we deduce that V_{i_0} contains a one-dimensional $\mathfrak{r}$-submodule V_0. Since $\mathfrak{g}_{ss} V_{i_0} = 0$, it follows that V_0 is a $\mathfrak{g}$-submodule, i.e., once again the same possibility 1) of Prop. 5.3 is realized.

Finally, let $\dim \tau_{j_0}^{i_0} > 1$ for some j_0. Let us show that in this case possibility 2) or Prop. 5.2 is realized. Indeed, up to renumbering of the subalgebras $\mathfrak{g}_j$, we may assume that $j_0 = 1$. Then $\dim \tau_2^{i_0} = \dots = \dim \tau_k^{i_0} = \dim \tau_{\mathfrak{r}}^{i_0} = 1$ and the image $T(\mathfrak{g})|_{V_{i_0}}$ is isomorphic to either $\mathfrak{g}_1$ or its trivial central extension $\mathfrak{c}(\mathfrak{g}_1)$. Moreover, $T(\mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_k)|_{V_{i_0}} = 0$ and $T(\mathfrak{r})|_{V_{i_0}} = \lambda$ is a character.

Since there are only three series of simple infinite dimensional Lie algebras with Weisfeiler filtration of depth one, it follows that a *faithful irreducible* representation τ of the Lie algebra $\mathfrak{h}$ contains a rank 1 operator if and only if for a character λ of $\mathfrak{h}$ we have

$$\begin{aligned}
\mathfrak{h} = \mathfrak{sl}(n) &\quad \text{and } \tau = \mathrm{id} \text{ or } \mathrm{id}^*;\\
\mathfrak{h} = \mathfrak{gl}(n) &\quad \text{and } \tau = \mathrm{id} \otimes \lambda \text{ or } \mathrm{id}^* \otimes \lambda;\\
\mathfrak{h} = \mathfrak{sp}(2n) &\quad \text{and } \tau = \mathrm{id} \cong \mathrm{id}^*;\\
\mathfrak{h} = \mathfrak{csp}(2n) &\quad \text{and } \tau = \mathrm{id} \otimes \lambda.
\end{aligned}$$

In our case the role of $\mathfrak{h}$ is played by the image $T(\mathfrak{g}_1)|_{V_{i_0}}$ and we see that $\mathfrak{g}_1$ is isomorphic to either $\mathfrak{sl}(n)$ or $\mathfrak{sp}(2n)$; while $T(\mathfrak{g}_1)|_{V_{i_0}}$ is either id or id^*.

Thus, for the role of W we take V_{i_0}. To complete the proof, it suffices to observe that the rank of any operator $T(g)$ is caclulated from the formula

$$\mathrm{rk}T(\mathfrak{g}) = \sum_{i=1}^{k} \mathrm{rk}T(\mathfrak{g})|_{V_i}$$

and, therefore, $T(\mathfrak{g})|_{V_i} = 0$ for $i \neq i_0$. □

5.4. The cases of Prop. 5.3: the two regular and two exceptional subcases. Now, return to the situation of interest to us: $\mathfrak{g}$ is an almost simple Lie superalgebra or a central extension thereof, T its faithful representation in a finite dimensional superspace V such that among the operators $T(g)$, $g \in \mathfrak{g}$, there is one of rank 1 with an even covector.

Consider the restriction of T onto the even part of $\mathfrak{g}$ — the Lie algebra $\mathfrak{g}_{\bar{0}}$. Thanks to Proposition 5.1 the representation of $\mathfrak{g}_{\bar{0}}$ in the subspace $V_{\bar{0}}$ must possess a rank 1 operator whereas the representation of $\mathfrak{g}_{\bar{0}}$ in the whole V has either a rank $(1, 0)$ operator or a rank $(1, 1)$ operator. Moreover, from Table 8.1 we see that $\mathfrak{g}_{\bar{0}}$ is always of the form $\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k \oplus \mathfrak{r}$, where the $\mathfrak{g}_i$ are simple and $\mathfrak{r}$ is a solvable Lie algebra ($\mathfrak{r}$ can be zero and almost always it is commutative). Hence, Prop. 5.3 is applicable to the $\mathfrak{g}_{\bar{0}}$-action in the spaces $V_{\bar{0}}$ and $V_{\bar{1}}$.

Let, as in Prop. 5.3, $\mathfrak{g}_{ss} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$. Consider, separately, the two cases corresponding to the possibilities of Prop. 5.3:

1) among the operators $T(\mathfrak{g}_{ss})$, there is no operator of rank $(1, 0)$ or $(1, 1)$;

2) among the operators $T(\mathfrak{g}_{ss})$, there is such an operator.

Case 1). As is clear from the proof of Prop. 5.3, we are in the setting of possibility 1) and the operator of rank $(1, 0)$ or $(1, 1)$ is in this case of the form $T(h)$, where $h \in \mathfrak{r}$. Moreover, due to the irreducibility of T the operator h can not belong to the center of the whole $\mathfrak{g}$.

This is possible in the following regular subcases:

$$\mathfrak{g} = \mathfrak{sl}(m|n) \text{ for } m \neq n,\ \mathfrak{gl}(m|n),\ \mathfrak{osp}(2|2n),\ \mathfrak{cosp}(2|2n)$$

and in the *exceptional subcases* to be considered separately:

a) $\mathfrak{psl}(2|2) \subset \mathfrak{g} \subset \mathfrak{der}\ \mathfrak{psl}(2|2)$ and $1 \leq \dim \mathfrak{g}/\mathfrak{psl}(2|2) \leq 2$;
b) the same as in a) but with $\mathfrak{cpsl}(2|2)$ instead of $\mathfrak{psl}(2|2)$.

5.4.1. The regular subcases. In these subcases $\mathfrak{g}$ possesses a $\mathbb{Z}$-grading of the form $\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}^1$, where $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}}$ and, therefore, $\mathfrak{g}_{\bar{1}} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^1$. Then, under a proper norming, the operator h acts as the grading operator:

$$[h, g] = \pm g \text{ for any } g \in \mathfrak{g}_{\pm 1}.$$

Let $W_{\bar{0}} = \mathbb{C}w_0$ be a one-dimensional $\mathfrak{g}_{\bar{0}}$-invariant subspace of $V_{\bar{0}}$ and $h \cdot w_0 = \lambda w_0$. Then

$$h(gw_0) = g(hw_0) + [h, g]w_0 = \lambda(gw_0) \pm (gw_0) = (\lambda \pm 1)(g \cdot w_0),$$

i.e., on each of the subspaces $W_{-1} = \mathfrak{g}^{-1}W$ and $W_1 = \mathfrak{g}^1 W$ the operator h acts as multiplication by a scalar.

Consider the action of $\mathfrak{g}_{ss}$ on $W_{\pm 1}$. If $g_0 \in \mathfrak{g}_{ss}$, $\xi \in \mathfrak{g}^{\pm 1}$, then

$$g_0(\xi w_0) = \xi(g_0 w_0) + [g_0, \xi] \cdot w_0 = [g_0, \xi]w_0.$$

Hence, each of the subspaces $W_{\pm 1}$ is $\mathfrak{g}_{\bar{0}}$-invariant and the representation of $\mathfrak{g}_{ss}$ in $W_{\pm 1}$ is isomorphic to the quotient representation of $\mathfrak{g}_{ss}$ in $\mathfrak{g}^{\pm 1}$.

Since in all the cases considered the representation of $\mathfrak{g}_{ss}$ in $\mathfrak{g}^{\pm 1}$ is irreducible, we see that either $W_{\pm 1} = 0$ or $\dim W_{\pm 1} = \dim \mathfrak{g}^{\pm 1} > 1$.

But then, to ensure that $\mathrm{rk}T(h) = 1$ or $(1, 1)$, we should have $\lambda = \pm 1$; moreover, if $\lambda + k \neq 0$, then $W_k = 0$ for $k = \pm 1$. Hence, $\mathfrak{g}_{\bar 1} W_{\bar 0} = W_{-\lambda}$ and $h|_{W_{-\lambda}} = 0$. Then $h|_{\mathfrak{g}^{\pm 1}.W_{-\lambda}} = \pm\mathrm{id}$ and $\mathfrak{g}^{\pm 1} W_{-\lambda} \subset V_{\bar 0}$.

But since $\mathrm{rk}T(h)|_{V_{\bar 0}} = 1$, we get:

$$\mathfrak{g}_\lambda W_{-\lambda} = W_{\bar 0}, \quad \mathfrak{g}_{-\lambda} W_{-\lambda} = 0 \text{ and } \mathfrak{g}^2_{\bar 1} W_{\bar 0} = W_{\bar 0}.$$

Therefore, $W_\lambda \oplus W_{\bar 0}$ is invariant with respect to the whole $\mathfrak{g}$; hence, $V = W_\lambda \oplus W_{\bar 0}$ and $\dim V = (1, n)$, where $n = \dim \mathfrak{g}^1 = \dim \mathfrak{g}^{-1}$.

Thus,

$$\mathfrak{g} \subset \mathfrak{gl}(1|n) \text{ and } \dim \mathfrak{g}_{\bar 1} = 2n = \dim \mathfrak{gl}(1|n)_{\bar 1}.$$

Therefore, either $\mathfrak{g} = \mathfrak{sl}(1|n)$ or $\mathfrak{gl}(1|n)$ and either $T = \mathrm{id}$ or $T = \mathrm{id}^*$ (see the agreement on the action of the center in Prop. 5.0).

5.4.2. The exceptional subcases. These cases are related with the simple Lie superalgebra $\mathfrak{psl}(2|2)$ whose Lie superalgebra of outer derivations is isomorphic to $\mathfrak{sl}(2)$.

Subcase a): $\mathfrak{psl}(2|2) \subset \mathfrak{g} \subset \mathfrak{der}\ \mathfrak{psl}(2|2)$ and $1 \leq \dim \mathfrak{g}/\mathfrak{psl}(2|2) \leq 2$;

Subcase b): $\mathfrak{cpsl}(2|2) \subset \mathfrak{g} \subset \mathfrak{der}\ \mathfrak{cpsl}(2|2)$ and $1 \leq \dim \mathfrak{g}/\mathfrak{cpsl}(2|2) \leq 2$.

Denote by z the central element of $\mathfrak{g}$ in subcase b).

Let $\mathfrak{r}_0 \subset \mathfrak{sl}(2)$ be a subalgebra of the Lie algebra of outer derivations of $\mathfrak{psl}(2|2)$ such that $1 \leq \dim \mathfrak{r}_0 \leq 2$. Let

$$r = \begin{cases} \mathfrak{r}_0 & \text{in subcase a)} \\ \mathfrak{r}_0 \oplus \mathbb{C}z & \text{in subcase b).} \end{cases}$$

Denote by $R_0 = \mathrm{id}_{\mathfrak{sl}(2)}|_{\mathfrak{r}_0}$ the restriction of the standard representation of $\mathfrak{sl}(2)$ onto $\mathfrak{r}_0$. Then $\mathfrak{g}_{\bar 0} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{r}_0$ and $\mathfrak{g}_{ss} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \cong \mathfrak{o}(4)$. Let χ_0 be the trivial character on $\mathbb{C}z$; then, as $\mathfrak{g}_{\bar 0}$-module, $\mathfrak{g}_{\bar 1}$ is isomorphic to $\mathrm{id} \otimes \mathrm{id} \otimes R$, where

$$R = \begin{cases} R_0 & \text{in subcase a)} \\ R_0 \otimes \chi_0 & \text{in subcase b).} \end{cases}$$

Set $X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}(2)$ and $Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2)$. Clearly, up to the choice of a basis in $\mathfrak{g}_{\bar 1}$, there are three essentially distinct possibilities for $\mathfrak{r}_0$:

1) $\mathfrak{r}_0 = Span(X, Y)$;
2) $\mathfrak{r}_0 = \mathbb{C}X$;
3) $\mathfrak{r}_0 = \mathbb{C}Y$.

Recall that we are looking for an irreducible representation T of $\mathfrak{g}$ in a superspace V such that $T(\mathfrak{g}_{\bar 0})$ contains an operator of rank $(1, 0)$ or $(1, 1)$ whereas $T(\mathfrak{g}_{ss})|_{V_{\bar 0}}$ has no such operators.

Then the operator of rank $(1, 0)$ or $(1, 1)$ is $T(r)$, where $r \in \mathfrak{r}_0$. Without loss of generality we may assume that either $r = X$ or Y.

If $r = X$, then the arguments of the regular case are applicable and we deduce that $\mathfrak{g}$ has no such representations.

Suppose that $r = Y$. Let, as in the general case, $W_{\bar 0} = \mathbb{C}w_0 \subset V_{\bar 0}$ be a one-dimensional $\mathfrak{g}_{\bar 0}$-invariant subspace. Then $T(\mathfrak{g}_{ss})w_0 = 0$, the representation T determines a character $\chi_{\lambda,\alpha}$ on $\mathfrak{r}$ from the formulas

$$T(z)w_0 = \chi_{\lambda,\alpha}(z)w_0 = \lambda w_0 \text{ for } \lambda \neq 0$$
$$T(Y)w_0 = \chi_{\lambda,\alpha}(Y)w_0 = \alpha w_0.$$

(Clearly, if $\dim \mathfrak{r}_0 = 2$, then $\alpha = 0$.)

Consider the $\mathfrak{g}_{\bar 0}$-module $\mathfrak{g}_{\bar 1}W_{\bar 0} \subset V_{\bar 1}$. It is isomorphic to a quotient of the $\mathfrak{g}_{\bar 0}$-module $\mathfrak{g}_{\bar 1} \otimes W_0 = \mathrm{id} \otimes \mathrm{id} \otimes (R \otimes \chi_{\lambda,\alpha})$. Therefore, the three cases are possible:

1) $\mathfrak{g}_{\bar 1}W_{\bar 0} = 0$. Then $W_{\bar 0}$ is a $\mathfrak{g}$-invariant subspace. Hence, $V = W_{\bar 0}$ and T is not faithful.

2) $\mathfrak{g}_{\bar 1}W_{\bar 0} \cong \mathfrak{g}_{\bar 1}\otimes W_0$. Then $\mathrm{rk}\, T(Y)|_{\mathfrak{g}_{\bar 1}\otimes W_0} = \begin{cases} 4 & \text{if } \alpha = 0 \\ 8 & \text{if } \alpha \neq 0. \end{cases}$ Both the possibilities contradict our assumption on the rank.

3) $\mathfrak{g}_{\bar 1}W_{\bar 0} \cong \mathrm{id} \otimes \mathrm{id} \otimes \chi_{\lambda,\alpha}$. If $\alpha \neq 0$, then $\mathrm{rk}\ T(Y)|_{\mathfrak{g}_{\bar 1}W_{\bar 0}} = 4$. Hence, $\alpha = 0$, i.e., $T(Y)|_{W_0} = T(Y)|_{\mathfrak{g}_{\bar 1}W_{\bar 0}} = 0$. Set

$$\mathfrak{g}_Y = \mathrm{Im}\ \mathrm{ad}Y \subset \mathfrak{g}_{\bar 1}. \tag{5.4.2.1}$$

Then

$$T(Y)W_{\bar 0} = T(Y)\mathfrak{g}_{\bar 1}W_{\bar 0} - \mathfrak{g}_{\bar 1}T(Y)W_{\bar 0} = 0.$$

Thus, w_0 is the highest weight vector in T.

