Arbeitstagung 1997

3. Arbeitstagung der neuen Serie

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

Germany

Mathematisches Institut der
Universität Bonn
Wegelerstraße 10
53115 Bonn

Germany

MPI 97-62

Inhalt

All the abstracts in this preprint had to be prepared in a very short time. The organizers of the Arbeitstagung would like to thank all the authors for their contributions.

Teilnehmerliste:

Amerik, E. (Bayreuth)
André, Yves (Paris VI)
Arone, Greg (U. of Chicago)
Ashikaga, Tadashi (Tohoku/MPI)
Askitas, Nikolaos (MPI)
Aus der Fünten, J. (Bonn)
Badescu, Lucian (Bukarest/MPI)
Ballmann, Werner (Bonn)
Baues, Hans Joachim (MPI)
Baumgartner, Martin (Bonn)
Beckert, Herbert (U. Leipzig)
Belabas, Karim (MPI)
Berndt, Rolf (Hamburg)
Besser, Amnon (MPI)
Bielawski, Roger (MPI)
Böcherer, Siegfried (Mannheim)
Bödigheimer, Carl-Friedrich (Bonn)
Borcherds, Richard (DPMMS Cambridge)
Bouard, Joël (Paris VII)
Bourguignon, Jean Pierre (I.H.E.S.)
Boxall, John (Caen)
Brin, Michael (U. of Maryland)
Bunke, Ulrich (Göttingen)
Cazaran, Jilyana (U. of Tasmania)
Chmutov, S. (MPI)
Cohen, Henri (Bordeaux I)
Cohen, Paula (Lille)
Cornelissen, Gunther (Gent)
Dais, Dimitrios (Tübingen)
Dasbach, Oliver (Düsseldorf)
Demchenko, Oleg (Grenoble)
Dombrowski, Peter (Köln)
Donaldson, Simon (Oxford)
Dulinski, Joachim (Aachen)
Dzhumadil'daev, Askar (Akad.Sci. Alma Ata)
Ebeling, Wolfgang (Hannover)
Eells, James (Cambridge/Warwick)
Erle, Dieter (Dortmund)
Fahl, Michael (Bonn)
Faltings, Gerd (MPI)
Feldhusen, Dirk (Bonn)
Fels, Marten (Bonn)
Fel'shtyn, A. (Greifswald)
Fialowski, Alice (Budapest/MPI)
Fischer, Gerd (Düsseldorf)
Florin, Nicolae (Humboldt-U.)
Friedlander, Eric (Northwestern U.)
Friedlander, Susan (U. of Illinois)
Friedrich, Reinhold (Bonn)
Franke, J. (Bonn)
Gajda, W. (Poznan/MPI)
Galicki, Krzysztof (U. of New Mexico/MPI)
Gangl, Herbert (IEM Essen)
van der Geer, Gerard (Amsterdam)
Gekhtman, Michael (MPI)
Gelimson (Augsburg)
Getzler, Ezra (Northwestern U.)
Ginzburg, Victor (U. of Chicago)
Goncharov, Alexander (MPI)
Gong, Donggeng (MPI)
Greuel, Gert-Martin (Kaiserslautern)
Grigoriev, Dima (Pennsylvania St.U.)
Gritsch, Ursula (Bonn)
Gritsenko, Valeri (POMI/MPI)
Grunewald, Fritz (Düsseldorf)
Gurevich, Alexander (Humboldt-U.)
Habegger, Nathan (Nantes)
Hahne, Christoph (IEM Essen)
Hambleton, Ian (McMaster U./MPI)
Hamenstädt, Ursula (Bonn)
Harbater, David (U. of Pennsylvania)
Harder, Günter (U. Bonn/MPI)
Häsemeyer, Christian (Bonn)
Hauser, Herwig (Innsbruck)
Hefer, Torsten (Bonn)
Heinze, Joachim (Springer-Verlag)
Henn, Hans-Werner (MPI)
Henne, Benedikt (Bonn)
Hertling, Claus (Bonn)
Hildebrandt, Stefan (Bonn)
Hirzebruch, Friedrich (MPI)
Höhn, Gerald (DPMMS Cambridge)
Holzapfel, R.-P. (Humboldt-U.)
Huebschmann, Johannes (Lille)
Huntebrinker, Wolfgang (Düsseldorf)
Husemöller, Dale (Haverford/MPI)
Jahnel, Jörg (Göttingen)
Jeschonnek, V. (Bonn)
Kabanov, Alexandre (Michigan St.U./MPI)
Kaenders, Rainer (Nijmegen)
Kaiser, Christian (Bonn)
Kankaanrinta, Marja (Helsinki)
Kantor, Jean-Michel (Paris VII)
Karcher, Hermann (Bonn)
Katzarkov, L. (UC Irvine)
Kaufmann, Ralph (MPI)
Kausz, Ivan (Köln)
Kazdan, Jerry (U. of Pennsylvania/Bonn)
Kellerhals, Ruth (Göttingen)
Kleinjung, Thorsten (MPI)
Knapp Karlheinz (Wuppertal)
Knörrer, H. (ETH Zürich)
Koepke, Peter (Bonn)
Köhler, Kai (Bonn)
Köhnken, Olaf (Bonn)
Kondo, Shigeyuki (Nagoya/MPI)
Kontsevich, Maxim (I.H.E.S.)
Koschorke, Ulrich (Siegen)
Kotschik, Dieter (Basel/MPI)
Kounchev, Ognyan (Duisburg)
Kramer, Jürg (Humboldt-U.)
Kramer, Wolfram (Bonn)
Kullmann, Peter (Basel/MPI)
Kurke, Herbert (Humboldt-U.)
Lang, Serge (Yale/MPI)
Leeb, Bernhard (Bonn)

Teilnehmerliste:

Lemaire, Luc (Brüssel)
Lieb, Ingo (Bonn)
Liu, Xiaobo (MPI)
Loeser, Francois (Paris VI)
Looijenga, Eduard (Utrecht)
Losik, Mark (MPI)
Löwe, Benedikt (Berkeley)
Lupton, G. (MPI)
Lustig, M. (Bochum)
Mahowald, Mark (Northwestern U.)
Manin, Yuri (MPI)
Markman, Eyal (U. of Massachusetts)
Matthes, Roland (Kassel)
Mayer, Karl Heinz (Dortmund)
Merkulov, Sergei (MPI)
Meyer, Dagmar (MPI)
Miller, Andrea (MPI)
Miller, Haynes (M.I.T.)
Millson, John (U. of Maryland)
Moatty, Laurent (MPI)
Monastyrski, M. (Moskau)
Moree, Pieter (MPI)
Müller, Werner (Bonn)
Nahm, Werner (Bonn)
Nastold, Hans-Joachim (Münster)
Neeman, Amnon (U. of Virginia)
Neumann, Walter (Melbourne/MPI)
Olivier, Reinhard (Bonn)
Ontaneda, Pedro (MPI)
Oort, Frans (Utrecht)
Parmentier, Serge (Lyon)
Pauly, Marc (Brüssel)
Pick, Lubos (Prag)
Pink, Richard (Mannheim)
Platonov, Vladimir (Düsseldorf)
Poletaeva, Elena (Michigan St.U./MPI)
Pop, Florian (Bonn)
Priplata, Christine (Düsseldorf)
Pukhlikov, Aleksandr (MPI)
Puschnigg, Michael (Heidelberg)
Raina, Ashok (T.I.F.R./MPI)
Raphael, Albrecht (Bonn)
Riedel, Jens-Erik (Bonn)
Rogmann, Sascha (Düsseldorf)
Rösgen, Michael (Bonn)
Rovinsky, M. (MPI)
Rudyak, Yuli (Heidelberg)
Runge, B. (Berlin)
Saito, K. (Kyoto)
Salomonsen, Gorm (Bonn)
Schappacher, Norbert (Strasbourg)
Scharf, Alois (Bonn)
Scharlau, Rudolf (Dortmund)
Schechtman, Vadim (MPI)
Schimmrigk, Rolf (Bonn)
Schlippe, Johannes (Bonn)
Schmalz, Gerd (Bonn)
Schmickler-Hirzebruch, Ulrike (Vieweg-Verlag)
Schmidt, Rolf (Hamburg)
Schoof, René (Rom II)
Schröder, Michael (Mannheim)
Schulze-Pillot, R. (Saarbrücken)
Schwachhöfer, Lorenz (U. Leipzig)
Sczech, Robert (Rutgers)
Shatz, Stephen (U. of Pennsylvania)
Sick, Oliver (Bonn)
Skorobogatov, Alexei (MPI)
Soibelman, Yan (MPI)
Spitzweck, Markus (Bonn)
Stahlke, Colin (Bonn)
Steinert, Bernd (Bonn)
Steinhauer, Mark (Bonn)
Stix, Jakob (Bonn)
Stoll, Michael (Düsseldorf)
Strauch, Matthias (Bonn)
Strooker, J.R. (Utrecht)
Teicher, Mina (Bar-Ilan U.)
Tschinkel, Yuri (U. of Illinois)
Turner, Paul (Aberdeen)
Tuschmann, W. (MPI Leipzig)
Vaillant, Boris (Bonn)
Vaintrob, Arkady (MPI)
Van de Ven, A. (Leiden)
Venkov, Alexei (Steklov St. Petersburg)
Violetta, Pilyugina (Greifswald)
Voevodski, Vladimir (Northwestern U./MPI)
Wajnryb, Bronislaw (Haifa)
Walgenbach, M. (Dortmund)
Walther, Hansjörg (Bonn)
Wang, Weiqiang (MPI)
Weselmann, Uwe (MPI)
Wolfart, Jürgen (Frankfurt)
Yang, Tonghai (U. of Michigan/MPI)
Yui, Noriko (Queen's Univ.)
Zagier, Don (MPI)
Zeidler, E. (MPI Leipzig)
Ziller, Wolfgang (Philadelphia)
Zuo, Kang (Kaiserlautern)

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

and

Mathematisches Institut
der Universität Bonn
Wegelerstraße 10
53115 Bonn

Program of the Mathematische Arbeitstagung 1997 (I)

Friday, June 20, 1997

3:30 – 4:15 p.m.	Opening and first program discussion
5:00 – 6:00 p.m.	S. Donaldson (Oxford) Kähler geometry, geometric invariant theory and Hamiltonian dynamics
8:00 – 11 (?) p.m.	Rector's Party Festsaal der Universität, Hauptgebäude (entrance from "Am Hof" street across from Bouvier bookstore)

Saturday, June 21, 1997

10:15 – 11:15 a.m.	L. Schwachhöfer (Leipzig) On the classification of holonomy groups
12:00 – 1:00 p.m.	A. Goncharov (MPI) Multiple polylogarithms at roots of unity and motivic Lie algebras
5:00 – 5:30 p.m.	V. Schechtman (MPI) Local structure of moduli spaces
5:30 – 6:00 p.m.	V. Ginzburg (Chicago) Hilbert schemes and reductive groups

Sunday, June 22, 1997

10:00 – 10:15 a.m.	Program discussion (II)
10:15 – 11:15 a.m.	R. Borcherds (Cambridge) Vertex algebras
12:00 – 1:00 p.m.	R. Pink (Mannheim) Motives and Hodge structures over function fields
5:00 – 6:00 p.m.	M. Kontsevich (I.H.E.S.) Deformation quantization

All lectures will take place in the "Großer Hörsaal", Wegelerstraße 10. There will be *tea breaks* on Saturday and Sunday from 11:15 to 12 a.m. and 4:15 to 5 p.m. At this time also *mail* will be distributed and you will have the opportunity to pay your *Tagungsbeitrag* of 30 marks. *Lists of participants* and other information will be available at the entrance of the lecture room. All participants are requested to put their name on the list!
All Arbeitstagung participants and those accompanying them are invited to the *Rector's Party* this evening.

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

and

Mathematisches Institut
der Universität Bonn
Wegelerstraße 10
53115 Bonn

Program of the Mathematische Arbeitstagung 1997 (II)

Monday, June 23, 1997

10:15 – 11:15 a.m.	Y. Tschinkel (U. of Illinois) Height zeta functions
12:00 – 9:00 p.m.	Boat trip to Engers Departure with "Carmen Sylva" next to the Kennedy Bridge

Tuesday, June 24, 1997

10:00 – 10:15 a.m.	Program discussion (III)
10:15 – 11:15 a.m.	B. Leeb (Bonn) Rigidity of symmetric spaces and Euclidean buildings
12:00 – 1:00 p.m.	D. Harbater (U. of Pennsylvania) Grothendieck-Teichmüller group and covers of $\mathbf{P}^1$
5:00 – 6:00 p.m.	L. Katzarkov (MPI) Complex surfaces and very interesting fundamental groups
7:30 – 9:30 p.m.	Chamber Music Concert: Moscow-Königsberg String Quartet Aula der Universität, Hauptgebäude (entrance from "Am Hof" street across from Bouvier bookstore)

Wednesday, June 25, 1997

10:15 – 11:15 a.m.	H. Miller (M.I.T.) Elliptic moduli in algebraic topology
12:00 – 1:00 p.m.	W. Nahm (Bonn) Branes

All lectures will take place in the "Großer Hörsaal", Wegelerstraße 10.

There are *tea breaks* on Tuesday and Wednesday from 11:15 to 12 a.m. and 4:15 to 5 p.m. At this time also *mail* will be distributed and you will have the opportunity to pay your *Tagungsbeitrag* of 30 marks.

Lists of participants and other information will be available at the entrance of the lecture room. All participants are requested to put their name on the list!

All Arbeitstagung participants and those accompanying them are warmly invited to the boat trip on Monday and to the chamber music program on Tuesday evening.

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

and

Mathematisches Institut
der Universität Bonn
Wegelerstraße 10
53115 Bonn

Program of the Mathematische Arbeitstagung 1997 (III)

Wednesday, June 25, 1997

4:45 – 5:00 p.m.	Program discussion (IV)
5:00 – 6:00 p.m.	E.Getzler (Northwestern U.) Counting elliptic curves in projective varieties

Thursday, June 26, 1997

10:15 – 11:15 a.m.	V. Gritsenko (MPI) Discriminants of $K3$-surfaces and Kac-Moody algebras
12:00 – 1:00 p.m.	F. Pop (Bonn) The anabelian conjecture
5:00 – 6:00 p.m.	M. Mahowald (Northwestern U.) Curves of higher genus and homotopy theory

All lectures will take place in the "Großer Hörsaal", Wegelerstraße 10.

There are *tea breaks* on Tuesday and Wednesday from 11:15 to 12 a.m. and 4:15 to 5 p.m. At this time also *mail* will be distributed and you will have the opportunity to pay your *Tagungsbeitrag* of 30 marks.

Lists of participants and other information will be available at the entrance of the lecture room. All participants are requested to put their name on the list!

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

and

Mathematisches Institut
der Universität Bonn
Wegelerstraße 10
53115 Bonn

Program of the Mathematische Arbeitstagung 1997 (IV)

Friday, June 27, 1997

10:15 – 11:15 a.m.	J. Millson (U. of Maryland) Artin groups which are not fundamental groups of quasi-projective varieties
12:00 – 1:00 p.m.	V. Voevodsky (MPI) Motivic homotopy theory
5:15 – 6:15 p.m. (Colloquium lecture)	W.D. Neumann (Melbourne) Hilbert's 3rd problem and 3-manifolds

All lectures will take place in the "Großer Hörsaal", Wegelerstraße 10.

There are *tea breaks* on Friday from 11:15 to 12 a.m. and 4:45 to 5:15 p.m. At this time also *mail* will be distributed and you will have the opportunity to pay your *Tagungsbeitrag* of 30 marks.

For *information material* see the desks at the entrance of the lecture room.

Vorträge

S.K. Donaldson The Mathematical Institute, Oxford

"Kähler Geometry, Hamiltonian dynamics and Geometric Invariant Theory"

1. The problem

Let (V, w_0) be a compact Kähler manifold: we seek a metric $w_0 + i\bar{\partial}\partial\phi$ in the same cohomology class whose scalar curvature R is constant.

Remarks
(i) In complex dimension 1 we have existence and (essential) uniqueness by the uniformization theorem.
(ii) In higher dimensions there are well-known obstructions (e.g. $\mathbb{CP}^2$ blown up at a point admits no solution)
(iii) If $c_1(V) = \lambda[w_0]$ then const. scal. curvature $\Leftrightarrow$ Kähler Einstein. This is the case which has been studied most with an enormous body of work due to Calabi, Yau, Tian and many others.

2. The conjecture

Suppose we have a line bundle $L \to V$ with $c_1(L) = [w_0]$. For $k >> 0$ we get an embedding by $| L^k |$ of V in $\mathbb{CP}^n$; thus a point $[v]$ in the relevant Hilbert Scheme S parametrising sub-varieties of $\mathbb{CP}^n$. The group $PGL(n+1), \mathbb{C})$ acts on S.

Conjecture ("Generalised Yau conjecture")
A constant scalar curvature metric exists if and only if $[v]$ is a "stable point" for the $PGL(n+1, \mathbb{C})$ action on S, and in that case the solution i) unique up to the action of holomorphic automorphisms. [This generalises a conjecture made by Yau in the Kähler Einstein case.]

3. Formal set-up

The main point of the talk is to describe a way of throwing this problem into a standard form in which one has:
i) A Kähler manifold Z and a group K acting on Z by Kähler isometries;
ii) A moment map $\mu : Z \to \text{Lie}(K)^*$ for the K-action.
iii) A complexification K^c of K and a holomorphic action of K^c on Z.

The general principle in this situation is that one expects to be able to make an identification

$$\mu^{-1}(0)/K = Z^s/K^c \qquad [*]$$

where $Z^s \subset Z$ is an open subset of "stable points". In finite dimensions this principle is due to Kempf-Ness-Kirwan ... There are familiar infinite-dimensional examples arising in Yang-Mills Theory, where K is a gauge group and Z a space of connections.

For the case at hand, consider a compact symplectic manifold (M, w) and assume for simplicity that $H^1(M) = 0$. Let J be the space of compatible almost-complex structures on M. The group G of symplectomorphisms of (M, w) acts on J, preserving a natural Kähler metric on J. The moment map is given by the "Hermitian scalar curvature", which reduces to the ordinary scalar curvature in the Kähler (i.e. integrable) case. The complexified group G^c does not really exist, as a group, but one can still make sense of its "orbits" in J. The conclusion of this discussion is that the conjecture in (2) above would follow if one could:

a) prove a version of the familiar identification [*] in this context, such that:
b) The notion of "stability" for the action on J matches up with the finite-dimensional, algebro-geometric one, via Hilbert schemes.

4. Geometry of space of Kähler metrics

The quotient space G^c/G is interpreted as the space of Kähler metrics, and a substitute for the 1-parameter subgroups in G^c is provided by the geodesics in G^c/G in the sense defined by S. Semmes. One natural question is: can any two points in G^c/G be joined by a geodesic? This corresponds to a Dirichlet problem for a degenerate Monge-Ampere equation on $V \times [0,1] \times S^1$. An affirmative answer to this geodesic question would imply the uniqueness of constant scalar curvature metrics (invoking a general convexity principle). One can also formulate an analogue of the "Hilbert criterion" for stability, using these geodesics in G^c/G.

The classification of holonomies of torsion free connections

Lorenz Schwachhöfer

Let M be a smooth connected manifold and let ∇ be a torsion free connection on TM. Such a connection provides a way of parallel translating tangent vectors along piecewise smooth paths; indeed, if γ is such a path from p to q then the parallel translation along γ induces a linear isomorphism $P_\gamma : T_pM \to T_qM$. For any point $p \in M$ we define the *holonomy group of* ∇ *at* p by

$$\mathcal{H}_p := \{P_\gamma \mid \gamma \text{ a } p\text{-based loop in } M\} \subset \mathrm{Aut}(T_pM).$$

It is well-known that the identity component of $\mathcal{H}_p$ is a closed Lie subgroup of $\mathrm{Aut}(T_pM)$ and that moreover, $\mathcal{H}_p \subset \mathrm{Aut}(T_pM)$ and $\mathcal{H}_q \subset \mathrm{Aut}(T_qM)$ are isomorphic subgroups where the isomorphism is induced by conjugation with some P_γ where γ is any path from p to q. Thus, if we fix a linear isomorphism $\imath : V \to T_pM$ where V is a vector space of the appropriate dimension, then the subgroup $\mathcal{H} := \imath(\mathcal{H}_p) \subset \mathrm{Aut}(V)$ is well-defined up to conjugation, independent of the choice of $\imath$ or p. $\mathcal{H}$ is called the *holonomy* of ∇.

Next, if $\pi : (\tilde{M}, \tilde{\nabla}) \to (M, \nabla)$ is the universal cover of M, then $\mathcal{H}_{\tilde{p}}^{\tilde{\nabla}} \cong (\mathcal{H}_p^{\nabla})_0$ and thus, after passing to the universal cover if necessary, we may assume that $\mathcal{H}$ is connected and hence a closed Lie subgroup. In this talk, we shall only be concerned with this case.

The holonomy problem which was posed by Cartan and Lichnerowicz, is then the following question:

Which irreducible connected closed Lie subgroups $H \subset \mathrm{Aut}(V)$ can occur as the holonomy group of a torsion free affine connection?

The notion of the holonomy group was introduced in the 1926 by Élie Cartan who used it to classify all Riemannian locally symmetric spaces. In fact, each symmetric space can be represented as $M = G/H$, and the isotropy group H coincides with the holonomy group $\mathcal{H}$. Thus, as a sub-problem to the holonomy problem, we get the classification of symmetric spaces. This task has been achieved by Cartan in 1926 for Riemannian symmetric spaces, and by Berger in 1957 for general symmetric spaces.

The next step was the *Ambrose-Singer Holonomy Theorem* which describes the Lie algebra $\mathfrak{hol}_p$ of $\mathcal{H}_p$ in terms of the curvature endomorphisms:

$$\mathfrak{hol}_p = \langle\{(P_\gamma R)(x_q, y_q) \mid x_q, y_q \in T_qM,\ \gamma \text{ a path from } q \text{ to } p.\}\rangle \subset \mathrm{End}(V).$$

This motivated Berger in 1955 to pose the following criterion. If $\mathfrak{h} \subset \mathrm{End}(V)$ is an irreducible Lie subalgebra then we define the space of formal curvatures

$$K(\mathfrak{h}) := \{R : \Lambda^2 V \to \mathfrak{h} \mid R(x,y)z + R(y,z)x + R(z,x)y = 0\}$$

and

$$\underline{\mathfrak{h}} := \{R(x,y) \mid x, y \in V, R \in K(\mathfrak{h})\}.$$

Thus, by the Ambrose-Singer Theorem, $\mathfrak{h}$ can occur as the Lie algebra of a holonomy group only if $\underline{\mathfrak{h}} = \mathfrak{h}$. A Lie subgroup whose algebra satisfies this criterion is called a *Berger group*. This splits the holonomy problem into two parts.

Problem A Classify all irreducible Berger groups.

Problem B For the irreducible Berger groups, decide if they can occur as holonomies.

Berger himself gave a complete solution to problem A in the case of metric representations, i.e. for those subgroups $H \subset O(V, <\,,\,>)$ where $<\,,\,>$ is a non-degenerate symmetric bilinear form on V. These correspond to the holonomies of (pseudo-)Riemannian connections. He also gave a partial classification of the remaining Berger groups. For all these entries, problem B was solved affirmatively in the following decades (until 1986). This is due to the efforts of many mathematicians, e.g. Calabi, Alekseevski, Bryant etc.

In the early 1990s, Bryant found several new Berger groups and was also able to solve problem B for these new holonomies. Other examples of Berger groups were found in joint work with Q.-S. Chi and S. Merkulov, and problem B was solved for these problems as well.

Finally, in joint work with S. Merkulov, we classified all possible Berger groups, and found yet some other new entries. This classification was obtained by direct methods; however, it was pointed out by W. Ziller that the list of possible holonomies is related to the classically known list of symmetric spaces. More precisely, the classification result of *complex* Berger groups may be stated as follows.

Classification Theorem Let $H_{\mathbb{C}} \subset \mathrm{Aut}(V_{\mathbb{C}})$ be a semi-simple irreducible complex Lie subgroup, and let $K \subset H_{\mathbb{C}}$ be the maximal compact subgroup.

1. If there is an irreducible real symmetric space of the form G/K, then $H_{\mathbb{C}}$ is a Berger group.
2. If there is a irreducible hermitian symmetric space of the form $G/(U(1)\cdot K)$, then both $H_{\mathbb{C}}$ and $\mathbb{C}^* \cdot H_{\mathbb{C}}$ are Berger groups.
3. If there is a irreducible quaternionic symmetric space of the form $G/(Sp(1)\cdot K)$, then $H_{\mathbb{C}}$ is a Berger group.
4. The items 1. – 3. yield a complete list of Berger groups, except for the following:
   (a) $G_2^{\mathbb{C}} \subset \mathrm{Aut}(\mathbb{C}^7)$,
   (b) $Spin(7, \mathbb{C}) \subset \mathrm{Aut}(\mathbb{C}^8)$,
   (c) $\mathbb{C}^* Sp(2, \mathbb{C}) \subset \mathrm{Aut}(\mathbb{C}^4)$.

Since real Berger subgroups remain Berger subgroups after complexification, it is not hard to obtain a complete list of real Berger subgroups from this list as well. Thus, this completely answers problem A.

There are several holonomy groups in this classification that had been previously unknown. These are precisely those which correspond to the quaternionic symmetric spaces. Thus, for these entries, we still need to solve problem B. This is done by the following method which had been established in joint work with Q.-S. Chi and S. Merkulov.

Let V be a finite dimensional vector space, $\mathfrak{h} \subset \mathrm{End}(V)$ a Lie sub-algebra, and let $H \subset \mathrm{Aut}(V)$ be the corresponding connected Lie group.

Let $W := \mathfrak{h} \oplus V$. Denote elements of $\mathfrak{h}$ and V by $A, B, \ldots$ and $x, y, \ldots$, respectively, and elements of W by $w, w', \ldots$. We may regard W as the semi-direct product of Lie algebras, i.e. we define a Lie algebra structure on W by the equation

$$[A + x, B + y] := [A, B] + A \cdot y - B \cdot x.$$

It is well-known that this induces a natural Poisson structure on the dual space W^* (the so-called *Kirillov bracket*) which we denote by $\{\ ,\ \}_K$. Now, we wish to perturb this Poisson structure as follows. Regarding elements $A + x, B + y \in W$ as linear functions on W^*, we define

$$\{A + x, B + y\}(p) := \{A + x, B + y\}_K(p) + \Phi(p)(x, y). \tag{1}$$

Here, $\Phi := \phi \circ pr$, where $pr : W^* \to \mathfrak{h}^*$ is the natural projection, and where $\phi : \mathfrak{h}^* \to \Lambda^2 V^*$ is a smooth map satisfying

1. ϕ is H-equivariant,
2. for every $p \in \mathfrak{h}^*$, the dual map $(d\phi_p)^* : \Lambda^2 V \to \mathfrak{h}$ is contained in $K(\mathfrak{h})$.

It is easy to check that these conditions on ϕ are equivalent to saying that the bracket in (1) is indeed Poisson.

Let $\pi : S \to U$ be a *symplectic realization* of an open subset $U \subset W^*$, i.e. S is a symplectic mainfold, π is a submersion which is compatible with the Poisson structures on S and P. For each $w \in W$, we define the vector fields

$$\xi_w := \#(\pi^*(w)) \in \mathfrak{X}(S),$$

where $w \in W \cong T^*W^*$ is regarded as a 1-form on W^*. Then the map $w \mapsto \xi_w$ is *pointwise injective*, i.e. $\Xi := \{\xi_w \mid w \in W\} \subset TS$ is a distribution on S whose rank equals the dimension of W. For the bracket relations, we compute

$$\begin{array}{lcll} [\xi_A, \xi_B] & = & \xi_{[A,B]} & \\ [\xi_A, \xi_x] & = & \xi_{A \cdot x} & \\ [\xi_x, \xi_y](s) & = & \xi_{d\Phi(p)^*(x,y)} & \text{where } p = \pi(s). \end{array} \tag{2}$$

This implies, of course, that the distribution ξ on S is *integrable*. Moreover, the first equation in (2) implies that the flow along the vector fields ξ_A induces a local H-action on S. Let $F \subset S$ be a maximal integral leaf of ξ. Clearly, F is H-invariant, and after shrinking F, we may assume that $M := F/H$ is a mainfold. Then F can be extended to a principal H-bundle over M, and the vector fields $\{\xi_x, x \in V\}$ define a connection on M whose holonomy is contained in H. In fact, for the exapmles we consider, we can always achieve that the holonomy is *all* of H.

We then get the following remarkable result.

Theorem Let $\mathfrak{h}$ and V be as before, and suppose there is a *quadratic polynomial map* $\phi : \mathfrak{h}^* \to \Lambda^2 V$ satisfying the conditions from above and such that its (linear) differential

$$d\phi : \mathfrak{h}^* \to K(\mathfrak{h})$$

is a linear isomorphism. Then every torsion free connection whose holonomy algebra is contained in $\mathfrak{h}$ comes from the above construction.

In particular, it turns out that this theorem applies to all the newly found holonomies. This has some remarkable consequences. For example, it means that the moduli space of connections with one of these holonomies is finite dimensional. Moreover, in many cases, there exist local symmetries, i.e. vector fields on M whose flow preserves the connections.

Multiple polylogarithms at roots of unity and motivic Lie algebras

A.B. Goncharov

June 26, 1997

1. Multiple polylogarithms. We define them via power series expansion:

$$Li_{n_1,...,n_m}(x_1,...,x_m) \quad = \quad \sum_{0<k_1<k_2<...<k_m} \frac{x_1^{k_1}x_2^{k_2}...x_m^{k_m}}{k_1^{n_1}k_2^{n_2}...k_m^{n_m}} \tag{1}$$

Here $w := n_1 + ... + n_m$ is called the weight and m the depth. The power series (1) generalize both Euler's classical polylogarithms $Li_n(x)$ (m=1), and multiple ζ-numbers ($x_1 = ... = x_m = 1$) :

$$\zeta(n_1,...,n_m) := \sum_{0<k_1<k_2<...<k_m} \frac{1}{k_1^{n_1}k_2^{n_2}...k_m^{n_m}} \qquad n_m > 1 \tag{2}$$

The multiple zeta numbers were invented and studied by L. Euler [E], then forgotten, resurrected by V. Drinfeld [Dr] (in the form of "Drinfeld integrals") and by D. Zagier [Z1], studied in [Z2], [G1-4], and appeared recently in computations in quantum field theory [B], [Kr].

The multiple polylogarithms were investigated in [G1-4]. I think the main reason to study them is the following: multiple polylogarithms are periods of mixed Tate motives (see s. 11 of [G2]). In this talk we study them at N-th roots of unity: $x_1^N = ... = x_m^N = 1$. Notice that $Li_1(x) = -\log(1-x)$, so if ζ_N is a primitive N-th root of 1, then $Li_1(\zeta_N)$ is a logarithm of a cyclotomic unit.

We suggest that the "higher cyclotomy theory" should study the motivic multiple polylogarithms at roots of unity.

2. Multiple ζ-values and the motivic Lie algebra of *Spec*$\mathbb{Z}$. Let Z_w be the $\mathbb{Q}$-vector space generated by the numbers $\zeta(n_1,...,n_m)$ of weight w. Then $Z_\bullet := \sum Z_w$ is obviously an algebra. Indeed,

$$\zeta(m)\cdot\zeta(n) = \sum_{k_1,k_2>0} \frac{1}{k_1^m k_2^n} = \zeta(m,n) + \zeta(m+n) + \zeta(n,m)$$

(split the sum over $k_1 < k_2$, $k_1 = k_2$ and $k_1 > k_2$).

Let $L_\bullet(\mathbb{Z})$ be the free graded Lie algebra generated by elements e_{2n+1} of degree $-2n-1$ ($n \geq 1$) and $UL_\bullet(\mathbb{Z})^*$ be the dual to its universal enveloping algebra.

Conjecture 0.1 *a) The weight provides a grading on the algebra* $Z_\bullet$.

b) One has an isomorphism of graded algebras

$$Z_\bullet \quad = \quad \mathbb{Q}[\pi^2] \otimes UL_\bullet(\mathbb{Z})^*, \quad (deg\pi^2 = 2)$$

The part a) means that relations between ζ's of different weight, like $\zeta(5) = \lambda\cdot\zeta(7)$ where $\lambda \in \mathbb{Q}$ are impossible.

This conjecture implies that for $d_k := dim Z_k$ then one should have $d_k = d_{k-2} + d_{k-3}$. Computer calculations of D.Zagier lead to this formula for $k \leq 12$. Just recently much more extensive calculations, also confirming it, were made by D. Broadhurst. Having the theory of mixed Tate motives over $\mathbb{Z}$ one can prove that $dim Z_k$ is not bigger then expected.

