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# Hitchin Systems - Symplectic Hecke Correspondence and Two-dimensional Version

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**Abstract**

The aim of this paper is two-fold. First, we define symplectic maps between Hitchin systems related to holomorphic bundles of different degrees. We call these maps the Symplectic Hecke Correspondence (SHC) of the corresponding Higgs bundles. They are constructed by means of the Hecke correspondence of the underlying holomorphic bundles. SHC allows to construct Bäcklund transformations in the Hitchin systems defined over Riemann curves with marked points. We apply the general scheme to the elliptic Calogero-Moser (CM) system and construct SHC to an integrable SL($N,\mathbb{C}$) Euler-Arnold top (the elliptic SL($N,\mathbb{C}$)-rotator). Next, we propose a generalization of the Hitchin approach to 2d integrable theories related to the Higgs bundles of infinite rank. The main example is an integrable two-dimensional version of the two-body elliptic CM system. The previous construction allows to define SHC between the two-dimensional elliptic CM system and the Landau-Lifshitz equation.

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# 1 Introduction

Nowadays many examples of integrable one-dimensional and two-dimensional models are known. The problem of listing all of them, up to some equivalence, was solved for some particular forms of two-dimensional models [1]. The recently developed conception of duality for one-dimensional models [2] can shed light on the classification problem in analogy with string theory. In spite of this progress we are still far from understanding the structure of this universe. Therefore, the classification of integrable systems, apart from solving any individual equation, continues to be an actual task. We will consider integrable systems that have the Lax or Zakharov-Shabat representations. In these cases the gauge transformations of the accompanying linear equations lead essentially to the same systems, though their equations of motion differ in a significant way. For example, the non-linear Schrödinger model is gauge equivalent to the isotropic Heisenberg magnetic [3]. In such a manner the integrable system should be classified up to gauge equivalence, though it is not the only equivalence principle in their possible classifications. The crucial and delicate point of this approach is the exact definition of allowed gauge transformations, and it will be discussed here.

We restrict ourselves to Hitchin systems [4] and their two-dimensional generalizations that we will construct. The Hitchin construction establishes relations between finite dimensional integrable systems and the moduli space of holomorphic vector bundles over Riemann curves. The phase space of the integrable system is the cotangent bundle to the moduli space and the dual variables $\Phi$ are called the Higgs fields. The pair $(E, \Phi)$, where $E$ is a holomorphic bundle, is called the Higgs bundle. The Lax representation arises immediately in this scheme as the equation of motion and the Lax operator is just the Higgs field defined on shell. The $C^\infty$ gauge transformations of the Lax pair define the equivalent holomorphic bundles. The different gauge fixing conditions give equivalent integrable systems.

We consider the generalization of the Hitchin systems based on the quasi-parabolic Higgs bundles [5], where the Higgs fields are allowed to have the first order poles at the marked points on the base curve. The gauge transformations preserve the flag structures that arise at the marked points. The corresponding integrable systems were considered in [6, 7, 8, 9]. We loosen the smoothness condition of the gauge transformations and allow them to have a simple zero or a pole at one of the marked points. This type of gauge transformations (the upper and lower Hecke correspondence (HC) or the modifications) is suggested by the geometric Langlands program. HC changes the degree of the underlying bundles on $\pm 1$. We assume, that HC is consistent with flag structures on the source and target bundles. It allows to choose a canonical form of the modifications. HC can be lifted as the symplectic correspondence (SHC) to the Higgs bundles. In this way SHC define a map of Hitchin systems related to bundles of different degrees. One can consider an arbitrary chain of consecutive SHC attributed to different marked points. If the resulting transformation preserves the degree of bundle, then it defines the Bäcklund transformations of the Hitchin system related to the initial bundle, or the integrable discrete time map [21].

As an example, we consider a trivial holomorphic $\mathrm{SL}(N, \mathbb{C})$-bundle $E^{CM}$ ($\deg(E^{CM}) = 0$) over an elliptic curve with a marked point. The corresponding quasi-parabolic Higgs bundle leads to the elliptic $N$-body Calogero-Moser system (CM system). The upper SHC defines a map of the Higgs bundle related to $E^{CM}$ to the Higgs bundle $(E^{rot}, \Phi^{rot})$ with $\deg(E^{rot}) = 1$. SHC is generated by the $N$-th order matrix $\Xi$ with theta-functions depending on coordinates of the particles as the matrix elements. The system $(E^{rot}, \Phi^{rot})$ is the integrable $\mathrm{SL}(N, \mathbb{C})$-Euler-Arnold top ($\mathrm{SL}(N, \mathbb{C})$-elliptic rotator). The Lax pair for this top was proposed earlier [10]. The consecutive upper and lower SHC define the Bäcklund transformations of the both systems. Another way to construct a Bäcklund transformation is achieved by applying $N$ consecutive upper modifications, since they lead to equivalent Higgs bundle.

In the second part of the paper we try to gain insight into interrelation between integrable theories in dimension one and two. It is known that some one-dimensional integrable systems can be extended to the two-dimensional case without sacrificing the integrability. For example, the Toda field theory comes from the corresponding Toda lattice. To understand this connection we apply the Hitchin construction to two-dimensional systems. For this purpose we consider infinite rank bundles over the Riemann curves with marked points. The transition group of the bundles is the central extended loop group $\hat{L}(\mathrm{GL}(N, \mathbb{C}))$. If the central charge vanishes the theory in essence becomes one-dimensional. In the two-dimensional situation the Higgs field is a $\mathrm{gl}(N, \mathbb{C})$ connection on a circle $S^1$. In addition, we put coadjoint orbits of $\hat{L}(\mathrm{GL}(N, \mathbb{C}))$ at the marked points and in this way introduce the quasi-parabolic structure on the Higgs bundle of infinite rank. The monodromy of the Higgs field is a generating function for the infinite number of conservation laws. The equations of motion on the reduced phase space are the Zakharov-Shabat equations. The similar class of the Hitchin type systems from a different point of view was introduced recently by Krichever [11]. We consider in detail the case of $\hat{L}(\mathrm{SL}(2, \mathbb{C}))$-bundle over an elliptic curve with $n$ marked points. The Higgs bundle corresponds to the two-dimensional version of the elliptic Gaudin system. For the 1 marked point case we come to the 2d two-body elliptic CM theory. The upper SHC is working in the two-dimensional situation as well. It leads to the map of the 2-body elliptic CM field theory to the Landau-Lifshitz equation.[3] To summarize we consider here the following diagram:

$$\begin{tikzcd} \boxed{\begin{array}{c} 2-\text{body elliptic } CM \\ \text{system} \end{array}} \arrow[r] \arrow[d] & \boxed{\mathrm{SL}(2,\mathbb{C})-\text{elliptic rotator}} \arrow[d] \\ \boxed{\begin{array}{c} 2-\text{body elliptic } CM \\ \text{field theory} \end{array}} \arrow[r] & \boxed{\text{Landau-Lifshitz equation}} \end{tikzcd}$$

Figure 1: Interrelation in integrable theories

In fact, the upper SHC can be applied to the $\mathrm{SL}(N,\mathbb{C})$ case. The quadratic Hamiltonian of the $N$-body elliptic CM field theory was constructed in [11], but the $\mathrm{SL}(N,\mathbb{C})$ generalization of the Landau-Lifshitz equation is unknown.

It should be mentioned that the quantum version of $\mathrm{SL}(N,\mathbb{C})$ SHC has appeared in a different context long ago [12]. It was defined as a twist transformation of the quantum $R$-matrices, and Hasegawa [13] has constructed such types of twists that transform the dynamical elliptic $R$-matrix of Felder [14] to the non-dynamical $R$-matrix of Belavin [15]. It was proved [16] that the dynamical $R$-matrix corresponds to the elliptic Ruijsenaars system [17]. The later is the relativistic deformation of the elliptic CM system. In this way the Hasegawa twist is the quantization of SHC we have constructed, since the elliptic CM system and the elliptic Ruijsenaars system are governed by the same $R$-matrix [18].

# 2 Hitchin systems in the Čech description

In this section we consider vector bundles with structure group $G = \mathrm{GL}(N,\mathbb{C})$, or any simple complex Lie group.

## 2.1 The moduli space of holomorphic quasi-parabolic bundles in the Čech description.

Let $E$ be a trivial rank $r$ holomorphic vector bundle over a Riemann curve $\Sigma_n$ with $n$ marked points. Consider a covering of $\Sigma_n$ by open disks $\mathcal{U}_a$, $a = 1, 2 \ldots$. Some of them may contain one marked point $w_\alpha$. The holomorphic structure on $E$ can be described by the differential $d''$. On $\mathcal{U}_a$ it can be represented as

$$d'' = \bar\partial_a + \bar A_a, \ \bar A_a = h_a^{-1}\bar\partial_a h_a, \quad \bar\partial_a = \frac{\partial}{\partial \bar z_a}$$

where $z_a$ is a local coordinate on $\mathcal{U}_a$ , and $h_a$ is a $C^\infty$ $G$-valued function on $\mathcal{U}_a$. It is a section of the local sheaf $\Omega^0_{C^\infty}(\Sigma_n, \mathrm{Aut}\ E)$.

[3]The equivalence of these models was pointed out by A.Shabat.

The transition functions $g_{ab} = h_a h_b^{-1}$ are defined on the intersections $\mathcal{U}_{ab} = \mathcal{U}_a \cap \mathcal{U}_b$. They are holomorphic since $\bar{A}_a = \bar{A}_b$ on $\mathcal{U}_{ab}$

$$g_{ab} \in \Omega^0_{hol}(\mathcal{U}_{ab}, \text{Aut } E)\,.$$

The transformation $h_a \to f_a h_a$ by a function holomorphic on $\mathcal{U}_a$ ($f_a \in \Omega^0_{hol}(\mathcal{U}_a, \text{Aut } E)$) does not change $\bar{A}_a$. Similarly, the transformation $h_b \to f_b h_b$ by $f_b \in \Omega^0_{hol}(\mathcal{U}_b, \text{Aut } E)$ does not change $\bar{A}_b$. Then the holomorphic structures described by the transition functions $g_{ab}$ and $f_a g_{ab} f_b^{-1}$ are equivalent. Globally we have the collection of transition maps

$$\mathcal{L}^C_\Sigma = \{g_{ab}(z_a) = h_a(z_a) h_b^{-1}(z_b(z_a)),\ z_a \in \mathcal{U}_{ab},\ a, b = 1, 2 \ldots, \}\,. \tag{2.1}$$

They define holomorphic structures on $E$ or $P = \text{Aut}E$ depending on the choice of the representations.