In subcase a) this means that T is one-dimensional and, therefore, not faithful.

In subcase b), let $c : \mathfrak{g}_{\bar 1} \times \mathfrak{g}_{\bar 1}$ be the cocycle that determines the central extension $\mathfrak{psl}(2|2) \longrightarrow \mathfrak{sl}(2|2)$. As is clear from Table 8.2,

$$c|_{\mathfrak{g}_{\bar 1}\times\mathfrak{g}_Y} = 0. \tag{5.4.2.2}$$

Let $\mathfrak{g}_{-Y} \subset \mathfrak{g}_{\bar 1}$ be a $\mathfrak{g}_{ss}$-invariant subspace complementary to $\mathfrak{g}_Y$. By (5.4.2.1), $\mathfrak{g}_{\bar 1}W_{\bar 0} = \mathfrak{g}_{-Y}W_{\bar 0}$. Take some nonzero $\xi \in \mathfrak{g}_Y$ and $\eta \in \mathfrak{g}_{-Y}$. Then

$$\xi \cdot \eta w_0 = -\eta\xi w_0 = [\xi, \eta]w_0 = c(\xi, \eta) \cdot \lambda w_0.$$

By (5.4.2.2), both Y and all the elements $\xi \in \mathfrak{g}_Y$ annihilate the space $\mathfrak{g}_{\bar 1}W_{\bar 0}$. Hence, $\mathfrak{g}_{\bar 1}W_{\bar 0}$ generates a $\mathfrak{g}$-invariant subspace in V which does not contain $W_{\bar 0}$. This contradicts irreducibility of T.

Conclusion: in none of the exceptional subcases $\mathfrak{g}$ has irreducible representations we are looking for.

5.5. Case 2). Due to Prop. 5.3, $\mathfrak{g}_{ss} = \mathfrak{g}_1 \oplus \mathfrak{g}_{comp}$, where $\mathfrak{g}_1 = \mathfrak{sl}(n)$ or $\mathfrak{sp}(2n)$ and the complementary Lie algebra $\mathfrak{g}_{comp}$ is either semisimple or zero. So $V = W \oplus U$ for (decomposable) $\mathfrak{g}_1$- and $\mathfrak{g}_{comp}$-modules W and U, where $T(\mathfrak{g}_1)|_U = 0$, the $\mathfrak{g}_{ss}$-module $W_{\bar 0}$ is isomorphic to either $\mathrm{id} \otimes \chi_0$ or $\mathrm{id}^* \otimes \chi_0$ and where the $\mathfrak{g}_{ss}$-module $W_{\bar 1}$ is either zero or isomorphic to either $\mathrm{id} \otimes \chi_0$ or $\mathrm{id}^* \otimes \chi_0$.

Consider the weight diagrams of $V_{\bar 0}$ and $V_{\bar 1}$ with respect to $\mathfrak{g}_{ss}$. Denote by $\varepsilon_1, \dots, \varepsilon_n$ the standard vectors that span the space dual to the Cartan subalgebra of $\mathfrak{g}_1$, see [OV], and by $\delta_1, \dots, \delta_N$ the similar vectors for $\mathfrak{g}_{comp}$. (Clearly, if $\mathfrak{g}_{ss}$ is simple and, therefore, $\mathfrak{g}_{comp} = 0$ these complementary vectors δ_i are absent.) Then the collection of weights (wrt $\mathfrak{g}_{ss}$) in each of $W_{\bar 0}$ and $W_{\bar 1}$ is as follows (with multiplicity one):

for $\mathfrak{g}_1 = \mathfrak{sl}(n)$: either $\varepsilon_1, \dots, \varepsilon_n$ or $-\varepsilon_1, \dots, -\varepsilon_n$.

for $\mathfrak{g}_1 = \mathfrak{sp}(2n)$: $\varepsilon_1, \dots, \varepsilon_n$ and $-\varepsilon_1, \dots, -\varepsilon_n$.

In the spaces $U_{\bar 0}$ and $U_{\bar 1}$ all the weights are of the form $\varphi(\delta_1, \dots, \delta_N)$, i.e., depend only on the δ's. The weight diagram of such a form will be referred to as a *desirable* one. Let α be an odd root of $\mathfrak{g}$ and ξ_α the corresponding root vector; let λ be a weight of V and v_λ the corresponding weight vector. Then $\xi_\alpha \cdot v_\lambda$ is a weight vector of weight $\lambda + \alpha$.

5.5.1. The scheme of our further considerations. 1°) We start with a $\mathfrak{g}_{\bar 0}$-invariant subspace $W_{\bar 0}$ whose weights we know.

2°) We apply $\mathfrak{g}_{\bar 1}$ to this subspace and get a subspace of $V_{\bar 1}$. By comparing the desirable weight diagram of $V_{\bar 1}$ with weights λ' of the form $\lambda + \alpha$ we get an information about possible weight diagrams of $\mathfrak{g}_{\bar 1}W_{\bar 0}$.

3°) By applying heading 2°) to a $\mathfrak{g}_{\bar 0}$-invariant subspace of $\mathfrak{g}_{\bar 1}W_{\bar 0}$, we deduce, as a result:

a) $\mathfrak{g}_{\bar 1}^2 W_{\bar 0} = W_{\bar 0}$ and, therefore, $V = W_{\bar 0} \oplus \mathfrak{g}_{\bar 1} W_{\bar 0}$;

b) the description of the weight diagram of V.

4°) Since the dimension of V is very low and its homogeneous components are irreducible with respect to $\mathfrak{g}_{\bar 0}$, the direct check and dimension considerations quickly bring about the result. If they fail, we compare the superspace V obtained, the two of whose highest weights with respect to $\mathfrak{g}_{\bar 0}$ are known, with the irreducible $\mathfrak{g}$-modules with the same highest weights. Since all such modules are quotients of the induced modules, it suffices to investigate the so-called singular vectors (see Appendix) of two induced modules only.

5.6. Case-by-case study.

5.6.1. $\mathfrak{g}_1 = \mathfrak{sp}(2n)$. Then either $\mathfrak{g} = \mathfrak{osp}(m|2n)$ or $\mathfrak{osp}(m|2n) \oplus \mathbb{C}z$. The weights λ of $W_{\bar 0}$ are $\pm\varepsilon_1, \dots, \pm\varepsilon_n$.

$\underline{m > 2}$. The odd roots are of the form $\alpha = \pm\varepsilon_i \pm \delta_j$ for $\mathfrak{osp}(2m|2n)$. For $\mathfrak{osp}(2m+1|2n)$, to the roots of the above form we must add the roots $\alpha = \pm\varepsilon_i$.

Thus, the sums $\lambda + \alpha$ which are desirable weights, are only those of the form:

$$\lambda' = \begin{cases} \pm\delta_j & \text{for the series } \mathfrak{osp}(2m|2n), \\ \pm\delta_j \text{ and } 0 & \text{for the series } \mathfrak{osp}(2m+1|2n). \end{cases}$$

By considering the sums λ'' of the form $\lambda' + \alpha$, we see that among them, the only desirable weights are those of the form $\pm\varepsilon_i$. Since their multiplicity in $V_{\bar 0}$ is equal to one, $\mathfrak{g}_{\bar 1}^2 W_{\bar 0} = W_{\bar 0}$; hence, $V = W_{\bar 0} \oplus \mathfrak{g}_{\bar 1} W_{\bar 0}$.

Consider the $\mathfrak{g}_{ss}$-module $\mathfrak{g}_{\bar 1} W_{\bar 0}$ more closely. Clearly, it is isomorphic to the quotient of $\mathfrak{g}_{\bar 1} \otimes W_{\bar 0}$. As $\mathfrak{g}_{ss}$-module, $\mathfrak{g}_{\bar 1}$ is $\mathrm{id}_{\mathfrak{sp}(2n)} \otimes \mathrm{id}_{\mathfrak{o}(m)}$ and $W_{\bar 0}$ is $\mathrm{id}_{\mathfrak{sp}(2n)} \otimes \chi_0$. Thus, $\mathfrak{g}_{\bar 1} \otimes W_{\bar 0}$ is the quotient of $\left(\mathrm{id}_{\mathfrak{sp}(2n)} \otimes \mathrm{id}_{\mathfrak{sp}(2n)}\right) \bigotimes \mathrm{id}_{\mathfrak{o}(m)}$ with the zero $\mathfrak{sp}(2n)$-action.

Now, observe that the tensor square of the standard $\mathfrak{sp}(2n)$-module is reducible:

$$\mathrm{id}_{\mathfrak{sp}(2n)} \otimes \mathrm{id}_{\mathfrak{sp}(2n)} = S^2(\mathrm{id}_{\mathfrak{sp}(2n)}) \oplus \Lambda^2(\mathrm{id}_{\mathfrak{sp}(2n)}),$$

moreover, $S^2(\mathrm{id}_{\mathfrak{sp}(2n)}) \cong \mathrm{ad}_{\mathfrak{sp}(2n)}$ (hence, irreducible) and $\Lambda^2(\mathrm{id}_{\mathfrak{sp}(2n)})$ is the direct sum of the trivial one-dimensional module and the complementary to it irreducible submodule. Therefere, the subspace of invariants in the tensor square $\mathrm{id}_{\mathfrak{sp}(2n)} \otimes \mathrm{id}_{\mathfrak{sp}(2n)}$ is one-dimensional. Therefore, the $\mathfrak{g}_{ss}$-module $\mathfrak{g}_{\bar 1} \otimes W_{\bar 0}$ can be either trivial or isomorphic to $\chi_0 \otimes \mathrm{id}_{\mathfrak{o}(m)}$.

Thus, $\dim V = (2n, 1)$ or $(2n, m)$. The dimension considerations (thanks to the faithfulness of the $\mathfrak{o}(m)$-action) show that only the second possibility can be realized; the representation T is, actually, $\Pi(\mathrm{id})$.

$\underline{m = 2}$. Then $\mathfrak{g}_{ss} = \mathfrak{sp}(2n)$. The weights λ of $W_{\bar 0}$ are $\pm\varepsilon_1, \dots, \pm\varepsilon_n$ and the odd roots are also $\pm\varepsilon_1, \dots, \pm\varepsilon_n$. Each root is of multiplicity 2 and the center of $\mathfrak{g}_{\bar 0}$ separates the odd roots into the negative and positive ones: $\mathfrak{g}_{\bar 1} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^1$; the $\mathfrak{g}_{ss}$-acton in $\mathfrak{g}^{-1}$ is id, same in $\mathfrak{g}^1$ is id^*.

The desirable weight λ' of the form $\alpha + \lambda$ is only $\lambda' = 0$. The subspace $\mathfrak{g}^{-1} W_{\bar 0}$ is the quotient of $\mathfrak{g}^{-1} \otimes W_{\bar 0} \cong \mathrm{id}^* \otimes \mathrm{id}$ and $\mathfrak{g}^1 W_{\bar 0}$ is the quotient of $\mathfrak{g}^1 \otimes W_{\bar 0} \cong \mathrm{id} \otimes \mathrm{id}$. As we have already noted, each of the modules $\mathrm{id}^* \otimes \mathrm{id}$ and $\mathrm{id} \otimes \mathrm{id}$ only contains a one-dimensional subspace of invariants. Thus,

$$\dim \mathfrak{g}_{\bar 1} W_{\bar 0} = 0, 1 \text{ or } 2.$$

The desirable weights of the form $\lambda' + \alpha$ are equal to $\pm\varepsilon_i$. Hence, $\mathfrak{g}_{\bar 1}^2 W_{\bar 0} = W_{\bar 0}$ and $V = W_{\bar 0} \oplus \mathfrak{g}_{\bar 1} W_{\bar 0}$.

As above, the dimension considerations imply that $T(\mathfrak{g}_{\bar 1}) = \mathfrak{gl}(2n|1)_-$. Hence, $\dim V = (2n|2)$ and $T = \Pi(\mathrm{id})$.

$\underline{m = 1}$. Then $\mathfrak{g}_{ss} = \mathfrak{sp}(2n)$ and $\mathfrak{g}_{\bar 1} \cong \mathrm{id}$, as $\mathfrak{g}_{ss}$-module. The same arguments as above imply that $\dim \mathfrak{g}_{\bar 1} W_{\bar 0} = 1$ and $V = W_{\bar 0} \oplus \mathfrak{g}_{\bar 1} W_{\bar 0}$. Thus, $T = \Pi(\mathrm{id})$.

5.6.2. $\mathfrak{g}_1 = \mathfrak{sl}(n)$**.** This case splits into subcases A) – F).

A) $\mathfrak{g} = \mathfrak{spe}(n)$, $\mathfrak{spe}(n) \oplus \mathbb{C}z$, $\mathfrak{as}$, $\mathfrak{pe}(n)$, $\mathfrak{pe}(n) \oplus \mathbb{C}z$. We will assume that $n > 3$ (since $\mathfrak{spe}(3) \cong \mathfrak{svect}(0|3)$, this case and all it relatives is considered in the study of vectorial Lie superalgebras).

For these Lie superalgebras $\mathfrak{g}_{ss} = \mathfrak{sl}(n)$ and the weights of $W_{\bar 0}$ are either $\lambda = \varepsilon_1, \dots, \varepsilon_n$ or $\lambda = -\varepsilon_1, \dots, -\varepsilon_n$. The odd roots are $\alpha = \pm(\varepsilon_i + \varepsilon_j)$ for $i \neq j$ and $2\varepsilon_i$. Observe that $\mathfrak{g}_{\bar 1}$ natuarally splits into the positive and negative part: $\mathfrak{g}_{\bar 1} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^1$.

The roots in $\mathfrak{g}^{-1}$ are $\alpha = -\varepsilon_i - \varepsilon_j$ while those in $\mathfrak{g}^1$ are $\alpha = \varepsilon_i + \varepsilon_j, 2\varepsilon_i$.

The weights λ' of the form $\lambda + \alpha$ that occure in the desirable diagram are as follows.

If $\lambda = \varepsilon_1, \dots, \varepsilon_n$, then $\lambda' = -\varepsilon_1, \dots, -\varepsilon_n$ and $\mathfrak{g}^1 W_{\bar 0} = 0$.

If $\lambda = -\varepsilon_1, \dots, -\varepsilon_n$, then $\lambda = \varepsilon_1, \dots, \varepsilon_n$ and $\mathfrak{g}^{-1} W_{\bar 0} = 0$.

Recall that the multiplicity of the roots of the form $\pm\varepsilon_i$ in $V_{\bar 1}$ can only be equal to 1. Therefore, $\dim \mathfrak{g}_{\bar 1} W_{\bar 0} = n$.

Similarly, the desirable weights λ'' of the form $\lambda' + \alpha$ are as follows.

If $\lambda' = \varepsilon_1, \dots, \varepsilon_n$, then $\lambda' + \alpha = -\varepsilon_1, \dots, -\varepsilon_n$ and $\mathfrak{g}_{\bar 1}^2 W_{\bar 0} = W_{\bar 0}$.

If $\lambda' = -\varepsilon_1, \dots, -\varepsilon_n$, then $\lambda + \alpha = \varepsilon_1, \dots, \varepsilon_n$ and $\mathfrak{g}_{\bar 1}^2 W_{\bar 0} = W_{\bar 0}$.