Example. The conjecture predicts that

$$Z_8 \stackrel{?}{=} < \pi^8, \zeta(3)\zeta(5), \zeta(3)^2\pi^2, \text{"a new number"} >_{\mathbb{Q}}$$

where for "a new number" one can take $\zeta(3,5)$.

This example suggets to look only on multiple ζ's modulo the products of them, i.e. to study the vector space $Z_{>0}/(Z_{>0})^2$. The conjecture means that one should have an isomorphism of graded spaces

$$Z_{>0}/(Z_{>0})^2 \quad = \quad L(\mathbb{Z})^*_\bullet \oplus < \pi^2 > \qquad (3)$$

The explicit structure of the isomorphism (3) is very mysterious. e^*_{2n-1} should correspond to $\zeta(2n-1)$, $3 \cdot [e_3, e_9]^* + [e_5, e_7]^*$ to $\zeta(3,9)$, but already to find what should correspond to $[e_3, e_9]^*$ we have to go to the depth 4(!).

Remark. Let $\pi_1^{(l)}(\mathbb{P}^1 \backslash \{0,1,\infty\})$ be the l-adic completion of the fundamental group. One has canonical homomorphism

$$\varphi^l : \quad Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow Out\pi_1^{(l)}(\mathbb{P}^1 \backslash \{0,1,\infty\}) \qquad (4)$$

It was studied by P.Deligne, Y.Ihara and others (see the beautiful talk [Ih] delivered by Y.Ihara in ICM-90, Kyoto and references there). Conjecture (0.1) is closely related to some conjectures/questions of P.Deligne [D] about the image of the map (4) and V. Drinfeld [Dr] about the structure of the pronilpotent version of the Grothendieck-Teichmuller group.

3. Multiple Dirichlet L-values and the cyclotomic quotient of the motivic Lie algebra of $Spec\mathbb{Z}[\zeta_N][\frac{1}{N}]$. In this section we are concerned with a "cyclotomic" generalization of the conjecture (0.1). Let ζ_N be a primitive N-th root of unity and $Z(N)_w$ the $\mathbb{Q}$-vector space generated by the numbers

$$\bar{L}i_{n_1,\dots,n_l}(\zeta_N^{\alpha_1}, \dots, \zeta_N^{\alpha_m}) := \quad \frac{1}{(2\pi i)^w} Li_{n_1,\dots,n_l}(\zeta_N^{\alpha_1}, \dots, \zeta_N^{\alpha_m})$$

Then $Z(N)_\bullet := \sum Z(N)_w$ is a *bifiltered* by the weight and by the depth.

Conjecture 0.2 *There exists a graded Lie algebra $C_\bullet(N)$ over $\mathbb{Q}$ such that one has an isomorphism of filtered (by the weight on the left and by the degree on the right) graded spaces*

$$Z(N)_{>0}/Z(N)^2_{>0} = C_\bullet(N)^*$$

The dual to the space $H^1_{(n)}(C_\bullet(N))$ of the degree n generators of the Lie algebra $C_\bullet(N)$ must be isomorphic to $K_{2n-1}(\mathbb{Z}[\zeta_N][\frac{1}{N}]) \otimes \mathbb{Q}$. (Here $H_{(n)}$ is the degree n part of H).

Examples. Assume that $N = p$ is a prime. Let $p = 1$. Then by the Borel theorem the only nontrivial modulo torsion K-groups are $K_{4n+1}(\mathbb{Z})$ which have rank 1 and correspond to $\bar{\zeta}(2n+1)$.

$p = 2$: generators should correspond to $(2\pi i)^{-1}\log 2, \bar{\zeta}(3), \bar{\zeta}(5), \dots$.

If $p > 2$ one has

$$rkK_{2n-1}(\mathbb{Z}[\zeta_p][\frac{1}{p}]) \otimes \mathbb{Q} = \frac{p-1}{2}, \qquad \text{for } p > 2$$

and generators should correspond to $\bar{L}i_n(\zeta_p^\alpha)$, $1 < \alpha \leq \frac{p-1}{2}$. The $\mathbb{Q}$-space they span is just the space spaned by $(2\pi i)^{-n}$ times the special values of Dirichlet L-functions of conductor p at $s = n$.

Remark. The conjecture (0.1) predicts that $C_\bullet(1) = L_\bullet(\mathbb{Z})$ is free. The main result of [G3] implies that this can not be true for prime $p > 3$ because

$$H^2_{(2)}(C(p)_\bullet, \mathbb{Q}) = H^1(X_1(p), \mathbb{Q})_+ \oplus \mathbb{Z}[\zeta_p]^* \otimes \mathbb{Q}$$

where $X_1(p)$ is the modular curve and $+$ means the coinvariants of the complex conjugation.

We will define the Lie coalgebra $C(N)_\bullet$ in the section 6 below. The conjectures suggest that one should be able to determine explicitely the coproduct in $C(N)_\bullet$ using the "natural" generators corresponding to multiple polylogarithms at N-th roots of unity. Below we will do this for the associate graded quotient of this Lie algebra with respect to the depth filtration. (For the Lie coalgebra itself see [G4]).

4. Properties of multiple polylogarithms. Set

$$\int_0^1 \frac{dt}{t-x_1} \circ \ldots \circ \frac{dt}{t-x_n} := \int_{0\leq t_1 \leq \ldots \leq t_n \leq 1} \frac{dt_1}{t_1 - x_1} \wedge \ldots \wedge \frac{dt_n}{t_n - x_n}$$

$$I_{n_1,\ldots,n_m}(a_1 : \ldots : a_m : a_{m+1}) := \int_0^{a_{m+1}} \underbrace{\frac{dt}{t-a_1} \circ \frac{dt}{t} \circ \ldots \circ \frac{dt}{t}}_{n_1 \text{ times}} \circ \quad \ldots \quad \circ \underbrace{\frac{dt}{t-a_m} \circ \frac{dt}{t} \circ \ldots \frac{dt}{t}}_{n_m \text{ times}}$$

The following theorem is the key to properties of multiple polylogarithms:

Theorem 0.3 $Li_{n_1,\ldots,n_m}(x_1, \ldots, x_m) = I_{n_1,\ldots,n_m}(1 : x_1 : x_1x_2 : \ldots : x_1\ldots x_m)$

If $x_i = 1$ we get the Kontsevich formula.

The double shuffle relations. Set

$$Li(x_1, \ldots, x_m | t_1, \ldots, t_m) := \sum_{n_i \geq 1} Li_{n_1,\ldots,n_m}(x_1, \ldots, x_m) t_1^{n_1 - 1} \ldots t_m^{n_m - 1}$$

$$I(a_1 : \ldots : a_m : a_{m+1} | t_1, \ldots, t_m) := \sum_{n_i \geq 1} I_{n_1,\ldots,n_m}(a_1 : \ldots : a_m : a_{m+1}) t_1^{n_1 - 1} \ldots t_m^{n_m - 1}$$

$$I^*(a_1 : \ldots : a_m : a_{m+1} | t_1, \ldots, t_m) := I(a_1 : \ldots : a_m : a_{m+1} | t_1, t_1 + t_2, \ldots, t_1 + \ldots + t_m)$$

For any $1 \leq k \leq n$ let $\Sigma_{k,n}$ be the subset of permutations of n letters $1, \ldots, n$ consisting of all shuffles of $\{1, \ldots, k\}$ and $\{k+1, \ldots, n\}$. It is very easy to see that

$$Li[x_1, \ldots, x_k | t_1, \ldots, t_k] \cdot Li[x_{k+1}, \ldots, x_n | t_{k+1}, \ldots, t_n] = \tag{5}$$

$$\sum_{\sigma \in \Sigma_{k,n}} Li[x_{\sigma(1)}, \ldots, x_{\sigma(n)} | t_{\sigma(1)}, \ldots, t_{\sigma(n)}] \quad + \quad \text{lower depth terms}$$

For example $Li_1(x)Li_1(y) = Li_{1,1}(x, y) + Li_{1,1}(y, x) + Li_2(xy)$.

Theorem 0.4

$$I^*[a_1 : \ldots : a_k : 1 | t_1, \ldots, t_k] \cdot I^*[a_{k+1} : \ldots : a_n : 1 | t_{k+1}, \ldots, t_n] = \tag{6}$$

$$\sum_{\sigma \in \Sigma_{k,n}} I^*[a_{\sigma(1)}, \ldots, a_{\sigma(n)} : 1 | t_{\sigma(1)}, \ldots, t_{\sigma(n)}]$$

Here is the simplest case: $I_1(x)I_1(y) = I_{1,1}(x,y) + I_{1,1}(y,x)$. Indeed,

$$\int_0^1 \frac{dt}{t-x} \cdot \int_0^1 \frac{dt}{t-y} \quad = \quad \int_0^1 \frac{dt}{t-x} \circ \frac{dt}{t-y} + \int_0^1 \frac{dt}{t-y} \circ \frac{dt}{t-x}$$

For multiple ζ's these are precisely the relations of Zagier.

Distribution relations.

Proposition 0.5 *For $|x_i| \leq 1$ one has*

$$Li(x_1^l, ..., x_m^l | t_1, ..., t_m) = \sum_{y_i^l = x_i^l} Li(y_1, ..., y_m | lt_1, ..., lt_m) \qquad (7)$$

5. The dihedral Lie coalgebra and multiple polylogarithms. Let G be a commutative group written multiplicatively. We will define a bigraded Lie coalgebra $\mathcal{D}_\bullet^\bullet(G) = \oplus_{n\geq 1}\mathcal{D}_\bullet^n(G)$. Let us first define a graded abelian group $\hat{\mathcal{D}}_\bullet^n(G)$. The group $\mathcal{D}_\bullet^n(G)$ will be its quotient.

Let C_{n+1} be the principal homogeneous space for the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$. Let $\mathbb{Z}[C_{n+1}]$ is the abelian group of $\mathbb{Z}$-valued functions on C_{n+1}, $\mathbb{Z}[C_{n+1}]^0$ the subspace of the functions with the sum zero and $\mathbb{Z}[C_{n+1}]_0$ is the quotient along the constant functions. Let $Pol^\bullet(\mathbb{Z}[C_{n+1}]_0)$ be the algebra of polynomial functions on $\mathbb{Z}[C_{n+1}]_0$. It is graded by the degree. Let D_{n+1} be the dihedral group of symmetries of the $(n+1)$-gon and $\chi_{n+1} : D_{n+1} \to \{\pm 1\}$ the character trivial on the cyclic subgroup and sending the involution to $(-1)^{n+1}$. Set

$$\mathcal{D}_\bullet^n(G) := \quad \Big(G[C_{n+1}]^0 \otimes_{\mathbb{Z}} Pol^\bullet(\mathbb{Z}[C_{n+1}]_0) \otimes_{\mathbb{Z}} \chi_{n+1}\Big)_{D_{n+1}}$$

The elements of the group $\hat{\mathcal{D}}_\bullet^n(G)$ can be presented by the *(extended) nonhomogeneous dihedral* words in G:

$$\{g_0, g_1, ..., g_n | t_0 : ... : t_n\} \qquad \text{such that} \quad g_0 \cdot ... \cdot g_n = 1,$$

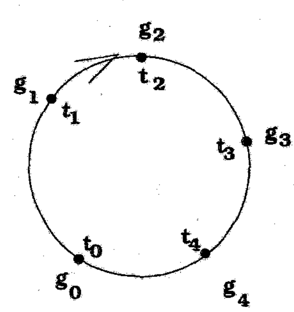
$$\{g_0, g_1, ..., g_n | t_0 : ... : t_n\} = \{g_0, g_1, ..., g_n | t + t_0 : ... : t + t_n\},$$

and (the dihedral symmetry)

$$\{g_0, ..., g_{n-1}, g_n | t_0 : t_1 : ... : t_n\} = \{g_1, ..., g_n, g_0 | t_1 : ... : t_n : t_0\}$$

$$\{g_0, ..., g_n | t_0 : ... : t_n\} = (-1)^{n+1}\{g_n, ..., g_0 | t_n : ... : t_0\}$$

One can picture the elements of $\hat{\mathcal{D}}_\bullet^n(G)$ as $n+1$ pairs $(g_0, t_0), ..., (g_n, t_n)$ located cyclically on an oriented circle:



We can parametrize the generators in a dual way, using the *(extended) homogeneous dihedral* words in G:

$$\{g_0 : g_1 : ... : g_n | t_0, ..., t_n\}, \quad t_0 + ... + t_n = 0$$

such that (homogeneuty in g_i's):

$$\{g \cdot g_0 : ... : g \cdot g_n | t_0, ..., t_n\} = \{g_0 : ... : g_n | t_0, ..., t_n\} \qquad \text{for any} \quad g \in G$$

and the dihedral symmetry holds.

The duality between the homogeneous and nonhomogeneous dihedral words is given by

$$\{g_0 : g_1 : ... : g_n | t_0, ..., t_n\} \longmapsto \{g_0^{-1}g_1, g_1^{-1}g_2, ..., g_n^{-1}g_0 | t_0 : t_0 + t_1 : ... : t_0 + ... + t_n\},$$

$$\{g_0, g_1, ..., g_n | t_0 : ... : t_n\} \longmapsto \{g_0 : g_0 g_1 : ... : g_0 ... g_n | t_1 - t_0, t_2 - t_1, ..., t_0 - t_n\}$$

Definition 0.6 *1. The double shuffle relations subspace of $\hat{\mathcal{D}}^n_\bullet(G)$ is generated by the elements*

$$\sum_{\sigma \in \Sigma_{k,n}} \{g_{\sigma(1)} : ... : g_{\sigma(n)} : g_{n+1} | t_{\sigma(1)}, ..., t_{\sigma(n)}, t_{n+1}\},$$

$$\sum_{\sigma \in \Sigma_{k,n}} \{x_0, x_{\sigma(1)}, ..., x_{\sigma(n)} | t_0 : t_{\sigma(1)} : ... : t_{\sigma(n)}\}$$

The distribution relations subspace of $\hat{\mathcal{D}}^n_\bullet(G)$ is generated by the elements

$$\{x_0^l, x_1^l, ..., x_n^l | t_0 : t_{\sigma(1)} : ... : t_{\sigma(n)}\} - \sum_{y_i^l = x_i^l} \{y_0, y_1, ..., y_n | l \cdot t_0 : ... : l \cdot t_n\}$$

(If G is finite we assume that l divides $|G|$).

2. $\mathcal{D}^n_\bullet(G)$ is the quotient of $\hat{\mathcal{D}}^n_\bullet(G)$ by the double shuffle and distribution subspaces.

Now we make the crucial step. Let us define a cobracket

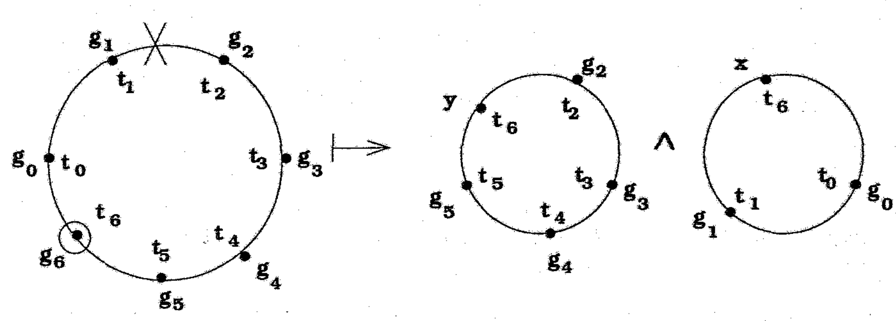
$$\delta : \hat{\mathcal{D}}^n_\bullet(G) \longrightarrow \sum_{k+l=n} \hat{\mathcal{D}}^k_\bullet(G) \wedge \hat{\mathcal{D}}^l_\bullet(G)$$

setting

$$\delta\{g_0, ..., g_n | t_0 : ... : t_n\} :=$$

$$\sum_{i=1}^{n-1} \sum_{j=0}^{n} \{g_{j+i+1}, ..., g_{j+n}, y_{ij} | t_{j+i+1} : ... : t_{j+n+1}\} \wedge \{x_{ij}, g_{j+1}, ..., g_{j+i} | t_j : ..., t_{j+i}\}$$

where indices are modulo $n+1$ and $x_{ij} g_{j+1} ... g_{j+i} = 1$, $y_{ij} g_{j+i+1} ... g_{j+n} = 1$. Each term of the formula corresponds to the following procedure: we choose an arc on the circle on the figure between the two neighboring distinguished points, and in addition choose a distinguished point different from the ends of the arc. Then we cut the circle in the choosen arc and in the choosen point, make two naturally oriented circles out of it, and then make the dihedral word on each of the circles out of the initial word in a natural way.



figure

There is a similar formula for the homogeneous dihedral words, just exchange g's and t's on the circle. For example restrictiong to $t_i = 0$ one has

$$\delta\{g_0 : g_1 : g_2\} \quad = \quad \{g_0 : g_1\} \wedge \{g_1 : g_2\} + \{g_1 : g_2\} \wedge \{g_2 : g_0\} + \{g_2 : g_0\} \wedge \{g_0 : g_1\}$$

Theorem 0.7 *δ provides the structure of bigraded Lie coalgebras on both $\hat{D}^{\bullet}_{\bullet}(G)$ and $\mathcal{D}^{\bullet}_{\bullet}(G)$. (i.e. $\delta^2 = 0$).*

6. $\mathcal{D}_{\bullet}(\mu_N)$ and motivic multiple polylogarithms. A mixed Hodge structure is called a Hodge Tate structure if all the Hodge numbers $h^{p,q}$ with $p \neq q$ vanish. The category of mixed $\mathbb{Q}$-Hodge Tate structures is canonically equivalent to the category of finite dimensional comodules over a certain graded pro-Lie coalgebra $\mathcal{L}^{HT}_{\bullet}$ over $\mathbb{Q}$. One can attach to the iterated integral related to $Li_{n_1,...,n_m}(x_1, ..., x_m)$ by theorem (0.3) an element $\tilde{L}i_{n_1,...,n_m}(x_1, ..., x_m) \in \mathcal{L}^{HT}_{w}$, the motivic multiple polylogarithm. See s.9 and s.11 of [G2] where $\mathcal{L}^{HT}_{\bullet}$ and an w-framed mixed Hodge Tate structure $\tilde{L}i_{n_1,...,n_m}(x_1, ..., x_m)$ related to the iterated integral are defined.

Definition 0.8 *$C(N)_w$ is the $\mathbb{Q}$-subspace of $\mathcal{L}^{H}_{w}$ generated by the motivic multiple polylogarithms at N-th roots of unity of weight w.*

Theorem 0.9 *$C(N)_{\bullet} := \oplus_{w \geq 1} C(N)_w$ is a Lie subcoalgebra in $\mathcal{L}^{H}_{\bullet}$.*

The category of mixed Tate motives over the scheme $S_N := Spec\mathbb{Z}[\zeta_n][\frac{1}{N}]$ is canonically equivalent to the category of finite dimensional comodules over a graded Lie coalgebra $L(S_N)_{\bullet}$ called the motivic Lie coalgebra of that scheme. $C(N)_{\bullet}$ is a subcoalgebra Lie in the motivic Lie algebra of the scheme S_N, but it does not coinside with it in general ([G3]).

Let us define elements $\{x_1, ..., x_m\}_{n_1,...,n_m}$ by

$$\{x_1, ..., x_m, x_{m+1} | t_1 : .... : t_m : 0\} =: \sum_{n_i > 0} \{x_1, ..., x_m\}_{n_1,...,n_m} t_1^{n_1-1} ... t_m^{n_m-1}$$

Let μ_N be the group of N-th roots of unity.

Theorem 0.10 *Assume that $x_i^N = 1$. Then the map*

$$\{x_1, ..., x_m\}_{n_1,...,n_m} \quad \longmapsto \quad \tilde{L}i_{n_1,...,n_m}(x_1, ..., x_m)$$

provides a morphism of the graded (by the weight) Lie coalgebras $\mathcal{D}^{\bullet}_{\bullet}(\mu_N) \to gr^{Depth} C(N)_{\bullet}$

It is easy to show that

$$\mathcal{D}^1_n(\mu_N) \otimes \mathbb{Q} = K_{2n-1}(S_N) \otimes \mathbb{Q} \qquad \text{if} \quad (n, N) \neq (1, 1)$$

The degree w part of the cohomology of the Lie coalgebra $\mathcal{D}^{\leq m}_{\bullet}(\mu_N)$ is related in a very mysterios way to cohomology of the local system with fiber $S^{w-m}(V_m)$ on the modular variety $\Gamma_1(N; m) \backslash SL_m(\mathbb{R}) / SO_m$. Here V_m is the standard SL_m-module. The simplest case $N = p, m = 2, w = 2$ is treated in [G3].

REFERENCES

[BGSV] Beilinson A.A., Goncharov A.A., Schechtman V.V., Varchenko A.N.: *Aomoto dilogarithms, mixed Hodge structures and motivic cohomolgy*, the Grothiendieck Feschtrift, Birkhouser, vol 86, 1990, p. 135-171.

[Br] Broadhurst D.J., *On the enumeration of irreducible k-fold sums and their role in knot theory and field theory* Preprint hep-th/9604128.

HILBERT SCHEMES AND REDUCTIVE GROUPS

VICTOR GINZBURG

This is a survey of a joint work currently in progress with Roman Bezrukavnikov. We will have some fascinating unexpected relations between Combinatorics, Algebraic Geometry and Lie theory. These relations are similar to the classically known relations between the combinatorics of symmetric functions, the geometry of conjugacy classes of $n \times n$ -matrices, and the representation theory of the general linear group $\mathrm{GL}_n(\mathbb{C})$. Our goal is to "double" the setup by replacing polynomials in n variables invariant under the action of the symmetric group S_n by polynomials in two n-tuples of variables invariant under the simultaneous S_n-action. As will be demonstrated below, this corresponds in geometry to replacing the set of conjugacy classes of matrices by the punctual Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^2)$.

Representation theoretic part of the story remains quite mysterious at the moment, but it is expected to be related to the theory of double-affine Hecke algebras [Ch] and to representation theory of some double-loop groups. It is therefore not accidental that our approach leads to a proof of a positivity conjecture concerning Macdonald's polynomials, since the latter were shown by Cherednik to be most adequately understood in terms of double-affine Hecke algebras.

This work has been strongly motivated by combinatorial ideas of Garsia and Haiman [GH], [H] and also by the Nakajima lectures [Na]. We are also grateful to M. Kapranov for very useful discussions.

1. HILBERT SCHEMES

Let X be a connected complex algebraic variety. We write $\mathcal{O}_X$ for the sheaf of regular functions on X. For $n = 1, 2, 3, \ldots$, let $\mathrm{Hilb}^n(X)$ denote the *punctual Hilbert scheme* of all ideals $I \subset \mathcal{O}_X$ such that $\mathcal{O}_X/I$ is a $\mathbb{C}$-algebra of dimension n. We will be mostly concerned below with the case $X = \mathbb{C}^k$, so that $\mathrm{Hilb}^n(\mathbb{C}^k)$ becomes the variety of all codimension n ideals in the polynomial algebra in k variables. An excellent survey on punctual Hilbert schems may be found in [Na].

BASIC PROPERTIES

1.1. There is a tautological algebraic vector bundle $\mathrm{Taut} \to \mathrm{Hilb}^n(X)$; its fiber over a point $I \in \mathrm{Hilb}^n(X)$ is the vector space $\mathcal{O}_X/I$ (here $I \subset \mathcal{O}_X$ is an ideal). Thus Taut is a rank n vector bundle whose fibers are $\mathcal{O}_X$-modules.

1.2. For $I \in \mathrm{Hilb}^n(X)$, the support of the $\mathcal{O}_X$-module $\mathcal{O}_X/I$ is a finite subscheme of X. Set-theoretically, this is a finite unordered collection of points of X. Each point occurs with a certain multiplicity so that we may think of $\mathrm{supp}(\mathcal{O}_X/I)$ as a zero-dimensional cycle in X of length (= sum of the multiplicities) n. The assignment $I \mapsto \mathrm{supp}(\mathcal{O}_X/I)$ gives a well-defined map $\pi : \mathrm{Hilb}^n(X) \to X^n/S_n$. The map

π is an isomorphism over the open dense subset of X^n/S_n formed by n-tuples of pairwise distinct points of X.

If X is smooth then X^n/S_n is a normal variety. Hence all fibers of π are connected and the pull-back of functions gives an isomorphism:

$$\pi^* : \mathcal{O}(X^n/S_n) \xrightarrow{\sim} \mathcal{O}(\mathrm{Hilb}^n(X)) \tag{1.1.1}$$

1.3. For a general smooth algebraic variety X, the Hilbert scheme $\mathrm{Hilb}^n(X)$ may be both singular and reducible. This never happens however in the special case of 2-dimensional varieties. One has the following fundamental result, see [ES], [F], [Na].

Theorem 1.3.1. *Let X be a smooth complex surface. Then*

(i) $\mathrm{Hilb}^n X$ *is a smooth irreducible variety of dimension* $2n$.

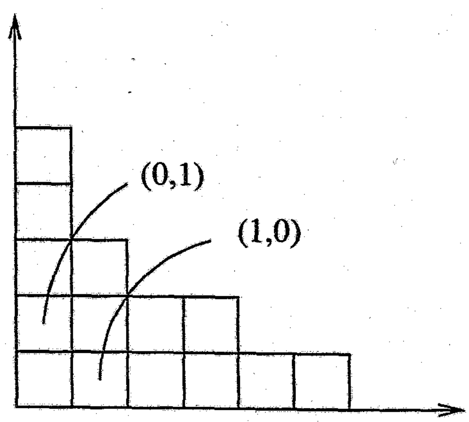
(ii) *The map* $\pi : \mathrm{Hilb}^n X \to X^n/S_n$ *is semi-small in the sense of Goresky-MacPherson.*

2. Hilbert scheme of a 2-plane

We assume throughout this section that $X = \mathbb{C}^2$, and write (x, y) for the coordinates on $\mathbb{C}^2$.

2.1. The group $T = \mathbb{C}^* \times \mathbb{C}^*$ acts on $X = \mathbb{C}^2$ by $(t_1, t_2) : (x, y) \mapsto (t_1 \cdot x, t_2 \cdot y)$. This induces a natural T-action on $\mathrm{Hilb}^n X$. The fixed points of the T-action on $\mathrm{Hilb}^n X$ form a finite set which is in $(1-1)$ -correspondence with the set of Young diagrams with n boxes. The correspondence is constructed as follows. Given a Young diagram

$\lambda =$



we enumerate the boxes in some order, and write (a_i, b_i) for the coordinates of the i-th box. This way we obtain n monomials:

$$x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, \dots, x^{a_n}y^{b_n} \in \mathbb{C}[x, y]$$

Define a point in $\mathrm{Hilb}^n(X)$ to be the ideal $I_\lambda \subset \mathbb{C}[x, y]$ spanned over $\mathbb{C}$ by all monomials in x and y with the exception of the above written ones. It is straightforward to see that I_λ is indeed an ideal, and that the I_λ's are exactly the fixed points of the T-action.

2.2. We have $X^n \simeq (\mathbb{C}^2)^n \simeq (\mathbb{C}^n)^2$. Write the coordinates on $\mathbb{C}^2$ in the form $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n)$, so that $\mathbb{C}[X^n] = \mathbb{C}[\mathbf{x}, \mathbf{y}]$, the polynomial algebra in $2n$ variables. The symmetric group S_n acts on $\mathbb{C}^{2n}$ by permutations of the $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ simultaneously, and we have

$$\mathcal{O}(X^n/S_n) = \mathbb{C}[\mathbf{x}, \mathbf{y}]^{S_n} = \boxed{\text{Symmetric polynomials under diagonal action}}$$

Combining this with isomorphism (1.1.1) one obtains

$$\mathcal{O}(\mathrm{Hilb}^n X) = \mathcal{O}(X^n/S_n) = \mathbb{C}[\mathbf{x}, \mathbf{y}]^{S_n} \tag{2.2.1}$$

2.3. Let $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{\mathrm{sign}}$ denote the subspace of skew-symmetric polynomials under the diagonal S_n -action. Given polynomials $\psi_1, \ldots, \psi_n$ in *two* variables we construct a skew-symmetric polynomial

$$\psi_1 \wedge \ldots \wedge \psi_n : (\mathbf{x}, \mathbf{y}) \mapsto \sum_{s \in S_n} (-1)^s \cdot \psi_1(x_{s(1)}, y_{s(1)}) \cdot \psi_2(x_{s(2)}, y_{s(2)}) \cdot \ldots \cdot \psi_n(x_{s(n)}, y_{s(n)}). \tag{2.3.1}$$

This way we obtain an isomorphism of vector spaces

$$\bigwedge^n \mathbb{C}[x, y] \xrightarrow{\sim} \mathbb{C}[\mathbf{x}, \mathbf{y}]^{\mathrm{sign}} \tag{2.3.2}$$

Recall the tautological rank n vector bundle Taut on $\mathrm{Hilb}^n X$ and introduce the line bundle

$$\mathcal{L} = \bigwedge^n \mathrm{Taut}$$

Proposition 2.3.3. *There is a natural isomorphism*

$$\Gamma(\mathit{Hilb}^n X, \mathcal{L}) \simeq \mathbb{C}[\mathbf{x}, \mathbf{y}]^{\mathrm{sign}}$$

We only indicate here how to construct a map from the RHS above to the LHS. By (3.2.2) it suffices to associate a global section of $\mathcal{L}$ to any element $\psi_1 \wedge \ldots \wedge \psi_n \in \mathbb{C}[\mathbf{x}, \mathbf{y}]^{\mathrm{sign}}$, see (3.2.1). To this end note that each $\psi_i \in \mathbb{C}[x, y]$ projects, for any ideal $I \subset \mathbb{C}[x, y]$, to an element in $\mathbb{C}[x, y]/I$, hence defines a regular section $\widetilde{\psi}_i \in \Gamma(\mathit{Hilb}^n X, \mathit{Taut})$. We then associate to $\psi_1 \wedge \ldots \wedge \psi_n$ the section

$$\widetilde{\psi}_1 \wedge \ldots \wedge \widetilde{\psi}_n \in \Gamma(\mathit{Hilb}^n X, \bigwedge^n \mathit{Taut}). \qquad \square$$

Observe that the space $\Gamma(\mathrm{Hilb}^n X, \mathcal{L})$ has a natural $\mathcal{O}(\mathrm{Hilb}^n X)$-module structure, while the space $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{\mathrm{sign}}$ has a natural $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{S_n}$-module structure. The isomorphism of Proposition 2.3.3 is compatible with these structures. Observe further that if $n > 1$ then the $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{S_n}$-module $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{\mathrm{sign}}$ is *not free* and has generically rank one. The meaning of the proposition above is that the line bundle $\mathcal{L}$ is a natural "resolution" of the module $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{\mathrm{sign}}$ by a locally-free sheaf. Specifically, we have the smooth resolution $\pi : \mathrm{Hilb}^n X \to X^n/S_n$, see 1.2, and proposition 2.3.3 means that the sheaf on X^n/S_n corresponding to the $\mathcal{O}(X^n/S_n)$-module $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{\mathrm{sign}}$ is canonically isomorphic to $\pi_* \mathcal{L}$. Moreover, one can show that $R^k \pi_* \mathcal{L} = 0$ for all $k > 0$.

2.4. To each Young diagram λ with n boxes we associate a skew-symmetric polynomial $\Delta_\lambda \in \mathbb{C}[\mathbf{x},\mathbf{y}]^{\text{sign}}$ by generalizing to $2n$ variables the construction of the Vandermonde determinant in n variables. Specifically the diagram λ gives rise, as explained in 2.1, a collection of non-negative integers $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$. Put

$$\Delta_\lambda(\mathbf{x},\mathbf{y}) = \det \begin{vmatrix} x_1^{a_1} y_1^{b_1} & x_2^{a_1} y_2^{b_1} & \ldots & x_n^{a_1} y_n^{b_1} \\ x_1^{a_2} y_1^{b_2} & x_2^{a_2} y_2^{b_2} & \ldots & x_n^{a_2} y_n^{b_2} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{a_n} y_1^{b_n} & x_2^{a_n} y_2^{b_n} & \ldots & x_n^{a_n} y_n^{b_n} \end{vmatrix}$$

Note that in the notation of (2.3.1) we have

$$\Delta_\lambda(\mathbf{x},\mathbf{y}) = x^{a_1} y^{b_1} \wedge x^{a_2} y^{b_2} \wedge \ldots \wedge x^{a_n} y^{b_n} \qquad (2.4.1)$$

The algebraic role of the polynomials Δ_λ is explained by the following

Lemma 2.4.2. *The set $\{\Delta_\lambda,\ \lambda$ is a diagram with n boxes$\}$ is a minimal set of generators of the $\mathbb{C}[\mathbf{x},\mathbf{y}]^{S_n}$-module $\mathbb{C}[\mathbf{x},\mathbf{y}]^{\text{sign}}$.*

We may also view Δ_λ as a regular section of the line bundle $\mathcal{L}$, see Proposition 2.3.3. Then we have

Lemma 2.4.3. *If λ, μ are two Young diagrams, then $\Delta_\lambda(I_\mu)$, the value of the section $\Delta_\lambda \in \Gamma(Hilb^n X, \mathcal{L})$ at the point $I_\mu \in Hilb^n X$, see 2.1, is non-zero iff $\lambda = \mu$, symbolically: $\Delta_\lambda(I_\mu) = \delta_{\lambda\mu}$ (Kronecker delta).*

3. Duality

We keep the assumption that $X = \mathbb{C}^2$ and write $\check{X}$ for the dual vector space. The symmetric algebra SX may be identified in a natural way with the algebra $D(X)$ of the *constant coefficient* differential operators on X. Thus we have

$$\mathbb{C}[\check{X}] \simeq SX \simeq D(X)$$

Therefore, we may view a point in $\mathrm{Hilb}^n(\check{X})$ as a codimension n ideal $I \subset D(X)$.