The definition of the holomorphic structures by the transition functions works as well in the case if $\deg(E) \neq 0$ ($G = \text{GL}(N, \mathbb{C})$). They should satisfy the cocycle condition

$$g_{ab}(z) g_{bc}(z) g_{ca}(z) = \text{Id}, \quad z \in \mathcal{U}_a \cap \mathcal{U}_b \cap \mathcal{U}_c\,,$$

and

$$g_{ab} = g_{ba}^{-1}\,. \tag{2.2}$$

The degree of the bundle $E$ is defined as the degree of the linear bundle $L = \det g$.

We choose an open subset of stable holomorphic structures $\mathcal{L}^{C,st}_\Sigma$ in $\mathcal{L}^C_\Sigma$. The gauge group $\mathcal{G}^{hol}_\Sigma$ acts as the automorphisms of $\mathcal{L}^{C,st}_\Sigma$

$$g_{ab} \to f_a g_{ab} f_b^{-1},\ f_a = f(z_a),\ f_b = f_b(z_b(z_a)),\ f \in \mathcal{G}^{hol}_\Sigma\,. \tag{2.3}$$

We prescribe the local behavior of the gauge transformations $\mathcal{G}^{hol}_\Sigma$ at the marked points. Let

$$P_1, \ldots, P_\alpha, \ldots, P_n$$

be parabolic subgroups of $G$ attributed to the marked points. Then we assume that

$$f_a = \begin{cases} \tilde{f}^{(0)}_\alpha + z_\alpha f^{(1)}_\alpha + \ldots, & \tilde{f}_\alpha \in P_\alpha, \text{ if } z_\alpha = z - w_\alpha,\ w_\alpha \text{ is a marked point,} \\ f^{(0)}_a + z_a f^{(1)}_a + \ldots, & f_a \in G \text{ if } a \neq \alpha,\ (\mathcal{U}_a \text{ does not contain a marked point})\,. \end{cases} \tag{2.4}$$

It follows from (2.3) that the left action of the gauge group at the marked points preserves the flags

$$E_\alpha(s) \sim P_\alpha \setminus G, \quad E_\alpha = F_1(\alpha) \supset \cdots \supset F_{s_\alpha}(\alpha) \supset F_{s_\alpha+1}(\alpha) = 0\,. \tag{2.5}$$

The moduli space of the stable holomorphic bundles $\mathcal{M}_n(\Sigma, G)$ with the quasi-parabolic structure at the marked points is defined in Ref.[19] as the factor space under this action

$$\mathcal{M}_n = \mathcal{G}^{hol}_\Sigma \backslash \mathcal{L}^{C,st}_\Sigma\,. \tag{2.6}$$

For $G = \text{GL}(N, \mathbb{C})$ we have a disjoint union of components labeled by the corresponding degrees $d = c_1(\det E)\,:\, \mathcal{M}_n(\Sigma, G) = \bigsqcup \mathcal{M}^{(d)}_n$.

The tangent space to $\mathcal{M}_n(\Sigma, G)$ is isomorphic to $h^1(\Sigma, \text{End}E)$. Its dimension can be extracted from the Riemann-Roch theorem and for curves without marked points ($n = 0$)

$$\dim h^0(\Sigma, \text{End}E) - \dim h^1(\Sigma, \text{End}E) = (1 - g) \dim G\,.$$

For stable bundles $h^0(\Sigma, \mathrm{End}E) = 1$ and

$$\dim \mathcal{M}_0(\Sigma, G) = (g-1)N^2 + 1$$

for $\mathrm{GL}(N, \mathbb{C})$, and

$$\dim \mathcal{M}_0(\Sigma, G) = (g-1)\dim G$$

for simple groups. For elliptic curves one has

$$\dim h^1(\Sigma, \mathrm{End}E) = \dim h^0(\Sigma, \mathrm{End}E),$$

and

$$\dim \mathcal{M}_0^d = \mathrm{g.c.d.}(N, d)\,. \tag{2.7}$$

In this case the structure of the moduli space for the trivial bundles (i.e. with $\deg(E) = 0$ and, for example, for bundles with $\deg(E) = 1$ are different. We use this fact below.

For the quasi-parabolic bundles we have

$$\dim \mathcal{M}_n^d = \dim \mathcal{M}_0^d + \sum_{\alpha=1}^{n} f_\alpha\,, \tag{2.8}$$

where $f_\alpha$ is the dimension of the flag variety $E_\alpha$. In particular, for $G = \mathrm{GL}(N, \mathbb{C})$, we get

$$f_\alpha = \frac{1}{2}\left(N^2 - \sum_{i=1}^{s_\alpha} m_i^2(\alpha)\right), \quad m_i(\alpha) = \dim F_i(\alpha) - \dim F_{i+1}(\alpha)\,. \tag{2.9}$$

The space $\mathcal{L}_\Sigma^C$ is a sort of a lattice 2d gauge theory. Consider the skeleton of the covering $\{\mathcal{U}_a,\ a = 1, \ldots\}$. It is an oriented graph whose vertices $V_a$ are some fixed inner points in $\mathcal{U}_a$ and edges $L_{ab}$ connect those $V_a$ and $V_b$ for whose $U_{ab} \neq \emptyset$. We choose an orientation of the graph, saying that $a > b$ on the edge $L_{ab}$ and put the holomorphic function $z_b(z_a)$ which defines the holomorphic map from $\mathcal{U}_a$ to $\mathcal{U}_b$. Then the space $\mathcal{L}_\Sigma^C$ can be defined by the following data. To each edge $L_{ab}$, $a > b$ we attach a matrix valued function $g_{ab} \in G$ along with $z_b(z_a)$. The gauge fields $f_a$ are living on the vertices $V_a$ and the gauge transformation is given by (2.3).

## 2.2 Hitchin systems.

The Hitchin systems in the Čech description can be constructed in the following way [20]. We start from the cotangent bundle $T^*\mathcal{L}_{\Sigma_n}^C$ to the holomorphic structures on $P = \mathrm{Aut}E$ (2.1). Now

$$T^*\mathcal{L}_{\Sigma_n}^C = \{\eta_{ab}, g_{ab}|\ \eta_{ab} \in \Omega_{hol}^{(1,0)}(\mathcal{U}_{ab}, (\mathrm{End}E)^*),\ g_{ab} \in \Omega_{hol}^0(\mathcal{U}_{ab}, P)\}\,. \tag{2.10}$$

This bundle can be endowed with a symplectic structure by means of the Cartan-Maurer one-forms on $\Omega_{hol}^0(\mathcal{U}_{ab}, P)$.

Let $\Gamma^{\beta\gamma} = \Gamma_a^b(\beta\gamma)$ be a path in $\mathcal{U}_{ab}$ with the end points in the triple intersections $\beta \in \mathcal{U}_{abc} = \mathcal{U}_a \cap \mathcal{U}_b \cap \mathcal{U}_c$, $\gamma \in \mathcal{U}_{abd}$. We can put the data (2.10) on the fat graph corresponding to the covering $\{\mathcal{U}_a\}$. Its edges are $\Gamma_a^b(\beta\gamma)$ and $\Gamma_b^a(\gamma\beta)$ with the opposite orientations. We assume that the covering is such that the orientation of edges defines the oriented contours around the faces $\mathcal{U}_a$. The fields $\eta_{ab}, g_{ab}$ are attributed to the edge $\Gamma_a^b(\beta\gamma)$, while $\eta_{ba}, g_{ba}$ to $\Gamma_b^a(\gamma\beta)$. The last pair is not independent because $g_{ab}^{-1} = g_{ba}$. Its counterpart in the dual space is

$$\eta_{ab}(z_a) = g_{ab}(z_a)\eta_{ba}(z_b(z_a))g_{ab}^{-1}(z_a)\,. \tag{2.11}$$

The symplectic structure is defined by the two-form

$$\omega^C = \sum_{\text{edges}} \int_{\Gamma_a^b(\beta\gamma)} D\mathrm{tr}\left(\eta_{ab}(z_a) D g_{ab} g_{ab}^{-1}(z_a)\right) . \tag{2.12}$$

Since $\eta_{ab}$ and $g_{ab}$ are both holomorphic in $\mathcal{U}_{ab}$, the integral is independent on the choice of the path $\Gamma_a^b$ within $\mathcal{U}_{ab}$.

The symplectic form is invariant under the gauge transformations (2.3) supplemented by

$$\eta_{ab} \to f_a \eta_{ab} f_a^{-1}. \tag{2.13}$$

The set of invariant commuting Hamiltonians on $T^*\mathcal{L}_\Sigma^C$ is

$$I_{j,k}^C = \sum_{\text{edges}} \int_{\Gamma_a^b(\beta\gamma)} \nu_{(j,k)}^C(z_a) \mathrm{tr}(\eta_{ab}^{d_j}(z_a)) , \quad (k = 1, \dots, n_j) \tag{2.14}$$

where $d_j$ are the orders of the basic invariant polynomials corresponding to $G$ and $\nu_{j,k}^C$ are $(1-d_j, 0)$-differentials. They are related locally to the $(1-j,1)$-differentials by $\nu_{j,k}^D = \bar{\partial}\nu_{j,k}^C$ and

$$n_j = h^1(\Sigma, \mathcal{T}^{\otimes(d_j-1)}) = \begin{cases} (2d_j-1)(g-1) + (d_j-1)n, & (j = 1, \dots, r) \\ g, & j = 1 \text{ for } G = \mathrm{GL}(N, \mathbb{C}) . \end{cases}$$

The total number of independent Hamiltonians is equal to

$$\sum_{j=1}^{N} n_j = \mathcal{M}_0^d + \frac{1}{2} r(r+1)n .$$

This number is greater than the dimension of the moduli space $\mathcal{M}_n^d$ (2.8). There are $rn$ highest weight integrals, $(j = r)$, that become Casimir elements of coadjoint orbits after the symplectic reduction, that we will consider below.