Thus, $V = W_{\bar 0} \oplus \mathfrak{g}_{\bar 1} W_{\bar 0}$ and $\dim V = (n|n)$. Hence, if $\mathfrak{g} = \mathfrak{spe}(n)$ or $\mathfrak{pe}(n)$, then either $T = \mathrm{id}$ or $T = \mathrm{id}^*$. Both these representations contain odd operators of rank 1, but for $T = \mathrm{id}^*$ the covector is odd.

For Lie superalgebras $\mathfrak{g} = \mathfrak{spe}(n) \oplus \mathbb{C}z$ and $\mathfrak{g} = \mathfrak{pe}(n) \oplus \mathbb{C}z$ we see that $T = \mathrm{id} \otimes \chi$, where χ is an arbitrary character of the center, and for $\mathfrak{g} = \mathfrak{as}$ there is the one-parametric family T_λ of $(4|4)$-dimensional representations that differ from each other by the value on the center.

B) $\mathfrak{g} = \mathfrak{psq}(n)$, $\mathfrak{sq}(n)$, $\mathfrak{pq}(n)$, $\mathfrak{q}(n)$. In these cases $\mathfrak{g}_{ss} = \mathfrak{sl}(n)$.

The odd roots are $\alpha = \varepsilon_i - \varepsilon_j$. Let, as always, λ denote a weight of $W_{\bar 0}$, let $\lambda' = \lambda + \alpha$ run over the weights of desirable form from $V_{\bar 1}$ and $\lambda'' = \lambda' + \alpha$ the weights of desirable form from $V_{\bar 0}$. Then

λ	λ'	λ''
$\varepsilon_1, \dots, \varepsilon_n$	$\varepsilon_1, \dots, \varepsilon_n$	$\varepsilon_1, \dots, \varepsilon_n$
$-\varepsilon_1, \dots, -\varepsilon_n$	$-\varepsilon_1, \dots, -\varepsilon_n$	$-\varepsilon_1, \dots, -\varepsilon_n$

Thus, $\mathfrak{g}_{\bar 1}^2 W_{\bar 0} = W_{\bar 0}$ and $\dim \mathfrak{g}_{\bar 1} W_{\bar 0} = n$. Hence, $\dim V = (n|n)$. But for $\mathfrak{g} = \mathfrak{psq}(n)$ and $\mathfrak{pq}(n)$ there are no such representations at all ([K4]), while for $\mathfrak{g} = \mathfrak{sq}(n)$ and $\mathfrak{q}(n)$ this is the standard representation which has no rank 1 operators.

C) $\mathfrak{g} = \mathfrak{sl}(n|m)$, $\mathfrak{gl}(n|m)$, $\mathfrak{psl}(n|n)$. Observe that for $n = m = 1$, the Lie superalgebra $\mathfrak{gl}(1|1)$ is solvable; hence, all its irreducible representations are induced [S]; the case considered in Step 4. Due to an isomorphism $\mathfrak{sl}(2|1) \cong \mathfrak{vect}(0|2)$ we can and will consider this case in Step 5B.

$m > 1$. Then $\mathfrak{g}_{ss} = \mathfrak{sl}(n) \oplus \mathfrak{sl}(m)$ and $\mathfrak{g}_{\bar 1} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^1$, where the roots of $\mathfrak{g}^{-1}$ are $\alpha_{-1} = -\varepsilon_i + \delta_j$ and the roots of $\mathfrak{g}^1$ are $\alpha_1 = \varepsilon_i - \delta_j$.

First, suppose $n > 2$. Denote by $\lambda'_{\pm 1}$ the desirable weights of the form $\lambda + \alpha_{\pm 1}$ and by $\lambda''_{\pm 1}$ the desirable weights of the form $\lambda'_{\pm 1} + \alpha_{\pm 1}$. We get:

λ	λ'_{-1}	λ'_1	λ''_{-1}	λ''_1
$\varepsilon_1, \dots, \varepsilon_n$	$\delta_1, \dots, \delta_n$	$\emptyset$	$\emptyset$	$\varepsilon_1, \dots, \varepsilon_n$
$-\varepsilon_1, \dots, -\varepsilon_n$	$\emptyset$	$\delta_1, \dots, \delta_n$	$-\varepsilon_1, \dots, -\varepsilon_n$	$\emptyset$

(5.6.2)

Thus, we see that $\mathfrak{g}_{\bar 1}^2 W_{\bar 0} = W_{\bar 0}$. Let us consider the $\mathfrak{g}_{ss}$-module $\mathfrak{g}_{\bar 1} W_{\bar 0}$ in more detail. First, let $W_{\bar 0}$, as $\mathfrak{g}_{ss}$-module, be $\mathrm{id} \otimes \chi_0$ (the weights are $\varepsilon_1, \dots, \varepsilon_n$). Then, as is clear from (5.6.2),

$\mathfrak{g}^1 W_{\bar 0} = 0$ and $\mathfrak{g}_{\bar 1} W_{\bar 0} = \mathfrak{g}^{-1} W_{\bar 0}$; hence, $\mathfrak{g}_{\bar 1} W_{\bar 0}$ is isomorphic to a quotient of the tensor product

$$\mathfrak{g}^{-1} \otimes W_{\bar 0} \cong (\mathrm{id} \otimes \mathrm{id}^*) \bigotimes (\mathrm{id} \otimes \chi_0)\,.$$

But the $\mathfrak{sl}(n)$-module id$\otimes$id* is the sum of the adjoint module and the trivial one-dimensional one. Since the $\mathfrak{sl}(n)$-action in $\mathfrak{g}^{-1} \otimes W_{\bar 0}$ is trivial, we deduce that $\dim \mathfrak{g}^{-1} \otimes W_{\bar 0} = (0|m)$ and $\dim V = (n|m)$.

Therefore,

$$T = \begin{cases} \mathrm{id} & \text{for } \mathfrak{g} = \mathfrak{sl}(n|m) \text{ and } \mathfrak{gl}(n|n);\\ \mathrm{id} \otimes \chi, \text{ where } \chi \text{ is an arbitrary character,} & \text{for } \mathfrak{g} = \mathfrak{gl}(n|m). \end{cases}$$

For $\mathfrak{g} = \mathfrak{psl}(n|n)$ there are no such representations: from dimension considerations and the fact that the irreducible $\mathfrak{g}$-module of the least dimension is the adjoint one [K4].

The case $W_{\bar 0} = \mathrm{id}^* \otimes \chi_0$ is similarly considered. We obtain:

$$T = \begin{cases} \mathrm{id}^* & \text{for } \mathfrak{g} = \mathfrak{sl}(n|m) \text{ and } \mathfrak{gl}(n|n);\\ \mathrm{id}^* \otimes \chi, \text{ where } \chi \text{ is an arbitrary character,} & \text{for } \mathfrak{g} = \mathfrak{gl}(n|m). \end{cases}$$

$\underline{n = 2}$. In this case by the study of the weight diagram only we cannot conclude that either $\mathfrak{g}^{-1} W_{\bar 0} = 0$ or $\mathfrak{g}^1 W_{\bar 0} = 0$. We will consider this case in sec. 5.6.3.

$\underline{m = 1 \text{ and } n > 2}$. Then $\mathfrak{g}_{ss} = \mathfrak{sl}(n)$ and $\mathfrak{g}_{\bar 1} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^1$, where the roots of $\mathfrak{g}^{-1}$ are $\alpha_{-1} = -\varepsilon_i$ and the roots of $\mathfrak{g}^1$ are $\alpha_1 = \varepsilon_i$. The list of desirable weights in this case is obtained from (5.6.2) by replacing $\delta_1, \dots, \delta_n$ with 0. The rest of our arguments are the same as in the above subcase.

5.6.3. Lemma . *Let the Lie superalgebra $\mathfrak{h} = \mathfrak{h}_{\bar 0} \oplus \mathfrak{h}_{\bar 1}$ be such that $\mathfrak{h}_{\bar 0} = \mathfrak{sl}(n)$ and $\mathfrak{h}_{\bar 1} = \mathrm{id}^*$, as $\mathfrak{sl}(n)$-module. Let V be an $\mathfrak{h}$-module and $W \subset V$ its $\mathfrak{sl}(n)$-invariant subspace isomorphic to* id *as $\mathfrak{sl}(n)$-module. Let, moreover, $\mathfrak{h}_{\bar 1} W$ be a one-dimensional trivial $\mathfrak{sl}(n)$-module. Then $\mathfrak{h}_{\bar 1}^2 W = 0$.*

Proof. By hypothesis, $\mathfrak{h}_{\bar 1}$ and W can be considered as dual linear spaces. Let $w_1, \dots, w_n$ be a basis of W and $h_1, \dots, h_n$ the dual basis of $\mathfrak{h}_{\bar 1}$. As we have already noticed, the $\mathfrak{sl}(n)$-module $\mathfrak{h}_{\bar 1} W$ is the quotient of the tensor product $\mathfrak{h}_{\bar 1} \otimes W$ modulo the adjoint submodule. The adjoint submodule is singled out as the kernel of the natural bilinear form $B(h \otimes w) = h(w)$ on $\mathfrak{h}_{\bar 1} \otimes W$. Then the generator u of $\mathfrak{h}_{\bar 1} W$ can be expressed as $u = h_1 w_1 = \dots = h_n w_n$. For $i \neq j$ we get

$$h_i u = h_i h_j w_j = -(-1)^{p(h_i)p(h_j)} h_j h_i w_j = 0. \quad \square$$

Corollary . 1) *The statement of Lemma holds also if $\mathfrak{h}_{\bar 1} = \mathrm{id}$ and $W = \mathrm{id}^*$.*

2) *For $n = 2$ the modules* id *and* id* *are isomorphic, so Lemma* 5.6.3 *holds also if $\mathfrak{h}_{\bar 1} \cong W \cong \mathrm{id}$ and $\mathfrak{h}_{\bar 1} \cong W \cong \mathrm{id}^*$.*

Thus, let $\mathfrak{g} = \mathfrak{sl}(2|m)$, $\mathfrak{gl}(2|m)$ for $m > 2$ or any almost simple Lie superalgebra related to $\mathfrak{psl}(2|2)$ except $\mathfrak{der}\, \mathfrak{psl}(2|2) \cong \mathfrak{osp}(4|2;0)$. Then $\mathfrak{g}_{ss} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(m)$ and it remains to consider the case when $\mathfrak{g}_1 = \mathfrak{sl}(2)$, i.e., the subspace $W_{\bar 0} \subset V$ is isomorphic to $\mathrm{id} \otimes \chi_0$ as $\mathfrak{g}_{ss}$-module.

Observe that the submodules $\mathfrak{g}^{-1} W_{\bar 0}$ and $\mathfrak{g}^1 W_{\bar 0}$ can not vanish simultaneously. Let, for example, $\mathfrak{g}^{-1} W_{\bar 0} \neq 0$. Then $\mathfrak{g}^{-1} W_{\bar 0} = \chi_0 \otimes \mathrm{id}_{\mathfrak{sl}(m)}$ and, thanks to Lemma 5.6.3, $(\mathfrak{g}^{-1})^2 W_{\bar 0} = 0$. But then $\mathfrak{g}^1(\mathfrak{g}^{-1} W_{\bar 0}) \neq 0$ due to irreducibility of the $\mathfrak{g}$-module V and our usual arguments appealing to the weight diagram show that $\mathfrak{g}^1(\mathfrak{g}^{-1} W_{\bar 0}) = W_{\bar 0}$. Moreover, $W_{\bar 0} \cong \mathrm{id}_{\mathfrak{sl}(2)} \otimes \chi_0$ arises as the quotient module of the tensor product

$$\begin{aligned}&\mathfrak{g}^1 \otimes (\mathfrak{g}^{-1} W_{\bar 0}) \cong \left(\mathrm{id}_{\mathfrak{sl}(2)} \otimes \mathrm{id}^*_{\mathfrak{sl}(m)}\right) \otimes \left(\chi_0 \otimes \mathrm{id}_{\mathfrak{sl}(m)}\right) = \\ &\mathrm{id}_{\mathfrak{sl}(2)} \bigotimes \left(\mathrm{id}^*_{\mathfrak{sl}(m)} \otimes \mathrm{id}_{\mathfrak{sl}(m)}\right).\end{aligned}$$

By applying Lemma 5.6.3 once again we deduce that $(\mathfrak{g}^1)^2(\mathfrak{g}^{-1}W_{\bar 0}) = \mathfrak{g}^1 W_{\bar 0} = 0$, i.e., $\dim V = (2|m)$ and $T(\mathfrak{g}) = \mathfrak{sl}(2|m)$ or $\mathfrak{gl}(2|m)$.

Let $\mathfrak{g} = \mathfrak{osp}(4|2n)$ for $n > 1$. Then

$$\begin{aligned}\mathfrak{g}_{\bar 0} &= \mathfrak{o}(4) \oplus \mathfrak{sp}(2n) \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sp}(2n),\\ \mathfrak{g}_{\bar 1} &\cong \mathrm{id}_{\mathfrak{sl}(2)} \otimes \mathrm{id}_{\mathfrak{sl}(2)} \bigotimes \mathrm{id}_{\mathfrak{sp}(2n)}.\end{aligned}$$

Let $\mathfrak{g}_1 = \mathfrak{sl}(2)$. By means of our routine arguments we deduce that

$$\mathfrak{g}_{\bar 1} W_{\bar 0} \cong \chi_0(\mathfrak{sl}(2)) \otimes \mathrm{id}_{\mathfrak{sl}(2)} \bigotimes \mathrm{id}_{\mathfrak{sp}(2n)} \text{ and } \mathfrak{g}_{\bar 1}^2 W_{\bar 0} = W_{\bar 0}.$$

This implies that $V = W_{\bar 0} \oplus \mathfrak{g}_{\bar 1} W_{\bar 0}$ and $\dim V = (2|4n)$.

Let us show that there are no such $\mathfrak{g}$-modules. Indeed, in the space dual to the Cartan subalgebra of $\mathfrak{o}(4)$, select basis elements $\varepsilon_1, \varepsilon_2$; for the roots of $\mathfrak{g}_1$ take $\pm 2\varepsilon_1$ and let the roots of the complementary to $\mathfrak{g}_1$ subalgebra in $\mathfrak{o}(4)$ be equal to $\pm 2\varepsilon_2$; let $\delta_1, \dots, \delta_n$ be the standard basis of the dual space to the Cartan subalgebra of $\mathfrak{sp}(2n)$. Then the weight basis of $\mathfrak{g}_{\bar 1} W_{\bar 0}$ consists of the vectors v_λ of weight $\lambda = \pm\varepsilon_2 \pm \delta_i$, whereas the weight basis of $W_{\bar 0}$ consists of $v_{\pm\varepsilon_1}$. Let $\xi_{\pm\varepsilon_1 \pm\varepsilon_2 \pm \delta_i}$ be an odd root vector of $\mathfrak{g}$. Then

$$v_{\pm\varepsilon_1} = \xi_{\pm\varepsilon_1 \pm\varepsilon_2 \pm\delta_i} \cdot v_{\mp\varepsilon_2 \mp\delta_i} \text{ and } \xi_\alpha \cdot v_\lambda = 0 \text{ if } \alpha + \lambda \neq \pm\varepsilon_1.$$

For $i \neq k$ we get

$$\begin{aligned}&\xi_{\varepsilon_1 \pm\varepsilon_2 \pm\delta_i} v_{\varepsilon_1} = 0;\\ &\xi_{-\varepsilon_1 \pm\varepsilon_2 \pm\delta_i} v_{\varepsilon_1} = \xi_{-\varepsilon_1 \pm\varepsilon_2 \pm\delta_i}(\xi_{\varepsilon_1 \pm\varepsilon_2 \pm\delta_k} v_{\mp\varepsilon_2 \mp\delta_k}) =\\ &-\xi_{\varepsilon_1 \pm\varepsilon_2 \pm\delta_k} \cdot \xi_{-\varepsilon_1 \pm\varepsilon_2 \pm\delta_i} v_{\mp\varepsilon_2 \mp\delta_k} = 0.\end{aligned}$$

The penultimate equality holds because $[\xi_{\varepsilon_1 \pm\varepsilon_2 \pm\delta_k}, \xi_{-\varepsilon_1 \pm\varepsilon_2 \pm\delta_i}] = 0$ and the latter is true since $\pm 2\varepsilon_2 \pm (\delta_k + \delta_i)$ is not a root of $\mathfrak{g}$. Thus, $\mathfrak{g}_{\bar 1} v_{\varepsilon_1} = 0$. Similarly, $\mathfrak{g}_{\bar 1} v_{-\varepsilon_1} = 0$, i.e., $\mathfrak{g}_{\bar 1} W_{\bar 0} = 0$ and $V = W_0$. Contradiction.