Let $\mathcal{H}\mathrm{ol}(X)$ be the (infinite dimensional) vector space of holomorphic functions on X. Given an ideal $I \subset D(X)$ we consider the system of differential equations

$$\{u\varphi = 0,\ u \in I\}. \qquad (3.1)$$

One can show that if I has codimension n in $D(X)$ then the space of homolorphic solutions $\varphi \in \mathcal{H}\mathrm{ol}(X)$ of (3.1) is an n-dimensional vector space $\mathcal{S}\mathrm{ol}_I$. The space $\mathcal{S}\mathrm{ol}_I$ is clearly stable under translations. Let $\mathrm{Gr}_n^{\mathrm{transl}}$ be the fixed point set of the group of translations acting on the Grassmannian $\mathrm{Gr}_n(\mathcal{H}\mathrm{ol}(X))$ of all n-dimensional subspaces in $\mathcal{H}\mathrm{ol}(X)$. One proves

Proposition 3.2. *The assignment $I \mapsto \mathcal{S}\mathrm{ol}_I$ establishes an isomorphism of $Hilb^n(X^*)$ with $\mathrm{Gr}_n^{\mathrm{transl}}$. The tautological rank n vector bundle on $\mathrm{Gr}_n(\mathcal{H}\mathrm{ol}(X))$ goes under the isomorphism to the dual of the tautological vector bundle Taut on $Hilb^n(X)$.*

The key point of our approach is a construction of a canonical holomorphic (non-algebraic) section:

$$\Delta \in \Gamma_{\text{hol}}(\text{Hilb}^n(\check{X}) \times \text{Hilb}^n(X), \check{\mathcal{L}} \boxtimes \mathcal{L}) \tag{3.3}$$

defined uniquely up to constant factor. To define Δ, one first proves the following holomorphic analogue of algebraic results (2.3.2) and 2.3.3

$$\bigwedge^n \mathcal{H}ol(X) \hookrightarrow \mathcal{H}ol(X^n)^{\text{sign}} \simeq \Gamma_{\text{hol}}(\text{Hilb}^n X, \mathcal{L}) \tag{3.4}$$

(note that the first map here is not surjective, as opposed to (2.3.2)). Hence, for any $I \in \text{Hilb}^n(\check{X})$, the one-dimensional space $\bigwedge^n \mathcal{S}ol_I \subset \bigwedge^n \mathcal{H}ol(X)$ gives via the composition (3.4) a line in $\Gamma_{\text{hol}}(\text{Hilb}^n X, \mathcal{L})$. As I varies, these lines form a line bundle on $\text{Hilb}^n(\check{X})$ which is isomorphic, due to Proposition 3.2, to $(\check{\mathcal{L}})^*$, the dual of the canonical line bundle $\check{\mathcal{L}}$ on $\text{Hilb}^n(\check{X})$ (here $\check{\mathcal{L}}$ is not the dual of $\mathcal{L}$ but just $\bigwedge^n$ Taut for $\text{Hilb}^n(\check{X})$). In this way, the canonical identity element of $\check{\mathcal{L}} \otimes (\check{\mathcal{L}})^*$ gives rise to an element of $\Gamma(\text{Hilb}^n(\check{X}), \check{\mathcal{L}} \otimes \Gamma(\text{Hilb}^n X, \mathcal{L}))$, that is to (3.3).

Write X^n_{reg} for the Zariski open subset of X^n formed by pairwise distinct n-tuples of points of X. We may view X^n_{reg}/S_n as a Zariski open subset of $\text{Hilb}^n X$. The restriction to X^n_{reg}/S_n of the tautological bundle on $\text{Hilb}^n X$, hence of the line bundle $\mathcal{L}$, has a canonical trivialisation. The meaning of the canonical section Δ in (3.3) is explained by the following:

Proposition 3.5. *(i) In the trivialization of $\check{\mathcal{L}} \boxtimes \mathcal{L}$ over $\check{X}^n_{reg}/S_n \times X^n_{reg}/S_n$ the section Δ is given by the formula*

$$\Delta(\check{v}, v) = \sum_{s \in S_n} (-1)^s \cdot e^{\langle s(\check{v}), v\rangle}, \qquad \forall \check{v} \in \check{X}^n_{\text{reg}}, \quad v \in X^n_{\text{reg}}$$

(ii) For any Young diagram λ with n boxes we have (see 2.1)

$$\Delta(I_\lambda, \bullet) = \Delta_\lambda(\bullet)$$

From the explicit formula in (i) we deduce

Corollary 3.6. *The section Δ is symmetric w.r.t. substitution $X \leftrightarrow \check{X}$.*

There seems to be an analogue of the above construction with $X = \mathbb{C}^2$ being replaced by the square of an elliptic curve. In that case the construction is closely related to the Mukai transform.

4. The $n!$-conjecture and Macdonald polynomials

Let $X = \mathbb{C}^2$ and $D(X^n)$ be the ring of constant coeffcient differential operators on X^n. For any polynomial $P \in \mathbb{C}[X^n]$ the $D(X^n)$-module $D(X^n) \cdot P$ generated by P has finite dimension over $\mathbb{C}$.

In 1993, A.M. Garsia and M. Haiman proposed the following, see [GH].

$n!$-conjecture.
For any Young diagram λ with n boxes

$$\dim_{\mathbb{C}} D(X^n] \cdot \Delta_\lambda = n!$$

In view of Proposition 3.5(ii) the $n!$-conjecture follows from the following more general result

Theorem 4.1. *For any $I \in Hilb^n(\check{X})$ the restriction of the section $\Delta(I,\bullet)$ to X^n_{reg} satisfies*

$$\dim_{\mathbb{C}} D(X^n)\cdot\Delta(I,\bullet) = n!$$

Idea of Proof. For generic I, i.e. for $I \in \check{X}^n_{\text{reg}}/S_n$, one calculates the dimension of $D(X^n)\cdot\Delta(I,\bullet)$ directly, using the explicit formula of Proposition 3.5(i). Now, let Diag $:= \text{Hilb}^n(\check{X}) \setminus \check{X}^n_{\text{reg}}/S_n$ be the special divisor. We must prove that the dimension of the vector space $D(X^n)\cdot\Delta(I,\bullet)$ does not drop when I becomes a point of Diag. In local coordinates, this amounts to showing that a certain determinant does not vanish at the divisor Diag. Consider now a Zariski open subset $\text{Diag}^{\text{reg}} \subset \text{Diag}$, the inverse image of the susbet of $\check{X}^n/S_n$ where exactly two among the elements of the n-tuple forming a point of $\check{X}^n/S_n$ coincide. We verify by hand that the determinant in question does not vanish at the points of Diag_{reg}, hence does not vanish on $\check{X}^n_{\text{reg}}/S_n \cup \text{Diag}_{\text{reg}}$. But $\text{Hilb}^n(\check{X}) \setminus (\check{X}^n_{\text{reg}}/S_n \cup \text{Diag}_{\text{reg}})$, the complement of this set, has codimension ≥ 2 in $\text{Hilb}^n(\check{X})$. Therefore the determinant, being a holomorphic function on the smooth variety, cannot vanish at a point of this complement as well. □

Garsia and Haiman came up with the $n!$-conjecture in an attempt to prove another conjecture concerning Macdonald's polynomials. In more detail, for any two complex parameters $q, t \in \mathbb{C}^*$, and a Young diagram λ with n boxes, Macdonald has introduced in [M] a so-called Macdonald's symmetric polynomial $H_\lambda(\mathbf{x}, q, t) \in \mathbb{C}[\mathbf{x}]^{S_n}$. The vector space of symmetric functions has a $\mathbb{C}$-basis formed by the "big elementary symmetric functions" $\mathbb{S}_\mu(\mathbf{x})$, see [M]. In particular, for any λ, one has an expansion

$$H_\lambda(\mathbf{x}, q, t) = \sum_{\mu<\lambda} K_{\lambda,\mu}(q,t)\cdot\mathbb{S}_\mu(\mathbf{x}) \tag{4.2}$$

where $K_{\lambda,\mu}$ turn out to be polynomials in q and t. In the special case $t = 1$ Macdonald's polynomials become the classical Hall polynomials, see [M] and the polynomials $K_{\lambda,\mu}(q,1)$ become the Kostka polynomials. The latter are known to have all coeffcients in $\mathbb{Z}_{\geq 0}$. Motivated by this, Macdonald proposed

Positivity conjecture [M]. *All coeffcients of the polynomials $K_{\lambda,\mu}(q,t)$ are in $\mathbb{Z}_{\geq 0}$.*

Integrality of the coefficients has been proved in a combinatorial way. Positivity turned out to be much more hard. In the case $t = 1$, the coefficients of the Kostka polynomial $K_{\lambda,\mu}(q,1) = \sum_n K_{\lambda,\mu}(n)\cdot q^n$ have a geometric interpretation as the multiplicity:

$$K_{\lambda,\mu}(n) = [H^n(\mathcal{B}_\lambda) : \chi_\mu] \tag{4.3}$$

where χ_μ is the irreducible representation of the group S_n corresponding to the Young diagram μ, $\mathcal{B}_\lambda$ is the variety of complete flags in $\mathbb{C}^n$ that are fixed by a nilpotent endomorphism with the Jordan form given by the diagram λ. Finally $H^n(\cdot)$ stands for the n-th cohomology group, considered as an S_n-module via the Springer construction, see e.g. [CG]. Formula (4.3) immediately implies that $K_{\lambda,\mu}(n) \in \mathbb{Z}_{\geq 0}$. It is likely that there is a similar geometric interpretation of the coefficients of the polynomials $K_{\lambda,\mu}(q,t)$ in terms of the cohomology of the fixed point variety of two commuting unipotent transformations of a certain infinite dimensional partial flag manifold (compare with Proposition 3.2).

Due to the absence of a geometric interpretation, Garsia and Haiman suggested an alternative algebraic approach to the Positivity conjecture. They observed that the vector space $D(X^n) \cdot \Delta_\lambda$ has a natural *bi-grading*

$$D(X^n) \cdot \Delta_\lambda = \bigoplus_{i,j \geq 0} \mathcal{H}_\lambda^{i,j} \tag{4.4}$$

induced by the bigrading on $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ by the total homogeneous bidegree in $\mathbf{x}$ and $\mathbf{y}$. Furthermore, each subspace $\mathcal{H}_\lambda^{i,j}$ is stable under the diagonal S_n-action on $\mathbb{C}[\mathbf{x}, \mathbf{y}]$. Garsia and Haiman introduced the generating functions:

$$\sum_{i,j \geq 0} [\mathcal{H}_\lambda^{i,j} : \chi_\mu] \cdot q^i t^j, \tag{4.5}$$

and observed that the Positivity conjecture would follow provided one knows that the polynomial (4.5) equals $K_{\lambda,\mu}(q,t)$. Indeed, Haiman proved

Theorem [H]. *If the $n!$-conjecture holds, then for any Young diagrams λ and μ, the polynomial (4.5) is equal to $K_{\lambda,\mu}(q,t)$.*

Thus, Theorem 4.1 combined with the theorem above proves the Positivity conjecture.

5. Hilbert quotients.

5.1. Let Γ be a finite group, let Y be a smooth complex algebraic variety with an algebraic Γ-action. The orbi-space Y/Γ has the natural structure of an algebraic variety, which is not smooth in general. We define a "resolution" (not necessarily *smooth*) $\widehat{Y/\Gamma} \to Y/\Gamma$, called the *Hilbert quotient*, as follows. Let N be the cardinality of the general Γ-orbit in Y, and let $Y^0 \subset Y$ be the union of all Γ-orbits of cardinality N. Any such Γ-orbit is an unordered N-tuple of distinct points of Y, hence may be viewed as a point in $\mathrm{Hilb}^N Y$. This way one gets an imbedding $Y^0/\Gamma \hookrightarrow \mathrm{Hilb}^N Y$. We define $\widehat{Y/\Gamma}$ to be the closure of the image of this imbedding.

5.2 Example. Let $Y = \mathbb{C}^2$ and let $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup. Then the space $\mathbb{C}^2/\Gamma$ has an isolated singularity at the origin, called a Kleinian singularity. For the proof of the following unpublished result of Ginzburg-Kapranov and Nakamura, see [Na].

Proposition. *The Hilbert quotient $\widehat{\mathbb{C}^2/\Gamma} \to \mathbb{C}^2/\Gamma$ is the minimal resolution of $\mathbb{C}^2/\Gamma$.*

This result has a close relation to the McKay correspondence, cf. [Na].

5.3. Let X be an algebraic variety and $n = 2, 3, \ldots$. We now apply the construction of 5.1 to $\Gamma = \Sigma_n$ acting naturally on $Y = X^n$ by permutations. We have $N = n!$ so that $\widehat{X^n/S_n} \subset \mathrm{Hilb}^{n!}(X^n)$. There exists a morphism $\nu\colon \widehat{X^n/S_n} \to \mathrm{Hilb}^n(X)$ such that the projection $\widehat{X^n/S_n} \to X^n/S_n$ factors as a composition

$$\widehat{X^n/S_n} \xrightarrow{\nu} \mathrm{Hilb}^n X \to X^n/S_n.$$

Here is one of our main results.

Theorem 5.3.1. *For any smooth complex surface X the morphism $\nu\colon \widehat{X^n/S_n} \to \mathrm{Hilb}^n X$ is an isomorphism.*

Idea of proof. Since the claim is local with respect to X, completing at a point of X one reduces the theorem to the special case $X = \mathbb{C}^2$. In this case we explicitly construct an inverse to the map ν as follows. First, we may replace X by $\check{X}$ and identify $D(X^n)$ with $\mathbb{C}[\check{X}^n]$. Then by Theorem 4.1 the assignment $I \mapsto D(X^n) \cdot \Delta(I, \bullet)$ associates to each point $I \in \mathrm{Hilb}^n(\check{X})$ an $n!$-dimensional cyclic $\mathbb{C}[\check{X}^n]$-module. The annihilation of the element $\Delta(I, \bullet)$ is an $n!$-codimensional ideal in $\mathbb{C}[\check{X}^n]$, hence a point in $\mathrm{Hilb}^{n!}(\check{X}^n)$. It is easy to see that the map $\mathrm{Hilb}^n(\check{X}) \to \mathrm{Hilb}^{n!}(\check{X}^n)$ is inverse to ν. $\square$

5.4. Write $K^{S_n}(X^n)$ for the S_n-equivariant topological K-group of X^n and $K(\mathrm{Hilb}^n X)$ for the ordinary (non-equivariant) K-group of the Hilbert scheme. Using Theorem 5.3.1 we can prove the following

Theorem 5.4.1. *For any smooth complex surface X there is a natural group homomorphism*

$$K^{S_n}(X^n) \to K(\mathrm{Hilb}^n X)$$

such that the induced map

$$\mathbb{Q} \otimes_{\mathbb{Z}} K^{S_n}(X^n) \to \mathbb{Q} \otimes_{\mathbb{Z}} K(\mathrm{Hilb}^n X)$$

is an isomorphism.

Remarks. (i) Although the K-groups above have natural ring structures the map in the theorem is *not* a ring homomorphism.
(ii) It would be interesting to compare the S_n-equivariant derived category of coherent sheaves on X^n with the derived category of coherent sheaves on $\mathrm{Hilb}^n X$.
(iii) I. Grojnowski has defined (see [Na] for details) on $\bigoplus_{n\geq 0} K(\mathrm{Hilb}^n X)$ the structure of a commutative and cocommutative Hopf algebra. Independently of this, G. Segal has constructed in [S] the structure of a commutative and cocommutative Hopf algebra on $\bigoplus_{n\geq 0} K^{S_n}(X^n)$. We verified (following a suggestion of E. Vasserot) that the morphism of Theorem 5.4.1 is compatible with these Hopf algebra structures.

6. Connection with reductive groups.

We indicate now how Theorem 4.1 has a natural interpretation and a generalisation in the framework of complex reductive groups.

6.1. Let G be a connected complex reductive group with Lie algebra $\mathfrak{g}$, let $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{g}$, and W the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. The group G acts on $\mathfrak{g}$ via the adjoint action, and we let $\mathbb{C}[\mathfrak{g}]^G$, resp. $\mathbb{C}[\mathfrak{h}]^W$, denote the algebra of G-invariant polynomials on $\mathfrak{g}$, resp. W-invariant polynomials on $\mathfrak{h}$. By the classical result of Chevalley, the restriction map $\mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{h}]$ induces an isomorphism

$$\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^W.$$

Inverting this isomorphism yields an imbedding $\mathbb{C}[\mathfrak{h}]^W \simeq \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$, hence gives raise to a morphism of the corresponding affine varieties

$$\mathfrak{g} = \mathrm{Specm}\,\mathbb{C}[\mathfrak{g}]^G \to \mathrm{Specm}\,\mathbb{C}[\mathfrak{h}]^W = \mathfrak{h}/W. \tag{6.1}$$

Now let $\mathfrak{g}^{\text{reg}} \subset \mathfrak{g}$ be the union of all G-orbits in $\mathfrak{g}$ of maximal dimension. Kostant [K] proved that the restriction of (6.1) to $\mathfrak{g}^{\text{reg}}$ induces a bijection

$$\mathfrak{g}^{\text{reg}}/G \xrightarrow{\sim} \mathfrak{h}/W. \tag{6.1.1}$$

6.2. We would like to have a "doubled" version of (6.1.1). To that end we consider the space $\mathfrak{g} \oplus \mathfrak{g}$ with the diagonal G-action, and also the space $\mathfrak{h} \oplus \mathfrak{h}$ with diagonal W-action. Notice that unlike what one had in 6.1 the G-saturation of the subset $\mathfrak{h}\oplus\mathfrak{h} \subset \mathfrak{g}\oplus\mathfrak{g}$ is *not* dense in $\mathfrak{g}\oplus\mathfrak{g}$. This suggests introducing the following *commuting variety*

$$Z = \{\, (x, y) \in \mathfrak{g} \oplus \mathfrak{g} \mid [x, y] = 0 \,\}.$$

Clearly Z is a closed G-stable subvariety of $\mathfrak{g} \oplus \mathfrak{g}$ containing $\mathfrak{h} \oplus \mathfrak{h}$, hence $G \cdot (\mathfrak{h} \oplus \mathfrak{h})$.

Theorem 6.2.1 [R]. *Z equals the closure of $G \cdot (\mathfrak{h} \oplus \mathfrak{h})$. In particular, Z is an irreducible variety and* $\dim Z = \dim \mathfrak{g} + \dim \mathfrak{h}$.

The following analogue of the Chevalley Restriction Theorem has been recently proved by T. Joseph [Jo].

Theorem 6.2.2. *The restriction $\mathbb{C}[Z] \to \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}]$ induces an isomorphism*

$$\mathbb{C}[Z]^G \xrightarrow{\sim} C[\mathfrak{h} \oplus \mathfrak{h}]^W.$$

As in 6.1 the above theorem gives a natural morphism

$$Z \to (\mathfrak{h} \oplus \mathfrak{h})/W. \tag{6.2.3}$$

Example. Let $G = \mathrm{GL}_n$. Then the map (6.1) is essentially given by assigning to an $n \times n$-matrix x its characteristic polynomial $z \mapsto \det(\mathrm{Id} - z \cdot x)$; the map (6.2.3) is essentially given by assigning to a pair (x, y) of commuting matrices the polynomial in two variables $z, w \mapsto \det(\mathrm{Id} - z \cdot x - w \cdot y)$.

6.3. Let $Z^{\text{reg}} \subset Z$ be the union of all G-orbits in Z of maximal dimension. The map (6.2.3) is constant along G-orbits but the induced map $Z^{\text{reg}}/G \to (\mathfrak{h} \oplus \mathfrak{h})/W$ is *not* a bijection.

To understand the situation consider the case $G = \mathrm{GL}_n$ first. Then $\mathfrak{h} = \mathbb{C}^n$ and $W = \Sigma_n$. We identify $\mathfrak{h} \oplus \mathfrak{h}$ with $\mathbb{C}^{2n} = X^n$, $(\mathfrak{h} \oplus \mathfrak{h})/W$ with X^n/S_n, where $X = \mathbb{C}^2$, as in 2.2.

Note further that given a pair (x, y) of commuting matrices we can make $\mathbb{C}^n$ into a $\mathbb{C}[X]$-module by letting a polynomial $P \in \mathbb{C}[X]$ act as the operator $P(x, y)$. Call a vector $v \in \mathbb{C}^n$ *cyclic* if $\mathbb{C}[X] \cdot v = \mathbb{C}^n$. Let $Z^0 \subset Z$ be the subset of pairs (x, y) that have a cyclic vector. The annihilation of such a vector is a codimension n ideal in $\mathbb{C}[X]$. Moreover, this ideal does not depend on the choice of the cyclic vector and does not change if (x, y) is replaced by a conjugate pair. Thus, associating to the G-orbit of $(x, y) \in Z^0$ the annihilator of a cyclic vector gives a bijection

$$Z^0/G \xrightarrow{\sim} \mathrm{Hilb}^n X \tag{6.3.1}$$

such that the map $Z^0/G \to (\mathfrak{h} \oplus \mathfrak{h})/W$ arising from (6.2.3) becomes identified via (6.3.1) with the standard projection $\mathrm{Hilb}^n X \to X^n/\Sigma_n$. It is easy to check that $Z^0 \subset Z^{\text{reg}}$. Thus we see that only the subset Z^0, but not Z^{reg}, has a nice quotient by the G-action.

6.4. There is no analogue of the notion of a cyclic vector, hence of the set Z^0, for a general reductive group G. We therefore have to proceed in a different way. To that end, given $(x,y) \in Z$, write $\mathfrak{a}(x,y)$ for the simultaneous centralizer of both x and y in $\mathfrak{g}$. If $(x,y) \in Z^{\text{reg}}$ then $\dim \mathfrak{a}(x,y) = \dim \mathfrak{h}$ and the collection of vector spaces $\mathfrak{a}(x,y)$ forms a G-equivariant vector bundle $\mathfrak{a}$ on Z^{reg}. We put $L = \bigwedge^{\operatorname{rk}\mathfrak{a}}(\mathfrak{a}^*)$, a line bundle on Z^{reg}. The graded space

$$A = \bigoplus_{k\geq 0} A_k, \qquad A_k = \Gamma\big(Z^{\text{reg}}, L^{\otimes k}\big)^G \tag{6.4.1}$$

has an obvious graded algebra structure, and for the reductive group G we define

$$\operatorname{Hilb}_G := \operatorname{Proj} A \quad (= \text{projective spectrum of } A).$$

This definition is motivated by the following result, which can be deduced from the construction of the Hilbert scheme of $X = \mathbb{C}^2$ by means of Geometric Invariant Theory, explained in [Na].

Theorem 6.4.2. *Let $G = \mathrm{GL}_n$, and $p\colon Z^0 \to \operatorname{Hilb}^n X$ the map constructed in 6.3. Then:*
(i) There is a natural isomorphism $L \simeq p^*\mathcal{L}$.
(ii) There is a natural isomorphism $\operatorname{Hilb}_G \simeq \operatorname{Hilb}^n(X)$ compatible with (i).

In the case of a general reductive group one has the following analogues of isomorphisms 2.2.1 and 2.3.3 respectively.

Lemma 6.4.3. *There are canonical isomorphisms*
(i) $A_0 = \mathcal{O}\big(Z^{\text{reg}}\big)^G \simeq \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}]^W$,
(ii) $A_1 = \Gamma\big(Z^{\text{reg}}, L\big)^G \simeq \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}]^{\text{sign}}$.

Note that by definition the line bundle L descends to a line bundle $\mathcal{L}$ on Hilb_G such that $\Gamma(\operatorname{Hilb}_G, \mathcal{L}) = A_1$. Note also that part (i) of the lemma gives a natural projection $\pi\colon \operatorname{Hilb}_G \to (\mathfrak{h}\oplus\mathfrak{h})/W$ induced by the algebra imbedding $A_0 \hookrightarrow A$.

6.5. Consider now the Hilbert quotient $\widehat{(\mathfrak{h}\oplus\mathfrak{h})/W}$. One constructs a map $\nu\colon \widehat{(\mathfrak{h}\oplus\mathfrak{h})/W} \to \operatorname{Hilb}_G$ making the following triangle commute

$$\begin{array}{ccc} \widehat{(\mathfrak{h}\oplus\mathfrak{h})/W} & \xrightarrow{\ \nu\ } & \operatorname{Hilb}_G \\ & \searrow \quad \swarrow_{\pi} & \\ & (\mathfrak{h}\oplus\mathfrak{h})/W & \end{array}$$

In the special case $G = \mathrm{GL}_n$ the map ν is nothing but the map $\nu\colon \widehat{X^n/S_n} \to \operatorname{Hilb}^n X$ introduced in 5.3.

The following is a "doubled" version of (6.1.1).

Main Conjecture. *For any reductive group G the variety Hilb_G is smooth and the map $\nu\colon \widehat{(\mathfrak{h}\oplus\mathfrak{h})/W} \to \operatorname{Hilb}_G$ is an isomorphism.*

If $G = \mathrm{GL}_n$, then by Theorem 6.4.2 the Main Conjecture reduces to Theorem 5.3.1 for $X = \mathbb{C}^2$. For $G = \mathrm{GL}_n$ we can also prove the following.

LOCAL STRUCTURE OF MODULI SPACES

Vadim Schechtman
Max-Planck-Institut für Mathematik,
Gottfried-Claren-Straße 26, 53225 Bonn

This is a report on a joint work with Vladimir Hinich.

1. Let X be a smooth algebraic variety over a field k of characteristic 0. Assume that X has no infinitesimal automorphisms, i.e. $H^0(X;\mathcal{T}_X)=0$ ($\mathcal{T}_X$ being the sheaf of vector fields). Let $\mathcal{M}=Spec(\hat{\mathcal{O}}_{\mathcal{M}})$ be the formal moduli space of deformations of X; $\hat{\mathcal{O}}_{\mathcal{M}}$ is a complete local k-algebra.

We have the *Kodaira-Spencer isomorphism*

$$T_{\mathcal{M};X}=H^1(X;\mathcal{T}_X) \qquad (KS)$$

Here $T_{\mathcal{M};X}=(\mathfrak{m}_{\mathcal{O}_{\mathcal{M}}}/\mathfrak{m}^2_{\mathcal{O}_{\mathcal{M}}})^*$ is the tangent space of $\mathcal{M}$ at X. We want to consider the

Problem. *Describe the whole algebra $\hat{\mathcal{O}}_{\mathcal{M}}$ in terms of X.*

Let

$$\mathcal{T}^\bullet_X=R\Gamma(X;\mathcal{T}_X):\ 0\longrightarrow\mathcal{T}^0\longrightarrow\mathcal{T}^1\longrightarrow\dots$$

be a complex computing the sheaf cohomology of $\mathcal{T}_X$. The last sheaf is a sheaf of Lie algebras, hence $\mathcal{T}^\bullet$ may be chosen to be a *differential graded Lie algebra*; it is a correctly defined object of the appropriate derived category of *Homotopy Lie Algebras*, [HS1].

Theorem 1. *One has a canonical isomorphism of k-algebras*

$$\hat{\mathcal{O}}_{\mathcal{M}}=[H_0^{Lie}(\mathcal{T}^\bullet_X)]^* \qquad (1)$$

The homology of a (dg) Lie algebra is a (dg) coalgebra. The dual space is an algebra. The isomorphism (1) is a generalization of the Kodaira-Spencer isomorphism.

This theorem is a just an example of a quite general fact; the similar results (with the same proof) hold true for other deformation problems. For example, we may wish to describe deformations of group representations, etc. Cf. [S2], [HS1].

I know two proofs of Theorem 1. The first one works in the case when $\mathcal{M}$ is smooth, and uses the *higher Kodaira-Spencer maps*, cf. [HS1]. The second one works in general situation. It uses certain very natural *sheaf property* of *Lie-Deligne functor*, and is decribed below.

2. Deligne groupoids. Let $\mathfrak{g}=\oplus_{i\geq 0}\,\mathfrak{g}^i$ be a nilpotent dg Lie algebra. Recall that *groupoid* is a category with all morphisms being isomorphisms. The *Deligne groupoid* $\mathcal{G}(\mathfrak{g}^\bullet)$ is defined as follows. Its objects are *Maurer-Cartan elements*

$$MC(\mathfrak{g}^\bullet):=\{y\in\mathfrak{g}^1|dy+\frac{1}{2}[y,y]=0\}$$

Let $\mathcal{G}(\mathfrak{g}^0)$ be the Lie group corresponding to the nilpotent Lie algebra $\mathfrak{g}^0$. The algebra $\mathfrak{g}^0$ acts on $MC(\mathfrak{g}^\bullet)$ by the rule

$$x \circ y = dx + [x, y], \quad x \in \mathfrak{g}^0, y \in MC(\mathfrak{g}^\bullet),$$

hence the group $\mathcal{G}(\mathfrak{g}^0)$ acts on $MC(\mathfrak{g}^\bullet)$. By definition,

$$Hom_{\mathcal{G}(\mathfrak{g}^\bullet)}(y, y') = \{g \in \mathcal{G}(\mathfrak{g}^0) | y' = gy\}$$

Morphisms are composed in the obvious way. Of course, this is a generalization of the Lie functor from Lie algebras to Lie groups.

3. Let us return to our deformation situation. The variety X defines a functor

$$Def_X : Art_k \longrightarrow Groupoids$$

where Art_k is the category of artinian k-algebras with residue field k. Namely, $Def_X(A)$ is the groupoid whose objects are flat deformations of X over A, and morphisms are isomorpshisms identical on X.

On the other hand, if $\mathfrak{g}^\bullet$ is a dg Lie algebra over k, it defines a functor

$$\mathcal{G}_{\mathfrak{g}^\bullet} : Art_X \longrightarrow Groupoids,$$

by

$$\mathcal{G}_{\mathfrak{g}^\bullet}(A) = \mathcal{G}(\mathfrak{m}_A \otimes \mathfrak{g}^\bullet)$$

where $\mathfrak{m}_A$ is the maximal ideal of A.

Example. Assume that $X = Spec(R)$ is affine. Then one sees immediately from the definitions (Grothendieck) that one has an isomorphism of functors

$$Def_X = \mathcal{G}_{\mathfrak{g}^\bullet} \tag{2}$$

where $\mathfrak{g}^\bullet = T_X = H^0(X; \mathcal{T}_X) = Der_k(R)$ considered as a dg Lie algebra concentrated in dimension 0.

Sometimes when (2) holds, the people say that the dg Lie algebra $\mathfrak{g}^\bullet$ *governs the deformations of X.*

Theorem 2. *Let X be arbitrary, and (2) holds for some $\mathfrak{g}^\bullet$. If $H^0(\mathfrak{g}^\bullet) = 0$ then*

$$\hat{O}_{\mathcal{M}} = (H_0^{Lie}(\mathfrak{g}^\bullet))^*$$

Proof. For an arbitrary $A \in Art_K$, we have

$$Hom_{Art_k}(\hat{O}_{\mathcal{M}}, A) = \pi_0(Def_X(A)) = \pi_0(\mathcal{G}(\mathfrak{m}_A \otimes \mathfrak{g}^\bullet)) =$$

$$= Hom_{alg}((H_0^{Lie}(\mathfrak{g}^\bullet))^*, A) \quad \triangle$$

4. Now, we know the Lie algebra $\mathfrak{g}^\bullet$ for affine varieties, and we want to know it for arbitrary ones, i.e. we want to glue them.

What is a vertex algebra?.