Perform the symplectic reduction with respect to the gauge action (2.3), (2.13) of $\mathcal{G}_{\Sigma_n}^{hol}$ (2.4). The moment map is

$$\mu_{\mathcal{G}_\Sigma^{hol}}(\eta_{ab}, g_{ab}) : \; T^*\mathcal{L}_\Sigma^C \to \mathrm{Lie}^*(\mathcal{G}_\Sigma^{hol}).$$

Here the Lie coalgebra $\mathrm{Lie}^*(\mathcal{G}_\Sigma^{hol})$ is defined with respect to the pairing

$$\sum_{\text{edges}} \int_{\Gamma_a^b(\beta\gamma)} \mathrm{tr}(\xi_a \epsilon_a), \quad \epsilon_a \in \mathrm{Lie}(\mathcal{G}_\Sigma^{hol}).$$

Then locally we have

$$\xi_a = \begin{cases} \left(z_a^{-1}\tilde{\xi}_a + z_a^{-2}\tilde{\xi}_a^{(-2)} + \dots\right) dz_a, & \tilde{\xi}_a \in \mathrm{Lie}^*(P_\alpha), \; (\mathcal{U}_a \text{ contains a marked point } w_\alpha) \\ \left(z_a^{-1}\xi_a^{(-1)} + z_a^{-2}\xi_a^{(-2)} + \dots\right) dz_a, & \xi_a^{(-1)} \in \mathrm{Lie}^*(G) (\mathcal{U}_a \text{ does not contain a marked point}) . \end{cases} \tag{2.15}$$

The canonical gauge transformations (2.3),(2.13) of the symplectic form (2.12) are generated by the Hamiltonian

$$F_{\epsilon^{hol}} = \sum_{\text{edges}} \int_{\Gamma_a^b(\beta\gamma)} \mathrm{tr}(\eta_{ab}(z_a)\epsilon_a^{hol}(z_a)) - \mathrm{tr}(\eta_{ab}(z_a) g_{ab}(z_a) \epsilon_b^{hol}(z_b(z_a)) g_{ab}(z_a)^{-1}) =$$

$$= \sum_a \int_{\Gamma_a} \sum_b \mathrm{tr}(\eta_{ab}(z_a)\epsilon_a^{hol}(z_a))\,,$$

where $\Gamma_a$ is an oriented contour around $\mathcal{U}_a$.

The non-zero moment is fixed in a special way at the neighborhoods of the marked points. Let $\tilde{G}_\alpha \subset P_\alpha$ be the maximal semi-simple subgroup of the parabolic group $P_\alpha$ defined at the marked point $w_\alpha$. We drop for a moment the index $\alpha$ for simplicity. We choose an ordering in the Cartan subalgebra $\mathfrak{h}$ $\in$Lie$(G)$, which is consistent with the embedding $P \subset G$. Let $\tilde{\mathfrak{h}} = \mathfrak{h} \cap \tilde{G}$ be the Cartan subalgebra in $\tilde{G}$. Consider the orthogonal decomposition of $\mathfrak{h}^*$

$$\mathfrak{h}^* = \tilde{\mathfrak{h}}^* + \mathfrak{h}'^*\,.$$

We fix a vector $p^{(0)} \in \mathfrak{h}^*$ such that it is a generic element in $\mathfrak{h}'^*$ and

$$\langle p^{(0)}, \tilde{\mathfrak{h}}^* \rangle = 0\,, \tag{2.16}$$

where $\langle\ ,\ \rangle$ is the Killing scalar product in $\mathfrak{h}^*$. Since $\mathfrak{h}'^*$ $\subset$Lie$^*(P)$, we can take $\mu_{\mathcal{G}_\Sigma^{hol}}$ in the form

$$\mu_{\mathcal{G}_\Sigma^{hol}} = \mu_0 = \sum_{\alpha=1}^{n} p_\alpha^{(0)} z_\alpha^{-1} dz_\alpha,\ p^{(0)} \in \mathfrak{h}'^*\,, \tag{2.17}$$

where $z_\alpha = z - w_\alpha$ is a local coordinate in $\mathcal{U}_\alpha$. The moment equation $\mu_{\mathcal{G}_\Sigma^{hol}} = \mu_0$ can be read off from $F_{\epsilon^{hol}}$. It follows from the definition of Lie$^*(\mathcal{G}_\Sigma^{0,hol})$ that $\eta_{ab}$ is the boundary value of some holomorphic or meromorphic one-form $H_a$ defined on $\mathcal{U}_a$ via

$$\eta_{ab}(z_a) = H_a(z_a),\ \text{for } z_a \in \mathcal{U}_{ab},\ H_a \in \Omega_{hol}^{(1,0)}(\mathcal{U}_a, \mathrm{End}^*(E))\,, \tag{2.18}$$

where

$$H_a = \begin{cases} z_a^{-1} p_\alpha^{(0)} + H_a^{(0)} + z_a H_a^{(1)} + \dots, & \text{if } \mathcal{U}_a \text{ contains a marked point } w_\alpha \\ H_a^{(0)} + z_a H_a^{(1)} + \dots, & \text{if } \mathcal{U}_a \text{ does not contain a marked point}\,. \end{cases}$$

The gauge fixing means that the transition functions $g_{ab}$ are elements of the moduli space $\mathcal{M}_n^d(\Sigma, E)$. Thus, the reduced system is a symplectic quotient

$$\mathcal{G}_\Sigma^{hol} \backslash\backslash T^* \mathcal{L}_\Sigma^C = \mathcal{G}_\Sigma^{hol} \backslash \mu^{-1}(\mu_0)\,.$$

We set off the zero modes $g_{\alpha b}^{(0)} \in \tilde{G}_\alpha$ of the transition functions in the symplectic form on the reduced space (see (2.12))

$$\omega^C = \sum_{\text{edges}} \int_{\Gamma_a^b(\beta\gamma)} D\mathrm{tr}\left(\eta_{ab}(z_a) D g_{ab} g_{ab}^{-1}(z_a)\right) + 2\pi i \sum_{\alpha=1}^{n} \sum_b D\mathrm{tr}\left(p_\alpha^{(0)} D g_{\alpha b}^{(0)} (g_{\alpha b}^{(0)})^{-1}\right)\,.$$

The last sum defines the Kirillov-Kostant symplectic forms on the set of coadjoint orbits $\mathcal{O}(n) = (\mathcal{O}_1, \dots \mathcal{O}_\alpha, \dots, \mathcal{O}_n)$, where

$$\mathcal{O}_\alpha = \{p_\alpha \in \mathrm{Lie}^*(G) \mid p_\alpha = (g_\alpha^{(0)})^{-1} p_\alpha^{(0)} g_\alpha^{(0)}\}\,. \tag{2.19}$$

Note that $\dim(\mathcal{O}_\alpha) = 2f_\alpha$ (2.9).

**Remark 2.1** *It is possible to construct another type of orbit $\mathcal{O}'_\alpha$ of the same dimension. There exist elements $p_\alpha^{'(0)}$ that belong to the complements of Lie$^*(\tilde{G}_\alpha)$ in Lie$^*(P_\alpha)$ such that the orbit*

$$\mathcal{O}'_\alpha = \{p_\alpha = (g_\alpha^{(0)})^{-1} p_\alpha^{'(0)} g_\alpha^{(0)}\}\,.$$

*is symplectomorphic to the cotangent bundles $T^*E_\alpha$ to the corresponding flags $E_\alpha$ (2.5), while $\mathcal{O}_\alpha$ (2.19) is a torsor over $\mathcal{O}'_\alpha$.*

## 2.3 Standard description of the Hitchin system.

The standard approach of the Hitchin systems [4] is based on the description of the holomorphic bundles in terms of the operator $d''$. The upstairs phase space has the form

$$T^*\mathcal{L}^D_{\Sigma_n} = \{\Phi, d'' \mid \Phi \in \Omega^{(1,0)}_{C^\infty}(\Sigma_n, \mathrm{End}^*\ E)\}\,, \tag{2.20}$$

where $\Phi$ is called the Higgs field. The symplectic form

$$\omega^D = \int_{\Sigma_n} \mathrm{tr}(D\Phi \wedge D\bar{A}) \tag{2.21}$$

is invariant under the action of the gauge group

$$\mathcal{G}^{C^\infty}_\Sigma = \{f \in \Omega^0_{C^\infty}(\Sigma_n, \mathrm{Aut}\ V)\},$$

$$\Phi \to f^{-1}\Phi f\,,\ \ \bar{A} \to f^{-1}\bar{\partial} f + f^{-1}\bar{A} f\,. \tag{2.22}$$

The gauge invariant integrals take the evident form (compare with (2.14))

$$I^D_{j,k} = \int_{\Sigma_n} \nu^D_{(j,k)} \mathrm{tr}(\Phi^{d_j})\,,\quad (k = 1, \ldots, n_j) \tag{2.23}$$

where $\nu^D_{j,k}$ are $(1-j,1)$-differentials on $\Sigma_n$. The symplectic reduction with respect to this action leads to the moment map

$$\mu\ :\ T^*\mathcal{L}^D_{\Sigma_n} \to \mathrm{Lie}^*(\mathcal{G}^{C^\infty}_\Sigma)\quad \mu = \bar{\partial}\Phi + [\bar{A}, \Phi]\,.$$

The Higgs field $\Phi$ is related to $\eta$ in a simple way

$$\eta_{ab} = h_a^{-1}\Phi h_a|_{\mathcal{U}_{ab}}\,,$$

and $\bar{A}_a = h_a^{-1}\bar{\partial}_a h_a$. The holomorpheity of $\eta$ is equivalent to the equation $\mu(\Phi, \bar{A}) = 0$, and $\Phi$ has the same simple poles as $H_a$ (2.18). For simplicity, we call $\eta$ the Higgs field. The bundle $E$ equipped with the one-form $\eta$ is called the Higgs bundle.