D) $\mathfrak{g} = \mathfrak{ab}_3$. Then $\mathfrak{g}_{\bar 0} = \mathfrak{g}_{ss} = \mathfrak{sl}(2) \oplus \mathfrak{o}(7)$. For this case we modify our notations somewhat. Instead of two vectors ε_1 and ε_2 we will use one vector, ε, normed so that the roots of $\mathfrak{sl}(2)$ be equal to $\pm 2\varepsilon$.

The weights of $W_{\bar 0}$ are $\lambda = \pm\varepsilon$; the odd roots are $\alpha = \pm\varepsilon + \frac{1}{2}(\pm\delta_1 \pm \delta_2 \pm \delta_3)$ and $\mathfrak{g}_{\bar 1} \cong \mathrm{id}_{\mathfrak{sl}(2)} \otimes spin_7$ as $\mathfrak{g}_{\bar 0}$-module.

The desirable weights are

$$\lambda' = \lambda + \alpha = \frac{1}{2}(\pm\delta_1 \pm \delta_2 \pm \delta_3) \text{ and } \lambda'' = \lambda' + \alpha = \pm\varepsilon.$$

Therefore, $\mathfrak{g}_{\bar 1}^2 W_{\bar 0} = W_{\bar 0}$ and $V = W_{\bar 0} \oplus \mathfrak{g}_{\bar 1} W_{\bar 0}$. Observe that $\mathfrak{g}_{\bar 1} W_{\bar 0}$ is the quotient of the $\mathfrak{g}_{\bar 0}$-module

$$\mathfrak{g}_{\bar 1} \otimes W_{\bar 0} \cong \left(\mathrm{id}_{\mathfrak{sl}(2)} \otimes \mathrm{id}_{\mathfrak{sl}(2)}\right) \bigotimes \left(\mathrm{spin}_7 \otimes \chi_0\right).$$

The dimension of the trivial subspace of the $\mathfrak{sl}(2)$-module $\mathrm{id} \otimes \mathrm{id}$ is equal to one and, since the $\mathfrak{sl}(2)$-action in $\mathfrak{g}_{\bar 1} W_{\bar 0}$ should be trivial, we deduce that $\dim \mathfrak{g}_{\bar 1} W_{\bar 0} = 1 \cdot \dim(\mathrm{spin}_7) = 8$, i.e., $\dim V = (2|8)$.

Our routine arguments demonstrate that $\mathfrak{g} = \mathfrak{ab}_3$ has no such irreducible representations.

E) $\mathfrak{g} = \mathfrak{ag}_2$. Then $\mathfrak{g}_{\bar 0} = \mathfrak{g}_{ss} = \mathfrak{sl}(2) \oplus \mathfrak{g}_2$ and $\mathfrak{g}_{\bar 1} \cong \mathrm{id}_{\mathfrak{sl}(2)} \otimes R(\pi_1)$, as $\mathfrak{g}_{\bar 0}$-module, where $R(\pi_1)$ is the irreducible representation of $\mathfrak{g}_2$ with the highest weight π_1, see [OV]. Recall that $\dim R(\pi_1) = 7$. The arguments similar to those in the above case show that the desirable representation should be of dimension $(2, 7)$ but $\mathfrak{ag}_2$ has no such irreducible representations, see [K5] and [KW].

F) $\mathfrak{g} = \mathfrak{osp}(4|2; \alpha)$. Then $\mathfrak{g}_{\bar 0} = \mathfrak{g}_{ss} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ and arguments similar to those used in the above two subcases yield that $\dim V = (2|4)$; there are no such irreducible $\mathfrak{g}$-modules for generic $\alpha \neq 1$.

Step 5B. Vectorial Lie superalgebras. The four cases

5.7. Let $\mathfrak{h}_0$ be a finite dimensional Lie superalgebra. Let z be either its outer differentiation or a trivial center. Set $\mathfrak{g}_0 = \mathfrak{h}_0 \Subset \mathbb{C} \cdot z$ (if z is central, this sum is direct).

Let $\mathfrak{h}_{-1} = \mathfrak{g}_{-1}$ be an irreducible finite dimensional $\mathfrak{g}_0$-module. Consider two Cartan prolongations:

$$\mathfrak{h} = \mathop{\oplus}_{i\geq -l} \mathfrak{h}_i = (\mathfrak{h}_{-1}, \mathfrak{h}_0)_* \text{ and } \mathfrak{g} = \mathop{\oplus}_{i\geq -l} \mathfrak{g}_i = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*.$$

5.7.0. Lemma . *Either $\mathfrak{g}_1 = \mathfrak{h}_1$, hence, $\mathfrak{g} = \mathfrak{d}\mathfrak{h}$ or $\mathfrak{g}_1 = \mathfrak{h}_1 \oplus \mathfrak{g}^*_{-1}$ (the direct sum of vector spaces).*

Proof. For any $F \in \mathfrak{g}_1$, $v \in \mathfrak{g}_{-1}$ the decomposition

$$[F, v] = \alpha_F(v)z + h,$$

where $h \in \mathfrak{h}_0$, determines a linear form $\alpha_F \in \mathfrak{g}^*_{-1}$. By the definition of Cartan prolongation, $\alpha_F = 0$ if and only if $F \in \mathfrak{h}_1$. Hence, as a $\mathfrak{g}_0$-module, the space $\mathfrak{g}_1/\mathfrak{h}_1$ is a quotient of the $\mathfrak{g}_0$-module $\mathfrak{g}^*_{-1}$ which is irreducible together with $\mathfrak{g}_{-1}$. This proves Lemma. □

1) $\mathfrak{h}_0 = \mathfrak{vect}(0|n)$, where $n \geq 3$, and $\mathfrak{g}_0 = \mathfrak{cvect}(0|n) = \mathfrak{h}_0 \oplus \mathbb{C} \cdot z$. (Recall that $\mathfrak{h}_0$ does not have any outer differentiation or nontrivial central extentions.) The rank 1 operator with even covector exists only for the following $\mathfrak{h}_0$-modules:

1a) $\mathfrak{h}_{-1} = \Pi(\Lambda(n)/\mathbb{C} \cdot 1) \Longrightarrow \mathfrak{h} = (\mathfrak{h}_{-1}, \mathfrak{h}_0)_* = \mathfrak{le}(n; n)$;

1b) $\mathfrak{h}_{-1} = Vol_0(0|n) \Longrightarrow \mathfrak{h} = (\mathfrak{h}_{-1}, \mathfrak{h}_0)_* = \mathfrak{svect}^\circ(1|n; n)$;

1c) $\mathfrak{h}_{-1} = \Pi(Vol_0(0|n)) \Longrightarrow \mathfrak{h} = (\mathfrak{h}_{-1}, \mathfrak{h}_0)_* = \mathfrak{b}_1{}^\circ(n; n)$;

1d) $\mathfrak{h}_{-1} = \Pi(Vol^\lambda(n)) \Longrightarrow \mathfrak{h} = (\mathfrak{h}_{-1}, \mathfrak{h}_0)_* = \mathfrak{b}_\lambda(n; n)$ for $\lambda \neq 0, 1$.

The element z acts in $\mathfrak{g}_{-1}$ as a nonzero scalar. We normalize z so that $\mathrm{ad} z|_{\mathfrak{g}_{-1}} = -1$ and use for such $\mathfrak{cvect}(0|n)$-modules the same notations as for $\mathfrak{vect}(0|n)$-modules.

5.7.1. Proposition . *Let $\mathfrak{g}_0 = \mathfrak{cvect}(0|n)$, where $n \geq 3$, and $\mathfrak{g}_{-1}$ a $\mathfrak{g}_0$-module of type* 1a) – 1d)*. Let $\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ be Cartan prolongation. Then in the respective cases we have:*

1a) *if $\mathfrak{g}_{-1} = \Pi(\Lambda(n)/\mathbb{C} \cdot 1)$, then $\mathfrak{g} = \mathfrak{vle}(4|3)$ for $n = 3$ and $\mathfrak{g} = \mathfrak{dle}(n; n)$ for $n > 3$;*

1b) *if $\mathfrak{g}_{-1} = Vol_0(0|n)$, then $\mathfrak{g} = \mathfrak{kas}^\xi(; 3\eta)$ for $n = 3$ and $\mathfrak{g} = \mathfrak{dsvect}^\circ(1|n; n)$ for $n > 3$;*

1c) *if $\mathfrak{g}_{-1} = \Pi(Vol_0(0|n))$, then $\mathfrak{g} = \mathfrak{d}\mathfrak{b}_1{}^\circ(n; n)$;*

1d) *if $\mathfrak{g}_{-1} = \Pi(Vol^\lambda(n))$, then $\mathfrak{g} = \mathfrak{d}\mathfrak{b}_\lambda(n; n)$ for $\lambda \neq 0, 1$.*

Proof. Observe that $\mathfrak{g}$ has a bigrading: each component $\mathfrak{g}_k$ has a $\mathbb{Z}$-grading by powers of ξ (call it ξ-grading) such that $\mathfrak{g}_0 = \mathop{\oplus}\limits_{i=-1}^{n-1} \mathfrak{g}_{0,i}$ and $z \in \mathfrak{g}_{0,0}$; we have $\mathfrak{g}_{-1} = \mathop{\oplus}\limits_{i=k}^{m} \mathfrak{g}_{-1,i}$, where $k = 0$ in cases 1b) – 1d) and $k = 1$ in case 1a); $m = n$ in cases 1a), 1d) and $m = n - 1$ in cases 1b), 1c).

By Lemma 5.7.0 if $\mathfrak{g} \neq \mathfrak{d}\mathfrak{h}$, then the subspace $\mathfrak{g}^*_{-1} \subset \mathfrak{g}_1$ contains the components of degree from $-m$ to $-k$ with respect to ξ-grading. In particular, $\mathfrak{g}_{1,-m} \neq 0$.

But in each case we have a subspace $W = \mathop{\oplus}\limits_{i=k}^{s} \mathfrak{g}_{-1,i} \subset \mathfrak{g}_{-1}$ such that $\mathfrak{g}_0$ does not contain operators commuting with the whole of W. Due to Lemma 4.2 this means that $\mathfrak{g}_{1,i} = 0$ for all $i < -1 - s$, i.e., the necessary condition for $\mathfrak{g} \neq \mathfrak{d}\mathfrak{h}$ is $-m \geq -1 - s$, i.e., $m \leq 1 + s$. Specifically,

1a) $s = 2, m = n \Longrightarrow n \leq 3$;

1b), 1c) $s = 1, m = n - 1 \Longrightarrow n \leq 3$;

1c) $s = 1, m = n \Longrightarrow n \leq 2$.

In cases 1a), 1b) for $n = 3$ we get exceptional Lie superalgebras $\mathfrak{vle}$ and $\mathfrak{kas}$.

The case 1c) for $n=3$ calls for subtler arguments. We have $\mathfrak{g}_{-1}=\mathop{\oplus}\limits_{i=0}^{2}\mathfrak{g}_{-1,i}$, where

$$\mathfrak{g}_{-1,0}=\mathbb{C}\Pi(1);\quad \mathfrak{g}_{-1,1}=\Pi(Span(\xi_j : j=1,2,3));\quad \mathfrak{g}_{-1,2}=\Pi(Span(\xi_j\xi_l : j,l=1,2,3)).$$

Suppose $\mathfrak{g}\neq\mathfrak{d}\mathfrak{h}$ and let $F=\Pi(\xi_1\xi_2)^*\subset\mathfrak{g}_1$, i.e.,

$$[F,\Pi(\xi_1\xi_2)]=z+h,\ \text{ where } h\in\mathfrak{g}_{0,0},\ \text{ i.e., } h=\sum_{i,j}a_{ij}\xi_i\partial_{\xi_j}.$$

Then: $F\in\mathfrak{g}_{1,-2}$ and, thus, $[F,\Pi(1)]=0$. By Jacobi identity,

$$0=[[F,\Pi(\xi_1\xi_2],\Pi(1)]=(z+h)\Pi(1)=-\Pi(1)+h(\Pi(1)).$$

Hence, $h(\Pi(1))=\Pi(1)$ and

$$\mathrm{div}h=-\sum_{i=1}^{3}a_{ii}=1. \tag{5.7.1.1}$$

Since $p(\Pi(\xi_1\xi_2))=\bar{1}$, we have

$$0=[[F,\Pi(\xi_1\xi_2)],\Pi(\xi_1\xi_2)]=(z+h)\Pi(\xi_1\xi_2)=-\Pi(\xi_1\xi_2)+h(\Pi(\xi_1\xi_2)).$$

Hence, $h(\Pi(\xi_1\xi_2))=\Pi(\xi_1\xi_2)$ and $a_{11}+a_{22}+\mathrm{div}h=1$ or

$$a_{11}+a_{22}=0. \tag{5.7.1.2}$$

Clearly, $[F,\Pi(\xi_3)]\in\mathfrak{g}_{0,-1}$, i.e., $[F,\Pi(\xi_3)]=\sum b_j\partial_{\xi_j}$. By Jacobi identity

$$\begin{gathered}0=[[F,\Pi(\xi_1\xi_2)],\Pi(\xi_3)]-[\Pi(\xi_1\xi_2),[F,\Pi(\xi_3)]]=\\ -\Pi(\xi_3)+h(\Pi(\xi_3))-(b_1\Pi(\xi_2)-b_2\Pi(\xi_1)).\end{gathered}$$

Hence, $a_{33}+\mathrm{div}h=1$, i.e., $a_{33}=0$. The last equality contradicts (5.7.1.1) and (5.7.1.2). □

2) $\mathfrak{h}_0=\mathfrak{svect}(0|n)$ for $n\geq 3$. The rank 1 operator with even covector exists only for the $\mathfrak{h}_0$-module: $\mathfrak{h}_{-1}=\Pi(T^0_0(\vec{0})$, hence, $\mathfrak{h}=(\mathfrak{h}_{-1},\mathfrak{h}_0)_*=\mathfrak{sle}^\circ(n;n)$.