Richard E. Borcherds, *
D.P.M.M.S., 16 Mill Lane, Cambridge, CB2 1SB, England.
reb@dpmms.cam.ac.uk, http://www.dpmms.cam.ac.uk/~reb

The answer to the question in the title is that a vertex algebra is really a sort of commutative ring. I will try to explain this in the rest of the talk, and show how to use this to generalize the idea of a vertex algebra to higher dimensions. The picture to keep in mind is that a commutative ring should be thought of as somehow related to quantum field theories in 0 dimensions, and vertex algebras are related in the same way to 1 dimensional quantum field theories, and we want to find out what corresponds to higher dimensional field theories. This talk is an exposition of the paper q-alg/9706008, which contains (some of) the missing details. There is also probably some overlap with unpublished notes of Soibelman, which he has promised will soon appear on the q-alg preprint server.

The relation of vertex algebras to commutative rings is obscured by the rather bad notation generally used for vertex algebras. Recall that for any element v of a vertex algebra V we have a vertex operator denoted by $V(v,z)$ taking V to the Laurent power series in V. I am going to change notation and write $V(v,z)u$ as $v^z u$. Let us see what several standard formulas look like in this new notation:

Old notation	New notation
$V(u,z)v$	$u^z v$
$V(V(a,x)b,y)c = V(a,x+y)V(b,y)c$	$(a^x b)^y c = a^{xy}(b^y c)$
$V(a,x)V(b,y)c = V(b,y)V(a,x)c$	$a^x b^y c = b^y a^x c$
$V(a,x)b = e^{xL_{-1}}(V(b,-x)a)$	$a^x b = (b^{x^{-1}} a)^x$
$V(1,x)b = b$	$1^x b = b$

The formulas on the right hand side are all easy to recognize: they are just standard formulas for a commutative ring acted on by a group G, where a,b,c are in the ring V, x,y,z are elements of the group G, and the action of $x \in G$ on $a \in V$ is denoted by a^x. This suggests that we should try to set things up so that vertex algebras are exactly the commutative rings objects over some sort of mysterious group-like thing G.

For simplicity we will work over a field of characteristic 0. This is not an important assumption; it just saves us from some minor technicalities about divided powers of derivations.

We will first look at the special case of vertex algebras such that all the vertex operators $V(a,x)$ are holomorphic. We show that such vertex algebras are the same as commutative algebras with a derivation D. The correspondence is given as follows. First suppose that V is a commutative algebra with derivation D. We define the vertex operator $V(a,x)$ by $V(a,x)b = \sum_{i\geq 0}(D^i a)bx^i/i!$. Conversely if V is a vertex algebra we define the product by $ab = V(a,0)b$ and the derivation by Da = coefficient of x^1 in $V(a,x)b$. (We cannot

* Supported by a Royal Society professorship.

really check that this turns commutative algebras into holomorphic vertex algebras and vice versa because we have not yet said exactly what the axioms for a vertex algebra are.)

In the new notation for vertex algebras above we would put $a^x = \sum_i x^i D^i a/i!$, $a^x b = \sum_i x^i D^i ab/i!$. Here we think of x as being an "element" of the one dimensional formal group $\hat{G}_a$. This formal group has as its formal group ring H the algebra of polynomials $k[D]$ and its coordinate ring is the ring of formal power series $k[[x]]$. (In characteristic 0 it does not matter whether we use Lie algebras or formal groups which are essentially equivalent, but in other characteristics formal groups are better than Lie algebras.) The (tensor) category of modules with a derivation is the same as the category of modules over the formal group ring H, so holomorphic vertex algebras are the same as the commutative ring objects in this category.

What is the difference between a commutative algebra over $\hat{G}_a$ and a (non holomorphic) vertex algebra? The only difference is that expressions like $a^x b^y c$ are no longer holomorphic in x, y but can have singularities; more precisely $a^x b^y c$ lies in $V[[x,y]][x^{-1}, y^{-1}, (x-y)^{-1}]$. In other words we can provisionally define a vertex algebra to be a module V such that we are given functions $a^x b^y c^z \cdots$ for each $a, b, c, \ldots \in V$ which behave just like the corresponding functions for commutative rings over $\hat{G}_a$, except that they are allowed to have certain sorts of singularities. Notice that we can no longer reconstruct a commutative ring structure on V by defining $ab = a^x b$ at $x = 1$, because $a^x b$ may have a singularity at $x = 1$.

The definition above is too vague to be useful, so we try to make it more precise. What we really want to do is to define some sort of category, whose multilinear maps are somehow allowed to have the sort of singularities above, and whose commutative ring objects are just vertex algebras. We first ask in what sort of categories we can define commutative ring objects. The obvious answer is tensor categories, such as the category of modules over $\hat{G}_a$ (or over any cocommutative Hopf algebra) but this turns out to be too restrictive. (We will implicitly assume that all categories are additive and have some sort of "symmetric" structure.) A tensor category requires that multilinear maps should be representable, but this is sometimes not the case for the categories we are interested in, and in any case this assumption is unnecessary. It is sufficient to assume that for each collection of objects $A_1, \ldots, A_n, B$ of the category we are given the space of multilinear maps $Multi(A_1, \ldots, A_n, B)$, and that these satisfy a large number of fairly obvious properties which I cannot be bothered to write down.

Unfortunately multilinear categories are not really the right objects either. The problem is the following. An expression like $a^x b^y c$ should live in a space like $V[[x,y][x^{-1}, y^{-1}, (x-y)^{-1}]$. However the expression $a^x(b^y c)$ does not naturally live in this space, but in the larger space $V[[y]][y^{-1}][[x]][x^{-1}]$, and $(a^{xy^{-1}} b)^y c$ lives in a different larger space. This makes it hard to compare these expressions in a clean way. The easiest way to solve this problem is to define it out of existence by using "relaxed multilinear categories". The key idea is that instead of just once space of multilinear maps $Multi(A_1, \ldots, A_n, B)$ we are given many different spaces $Multi_p(A_1, \ldots, A_n, B)$ of multilinear maps, parameterized by trees p with a root (corresponding to B) and n leaves, corresponding to $A_1, \ldots, A_n$. We should also have some extra structures, consisting of maps between different spaces of multilinear maps corresponding to collapsing maps between trees, and a composition of multilinear maps taking multilinear maps of types $p_1, \ldots, p_n, p$ to a multilinear map of type $p(p_1, \ldots, p_n)$. (Here $p(p_1, \ldots, p_n)$ is the tree obtained by attaching $p_1, \ldots, p_n$ to the leaves of p.) For details see my paper or Soibelman's notes, or better still work them out for yourself.

The main point is that in a relaxed multilinear category it is still possible to define commutative associative algebras. Joyal has pointed out that the definition of associative algebras in a relaxed multilinear category is strikingly similar to the definition of an A_∞ algebra; for example, the cells of the complexes used to define A_∞ algebras are parameterized by rooted trees with n leaves, and the boundary maps correspond to the collapsing maps between trees.

One way of constructing relaxed multilinear categories is as the representations of "vertex groups". A vertex group can be thought of informally as a group together with certain sorts of allowed singularities of functions on the group. More precisely a vertex group is given by a cocommutative Hopf algebra H, which we thing of as its group ring, together with an algebra of "singular functions" K over the "coordinate ring" H^* of H. The axioms for a vertex group say that K behaves as if it were the ring of meromorphic functions over the "group" G; for example, the ring of meromorphic functions is acted on by left and right translations, so K should have good left and right actions of H. A typical example of a vertex group is to take $H = k[D]$, $H^* = k[[x]]$ (so that H is the formal group ring of $\hat{G}_a$), and to take K to be the quotient field $k[[x]][x^{-1}]$ of H^*, which we can think of as the field of rational functions on the formal group $\hat{G}_a$.

We can construct a relaxed multilinear category from a vertex group roughly as follows. The underlying category is the same as that of the Hopf algebra of the vertex group. However the spaces of multilinear maps are different. Rather than define these in general, which is a bit complicated, we will just look at one example. We take G to be the vertex group above (whose commutative rings are vertex algebras), and take 3 G-modules A, B, and C. Then the space of bilinear maps from A, B to C is defined to be the ordinary space of bilinear maps from $A \times B$ to $C[[x, y]][(x-y)^{-1}]$ which are invariant under an action of G^3. (The easiest way to work out what the action of G^3 should be is to see what it has to be for the invariant bilinear maps from $A \times B$ to $C[[x, y]]$ taking $a \times b$ to $\sum_{i,j} f(D^i a, D^j b)x^i y^j / i!j!$ to be the same as invariant maps f from $A \times B$ to C.)

We summarize what we have done so far:

1 We have introduced "vertex groups"'.

2 The modules over a vertex group form a "relaxed multilinear category".

3 The commutative ring objects over the simplest nontrivial vertex group are exactly vertex algebras.

Now that we have set up this machinery, it is easy to find higher dimensional analogues of vertex algebras: all we have to do is look at commutative algebras over higher dimensional vertex groups G; we will call these vertex G algebras. As an example we will construct vertex algebras related to free quantum field theories in higher dimensions. (Can one construct vertex algebras corresponding to nontrivial quantum field theories in higher dimensions? At the moment this is just a daydream, as it is too vague to be called a conjecture.)

We first need to construct a suitable vertex group G. We take its underlying Hopf algebra H to be the polynomial algebra $\mathbf{R}[D_0, \ldots, D_n]$ where $D_i = \partial/\partial x_i$, which we think of as the universal enveloping algebra of the Lie algebra of translations of spacetime (with $x_0 = t$). The dual H^* is then $\mathbf{R}[[x_0, \ldots, x_n]]$, which we think of as the algebra of functions on spacetime. We define K to be $H^*[(x_0^2 - x_1^2 - \cdots x_n^2)^{-1}]$, which we think of as the algebra of functions on spacetime which are allowed to have singularities (poles) on the light cone.

Now we define the vertex G algebra V. The underlying space of V is the universal commutative H-algebra generated by an element ϕ, so $V = \mathbf{R}[\phi, D_0\phi, D_1\phi, \ldots, D_0^2\phi, \ldots]$ is a ring of polynomials in an infinite number of variables. We think of V as the ring of classical fields generated by ϕ, and it is a (holomorphic) vertex G algebra as it is a commutative ring acted on by H. We will turn it into a a nontrivial vertex G algebra by "deforming" this trivial vertex G algebra structure. (In general, for vertex G-algebras, quantization means deforming the structure on some commutative ring to turn it into a vertex G algebra.)

To do this we recall the following method of constructing commutative rings: if V is a space acted on by commuting operators v_n, and if V is generated by an element $1 \in V$ by the action of these operators, then V has a unique commutative ring structure such that 1 is the identity and the actions of all the operators are given by multiplication by elements of V. (Proof: easy exercise.) A similar theorem holds for vertex algebras (as was proved by Frenkel, Kac, Rado, and Wang). We will make V into a vertex G algebra by finding a vertex operator $\phi(x) = \phi(x_0 \ldots, x_n)$ acting on V such that $\phi(x)\phi(y) = \phi(y)\phi(x)$ and applying the construction above.

To construct $\phi(x)$, we first put

$$\phi^+(x) = \sum_i D^i \phi x^i / i!.$$

The vertex algebra structure on V defined by this vertex operator is just the commutative ring structure on V, so we need to deform ϕ^+. We define $\phi^-(x)$ to be the unique G-invariant derivation from V to $V[x][(x_0^2 - x_1^2 \cdots)^{-1}]$ taking ϕ to some even function $\Delta(x)$ (called the propagator). This is uniquely defined by the universal property of V. Finally we put

$$\phi(x) = \phi^+(x) + \phi^-(x).$$

It is easy to check that $\phi(x)$ and $\phi(y)$ commute, as $\phi^+(x)$ and $\phi^+(y)$ commute, $\phi^-(x)$ and $\phi^-(y)$ commute, and $[\phi^-(x), \phi^+(y)] = -[\phi^+(x), \phi^-(y)] = \Delta(x - y)$. Therefore we can make V into a commutative vertex G-algebra.

Notice that in quantum field theory, $\phi(x)$ means the value of some operator valued distribution ϕ at some point x of a manifold. On the other hand, for vertex G algebras, $\phi(x)$ should be thought of as the action of a "group" element x on an element ϕ of a vertex G algebra. Several other concepts in quantum field theory can also be translated into vertex algebra theory; for example, the correlation functions are $Tr(\phi(x)\phi(y)\phi(z)\cdots)$, where Tr is some G invariant linear function on V. Some concepts are not so easy to extend; for example, vertex G algebras do not seem to be able to cope with arbitrary curved spacetimes other than Lie groups, and it can be difficult to reconstruct a Hilbert space (this includes as a very special case the problem of deciding which representations of the Virasoro algebra are unitary).

Motives and Hodge structures over function fields

Richard Pink

Universität Mannheim, D–68131 Mannheim

e-mail: pink@math.uni-mannheim.de

June 22, 1997

1. Motivation: The Mumford-Tate conjecture.

Alexandre Grothendieck's concept of motives was intended as a formal framework combining the many different aspects of algebraic varieties in a single theory. Thus to an algebraic variety X over a number field $K \subset \mathbb{C}$ are associated, among others, the rational mixed Hodge structure $H := H^n(X(\mathbb{C}), \mathbb{Q})$ and the ℓ-adic cohomology group $H^n(X_{\bar{K}}, \mathbb{Q}_\ell) \cong H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ as a a continuous Galois representation. Let $G_\infty \subset \mathrm{Aut}(H)$ be the Hodge group associated to this mixed Hodge structure, and $\Gamma_\ell \subset \mathrm{Aut}(H)(\mathbb{Q}_\ell)$ the image of $\mathrm{Gal}(\bar{K}/K)$. The Mumford-Tate conjecture asserts that Γ_ℓ is commensurable to $G_\infty(\mathbb{Z}_\ell)$, that is, their intersection is open in each of these two groups. The importance and the beauty of this statement lies in the fact that it relates two groups which are constructed in completely different ways and thus reflect very different properties of X, i.e. analytic resp. arithmetic ones.

While there has been some progress on this conjecture when X is an abelian variety, the general case remains completely open. In the analogous case of motives over function fields, however, it is now possible to prove such a conjecture in reasonable generality. The aim of this talk is to explain the necessary theory of Hodge structures and Hodge groups associated to motives over function fields.

My motivation to deal with the function field case is twofold. On the one hand I believe that definitions, theorems, and methods of proof in this area are interesting in themselves and often quite beautiful. On the other hand I hope that the study of function field analogues can provide us eventually with new ideas that re-fertilize the arithmetic over number fields.

2. Drinfeld modules.

In the following we fix a finite field $\mathbb{F}_q$ with q elements, and set $A := \mathbb{F}_q[t]$. This ring will play the role that $\mathbb{Z}$ plays in the number field case. Instead of $\mathbb{Q}$ we work with the rational function field $F := \mathbb{F}_q(t)$, and the completion $\mathbb{R}$ is replaced by $F_\infty := \mathbb{F}_q((t^{-1}))$. In all this theory, these rings may be replaced by finite extensions. As an analogue of $\mathbb{C}$ we take the completion of the algebraic closure of $\mathbb{F}_q((\theta^{-1}))$, denoted $\mathbb{C}_q$. Here θ is a new variable which in this section will be identified with t, but not afterwards. The field $\mathbb{C}_q$ is the basis for non-archimedean analysis in equal characteristic. Note that it has infinite degree over F_∞.

Now consider an A-lattice $\Lambda \subset \mathbb{C}_q$ of rank $r \geq 1$, that is, a discrete A-submodule which is isomorphic to A^r. Let us fix such an isomorphism. One can form the quotient "$\mathbb{C}_q/\Lambda$" in the following sense. One defines formally

$$e_\Lambda(X) := X \cdot \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{X}{\lambda}\right),$$

proves that this converges to an $\mathbb{F}_q$-linear power series, i.e., one of the form

$$e_\Lambda(X) = X + e_1 X^q + e_2 X^{q^2} + e_3 X^{q^3} + \ldots ,$$

shows that it converges on all of $\mathbb{C}_q$ and, finally, that the following sequence is exact:

$$0 \longrightarrow \Lambda \longrightarrow \mathbb{C}_q \xrightarrow{e_\Lambda} \mathbb{C}_q \longrightarrow 0 .$$

Multiplication by t then induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{C}_q & \xrightarrow{e_\Lambda} & \mathbb{C}_q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \Phi} & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{C}_q & \xrightarrow{e_\Lambda} & \mathbb{C}_q & \longrightarrow & 0 \end{array}$$

and one proves that Φ is a polynomial

$$\Phi(X) = \theta X + \Phi_1 X^q + \ldots + \Phi_r X^{q^r}$$

with $\Phi_r \neq 0$. In this way the quotient "$\mathbb{C}_q/\Lambda$" has been endowed with an algebraic structure over $\mathbb{C}_q$. This object makes up an algebraic Drinfeld module of rank r.

Let us now assume that this Drinfeld module is defined over a finitely generated subfield $\mathbb{F}_q(\theta) \subset K \subset \mathbb{C}_q$, that is, that all coefficients $\Phi_i \in K$. For any prime polynomial $\wp(t) \in A$ und any $n \geq 0$ we then have

$$\mathrm{Kern}(\wp(\Phi)^n : \bar{K} \to \bar{K}) = \mathrm{Kern}(\wp(\Phi)^n : \mathbb{C}_q \to \mathbb{C}_q) \cong \wp^{-n}\Lambda/\Lambda \cong (A/\wp^n A)^r .$$

Here $\bar{K}$ denotes the algebraic closure of K in $\mathbb{C}_q$, and the Galois action corresponds to a homomorphism

$$\mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}_r(A/\wp^n A) .$$

In the limit these homomorphisms fit together to a homomorphism

$$\mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}_r(A_\wp)$$

and we are interested in its image $\Gamma_\wp$.

We know a priori that all endomorphisms of Φ are defined over a finite extension of K and therefore commute with an open subgroup of $\Gamma_\wp$. Viewing these endomorphisms as subring of the matrix ring

$$\mathrm{End}_{\bar{K}}(\Phi) \cong \{x \in \mathbb{C}_q \mid x\Lambda \subset \Lambda\} \hookrightarrow \mathrm{End}_A(\Lambda) \cong \mathcal{M}_{r\times r}(A) ,$$

we can look at their centralizer

$$G_\infty := \mathrm{Cent}_{\mathrm{GL}_{r,F}}\left(\mathrm{End}_{\bar{K}}(\varphi)\right) .$$

In the generic case $\mathrm{End}_{\bar{K}}(\varphi) = A$ we have, of course, $G_\infty = \mathrm{GL}_{r,F}$; in general G_∞ is a form of $\mathrm{GL}_{r'}$ for $r'|r$.

Satz: $\Gamma_\wp$ *and* $G_\infty(A_\wp)$ *are commensurable. (see [6])*

This result is a precise analogue of the usual Mumford-Tate conjecture, with one important difference: Here the group G_∞ is defined only ad hoc and does not result from a general theory of Hodge structures. I will now show how to fill this gap.

One central requirement for such a theory is the invariance under tensor products. More precisely: Hodge structures should possess tensor products, and the desired functor associating Hodge structures to (certain) motives should be compatible with tensor products. Tensor products of Drinfeld modules are special cases of Anderson's uniformizable t-motives, so it would be best to have a theory applicable to all of these.

3. Anderson's t-motives.

Anderson made the fundamental and rather subtle observation that the two distinct roles of the variable t, once in the ring of coefficients $A = \mathbb{F}_q[t]$, once as element of the base field $\mathbb{C}_q$, ought to be separated. We have already replaced t by θ in its second meaning. Now take $d \geq 1$ and let $t \in \mathcal{M}_{d\times d}(\mathbb{C}_q)$ be a quadratic matrix whose only eigenvalue is θ, i.e. with

$$(\dagger) \qquad (t-\theta)^n = 0 \qquad \text{for all } n \gg 0.$$

Then we obtain a natural action of A on the vector space $\mathbb{C}_q^d$, and we consider an A-lattice $\Lambda \subset \mathbb{C}_q^d$, discrete and free of finite type over A. In general we cannot write down a series e_Λ as in the preceding section; instead we postulate its existence. For any vector $X \in \mathbb{C}_q^d$ let ${}^\sigma X$ denote taking the q^{th} power in each coefficient. We suppose given a power series

$$e(X) = X + e_1 \cdot {}^\sigma X + e_2 \cdot {}^{\sigma^2} X + \ldots$$

and a polynomial

$$\Phi(X) = \Phi_0 \cdot X + \Phi_1 \cdot {}^\sigma X + \ldots + \Phi_m \cdot {}^{\sigma^m} X$$

with $e_i,\ \Phi_i \in \mathcal{M}_{d\times d}(\mathbb{C}_q)$ such that we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{C}_q^d & \xrightarrow{e} & \mathbb{C}_q^d & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \Phi} & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{C}_q^d & \xrightarrow{e} & \mathbb{C}_q^d & \longrightarrow & 0\,. \end{array}$$

Essentially this makes up a uniformizable t-motive in the sense of Anderson [1]. (Here I neglect a certain technical assumption. Strictly speaking, the object thus constructed is called a uniformizable t-module, and the term t-motive is reserved for a certain equivalent dual description.) For instance, every Drinfeld module corresponds to a uniformizable t-motive.

Now we reencode the information in the lattice $\Lambda \subset \mathbb{C}_q^d$ in a way that is suited for tensor products. Condition (†) implies a natural map on the right hand side in the sequence

$$0 \longrightarrow \mathfrak{q} \longrightarrow \Lambda \otimes_A \mathbb{C}_q[\![t-\theta]\!] \longrightarrow \mathbb{C}_q^d \longrightarrow 0\,.$$

By Anderson this map is surjective. Thus the main information is contained in the kernel $\mathfrak{q}$, which by construction is a $\mathbb{C}_q[\![t-\theta]\!]$-lattice, i.e., a finitely generated $\mathbb{C}_q[\![t-\theta]\!]$-submodule containing a basis of the vector space $\Lambda \otimes_A \mathbb{C}_q(\!(t-\theta)\!)$.

The second ingredient of the desired Hodge structures is the weight filtration. There is no particular technical difficulty involved in working with mixed objects instead of pure ones. Besides, one reason for this greater generality is the fact that pure t-motives may degenerate into mixed t-motives. If m, n are positive integers, the t-motive is called pure of weight $\mu = -\frac{m}{n}$ if and only if after suitable reparametrization we can write

$$\Phi^n(X) = \ldots + \Phi_{n,m} \cdot {}^{\sigma^m}X$$

with $\det(\Phi_{n,m}) \neq 0$. For example a Drinfeld module of rank r is pure of weight $-\frac{1}{r}$. (My convention differs from Anderson's by a minus sign.) An arbitrary t-motive is called mixed if and only if it possesses an increasing weight filtration $W_\bullet$, indexed by rational numbers, such that each graded piece of weight μ is a pure t-motive of weight μ. If the t-motive is mixed and uniformizable, then each pure constituent is uniformizable, hence the lattice Λ inherits the weight filtration, again denoted by $W_\bullet$.

Now we have collected all the necessary ingredients for the desired Hodge structures. In order to have an F-linear theory we put $H := \Lambda \otimes_A F$, then the uniformizable t-motive up to isogeny determines the data $\underline{H} = (H, W_\bullet, \mathfrak{q})$. The tensor product of two such triples $\underline{H}_i = (H_i, W_\bullet, \mathfrak{q}_i)$ is defined as $(H, W_\bullet, \mathfrak{q})$ with $H := H_1 \otimes_F H_2$, the weight filtration

$$W_\mu(H_1 \otimes_F H_2) := \sum_{\mu_1+\mu_2=\mu} W_{\mu_1}H_1 \otimes_F W_{\mu_2}H_2\,,$$

and $\mathfrak{q} := \mathfrak{q}_1 \otimes_{\mathbb{C}_q[\![t-\theta]\!]} \mathfrak{q}_2$. This definition is compatible with the tensor product of t-motives as defined by Anderson.

4. Mixed $\mathbb{F}_q(t)$-Hodge structures.

Observe that the inclusion $A = \mathbb{F}[t] \hookrightarrow \mathbb{C}_q[[t-\theta]]$ extends naturally to an inclusion $F \subset F_\infty \hookrightarrow \mathbb{C}_q[[t-\theta]]$.

Definition: *A mixed F-pre-Hodge structure $\underline{H} = (H, W_\bullet, \mathfrak{q})$ consists of a finite dimensional F-vector space H, an increasing filtration by F-subspaces $W_\mu H$, indexed by $\mu \in \mathbb{Q}$ and called weight filtration, and a $\mathbb{C}_q[[t-\theta]]$-lattice $\mathfrak{q} \subset H \otimes_F \mathbb{C}_q((t-\theta))$.*

Homomorphisms of such objects are homomorphisms of the underlying F-vector spaces that are compatible with the filtrations and lattices. This category is F-linear but not abelian, so we want to restrict attention to a suitable subcategory. Note also that when $\underline{H}$ comes from a uniformizable t-motive, we have not yet used the discreteness of Λ. This property is related to the following numerical condition. For every F_∞-subspace $H'_\infty \subset H_\infty := H \otimes_F F_\infty$ consider the lattices $\mathfrak{q}' := \mathfrak{q} \cap (H'_\infty \otimes_{F_\infty} \mathbb{C}_q((t-\theta)))$ and $\mathfrak{p}' := H'_\infty \otimes_{F_\infty} \mathbb{C}_q[[t-\theta]]$, and put

$$\deg_{\mathfrak{q}}(H'_\infty) := \dim_{\mathbb{C}_q}\Big(\frac{\mathfrak{q}'}{\mathfrak{p}' \cap \mathfrak{q}'}\Big) - \dim_{\mathbb{C}_q}\Big(\frac{\mathfrak{p}'}{\mathfrak{p}' \cap \mathfrak{q}'}\Big) .$$

This number measures the size of $\mathfrak{q}'$. On the other hand set

$$\deg_W(H'_\infty) := \sum_{\mu\in\mathbb{Q}} \mu \cdot \dim_{F_\infty} \mathrm{Gr}^W_\mu (H'_\infty) .$$

Definition: *A mixed F-pre-Hodge structure $\underline{H} = (H, W_\bullet, \mathfrak{q})$ is called a mixed F-Hodge structure if and only if for every H'_∞ we have*

$$\deg_{\mathfrak{q}}(H') \leq \deg_W(H') ,$$

with equality whenever $H' = W_\mu H$ for some $\mu \in \mathbb{Q}$. The full subcategory of all mixed F-Hodge structures is denoted $\mathcal{H}odge_F$.

A closer look at the pure case shows that this condition is rather similar to the usual semistability condition of vector bundles.

Satz: *$\mathcal{H}odge_F$ is a neutral tannakian category over F.*

The proof is modeled on similar statements for vector bundles or filtered modules. The hardest part is to show that the semistability condition is invariant under tensor product. The term "neutral" refers to the tautological fiber functor $\underline{H} \mapsto H$. We also have:

Satz: *The above construction defines a tensor functor from the category of uniformizable mixed t-motives over $\mathbb{C}_q$ up to isogeny to the category $\mathcal{H}odge_F$. This functor is exact, F-linear, fully faithful, and its essential image is closed under taking subquotients.*

The last two statements amount to an analogue of the Hodge conjecture.

5. The Hodge group.

For any object $\underline{H}$ of $\mathscr{Hodge}_F$ let $\langle\!\langle \underline{H} \rangle\!\rangle$ denote the smallest abelian full subcategory of $\mathscr{Hodge}_F$ that contains $\underline{H}$ and is closed under tensor product, dualization, and subquotients. By general tannakian theory there is a well-defined algebraic subgroup $G_{\underline{H}} \subset \mathrm{Aut}_F(H)$ such that $\langle\!\langle \underline{H} \rangle\!\rangle$ is equivalent to the category of finite dimensional representations of $G_{\underline{H}}$ over F. This group is called the Hodge group of $\underline{H}$. In the case of a Drinfeld module our original expectations are confirmed:

Satz: *If $\underline{H}$ is associated to the Drinfeld module of Section 2, we have $G_{\underline{H}} = G_\infty$.*

More generally, suppose that the coefficients of the t-motive Φ in Section 3 are contained in a finitely generated extension $K \subset \mathbb{C}_q$ of F. As in Section 2 we obtain a Galois representation

$$\mathrm{Gal}(\bar{K}/K) \twoheadrightarrow \Gamma_\wp \subset \mathrm{Aut}_{A_\wp}(\Lambda \otimes_A A_\wp) \cong \mathrm{GL}_r(A_\wp) .$$

Satz: *$\Gamma_\wp$ is commensurable to a Zariski dense subgroup of $G_\infty(A_\wp)$.*

This is proved by combining the above analogue of the Hodge conjecture with a theorem of A. Tamagawa amounting to a strong form of the Tate conjecture for t-motives. I also expect to determine $\Gamma_\wp$ up to commensurability, but the precise statement will be somewhat technical.

Bibliography.

[1] Anderson, G., t-motives *Duke Math. J.* **53** (1986), 457–502.

[2] Deligne, P., J. S. Milne, Tannakian Categories, in: *Hodge Cycles, Motives, and Shimura Varieties*, P. Deligne et al. (Eds.), ch. VI, LNM **900**, Berlin etc.: Springer (1982), 101–228.

[3] Goss, D., L-series of t-motives and Drinfeld Modules, in: *The Arithmetic of Function Fields*, Goss, D., et al. (Eds.) Proceedings of the workshop at Ohio State University 1991. Berlin: Walter de Gruyter (1992), 313–402.

[4] Goss, D., *Basic Structures of Function Field Arithmetic*, Ergebnisse **35**, Berlin etc.: Springer (1996).

[5] Pink, R., Compact subgroups of linear algebraic groups, Preprint Mannheim, (August 1996), 58 S.

[6] Pink, R., The Mumford-Tate conjecture for Drinfeld modules, Preprint Mannheim, (Sept. 1996), 26 S., to appear in *Publ. RIMS, Kyoto University* **33**, 3.

[7] Pink, R., ℓ-adic algebraic monodromy groups, cocharacters, and the Mumford-Tate conjecture, Preprint Mannheim, (March 1997), 54p.

[8] Pink, R., Hodge structures over function fields, in preparation.

DEFORMATION QUANTIZATION

Maxim Kontsevich

I.H.E.S., 35 Route de Chartres, Bures-sur-Yvette 91440, France; email: maxim@ihes.fr

1. Star-products

Let A be the algebra over $\mathbf{R}$ of functions on C^∞-manifold X. *Star-product* on X is a structure of an associative algebra over $\mathbf{R}[[\hbar]]$ on $A[[\hbar]] := A\widehat{\otimes}_{\mathbf{R}}\mathbf{R}[[\hbar]]$ such that for any $f, g \in A \subset A[[\hbar]]$ the "new " product, denoted by $f \star g$, is given by the formula

$$f \star g = fg + \hbar B_1(f \otimes g) + \hbar^2 B_2(f \otimes g) + \ldots \in A[[\hbar]]$$

where $B_i : A \otimes A \longrightarrow A,\ i \geq 1$ are bidifferential operators on X. Associativity of the star-product

$$(f \star g) \star h = f \star (g \star h) \quad \forall f, g, h \in A[[\hbar]]$$

is a non-trivial quadratic constraint on $(B_i)_{i\in\mathbf{N}}$.

There is an action of infinite-dimensional group $G \subset Aut_{\mathbf{R}[[\hbar]]-mod}(A[[\hbar]])$:

$$G := \{\text{maps } f \mapsto f + \hbar D_1(f) + \hbar^2 D_2(f) + \ldots \mid D_i \text{ are differential operators on } X\}$$

on the set of star-products.

It is easy to see that by G-action one can kill symmetric part of bidifferential operator B_1. Thus, we can assume that B_1 is antisymmetric, $B_1(f \otimes g) = -B_1(g \otimes f)$. It follows from the associativity that B_1 is a bi-derivation, i.e. a bivector field, and also it satisfies the Jacobi identity. The conclusion is that B_1 gives a Poisson structure on X.

Any symplectic manifold (a natural object in classical mechanics) carries non-degenerate Poisson structure and could correspond via term B_1 to a non-commutative algebra (observables in quantum mechanics). It was one of motivations 20 years ago for Bayen, Flato, Fronsdal, Lichnerowitz and Sternheimer to start the study of star-products. A star-product modulo the gauge equivalence (G-action) is called a deformation quantization of manifold X. In 80-ies De Wilde, Lecompte and (later) Fedosov constructed a canonical gauge equivalence class of star-products on all symplectic manifolds. We think nevertheless that the whole line of ideas was based on slightly unnatural assumptions. First of all, the Euler-Lagrange equation in classical mechanics gives a closed 2-form, not a Poisson bracket. In degenerate cases one can not relate 2-forms and bivector fields. Also, there is no intrinsic reason in quantum mechanics to have an associative algebra of observables.

In the next section we will describe all deformation quantizations in geometrical terms. The proof of our main result is based on ideas from string theory. It seems that associative algebras are most closely related with open string theories, not with the quantum mechanics.