## 2.4 Modified Čech description of the moduli space.

We modify the Čech description of the moduli space of $\mathrm{GL}(N,\mathbb{C})$-vector bundles in the following way. Consider a formal (or rather small) disk $D$ embedded into $\Sigma$ in such way that its center maps to the point $w$.

Consider first the case of $G = \mathrm{PGL}(N,\mathbb{C})$-bundles. The moduli space $\mathcal{M}^d_n$ is the quotient of the space $\mathcal{G}_{D^*}$ of $G$-valued functions $g$ on the punctured disk $D^*$ by the right action of the group $\mathcal{G}_{out}$ of $G$-valued holomorphic functions on the complement to $w$ and by the left action of the group $\mathcal{G}_{int}$ of $G$-valued holomorphic functions on the disk:

$$\mathcal{M}^d_n = \mathcal{G}_{int}\backslash\mathcal{G}_{D^*}/\mathcal{G}_{out},\quad g \to h_{int} g h_{out}\,.$$

We assume that these transformations preserve the quasi-parabolic structure of the vector bundle $E$.

Now consider $\mathrm{GL}(N,\mathbb{C})$-bundles. The group $\mathrm{GL}(N,\mathbb{C})$ is not semi-simple. One has an action of the Jacobian $Jac(\Sigma)$ on the moduli space of vector bundles by the tensor multiplication, and the quotient is equal to the space of $\mathrm{PGL}(N,\mathbb{C})$-bundles. This follows from the exact sequence

$$0 \to \mathcal{O}^* \to \mathrm{GL}(N,\mathbb{C})(\mathcal{O}) \to \mathrm{PGL}(N,\mathbb{C})(\mathcal{O}) \to 0\,.$$

Hence locally the moduli space of vector bundles is product of the Jacobian of the curve and moduli space of $\mathrm{PGL}(N,\mathbb{C})$-bundles. We associate to the pair $(g,L)$ the bundle which is equal to $\mathbb{C}^N \otimes L$ on the complement of a point, and the transition function on the punctured disk is $g$.

Assume for simplicity that there is only one marked point and it coincides with the center of $D^*$. Let $z$ be the local coordinate on $D^*$. Then the gauge group $\mathcal{G}_{D^*}$ can be identify with the loop group $L(\mathrm{GL}(N,\mathbb{C}))$. A parabolic subgroup of $L(\mathrm{GL}(N,\mathbb{C}))$ has the form

$$\mathcal{G}_{int} \sim P\cdot \exp L^+(\mathrm{gl}(N,\mathbb{C})), \quad L^+(\mathrm{gl}(N,\mathbb{C})) = \sum_{j>0} \mathfrak{g}_j z^j, \mathfrak{g}_j \in \mathrm{gl}(N,\mathbb{C})\,,$$

where $P$ is a parabolic subgroup in $\mathrm{GL}(N,\mathbb{C})$ . The quotient $\mathcal{G}_{int}\backslash\mathcal{G}_{D^*}$ is the infinite-dimensional flag variety, corresponding to the finite-dimensional flag $E(s)$ (see (2.5))

$$\cdots \supset F_s z^{-1} + \sum_{j\geq 0} E z^j \supset \sum_{j\geq 0} E z^j \supset F_1 + \sum_{j>0} E z^j \supset \cdots \tag{2.24}$$

$$\supset F_s + \sum_{j>0} E z^j \supset \sum_{j>1} E z^j \supset \cdots\,.$$

The moduli space $T^*\mathcal{M}_n^d$ of the $\mathrm{GL}(N,\mathbb{C})$ Higgs bundles can be locally identified with the Hamiltonian quotient

$$\mathcal{G}_{in}\backslash\backslash T^*\mathcal{G}_{D^*} \times T^*Jac(\Sigma)//\mathcal{G}_{out}\,.$$

The cotangent bundle of $T^*\mathcal{G}_{D^*}$ is identified with the space of pairs $(g,\eta)$, where $\eta$ is a $\mathrm{Lie}^*(G)$-valued one-form. The canonical one-form is equal to $\mathrm{res}_w(\mathrm{tr}(\eta D g g^{-1}))$. The second component $T^*Jac(\Sigma)$ is the pair $(t,L)$, where $L$ is a point of $Jac(\Sigma)$ and $t$ is the corresponding co-vector. The canonical one-form is $\langle t, DL\rangle_{Jac}$ and the brackets denote the pairing between vectors and co-vectors on the Jacobian.

The group $\mathcal{G}_{out}$ acts as $(g,\eta)\to(gh_{out},\eta)$. The corresponding momentum constraint can be reformulated as the following condition: $g\eta g^{-1}$ is the restriction of some $\mathrm{Lie}^*(G)$-valued form on the complement to $w$. The group $\mathcal{G}_{int}$ acts as $(g,\eta)\to(h_{int}g, h_{int}\eta h_{int}^{-1})$. The momentum constraint means that $\eta$ is holomorphic in $\mathcal{U}_w$ if $w$ is a generic point, or it has the first order pole if $w$ is a marked point.

# 3 Symplectic Hecke correspondence

In this section we consider only $\mathrm{GL}(N,\mathbb{C})$-bundles.

## 3.1 Hecke correspondence.

Let $E$ and $\tilde{E}$ be two quasi-parabolic bundles over $\Sigma$ of the same rank. Assume that there is a map $\Xi^+ : E \to \tilde{E}$ such that it is an isomorphism on the complement to $w$ and it has one-dimensional cokernel at $w$ :

$$0 \to E \stackrel{\Xi^+}{\to} \tilde{E} \to \mathbb{C}|_w \to 0\,.$$

**Definition 3.1** *The upper Hecke correspondence (HC) at the point $w\in\Sigma$ of order 1 is an auto-correspondence $\mathfrak{H}_w^+$ on $\mathcal{M}$ related to the map $\Xi^+$.*

The correspondence $\mathfrak{H}_w^+$ has components placed only at $\mathcal{M}^{(d)}\times\mathcal{M}^{(d+1)}$. On the complement to the point $w$ consider the map

$$E \stackrel{\Xi^-}{\leftarrow} \tilde{E}\,,$$

such that $\Xi^-\Xi^+$ =Id. It defines the lower HC $\mathfrak{H}_w^-$ at the point $w$.

Let $w=w_\alpha$ be a marked point and the corresponding flag $E_\alpha$ (2.5) has a one-dimensional subspace $F_{s_\alpha}$. In this case the HC $\mathfrak{H}^\pm$ can be chosen in the following way. Let

$$\tilde{E}_\alpha=\tilde{F}_1(\alpha)\supset\cdots\supset\tilde{F}_{s_\alpha}(\alpha)\supset\tilde{F}_{s_\alpha+1}(\alpha)=0$$

be the image of the flag variety $E_\alpha$ under the action of $\Xi^+_\alpha$. In agreement with Definition 3.1 the map $F_{s_\alpha}(\alpha)\to\tilde{F}_1(\alpha)\sim\tilde{E}_\alpha$ has a one-dimensional cokernel. Moreover, we assume that $\Xi^+_\alpha$ acts as

$$\Xi^+_\alpha\ :\ F_1(\alpha)\to\tilde{F}_2(\alpha),\ldots,F_{s_\alpha}(\alpha)\to\tilde{F}_1(\alpha).$$

Then $\Xi^+_\alpha$ can be gauge transformed to the canonical form

$$\Xi^+_\alpha=\left(\begin{array}{cc}0 & \mathrm{Id}_{N-1}\\ z_\alpha & 0\end{array}\right). \tag{3.1}$$

It is just the Coxeter transformation in the loop algebra $L(\mathrm{gl}(N,\mathbb{C}))$, that has been defined on the punctured disk $D^*_\alpha\subset\mathcal{U}_\alpha$ in Subsection 2.4. The Coxeter transformation provides the map $E^d\to\tilde{E}^{d+1}$. On the infinite-dimensional flag variety chain (2.24) it acts as the left shift operator. In a similar way the lower HC can be transformed to the form

$$\Xi^+_\alpha=\left(\begin{array}{cc}0 & z_\alpha^{-1}\\ \mathrm{Id}_{N-1} & 0\end{array}\right). \tag{3.2}$$

In this case $E^{d+1}\to\tilde{E}^d$ and the flag variety is shifted to the left.

### 3.2 Symplectic Hecke correspondence

. Consider two Higgs bundles $(E,\eta)$ and $(\tilde{E},\tilde{\eta})$, where $E,\tilde{E}$ are in the HC $\mathfrak{H}_a$ at $w_a\in\Sigma$.

**Definition 3.2** *The upper symplectic Hecke correspondence (SHC) is the map of the Higgs bundles*

$$\mathfrak{S}^+_a:\ (E,\eta)\to(\tilde{E},\tilde{\eta})\,,$$

*where $E$ and $\tilde{E}$ are in Hecke correspondence $\mathfrak{H}^+_{w_a}$.*

The lower SHC $\mathfrak{S}^-_a$ is defined in a similar way.

Let $w_\alpha$ be a marked point. The Higgs field $\eta$ has the first order poles at $w_\alpha$ and the residue $p^{(0)}_\alpha$ of $\eta$ defines an orbit $\mathcal{O}_\alpha$. Then the correctness of the definition in this case follows from the following

**Lemma 3.1** *The gauge transforms $\Xi^\pm_\alpha$ corresponding to $\mathfrak{H}^\pm_\alpha$*
- *do not change singularity of the Higgs field at $w_\alpha$;*
- *are symplectic.*

*Proof.*
The choice of $p^{(0)}_\alpha$ (2.16) is consistent with the canonical forms (3.1),(3.2) of $\Xi^\pm_\alpha$ and their action does not change the order of the pole. The action is symplectic since $\Xi^\pm_\alpha$ depends only on $p^{(0)}_\alpha$.
□

### 3.3 SHC and skew-conormal bundles.

For any smooth correspondence $Z$ between equi-dimensional varieties $X$ an $Y$ we define a *skew-conormal bundle* $\mathcal{SN}^*Z$ of $Z$ as follows. Let

$$\nu = (\nu_X, \nu_Y) \in T^*_z(X \times Y) = T^*_x X \oplus T^*_y Y$$

be a co-vector attached to a point $z = (x, y) \in Z \subset X \times Y$. It belongs to the fiber $\mathcal{SN}^*_z Z$ of the skew-conormal bundle $\mathcal{SN}^*Z$ at the point $z$ iff for any vector $v = (v_X, v_Y)$ tangent to $Z$

$$\nu_X(v_X) = \nu_Y(v_Y)\,.$$

Note that for the conormal bundle one has the opposite sign: $\nu_X(v_X) = -\nu_Y(v_Y)$.