We know that $\mathfrak{g}_0$ has no nontrivial central extentions, but it has an outer derivation, $D=\sum\xi_i\partial_{\xi_i}\in\mathfrak{vect}(0|n)$. So we have to consider the following cases:

2a) $\mathfrak{g}_0=\mathfrak{h}_0\oplus\mathbb{C}z$, $\mathfrak{g}_{-1}=\mathfrak{h}_{-1}$ as a linear space and $\mathfrak{g}_{-1}=\mathfrak{h}_{-1}\otimes(-1)$ as $\mathfrak{g}_0$-module;

2b) $\mathfrak{g}_0=\mathfrak{h}_0\Subset\mathbb{C}D$, $\mathfrak{g}_{-1}=\mathfrak{h}_{-1}$ as a linear space, where D acts as $\sum\xi_i\partial_{\xi_i}+\lambda\cdot 1$;

2c) $\mathfrak{g}_0=(\mathfrak{h}_0\oplus\mathbb{C}z)\Subset\mathbb{C}D$, $\mathfrak{g}_{-1}=\mathfrak{h}_{-1}$ as a linear space, where z acts as multiplication by -1 and D as $\sum\xi_i\partial_{\xi_i}$.

2d) $\mathfrak{g}_0=\mathfrak{h}_0\Subset\mathbb{C}(az+bD)$ for $a,b\in\mathbb{C}$ and $\mathfrak{g}_{-1}=\mathfrak{h}_{-1}$.

5.7.2. Proposition . *Let $(\mathfrak{g}_{-1},\mathfrak{g}_0)$ be one of the pairs 2a) – 2c) and $\mathfrak{g}=(\mathfrak{g}_{-1},\mathfrak{g}_0)_*$ its Cartan's prolong. Then in 2b) for $n=3$, $\lambda=0$ we have $\mathfrak{g}=\mathfrak{le}(3)$. In all other cases, $\mathfrak{g}=\mathfrak{d}\mathfrak{h}$.*

Proof. In all cases, $\mathfrak{g}_{-1}=\mathop{\oplus}\limits_{i=1}^{n-1}\mathfrak{g}_{-1,i}$. Due to Lemma 5.7.0, if $\mathfrak{g}\neq\mathfrak{d}\mathfrak{h}$, then $\mathfrak{g}_{1,-n+1}\neq 0$. On the other hand, $\mathfrak{g}_0=\mathop{\oplus}\limits_{i=-1}^{n-2}\mathfrak{g}_{-1,i}$ and $\mathfrak{g}_0$ does not contain elements that commute with the subspace $W=\mathfrak{g}_{-1,1}\oplus\mathfrak{g}_{-1,2}$. Hence, by Lemma 4.2, $\mathfrak{g}_{1,s}=0$ for all s such that $s+2<-1$. Thus, $-n+1\geq -3$, i.e., $n\leq 4$.

(Observe that $\mathfrak{svect}(0|3)\cong\mathfrak{spe}(3)$ and our Proposition is a particular case of 5.6.2.)

Let us consider the case $n=4$ more carefully. We have $\mathfrak{g}_{-1}=\mathop{\oplus}\limits_{i=1}^{3}\mathfrak{g}_{-1,i}$, where

$$\begin{aligned}&\mathfrak{g}_{-1,1}=Span(\Pi(\xi_i))_{i=1}^4;\ \mathfrak{g}_{-1,2}=Span(\Pi(\xi_i\xi_j))_{i=1}^4;\\ &\mathfrak{g}_{-1,3}=Span(\Pi(\xi_1\xi_2\xi_3)\}.\end{aligned}$$

Suppose $\mathfrak{g}_1 \neq \mathfrak{h}_1$. In all cases 2a) – 2d) there should exist elements $F \in \mathfrak{g}_{1,-3}$ and $\alpha z + \beta D \in \mathfrak{g}_0$ such that

$$g = [F, \Pi(\xi_1\xi_2\xi_3)] = -\alpha \cdot 1 + \beta D + h,$$

where $\alpha, \beta \in \mathbb{C}$, $\alpha\beta \neq 0$ and $h \in \mathfrak{h}_{0,0} = \mathfrak{sl}(4)$.

By grading reasons, $[F, \mathfrak{g}_{-1,1}] = 0$. Due to the Jacobi identity we have

$$h|_{\mathfrak{g}_{-1,1}} = \alpha \cdot 1 - \beta D|\mathfrak{g}_{-1,1} = (\alpha - \beta) \cdot 1.$$

Since $h \in \mathfrak{sl}(4)$, it follows that $\alpha - \beta = 0$ and $h = 0$. Without loss of generality we may assume that $\alpha = \beta = 1$. Then $\mathrm{ad} g|_{\mathfrak{g}_{-1,2}} = -1 + D|_{\mathfrak{g}_{-1,2}} = 1$. Hence, $\mathfrak{g}_{0,0} = \mathfrak{gl}(4) \cong \mathfrak{co}(6)$ and, as $\mathfrak{g}_{0,0}$-modules, $\mathfrak{g}_{-1,2} \cong \mathrm{id}_{\mathfrak{o}(6)} \otimes 1$. Note that $[\mathfrak{g}_{-1,2}, \mathfrak{g}_{1,-2}] \subset \mathfrak{g}_{0,0}$. Set

$$V = \{F \in \mathfrak{g}_{1,-2} \mid [F, \mathfrak{g}_{-1,2}] = 0\}.$$

Then the quotient $\mathfrak{g}_{1,-2}/V$ naturally embeds into the first component of Cartan prolong $(\Pi(\mathrm{id}), \mathfrak{co}(6))_* = \mathfrak{oh}(0|6)$. Due to Lemma 5.7.0 this contradicts the assumption $\mathfrak{g}_1 \neq \mathfrak{h}_1$. □

<u>3) $\mathfrak{h}_0 = \widetilde{\mathfrak{svect}}(0|n)$ for $n \geq 3$.</u> The rank 1 operator with even covector exists only for the $\mathfrak{h}_0$-module $\mathfrak{h}_{-1} = \frac{\Pi(\tilde{Vol}(0|n))}{\mathbb{C}(b+\alpha\xi_1\dots\xi_n)vol(\xi)}$. Hence, $\mathfrak{h} = (\mathfrak{h}_{-1}, \mathfrak{h}_0)_* = \widetilde{\mathfrak{sb}}_\mu(2^{n-1} - 1|2^{n-1})$ with $\mu = \frac{a}{b}$.

We know that $\mathfrak{h}_0$ does not have any nontrivial central extentions or outer differentiations. Let $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathbb{C}z$, then $\mathfrak{g}_{-1} = \mathfrak{h}_{-1}$, as $\mathfrak{h}_0$-modules, and let z act on $\mathfrak{g}_{-1}$ by multiplication by -1.

5.7.3. Proposition . $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{oh}$.

Proof. The components $\mathfrak{g}_k$ have the filtration by powers of ξ. Consider the corresponding bigraded Lie superalgebra $\mathrm{gr}\mathfrak{g}$. If $\mathfrak{g}_1 \neq \mathfrak{h}_1$, then, due to Lemma 5.7.0 $(\mathrm{gr}\mathfrak{g})_1 = (\mathrm{gr}\mathfrak{h})_1 \oplus (\mathrm{gr}\mathfrak{g})^*_{-1}$. But $(\mathrm{gr}\mathfrak{g})_{-1} = \bigoplus\limits_{i=0}^{2n-1} \mathfrak{g}_{-1,i}$. Hence, $(\mathrm{gr}\mathfrak{g})_{1,-2n+1} \neq 0$.

On the other hand, $(\mathrm{gr}\mathfrak{g})_0 = \bigoplus\limits_{i=-1}^{2n-2} \mathfrak{g}_{0,i}$ and $\mathrm{gr}\mathfrak{g}_0$ does not contain elements that commute with the subspace $W = \bigoplus\limits_{i=0}^{1} \mathfrak{g}_{-1,i}$. Hence, $(\mathrm{gr}\mathfrak{g})_{1,s} = 0$ for all s such that $s+1 < -1$. Therefore, $-2n + 1 + 1 \geq -1$ or $2n \leq 3$. This contradicts to the fact that $n \geq 2$. □

<u>4) $\mathfrak{h}_0 = \mathfrak{h}^o(0|n)$ for $n \geq 5$ and related subcases.</u> As follows from the description of the irreducible representations of these Lie superalgebras, see [Sha1], [Sha2], no induced representation has a rank 1 operator; among the exceptional representations only the representation in the superspace of functions could have qualified, but, it has no rank 1 operators either.

§6. Step 6. A particular case: case 3) of Corollary 3.1

Let $\mathfrak{g}_{-1} = \Lambda(\xi_1, \xi_2)$ or $\Pi(\Lambda(\xi_1, \xi_2))$; let $\mathfrak{g}_0 \subset \mathfrak{hei}(0|4) \Subset \mathfrak{o}(4)$, where

$$\begin{array}{rcl} \mathfrak{hei}(0|4) & = & Span(1, \xi_1, \xi_2, \partial_1, \partial_2); \\ \mathfrak{sl}_1(2) & = & Span(\xi_1\partial_2, \xi_2\partial_1, \xi_1\partial_1 - \xi_2\partial_2); \\ \mathfrak{sl}_2(2) & = & Span(\partial_1\partial_2, \xi_1\xi_2, \xi_1\partial_1 + \xi_2\partial_2 - 1). \end{array}$$

We realize the elements of $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ with vector fields, i.e., we embed $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ into $\mathfrak{vect}(2|2)$. Redenote the respective vectors of the monomial basis of $\mathfrak{g}_{-1} = \Pi(\Lambda(2))$:

$$\pi(\xi_1) = \partial_x, \ \pi(\xi_2) = \partial_y; \ \pi(1) = \partial_\eta, \ \pi(\xi_1\xi_2) = \partial_\zeta.$$

To the basis vectors from $\mathfrak{g}_0$ we assign the following vector fields of degree 0 that act in the space of functions in x, y, η, ζ:

$$\begin{array}{rcl}
\mathfrak{hei}(0|4): \quad -1 & \mapsto & x\partial_x + y\partial_y + \eta\partial_\eta + \zeta\partial_\zeta \\
\xi_1 & \mapsto & \eta\partial_x - y\partial_\zeta \\
\xi_2 & \mapsto & \eta\partial_y + x\partial_\zeta \\
\partial_1 & \mapsto & \zeta\partial_y - x\partial_\eta \\
\partial_2 & \mapsto & -\zeta\partial_x - y\partial_\eta
\end{array}$$

$$\begin{array}{lrcl}
\mathfrak{sl}_1(2): & \xi_1\partial_2 & \mapsto & -y\partial_x \\
& \xi_2\partial_1 & \mapsto & -x\partial_y \\
& \xi_1\partial_1 - \xi_2\partial_2 & \mapsto & y\partial_y - x\partial_x
\end{array}
\qquad
\begin{array}{lrcl}
\mathfrak{sl}_2(2): & \partial_2\partial_1 & \mapsto & -\zeta\partial_\eta \\
& \xi_1\xi_2 & \mapsto & -\eta\partial_\zeta \\
& \xi_1\partial_1 + \xi_2\partial_2 - 1 & \mapsto & -\eta\partial_\eta - \zeta\partial_\zeta
\end{array}$$

We see that this realization of $\mathfrak{g}_0$ is invariant under the changes of parity in $\mathfrak{g}_{-1}$. So, both $\mathfrak{g}_{-1} = \Lambda(2)$ and $\mathfrak{g}_{-1} = \Pi(\Lambda(2))$ bring about the same result (isomorphic algebras $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$).

Consider the following subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$:

$$\begin{array}{l}
\mathfrak{h}_0 = \mathfrak{hei}(0|4) \Subset \mathfrak{sl}_1(2) = \\
Span(1, \xi_1, \xi_2) \Subset Span(\partial_1, \partial_2, \xi_1\partial_2, \xi_2\partial_1, \xi_1\partial_1 - \xi_2\partial_2) \cong \\
(\Lambda(2) \setminus \mathbb{C} \cdot \xi_1\xi_2) \Subset \mathfrak{svect}(0|2).
\end{array}$$

We see that the Cartan's prolong $(\mathfrak{g}_{-1}, \mathfrak{h}_0)_*$ is isomorphic to $\mathfrak{b}_\infty(2;2)$.

The direct calculations show that

$$(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \cong (\mathfrak{g}_{-1}, \mathfrak{h}_0)_* \oplus \mathfrak{sl}_2(2) \text{ and } (\mathfrak{g}_{-1}, \mathfrak{hei}(0|4))_* \cong \mathfrak{g}_{-1} \oplus \mathfrak{hei}(0|4).$$

So a necessary condition for simplicity, i.e., $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$, implies that $\mathfrak{g}_0 = \mathfrak{h}_0$ and $\mathfrak{g} \cong \mathfrak{b}_{a,a}(2;2) \cong \mathfrak{b}_\infty(2;2)$.

§7. Step 7. $\mathbb{Z}$-gradings of vectorial Lie superalgebras

Let $\mathfrak{g} = \bigoplus\limits_{i \geq -d} \mathfrak{g}_i$ be a $\mathbb{Z}$-graded vectorial superalgebra. By a *Weisfeiler regrading* (or just regrading) of $\mathfrak{g}$ we understand the passage from $\mathfrak{g}$ to the Lie superalgebra $\mathfrak{h} = \bigoplus\limits_{i \geq -D} \mathfrak{h}_i$ isomorphic to $\mathfrak{g}$ as abstract Lie superalgebra and such that $\mathfrak{h}$ is transitive; $\mathfrak{h}^0 = \bigoplus\limits_{i \geq 0} \mathfrak{h}_i$ is a maximal Lie superalgebra of finite codimension; $D < \infty$, i.e., $\mathfrak{h}$ is of finite depth.

In what follows in this section $\mathfrak{h}$ denotes a Lie superalgebra obtained from $\mathfrak{g}$ by a regrading. Clearly, not every two gradings on the same algebra determine a bigrading; fortunately, it suffices for our purposes to consider only compatible gradings for which (7.1.1) is well-defined by definition.

For the proof of the next three Lemmas see [Sh14].

7.1.1. Lemma . *Let $\mathfrak{h} = \oplus_{i \geq -D} \mathfrak{h}_i$ be a W-regrading of the Lie superalgebra $\mathfrak{g} = \oplus_{j \geq -d} \mathfrak{g}_j$ that determines a bigrading:*

$$\mathfrak{h}_i = \bigoplus_{j \geq m(i)}^{M(i)} \mathfrak{h}_{i,j}, \text{ where } \mathfrak{h}_{i,j} = \mathfrak{h}_i \cap \mathfrak{g}_j. \tag{7.1.1}$$

Suppose $\dim(\mathfrak{h}_0 \cap \mathfrak{g}_-) = 0|k$. *Then the number $M = M(-1) - m(-1) + 1$ of the summands $\mathfrak{h}_{-1,j}$ in the decomposition* (7.1.1) *of $\mathfrak{h}_{-1}$ does not exceed the number of homogeneous components with respect to the $\mathbb{Z}$-grading of $\Lambda(\mathfrak{h}_0 \cap \mathfrak{g}_-)$.*

In particular, if $\mathfrak{h}_0 \cap \mathfrak{g}_-$ is homogeneous with respect to the $\mathbb{Z}$-grading in $\mathfrak{g}$, i.e., $\mathfrak{h}_0 \cap \mathfrak{g}_- \subset \mathfrak{g}_j$ for some j, then $M \leq k + 1$.