2. Classification of star-products

Theorem. *For any manifold X one can canonically identify the set of gauge equivalence classes of star-products on X with the following set:*

$$\{\alpha(\hbar) \mid \alpha(\hbar) = \alpha_1\hbar + \alpha_2\hbar^2 + \ldots \in \Gamma(\wedge^2 T_X)[[\hbar]],\ [\alpha(\hbar), \alpha(\hbar)] = 0\}/\widetilde{G}$$

where $[\cdot,\cdot]:\Gamma(\wedge^2 T_X)\otimes\Gamma(\wedge^2 T_X)\longrightarrow\Gamma(\wedge^3 T_X)$ is the Schouten-Nijenhuis bracket on polyvector fields, and $\widetilde{G}$ is the group of formal paths starting at id_X in the diffeomorphism group of X:

$$\widetilde{G} := Maps\left((Spec(\mathbf{R}[[\hbar]],0)\longrightarrow(Diff(X),id_X)\right) \ .$$

We remind that bivector field $\alpha\in\Gamma(\wedge^2 T_X)$ gives a Poisson structure on X iff $[\alpha,\alpha]=0$.

As an immediate corrolary of our theorem we conclude that any Poisson structure $\alpha_1\in\Gamma(\wedge^2 T_X)$ is canonically quantazible. The deformation quantization corresponds to the path $\alpha(\hbar):=\alpha_1\hbar$.

3. Explicit formula for $X=\mathbf{R}^N$

Let $\alpha=\sum\alpha^{ij}(x)\partial_i\wedge\partial_j$ be a Poisson structure on $\mathbf{R}^N$, $\partial_i=\partial/\partial x^i$, $i=1,\dots,N$. First few terms for the star-product corresponding to α are following:

$$f\star g = fg+\hbar\sum_{i,j}\alpha^{ij}\,\partial_i f\,\partial_j g+\frac{\hbar^2}{2}\sum_{i,j,k,l}\alpha^{ij}\,\alpha^{kl}\,\partial_i\partial_k f\,\partial_j\partial_l g+$$

$$+\frac{\hbar^2}{3}\sum_{i,j,k,l}\alpha^{ij}\,\partial_i\alpha^{kl}\,(\partial_j\partial_k f\,\partial_l g-\partial_j\partial_k g\,\partial_l f)+O(\hbar^3)\ .$$

In the full formula terms are naturally labeled by certain oriented graphs. It is convenient to encode graphs of degree $n\in\mathbf{Z}_{\geq 0}$ (giving terms proportional to $\hbar^n$) by two maps

$$a_1,\,a_2:\{1,\dots,n\}\longrightarrow\{1,\dots,n+2\}$$

such that for any $k\in\{1,\dots,n\}$ three numbers $k,a_1(k),a_2(k)$ are pairwise distinct.

Graph Γ associated with (a_1,a_2) has $n+2$ enumerated vertices. First n vertices correspond to bivector field α, the $(n+1)$-st vertex corresponds to function f, and the $(n+2)$-nd vertex corresponds to function g. Edges of Γ are oriented. The complete list of edges is

$$\{k\longrightarrow a_1(k),\ k\longrightarrow a_2(k)\,|\,k=1,\dots,n\}\ .$$

The expression in our formula corresponding to Γ is

$$B_\Gamma(f,g;\alpha):=\sum_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n\\ 1\leq i_*,j_*\leq N}}\prod_{k=1}^{n}\left[\left(\prod_{l_1:a_1(l_1)=k}\partial_{i_{l_1}}\right)\cdot\left(\prod_{l_2:a_2(l_2)=k}\partial_{j_{l_2}}\right)\alpha^{i_k j_k}\right]\cdot$$

$$\cdot\left[\left(\prod_{l_1:a_1(l_1)=n+1}\partial_{i_{l_1}}\right)\cdot\left(\prod_{l_2:a_2(l_2)=n+1}\partial_{j_{l_2}}\right)f\right]\cdot\left[\left(\prod_{l_1:a_1(l_1)=n+2}\partial_{i_{l_1}}\right)\cdot\left(\prod_{l_2:a_2(l_2)=n+2}\partial_{j_{l_2}}\right)g\right]$$

In short, functions $B_\Gamma(f, g; \alpha)$ are all possible $GL(N, \mathbf{R})$-invariant expressions constructed form partial derivatives of functions f, g and of coefficients of bivector field α by contractions of upper and lower indices, without making an assumption that $[\alpha, \alpha] = 0$.

The general formula for the star-product is

$$f \star g = \sum_{n \geq 0} \frac{\hbar^n}{n!} \sum_{\substack{\text{graphs } \Gamma \\ \text{of degree } n}} c_\Gamma \cdot B_\Gamma(f, g; \alpha)$$

where $c_\Gamma \in \mathbf{R}$ are constants defined in the next section. The associativity of star-product follows form certian non-homogeneous quadratic relations between numbers c_Γ.

4. Integral formula for c_Γ

Let $\mathcal{H} = \{z \in \mathbf{C} |\, Im\, z > 0\}$ be the standard upper half-plane, $\overline{\mathcal{H}} := \mathcal{H} \cup \mathbf{R} \subset \mathbf{C}$ be its closure in $\mathbf{C}$. We define a map

$$\phi : \overline{\mathcal{H}} \times \overline{\mathcal{H}} \setminus \text{diagonal} \longrightarrow \mathbf{R}/2\pi\mathbf{Z}$$

by the formula

$$\phi(z, w) = Arg(z - w) - Arg(\overline{z} - w) \ .$$

The meaning of this formula is that $\phi(z, w)$ is equal to the angle between lines (z, w) and $(z, +i\infty)$ in the Lobachevsky geometry.

The value of c_Γ is given by the following integral:

$$c_\Gamma = \frac{1}{(8\pi^2)^n} \int_{\substack{(z_1, \dots, z_n) \in \mathcal{H}^n \\ z_i \neq z_j \text{ for } i \neq j}} \bigwedge_{k=1}^{n} \left(d\phi(z_k, z_{a_1(k)}) \wedge d\phi(z_k, z_{a_2(k)}) \right)$$

where we define z_{n+1}, z_{n+2} as points $0, 1 \in \overline{\mathcal{H}}$ respectively.

The integral from above is absolutely convergent. Probably, all numbers c_Γ are rational, although we cannot prove or disprove this statement at present.

The proof of quadratic relations between numbers c_Γ is essentially an application of the Stockes formula. In order to clarify the combinatorics of the proof, and also to construct star-products on general Poisson manifolds, we have to introduce general notions and constructions from the deformation theory.

5. Deformation theory and quasi-isomorphisms

Let $\mathbf{g}^*$ be a differential $\mathbf{Z}$-graded Lie algebra (DGLA) over field k of characteristic 0. The deformation functor $Def_{\mathbf{g}^*}$ associates with any finite-dimensional Artin algebra $\mathbf{A}$ over k the following set:

$$\{\alpha \in \mathbf{g}^1 \otimes m_\mathbf{A} |\, d\alpha + \frac{1}{2}[\alpha, \alpha] = 0 \in \mathbf{g}^2 \otimes m_\mathbf{A}\}/G_{(\mathbf{A})}$$

where $m_{\mathbf{A}} \subset \mathbf{A}$ is the maximal ideal of $\mathbf{A}$, and group $G_{(\mathbf{A})}$ is the nilpotent group associated with the nilpotent Lie algebra $\mathbf{g}^0 \otimes m_{\mathbf{A}}$. The action of $G_{(\mathbf{A})}$ on the set of solutions of the Maurer-Cartan equation is in infinitesimal form

$$\dot{\alpha} = d\gamma + [\gamma, \alpha], \;\; \gamma \in \mathbf{g}^0 \otimes m_{\mathbf{A}} \;\; .$$

One of most familiar examples is the deformation theory of complex structures on a complex manifold M. In this case $k = \mathbf{C}$, and the DGLA controlling the deformation theory is

$$\mathbf{g}^* = \bigoplus_{k \geq 0} \mathbf{g}^k, \;\; \mathbf{g}^k = \Gamma(M, T^{1,0} \otimes \wedge^k \left((T^{0,1})^{dual}\right)$$

with the differential equal to the usual $\overline{\partial}$-operator and with Lie bracket coming from the usual Lie bracket on vector fields and from the cup-product on differential forms. The set $Def_{\mathbf{g}^*}(\mathbf{A})$ is the set of equivalence classes of flat morphisms of complex analytic spaces $\widetilde{M} \longrightarrow Spec(\mathbf{A})$, endowed with an identification of the special fiber $\widetilde{M} \underset{Spec(\mathbf{C})}{\times} Spec(\mathbf{A})$ with M.

Let $\mathbf{g}_1^*, \mathbf{g}_2^*$ are two DGLAs. We are going to introduce a structure (a *quasi-isomorphism* between $\mathbf{g}_1^*$ and $\mathbf{g}_2^*$) which identifies deformation functors $Def_{\mathbf{g}_1^*}$ and $Def_{\mathbf{g}_2^*}$.

Definition. *An L_∞-morphism $\mathcal{T}$ from $\mathbf{g}_1^*$ to $\mathbf{g}_2^*$ is an homomorphism of differential graded cocommutative coassociative coalgebras*

$$\mathcal{T} : \bigoplus_{k \geq 1} Sym^k(\mathbf{g}_1^*[1]) \longrightarrow \bigoplus_{k \geq 1} Sym^k(\mathbf{g}_2^*[1]) \;\; .$$

In the formula from above symmetric powers are constructed in the tensor category of $\mathbf{Z}$-graded vector spaces (i.e. using the Koszul rule of signs). The graded space $\mathbf{g}^*[1]$ is obtained from $\mathbf{g}^*$ by the shift of degrees by 1:

$$(\mathbf{g}[1])^n := \mathbf{g}^{n+1} \;\; .$$

The differential in the "chain complex" $C_*(\mathbf{g}^*) := \bigoplus_{k \geq 1} Sym^k(\mathbf{g}^*[1])$ of any DGLA $\mathbf{g}^*$ is defined by usual formula using the differential and the Lie bracket in $\mathbf{g}^*$. Geometrically, one can think about coalgbera $C_*(\mathbf{g}^*)$ as of an object encoding an infinite-dimensional formal $\mathbf{Z}$-graded supermanifold. The reason is that the dual space to $C_*(\mathbf{g}^*)$ is the algebra of formal power series. The differential on $C_*(\mathbf{g}^*)$ can be viewed as an odd vector field Q on a supermanifold such that $[Q, Q] = 0$. An L_∞-morphim gives a Q-equivariant map between formal supermanifolds.

One can reformulate the definition of the deformation functor in geometrical terms (i.e. for odd vector field Q). Any L_∞-morphim induces a natural transformation between deformation functors.

Definition. *An L_∞-morphism $\mathcal{T}$ from $\mathbf{g}_1^*$ to $\mathbf{g}_2^*$ is called a quasi-isomorphism iff its component $\mathcal{T}^{(1,1)}$ which maps $\mathbf{g}_1^*[1]$ to $\mathbf{g}_2^*[1]$ is a quasi-isomorphim of complexes.*

Below we state a well-known result in slightly new form:

Theorem. *Any quasi-isomorphism induces an isomorphism between deformation functors.*

6. Formality

Let X be a manifold, A be the algebra of functions on X. We define two DGLAs over $\mathbf{R}$ associated with X. The first algebra $D^*(X)$ is related with the deformation quantization. For each $n \geq -1$ we define $D^n(X)$ by the formula

$$\{\Phi : A^{\otimes(n+1)} \longrightarrow A \,|\, \Phi(f_0 \otimes f_1 \otimes \ldots \otimes f_n) \text{ is a polydifferential operator in } f_*\} .$$

The differential and the bracket in $D^*(X)$ are given by standard formulas for the differential and the bracket in the Hochschild complex. We define a bilinear operation $(\Phi_1, \Phi_2) \longrightarrow \Phi_1 \circ \Phi_2$ on $D^*(X)$ for $\Phi_1 \in D^m(X)$ and $\Phi_2 \in D^n(X)$

$$(\Phi_1 \circ \Phi_2)(f_0 \otimes \ldots \otimes f_{n+m}) := \sum_{k=0}^{m} \pm \Phi_1(f_1 \otimes \ldots \otimes \Phi_2(f_k \otimes \ldots \otimes f_{k+n}) \otimes \ldots \otimes f_{n+m}) .$$

The Lie bracket in $D^*(X)$ is defined as

$$[\Phi_1, \Phi_2] = \Phi_1 \circ \Phi_2 - (-1)^{mn} \Phi_2 \circ \Phi_1$$

and the differential as

$$d\Phi = [m_X, \Phi]$$

where $m_X \in D^1(X)$ is the product in A: $m_X(f_0 \otimes f_1) = f_0 f_1$.

The second DGLA is denoted by $T^*(X)$. It is simply the cohomology of $D^*(X)$ with respect to the differential in $D^*(X)$. The differential in $T^*(X)$ is defined to be zero. By a version of Hochschild-Kostant-Rosenberg theorem graded components of $T^*(X)$ are spaces of polyvector fields:

$$T^n(X) = \Gamma(X, \wedge^{n+1} T_X), \quad n \geq -1$$

and the bracket in $T^*(X)$ is the usual Schouten-Nijenhuis bracket.

Theorem. *For any manifold X two DGLAs $D^*(X)$ and $T^*(X)$ are quasi-isomorphic.*

Solutions of the Maurer-Cartan equation in $D^*(X)$ parametrized by $Spec(\mathbf{R}[[\hbar]])$ are exactly star-products on X. Solutions in $T^*(X)$ are Poisson structures. Thus, we get a canonical quantization for arbitrary Poisson structure.

Usually, a differential graded algebra quasi-isomorphic to its cohomology algebra, is called *formal*. For example, the de Rham complex on any Kähler manifold is formal. Our result means that DGLA $D^*(X)$ is formal.

7. Few words about the proof

Firts of all, using a generalization of the construction with graphs as in sections 3,4 we construct an explicit quasi-isomorphism from $T^*(\mathbf{R}^N)$ to $D^*(\mathbf{R}^N)$ for any N. The check of relevant identities uses the Stockes formula on certain compactifications of configurations spaces of $\overline{\mathcal{H}}$, and the following lemma:

Lemma. *Let M be a complex algebraic variety of dimension $d \geq 1$, anf $f_1, \ldots, f_{2d}$ be non-zero rational functions on M. Then the integral*

$$\int_{M(\mathbf{C})} \bigwedge_{k=1}^{2d} d\,\mathrm{Arg}(f_k)$$

is absolutely convergent and equal to zero.

This lemma is used in the study of certain degenerations when several points on $\mathcal{H}$ move close to each other. The main step in the proof of the lemma is the following identity:

$$\bigwedge_{k=1}^{2d} d\,\mathrm{Arg}(f_k) = \bigwedge_{k=1}^{2d} d\,\mathrm{Log}|f_k| \ .$$

The next step is to introduce a Gelfand-Fuks cocyle of the Lie algebra of formal vector fields with coefficients in a module responsible for L_∞-morphisms form $T^*(\mathbf{R}^N)$ to $D^*(\mathbf{R}^N)$. Fortunately, it can be done in essentially the same manner as for an individual L_∞-morphism. Vanishing of some integral over a configuration space of $\mathcal{H}$ guarantees that this cocycle is a relative cocycle with respect to the Lie algebra $gl(N,\mathbf{R}) \subset Vect(\mathbf{R}^n)$. The rest is a generalization of standard constructions of characteristic classes associated with Gelfand-Fuks cohomology.

8. Applications

There many of them. For example, any quadratic Poisson bracket on a finite-dimensional vector space admits a canonical quantization to a graded algebra with quadratic relations. It gives the positive answer to one of questions posed by Drinfeld.

Here is another application (which needs in fact an additional work with graphs and integrals):

Theorem. *Let $\mathbf{g}$ be a Lie algebra in a tensor category $\mathcal{C}$ which is a "finite-dimensional" object of $\mathcal{C}$, i.e. the dual object $\mathbf{g}^{dual}$ exists and $(\mathbf{g}^{dual})^{dual} = \mathbf{g}$. Then the center of the universal enveloping algebra $Z(\mathcal{U}\mathbf{g}) = (\mathcal{U}\mathbf{g})^{\mathbf{g}}$ is isomorphic as an algebra in $\mathcal{C}$ to the algebra $(\oplus_{k\geq 0} Sym^k(\mathbf{g}))^{\mathbf{g}}$ of ad^*-invariant polynomials on $\mathbf{g}^{dual}$.*

In the classical case of the category of vector spaces this fact was proven by Duflo using at certain essential step the classification theory of Lie algebras. In fact, the isomorphism in our theorem is the one predicted by Kirillov and Duflo, and involves a kind of Todd class for elements of finite-dimensional Lie algebras. The analogous statement for Lie superlalgebras was unknown. Now we can say finally that the orbit method has a solid background.

A parallel new theorem in algebraic geometry is

Theorem. *Let M be smooth algebraic variety over a field of characteristic zero. Then the graded algebra $Ext^*_{M\times M}(\mathcal{O}_{diag}, \mathcal{O}_{diag})$ is isomorphic to $\oplus H^*(M, \wedge^* T_X)$.*

Another application is to the Mirror Symmetry, but we will not try to explain it here.

Height ζ-functions

Yuri Tschinkel

University of Illinois at Chicago, U.S.A.

Let X be a non-singular quasi-projective algebraic variety over a number field F and $\mathcal{L} = (L, \{\|\cdot\|_v\})$ a metrized line bundle on X. It defines a height function

$$H_{\mathcal{L}} \,:\, X(F) \to \mathbf{R}_{>0}$$

from the set of F-rational points to the real numbers and a counting function

$$N(X, \mathcal{L}, B) := \{x \in X(F) \mid H_{\mathcal{L}}(x) \leq B\}.$$

We are interested in the asymptotic behavior of $N(X, \mathcal{L}, B)$ for $B \to \infty$. The program of studying such asymptotics was initiated by Yu. I. Manin who proposed conjectures about their relationship with geometric invariants of the variety X.

In praxis, such asymptotics can be established by means of a Tauberian theorem provided one knows the analytic properties of the *height ζ-function* defined by the following series

$$\zeta(X, \mathcal{L}, s) := \sum_{x \in X(F)} H_{\mathcal{L}}(x)^{-s}.$$

This series converges to a holomorphic function for appropriate X and $\mathcal{L}$ and $\mathrm{Re}(s) \gg 0$ and our goal is to describe the location and multiplicity of its first pole as well as the leading coefficient at this pole, at least for the following class of varieties.

Let G be a semi-simple simply connected split algebraic group over F and $P \subset G$ a parabolic subgroup. Let $\eta \,:\, P \to T$ be a homomorphism from P to an algebraic torus T and X some equivariant compactification of T. This defines an action of P on $T \times G$. We denote by $Y_\eta := X \times^P G$ the quotient and by $Y^0 = T \times^P G$. The variety Y_η has a structure of a toric bundle over the flag variety $P\backslash G$. The following theorem is joint work with M. Strauch.

Theorem [4] *Let L be a line bundle on Y_η such that its class is contained in the interior of the cone of effective divisors $\Lambda_{\mathrm{eff}}(Y_\eta) \subset \mathrm{Pic}(Y_\eta)_{\mathbf{R}}$. There exists a metrization $\mathcal{L}$ of L such that the height zeta function has the following representation*

$$\zeta(Y^0, \mathcal{L}, s) = \frac{c(\mathcal{L})}{(s - a(L))^{b(L)}} + \frac{g(s)}{(s - a(L))^{b(L)-1}}$$

with $a(L) > 0$, $c(\mathcal{L}) \neq 0$, $b(L) \in \mathbf{N}$ and some function $g(s)$ which is holomorphic for $\mathrm{Re}(s) > 1 - \delta$ *(for some $\delta > 0$).*

The rest of the talk was a report on my joint work with V. Batyrev [2], where we defined the constants a, b and c in a more general framework. It turns out that for general ample metrized line bundles $\mathcal{L}$ the constant $c(\mathcal{L})$ is given as an infinite (converging!) sum over positive real numbers labeled by rational points contained in a base of a certain fibration, where each summand is a product of 3 numbers: an invariant of the cone of effective divisors, a cohomological invariant and a Tamagawa number of the corresponding fiber.

References

[1] V. Batyrev, Yu. I. Manin, *Sur le nombre des points rationnels de hauteur bornée des variétés algebriques*, Math. Ann. 286 (1990), 27-43.

[2] V. Batyrev, Yu. Tschinkel, *Saturated varieties*, Preprint (1997).

[3] E. Peyre, *Hauteurs et nombres de Tamagawa sur les variétés de Fano*, Duke Math. J. 79 (1995), 101-218.

[4] M.Strauch, Yu. Tschinkel, *Height zeta functions of toric bundles over flag varieties*, Preprint MPI (1997).

A characterization of irreducible symmetric spaces and Euclidean buildings of higher rank by their asymptotic geometry

Bernhard Leeb (Univ. Bonn)

The results presented here address the general

Question: *To what extent are the asymptotic properties of a Hadamard space reflected in its geometry?*

Hadamard manifolds are simply-connected complete Riemannian manifolds of nonpositive sectional curvature. Prominent examples are Riemannian (globally) symmetric spaces of noncompact type, e.g. $SL(3,R)/SO(3)$, but many more examples occur as universal covers of closed nonpositively curved manifolds, for instance, of Haken 3-manifolds. Not the notion of sectional curvature itself, however the notion of an upper curvature bound can be expressed purely by inequalities involving the distances between finitely many points but no derivatives of the Riemannian metric, and hence generalizes from the narrow world of Riemannian manifolds to a wide class of metric spaces. The natural generalization of Hadamard manifolds are Hadamard spaces, i.e. complete geodesic metric spaces which are nonpositively curved in the (global) sense of distance comparison. Hadamard spaces comprise besides Hadamard manifolds a large class of interesting singular spaces, among them Euclidean buildings (the non-Archimedean cousins of symmetric spaces), many piecewise Euclidean or Riemannian complexes which occur, for instance, in geometric group theory, certain branched covers of Hadamard manifolds etc. Hadamard spaces received much attention in the last decade, notably with view to geometric group theory, a main impetus coming from Gromov's work on hyperbolic groups and asymptotic invariants of infinite groups.

We recall that a fundamental feature of a Hadamard space is the convexity of its distance function with the drastic consequences such as uniqueness of geodesics and in particular contractibility. This illustates that at least basic geometric objects such as geodesics are rather well-behaved, which gets the foot in the door for a more advanced geometric understanding. The importance of the geometry of nonpositive curvature lies in the coincidence that one has a rich supply of interesting examples reaching into many different branches of mathematics (like geometric group theory, representation theory, arithmetic) and, at the same time, these spaces share simple basic geometric properties which makes them understandable to a certain extent and in a uniform way.

Let us now describe the asymptotic information which we consider. The *ideal boundary* $\partial_\infty X$ of a Hadamard space X is defined as the set of equivalence classes of asymptotic geodesic rays. (Two rays are called asymptotic if they remain at bounded distance.) The topology on X extends to a natural *cone topology* on the geometric completion $\bar{X} = X \cup \partial_\infty X$ which is compact if and only if X is locally compact. The ideal boundary points $\xi \in \partial_\infty X$ can be thought of as the ways to go straight to infinity. It is fair to say that the topological type of $\partial_\infty X$ is not a very strong invariant, for example it is a $(n-1)$-sphere for any n-dimensional Hadamard manifold.

For us, another structure on $\partial_\infty X$ will be particularly important, namely the *Tits (angle) metric* introduced in full generality by Gromov in [BGS]. For two points $\xi_1, \xi_2 \in$

$\partial_\infty X$ at infinity their Tits angle $\angle_{Tits}(\xi_1, \xi_2)$ measures the maximal visual angle $\angle_x(\xi_1, \xi_2)$ under which they can be seen from a point x inside X, or equivalently, it measures the asymptotic linear rate at which unit speed geodesic rays ρ_i asymptotic to the ideal points ξ_i diverge from each other. If X has a strictly negative upper curvature bound the *Tits boundary* $\partial_{Tits}X = (\partial_\infty X, \angle_{Tits})$ is a discrete metric space and only of modest interest. On the other hand, if X contains flats, that is convex subsets isometric to Eulidean space, then their ideal boundaries are unit spheres in $\partial_{Tits}X$. As a guiding principle, the non-triviality of the Tits metric is related to the presence of extremal curvature zero in X which is the source for the kind of rigidity phenomena we consider here. The Tits metric together with the cone topology on $\partial_\infty X$ are the *asymptotic data* relevant for us.

Our main result is the following characterization of symmetric spaces and Euclidean buildings of higher rank as Hadamard spaces with spherical building boundary. All symmetric spaces in this note are assumed to have non-compact type. A Hadamard space is *geodesically complete* if every segment can be extended to a complete geodesic.

Main Theorem. *Let X be a locally compact and geodesically complete Hadamard space and assume that $\partial_{Tits}X$ is a thick irreducible spherical building of dimension $r - 1 \geq 1$. Then X is an irreducible Riemannian symmetric space or a Euclidean building of rank r.*

In the smooth case, i.e. for Hadamard manifolds, this follows from work of Ballmann and Eberlein, or else from arguments of Gromov [BGS], Burns and Spatzier. There is a dichotomy into two cases, according to whether geodesics in X branch or not. In the absence of branching the ideal boundaries are very symmetric because the "reflection" at any point $x \in X$ yields an involution ι_x of $\partial_\infty X$ which is a topological spherical building automorphism. One then can adapt arguments from Gromov in the proof of his rigidity theorem [BGS]. Our main contribution lies in the case of geodesic branching. There the ideal boundary admits in general no non-trivial symmetries at all and another approach is needed.

If, besides the Tits geometry, we take into account also the cone topology on $\partial_\infty X$ it turns out that the spaces in consideration are completely determined by their asymptotic data up to a scale factor. A *boundary isomorphism* is a map of ideal boundaries which simultaneously is a cone topology homeomorphism and a Tits isometry, and a *homothety* is a map between metric spaces which multiplies all distances by the same factor.

Addendum. *Let X_1 and X_2 be spaces as in the Main Theorem. Then any boundary isomorphism $\partial_\infty X_1 \to \partial_\infty X_2$ is induced by a homothety $X_1 \to X_2$.*

This result follows from Tits classification of automorphisms of spherical buildings in the cases when X has many symmetries, e.g. when it is a Riemannian symmetric space or a Euclidean building associated to a simple algebraic group over a local field with non-Archimedean valuation, and in particular if $rank(X) \geq 3$. However his results don't cover the cases when X is a rank-2 Euclidean building with small isometry group. Our methods provide a uniform proof in all cases and in particular a direct argument in the cases with high symmetry.

A major motivation for this work was Mostow's Strong Rigidity Theorem for locally symmetric spaces, namely the special case for compact quotients of irreducible symmetric spaces of higher rank:

Theorem (Mostow). *Let M and M' be locally symmetric spaces whose universal covers are irreducible symmetric spaces of rank ≥ 2. Then any isomorphism $\pi_1(M) \to \pi_1(M')$ of fundamental groups is induced by a homothety $M \to M'$.*

It is natural to ask whether this rigidity of locally symmetric spaces persists in the wider class of closed manifolds of nonpositive sectional curvature. This is "true" and the content of Gromov's Rigidity Theorem [BGS]. As an application of our results we present an extension of Mostow's theorem and Prasad's analogue for compact quotients of Euclidean buildings to the larger class of singular nonpositively curved (orbi)spaces:

Application. *Let X be a locally compact and geodesically complete Hadamard space. Suppose furthermore that X_{model} is a symmetric space or a thick Euclidean building all of whose irreducible factors have rank ≥ 2. If the same finitely presented group Γ acts cocompactly and properly discontinuously on X and X_{model} then, after suitably rescaling the metrics on the irreducible factors of X_{model}, there is a Γ-equivariant isometry $X \to X_{model}$.*

This means that among (possibly singular) geodesically complete compact spaces of nonpositive curvature (this time in the local sense) quotients of irreducible higher rank symmetric spaces or Eulidean buildings are determined by their homotopy type up to a scale factor.

Example: *On a locally symmetric space with irreducible higher-rank universal cover there exists no piecewise Euclidean singular metric of nonpositive curvature.*

References:

[BGS] W. Ballmann, M. Gromov and V. Schroeder, *Manifolds of Nonpositive Curvature,* Birkhäuser 1985.

[KL] B. Kleiner and B. Leeb, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings,* preprint, Bonn, april 1, 1996, to appear in Publications IHES.

[L] B. Leeb, *A characterization of irreducible symmetric spaces and Euclidean buildings of higher rank by their asymptotic geometry,* Preprint, Bonn, june 1997.

Grothendieck-Teichmüller group and covers of genus 0 moduli

David Harbater, University of Pennsylvania

Every covering space of the sphere with three points deleted can be realized as a morphism of complex curves $X \to \mathbf{P}^1 - \{0,1,\infty\}$, and in fact as a cover of curves over $\overline{\mathbf{Q}}$. Thus such covers are acted upon by $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Explicitly, a Galois cover $X \to \mathbf{P}^1 - \{0,1,\infty\}$ has topological branch cycle description (a,b,c), for some generators a,b,c whose product is 1, and which are given up to uniform conjugacy. If $\sigma \in G_{\mathbf{Q}}$, then the induced cover $X^\sigma \to \mathbf{P}^1 - \{0,1,\infty\}$ has topological branch cycle description (a',b',c'), for some generators a',b',c' that are respectively conjugate to $a^{\chi(\sigma)}, b^{\chi(\sigma)}, c^{\chi(\sigma)}$, where χ is the cyclotomic character. Unfortunately, we do not know the elements a',b',c' up to *uniform* conjugacy, and so do not know the action of $G_{\mathbf{Q}}$ on covers.

Since an element of $G_{\mathbf{Q}}$ can move base points, there is only a natural *outer* action of $G_{\mathbf{Q}}$ on covers, and hence on $\pi_1(\mathbf{P}^1 - \{0,1,\infty\}) = \hat{F}_2$. In a well-known paper in 1980, Belyi showed that every $\overline{\mathbf{Q}}$-curve is a cover of $\mathbf{P}^1$ branched only at $\{0,1,\infty\}$. This implies that the homomorphism $G_{\mathbf{Q}} \to \mathrm{Out}(\hat{F}_2)$ is an inclusion. Later, in his *Esquisse*, Grothendieck suggested that $G_{\mathbf{Q}}$ be determined explicitly as a subgroup of $\mathrm{Out}(\hat{F}_2)$. Moreover, he suggested that rather than generalizing $\mathbf{P}^1 - \{0,1,\infty\}$ to $\mathbf{P}^1 - \{n \text{ points}\}$, we regard $\mathbf{P}^1 - \{0,1,\infty\}$ as the moduli space $M_{0,4}$ of genus 0 curves with four ordered marked points — and that we determine $G_{\mathbf{Q}}$ as a subgroup of $\mathrm{Out}(\hat{\Gamma}_{0,n})$ for all n (where $\hat{\Gamma}_{0,n} = \pi_1(M_{0,n})$).

In his 1980 paper, Belyi had also observed that the above triple (a',b',c') can be chosen in a unique, compatible way so that $a' = a^{\chi(\sigma)}$ and b' is conjugate to $b^{\chi(\sigma)}$ by a (unique) element f_σ of the commutator subgroup $\hat{F}_2'$. Thus to each $\sigma \in G_{\mathbf{Q}}$ there is assigned a pair $(\lambda_\sigma, f_\sigma) \in \hat{\mathbf{Z}}^* \times \hat{F}_2'$ that gives the outer action of σ on $\hat{F}_2$. (Here $\lambda_\sigma = \chi(\sigma)$.) So the outer action of $G_{\mathbf{Q}}$ lifts to a true action on $\hat{F}_2 = \langle x,y,z \mid xyz = 1\rangle$, by $x \mapsto x^\lambda, y \mapsto f^{-1}y^\lambda f$.

Later, motivated by considerations of Hopf algebras, Drinfeld considered three relations on pairs (λ, f) as above, and said that these were satisified by pairs coming from $G_{\mathbf{Q}}$. He also introduced the Grothendieck-Teichmüller group $\widehat{GT}$ of all pairs $(\lambda, f) \in \hat{\mathbf{Z}}^* \times \hat{F}_2'$ which give an automorphism of $\hat{F}_2$ by the action as above on x, y, and which satisfy the three conditions (I), (II), (III). (Here, conditions (I) and (II) are conditions in $\hat{F}_2 = \hat{\Gamma}_{0,4}$ and (III) is a condition in $\hat{\Gamma}_{0,5}$. If we just use (I) and (II), then we get a larger group, $\widehat{GT}_0$.) Moreover he proposed an action of $\widehat{GT}$ on the groups $\hat{\Gamma}_{0,n}$; and suggested that $\widehat{GT}$ is (in a suitable sense) the automorphism group of the "Teichmüller tower" of genus 0 fundamental groups $\hat{\Gamma}_{0,n}$ (or perhaps, fundamental groupoids). These assertions, except for the one on the tower, were later verified by Ihara. Thus $\widehat{GT}$ is a group-theoretic approximation of $G_{\mathbf{Q}}$ from above, and the hope is that group theory and geometry can then shed light on $G_{\mathbf{Q}}$. In fact, Ihara asked if in fact the inclusion $G_{\mathbf{Q}} \subset \widehat{GT}$ is an isomorphism.