The total space of the skew-conormal bundle is a Lagrangian subvariety of the total space of the cotangent bundle $T^*(X \times Y)$ with respect to the symplectic form $\omega_X - \omega_Y$, where $\omega$ denotes the canonical symplectic form on the cotangent bundle. So, the skew-conormal bundle of a correspondence is rather close to the graph of a symplectic map between cotangent bundles.

**Proposition 3.1** *The graph of the SHC $\mathfrak{S}_w$ is isomorphic to the skew-conormal bundle $\mathcal{SN}^*\mathfrak{H}_w$ of the usual Hecke correspondence $\mathfrak{H}_w$.*

*Proof.*
As it was explained in Subsection 2.4, a $\mathrm{GL}(N,\mathbb{C})$-bundle $E$ is determined by the pair $(g, L)$ in a neighborhood of a point $w \in \Sigma$. An upper HC of $E$ corresponds to $(\tilde{g}, L)$, where $\tilde{g} = \Xi g$ and

$$\bar{\partial}\Xi = 0 \text{ in } \mathcal{U}_w, \quad \mathrm{ord}_w(\det(\Xi)) = 1\,. \tag{3.3}$$

Therefore, the skew-conormal bundle $\mathcal{SN}^*\mathfrak{H}_w$ of the HC $\mathcal{SN}^*\mathfrak{H}_w$ can be described by the data

$$(g, \tilde{g}; \eta, \tilde{\eta}; t, \tilde{t}, L), \;\; \tilde{g} = \Xi g\,,$$

where $\Xi$ satisfies (3.3), and

$$\langle t, DL\rangle_{Jac} = \langle \tilde{t}, DL\rangle_{Jac} \tag{3.4}$$

$$\mathrm{res}_w(\mathrm{Tr}(\tilde{\eta} D\tilde{g}\tilde{g}^{-1})) = \mathrm{res}_w(\mathrm{Tr}(\eta D g g^{-1})) \tag{3.5}$$

for any variations of $g$ and $\tilde{g}$, that preserve properties of $\Xi = g^{-1}\tilde{g}$. The first condition (3.4) means that $t = \tilde{t}$.

The condition (3.5) can be rewritten as

$$\mathrm{res}_w\left(\mathrm{tr}(\tilde{\eta} D\Xi\Xi^{-1} + (\Xi^{-1}\tilde{\eta}\Xi - \eta)g^{-1}D\,g)\right) = 0\,.$$

Since variations of $g$ and $\Xi$ are independent, both terms in the last expression must vanish separately:

$$\mathrm{res}_w(\mathrm{tr}(\tilde{\eta} D\Xi\Xi^{-1})) = 0\,, \tag{3.6}$$

$$\mathrm{res}_w\left(\mathrm{tr}(\Xi^{-1}\tilde{\eta}\Xi - \eta)g^{-1}D\,g)\right) = 0\,. \tag{3.7}$$

Consider first the case when $w$ is not a marked point. Then we will demonstrate that (3.6) means that $\eta' = \Xi^{-1}\tilde{\eta}\Xi$ is holomorphic in $w$. Consider the value of $\Xi$ at zero, this matrix has rank $N-1$. Denote by $K$ its kernel and by $I$ its image. An essential variation of $\Xi$ corresponds to the variations of its image, so it is a map $I \to \mathbb{C}^N/I$. This variation corresponds to the right action: $D\,\Xi = \Xi\epsilon$. The singular part $\Xi^{-1}_{sing}$ of $\Xi^{-1}$ at $w$ is an operator of rank 1. Its kernel equals

$I$ and its image equals $K$, so the singular part $\eta'_{sing}$ of $\eta'$ is a map from $\mathbb{C}^N/\mathrm{Ker}(\Xi^{-1}_{sing}) = \mathbb{C}^N/I$ to $\mathrm{Im}(\Xi^{-1}_{sing}) = I$. The first condition can be rewritten as:

$$0 = \mathrm{res}_w(\mathrm{tr}(\tilde{\eta} D\Xi\Xi^{-1})) = \mathrm{res}_w(\mathrm{tr}(\eta'\Xi^{-1}D\Xi)) = \mathrm{tr}(\eta'_{sing}\epsilon)$$

for any $\epsilon \in \mathrm{Hom}(I, \mathbb{C}^r/I)$. The space $\mathrm{Hom}(I, \mathbb{C}^r/I)$ is dual to $\mathrm{Hom}(\mathbb{C}^r/I, I)$, so $\eta'_{sing}$ vanishes and $\eta'$ is holomorphic.

Note that $\eta'$ determines some Higgs field for $g$. Indeed, it is holomorphic in $\mathcal{U}_w$ and $g^{-1}\eta' g = \tilde{g}^{-1}\tilde{\eta}\tilde{g}$ is the restriction of some one-form on the complement to $w$. As the canonical one-form $\mathrm{tr}(\eta D g g^{-1})$ is non-degenerate on $T^*\mathcal{M}^d_n$, from the second condition (3.7) we conclude that $\eta' = \eta$.

If $w$ is a marked point then $\Xi$ is fixed and it maps the Higgs field $\eta$ into the Higgs field $\tilde{\eta}$ (Lemma 3.1). There is no variation of $\Xi$ and we immediately have that again $\eta' = \eta$. $\square$

### 3.4 Bäcklund transformation.

The gauge transformations $\Xi^{\pm}_{\alpha}$ related to the SHC $\mathfrak{S}^{\pm}_{\alpha}$ depend on the marked point $w_\alpha$ and the choice of the eigenvectors $e^{\pm}$ of $p^{(0)}_{\alpha}$. Then the gauge transformations define the maps of the Hitchin systems

$$\mathfrak{S}^{+}_{\alpha} \sim \xi^{e^+_j,\alpha} : \; T^*\mathcal{M}^{(d)}(\Sigma_n, G) \to T^*\mathcal{M}^{(d+1)}(\Sigma_n, G)\,, \tag{3.8}$$

$$\mathfrak{S}^{-}_{\alpha} \sim \xi_{e^-_j,\beta} : \; T^*\mathcal{M}^{(d)}(\Sigma_n, G) \to T^*\mathcal{M}^{(d-1)}(\Sigma_n, G)\,. \tag{3.9}$$

Consider consecutive upper and lower modifications

$$\xi^{e_{j_1},\alpha_1}_{e_{j_2},\alpha_2} = \xi^{e_{j_1},\alpha_1} \cdot \xi_{e_{j_2},\alpha_2}\,. \tag{3.10}$$

Since $\deg(E)$ does not change it is a symplectic transform $T^*\mathcal{M}^d_n(\Sigma, E)$. In this way $\xi^{e_{j_1},\alpha_1}_{e_{j_2},\alpha_2}$ maps solutions of the Hitchin hierarchy into solutions.

**Corollary 3.1** *The map (3.10) is* the Bäcklund transformation, *parameterized by a pair of marked points* $(w_{\alpha_1}, w_{\alpha_2})$ *and a pair of the eigenvectors* $(e_{j_1}, e_{j_2})$.

We can generalize (3.10) as

$$\xi^{e_{j_1},\alpha_{j_1};\ldots;e_{j_s},\alpha_{j_s}}_{e_{i_1},\alpha_{i_1};\ldots;e_{i_s},\alpha_{i_s}} = \xi^{e_{j_1},\alpha_{j_1}} \cdot \xi_{e_{i_1},\alpha_{i_1}} \cdots\,.$$

For the generic orbits we have $N$ parameters at a marked point. Because the Bäcklund transformation is a canonical one we can consider a discrete Hamiltonian system defined on the phase space $T^*\mathcal{M}^d_n(\Sigma, E)$. They pairwise commute and in terms of the angle variables generate a lattice in the Liouville torus [21, 22]. In our case the dimension of the Liouville torus is equal to $\dim \mathcal{M}^d_n$ (2.8), but the lattice we have constructed has in general a smaller dimension.

Note that when $\Sigma_n$ is an elliptic curve, the Hitchin systems corresponding to $d = kN$ and $d = 0$ ($d = \deg(V)$) are equivalent. Hence, in this case one can construct some Bäcklund transformations by applying the upper SHC $N$ times.

# 4 Elliptic CM system - elliptic $\mathrm{SL}(N,\mathbb{C})$-rotator correspondence

## 4.1 Elliptic CM system

. The elliptic CM system was first introduced in the quantum version [23]. It is defined on the phase space

$$\mathcal{R}^{CM}=\left(\mathbf{v}=(v_1,\dots,v_N),\ \mathbf{u}=(u_1,\dots,u_N),\quad \sum_j v_j=0,\ \sum_j u_j=0\right)\,, \tag{4.1}$$

with the canonical symplectic form

$$\omega^{CM}=(D\mathbf{v}\wedge D\mathbf{u})\,.$$

The second order with respect to the momenta $\mathbf{v}$ Hamiltonian is

$$H_2^{CM}=\frac{1}{2}\sum_{j=1}^{N}v_j^2+\nu^2\sum_{j>k}\wp(u_j-u_k;\tau)\,.$$

It was established in [7, 24] that the elliptic CM system can be derived in the Hitchin approach. The Lax operator $L^{CM}$ is the reduced Higgs field $\eta$ over the elliptic curve

$$E_\tau=\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$$

with a marked point $z=0$. In this way the phase space $\mathcal{R}^{CM}$ is the space of pairs

($SL_N$-bundle $V$ over $E_\tau$, the Higgs field $L^{CM}$ on this bundle (4.4)).