7.1.2. Lemma . *If $\mathfrak{h}_0 \cap \mathfrak{g}_- = 0$, then $\mathfrak{h}_0 \subset \mathfrak{g}_0$ and $\mathfrak{h}_{-1}$ is homogeneous.*

7.1.3. Lemma . *If $\mathfrak{h}_0 \cap \mathfrak{g}_- = 0$ and there exists a nonzero $x \in \mathfrak{g}_{-1} \cap \mathfrak{h}_-$, then the gradings of $\mathfrak{h}$ and $\mathfrak{g}$ coincide.*

7.2. How to describe all the W-regradings.

1) Determine $\mathfrak{h}_0 \cap \mathfrak{g}_{-1}$.

2) Construct a "minimal" (better, though informally speaking, "most tightly compressed") regrading with the given intersection, i.e., such that preserves in $\mathfrak{h}_{-1}$ all the elements of $\mathfrak{g}_{-1}$ except for those that have to go away in view of the condition on the intersection.

3) If the "minimal" regrading is a Weisfeiler one, then with the help of Lemmas 7.1.2 and 7.1.3 we prove that any other W-regrading with the given intersection coincides with the minimal one.

4) If the "minimal" regrading is not a Weisfeiler one, then with the help of Lemma 7.1.1 we prove that there are no W-regradings with the given intersection.

7.3. Theorem . *Any Weisfeiler regrading $\mathfrak{h}$ of the simple vectorial Lie superalgebra is a regrading from Tables I.2.2.*

Let $1 \leq r \leq m$, let $\varphi(s) = \{i_1, \dots, i_r\}$ be a set of distinct indices from 1 to m. Permuting the indeterminates, if necessary, we may and will assume that $\varphi(r) = \{1, \dots, r\}$. Set $\deg \xi_1 = \dots = \deg \xi_r = 0$ and define the remaining degrees from Tables I.2.2. In Tables I.2.1 and I.2.2 the terms $\mathfrak{h}_{-1}$, $\mathfrak{h}_0$ are indicated for the regraded algebras $\mathfrak{h}$ for the series and the exceptions, respectively.

Proof. 1) Suppose $\mathfrak{g} = \mathfrak{vect}(n|m) = \mathfrak{vect}(x, \xi)$ or $\mathfrak{g} = \mathfrak{svect}(n|m) = \mathfrak{svect}(x, \xi)$. Then $\mathfrak{g}_- = \mathfrak{g}_{-1} = Span(\partial_{x_1}, \dots, \partial_{x_n}, \partial_{\xi_1}, \dots, \partial_{\xi_n})$. Since the depth of $\mathfrak{h}$ if finite, $\deg x_i > 0$ for all i. Hence, $\partial_{x_i} \in \mathfrak{h}_- \cap \mathfrak{g}_{-1}$.

If $\deg \xi_j \neq 0$ for all j, then $\mathfrak{h}_0 \cap \mathfrak{g}_- = 0$ and by Lemma 7.1.3 $\mathfrak{h} = \mathfrak{g}$.

2) Suppose now that $\deg \xi_i = 0$ for $i \leq r$ and $\deg \xi_i \neq 0$ for $i > r$. Let us denote the regraded superalgebras by $\mathfrak{r} = \mathfrak{vect}(n|m; r)$ or $\mathfrak{r} = \mathfrak{svect}(n|m; r)$. Then

$$\mathfrak{r}_{-1} = Span(\partial_{x_1}, \dots, \partial_{x_n}; \partial_{\xi_j} \mid j > r) \otimes \Lambda(\xi_i \mid i \leq r).$$

Let us show that $\mathfrak{h} \cong \mathfrak{r}$.

Since $\partial_{x_i} \in \mathfrak{r}_{-1}$, then by Lemma 7.1.3 it suffices to show that $\mathfrak{r}_{-1} \cap \mathfrak{h}_0 = 0$. By definition of $\mathfrak{h}$ we have

$$\big(Span(\partial_x, \partial_{\xi_j}) \otimes 1\big) \cap \mathfrak{h}_0 = 0 \text{ for } j > r.$$

But $\partial_{\xi_i} \in \mathfrak{h}_0$ for $i < r$. Since $\mathfrak{h}_0$ is a subalgebra, the following implication holds:

if any element from $\mathfrak{r}_{-1}$ belongs to $\mathfrak{h}_0$, then the elements from $Span(\partial_x, \partial_{\xi_j})_{j>r}$ belong to $\mathfrak{h}_0$.

Therefore, $\mathfrak{r}_{-1} \cap \mathfrak{h}_0 = 0$.

3) For the other series take into account that the preserved form (ω_0, ω_1, α_0, α_1, $\tilde{\alpha_1}$, etc.) should be homogeneous. For the series $\mathfrak{h}$ and $\mathfrak{k}$ it is convenient to split the indeterminates into dual pairs: the even ones, $p_1, \dots, p_m$ and $q_1, \dots, q_m$ and the odd ones: $\xi_1, \dots, \xi_k$ and $\eta_1, \dots, \eta_k$; the remaining odd coordinates being the θ's. Then the Poisson bracket takes the form

$$\{f, g\}_{P.B.} = \sum_{i \leq m} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right)$$
$$-(-1)^{p(f)} \left[\sum_{i \leq k} \left(\frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i} \frac{\partial g}{\partial \xi_i} \right) + \sum_{j \leq n-2k} \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial \theta_i} \Big) \right].$$

Appropriate modifications for contact series are obvious.

Observe that the cases $\mathfrak{k}(1|2n; n)$ and $\mathfrak{m}(n; n)$ are obtained from the general considerations given above by "compressing" the grading: it is clear that for a non-compressed grading $\mathfrak{h}_{2a+1} = 0$ for all a.

The case of the exceptional grading of $\mathfrak{b}_\lambda(2)$ is considered in sec. I.2.2. □

W-regradings of $\mathfrak{ksle}(9|6)$. Though description of all possible regradings of $\mathfrak{ksle}(9|6)$ in [CK2] is correct, there is only given an outline of the idea (any regrading is determined by the degrees of the indeterminates the vectorial Lie algebra is expressed with) and the final result. Since some of the regradings of $\mathfrak{kas}$ were overlooked in [CK2], we give a detailed proof of the completeness of the list of regradings of $\mathfrak{ksle}(9|6)$; unfortunately, this was the place in the proof in [Sh14] written by analogy and we missed $\mathfrak{ksle}(9|11)$.

There are two "least codimension realizations" for $\mathfrak{ksle}(9|6)$ and to describe regradings is more convenient starting with Kac's realization: $\mathfrak{g}_{\bar 0} = \mathfrak{svect}(5)$; $\mathfrak{g}_{\bar 1} = \Pi(d\Omega^1)$, cf. sect. .

Set $\mathfrak{g}_{-2} = Span(\partial_1, \dots, \partial_5)$ and $\mathfrak{g}_{-1} = Span(dx_i dx_j \mid 1 \leq i < j \leq 5)$.

Let $\mathfrak{h}$ be a regrading of $\mathfrak{g}$. Then, clearly, $\mathfrak{g}_{-2} \subset \mathfrak{h}_-$ and $\mathfrak{h}_0 \cap \mathfrak{g}_- \subset \mathfrak{g}_{-1}$.

Since for any permutation $\sigma \in S_5$ we have

$$[dx_{\sigma(1)}dx_{\sigma(2)}, dx_{\sigma(3)}dx_{\sigma(4)}] = (-1)^{\mathrm{sign}(\sigma)}\frac{\partial}{\partial x_{\sigma(5)}},$$

it follows that $dx_{\sigma(1)}dx_{\sigma(2)}$ and $dx_{\sigma(3)}dx_{\sigma(4)}$ can not simultaneously appear in $\mathfrak{h}_0$ after the regrading. Hence, $\dim \mathfrak{h}_0 \cap \mathfrak{g}_- \leq 4$.

So we have to consider the following cases:

1) $\dim \mathfrak{h}_0 \cap \mathfrak{g}_- = 1$. In this case $\mathfrak{h}_0 \cap \mathfrak{g}_- \simeq Span(dx_1dx_2)$.

2) $\dim \mathfrak{h}_0 \cap \mathfrak{g}_- = 2$. In this case $\mathfrak{h}_0 \cap \mathfrak{g}_- \simeq Span(dx_1dx_2, dx_1dx_3)$.

3) $\dim \mathfrak{h}_0 \cap \mathfrak{g}_- = 3$. In this case either (a) $\mathfrak{h}_0 \cap \mathfrak{g}_- \simeq Span(dx_1dx_2, dx_1dx_3, dx_1dx_4)$ or (b) $\mathfrak{h}_0 \cap \mathfrak{g}_- \simeq Span(dx_1dx_2, dx_1dx_3, dx_2dx_3)$.

4) $\dim \mathfrak{h}_0 \cap \mathfrak{g}_- = 4$. In this case $\mathfrak{h}_0 \cap \mathfrak{g}_- \simeq Span(dx_1dx_2, dx_1dx_3, dx_1dx_4, dx_1dx_5)$.

The standard arguments (sec. 7.2 or based on Lemma 7.1.3 in [Sh14]) show that in the cases 2) and 3a) $\mathfrak{h}$ is not a W-regrading, whereas in case 1) we obtain $\mathfrak{ksle}(9|11)$, in case 3b) we obtain $\mathfrak{ksle}(11|9)$ and in case 4) we obtain $\mathfrak{ksle}(9|6)$. The corresponding W-regradings listed in [CK2] are:

1) $\deg x_1 = \deg x_2 = 3$, $\deg x_3 = \deg x_4 = \deg x_5 = 2$; $\deg d = -3$;

3b) $\deg x_1 = \deg x_2 = 1$, $\deg x_3 = \deg x_4 = \deg x_5 = 2$; $\deg d = -2$;

4) $\deg x_i = 1$ for $i = 1, \dots, 4$, $\deg x_5 = 2$; $\deg d = -\frac{3}{2}$.

§8. Tables

$\mathfrak{g}$	$\mathfrak{g}_{\bar 0}$	$R_{\bar 1}$	$\mathfrak{g}_{\bar 0}:\mathfrak{g}_{\bar 1}$
$\mathfrak{sl}(n\vert 1)$	$\mathfrak{sl}(n)\oplus\mathbb{C}h$	$\pm(\varepsilon_i-\delta)$	$(\mathrm{id}\otimes\chi_0)\oplus(\mathrm{id}\otimes\chi_0)^*$
$\mathfrak{sl}(n\vert m)$ $n\neq m$	$\mathfrak{sl}(n)\oplus\mathfrak{sl}(m)\oplus\mathbb{C}h$	$\pm(\varepsilon_i-\delta_j)$	$\left(\mathrm{id}_{\mathfrak{sl}(n)}\otimes\mathrm{id}^*_{\mathfrak{sl}(m)}\otimes\chi_0\right)\oplus\left(\mathrm{id}_{\mathfrak{sl}(n)}\otimes\mathrm{id}^*_{\mathfrak{sl}(m)}\otimes\chi_0\right)^*$
$\mathfrak{psl}(n\vert n)$ $n>2$	$\mathfrak{sl}(n)\oplus\mathfrak{sl}(n)$	$\pm(\varepsilon_i-\delta_j)$	$\left(\mathrm{id}_{\mathfrak{sl}(n)}\otimes\mathrm{id}^*_{\mathfrak{sl}(n)}\right)\oplus\left(\mathrm{id}_{\mathfrak{sl}(n)}\otimes\mathrm{id}^*_{\mathfrak{sl}(n)}\right)^*$
$\mathfrak{osp}(1\vert 2n)$	$\mathfrak{sp}(2n)$	$\pm\delta_j$	$\mathrm{id}_{\mathfrak{sp}(2n)}$
$\mathfrak{osp}(2\vert 2n)$	$\mathfrak{sp}(2n)\oplus\mathbb{C}h$	$\pm\varepsilon\pm\delta_j$	$(\mathrm{id}_{\mathfrak{sp}(2n)}\otimes\chi_1)\oplus\left(\mathrm{id}^*_{\mathfrak{sp}(n)}\otimes\chi_{-1}\right)$
$\mathfrak{osp}(m\vert 2n)$ $m>2$	$\mathfrak{sp}(2n)\oplus\mathfrak{o}(m)$	$\pm\varepsilon_i\pm\delta_j$ if $m=2k$ $\pm\varepsilon_i\pm\delta_j$ and $\pm\delta_j$ if $m=2k+1$	$\mathrm{id}_{\mathfrak{sp}(2n)}\otimes\mathrm{id}_{\mathfrak{o}(m)}$
$\mathfrak{osp}_\alpha(4\vert 2)$	$\mathfrak{sl}_1(2)\oplus\mathfrak{sl}_2(2)\oplus\mathfrak{sl}_3(2)$	$\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3$	$\mathrm{id}_1\otimes\mathrm{id}_2\otimes\mathrm{id}_3$
$\mathfrak{psq}(n)$	$\mathfrak{sl}(n)$	$\pm(\varepsilon_i-\varepsilon_j)$ and 0	ad
$\mathfrak{ag}_2$	$\mathfrak{sl}(2)\oplus\mathfrak{g}_2$	$\pm\varepsilon_i\pm\delta;\pm\delta$ $i=1,2,3;\ \sum\varepsilon_i=0$	$\mathrm{id}_{\mathfrak{sl}(2)}\otimes R(\pi_1)$
$\mathfrak{ab}_3$	$\mathfrak{sl}(2)\oplus\mathfrak{o}_7$	$\frac{1}{2}(\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\delta)$	$\mathrm{id}_{\mathfrak{sl}(2)}\otimes\mathrm{spin}_7$

Table 8.1. Odd roots of simple Lie superalgebras of non-vectorial type and their relatives.

$\mathfrak{g}$	$\mathfrak{psl}(n\vert n)$, $n>2$	$\mathfrak{psl}(2\vert 2)$	$\mathfrak{psq}(n)$
$\mathfrak{eg}$	$\mathfrak{sl}(n\vert n)$	$\begin{cases}\mathfrak{sl}(2\vert 2)\\ \mathfrak{spo}(0\vert 4)\end{cases}$	$\mathfrak{sq}(n)$
$\mathfrak{der}\ \mathfrak{eg}$	$\mathfrak{gl}(n\vert n)$	$\begin{cases}\mathfrak{gl}(2\vert 2)\\ \mathfrak{po}(0\vert 4)\oplus\mathbb{C}E\end{cases}$	$\mathfrak{q}(n)$

Table 8.2. Central extensions and outer derivations of simple Lie superalgebras of non-vectorial type and their relatives.

§9. Appendix. Irreducible representations of simple finite dimensional Lie superalgebras of vector fields and their relatives

9.1. Induced representations. Consider $\mathcal{L}=\mathfrak{vect}(0|m)$ and its subsuperalgebras $\mathfrak{svect}(0|m)$ and $\mathfrak{g}=\mathfrak{h}^o(0|m)$ in their standard filtration.