The above discussion contains four main questions:

Question 1: How does $G_{\mathbf{Q}}$ act on covers?

Question 2: What is the image of $G_{\mathbf{Q}}$ in $\mathrm{Out}\pi_1$?

Question 3: Is $\widehat{GT}$ the automorphism group of the genus 0 Teichmüller tower? of the full Teichmüller tower?

Question 4: Is $\widehat{GT} = G_{\mathbf{Q}}$?

Recent work of Leila Schneps and the speaker approach these problems by "sharpening the focus" to the good elements of $\mathrm{Out}\hat{\Gamma}_{0,n}$, viz. to those that act like elements of $G_{\mathbf{Q}}$. Specifically, the action of any element $\omega \in G_{\mathbf{Q}}$ has the following two properties:

(i) There is a $\lambda \in \hat{\mathbf{Z}}^*$ such that for each generator $x \in \hat{\Gamma}_{0,n}$, the image of x is conjugate to x^λ.
(ii) The action is unaffected by permuting the deleted points, i.e. it commutes with the natural outer action of S_n.

We denote the subgroup of $\mathrm{Out}\hat{\Gamma}_{0,n}$ satisfying these properties by $\mathrm{Out}^\#\hat{\Gamma}_{0,n}$. Thus $G_{\mathbf{Q}} \hookrightarrow \mathrm{Out}^\#\hat{\Gamma}_{0,n}$. Concerning Question 2, are they isomorphic? Since we also have that $G_{\mathbf{Q}} \hookrightarrow \widehat{GT}$, we can also ask (in light of Question 4): Is $\widehat{GT} \simeq \mathrm{Out}^\#\hat{\Gamma}_{0,n}$? In fact, we have

Theorem. *For $n \geq 5$, there is a natural isomorphism $\mathrm{Out}^\#\hat{\Gamma}_{0,n} \simeq \widehat{GT}$, and for $n = 4$ there is a natural isomorphism $\mathrm{Out}^\#\hat{\Gamma}_{0,4} \simeq \widehat{GT}_0$.*

This gives a natural geometric interpretation of GT (and $\widehat{GT}_0$), without the mysterious cocycle relations (I) - (III). Also, it answers Question 2 for $\widehat{GT}$, rather than for $G_{\mathbf{Q}}$, and answers a version of Question 3, viz. asserting that $\widehat{GT}$ is the group of "nice" outer automorphisms of the genus 0 Teichmüller tower.

The strategy is to show that Belyi's lift of the outer $G_{\mathbf{Q}}$-action to a true action extends to a lift of the (tautological) outer $\mathrm{Out}^\#\hat{\Gamma}_{0,4}$-action to a true action. But there is also an extension of the lift from $G_{\mathbf{Q}}$ to $\widehat{GT}_0$. After showing that condition (I) corresponds to commutation with $(12) \in S_3$ and that (II) corresponds to commutation with $(123) \in S_3$, it follows that $\mathrm{Out}^\#\hat{\Gamma}_{0,4} \simeq \widehat{GT}_0$. The parallel fact for $\hat{\Gamma}_{0,5}$ has a similar (but more involved) proof, relying on Nakamura's analog of Belyi's lift for $\hat{\Gamma}_{0,5}$, and proving that (III) corresponds to commutation with $(12345) \in S_5$. The case for general n is proven by reducing to the case $n = 5$.

As an application of the above, it is possible to obtain information about the field of moduli of a G-Galois cover of $\mathbf{P}^1 - \{0, 1, \infty\}$. Namely, the $G_{\mathbf{Q}}$-orbit of a cover is contained in the $\widehat{GT}$-orbit, which in turn is contained in the $\widehat{GT}_0$-orbit. The sizes of the latter orbits bound that of the former, and so bound the degree (and Galois group) of the field of moduli (and sometimes even produce an overfield of the field of moduli). Of course, it would not seem possible to compute the $\widehat{GT}$- or $\widehat{GT}_0$-orbits directly from the definition, but the above theorem makes this possible. Namely, the action of $\widehat{GT}_0$ on the orbit of the cover factors through $\mathrm{Out}G$ (where we may need to enlarge G, temporarily, to get a well-defined map to $\mathrm{Out}G$), and the image is contained in a subgroup $\mathrm{Out}^\#(G) \subset \mathrm{Out}(G)$ that can be computed explicitly. In fact, by working with groups between $\hat{F}_2$ and G, one can obtain the exact image of $\widehat{GT}_0$, and hence the $\widehat{GT}_0$-orbit. A similar, but more involved approach can be used for $\widehat{GT}$. If, for example, the $\widehat{GT}$-orbit of a cover has just one element, then the cover has field of moduli $\mathbf{Q}$ (and so is equal to $\mathbf{Q}$, under mild conditions on G). And in general, the closer that $\widehat{GT}$ is to $G_{\mathbf{Q}}$, the better the information that will be obtained about the $G_{\mathbf{Q}}$-orbit and the field of moduli of a cover.

This returns the discussion to Question 4, which, given the above, is equivalent to asking if the image in Question 2 is $\mathrm{Out}^\#\hat{\Gamma}_{0,n}$. This can be approached via Question 3, in the case of the tower of *all* $M_{g,n}$'s. Here, higher genus curves can be built up from lower genus ones by pasting, for example with respect to Thurston's "pants decompositions." Algebraically, this can be viewed as forming a reducible curve of higher arithmetic genus, and then using formal patching to deform to a smooth curve of that genus. In fact, the above groups $\mathrm{Out}^{\#}\hat{\Gamma}_{0,n}$ can be generalized to subgroups $\mathrm{Out}^{\#}\hat{\Gamma}_{g,n} \subset \mathrm{Out}\hat{\Gamma}_{g,n}$ of "nice" outer automorphisms, such that the groups map to each other in a tower corresponding to the ways that a higher genus curve can be built of lower genus ones. (The definition includes the analogs of (i) and (ii) above, along with a third condition to guarantee that the action is "local" on the topological surface.) One expects these groups to stabilize for g, n sufficiently large, and the limiting group is a natural candidate for $G_{\mathbf{Q}}$.

To carry over the ideas from the genus 0 case, one needs a lift of the outer $G_{\mathbf{Q}}$-action on $\hat{\Gamma}_{g,n}$ to a true action. At the moment, this is unknown in general. But recently, Nakamura has constructed such a lift for the case $n = 1$, using formal patching to increase the genus. It turns out that the lifts of $G_{\mathbf{Q}}$ satisfy not only (I) - (III), but also two other relations. Thus $G_{\mathbf{Q}}$ is contained in the group $\mathbf{\Gamma} \subset \widehat{\mathbf{GT}}$ given by all five relations, and if the two new relations are not implied by the old three, then $G_{\mathbf{Q}}$ is strictly smaller than $\widehat{GT}$ (though it is perhaps equal to $\mathbf{\Gamma}$). One hopes that the case of $\hat{\Gamma}_{g,n}$ for $n > 1$ will not produce infinitely many such additional relations, as $n \to \infty$; and Grothendieck had said that the all the relations ought to be contained in information coming from $(g, n) = (0,4), (0,5), (1,1), (1,2)$. At least, it is now known that $\mathbf{\Gamma}$ does have an outer action on all the $M_{g,1}$'s (Nakamura, Lochak, Schneps); and it appears that $\mathbf{\Gamma}$ embeds into $\mathrm{Out}^{\#}\hat{\Gamma}_{g,n}$. (This has been so far verified on a "large" subgroup of $\mathbf{\Gamma}$, by Lochak and Schneps.) So perhaps $\mathrm{Out}^{\#}\hat{\Gamma}_{g,n}$ is isomorphic to $G_{\mathbf{Q}}$ for large g, n, even if $\widehat{GT}$ is not — corresponding to the hope that $G_{\mathbf{Q}}$ can be recovered by its actions on the fundamental groups of *all* the moduli spaces of curves.

Finally, the above theorem in genus 0 suggests a possible geometric approach to the Shafarevich Conjecture. Namely, this conjecture states that the absolute Galois group of K^{cycl} is a free profinite group, where K is any global field and K^{cycl} is its maximal cyclotomic extension. In the geometric case, where K is the function field of a curve over a finite field, this has been proven independently by the speaker and Florian Pop, using patching to construct covers with desired properties. (The point is that after adjoining all the roots of unity, the base curve is over the algebraic closure of the finite field, and so patching and specialization can be applied.) But in the arithmetic case, where K is a number field, the techniques of geometry do not directly apply, and the problem remains open. But under the inclusion $G_{\mathbf{Q}} \hookrightarrow \widehat{GT} \simeq \mathrm{Out}^{\#}\hat{\Gamma}_{0,5}$, the absolute Galois group of $\mathbf{Q}^{\mathrm{cycl}} = \mathbf{Q}^{\mathrm{ab}}$ injects into the subgroup $(\mathrm{Out}^{\#}\hat{\Gamma}_{0,5})^1 \in \mathrm{Out}^{\#}\hat{\Gamma}_{0,5}$ for which $\lambda = 1$. The question of whether $(\mathrm{Out}^{\#}\hat{\Gamma}_{0,5})^1$ is free is geometric and group-theoretic, rather than arithmetic. And if it is free, and if $\mathbf{Q}^{\mathrm{ab}}$ is an open subgroup of $(\mathrm{Out}^{\#}\hat{\Gamma}_{0,5})^1$, then $\mathbf{Q}^{\mathrm{ab}}$ would also be free. Thus, following the general philosophy of this talk, it may be possible to reduce this arithmetic problem to a geometric one, where many other techniques can be brought to bear.

Infinite torsion groups and algebraic surfaces.

L. Katzarkov

1 Introduction

The question of characterizing the universal covering of a smooth projective variety X is a difficult question. Central to the subject is a question asked by Shafarevich if the universal covering of a smooth projective variety X, $\widetilde{X}$ is holomorphically convex. A complex manifold M is holomorphically convex if for every infinite discrete subset S of M there exists a holomorphic function on M that is unbounded on S. T.Napier has studied the question when (unramified, infinite) coverings of a complex surface are holomorphically convex and has shown [7] that one of the basic obstructions to the holomorphic convexity is the existence of connected noncompact analytic sets all of whose irreducible components are compact. We will call such an analytic set an infinite chain.

We give an explanantion of the infinite chain condition. Let X be an algebraic surface and D is an effective divisor in X that is a connected reducible curve having only rational curves as irreducible components. We can produce a counterexample to the Shafarevich conjecture if we can arrange that $\mathrm{im}[\pi_1(Y) \to \pi_1(X))]$ is infinite. Deligne [3] has shown that it is impossible to achieve this on the level of homology since we have

$$\mathrm{im}[H_1(D,\mathbb{Q}) \to H_1(X,\mathbb{Q})] = \mathrm{im}[H_1(D',\mathbb{Q}) \to H_1(X,\mathbb{Q})] = \mathrm{im}[\cup H_1(\mathbb{P}^1,\mathbb{Q}) \to H_1(X,\mathbb{Q})] = 0.$$

(Here we denote by D' the desingularization of D.)

Using the results of Lassel and Ramachandran [4] and of Simpson (see e.g. [9]) one can prove the following:

Theorem 1.1 *Let $\rho : \pi_1(X) \to GL(N,\mathbb{C})$ be a linear representation of the fundamental group of an algebraic surface X and let $Y = \cup D_i$ is a divisor in X. If the restriction of ρ on $\pi_1(D_i)$ is trivial then the restriction of ρ on $\pi_1(Y)$ is a finite group.*

This theorem implies (see e.g. [5]):

Theorem 1.2 *Let $\pi_1(X) \in GL(N,\mathbb{C})$ be the fundamental group of an algebraic surface X. Then the Shafarevich conjecture is correct for X.*

Simpson's technique of mixed twistor structures (MTS) [8] allows us to prove much stronger results which a subject of a recent work by Katzarkov, Pantev and Ramachandran [6]. It is plausible that (MTS) will give a prove for X and algebrai c surface with residually finite torsion free fundamental group. It will be interesting to see some examples of group outside this class. The questions about the existence of smooth projective varieties with nonresidually fundamental group was first ask ed by Serre. The first example like that belongs to Toledo. Similar examples were given by Catanese, Kollár and Nori. These groups are extensions of lattices by abelian groups.

2 The construction

Recently some examples of interesting fundamental groups were suggested by F.Bogomolov and L. Katzarkov [1]. These examples are potential examples of nonresidually finite fundamental groups with infinite torsion. It was shown in [1] that these examples are related to some group theoretic conjectures whose positive answer will give a counterexample to the Shafarevich conjecture. We first describe the construction locally. We begin with some well known results on degeneration of curves. Let X_D be a smooth complex surface fibered over a disc D. We assume that f ibers over a punctured disc $D^* = D - 0$ are smooth curves of genus g and the projection $t : X_D \to D$ is a complex Morse function. In particular the fiber X_0 over $0 \in D$ has only quadratic singular points and it has no multiple components. Denote by P the set of singular points of X_0 and by $T : \pi_1(X_t) \to \pi_1(X_t)$ the monodromy transformation acting on fundamental group $\pi_1(X_t)$ of a generic fiber X_t. This monodromy action can be described in terms of Dehn twists. Obviously it induces an action on $H^1(X_t, \mathbb{Z})$. The following proposition describes completely the topology of X_D and the projection $t : X_D \to D$. We define a natural topologic al contraction $cr : X_D \to X_0$. Observe that $cr^{-1}(P)$ is an union S of circles S_i. The restriction of cr on $X_t - S$ is an isomorphism.

Definition 2.1 *We will call the the free homotopy loop S_i in a first homotopy group of X_t a geometric vanishing cycle.*

We need the following easy lemma.

Lemma 2.1 *1)The monodromy transformation T acts via a unipotent transformation T_H on the homology group $H_1(X_t, \mathbb{Z})$.*

2) $(1 - T_H)^2 = 0$.

3) $(1 - T_H^N) = 0(mod N)$ for any N.

Lemma 2.2 *The surface X_U contracts to the central fiber X_0.*

Indeed the fiberwise contraction of X_D to X_0 can be lifted into a contraction of X_U.

Remark 2.1 *The fundamental group* $\pi_1(X_D - P) = \pi_1(X_D) = \pi_1(X_0)$ *since the singular points of the fiber* X_0 *are nonsingular points of* X_D. *The analogous statement is not true however for* X_U.

The proof of the following theorem can be found in [1].

Theorem 2.1 *The fundamental group* $\pi_1(X_U - P)$ *is equal to the quotient of* $\pi_1(X_t) = \pi_g$ *by a normal subgroup generated by the elements* s_i^N.

Let us denote by G_X the fundamental group of $X_U - P$ and by $\widetilde{X}$ its universal covering. Here we start to develop the idea of the construction. Namely we show that we can work with open surfaces and get results concerning the universal cove rings of the closed surfaces. The advantage is that we get bigger variety of fundamental groups.

Theorem 2.2 *There is a natural* G_X*-invariant imbedding of* $\widetilde{X}$ *into a smooth surface* $\widetilde{X}_U$ *with* $\widetilde{X}_U/G_X = X_U$. *The complement of* $\widetilde{X}$ *in* $\widetilde{X}_U$ *consists of a discrete subset of points.*

Now we globalize the construction. We will show that we can do the construction described above globally in a compact surface. Let X be smooth surface with a proper map to a smooth projective curve C. We assume that the map $f : X \to C$ is described locally by a set of holomorphic Morse functions. Thus the generic fiber is a smooth curve $X_t, t \in C$ of genus g. As in the previous section we denote the fundamental group of X_t by π_g. Abusing the notations now we denote by P the set of all singular po ints of the fibers and by P_C the set of points in C corresponding to the singular fibers. Thus $f(P) = P_C$ and all the points in P are singular double points. The principal difference of the global situation lies in the presence of the global monodromy group which is the image of $\pi_1(C - P_C)$ in the mapping class group $Map(g)$. We denote this group by M_X. Let us chose an integer N and consider a base change $h : R \to C$ where R such that the map h is N-ramified at all the preimages of the points from P_C in R. Now consider a surface S obtained via a base change $h : R \to C$. We have the finite map $h' : S \to X$ defined via h and the projection $z : S \to R$ with a generic fiber $S_t = X_{h(t)}$. The surface S is singular with the set of singular points equal to $h^{-1}(P) = Q$ and the set of singular fibers over the points of $h^{-1}P_C = P_R$. The monodromy group M_S of the family S is subgroup of finite index of the group M_X.

Theorem 2.3 *The fundamental group* $\pi_1(S - Q)$ *is an extension of* $\pi_1(R)$ *by the quotient of* π_g *by a normal subgroup generated by the orbits of* $M_S(s_i^N)$.

Now we move to the second step of our construction, modifying $S - Q$ so that we get a smooth compact surface S^N with almost the same fundamental group. Let us assume that $N \geq 2$.

Lemma 2.3 *There exists a smooth projective surface* S^N *with a finite map* $f : S^N \to S$ *such that the image of the homomorphism* $f_* : \pi_1(S^N) \to \pi_1(S-Q)$ *is a subgroup of finite index in* $\pi_1(S-Q)$.

3 Applications

This is the end of the construction and we suggest some possible applications. To find a counterexample to the Shafarevich conjecture we need to control the existence of an infinite connected chain of compact curves in the universal covering of S^N. In other words we need to control the image of the fundamental groups of the irreducible components of the reducible singular fibers in the fundamental group of the whole surface S^N. The above construction shows that we need to look at the i mage of the fundamental groups of the open irreducible components of the reducible singular fibers in the fundamental group of the open surface $S - Q$. We can construct a smooth family of curves $S \to R$ with the following properties:

1) The fundamental group of R_0 surjects on the components group of the moduli space M_g^L - the mapping class group $Map(g)$. Here R_0 is the open curve that parameterizes only the smooth fibers.

2) $S \to R$ has a section $s : R \to S$.

3) All singular fibers have singularities that are ordinary double points.

It easy to arrange so that in the above family of curves of genus $g = 2k$ there is a fiber which consists of two components each of genus k that intersect in one point.

Definition 3.1 *Denote by $\pi_g/(x^N = 1)$ the quotient of the fundamental group of a Riemann surface of genus g by the group generated by the $N-th$ powers of the images of the primitive elements in π_g under the action of a subgroup of a fini te index in $Map(g)$.*

The free group on $2k$ generators $\mathbb{F}^{2k}$ embeds in a standard way in the fundamental group of a Riemann surface of genus $2k$ punctured at one point.

Definition 3.2 *We will denote by $P(2k)^N$ the quotient of $\mathbb{F}^{2k}$ by the subgroup generated in $\mathbb{F}^{2k}$ by the orbits of the N powers of embedded loops in the fundamental group of a Riemann surface of genus $2k$ punctured in one point.*

Applying the above construction we get that the image of the fundamental group of every component in the fundamental group of $S - Q$ is equal to $P^N(k)$. The image of the fundamental group of a generic fiber in the fundamental group of $S - Q$ is equal to $\pi_g/(x^N = 1)$. We formulate the following question.

Question Are there such a $2k$ and N such that $P^N(2k)$ is a finite group and $\pi_{2k}/(x^N = 1)$ is an infinite group?

If the answer of the above question is affirmative then we get a counterexample to the Shafarevich conjecture. Observe that irreducible components in the construction above could be made of different genus. This gives us much bigger variety of examples. Each of these examples produces hard (according to Olshanskii and Zelmanov) group theoretic question similar to the one we have formulated above. The Shafarevich conjecture in particular implies that all these questions have negative answer. It is also worth noticing that the surfaces S^N described above most probably have nonresisually finite fundamental groups as it was conjectured:

Conjecture 3.1 *(Zelmanov) The group $\pi_{2k}/(x^N = 1)$ is not residually finite.*

4 Some arithmetic and symplectic reflections

Observe that the construction defined above extends to the symplectic category where it could lead to interesting examples of symplectic four dimensional manifolds. Using this construction Bogomolov and Katzarkov have defined in [2] an obstruction to a symplectic Lefschetz pencil being a Kähler Lefschetz pencil. Let us begin with a symplectic fourfold X. Consider the corresponding Lefschetz pencil with reducible fiber and apply to it the construction. It is shown in [2] that the construction goes through and for a fix integer N we get a symplectic fourfold S^N. Let ρ be a representation $\rho : \pi_1(S^N) \to GL(n, \mathbb{C})$ such that $Im(\rho)$ is not virtually equal to $\mathbb{Z}$. Denote by Y_i the components of the preimage of the reducible fiber of S in S^N and denote by F the general fiber of S^N. Denote by Γ the image of $\pi_1(F)$ in $\pi_1(S^N)$ and by Γ_i the images of the fundamental groups of Y_i in $\pi_1(S^N)$. If the restrictions of ρ on Γ and Γ_i for all i are both finite or infinite we will say that the obstruction $O(X)^{N,n}$ is equal to zero and to one otherwise.

Proposition 4.1 *If X is a Kähler surface then $O(X)^{N,n}$ is trivial for every pair N, n.*

Indeed otherwise we will have that covering of S^N with linear Galois group contains infinite chain of compact curves so it cannot be holomorphically convex.

Remark 4.1 *The above obstruction does not distinguish in general Kähler surfaces from symplectic fourfolds. It distinguishes only symplectic from Kähler Lefschetz pencils.*

All these suggest that holomorphic convexity of the universal coverings of projective surfaces could have a motivic character so we formulate an arithmetic variant of the Shafarevich conjecture. Let K be a number field with a ring of integers O_K. Start with a projective semi-stable curve C_K over K. Since K is not algebraically closed we can find a nontrivial map $C \to \mathrm{Spec}(O_K)$ whose generic fiber is isomorphic to C_K. Due to the semi-stable reduction theorem we may assume (after passing to a finite extension of K if necessary) that C is also semi-stable. This means that C is a normal variety with semi-stable fibers consisting of normally intersecting divisors. For any finite nonramified geometric extension L of $K(C)$ we have a model C_L with a finite map onto C. As it follows from the above discussion the fibers of this new model C_L are uniquely determin ed by C. We formulate an arithmetic version of the Shafarevich conjecture.

Question Is there a constant $J(K(C))$ such that the number of components of any fiber of C_L is bounded by $J(K(C))$ for any finite extension L of $K(C)$ contained in $K(C)^{nr}$.

There exists some evidence for the above arithmetic conjecture - it is true if we consider only abelian coverings. Bogomolov proved that the torsion group of an abelian variety A is finite if the latter is defined over an infinite nonramified extension of K (see e.g. R.Coleman's paper in Duke Math. Journal, 54 (1987), p. 615).

References

[1] **F.Bogomolov and L.Katzarkov,** *Complex projective surfaces and infinite groups* , preprint, 1995.

[2] **F. Bogomolov and L.Katzarkov,** *Symplectic four manifolds and projective surfaces* , preprint, 1996.

[3] **P.Deligne,** *Structures de Hodge mixtes réelles,* Proceedings of Symposia in Pure Mathematics, vol. **55** (1994), Part I, 509-514.

[4] **B.Lasell and M.Ramachandran,** *Observations on harmonic maps and singular varieties,* Ann. Sci. École Norm. Sup. **29**, 1996, 135-148.

[5] **L.Katzarkov,** *On the Shafarevich maps,* to appear in Proceedings of AMS Alg. Geometry , S. Cruz, 1995.

[6] **L.Katzarkov, T.Pantev and M.Ramachandran,** *Geometric factorization of linear fundamental groups,* in preparation.

[7] **T.Napier,** *Convexity properties of coverings of smooth projective varieties,* Math. Ann. **286** 1990, 433-480.

[8] **C.T.Simpson,** *Mixed twistor structures.* alg-geom/9705006.

[9] **C.T.Simpson,** *The Hodge filtration on the non-abelian cohomology.* to appear in Proceedings of AMS Alg. Geometry , S. Cruz, 1995.

L. Katzarkov, UC Irvine.
lkatzark@math.uci.edu

Elliptic Moduli in Algebraic Topology

Haynes Miller
MIT

This is a report on joint work in progress with M. J. HOPKINS.

The theme. Recently topology has been interacting in new ways with algebra via the following pull-back diagram.

$$\begin{array}{ccc} \text{topology} & \longleftarrow & \text{new stuff} \\ \downarrow & & \downarrow \\ \text{sets} & \longleftarrow & \text{algebra} \end{array}$$

The "new stuff" here forms a natural enrichment of the algebra under it. The algebra today is moduli of elliptic curves.

Here's the pattern. If I is some shape of diagram—some small category—and D is a functor from it to groups, I can form $\lim D$, the group of of compatible families of elements of the $D(i)$. For example a pull-back is a limit, as is the fixed point set of a group action.

Now suppose that $D_* = \pi_*(X)$, for a diagram of *spaces*. A characteristic feature of homotopy theory is that one can form a modification of the notion of limit which is both more homotopy-invariant and more interesting: the *homotopy limit* $\operatorname{holim} X$. For example,

$$\operatorname{holim}\left\{ \begin{array}{ccc} & & X \\ & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array} \right\} = \left\{ \begin{array}{cc} & x \\ & \\ y & (\omega : g(y) \sim f(x)) \end{array} \right\}$$

If $X = * = Y$ then the holim is empty if they land in different path components, and homotopy equivalent to the space of pointed loops on Z if the are in the same component. The homotopy limit of a group action is the *homotopy fixed point set* $\operatorname{map}_G(EG, X)$. The homotopy limit is *not* the limit in the homotopy category, but rather a more sophisticated construction which in general depends not just on the image of the diagram in the homotopy category but rather on the diagram of spaces itself. On the other hand, unlike the actual limit, a map of diagrams which is a homotopy equivalence on each object induces a homotopy equivalence of the homotopy inverse limits.

There is a map

$$\pi_* \operatorname{holim} X \to \lim \pi_* X$$

which is in general neither injective nor surjective. It's the edge homomorphism of a spectral sequence.

Model example. In this work we'll replace spaces by *spectra*. These objects should be better-known outside Topology since they they make life so much easier. They behave like spaces which can have homotopy groups in negative dimensions. In compensation the homotopy in each dimension is an abelian group. They represent cohomology theories. For example, topological K-theory is represented by a spectrum K. $\pi_n K = \tilde{K}(S^n)$, which is zero for n odd and $\mathbb{Z}$ for n even. In fact K-theory is a "ring-spectrum," and $\pi_* K = \mathbb{Z}[u^{\pm 1}]$. It is a "periodic ring spectrum," in the sense that it has no odd homotopy and has a unit in dimension 2 (the Bott class).

Periodic ring spectra represent computable cohomology theories. For example if E is a periodic ring spectrum then

$$E^0(\mathbb{CP}^\infty) \cong E^0[[x]]$$

so there is a "first Chern class" or Euler class for complex line bundles (NOVIKOV and QUILLEN). It satisfies a product law

$$e(L_1 \otimes L_2) = F(e(L_1), e(L_2)).$$

The power series F is a *formal group law*. For example we can take $e(L) = 1 - L$ in the case of K-theory, and then the formal group law is the *multiplicative formal group* $G_m(x, y) = x + y - xy$. I should point out that the formal group *law* depends upon the parameter x, while the formal group is canonically determined by E.

Complex conjugation acts on this spectrum; this is a diagram of spectra. $Tu = -u$ so the fixed subring is $\mathbb{Z}[u^{\pm 2}]$. On the other hand

$$\operatorname*{holim}_{\mathbb{Z}/2} K = KO.$$

There is a natural map $\pi_* KO \to (\pi_* K)^{\mathbb{Z}/2}$, which is neither onto (u^{4k+2} is not in the image) nor one-to-one (there is 2-torsion in $\pi_{8k+1,2} KO$). There is a spectral sequence

$$H^*(\mathbb{Z}/2; \pi_* K) \Longrightarrow \pi_* KO.$$

Why spectra? Perhaps I should say a word about why you should care about spectra. I'll motivate from index theory. A *genus* is an additive and multiplicative bordism invariant of manifolds with some geometric structure; for example a complex structure, or, better, a complex structure on the normal bundle: a *U-manifold*. NOVIKOV and MILNOR showed that a genus on U-manifolds with values in a $\mathbb{Q}$-algebra is determined by its values on $\mathbb{CP}^n$. The *Todd genus* for example is such that

$$\mathrm{Td}(\mathbb{CP}^n) = 1$$

for every n. According to Thom, the ring of bordism classes of U-manifolds is isomorphic to the homotopy ring of a spectrum, the unitary Thom spectrum MU. The Todd genus thus defines a homomorphism of graded rings $\pi_*(MU) \to \mathbb{Z}[u^{\pm 1}]$.

HIRZEBRUCH's book is devoted to showing that for complex manifolds this coincides with the *arithmetic genus*, i.e. (after HIRZEBRUCH) the alternating sum of the dimensions of the cohomology groups (which are finite dimensional) of the Dolbeault complex

$$0 \longrightarrow C^\infty(M) \xrightarrow{\bar\partial} C^\infty(\bar{T}^* M) \xrightarrow{\bar\partial} C^\infty(\Lambda^2 \bar{T}^* M) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} C^\infty(\Lambda^n \bar{T}^* M) \longrightarrow 0.$$

Now suppose that $E \downarrow X$ is a *family* of complex manifolds. Any genus gives us a *number* for each point in X—locally the same number, of course. But the Dolbeault complex gives us more: the cohomology groups form *vector bundles* over X, and their alternating sum is an element of $K(X)$. By pairing the Dolbeault complex with a vector bundle over E you get a map $K(E) \to K(X)$.

On the other hand, the fact that complex vector bundles have K-theory Euler classes leads to a map of ring spectra $MU \to K$. This map induces the Todd genus in homotopy, and also constructs from the bundle $E \to B$ a *Gysin map* $K(E) \to K(B)$. The index theorem for families identifies these two covariant maps.

The homotopy type of the spectrum reflects the geometry (vector bundles here), and the orientation $MU \to K$ reflects the analysis (index theory).

Elliptic curves. The next analogue is much more interesting. Here the diagram is indexed by a certain category of elliptic curves, which I review in pedestrian form.

Look at a smooth cubic plane projective curve E over a field k, and suppose that $o \in E(k)$. The k-valued points form a group by requiring that the sum of the three points at which E meets a line is o. This curve can be normalized so that $o = [0,1,0]$ is the unique point at infinity and that the line at infinity is tangent to E. By scaling x and y appropriately this curve is given by a Weierstrass equation

$$E: \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \qquad a_i \in R.$$

I've written R here because this equation makes sense over any ring R. Smoothness is equivalent to a certain polynomial, the discriminant Δ, being a unit in R. There are still some coordinate changes that preserve this form. (I'll omit scaling, which contributes a grading to everything.)

$$\begin{aligned} x &= x' + r \\ y &= y' + sx' + t \end{aligned}$$

The set of Weierstrass equations is a functor of R which is representable by the ring

$$A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, \Delta^{-1}].$$

The group of coordinate changes is represented by a Hopf algebra with underlying ring

$$S = \mathbb{Z}[r, s, t]$$

which co-acts on A: $\psi : A \to A \otimes S$.

We can form a category of Weierstrass curves E/R, with maps $E/R \to E'/R'$ given by a ring homorphism $f : E \to E'$ and a coordinate change $fE \to E'$. Actually, it is better to form the assocated stack $\mathcal{E}ll$ in the flat topology on affine schemes, and we do so.

Now $E/R \mapsto R$ is a functor on this category, and we may form the limit. This will give natural invariants of elliptic curves in the ground-ring: i.e., polynomials in a_i, Δ, which are left fixed by coordinate changes. This ring of "integral modular forms" can also be thought of as the ring

$$H^0(S; A)$$

of primitive elements for the coaction of S by coordinate changes, and was computed by TATE and DELIGNE:

$$H^0 = \mathbb{Z}[c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 = 12^3\Delta).$$

Topological modular forms. To begin with one wishes to associate a spectrum to an elliptic curve. I do not know how to do this in general, but if E/R satisfies a simple flatness condition then it can be done. The flatness condition is that the map

$$A \xrightarrow{\psi} A \otimes S \xrightarrow{``E"\otimes 1} R \otimes S \tag{1}$$

should be flat. For example the universal case $R = A$ works, as does the "Legendre curve"

$$y^2 = x(x-1)(x-\lambda) \qquad \text{over} \qquad \mathbb{Z}[1/2, \lambda^{\pm 1}, (1-\lambda)^{-1}].$$

Theorem I. There is a lift in

$$\begin{array}{ccc} & & \begin{pmatrix}\text{Periodic} \\ \text{Ring Spectra}\end{pmatrix} \\ & \nearrow & \downarrow \\ \mathcal{E} : \mathcal{E}ll_{\text{flat}} & \underset{\substack{\text{completion} \\ \text{along } o}}{\longrightarrow} & (\text{Formal Groups}) \end{array}$$

This is based on the work of QUILLEN, relating complex cobordism to the theory of formal groups. PETER LANDWEBER used this to give a general prescription for constructing a spectrum from a formal group. The theorem of PIERRE CONNER and ED FLOYD relating K-theory to complex bordism was the motivating example. The first example of such a construction starting with an elliptic curve was due long ago to JACK MORAVA, and more recently to LANDWEBER, DOUG RAVENEL, and BOB STONG. The possibility of a more general construction was perceived by JENS FRANKE. The final touches rely on recent work of MARK HOVEY and NEIL STRICKLAND. (As a technical point, one must restrict still further to insure that the completion at o is a formal group in the usual sense, rather than just locally so, but we won't belabor the issue.)