The bundle is determined by the transition functions (the multipliers)

$$Id_N:\quad z\to z+1\,, \tag{4.2}$$

$$\mathbf{e}(\mathbf{u})=\mathrm{diag}(\mathbf{e}(u_1),\dots,\mathbf{e}(u_N)):\quad z\to z+\tau\,,$$

where $\mathbf{e}$ is defined in (A.1). The Lax operator $L^{CM}(z)$ is the quasiperiodic one-form

$$L^{CM}(z+1)=L^{CM}(z),\quad L^{CM}(z+\tau)=\mathbf{e}(-\mathbf{u})L^{CM}(z)\mathbf{e}(\mathbf{u})\,. \tag{4.3}$$

It is the $N$-th order matrix

$$L^{CM}=P+X,\ \text{where } P=\mathrm{diag}(v_1,\dots,v_N),\quad X_{jk}=\nu\phi(u_j-u_k,z)\,, \tag{4.4}$$

and $\phi$ is defined as (B.5). It has the first order pole at the marked point with the residue in the minimal orbit

$$\mathrm{Res}_{z=0}(L^{CM}(z))=L^{CM}_{-1}=\nu\begin{pmatrix}0&1&\cdots&1\\1&0&\cdots&1\\\vdots&\vdots&\ddots&\vdots\\1&1&\cdots&0\end{pmatrix}, \tag{4.5}$$

so the column-vector $e_1=(1,1,\cdots,1)$ is an eigen-vector $e_1$

$$L^{CM}_{-1}e_1=(N-1)\nu e_1. \tag{4.6}$$

There is also an $(N-1)$-dimensional eigen-subspace $T_{N-1}$ corresponding to the degenerate eigen-value $-\nu$

$$L_{-1}^{CM}e_{\mathbf{a}} = -\nu e_{\mathbf{a}}, \quad e_{\mathbf{a}} = (a_1, \ldots, a_N), \ (\sum_n a_n = 0)\,. \tag{4.7}$$

The $M^{CM}$ operator corresponding to $H_2^{CM}$ has the form

$$M^{CM} = -D + Y, \text{ where } D = \mathrm{diag}(Z_1, \ldots, Z_N), \quad Y_{jk} = y(u_j - u_k, z)\,, \tag{4.8}$$

$$Z_j = \sum_{k \neq j} \wp(u_j - u_k), \quad y(u,z) = \frac{\partial \phi(u,z)}{\partial u},.$$

### 4.2 The elliptic $\mathrm{SL}(N,\mathbb{C})$-rotator.

The elliptic $\mathrm{SL}(N,\mathbb{C})$-rotator is an example of the Euler-Arnold top [16]. It is defined on a coadjoint orbit of $\mathrm{SL}(N,\mathbb{C})$:

$$\mathcal{R}^{rot} = \{\mathbf{S} \in \mathrm{sl}(N,\mathbb{C}), \quad \mathbf{S} = g^{-1}\mathbf{S}^{(0)}g\}\,. \tag{4.9}$$

The phase space $\mathcal{R}^{rot}$ is equipped with the Kirillov-Kostant symplectic form

$$\omega^{rot} = \mathrm{tr}(\mathbf{S}^{(0)}Dgg^{-1}Dgg^{-1})\,.$$

The Hamiltonian is defined as

$$H^{rot} = -\frac{1}{2}\mathrm{tr}(\mathbf{S}J(\mathbf{S}))\,, \tag{4.10}$$

where $J$ is a linear operator on $\mathrm{Lie}(\mathrm{SL}(N,\mathbb{C}))$. The inverse operator is called the inertia tensor. The equation of motion takes the form

$$\partial_t \mathbf{S} = [J(\mathbf{S}), \mathbf{S}]. \tag{4.11}$$

We consider here a special form $J$, that provides the integrability of the system. Let

$$J(\mathbf{S}) = \mathbf{J}\cdot\mathbf{S} = \sum_{mn} J_{mn}S_{mn}\,,$$

where $\mathbf{J}$ is a $N$-th order matrix

$$\mathbf{J} = \{J_{mn}\} = \left\{\wp\left[\begin{array}{c} m \\ n \end{array}\right]\right\}, \quad (m,n = 1,\ldots,N), \quad \wp\left[\begin{array}{c} m \\ n \end{array}\right] = \wp(\frac{m+n\tau}{N};\tau)\,. \tag{4.12}$$

We write down (4.11) in the basis of the sin-algebra $\mathbf{S} = S_{mn}E_{mn}$ (see (A.4))

$$\partial_t S_{mn} = \frac{N}{\pi}\sum_{k,l} S_{k,l}S_{m-k,n-l}\wp\left[\begin{array}{c} k \\ l \end{array}\right]\sin\frac{\pi}{N}(kn - ml)\,. \tag{4.13}$$

The elliptic rotator is a Hitchin system [7]. We give a proof of this statement.

**Lemma 4.1** *The elliptic* $\mathrm{SL}(N,\mathbb{C})$*-rotator is a Hitchin system corresponding to the* $\mathrm{GL}(N,\mathbb{C})$*-bundle* $E$ *(deg(E)=1) over the elliptic curve* $E_\tau$ *with the marked point* $z=0$.

*Proof.*

It can be proved that (4.13) is equivalent to the Lax equation. The Lax matrices in the basis of the sin-algebra take the form

$$L^{rot} = \sum_{m,n} S_{mn}\varphi\left[\begin{array}{c} m \\ n \end{array}\right](z)E_{mn}, \quad \varphi\left[\begin{array}{c} m \\ n \end{array}\right](z) = \mathbf{e}(-\frac{nz}{N})\phi(-\frac{m+n\tau}{N};z), \tag{4.14}$$

$$M^{rot} = \sum_{m,n} S_{mn}f\left[\begin{array}{c} m \\ n \end{array}\right](z)E_{mn}, \quad f\left[\begin{array}{c} m \\ n \end{array}\right](z) = \mathbf{e}(-\frac{nz}{N})\partial_u\phi(u;z)|_{u=-\frac{m+n\tau}{N}}. \tag{4.15}$$

They lead to the Lax equation for the matrix elements

$$\partial_t S_{mn}\varphi\left[\begin{array}{c} m \\ n \end{array}\right](z) = \sqrt{-1}\sum_{k,l} S_{m-k,n-l}S_{kl}\varphi\left[\begin{array}{c} m-k \\ n-l \end{array}\right](z)f\left[\begin{array}{c} k \\ l \end{array}\right](z)\sin\frac{\pi}{N}(nk-ml).$$

Using the Calogero functional equation (B.27) we rewrite it in the form (4.13). Since

$$\frac{1}{N}\mathrm{tr}(L^{rot})^2 = -2H^{rot} + \mathrm{tr}\mathbf{S}^2\wp(z),$$

$H^{rot}$ is the Hitchin quadratic integral.

The Lax operator satisfies the Hitchin equation

$$\bar{\partial}L^{rot} = 0, \quad \mathrm{Res}L^{rot}|_{z=0} = 2\pi\sqrt{-1}\mathbf{S}$$

and is quasiperiodic

$$L^{rot}(z+1) = Q(\tau)L^{rot}(z)Q^{-1}(\tau), \tag{4.16}$$

$$L^{rot}(z+\tau) = \tilde{\Lambda}(z,\tau)L^{rot}(z)(\tilde{\Lambda}(z,\tau))^{-1}, \tag{4.17}$$

where $\tilde{\Lambda}(z,\tau) = -\mathbf{e}(\frac{-z-\frac{1}{2}\tau}{N})\Lambda$ and the matrices $Q$ and $\Lambda$ are defined in (A.2),(A.3). The transition functions

$$Q(\tau): \quad z \to z+1, \tag{4.18}$$

$$\tilde{\Lambda}(z,\tau): \quad z \to z+\tau \tag{4.19}$$

define $\mathrm{GL}(N,\mathbb{C})$-bundle over $E_\tau$ with $\deg(V) = 1$. Thus the phase space of the $\mathrm{SL}_N$-rotator is the space of the Higgs fields $L^{rot}$ on the bundle determined by multipliers $Q$, $\tilde{\Lambda}$ with the first order singularities at zero. □

### 4.3 A map $\mathcal{R}^{CM} \to \mathcal{R}^{rot}$.

We construct a map from the phase space of the elliptic CM system $\mathcal{R}^{CM}$ into the phase space of the $SL_N$-rotator $\mathcal{R}^{rot}$. The $SL_N$-rotator is living on the most degenerate orbit corresponding to $L^{CM}_{-1}$ (4.5). The phase space of CM systems with spins is mapped into the general coadjoint orbits. This generalization is straightforward. In this way, for $N = 2$ we describe the upper horizontal arrow in Fig.(1.1).

The map is defined as the conjugation of $L^{CM}$ by some matrix $\Xi(z)$:

$$L^{rot} = \Xi \times L^{CM} \times \Xi^{-1}. \tag{4.20}$$

It follows from comparing (4.3) with (4.16) and (4.17) that $\Xi$ must intertwine the multipliers of bundles:

$$\Xi(z+1,\tau) = Q \times \Xi(z,\tau)\,, \tag{4.21}$$

$$\Xi(z+\tau,\tau) = \tilde{\Lambda}(z,\tau) \times \Xi(z,\tau) \times \mathrm{diag}(\mathbf{e}(u_j))\,. \tag{4.22}$$

The matrix $\Xi(z)$ degenerates at $z=0$, and the column-vector $(1,\cdots,1)$, in accordance with Lemma 3.1, should belong to the kernel of $\Xi(0)$. In this case, $\Xi \times L^{CM} \times \Xi^{-1}$ has a first order pole at $z=0$.

Consider the following $(N\times N)$- matrix $\tilde{\Xi}(z,u_1,\ldots,u_N;\tau)$ :

$$\tilde{\Xi}_{ij}(z,u_1,\ldots,u_N;\tau) = \theta\left[\begin{array}{c} \frac{i}{N}-\frac{1}{2} \\ \frac{N}{2} \end{array}\right](z-Nu_j,N\tau), \tag{4.23}$$

where $\theta\left[\begin{array}{c} a \\ b \end{array}\right](z,\tau)$ is the theta function with a characteristic (B.31). Sometimes we omit nonessential arguments of $\Xi$ for brevity.