Let us prepare a supply of finite dimensional irreducible $\mathcal{L}$-modules. Take an irreducible L_0-module L^χ with highest weight χ; assume that the highest weight vector (as well as the whole L^χ) is even. Set $\mathcal{L}_1L^\chi=0$; this allows one to define the $\mathcal{L}$-module by setting

$$I(\chi)=\mathrm{ind}^{\mathcal{L}}_{\mathcal{L}_0}(L^\chi)=U(\mathcal{L})\otimes_{\mathcal{L}_0}L^\chi.$$

It turns out that for the generic χ the modules $I(\chi)$ are irreducible; the exceptional — reducible — modules line up into a sequence which is exact for $\mathfrak{vect}(0|m)$ and $\widetilde{\mathfrak{svect}}(0|m)$ and "almost exact" for $\mathfrak{svect}(0|m)$, see [BL1].

To formulate the answer graphically, let us pass from the induced modules $I(\chi)$ to their duals, the coinduced modules $T(\chi)=(I(\chi\otimes(-\mathrm{tr})))^*$, where tr is the one-dimensional $\mathfrak{gl}=\mathcal{L}_0$-module determined by the trace, see sec. 0.6 and [BL1].

Theorem . *Let $\mathcal{L} = \mathfrak{vect}(0|m)$, $\mathfrak{svect}(0|m)$ or $\widetilde{\mathfrak{svect}}(0|m)$. Then the $\mathcal{L}$-modules $T(\chi)$ are irreducible except for the cases when L^χ is isomorphic to one of the following: $S^k(\mathrm{id})$ or $S^k(\mathrm{id}^*) \otimes (-\mathrm{tr})$, where* id *is the identity representation of L_0. These exceptional modules, denoted by Ω^k and Σ_{-k}, respectively, are interpreted as the spaces of k-th degree differential or, respectively, $(-k)$-th degree integral forms on the $(0|m)$-dimensional supermanifold. The exterior differential and the Berezin integral unite the exceptional modules in one (infinite in both directions) exact sequence:*

$$\dots \xrightarrow{d} \Sigma_{-k} \xrightarrow{d} \dots \xrightarrow{d} \Sigma_{-1} \xrightarrow{d} \Sigma_0 \xrightarrow{\int} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega^k \xrightarrow{d} \dots$$

and, therefore, the kernels (or the images) of the avove maps are the only irreducible submodules of the exceptional modules if $\mathcal{L}$ is of series $\mathfrak{vect}$ or $\widetilde{\mathfrak{svect}}$.

For the $\mathcal{L} = \mathfrak{svect}(0|m)$, since $\Sigma_0 \cong \Omega^0$, the module of functions, Ω^0, has an additional submodule $T_0^0(\vec{0})$ of the functions with integral zero.

We see that except for $m = 2$ (which was already considered since $\mathfrak{vect}(0|2) \simeq \mathfrak{sl}(1|2) \simeq \mathfrak{osp}(2|2)$) an operator of rank 1 can only occure in the modules $T(\chi)$ coinduced from the 1-dimensional module, or in a quotient of such. Or in the modules obtained from such by applying the change of parity. The corresponding cases are listed in tables of sec. I.2.1. For example, the corresponding rank 1 operators from $\mathfrak{vect}(0|m)$ and $\widetilde{\mathfrak{svect}}(0|m)$ are

$$\xi_1 \dots \xi_m \frac{\partial}{\partial \xi_i} \text{ realized in } \Lambda(\xi) \text{ and even in } Vol(\xi)^\lambda \text{ for any } i \text{ if } m \text{ is even or}$$
$$\text{in } \Pi(\Lambda(\xi)) \text{ or } \Pi(Vol(\xi)^\lambda) \text{ for any } i \text{ if } m \text{ is odd.}$$

In the divergence-free case we take $\xi_1 \dots \xi_{m-1} \frac{\partial}{\partial \xi_m}$ and analogous operators.

Several of our exceptional vectorial superalgebra exist due to the fact that adding the trivial center to the above-considered $\mathcal{L}$ we get in certain small dimensions new nontrivial Cartan prolongs or mk-prolongs. For the details see [Sh14].

If $\mathcal{L} = \mathfrak{h}^o(0|m)$ realised preserving the 2-form ω, then the exceptional modules are still realized in the spaces of the differential forms (since $\Sigma_{-i} \simeq \Omega^i$ thanks to nondegeneracy of ω). The complete description of irreducible modules over $\mathcal{L} = \mathfrak{h}^o(0|m)$ and all its relatives — central extensions (i.e., finite dimensional Poisson Lie superalgebras) and superalgebras of derivations — is performed by Shapovalov [Sha1], [Sha2]. Except for the cases of small dimension when an occasional isomorphism occurs (as $\mathfrak{h}^o(0|4) \simeq \mathfrak{psl}(2|2)$) and, therefore, are already considered, there are no operators of rank 1.

§10. Appendix. On primitive Lie superalgebras

Recall that a filtered Lie (super)algebra $\mathfrak{g}$ (or an associated graded algebra) is called *primitive* if it possesses a maximal subalgebra of finite codimension which does not contain an homogeneous ideal of $\mathfrak{g}$.

Let $\mathfrak{r} \subset \mathfrak{vect}(\xi)$ be a subalgebra such that $pr_{-1}(\mathfrak{r}) = \mathfrak{vect}_{-1}(\xi)$, where pr_{-1} is the projection of $\mathfrak{vect}(\xi)$ onto $\mathfrak{vect}_{-1}(\xi) = Span(\frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_n})$.

Let $\mathfrak{h} = \underset{i \geq -d}{\oplus} \mathfrak{h}_i$ be a *primitive* Lie superalgebra (with $\underset{i \geq 0}{\oplus} \mathfrak{h}_i$ being its maximal subalgebra) such that $\mathfrak{h}_{-1}$ is an irreducible $\mathfrak{h}_0$-module and $\mathfrak{h}_-$ is generated by $\mathfrak{h}_{-1}$. Consider the following two Lie superalgebras:

$$\mathfrak{g}_{min} = \mathfrak{h} \otimes \Lambda(n) \Subset \mathfrak{r} \text{ and } \mathfrak{g}_{max} = (\mathfrak{h}_{-1} \otimes \Lambda(n), \mathfrak{h}_0 \otimes \Lambda(n) \Subset \mathfrak{r})_*, \qquad (*)$$

where $(V,\mathfrak{g})_*$ denote the Cartan prolong of $(V,\mathfrak{g})$. It is not difficult to see that any Lie superalgebra $\mathfrak{p}$ sandwiched between $\mathfrak{g}_{min}$ and $\mathfrak{g}_{max}$, i.e., such that $\mathfrak{g}_{min}\subset\mathfrak{p}\subset\mathfrak{g}_{max}$, is primitive; $\mathfrak{g}_{min}$ and $\mathfrak{g}_{max}$ are also primitive.

We conjecture that every primitive $\mathfrak{p}$ is of the form $\mathfrak{g}_{min}$ for a suitable $\mathfrak{h}$. Be it so or not, there are indescribably many primitive Lie superalgebras.

§11. Appendix. On Lie superalgebra $E(2,2)$ from [K7]

While this paper was being written and [Sh14] refereed, there appeared a proof of a part of our classification. In it, V. Kac [K7] states, among other things, that there exist six infinite dimensional exceptional simple $\mathbb{Z}$-graded Lie superalgebras. In addition to the five (considered as abstract Lie superalgebras) exceptions described already in [Sh5] Kac distinguished one more "exception" — the Lie superalgebra $E(2,2)$ in his notations.

Let us prove that, $E(2,2)$ is, actually $\mathfrak{b}^\circ_\infty(2;2)$ or, in Kac's notations, "$SKO(2,3;1)$ with subprincipal gradation". So it is not an exception at all. Indeed, by definition, $E(2,2)$ is the Cartan prolong $\mathfrak{g}=(\mathfrak{g}_{-1},\mathfrak{g}_0)_*$, where "the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ gives rise to the linear Lie superalgebra $spin^\circ_4$." This means that $\mathfrak{g}_0$ is the semidect sum $\mathfrak{g}_0=\mathfrak{hei}(0|4)\in\mathfrak{sl}(2)$ and $\mathfrak{g}_{-1}$ is the space of unique irreducible representation of $\mathfrak{hei}(0|4)$ for which the central element corresponds to the identity operator. Hence, we may identify $\mathfrak{g}_{-1}$ either with $\Lambda(2)=\Lambda(\xi_1,\xi_2)$ or (due to uniqueness) with $\Pi(\Lambda(2))=\Pi(\Lambda(\xi_1,\xi_2))$. The second variant is more convenient for our purposes.

The action of $\mathfrak{hei}(0|4)$ is given by operators $(1,\xi_1,\xi_2,\partial_{\xi_1},\partial_{\xi_2})$, while $\mathfrak{sl}(2)$ acts via the standard representation on the even component of $\mathfrak{g}_{-1}$ and via the trivial representation on the odd component of $\mathfrak{g}_{-1}$.

Now what is $\mathfrak{b}^\circ_\infty(2;2)$? It is the Cartan prolong $\mathfrak{h}=(\mathfrak{h}_{-1},\mathfrak{h}_0)_*$, where $\mathfrak{h}_{-1}=\Pi(\Lambda(2))=\Pi(\Lambda(\eta_1,\eta_2))$ and

$$\mathfrak{h}_0=Span(1,\eta_1,\eta_2)\in\mathfrak{svect}(0|2)\cong(Span(1,\eta_1,\eta_2)\in Span(\partial_{\eta_1},\partial_{\eta_2}))\in\mathfrak{sl}(2)\cong\mathfrak{g}_0.$$

The subalgebra $\mathfrak{sl}(2)$ acts via the standard representation on the subspace $Span(\Pi(\eta_1),\Pi(\eta_2))$, i.e., on the even component of $\mathfrak{h}_{-1}$ and via the trivial representation on the subspace $Span(\Pi(1),\Pi(\eta_1\eta_2))$, i.e., on the odd component of $\mathfrak{g}_{-1}$.

We get the isomorphism $\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\cong\mathfrak{h}_{-1}\oplus\mathfrak{h}_0$, hence, the isomorphism $\mathfrak{g}\cong\mathfrak{h}$.

In a recent paper [CK2] the result of this section is briefly stated but the description of $\mathfrak{g}_0$ in [CK2] should still be corrected.

References

[ALSh] Alekseevsky D., Leites D., Shchepochkina I., Examples of simple Lie superalgebras of vector fields. C. r. Acad. Bulg. Sci., v. 34, N 9, 1980, 1187–1190 (in Russian)

[B1] Berezin F., Representations of the supergroup $U(p,q)$. Functional Anal. Appl. 10 (1976), no. 3, 221–223

[B2] Berezin F., *Introduction to superanalysis.* Edited and with a foreword by A. A. Kirillov. With an appendix by V. I. Ogievetsky. Translated from the Russian by J. Niederle and R. Kotecký. Translation edited by Dimitri Leites. Mathematical Physics and Applied Mathematics, 9. D. Reidel Publishing Co., Dordrecht, 1987. xii+424 pp.

[BL] Bernstein, J.N., Leites, D.A., Integration of differential forms on supermanifolds, Functional Anal. Appl. 11 (1977), 219–221.

[BL1] Bernstein, J.N., Leites D.A., Irreducible representations of Lie superalgebras of type W and S, C. R. de l'Acad. de Sci. Bulgare 32 (1979), 277–278. (in Russian; English translation in: Selecta Math. Soviet. 3 (1983/84), no. 1, 63–68)

[BL2] Bernstein J., Leites D., Invariant differential operators and irreducible representations of Lie superalgebras of vector fields. Sel. Math. Sov., v. 1, no. 2, 1981, 143–160

[CK1] Cheng S., Kac V., Generalized Spencer cohomology and filtered deformations of $\mathbb{Z}$-graded Lie superalgebras. Adv. Theor. Math. Phys. 2 (1998), no. 5, 1141–1182; math.RT/9805039

[CK2] Cheng S., Kac V., Structure of some $\mathbb{Z}$-graded Lie superalgebras of vector fields, Transformation groups, v. 4, 1999, 219–272

[D] Deligne P. et al (eds.) *Quantum fields and strings: a course for mathematicians.* Vol. 1, 2. Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996–1997. AMS, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1999. Vol. 1: xxii+723 pp.; Vol. 2: pp. i–xxiv and 727–1501

[DGS] Donin J., Gurevich D., Shnider S., Quantization of function algebras on semisimple orbits in $\mathfrak{g}^*$, q-alg/9607008

[Di] Dixmier J. *Enveloping algebras.* Revised reprint of the 1977 translation. Graduate Studies in Mathematics, 11. American Mathematical Society, Providence, RI, 1996. xx+379 pp.

[FLS] Feigin, B. L.; Leites, D. A.; Serganova, V. V. Kac-Moody superalgebras. Group theoretical methods in physics, Vol. 1–3 (Zvenigorod, 1982), 631–637, Harwood Academic Publ., Chur, 1985

[FLSf] Fulp R., Lada T., Stasheff J., Noether's variational theorem II and the BV formalism, math.QA/0204079

[FK] Freund P., Kaplansky I., Simple supersymmetries. J. Math. Phys. 17 (1976), no. 2, 228–231

[G] Grozman P., Classification of bilinear invariant operators on tensor fields. Functional Anal. Appl. 14 (1980), no. 2, 127–128; id., ESI-preprint 1114 (http://www.esi.ac.at)

[GL1] Grozman P., Leites D., Defining relations for Lie superalgebras with Cartan matrix. hepth-9702073; Czech. J. Phys., Vol. 51, 2001, No. 1, 1–22

[GL2] Grozman P., Leites D., Lie superalgebras of supermatrices of complex size. Their generalizations and related integrable systems. In: by E. Ramírez de Arellano, M. V. Shapiro, L. M. Tovar and N. L. Vasilevski (eds.) *Proc. Internatnl. Symp. Complex Analysis and related topics*, Mexico, 1996, Birkhauser Verlag, 1999, 73–105

[GLS1] Grozman P., Leites D., Shchepochkina I., Lie superalgebras of string theories., hep-th 9702120; Acta Mathematica Vietnamica, v. 26, 2001, no. 1, 27–63

[GLS2] Grozman P., Leites D., Shchepochkina I., Analogs of Riemannian tensor on supermanifolds, Theor. and Math. Physics, to appear

[GLS3] Grozman P., Leites D., Shchepochkina I., Invariant differential operators on supermanifolds and The Standard Model. In: Olshanetsky M., Vainshtein A., (eds.) *M. Marinov memorial volume*, World Sci., 2002, to appear math.RT/0202193

[GLS4] Grozman P., Leites D., Shchepochkina I., Defining relations for the exceptional Lie superalgebras of vector fields. In: C. Duval, L. Guieu and V. Ovsienko (eds.) *The orbit method in geometry and physics* (A. A. Kirillov Festschrift), Progress in Mathematics, Birkhauser, 2002, to appear math.RT/0202025

[GPS] Gomis J., Paris J., Samuel S., Antibracket, Antifields and Gauge-Theory Quantization, Phys. Rept. 259 (1995) 1–191

[GQS] Guillemin, V.; Quillen, D.; Sternberg, S., The classification of the complex primitive infinite pseudogroups. Proc. Nat. Acad. Sci. U.S.A. 55, 1966, 687–690; id., The classification of the irreducible complex algebras of infinite type. J. Analyse Math., 18, 1967, 107–112

[J] Joseph A., Infinite dimensional Lie algebras in mathematics and physics. *Groups theoretical methods in Physics* (Proc. Third Internat. Colloq., Centre Phys. Théor., Marceille, 1974), vol 2, 582–665

[GSW] Green M., Schwarz J., Witten E. *Superstring theory*, vv.1, 2. Second edition. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1988. x+470 pp., xii+596 pp.