Now we'd like to form a homotopy limit of this diagram, but a diagram up to homotopy is not sufficiently rigid to make this construction (even though the homotopy limit is homotopy invariant!). We have to lift further to some category of spectra and real maps rather than homotopy classes of maps between them. For this it turns out to be useful to use the ring-structure. There is a category of "A_∞-ring spectra," which is a topological version of the theory of associative rings. It forms a topological model category. It is due in different forms to a large group of people. The result I am reporting on is the sort of application one can make of the technical work on spectra and could not be done without it.

Three obstruction theories. First, there is an obstruction theory for the existence of an A_∞ structure. To make the obstructions vanish I must restrict the elliptic curve further, requiring that the map (1) should be not just flat but "étale." Etale means that in a homotopical sense there are no relative differentials; so that R can't be too big relative to A. In fact the universal Weierstrass curve itself is not étale, but the Legendre curve and enough other examples are.

Theorem II: Objects. For E/R étale, $\mathcal{E}^{E/R}$ admits an essentially unique A_∞ structure.

Here we rely on an obstruction theory developed by ALAN ROBINSON and more recently and in different form by CHARLES REZK and by HOPKINS and PAUL GOERSS.

Theorem III: Morphisms. There is a lift in

$$\begin{array}{ccc} & & \mathrm{Ho}(\mathrm{Periodic}A_\infty) \\ & \nearrow & \downarrow \\ \mathcal{E}:\mathcal{E}ll_{\text{ét}} & \longrightarrow & (\text{Periodic Ring Spectra}) \end{array}$$

and the lift is fully faithful.

The final job is to lift the diagram from $\mathrm{Ho}A_\infty$ to a diagram in A_∞ itself. For this a key observation is that any $f : E/R \to E'/R'$ induces a homotopy equivalence

$$A_\infty(\mathcal{E}^{E/R}, \mathcal{E}^{E'/R'})_f \xleftarrow{f^*} A_\infty(\mathcal{E}^{E/R}, \mathcal{E}^{E/R})_1 :$$

the diagram is *centric*. This allows us to apply an obstruction theory developed by BILL DWYER and DAN KAN for use in the theory of p-compact groups. It leads to

Theorem IV: Associativity. There is an essentially unique lift

$$\begin{array}{ccc} & & A_\infty \\ & \nearrow & \downarrow \\ \mathcal{E}:\mathcal{E}ll_{\text{ét}} & \longrightarrow & \mathrm{Ho}A_\infty. \end{array}$$

This rigid diagram *is* elliptic cohomology. It is the sort of structure which would emerge naturally from a construction involving geometric cocycles (analogous to vector bundles).

Now, finally, I can take

$$TMF = \operatorname*{holim}_{\mathcal{E}ll_{\text{et}}} \mathcal{E}$$

in the category A_∞. The result, like KO, is an A_∞ ring spectrum. There is a spectral sequence

$$H^*(S; A) \Longrightarrow \pi_* TMF$$

whose edge homomorphism

$$\pi_* TMF \to H^0$$

is neither one-to-one nor onto. There is nontrivial higher cohomology (all killed by 24). There are nontrivial differentials (on Δ, for example). So Δ is not a *topological* modular form, though 24Δ and Δ^{24} are. Δ^{24} is a unit, giving TMF a periodicity of degree 24^2. The homotopy of TMF (which is hardest to understand at the prime 2) has been studied in detail by HOPKINS and MARK MAHOWALD. Indeed, MAHOWALD was aware of the completion at 2 of this spectral sequence some twenty-five years ago.

The Witten genus and further questions. The action by complex conjugation on K leads to a variant of the Todd genus $MU \to K$ with values in KO, no longer on U-manifolds but rather on Spin manifolds: the $\hat{A}$ genus, or, more subtly, the "Atiyah invariant" $\alpha : MSpin \longrightarrow KO$. This genus also has an index-theory interpretation, by means of the Dirac operator.

It seems that there should be an analogue for TMF. Witten produced a genus which takes values in modular forms on "String manifolds," that is, manifolds whose structure group reduces to the next connective cover of $Spin(n)$. This amounts to $p_1 = 0$. The connective covering group is a good topological group but is no longer finite dimensional. (It's a bundle over $Spin(n)$ with $\mathbb{C}\mathrm{P}^\infty$ as fiber.) An analogue of the Clifford algebra approach would be nice.

MATTHEW ANDO, HOPKINS, and NEIL STRICKLAND have shown that there is a *canonical* ring-spectrum map

$$MStr \to \mathcal{E}^{E/R}$$

for any flat object E/R (or indeed for any "elliptic spectrum"). The proof uses the "theorem of the cube." This is a beautiful result, but doesn't quite do what we want. We would like to lift this to a map from the constant diagram $MStr$ to the diagram $\mathcal{E}$ in A_∞. If this can be done, then we get an orientation

$$\omega : MStr \longrightarrow TMF$$

enriching the Witten genus. This would put interesting restrictions on the image of the Witten genus and also define a whole batch of new torsion invariants for string manifolds.

The motivating question remains: what is the corresponding theory of geometric cocycles? What is the analogue of index theory?

Branes

Werner Nahm
Physikalisches Institut der Universität

Branes are objects in string theory/ M-theory/ F-theory (which are all aspects of the same mathematical structure). A p-brane extends in (p+1) spacetime dimensions. More precisely, solitonic p-branes have a (p+1)-dimensional core, but include a description of spacetime deformation around this core. Membranes are 2-branes, strings 1-branes, particles 0-branes, instantons (-1)-branes.

To construct a solitonic p-brane, consider first the massless fields of a string or membrane theory in a flat spacetime. These include gravitons and various kinds of gauge fields. The corresponding fields can take non-zero values when the string motion in the corresponding background remains unobstructed. This yields equations for the background fields (modifications of Einstein and Yang-Mills equation). To find solutions of minimal energy (BPS configurations), one has to solve stronger first order equations.

As an example consider the 11-dimensional theory which has membranes and 5-branes (Duff, Stelle, Gueven). The background fields are the metric and a closed 4-form K. The 5-brane carries a charge proportional to the integral of K along a four sphere wrapping around it, the membrane analogously for $*K$. The product of the 2-brane and 5-brane charges is constrained by an analog of Dirac's quantization condition for electric and magnetic charges.

For a given background, string theory yields a quantum field theory. When the parameters of such a theory are varied, heavy semiclassical solitons may become light and are better described by local fields. For example, the soliton of the two dimensional sine-Gordon equation

$$(\partial_t^2 - \partial_x^2)\phi = sin(\phi)$$

corresponds to a quantum field which describes a 2π jump of ϕ which is localized at a point. For p-branes with $p > 0$ one can expect analogous localized objects.

Such configurations turned up for open strings in R^n/Λ, where Λ is a lattice (Polchinsky). Free strings are described by quantizing the equation

$$(\partial_t^2 - \partial_x^2)\phi(x,t) = 0 ,$$

$\phi(x,t) \in R^n/\Lambda$. The Hilbert space is a tensor product of a Fock space given by the non-zero Fourier modes and a zero mode factor. For closed strings, the latter corresponds to the classical solution $\phi = \lambda x + \mu t$, where $\lambda \in \Lambda$. By momentum quantization, μ must belong to the dual lattice Λ^v. Correspondingly, the Hilbert space gets a factor $l^2(\Lambda) \otimes l^2(\Lambda^v)$. The lattices Λ and Λ^v yield isomorphic quantum theories (T-duality). For the classical solutions, this corresponds to an interchange of x and t.

For the standard open strings with Neumann boundary conditions, the zero modes are given by $\phi = \mu t$ and the Hilbert space has a zero mode factor $l^2(\Lambda^v)$. An isomorphic theory is obtained by $\phi \in R^n/\Lambda^v$. with Dirichlet boundary conditions $\phi(0,t) = \phi(1,t) = const.$ One also may obtain isomorphic quantum theories by restricting some components of $\phi(0,t)$ and $\phi(1,t)$ to a hyperplane in the target space and to use Neumann boundary conditions for the others. Such hyperplanes are called $d - branes$. As for the solitonic branes, the background configuration can be deformed to yield moving d-branes. Again, such d-branes may carry charges. For a given string theory, a $(p+2) - form$ gauge field K indicates the existence of p-branes. Similar considerations determine which $p_1 - branes$ can end on $p_2 - branes$.

Apart from their applications in string theory itself, branes can be used to obtain quantum field theories and their moduli spaces. In particular, one can put a p-brane in some background and study the induced quantum field theory on the (p+1)-dimensional world sheet. More generally, one can consider some parallel branes with distances small compared to the length scales under study, since this configuration behaves as a single spacetime. This procedure supplements the Kaluza-Klein construction of field theories by compactifying some of the space dimensions. It has the advantage that it easily yields mass parameters.

When $p+1$ is the spacetime dimension, p-branes have no dynamics, but label the ends of open strings (Chan-Paton factors). For n distinct labels and oriented strings, one obtains $U(n)$ gauge fields, as noticed in the early days of string theory. The symmetry can be broken by separating the branes, with a breaking scale proportional to their distances. This is the mechanism which has been used by Connes and Lott to describe the broken $U(1) \times SU(2)$ invariance of the standard model. As in their case, one obtains a geometric interpretation of the leading term of the β-function, in other words the logarithmic divergence of 1-loop Feynman graphs.

Another application yields a conception interpretation of the ADHMN description of BPS monopoles (Diaconescu). For an $SU(2)$ monopole of charge k in 3+1 spacetime dimensions one needs two 3-branes joined by k superimposed strings. Along the strings one finds the expected differential equation of three $k \times k$ matrices. For $SU(n)$ monopoles one needs n of the 3-branes joined by a varying number of strings. This corresponds to a differential equation for matrices of varying size, with suitable continuity conditions at the locations of the 3-branes, exactly as in the ADHMN formalism.

Many of the applications are still at a purely heuristic level, but they certainly are promising and have uncovered a new aspect of string theory.

COUNTING ELLIPTIC CURVES IN PROJECTIVE SPACES

E. GETZLER

There are two possible ways of counting genus g curves in a projective variety (except in genus 0): we may fix the modulus of the curve, or allow it to vary. In this talk, we concentrate in the second of these problems.

Thus, we are "integrating" over $\mathcal{M}_g$. Think of $\mathcal{M}_g$ as the space of (conformal classes of) metrics on a topological surface of genus g. Then the problem of integrating over $\mathcal{M}_g$ is a model for the problem of integrating over the space of metrics on space-time, which arises in the (unsolved) problem of quantizing gravity. Some of the ideas in this talk first arose in the study of gravity in two dimensional space-times.

Let $\mathcal{M}_{g,n}(V)$ be the moduli space of data $(f : C \to V, z_1, \dots, z_n)$, where C is a smooth projective curve of genus g, V is a smooth projective variety of dimension D, f is an algebraic map, and z_i are distinct points of C. The tangent space of $\mathcal{M}_{g,n}(V,\beta)$ at such a map is given by the zeroth cohomology group of the complex

$$TC(-Z_1, \dots, -Z_n)[1] \xrightarrow{df} f^*TV,$$

which has Euler characteristic

$$(D-3)(1-g) - \int_C f^*K_V + n.$$

Let $\mathrm{NE}_1(V)$ be the semigroup

$$\mathrm{NE}_1(V) = \mathrm{ZE}_1(V)/\text{numerical equivalence},$$

where $\mathrm{ZE}_1(V)$ is the semigroup of effective 1-cycles on V. $\mathcal{M}_{g,n}(V)$ may be partitioned according to degree of the map f:

$$\mathcal{M}_{g,n}(V) = \coprod_{\beta \in \mathrm{NE}_1(V)} \mathcal{M}_{g,n}(V,\beta),$$

where $\mathcal{M}_{g,n}(V,\beta) = \{[f : C \to V, z_i] \mid f_*[C] = \beta\}$.

We have evaluation maps

$$\begin{array}{ccc} \mathcal{M}_{g,n}(V,\beta) & \xrightarrow{\ e\ } & V^n \\ {\scriptstyle p}\big\downarrow & & \\ \mathcal{M}_{g,n} & & \end{array}$$

where the horizontal arrow is given by evaluation of f at the points z_i, while the vertical arrow forgets V and f. Given differential forms α_i on V dual to cycles Z_i, consider the problem of calculating

$$\#\{[f : C \to V, z_i] \in \mathcal{M}_{g,n}(V,\beta) \mid f(z_i) \in Z_i\}.$$

If $\mathcal{M}_{g,n}(V,\beta)$ is compact and compact, this should equal the integral of $e^*(\alpha_1 \boxtimes \cdots \boxtimes \alpha_n)$ over $\mathcal{M}_{g,n}(V,\beta)$.

This program has many difficulties, all of which have been resolved in the last few years:

1. $\mathcal{M}_{g,n}(V,\beta)$ is not compact: Kontsevich has shown how to compactify it;
2. $\mathcal{M}_{g,n}(V,\beta)$ is a stack, not a variety — this is a minor problem, but forces us to work over a ring of characteristic zero;
3. $\mathcal{M}_{g,n}(V,\beta)$ may have excess dimension, and even be singular — Kontsevich proposed a solution to this, the definition of a "virtual fundamental class" of dimension $(D-3)(1-g)-K_V\cdot\beta+n$, which has been carried out by Behrend and Fantecchi, and Li and Tian, and by Li and Tian in the symplectic setting.

Finally, we obtain the so-called Gromov-Witten invariants, which are linear forms

$$\langle I^V_{g,n,\beta}\rangle : H^*(V^n,\mathbb{Q}) \to \mathbb{Q}.$$

These satisfy a number of axioms (see Behrend and Manin), and are the solution of a generalized enumerative problem. However, in general, the answer must be corrected, since Kontsevich's compactification has ghost components, which do not correspond to embedded rational curves. (Along these components, the curve C has double points, and some components of C may be contracted to a point by f.) This problem of correction has been solved in three cases: for del Pezzo surfaces, it does not arise (roughly, because the normal bundle of a map from a curve to a surface is a line bundle, and hence its H^1 may be controlled by its degree), for $g=0$ and flag varieties, the moduli problem is unobstructed, while for $g=1$ and $V=\mathbb{CP}^3$, we have shown in joint work with Pandharipande that the true enumerative invariant is

$$\langle I^{\mathbb{CP}^3}_{1,n,kH}\rangle + \frac{2k-1}{12}\langle I^{CP^3}_{0,n,kH}\rangle.$$

Our approach to calculating the Gromov-Witten invariants $\langle I^{\beta}_{1,n,\beta}\rangle$ is to find a relation among cycles in the homology of $\overline{\mathcal{M}}_{1,4}$. There is a generalization of the Gromov-Witten invariants, which takes values in $H^*(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$, and which is determined on Knudsen's strata by the classes $\langle I^V_{g,n,\beta}\rangle$. Thus, our relation gives relations among Gromov-Witten invariants which, when there are no primitive cohomology classes of degree greater than 2 in V, reduce the calculation $\langle I^V_{1,n,\beta}\rangle$ to that of

$$\langle I^V_{1,1,\beta}\rangle : H^{2(i+1)}(V,\mathbb{Q}) \to \mathbb{Q}, \quad 0\le i = -K_V\cap\beta < D,$$

together with the rational Gromov-Witten invariants.

Using mixed Hodge theory, we have proved that the cycles $[\overline{\mathcal{M}}(G)]$, as G ranges over all stable graphs of genus 1 and valence n, span the even dimensional homology of $\overline{\mathcal{M}}_{1,n}$, and that our new relation, together with those already known in genus 0, generate all relations among these cycles. This result is the analogue, in genus 1, of a theorem of Keel in genus 0.

We may introduce a filtration on Gromov-Witten invariants with respect to which the leading order of our new relation takes a relatively simple form; by analogy with the case of differential operators, we call this leading order relation the symbol of the full relation. Unfortunately, the nature of this relation remains rather mysterious: we obtained it as the null-vector of an intersection matrix.

We define a total order on the symbols $\langle I^V_{g,n,\beta}\rangle$ by setting $\langle I^V_{g',n',\beta'}\rangle \prec \langle I^V_{g,n,\beta}\rangle$ if $g' < g$, or $g' = g$ and $n' < n$, or $g' = g$, $n' = n$ and $\beta = \beta' + \beta''$ where $\beta'' \in \mathrm{NE}_1(V)$ is non-zero. Thus, knowledge of the symbol determines relations among Gromov-Witten invariants such that the error in the relation on $\langle I^V_{g,n,\beta}\rangle$ involves invariants $\langle I^V_{g',n',\beta'}\rangle$ with $\langle I^V_{g',n',\beta'}\rangle \prec \langle I^V_{g,n,\beta}\rangle$. (Here, we must of course exclude $\langle I^V_{0,3,0}\rangle$.) We use the symbol $\sim$ to denote this equivalence relation.

Abbreviating $\langle I^V_{1,n,\beta}(\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_n)\rangle$ to $\{\alpha_1,\alpha_2\}$, we have

$$\begin{aligned}\Psi(a,b,c,d) &= \{a\cup b, c\cup d\} + \{a\cup c, b\cup d\} + \{a\cup d, b\cup c\}\\ &- \{a, b\cup c\cup d\} - \{b, a\cup c\cup d\} - \{c, a\cup b\cup d\} - \{d, a\cup b\cup c\} \sim 0.\end{aligned}$$

Our new relation is closely related to a relation in $A_2(\overline{\mathcal{M}}_3)\otimes\mathbb{Q}$ discovered by Faber (Lemma 4.4 of Faber); the image of his relation in $H_4(\overline{\mathcal{M}}_3,\mathbb{Q})$ under the cycle map is the same as the push-forward of our relation under the map $\overline{\mathcal{M}}_{1,4} \to \overline{\mathcal{M}}_3$ obtained by contracting the 4 tails pairwise. Pandharipande has found a direct geometric proof of the our relation, showing that it is a rational equivalence, by means of an auxilliary moduli space of admissible covers of $\mathbb{CP}^1$.

Let us illustrate our results with the case of the projective plane. The genus 0 and genus 1 potentials of $\mathbb{CP}^2$ equal

$$\begin{aligned}F_0(\mathbb{CP}^2) &= \frac{1}{2}(t_0t_1^2 + t_0^2t_2) + \sum_{n=1}^{\infty} N_n^{(0)} q^n e^{nt_1}\frac{t_2^{3n-1}}{(3n-1)!},\\ F_1(\mathbb{CP}^2) &= -\frac{t_1}{8} + \sum_{n=1}^{\infty} N_n^{(1)} q^n e^{nt_1}\frac{t_2^{3n}}{(3n)!},\end{aligned}$$

where t_0, t_1 and t_2 are formal variables, of degree -2, 0 and 2 respectively, dual to the classes $1 \in H^0(\mathbb{CP}^2,\mathbb{Q})$, $\omega \in H^2(\mathbb{CP}^2,\mathbb{Q})$ and $\omega^2 \in H^4(\mathbb{CP}^2,\mathbb{Q})$ respectively, and $N_n^{(0)}$ and $N_n^{(1)}$ are the number of rational, respectively elliptic, plane curves of degree n which meet $3n-1$, respectively $3n$, generic points. Kontsevich and Manin establish the recursion

$$N_n^{(0)} = \sum_{n=i+j} \left(\tbinom{3n-4}{3i-2} i^2j^2 - i^3j\tbinom{3n-4}{3i-1}\right) N_i^{(0)} N_j^{(0)},$$

which, together with the initial condition $N_1^{(0)} = 1$, determines the coefficients $N_n^{(0)}$. Our method gives the recursion

$$\begin{aligned}6N_n^{(1)} &= \sum_{n=i+j+k} \tbinom{3n-2}{3j-1,3k-1} ij^3k^3(2i-j-k) N_i^{(1)} N_j^{(0)} N_k^{(0)}\\ &+ 2\sum_{n=i+j}\left(\tbinom{3n-2}{3i} ij^2(8i-j) - \tbinom{3n-2}{3i-1} 2(i+j)j^3\right) N_i^{(1)} N_j^{(0)}\\ &- \frac{1}{24}\left(\sum_{n=i+j}\tbinom{3n-2}{3i-1}(n^2-3n-6ij)i^3j^3 N_i^{(0)}N_j^{(0)} + 6n^3(n-1)N_n^{(0)}\right).\end{aligned}$$

TABLE 1. Rational and elliptic Gromov-Witten invariants of $\mathbb{CP}^2$

n	$N_n^{(0)}$	$N_n^{(1)}$
1	1	0
2	1	0
3	12	1
4	620	225
5	87 304	87 192
6	26 312 976	57 435 240

In Table 1, we list the first few coefficients $N_n^{(1)}$; for comparison, we also include the corresponding rational Gromov-Witten invariants.

Recently, Eguchi, Hori and Ziong have proposed a bold conjecture, generalizing the conjectured of Witten and proved by Kontsevich, that the Gromov-Witten invariants of a point ("in the large phase space") are the highest weight vector for a certain Virasoro algebra. Their conjecture implies in particular the recursion

$$N_n^{(1)} = \tfrac{1}{12}\tbinom{n}{3}N_n^{(0)} + \tfrac{1}{9}\sum_{n=i+j}\tbinom{3n-1}{3i-1}(3i^2-2i)jN_i^{(0)}N_j^{(1)},$$

which is far simpler than ours. Pandharipande has proved that this recursion is a formal consequence of our recursion, but otherwise, their conjecture remains very mysterious.

REFERENCES

[1] K. Behrend, *Gromov-Witten invariants in algebraic geometry,* Invent. Math. **127** (1997), 601–617.

[2] K. Behrend and B. Fantechi, *The intrinsic normal cone,* Invent. Math. **128** (1997), 45–88.

[3] K. Behrend and Yu. Manin, *Stacks of stable maps and Gromov-Witten invariants,* Duke Math. J. **85** (1996), 1–60.

[4] T. Eguchi, K. Hori and C.-S. Xiong, *Quantum cohomology and the Virasoro algebra,* hep-th/9703086.

[5] C. Faber, *Chow rings of moduli spaces of curves. I: The Chow ring of $\overline{\mathcal{M}}_3$*, Ann. Math. **132** (1990), 331-419.

[6] E. Getzler, *Intersection theory on $\overline{\mathcal{M}}_{1,4}$ and elliptic Gromov-Witten invariants,* MPI Preprint 96-161. To appear, J. Amer. Math. Soc. alg-geom/9612004.

[7] S. Keel, *Intersection theory of moduli spaces of stable n-pointed curves of genus zero,* Trans. Amer. Math. Soc. **330** (1992), 545–574.

[8] M. Kontsevich and Yu. Manin, *Gromov-Witten classes, quantum cohomology and enumerative geometry,* Commun. Math. Phys. **164** (1994), 525–562.

[9] J. Li and G. Tian, *Virtual moduli cycles and GW-invariants,* alg-geom/9602007.

[10] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds,* alg-geom/9608032.

[11] R. Pandharipande, *A geometric realization of Getzler's relation,* hep-th/9705016.

[12] E. Witten, *On the structure of the topological phase of two-dimensional gravity,* Nucl. Phys. **B340** (1990), 281–332.

NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208-2730, USA
E-mail address: getzler@math.nwu.edu

DISCRIMINANTS OF K3 SURFACES AND KAC–MOODY ALGEBRAS

V. Gritsenko

In this talk we give review of our joint results with **V. Nikulin**. For more results, details and conjectures see our paper "*K3 surfaces, Lorentzian Kac–Moody algebras and mirror symmetry*" in Math. Res. Let. vol. 3 (1996) pp. 211–229 and our preprint of RIMS of Kyoto University N 1129 (1997) "*The arithmetic mirror symmetry and Calabi–Yau manifolds*" (alg-geom/9612002).

1. Mirror symmetry for K3 surfaces. Let X be a K3 surface (i.e. a regular compact complex-analytic surface with trivial canonical class). Its Picard lattice $\mathrm{Pic}(X)$ is an even hyperbolic lattice, i.e. a free $\mathbb{Z}$-module with an integral even symmetric bilinear form of signature $(1, n)$. The orthogonal complement of the Picard lattice in $H^2(X, \mathbb{Z})$

$$T_X = (\mathrm{Pic}(X))^{\perp}_{H^2(X,\mathbb{Z})}$$

is called the lattice of transcendental cycles (transcendental lattice) of the K3 surface X.

We consider two families of K3 surfaces.

(A) S is a Picard lattice of some K3 surface. The family

$$\mathcal{M}_S = \{ \text{ K3 surface } X \mid S \subset \mathrm{Pic}(X)\}$$

has dimension $20 - \dim S$. A general member X of this family has the Picard lattice S.

(B) We choose a primitive isotropic element $c \in T$ and we put $S = (c^{\perp}_{T})/\mathbb{Z}c$ for the lattice of transcendental cycles T of some K3 surface. We define the second family of dimension $\dim S$ as

$$\mathcal{M}_T = \{ \text{ K3 surface } X \mid T_X \subset T\}.$$

A general member X of this family has $T_X = T$. These two families $\mathcal{M}_S$ and $\mathcal{M}_{T^{\perp}}$ are called dual (or mirror symmetric, or mirror). This is how mirror symmetry for K3 surfaces had first appeared in the papers of H. Pinkham, I. Dolgachev–V. Nikulin and V. Nikulin. It was inspired by explanation of the Arnol'd Strange Duality (1974) for exceptional unimodular critical points. See the modern review of I. Dolgachev (alg-geom/9502005) about this subject.

The main aim of my talk is to show that *the mirror symmetry for K3 surfaces is not exhausted by the duality* $(\mathcal{M}_S, \mathcal{M}_T)$. It turns out that in some cases "geometry" of irreducible and effective classes of divisors of general $X \in \mathcal{M}_S$ is related with interesting automorphic forms (*automorphic discriminants*) on the dual family $\mathcal{M}_T$. This relation involves generalized Lorentzian Kac–Moody Lie superalgebras.

2. Discriminant of $\mathcal{M}_T$.

The moduli space $\mathcal{M}_T$ is a quotient of a symmetric domain of type IV by some arithmetic group.

Let

$$\Omega(T) = \{\mathbb{C}\omega \subset T \otimes \mathbb{C} \mid \omega^2 = 0,\ \omega \cdot \overline{\omega} > 0\}_0$$

be the (connected) homogeneous domain of type IV. We introduce a quadratic divisor $\mathcal{H}_\delta = \{\mathbb{C}\omega \in \Omega(T) \mid \omega \cdot \delta = 0\}$. The discriminant is defined as a union of some rational quadratic divisors

$$D(T) = \bigcup_{\delta \in T,\ \delta^2 = -2} \mathcal{H}_\delta.$$

For a connected component of the moduli space of K3 surfaces with a condition on the transcendental cycles we have $(\mathcal{M}_T)_0 = O(T)' \setminus \big(\Omega(T) - D(T)\big)$ for an appropriate subgroup $O(T)'$ of finite index of $O(T)$.

Problem: To construct a holomorphic modular form F with respect to $O(T)'$ such that $\mathrm{div}(F) = D(T)$.

We use two different methods to construct such modular forms. The first one was proposed by the author in 1988 as a generalization of Maaßspezialschar (see my more recent paper on this subject "*Modular forms and moduli spaces of Abelian and K3 surfaces*" in St.Petersburg Math. Jour. vol. 6 (1994) pp. 1179–1208).

3. Arithmetic Lifting.

The boundary of the Satake compactification of the quotient $O(T)' \setminus \Omega(T)$ consists of two types of boundary components: points and curves. An isotropic vector $c \in T$ defines a zero dimensional component (see the definition of $\mathcal{M}_T$ and $\mathcal{M}(S)$ above). A boundary curve is determined by an isotropic plane P or, equivalently, by a maximal parabolic subgroup Γ_P of $O'(T)$. Let Γ_P^J be the Jacobi subgroup of Γ_P. The arithmetic lifting is an embedding of the modular forms $\mathfrak{M}_k(\Gamma_P^J, \chi)$ of weight k with a character with respect to the Jacobi group Γ_P^J (i.e. of some Jacobi modular forms) into the space of modular forms with respect to $O'(T)$. This construction can be represented as follows.

Let $f \in \mathfrak{M}_k(\Gamma_P^J, \chi)$ be a Jacobi form. Then

$$\mathrm{Lift}(f) = f|_k\, L_{SL_2}^{Hecke}(k, \chi_0)$$

where $L_{SL_2}^{Hecke}(k, \chi_0)$ is a formal Dirichlet series with Euler product (a formal Hecke L-function of SL_2-type) over the Hecke ring of the parabolic subgroup Γ_P^J. It means that we know the Fourier expansion of $\mathrm{Lift}(f)$ at the one dimensional cusp P. This is the so-called Fourier–Jacobi expansion.

In what follows we restrict ourselves to the case of the moduli $\mathcal{M}_T$ of dimension three. This case is closely related with the moduli spaces of polarized Abelian and Kummer surfaces (see our joint paper with K. Hulek "*Commutator coverings of Siegel threefolds*" in alg-geom/9702007 for more details).

We explain the main construction on the following example. Let $S \cong U(8) \oplus \langle -2 \rangle$ and $T \cong U(8) \oplus U(8) \oplus \langle -2 \rangle$. We consider a general member X of the family $\mathcal{M}_S$ ($\dim(\mathcal{M}_S) = 17$). An element $h \in S$ is called *irreducible* if it contains an irreducible curve on X. An element $h \in S$ is called *effective* if it is a finite sum of irreducible elements. The set $\Delta^{\mathrm{ir}}_{-2} = \{\delta \in \Delta^{\mathrm{ir}} \mid \delta^2 = -2\}$ of classes of the non-singular irreducible rational curves (the exceptional curves) on X consists of four elements $\{e_1, e_2, e_3, e_4\}$. They generate the lattice S and their intersection matrix is

$$(e_i \cdot e_j) = \begin{pmatrix} -2 & 2 & 6 & 2 \\ 2 & -2 & 2 & 6 \\ 6 & 2 & -2 & 2 \\ 2 & 6 & 2 & -2 \end{pmatrix}. \qquad \text{(CM)}$$

The K3 surface X can be realized as intersection of three quadrics in $\mathbb{P}^5$. The corresponding linear system is defined by the element 4ρ where $\rho = (e_1 + e_3)/4 = (e_2 + e_4)/4$. For this embedding, all four non-singular rational curves on X have degree 4. The curves $e_1 + e_3$ and $e_2 + e_4$ give two hyperplane sections of X.

Since S is hyperbolic, the cone

$$V(S) = \{x \in S \otimes \mathbb{R} \mid x^2 > 0\}$$

is the union of two half cones $\pm V^+(S)$ where $V^+(S)$ contains the class of a hyperplane section. It is easy to see that

$$\mathrm{NEF}(S) = \{h \in S \mid h \in \overline{V^+(S)} - \{0\} \text{ and } h \cdot \Delta^{\mathrm{ir}}_{-2} \geq 0\}$$

the set of all numerically effective divisors of X.

There is another homogeneous description of the both sets $\mathrm{NEF}(S)$ and $\Delta^{\mathrm{ir}}_{-2}$ of S in terms of the group $W(S) \subset O(S)$ generated by all reflections $s_\delta : x \mapsto x + (x \cdot \delta)\delta$, $x \in S$ of the lattice S in elements $\delta \in S$ with $\delta^2 = -2$. The real convex cone $\mathbb{R}_+\mathrm{NEF}(S)$ is a fundamental domain (the Weyl chamber) for the group $W(S)$ acting in $V^+(S)$ with the set of orthogonal vectors $\Delta^{\mathrm{ir}}_{-2}$.

There are two realizations of $\Omega(T)$ as a tube domain. They are the complexification of the cone $V^+(S)$ and the Siegel upper half plane $\mathbb{H}_2$

$$\Omega(T) \cong S \otimes \mathbb{R} + iV^+(S) \cong \mathbb{H}_2 = \{Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}, \quad \mathrm{Im}(Z) > 0\}.$$

The orthogonal group $PO'_0(T)$ is isomorphic to the group $\Gamma_2/\{\pm 1_4\}$ where Γ_2 is the integral symplectic group of the skew-symmetric form with elementary divisors $(1, 2)$. (The space $\Gamma_2 \setminus \mathbb{H}_2$ is the moduli space of $(1, 2)$-polarized Abelian surfaces.) Thus we can define a modular form with respect to $O'(T)$ as a Siegel modular form.