**Lemma 4.2** *The matrix $\tilde{\Xi}$ is transformed under the translations $z\to z+1$, $z\to z+\tau$ and $u_j\to u_j+1$, $u_j\to u_j+\tau$ as :*

$$\tilde{\Xi}(z+1,\tau) = -Q\times\tilde{\Xi}(z,\tau)\,, \tag{4.24}$$

$$\tilde{\Xi}(z+\tau,\tau) = \tilde{\Lambda}(z,\tau)\times\tilde{\Xi}(z,\tau)\times\mathrm{diag}(\mathbf{e}(u_j)), \tag{4.25}$$

$$\tilde{\Lambda}(z,\tau) = -\mathbf{e}\left(-\frac{\tau}{2N}-\frac{z}{N}\right)\Lambda\,;$$

$$\tilde{\Xi}(u_j+1,;\tau) = \tilde{\Xi}(u_j;\tau)\times\mathrm{diag}(1,\cdots,(-1)^N,\cdots,1)\,, \tag{4.26}$$

$$\tilde{\Xi}(u_j+\tau;\tau) = \tilde{\Xi}(u_j;\tau)\times\mathrm{diag}(1,\cdots,(-1)^N\mathbf{e}\left(-\frac{N\tau}{2}+z-Nu_j\right)\cdots,1)\,. \tag{4.27}$$

*Proof.*
The statement of the lemma follows from the properties of the theta functions with characteristics (B.33)-(B.35). □

Now we assume that $\sum u_j = 0$, so $u_N$ is no more an independent variable, but it is equal to $-\sum_{j=1}^{N-1} u_j$.

**Lemma 4.3** *The determinant of the matrix $\tilde{\Xi}$ has the form*

$$\det\tilde{\Xi} = C(\tau)\vartheta(z,\tau)\prod_{j<k}\vartheta(u_j-u_k,\tau)\,, \tag{4.28}$$

*for some non-zero function $C(\tau)$, where $\vartheta$ is defined in (B.1).*

*Proof.*
It is easy to check that the lhs and the product of the theta function on the rhs are quasiperiodic with respect to translations $z\to z+1$, $z\to z+\tau$ and $u_j\to u_j+1$, $u_j\to u_j+\tau$, $j<N$ with multiplicators $-1$, $-\mathrm{e}(-\frac{\tau}{2}-z)$, $1$ and $\mathrm{e}(-N\tau-u_j+u_N)$ respectively. So, their quotient is an elliptic function in the variables $z$, $u_j$, $(j=1,\ldots,N-1$. As a function in the variable $z$, this quotient may have a unique pole of first order at zero, so it does not depend on the variable $z$.

The determinant is an antisymmetric function of the columns of a matrix, so the lhs vanishes if $u_j = u_k$; hence the quotient has no poles as a function in the $u_j$ and does not depend on this variable. Consequently, the quotient $C(\tau)$ depends only on $\tau$.

One can calculate the leading term of $C(\tau)$ when $\tau$ tends to $\sqrt{-1}\infty$: the leading term of $\tilde{\Xi}_{ij}$ equals to

$$\begin{cases} \mathbf{e}\left(\left(\frac{i}{N}-\frac{1}{2}\right)^2\frac{N\tau}{2}+\left(\frac{i}{N}-\frac{1}{2}\right)\left(z+Nu_j+\frac{N}{2}\right)\right), & i\neq N\,, \\ \mathbf{e}\left(\left(\frac{1}{2}\right)^2\frac{N\tau}{2}+\left(\frac{1}{2}\right)\left(z+Nu_j+\frac{N}{2}\right)\right)\times\left(1+(-1)^N\mathbf{e}(-z-Nu_j)\right), & i=N\,. \end{cases}$$

So the leading term of the determinant equals

$$\mathbf{e}\left(\frac{(N^2+2)}{24}\tau+\frac{1}{2}\left(z+\frac{N}{2}\right)-\frac{N}{2}\sum u_j\right)$$
$$\times\det\left(\mathbf{e}(iu_j)\right)+(-1)^N\mathbf{e}(-z)\det(\mathbf{e}((i-N\delta_{i,N})u_j))\ .$$

Both determinants are Vandermonde determinants, and we get as an answer (by taking into account the condition $\sum u_j = 0$):

$$(-\sqrt{-1})^N\mathbf{e}\left(\frac{(N^2+2)}{24}\tau-\frac{1}{2}z\right)(1-\mathbf{e}(z))\prod_{j>k}(\mathbf{e}(u_j)-\mathbf{e}(u_k)).$$

On the other hand, the leading term of the theta function $\vartheta(x,\tau)$ is equal to

$$-\sqrt{-1}\mathbf{e}\left(\frac{1}{8}\tau-\frac{1}{2}x\right)(1-\mathbf{e}(x)).$$

Therefore, the leading term of the product of the theta functions on the rhs is equal to

$$(-\sqrt{-1})^{N(N-1)/2+1}\mathbf{e}\left(\frac{N(N-1)/2+1}{8}\tau-\frac{1}{2}z\right)(1-\mathbf{e}(z))\prod_{j>k}(\mathbf{e}(u_j)-\mathbf{e}(u_k)).$$

Hence, its quotient has the leading term

$$(\sqrt{-1})^{\frac{(N-2)(N-1)}{2}}\mathbf{e}\left(-\frac{(N-2)(N-1)}{48}\tau\right).$$

Thus, the leading term of $C(\tau)$ not being zero, and $C(\tau)$ does not vanish.

For general $\tau$, $z\neq m\tau+n$ and $u_j-u_k\neq m\tau+n$ $(m,n\in\mathbb{Z})$ the matrix $\tilde{\Xi}$ is invertible, but for $z=0$ it degenerates. $\square$

**Lemma 4.4** *The kernel of $\tilde{\Xi}$ at $z=0$ is generated by the following column-vector :*

$$\left\{(-1)^l\prod_{j<k;j,k\neq l}\vartheta(u_k-u_j,\tau)\right\},\quad l=1,2,\cdots,N.$$

*Proof.*
We must prove that for any $i$ the following expression

$$\sum_{l=1}^{N}(-1)^l\theta\begin{bmatrix} \frac{i}{N}-\frac{1}{2} \\ \frac{N}{2} \end{bmatrix}(z-Nu_l,N\tau)\prod_{j<k;j,k\neq l}\vartheta(u_k-u_j,\tau) \tag{4.29}$$

vanishes. First, the symmetric group $S_N$ acts on $\mathbf{u}$ by permutation of $u_1,\dots,u_N$ and (4.29) is antisymmetric with respect to the $S_N$ action. Hence it vanishes on the hyperplanes $u_i=u_j$. As a function on $u_1$, (4.29) has $2N$ zeroes: $N-2$ zeroes $u_1=u_k$, $k\neq 1,N$, $N-2$ zeroes $u_N=u_k$, $k\neq 1,N$ and four zeroes $u_1=u_N$ (the last equation is $2u_1=-\sum_{j=2}^{N-1}u_j$).

Second, (4.29) is quasiperiodic with respect to the shifts $u_1\to u_1+1$, $u_1\to u_1+\tau$ with multiplicators 1 and $\mathbf{e}\left(-(N-1)\tau-(N-1)(u_1-u_N)\right)$. Any quasiperiodic function with such multiplicators is either zero or has $2N-2$ zeroes. Since our expression vanishes in $2N$ points it vanishes identically. $\Box$

It follows from the previous lemmas that the matrix

$$\Xi(z)=\tilde{\Xi}(z)\times\mathrm{diag}\left((-1)^l\prod_{j<k;j,k\neq l}\vartheta(u_k-u_j,\tau)\right) \tag{4.30}$$

is the singular gauge transform from Lemma 2.1 that maps $L^{CM}$ to $L^{rot}$. This transformation leads to the symplectic map

$$\mathcal{R}^{CM}\to\mathcal{R}^{rot},\quad (\mathbf{v},\mathbf{u})\mapsto\mathbf{S}. \tag{4.31}$$

Consider in detail the case $N=2$. Let

$$\mathbf{S}=S_a\sigma_a,$$

where $\sigma_a$ denote the sigma matrices subject to the commutation relations

$$[\sigma_a,\sigma_b]=2\sqrt{-1}\varepsilon_{abc}\sigma_c\,.$$

Then the transformation has the form

$$\left\{\begin{array}{l} S_1=-v\frac{\theta_{10}(0)}{\vartheta'(0)}\frac{\theta_{10}(2u)}{\vartheta(2u)}-\nu\frac{\theta_{10}^2(0)}{\theta_{00}(0)\theta_{01}(0)}\frac{\theta_{00}(2u)\theta_{01}(2u)}{\vartheta^2(2u)}\,, \\ S_2=-v\frac{\theta_{00}(0)}{\sqrt{-1}\vartheta'(0)}\frac{\theta_{00}(2u)}{\vartheta(2u)}-\nu\frac{\theta_{00}^2(0)}{\sqrt{-1}\theta_{10}(0)\theta_{01}(0)}\frac{\theta_{10}(2u)\theta_{01}(2u)}{\vartheta^2(2u)}\,, \\ S_3=-v\frac{\theta_{01}(0)}{\vartheta'(0)}\frac{\theta_{01}(2u)}{\vartheta(2u)}-\nu\frac{\theta_{01}^2(0)}{\theta_{00}(0)\theta_{10}(0)}\frac{\theta_{00}(2u)\theta_{10}(2u)}{\vartheta^2(2u)}\,. \end{array}\right. \tag{4.32}$$

Formulae of this kind were obtained in [12].