[K0] Kac V. G., Some algebras related to the quantum field theory, XI-th All-Union Algebr. Coll., Kishinev, 1971, 140–141 (in Russian)

[K1] Kac V. G., Classification of simple Lie superalgebras. Funktsional. Anal. i prilozheniya 9, n. 3, 1975, 91–92 (in Russian)

[K1C] Kac V. G., Letter to editors, Funkcional. Anal. i prilozheniya 10, n. 2, 1976, 93 (in Russian)

[K1.5] Kac, V. G. A sketch of Lie superalgebra theory. Comm. Math. Phys. 53 (1977), no. 1, 31–64

[K2] Kac V. G., Lie superalgebras. Adv. Math. v. 26, 1977, 8–96

[K3] Kac V. G., *Infinite Dimensional Lie Algebras.* 3rd ed. Cambridge Univ. Press, Cambridge, 1992

[K4] Kac V., Some remarks on Lie superalgebra theory. ca. 1978 (preprint)

[K5] Kac V., Characters of typical representations of classical Lie superalgebras. Comm. Algebra 5 (1977), no. 8, 889–897; id., Representations of classical Lie superalgebras. Differential geometrical methods in mathematical physics, II (Proc. Conf., Univ. Bonn, Bonn, 1977), pp. 597–626, Lecture Notes in Math., 676, Springer, Berlin, 1978

[K6] Kac V., Classification of infinite-dimensional simple linearly compact Lie superalgebras. Talk at a Conference dedicated to S. Novikov's 60-th birthday, Steklov Math. Inst., Moscow, May, 1998; ESI-preprint 605 (http://www.esi.ac.at)

[K7] Kac V., Classification of infinite-dimensional simple linearly compact Lie superalgebras. Adv. Math. 139 (1998), no. 1, 1–55

[K8] Kac V., Superconformal algebras and transitive group actions on quadrics. Comm. Math. Phys. 186 (1997), no. 1, 233–252; Erratum: Comm. Math. Phys. 217 (2001), no. 3, 697–698

[K9] Kac V., Classification of infinite-dimensional simple groups of supersymmetry and quantum field theory. math.QA/9912233. GAFA 2000 (Tel Aviv, 1999). Geom. Funct. Anal. 2000, Special Volume, Part I, 162–183

[KvdL] Kac, V. G., van de Leur, J. W. On classification of superconformal algebras. In: S. J. Gates, Jr., C. R. Preitschopf and W. Siegel (eds.) *Strings '88* Proceedings of the workshop held at the University of Maryland, College Park, Maryland, May 24–28, 1988. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989, 77–106

[KR] Kac, V. G., Rudakov A., Representations of the Exceptional Lie superalgebra E(3,6): I. Degeneracy conditions, math-ph/0012049; Representations of the Exceptional Lie superalgebra E(3,6): II. Four series of degenerate modules, math-ph/0012050

[KW] Kac V., Wakimoto M., Integrable highest weight modules over affine superalgebras and number theory. Lie theory and geometry, Progr. Math., 123, Birkhäuser Boston, Boston, MA, 1994, 415–456

[Kap] Kaplansky I., Superalgebras. Pacific J. Math. 86 (1980), no. 1, 93–98 (id., "Graded Lie algebras I, II", preprints, Univ. Chicago, Chicago, III., 1975)

[KV] Konstein, S. E.; Vasiliev, M. A. Supertraces on the algebras of observables of the rational Calogero model with harmonic potential. J. Math. Phys. 37 (1996), no. 6, 2872–2891; Konstein, S. E. Supertraces on some deformations of Heisenberg superalgebra. Supersymmetry and quantum field theory (Kharkov, 1997), Lecture Notes in Phys., 509, Springer, Berlin, 1998, 305–311; Konstein, S. E. An example of simple Lie superalgebra with several invariant bilinear forms, math-ph/0112063

[Ko1] Kotchetkoff Yu. Déformations de superalgébres de Buttin et quantification. C. R. Acad. Sci. Paris, ser. I, 299: 14 (1984), 643–645

[Ko2] Kochetkov Yu. Yu., Singular vectors and invariant operators. (Russian) Uspekhi Mat. Nauk 43 (1988), no. 5(263),201–202; translation in Russian Math. Surveys 43 (1988), no. 5, 242–24; id., Singular vectors, invariant operators and discrete modules for the Lie algebra of Hamiltonian vector fields $H(n)$. (Russian) Uspekhi Mat. Nauk 44 (1989), no. 5(269),167–168; translation in Russian Math. Surveys 44 (1989), no. 5, 201–202

[Ko3] Kochetkov Yu. Yu., Induced irreducible representations of Leites superalgebras. In: Onishchik (ed.) *Problems in group theory and homological algebra*, Yaroslav. Gos. Univ., Yaroslavl, 1983, 120–123 (Russian); Kochetkov, Yu. Yu., Discrete modules for Lie superalgebras of the series Le. Onishchik (ed.) *Problems in group theory and homological algebra*, Yaroslav. Gos. Univ., Yaroslavl, 1989, 142–148 (Russian)

[Ko4] Kochetkov Yu. Yu., Deformations of Lie superalgebras. VINITI depositions (cited in [L6])

[Kos] Kostant B., Quantization and unitary representations. I. Prequantization. Lectures in modern analysis and applications, III. Lecture Notes in Math., Vol. 170, Springer, Berlin, 1970, 87–208

[L1] Leites D., New Lie superalgebras and mechanics. Soviet Math. Doklady, v. 18, n. 5, 1977, 1277–1280

[L2] Leites D., Lie superalgebras. In: *Modern Problems of Mathematics. Recent developments*, v. 25, VINITI, Moscow, 1984, 3–49 (in Russian; English translation in: J. Soviet Math., v. 30 (6), 1985, 2481–2512)

[L3] Leites D., *Supermanifold theory*, Karelia Branch of the USSR Acad. Sci., Petrozavodsk, 1983, 200 pp. (in Russian) (expanded in [L5])

[L4] Leites D., Clifford algebra as a superalgebra and quantization. Theor. Math. Phys., v. 58, N2, 1984, 229–232

[L5] Leites D. (ed.) *Seminar on Supermanifolds*, Reports of Stockholm University, 30/1988-13 and 31/1988-14

[L6] Leites D., *Quantization. Supplement* 3. In: F. Berezin, M. Shubin. *Schrödinger equation*, Kluwer, Dordrecht, 1991, 483–522

[L7] Leites D., Selected problems of supermanifold theory, 1–21 (Duke Math. J., v. 54, no. 2, 649–656

[L8] Leites D., The Riemann tensor for nonholonomic manifolds. math.RT/0202213 Homology, Homotopy and Applications, vol 4 (2), 2002, 397–407

[LS] Leites D., Serganova V., Metasymmetry and Volichenko algebras. Phys. Lett. B 252 (1990), no. 1, 91–96; Symmetries wider than supersymmetries. In: S. Duplij and J. Wess (eds.) *Noncommutative structures in mathematics and physics*, Proc. NATO Advanced Research Workshop, Kiev, 2000. Kluwer, 13–30

[LS1] Leites D., Serganova V., Defining relations for classical Lie superalgebras. I. Superalgebras with Cartan matrix or Dynkin-type diagram. Topological and geometrical methods in field theory (Turku, 1991), 194–201, World Sci. Publishing, River Edge, NJ, 1992

[LSe] Leites D., Sergeev A., Orthogonal polynomials of discrete variable and Lie algebras of complex size matrices. In: Proceedings of M. Saveliev memorial conference, MPI, Bonn, February, 1999, MPI-1999-36, 49–70; Theor. and Math. Physics, v. 123, no. 2, 205–236 (Russian), 582–609

[LSh1] Leites D., Shchepochkina I., Toward classification of simple vectorial Lie superalgebras. In: [L5], 31/1988-14, 235-278; Leites D., Toward classification of classical Lie superalgebras. In: Nahm W., Chau L. (eds.) *Differential geometric methods in theoretical physics* (Davis, CA, 1988), NATO Adv. Sci. Inst. Ser. B Phys., 245, Plenum, New York, 1990, 633–651

[LSh2] Leites D., Shchepochkina I., Quivers and Lie superalgebras, Czech. J. Phys. vol 47, n 12, 1997, 1221–1229; id. Quivers and Lie superalgebras. In: Eisenbud D., Martsinkovsky A., Weyman J., (eds.) *Commutative Algebra, Representation theory and Combinatorics. Conference in honor of D. Buchsbaum*, Northeastern U., Boston, October 18-20, 1997, 67

[LSh3] Leites D., Shchepochkina I., How to quantize antibracket, preprint ESI-875 (www.esi.ac.at); Theor. and Math. Physics, v. 126, no. 3, 339–369

[LSh4] Leites D., Shchepochkina I., Automorphisms and real forms of vectorial Lie superalgebras, in preparation

[LS5] Leites D., Shchepochkina I., Deformations of simple vectorial Lie superalgebras, in preparation

[M] Mathieu O., Class of simple graded Lie algebras of finite growth, Inv. Math., 108, 1992, 455–519

[O] Ochiai, T., Classification of the finite nonlinear primitive Lie algebras. Trans. Amer. Math. Soc. 124, 1966, 313–322

[OV] Onishchik A., Vinberg E., *Algebraic groups and Lie groups*, Springer, 1987

[R1] Rudakov, A.N., Irreducible representations of infinite-dimensional Lie algebras of the Cartan type, Mathematics of the USSR-Izvestia 8 (1974), 835–866.

[R2] Rudakov, A.N., Irreducible representations of infinite-dimensional Lie algebras of type S and H, Mathematics of the USSR-Izvestia 9 (1975), 465–480.

[Sha1] Shapovalov, A.V., Finite-dimensional irreducible representations of Hamiltonian Lie superalgebras, Mathematics of the USSR-Sbornik, 35, 1979, 541–554

[Sha2] Shapovalov, A. V. Invariant differential operators and irreducible representations of finite-dimensional Hamiltonian and Poisson Lie superalgebras. (Russian) Serdica, 7, 1981, no. 4, 337–342

[SNR] Scheunert, M.; Nahm, W.; Rittenberg, V. Classification of all simple graded Lie algebras whose Lie algebra is reductive. I, II. Construction of the exceptional algebras. J. Mathematical Phys. 17 (1976), no. 9, 1626–1639, 1640–1644

[Sch] Scheunert M., *The theory of Lie superalgebras. An introduction*. Lecture Notes in Mathematics, 716. Springer, Berlin, 1979. x+271 pp.

[Sh3] Shchepochkina I., Exceptional simple infinite dimensional Lie superalgebras. (Russian) C. R. Acad. Bulgare Sci. 36 (1983), no. 3, 313–314

[Sh5] Shchepochkina I., The five exceptional simple Lie superalgebras of vector fields, hep-th 9702121; Five simple exceptional Lie superalgebras of vector fields. (Russian) Funktsional. Anal. i Prilozhen. 33 (1999), no. 3, 59–72, 96 translation in Funct. Anal. Appl. 33 (1999), no. 3, 208–219 (a preliminary version of [Sh14])

[Sh14] Shchepochkina I., Five exceptional simple Lie superalgebras of vector fields and their fourteen regradings. Represent. Theory, v. 3, 1999, 3 (1999), 373–415

[Sh-M] Shchepochkina I., Maximal subalgebras of classical Lie superalgebras. In [L5], 32/1988-15, 1–43; In: C. Duval, L. Guieu and V. Ovsienko (eds.) *The orbit method in geometry and physics* (A. A. Kirillov Festschrift), Progress in Mathematics, Birkhäuser, to appear (a shorter version: hep-th 9702122)

[ShP] Shchepochkina I., Post G., Explicit bracket in an exceptional simple Lie superalgebra. Internat. J. Algebra Comput. 8 (1998), no. 4, 479–495 (physics/9703022)

[Se1] Serganova V., Characters of irreducible representations of simple Lie superalgebras. Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 583–593 (electronic)

[Se2] Serganova V., Classification of simple real Lie superalgebras and symmetric superspaces. (Russian) Funktsional. Anal. i Prilozhen. 17 (1983), no. 3, 46–54

[Se3] Serganova V., Automorphisms of simple Lie superalgebras. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 48 (1984), no. 3, 585–598; Feigin, B. L.; Leites, D. A.; Serganova, V. V. Kac-Moody superalgebras. Group theoretical methods in physics, Vol. 1–3 (Zvenigorod, 1982), 631–637, Harwood Academic Publ., Chur, 1985

[S] Sergeev A., Irreducible representations of solvable Lie superalgebras, In: [L5], 22/1988, 1–12; math.RT/9810109 and Represent. Theory 3 (1999), 435–443

[St] Sternberg S., *Lectures on differential geometry*, Chelsey, 2nd edition, 1985

[Sud] DeWitt B., van Nieuwenhuizen P., Explicit construction of the exceptional superalgebras $F(4)$ and $G(3)$. J. Math. Phys. 23 (1982), no. 10, 1953–1963; Sudbery A., Octonionic description of exceptional Lie superalgebras. J. Math. Phys. 24 (1983), no. 8, 1986–1988

[T] Tanaka N., On differential systems, graded Lie algebras and pseudogroups. J. Math. Kyoto Univ. 10, 1970, 1–82; id, On infinitesimal automorphisms of Siegel domains, J. Math. Soc. Japan 22, 1970, 180–212; id., On the equivalence problems associated with simple graded Lie algebras. Hokkaido Math. J., 8 (1979), no. 1, 23–84

[vdL] van de Leur J., A classification of contragredient Lie superalgebras of finite growth. Comm. Algebra 17 (1989), no. 8, 1815–1841

[W] Weisfeiler B., Infinite dimensional filtered Lie algebras and their relation with the graded Lie algebras, Funktsional. Anal. i prilozhen. 2, n.1, 1968, 94–95 (in Russian)

[WZ] Wess, J., Zumino, B., *Supergauge transformations in four dimensions.* Nuclear Phys. **B70** (1974), 39–50; Wess J., *Supersymmetry/supergravity. Concepts and trends in particle physics* (Schladming, 1986), 29–58, Springer, Berlin, 1987; Wess J., *Introduction to supersymmetric theories. Frontiers in particle physics '83* (Dubrovnik, 1983), 104–131, World Sci. Publishing, Singapore, 1984; Wess J., Bagger J., *Supersymmetry and supergravity.* Second edition. Princeton Series in Physics. Princeton University Press, Princeton, NJ, 1992. x+259 pp.

[WL] Woit P., String Theory: An Evaluation, physics/0102051; Larsson T., Symmetries of Everything, math-ph/0103013

[Y] Yamaguchi K., Differential systems associated with simple graded Lie algebras. *Progress in differential geometry*, Adv. Stud. Pure Math., 22, Math. Soc. Japan, Tokyo, 1993, 413–494

[1]Department of Mathematics, University of Stockholm, Kräftriket hus 6, SE-106 91, Stockholm, Sweden; mleites@matematik.su.se and [2]Independent University of Moscow, Bolshoj Vlasievsky per, dom 11, RU-121 002 Moscow, Russia; Ira@Paramonova.mccme.ru