To construct a modular form as the arithmetic lifting we use the Jacobi theta-series

$$\vartheta(\tau, z) = -q^{1/8} r^{-1/2} \prod_{n \geq 1} (1 - q^{n-1} r)(1 - q^n r^{-1})(1 - q^n)$$

where we put $q = \exp(2\pi i \tau)$, $r = \exp(2\pi i r)$, $p = \exp(2\pi i \omega)$. Let us define

$$\Delta_2(Z) = \mathrm{Lift}(\eta(\tau)^3 \vartheta(\tau, z)) \mathfrak{M}_2^{cusp}(\Gamma_2, \chi_4)$$

is a Γ_2-cusp form of weight 2 with a character of order 4. This function has the following Fourier expansion at zero-dimensional cusp

$$\Delta_2(Z) = \sum_{N \geq 1} \sum_{\substack{2mn - l^2 = N^2 \\ n, m \equiv 1 \bmod 4 \\ n > 0,\ l \equiv 1 \bmod 2,}} (-1)^{\frac{l+1}{2}} \left(\frac{-4}{N}\right) N \sum_{d \mid (n,l,m)} \left(\frac{-4}{d}\right) q^{n/4} r^{l/2} p^{m/2}.$$

We remark that this is the first example of a Siegel cusp form with elementary Fourier coefficients. Using a formula for a character χ_4 it is easy to see that

$$\Delta_2(w.\mathfrak{z}) = (-1)^{det(w)}\Delta_2(\mathfrak{z})$$

for arbitrary $w \in W(S)$ with $\mathfrak{z} \in S \otimes \mathbb{R} + iV^+(S)$. Thus Div($Delta_2$) contains the discriminant $D(T)$ for the lattice T fixed above. Using the fact that the weight of Δ_2 is very small one can prove that $\mathrm{Div}(\Delta_2) = D(T)$ and the multiplicities of zero is one along any component. Moreover using the anti-invariantness of the modular forms with respect to the Weyl group $W(S)$ we can rewrite the Fourier expansion in the following form

$$\Delta_2(\mathfrak{z}) = \sum_{w\in W(S)} \det(w) \left(\exp(2\pi i(w(\rho)\cdot\mathfrak{z})) - \sum_{a\in NEF(S)} m(a)\exp(2\pi i(w(\rho+a)\cdot\mathfrak{z})) \right).$$

It follows that the Fourier expansion of $\Delta_2(\mathfrak{z})$ at the zero dimensional cusp c (see the definition of $\mathcal{M}(S)$) defines *a generalized Kac–Moody superalgebra* $\mathfrak{g}(S,\ {}_s\Delta)$. It is defined by a set ${}_s\Delta$ of simple roots which consists of a set of simple real (even) roots ${}_s\Delta^{\mathrm{re}} = \Delta^{\mathrm{ir}}_{-2}(S)$ and a set of simple imaginary roots ${}_s\Delta^{\mathrm{im}}$ divided in a set of even simple imaginary roots ${}_s\Delta^{\mathrm{im}}_{\bar{0}}$ and a set of simple odd imaginary roots ${}_s\Delta^{\mathrm{im}}_{\bar{1}}$. By definition, the sequences of imaginary roots are

$${}_s\Delta^{\mathrm{im}}_{\bar{0}}(\text{resp. } {}_s\Delta^{\mathrm{im}}_{\bar{1}}) = \{|m(a)|\text{-times } a \mid a\in \mathrm{NEF}(S),\ a^2>0,\ m(a)>0 \ (\text{resp. } \ m(a)<0)\}$$

(plus some more complicated condition if $a^2=0$). The *generalized Kac–Moody superalgebra* (without odd real simple roots) $\mathfrak{g}(S,\ {}_s\Delta)$ is a Lie superalgebra generated by h_r, e_r, f_r where $r \in {}_s\Delta$. All h_r are even, e_r, f_r are even (respectively odd) if r is even (respectively odd). The algebra has the following defining relations:

(1) The map $r \mapsto h_r$ for $r \in {}_s\Delta$ gives an embedding of $S\otimes\mathbb{R}$ into $\mathfrak{g}(S, {}_s\Delta)$ as an abelian subalgebra (it is even since all h_r are even). In particular, all elements h_r commute.

(2) $[h_r, e_{r'}] = -(r\cdot r')e_{r'}$, and $[h_r, f_{r'}] = (r\cdot r')f_{r'}$.

(3) $[e_r, f_{r'}] = h_r$ if $r=r'$, and is 0 if $r\neq r'$.

(4) $(\mathrm{ad}\ e_r)^{1+r\cdot r'}e_{r'} = (\mathrm{ad}\ f_r)^{1+r\cdot r'}f_{r'} = 0$ if $r\neq r'$ and $r\in {}_s\Delta^{\mathrm{re}}$.

(5) If $r\cdot r'=0$, then $[e_r, e_{r'}] = [f_r, f_{r'}] = 0$.

The algebra $\mathfrak{g}(S,\ {}_s\Delta)$ is called *automorphic correction* of the Lorentzian Kac–Moody algbera with the set of real simple roots ${}_s\Delta^{\mathrm{re}}$. This superalgebra is graded by S as follows

$$\mathfrak{g}(S,\ {}_s\Delta) = \Big(\bigoplus_{\alpha\in \mathrm{EF}(S)_{\geq -2}} \mathfrak{g}_\alpha\Big) \oplus \mathfrak{g}_0 \oplus \Big(\bigoplus_{\alpha\in \mathrm{EF}(S)_{\geq -2}} \mathfrak{g}_{-\alpha}\Big).$$

where $\mathrm{EF}(S)_{\geq -2}$ is the set of all effective elements of S such that $\alpha^2 \geq -2$. Thus according to the Weyl–Kac–Borcherds denominator formula for $\mathfrak{g}(S,\ {}_s\Delta)$ we obtain that

$$\Delta_2(\mathfrak{z}) = \exp(2\pi i(\rho\cdot z)) \prod_{\alpha\in \mathrm{EF}(S)_{\geq -2}} (1-\exp(2\pi i(\alpha\cdot z)))^{\mathrm{mult}\ \alpha}.$$

5. Borcherds Lifting. How to find a formula for mult α in the last formula? The answer is given by a variant of Borcherds construction of the automorphic products (see Borcherd's paper *Automorphic forms on $O_{s+2,2}(R)$ and infinite products* in Invent. Math. vol. 120 (1995) pp. 161–213). It gives us the second method of construction of automorphic discriminants. In genral the Borcherds lifting can be represented as follows. Let ϕ be a nearly-holomorphic Jacobi form $\phi \in \mathfrak{M}_0(\Gamma_P^J)$ of weight 0. Then

$$\mathrm{B}(\phi) = \Psi(\mathfrak{z}) \exp(-\phi|_0\, L_{SL_2}^{Hecke}(0))$$

where $L_{SL_2}^{Hecke}$ is the same formal Hecke L-function we considered in the arithmetic lifting and $\Psi(\mathfrak{z})$ is a meromorphic Jacobi form.

For the modular forms Δ_2 we should take the weak Jacobi form

$$\phi_{0,2}(\tau, z) = \frac{\vartheta(\tau, 2z)^4}{\vartheta(\tau, z)^3\vartheta(\tau, 3z)} + \frac{\vartheta(\tau, 4z)\vartheta(\tau, z)}{\vartheta(\tau, 2z)\vartheta(\tau, 3z)} = \sum_{n,l} c(n, l)\, q^n r^l.$$

Then we can prove that

$$\Delta_2(Z) = q^{1/4} r^{1/2} p^{1/2} \prod_{\substack{n,\, l,\, m\in\mathbb{Z}\\ (n,l,m)>0}} (1 - q^n r^l p^{2m})^{c(nm,l)}$$

where $(n, l, m) > 0$ means that $n \ge 0$, $m \ge 0$, l is an arbitrary integral if $n+m > 0$, and $l > 0$ if $n = m = 0$. It gives the formula for multiplicities of all positive roots of the Kac–Moody algbera defined above.

6. Classification. Using some evidence from the classification theory of hyperbolic lattices we can formulate the following conjecture

Conjecture. *Let us assume that for an even integral lattice T of signature $(2, m)$ there exist automorphic form with divisor equals to the discriminant $D(T)$ of the moduli space $\mathcal{M}_T$ of K3 surfaces with a condition on the transcendental lattice T Then*

a) For any primitive isotropic $c \in T$ such that $c_T^\perp$ contains an element with square -2, the hyperbolic lattice $S = c_T^\perp/\mathbb{Z}c$ is 2-reflective.

b) The set of such lattices T of rk $T \ge 5$ is finite.

Roughly speaking a lattice S is 2-reflective if the Weyl group $W(S)$ has a finite index in $O(T)$ (see our joint preprint with V. Nikulin alg-geom/9610022 for more details).

The first multi-dimensional automorphic form which defines a generalized Kac–Moody algebra with an automorphic form as denominator function was found by R. Borcherds in his paper cited above. He constructed it for the even unimodular lattice T of rank 28. Then S is the even unimodular hyperbolic lattice of rank 26 and the corresponding generalized Kac–Moody Lie algebra is the fake monster Lie algebra. In this case the number of real symple roots is infinite. It is expected that this case is the most multi-dimensional. This case does not correspond to K3 surfaces but considering a primitive sublattice $T_1 \subset T$ with two positive squares and restriction of the Borcherds form to $\Omega(T_1)$, one can construct other examples. (Usually such restrictions have rather complicated structure of its divisor.) Considering this restriction, R. Borcherds found the form Φ which gives the discriminat for $T = U(2) \oplus U \oplus E_8(-2)$ (we denote by $K(t)$ a lattice K with the form multiplied by $t \in \mathbb{Q}$). See Borcherds paper "*The moduli space of Enriques surfaces and the fake monster Lie superalgebra*" in Topology vol. 35 (1996) pp. 699–710. This case corresponds to the moduli space of K3 surfaces which cover twice Enriques surfaces. Considering orthogonal complement to $c \in U(2)$, we get the mirror symmetric family with $S = U \oplus E_8(-2)$. This corresponds to K3 surfaces with involution having the set of fixed points equals to union of two elliptic curves.

It would be very interesting to make a classification of all automorphic discriminants. We hope to do it at least in the case when rank (T)=5. Some steps in this direction have been made in our joint preprints with V. Nikulin *Automorphic forms and Lorentzian Kac–Moody algebras I, II* (alg-geom/9610022 and 9611028). In particular we construct automorphic discriminants and corresponding Kac–Moody algebras (when they exist) for the lattices

$$T = U \oplus U \oplus < -2k > \quad k = 2,\ 3,\ 4,$$

and

$$T = U(k) \oplus U(k) \oplus < -2 > \quad k = 1, \ldots, 7,\ 8,\ 10,\ 12,\ 16.$$

At the end of this talk we remark that there are some connections between the subject we disscussed above and the theory of mirror symmetry of Calabi–Yau manifolds. See, for example, the papers of T. Kawai, G.L. Cardoso, G. Curio, D. Lüst where Siegel modular forms constructed in our papers mentioned above (f.e. the modular form Δ_2) provide the heterotic perturbative Wilsonian gravitational coupling. The Borcherds automorphic form for $T = U \oplus U(2) \oplus E_8(-2)$ was recently used by J. Harvey and G. Moore.

St. Petersburg Department Steklov Mathematical Institute Fontanka 27, 191011 St. Petersburg, Russia
and MPI für Mathematik, Bonn
E-mail address: gritsenk@mpim-bonn.mpg.de; gritsenk@pdmi.ras.ru

Anabelian Geometry

by F. Pop, Universität Bonn

–Report on new and recent results–

Let X be an integral scheme, and $\pi_1(X,\overline{x})$ its étale fundamental group, which we simply denote $\pi_1(X)$ hence leaving away the geometric point $\overline{x}$. We say that X is *anabelian* if X is functorially encoded in $\pi_1(X)$ viewed as a profinite group. In particular, if X and Y are anabelian schemes, then

$$\mathrm{Isom}(X^{\mathrm{i}}, Y^{\mathrm{i}}) \cong \mathrm{Isomout}\big(\pi_1(X), \pi_1(Y)\big)$$

where the superscript $^{\mathrm{i}}$ means up to pure inseparable covers (which are not "seen" by π_1) and Isomout means outer Isomorphisms.

We say that a *category* $\mathcal{A}$ of anabelian schemes is itself *anabelian,* if for all X, Y in $\mathcal{A}$ one has

$$\mathrm{Hom}^{\mathrm{dom}}(X^{\mathrm{i}}, Y^{\mathrm{i}}) \cong \mathrm{Homout}^{\mathrm{open}}\big(\pi_1(X), \pi_1(Y)\big).$$

where $\mathrm{Hom}^{\mathrm{dom}}$ denotes dominant morphisms of schemes and $\mathrm{Homout}^{\mathrm{open}}$ denotes outer Morphisms of profinite groups with open image.

One can also work relatively to a base scheme S, and define S-anabelian objects, respectively an S-anabelian category of S-anabelian schemes.

It was conjectured by Grothendieck that the *finitely generated infinite fields,* as well as the *hyperbolic curves over finitely generated fields* are anabelian, even more, that the *category of all such schemes is itself anabelian.* Here, for a given field k, a hyperbolic curve over k is by definition a smooth, geometrically integer curve over k.

The following new results were mentioned:

1) Birational results:

Let k be an arbitrary field, $\ell \neq \mathrm{char}(k)$ a rational prime number. Let k' be the maximal pro-ℓ abelian extension of $k[\mu_{\ell^\infty}]$, and $k'' := (k')'$. Then $k''|k$ is a Galois extension, and let G''_k denote its Galois group. One has:

Theorem [Pop].

(1) Global version: *Let K be a finitely generated field of absolute transcendence degree ≥ 2, and $\ell \neq \mathrm{char}(K)$. Then K is functorially encoded in G''_K. In particular, such fields K are anabelian schemes.*

(2) Local version: *Let k be a local field, $\ell \neq \mathrm{char}(k)$. The every finitely generated field $K|k$ is encoded in $G''_K \to G''_k$. In particular, the finitely generated fields over k are k-anabelian.*

This extends the famous result of Neukirch, Ikeda, Uchida, Iwasawa concerning global fields.

2) Results concerning curves:

Theorem [Tamagawa].
The hyperbolic curves over finitely generated fields are anabelian schemes. In particular, if X and Y are such curves, then

$$\mathrm{Isom}(X^{\mathrm{i}}, Y^{\mathrm{i}}) \cong \mathrm{Isomout}\big(\pi_1(X), \pi_1(Y)\big).$$

The fact that the hyperbolic curves over finite fields are anabelian schemes is quite unexpected.

Finally, the very impressive (and very recent) results by Mochizuki: We say that a field k is sub-p-adic, if k is contained in some finitely generated extension of $\mathbb{Q}_p$. Let X be a geometrically integral scheme over k, and denote $\overline{X} = X \times_k \overline{k}$ the base change to $\overline{k}$. If Δ_X is the maximal pro-p quotient of $\pi_1(\overline{X})$, then $D = \ker\big(\pi_1(\overline{X}) \to \Delta_X\big)$ is a normal subgroup in $\pi_1(X)$. Setting $\Pi_X = \pi_1(X)/D$ we get a projection $\Pi_X \to G_k$ with kernel Δ_X.

Theorem [Mochizuki].

(1) *With the above notations, every hyperbolic curve X over k is functorially encoded in $\Pi_X \to G_k$, and for X, Y hyperbolic curves over k one has:*

$$\mathrm{Hom}_k^{\mathrm{dom}}(X, Y) \cong \mathrm{Homout}_{G_k}^{\mathrm{open}}(\Pi_X, \Pi_Y).$$

In particular, the category of all hyperbolic curves over k is anabelian.

(2) *If $K|k$ is a finitely generated field extension, then K is functorially encoded in $\Pi_K \to G_k$, and for K, L finitely generated fields over k one has:*

$$\mathrm{Hom}_k(L, K) \cong \mathrm{Homout}_{G_k}^{\mathrm{open}}(\Pi_K, \Pi_L).$$

In particular, the category of all finitely generated regular field extensions k is k-anabelian.

This answers the question of Grothendieck completely in the case of schemes of dimension ≤ 1 in characteristic zero. The result is even stronger than conjectured by Grothendieck in two directions: First, one works with Π_X instead of the full $\pi_1(X)$ and second, over the p-adics instead of the rationals.

Curves of higher genus and homotopy theory

Mark Mahowald

This is an abstract of a talk based on joint work with Mike Hopkins and Vassily Gorbonov.

Let (k, Γ) be a pair consisting of a formal group Γ and a field k of characteristic $p > 0$. A deformation of (k, Γ) to a complete local ring B (with maximal ideal m) is a pair (G, i) consisting of a formal group G over B and a map $i : k \to B/m$ such that $i^*\Gamma = \pi^*G$ where $\pi : B \to B/m$.

Lubin and Tate proved that there is universal deformation ring, E_{n*}. If Γ has height n then $E_{n*} = WF_{p^n}[[u_1, \cdots u_{n-1}]]$. If we consider a hight n formal group over F_{p^n}, then the group of strict automorphisms of Γ is called the Morava stabilzer group, S_n. If $(p-1)|n$ then S_n contains a finite subgroup of prime p power order. Let G_n be a maximal finite subgroup. We have the following:

- There is a spectrum E_n with $E_{n*} = WF_{p^n}[[u_1 \cdots u_{n-1}]][u, u^{-1}]$ with homotopy dimension of $u = -2$. (Landweber exact functor theorem)
- G_n acts on E_n. (Hopkins and Miller)
- $EO_n = (E_n)^{hG_n}$. There is a spectral sequnce computing EO_{n*} with $E_2 = (H^*(G_n; E_{n*})^{Gal}$.

When $n = 2$ ellpitic curves over F_{p^2} with $p = 2$ or 3 are sources of such formal groups and the role of ellptic curves in the construction of EO_2 is rather well understood. (Compare the talk by Haynes Miller in these same abstracts.) The goal here is to show that the universal lift of a formal group of height n over F_p comes as a summand in the formal completion of the Jacobian variety of a certain curve with p marked points. The action of G_n on the curve can be expressed in terms of the action on the marked points. This leads to a precise description of the Lubin-Tate action of G_n on E_{n*}.

We will call the plane curve C_p over $E = Z_p[[u_1, \cdots, u_{p-2}]]$ defined by

$$y^{p-1} = x^p + u_1 x^{p-1} + \cdots + u_{p-2}x^2 + (-1 - \Sigma_{i=1}^{p-2} u_i)x$$

the generalized Lengendre curve. Let $b = -1 - \Sigma_{i=1}^{p-2}$.

Let $I = (p, u_1, \cdots, u_{p-2})$. Then the reduction of the curve C_p to $F_p = E/I$ is the curve $\bar{C}_p$ defined by $y^{p-1} = x^p - x$. The roots of the right side are just $(0, \cdots, p-1)$ *mod* p. Hensel's lemma gives a lift of these roots to E and we can write the equation as

$$y^{p-1} = x(x-1)(x-e_1)\cdots(x-e_{p-2})$$

where $e_i = (i+1)\ mod\ I$.

Let $n = p-1$. We see that C_p is an n covering of a projective line with p marked points. Z/p acts on $\bar{C}_p$ via

$$x \mapsto x+1$$

$$y \mapsto y$$

and Z/n^2 acts by

$$x \mapsto \rho^n x$$

$$y \mapsto \rho^p y$$

and so G_n is a semi direct product of Z/p and Z/n^2.

Proposition 1 *For all primes p, C_p and $\bar{C}_p$ are non-singular curves of genus $g = (p-1)(p-2)/2$. The differentials $\omega_{i,j} = x^i\ dx/y^j$ for $0 \le i \le j \le p-3$ form a basis of the space of the holomorphic differentials.*

Over the the Jacobian variety we have a formal group F of dimension g.

Theorem 1 *For all primes p, F splits into $p-2$ summands of dimension $1, 2, \cdots, p-2$ respectively. The differentials dx/y corresponds to a summand of F of dimension one. This defines a one dimensional formal group F_{C_p} of height $p-1$.*

Theorem 2 *The formal group law F_{C_p} is a universal Lubin-Tate lift of a formal group of height n over F_p.*

Artin groups, projective arrangements and fundamental groups of smooth complex varieties

John J. Millson (U. Maryland)

What follows is a short account of my joint work [2] with Michael Kapovich. Our work concerns Serre's problem of determining which finitely presented groups are fundamental groups of smooth (not necessarily compact) complex algebraic varieties. The first examples of finitely presented groups which were not fundamental groups of smooth complex algebraic varieties were given by Morgan [5]. We find a new class of examples which consists of Artin groups. We now recall the definition of an Artin group we will use here.

Let Λ be a finite graph where two vertices v, w are connected by at most one edge e_{vw}, there are no loops (i.e. no vertex is connected by an edge to itself) and to each edge e is assigned an integer $\epsilon(e) \geq 2$. Let $\mathcal{V}(\Lambda)$ denote the set of vertices of Λ. The Artin group G^a_Λ is generated by $\{g_v : v \in \mathcal{V}(\Lambda)\}$ subject to the relations

$$\underbrace{g_v g_w g_v g_w \cdots}_{\epsilon(e_{vw}) \text{ factors}} = \underbrace{g_w g_v g_w g_v \cdots}_{\epsilon(e_{vw}) \text{ factors}}$$

as v, w vary over pairs of vertices of Λ which are connected by an edge. We warn the reader that our definition of Artin group is somewhat different from the usual one.

The result that there are Artin groups which are not the fundamental groups of smooth complex varieties is surprising because the basic examples of Artin groups, the Artin groups corresponding to finite Coxeter groups, are fundamental groups of smooth complex quasi-projective varieties. Also free groups and free abelian groups are Artin groups which are fundamental groups of smooth complex quasi-projective varieties.

Our results on Serre's problem follow by combining Theorems 1 and 2 below:

Theorem 1. *For any affine scheme S of finite type over Z there exists an Artin group G such that a Zariski open subset U of S is biregularly isomorphic to a Zariski open subset U in the character variety $Hom(G, PO(3))//PO(3)$. The subset U contains all real points of S.*

Roughly speaking, Theorem 1 implies that any algebraic germ (S, s_0) defined over Z can be realized inside a character variety of an Artin group. In [2] it is also proved that G may be chosen so that the basepoint s_0 of the germ corresponds to a representation ρ_0 which is irreducible and has *finite image.*

We combine Theorem 1 with the following theorem, a variant of a theorem of Hain [1].

Theorem 2. *Suppose M is a (not necessarily compact) connected smooth algebraic variety, G is a reductive algebraic group and $\rho : \pi_1(M) \to G$ is a representation with finite image. Then the germ $(Hom(\pi_1(M)), \rho)$ is a weighted-homogeneous cone with generators of weights 1 and 2 and relations of weights 2,3 and 4. Suppose further that there is a local cross section through ρ to the $Ad(G)$-orbits in $Hom(\pi_1(M), G)$. Then the quotient germ $(Hom(\pi_1(M), G)//G, [\rho])$ is a weighted-homogeneous cone with generators of weights 1 and 2 and relations of weights 2,3 and 4.*

Here we use the following

Definition. Let X be a real or complex analytic space, $x \in X$ and G a Lie group acting on X. We say that there is a local cross section through x to the G-orbits if there is a G-invariant open neighborhood U of x and a closed analytic subspace $S \subset U$ such that the natural map $G \times S \to U$ is an isomorphism of analytic spaces.

Proofs.

Theorem 1 is proved by showing that the character variety $Hom(G^a_\Lambda, PO(3))//PO(3)$ is isomorphic to a moduli space of line arrangements in the projective plane P^2. (The graph Λ is chosen to be bipartite and determines the abstract incidence relations of the arrangement.) We then prove a scheme theoretic version of a theorem of Mnev [4] to the effect that all affine schemes of finite type over Z occur as Zariski open subschemes of moduli spaces of line arrangements in P^2.

Theorem 2 is proved by giving the controlling differential graded Lie algebras $L^\cdot$ for deformations of ρ the structure of a mixed Hodge differential graded Lie algebra with the appropriate weights. The weight filtration induces a grading of the complete local ring $R_{L^\cdot}$ associated to $L^\cdot$, see [3].

References:

[1] R. Hain, in preparation.

[2] M. Kapovich and J.J. Millson, *On representation varieties of Artin groups, projective arrangements and fundamental groups of smooth complex algebraic varieties,* preprint.

[3] J.J. Millson, *Rational homotopy theory and deformation problems from algebraic geometry,* Proc. ICM 1990, vol. I, 549-558.

[4] N. Mnev, *The universality theorems on the classification problem of configuration varieties and convex polytope varieties,* Springer Lecture Notes in Math. 1346 (1988), 527-543.

[5] J. Morgan, *The algebraic topology of smooth algebraic varieties,* Publ. Math. IHES 48 (1978), 137-204.

Motivic homotopy theory

V. Voevodsky.

Consider the category Sm/k of smooth algebraic varieties over a field k. This category has a distinguished object $\mathbf{A}^1$ called the affine line over k. Let us say that two morphisms $f, g : X \to Y$ are elementary $\mathbf{A}^1$-homotopic if there is a morphism $X \times \mathbf{A}^1 \to Y$ whose restriction to $X \times \{0\}$ is f and to $X \times \{1\}$ is g. Let us say that f and g are $\mathbf{A}^1$-homotopic if they are equivalent with respect to the minimal equivalence relation generated by the relation of being elementary $\mathbf{A}^1$-homotopic. *Motivic homotopy theory* is the homotopy theory of smooth algebraic varieties (or more generally of smooth schemes over a base scheme S) which is based on the notion of $\mathbf{A}^1$-homotopy between morphisms.

The category Sm/k is very inconvenient for the purposes of homotopy theory. Indeed most of the standard constrcutions of the ordinary homotopy theory such as smash products, suspensions etc. can not be performed in Sm/k since the required colimits do not exist in this category. To deal with this problem one replaces Sm/k with the category $PreShv(Sm/k)$ of presheaves on Sm/k (i.e. of contravariant functors from Sm/k to the category of sets). Since any presheaf is a colimit of representable presheaves passing from Sm/k to $PreShv(Sm/k)$ we just formally add all colimits to Sm/k. For a number of technical reasons it is convenient to further enlarge the category $PreShv(Sm/k)$ and work with the category $\Delta^{op}PreShv(Sm/k)$ of simplicial presheaves on Sm/k.

One defines a closed model structure on $\Delta^{op}PreShv(Sm/k)$ which is called the $\mathbf{A}^1$-closed model strcuture in two steps.

First let us say that a morphism $f : \mathcal{X} \to \mathcal{Y}$ is a Nisnevich weak equivalence if for any smooth X over k and a point x of X the map of simplicial sets

$$\mathcal{X}(Spec\mathcal{O}^h_{X,x}) \to \mathcal{Y}(Spec\mathcal{O}^h_{X,x})$$

where $\mathcal{O}^h_{X,x}$ is the henselisation of the local ring of X in x is a weak equivalence of simplicial sets. The general theory of simplicial presheaves on sites developed by Joyal and Jardine implies that there is a closed model structure on $\Delta^{op}PreShv(Sm/k)$ where all monomorphisms are cofibrations and Nisnevich weak equivalences are weak equivalences. Denote the corresponding homotopy category by $\mathcal{H}_{Nis}(k)$.

An object $\mathcal{X}$ in $\mathcal{H}_{Nis}(k)$ is called $\mathbf{A}^1$-local if for any simplicial presheaf $\mathcal{Y}$ the map $Hom_{\mathcal{H}_{Nis}(k)}(\mathcal{Y}, \mathcal{X}) \to Hom_{\mathcal{H}_{Nis}(k)}(\mathcal{Y} \times \mathbf{A}^1, \mathcal{X})$ induced by the projection $\mathcal{Y} \times \mathbf{A}^1 \to \mathcal{Y}$ is a bijection. A morphism $\mathcal{Y} \to \mathcal{Y}'$ is called an $\mathbf{A}^1$-weak equivalence if for any $\mathbf{A}^1$-local $\mathcal{X}$ the map

$$Hom_{\mathcal{H}_{Nis}(k)}(\mathcal{Y}', \mathcal{X}) \to Hom_{\mathcal{H}_{Nis}(k)}(\mathcal{Y}, \mathcal{X})$$

is a bijection. As was shown by F. Morel and myself there is a proper simplicial closed model strcuture on $\Delta^{op}PreShv(Sm/k)$ where all monomorphisms are cofibrations and $\mathbf{A}^1$-weak equivalences are weak equivalences. The corresponding homotopy category $\mathcal{H}(k)$ is called the $\mathbf{A}^1$-homotopy category of (smooth) schemes over k.

Together with the unstable homotopy category $\mathcal{H}(k)$ one can also define the stable homotopy category $\mathcal{SH}(k)$. More generally to any E_∞-ring spectrum $\mathbf{R}$ over $\Delta^{op}PreShv(Sm/k)$ one can assign the stable homotopy category $\mathcal{SH}(k, \mathbf{R})$ of modules over $\mathbf{R}$. If R is a commutative ring it turns out that there are two different Eilenberg-Maclane spectra which can be associated to R. The corresponding generalized cohomology theories are called fake motivic cohomology with coefficients in R and motivic cohomology with coefficients in R respectively. Motivic cohomology with coefficients in $\mathbf{Z}$ are canonically isomorphic for smooth varieties to the higher Chow groups introduced by S. Bloch. We do not know at the moment any description of the fake motivic cohomology with coefficients in $\mathbf{Z}$ in terms of algebraic cycles.

Hilbert's 3rd Problem and 3-manifolds

Walter Neumann

In this talk I described the history of the scissors congruence problem and the current state of knowledge (due primarily to Bloch, Bokstedt, Brun, Dupont, Parry, Sah and Suslin) about the scissors congruence groups $\mathcal{P}(\mathbb{H}^3)$ and $\mathcal{P}(\mathbb{S}^3)$ for 3-dimensional hyperbolic and spherical geometry.

Namely, if $\mathbb{X}$ is either of these geometries and $\mathrm{Ker}\delta(\mathbb{X})$ is the kernel of the Dehn invariant $\delta: \mathcal{P}(\mathbb{X}) \to \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q}$ then it is known that the image of $vol: \mathrm{Ker}\delta(\mathbb{X}) \to \mathbb{R}$ is countable, that $\mathcal{P}(\mathbb{X})$ is uniquely divisible (and hence a $\mathbb{Q}$-vector space, and that the $\mathbb{Q}$-vector-subspace $\mathrm{Ker}\delta(\mathbb{X})$ is infinite dimensional. It follows that the first of the following two "standard conjectures" implies the second.

Conjecture (Sufficiency of Dehn invariant). *The volume map vol is injective on $Ker\delta(\mathbb{X})$.*

Conjecture (Rigidity). *$Ker\delta(\mathbb{X})$ is $\mathbb{Q}$-vector subspace of $\mathcal{P}(\mathbb{X})$ of countably infinite dimension.*

The first of these conjectures is generally considered to be very difficult, and unlikely to be resolved in the foreseeable future. It includes as a very special case a conjecture of Milnor about relations between values of the dilogarithm at roots of unity and is closely related to a conjecture of Zagier about relations among special values of the dilogarithm. The second conjecture is equivalent to the rigidity conjecture for $K_3(\mathbb{C})$ that $K_3^{ind}(\overline{\mathbb{Q}}) = K_3^{ind}(\mathbb{C})$, where $\overline{\mathbb{Q}}$ is algebraic numbers.

A hyperbolic 3-manifold represents a class in $\mathcal{P}(\mathbb{H}^3)$ but this is a slightly unsatisfactory invariant because scissors congruence is orientation insensitive. However, there is an "orientation sensitive" version of scissors congruence for which the kernel of Dehn invariant is the Bloch group $\mathcal{B}_\mathbb{C}$ of Bloch, Wigner and Suslin. Any hyperbolic 3-manifold M represents a class $\beta(M)$ in this group. This is a very computable invariant (at least modulo torsion) and it gives a lot of interesting information. A program "snap" written by Oliver Goodman at Melbourne computes this invariant and other arithmetic data.

This "orientation sensitive" scissors congruence is still unsatisfactory, for instance in that the invariant $\beta(M)$ determines Chern-Simons invariant, but only modulo $\mathbb{Q}$.

The Bloch group is defined in terms of $\mathbb{C} - \{0, 1\}$. A simple modification of the definiton to use instead a $\mathbb{Z} \times \mathbb{Z}$ cover of $\mathbb{C} - \{0, 1\}$ leads to an "extended Bloch group." This extended Bloch group is a $\mathbb{Q}/\mathbb{Z}$ extension of $\mathcal{B}_\mathbb{C}$ and is naturally isomorphic (modulo a possible kernel and cokernel of order dividing 4) to $H_3(PSL(2,\mathbb{C})^\delta)$. Any hyperbolic 3-manifold still represents a class in this extended group in a way that is directly computable from ideal triangulation. In particular one obtains a satisfactory "simplicial" computation of Chern-Simons invariant.

This is joint work with Jun Yang.