## 4.4 Bäcklund transformations in the CM systems.

We now use the map (4.31) to construct the Bäcklund transformation in the CM systems

$$\xi:\ (\mathbf{v},\mathbf{u})\to(\tilde{\mathbf{v}},\tilde{\mathbf{u}})\,.$$

Let the Lax matrix depends on the new coordinates and momenta $L=L(\tilde{\mathbf{v}},\tilde{\mathbf{u}})$. Consider the upper modification $\Xi(z)$ (4.30). To construct the Bäcklund transformation $\xi$, we map $(\mathbf{v},\mathbf{u})$ and $(\tilde{\mathbf{v}},\tilde{\mathbf{u}})$ to the same point $\mathbf{S}\in\mathcal{R}^{rot}$:

$$
\begin{array}{ccccc}
 & & \boxed{L^{rot}(\mathbf{S})} & & \\
 & \nearrow \Xi(z,\mathbf{u}) & & \nwarrow \Xi(z,\tilde{\mathbf{u}}) & \\
\boxed{L^{CM}(\mathbf{v},\mathbf{u})} & & \stackrel{\xi}{\longrightarrow} & & \boxed{L^{CM}(\tilde{\mathbf{v}},\tilde{\mathbf{u}})}
\end{array}
$$

In this way we reproduce implicitly the general formula (3.10) for the Bäcklund transformations. This transformation defines an integrable discrete time dynamics of a CM system. One example of this dicretization was proposed in [25]. It can be supposed to correspond to $\xi$.

Another way to construct new solutions from $(\mathbf{v},\mathbf{u})$ is to act by $N$ consecutive upper modifications

$$
\Xi(N) = D_N \Xi_N \cdots \Xi_j \cdots \Xi_2 \cdot \Xi \,. \tag{4.33}
$$

Here the matrices $\Xi_j,\ j = 2,\dots,N$, satisfy the quasi-periodicity conditions

$$
\Xi_j(z+\tau) = \tilde{\Lambda}^j \Xi_j(z) \tilde{\Lambda}^{1-j} \,,
$$

and $D_N$ is an arbitrary diagonal matrix. We come back to the $N$-dimensional moduli space $\mathcal{M}^{(N)}$ (see (2.5)) and to the map

$$
L^{CM}(\mathbf{v},\mathbf{u}) \stackrel{\Xi(N)}{\longrightarrow} L^{CM}(\tilde{\mathbf{v}},\tilde{\mathbf{u}}) \,.
$$

If we break the chain (4.33) on a step $k < L$, then we obtain the map

$$
L^{CM} \to L^{rot,k} \,,
$$

where $L^{rot,k}$ is the Lax operator for the elliptic rotator related to the holomorphic bundle of degree $k$. It satisfies the quasi-periodicity condition (4.16) and

$$
L^{rot,k}(z+\tau) = \tilde{\Lambda}^j L^{rot,k}(z) \tilde{\Lambda}^{-j}
$$

instead of (4.17).

# 5 Hitchin systems of infinite rank

Here we generalize the derivation of finite-dimensional integrable systems in the form (2.20)-(2.23) on two-dimensional integrable field theories.

## 5.1 Holomorphic $\hat{L}(\mathrm{GL}(N,\mathbb{C}))$-bundles.

Let $L(\mathrm{gl}(N,\mathbb{C}))$ be the loop algebra of $C^\infty$-maps $L(\mathrm{gl}(N,\mathbb{C})) :\ S^1 \to \mathrm{gl}(N,\mathbb{C})$, and $\hat{L}(\mathrm{gl}(N,\mathbb{C}))$ be its central extension with the multiplication

$$
(g,c) \times (g',c') = \left(gg', cc' \exp \mathcal{C}(g,g')\right) , \tag{5.1}
$$

where $\exp \mathcal{C}(g,g')$ is a 2-cocycle of $\hat{L}(\mathrm{GL}(N,\mathbb{C}))$ providing the associativity of the multiplication.

Consider a holomorphic bundle $V$ of an infinite rank over a Riemann curve $\Sigma_n$ with $n$ marked points. The bundle is defined by the transition functions from $\hat{L}(\mathrm{GL}(N,\mathbb{C}))$. Its fibers are isomorphic to the Lie algebra $\hat{L}(\mathrm{gl}(N,\mathbb{C}))$. The holomorphic structure on $V$ is defined by the operator

$$
d'' : \Omega^{(0)}_{C^\infty}(\Sigma_n, \mathrm{End}\, V) \to \Omega^{(0,1)}_{C^\infty}(\Sigma_n, \mathrm{End}\, V) \,.
$$

It has two components $d^{''}=d^{''}_{\bar{A}}+d^{''}_{\lambda}$. The first component is

$$d^{''}_{\bar{A}}:\Omega^{(0)}_{C^\infty}(\Sigma_n,L(\mathrm{gl}(N,\mathbb{C})))\to\Omega^{(0,1)}_{C^\infty}(\Sigma_n,L(\mathrm{gl}(N,\mathbb{C})))\,.$$

Locally

$$d^{''}_{\bar{A}}=\bar{\partial}+\bar{A},\quad \bar{\partial}=\partial_{\bar{z}},\quad \bar{A}=\bar{A}(x,z,\bar{z}),\ x\in S^1.$$

The second component is defined by the connection $d^{''}_{\lambda}$ on a trivial linear bundle $\mathcal{L}$ on $\Sigma_n$, given by

$$d^{''}_{\lambda}=\bar{\partial}+\lambda\,.$$

The field $\lambda$ is a map from $\Sigma_n$ to the central element of the Lie algebra $\hat{L}(\mathrm{gl}(N,\mathbb{C}))$. A local section $\sigma$ of $V$ is holomorphic if $d^{''}\sigma=0$. The sections allow to define the transition functions. We assume that $\bar{A}$ and $\lambda$ are smooth at the marked points.

In addition we define $n$ copies of the central extended loop groups located at the marked points

$$\hat{L}G_\alpha=(g_\alpha(x),c_\alpha),\ G_\alpha=\mathrm{GL}(N,\mathbb{C}),\ (\alpha=1,\ldots,n),\ x\in S^1,$$

with the multiplication (5.1).

Thus, we have the set $\mathcal{R}$ of fields playing the role of the "coordinate space":

$$\mathcal{R}=\left\{\bar{A},\lambda,(g_1,c_1),\ldots,(g_n,c_n)\right\}. \tag{5.2}$$

## 5.2 Gauge symmetries.

Let $\mathcal{G}$ be the group of automorphisms of $\mathcal{R}$ (the gauge group).

$$\mathcal{G}=C^\infty\mathrm{Map}(\Sigma_n\to\hat{L}(\mathrm{GL}(N,\mathbb{C})))=\{f(z,\bar{z},x),s(z,\bar{z})\}\,,$$

where $f(z,\bar{z},x)$ takes values in $\mathrm{GL}(N,\mathbb{C})$, and $s(z,\bar{z})$ is the map to the central element of $\hat{L}(\mathrm{GL}(N,\mathbb{C})))$. The multiplication is pointwise with respect to $\Sigma_n$

$$(f_1,s_1)\times(f_2,s_2)=(f_1f_2,s_1s_2\exp\mathcal{C}(f_1,f_2))\ ,$$

where $\exp\mathcal{C}(f_1,f_2)$ is a map from $\Sigma_n$ to the 2-cocycle of $\hat{L}(\mathrm{GL}(N,\mathbb{C}))$.

Let $(f_\alpha=f_\alpha(x),s_\alpha)$ be the value of the gauge fields at the marked point $w_\alpha$. The action of $\mathcal{G}$ on $\mathcal{R}$ takes the following form

$$\bar{A}\to f^{-1}\bar{\partial}f+f^{-1}\bar{A}f, \tag{5.3}$$

$$\lambda\to\lambda+s^{-1}\bar{\partial}s+\oint\mathrm{tr}(\bar{A}f^{-1}\partial_x f)dx, \tag{5.4}$$

$$c_\alpha\to c_\alpha s_\alpha,\ g_\alpha\to g_\alpha f_\alpha. \tag{5.5}$$

The quotient space $\mathcal{N}=\mathcal{R}/\mathcal{G}$ is the moduli space of infinite rank holomorphic bundles over Riemann curves with marked points.

### 5.3 Phase space.

The cotangent space to $\mathcal{R}$ has the following structure. Consider the analog of the Higgs field $\Phi \in \Omega^{(1,0)}_{C^\infty}(\Sigma_n, (\text{End } V)^*)$. It is a one-form $\Phi$ on $\Sigma_n$ taking values in the Lie coalgebra $L^*(\text{gl}(N,\mathbb{C}))$. Let $k$ be a scalar one-form on $\Sigma_n$, $k \in \Omega^{(1,0)}_{C^\infty}(\Sigma_n)$. It is dual to the field $\lambda$. At the marked points we have the Lie coalgebras $\text{Lie}^*(G_\alpha) \sim L(\text{gl}(N,\mathbb{C}))$ along with the central elements $r_\alpha$, dual to $c_\alpha$. Thus the cotangent bundle $T^*\mathcal{R}$ contains the fields

$$T^*\mathcal{R} = \left\{(\bar{A},\Phi),(\lambda,k);(p_1,r_1),\ldots,(p_n,r_n)\right\}\,. \tag{5.6}$$

There is a canonical symplectic structure on $T^*\mathcal{R}$. For $F \in \Omega^{(1,0)}_{C^\infty}(\Sigma_n, (\text{End } V)^*)$ and $G \in \Omega^{(0,1)}_{C^\infty}(\Sigma_n, L(\text{gl}(N,\mathbb{C})))$ define the pairing

$$< F|G >= \int_{\Sigma_n}\oint \text{tr}(FG)dx.$$

Then

$$\omega =< D\Phi \wedge D\bar{A} > + \int_{\Sigma_n} Dk\wedge D\lambda + \sum_{\alpha=1}^{n}\omega_\alpha\,, \tag{5.7}$$

where $\omega_\alpha$ is a canonical form on $T^*\hat{L}(G_\alpha)$. It is constructed in the canonical way by means of the Maurer-Cartan form on $\hat{L}(G_\alpha) = \{g_\alpha, c_\alpha\}$. The result is

$$\omega_\alpha = \oint_{S^1_\alpha} \text{tr}(D(p_\alpha g_\alpha^{-1})Dg_\alpha) + D(r_\alpha c_\alpha^{-1})Dc_\alpha + \frac{r_\alpha}{2}\oint_{S^1_\alpha}\text{tr}\left(g_\alpha^{-1}Dg_\alpha\partial_x(g_\alpha^{-1}Dg_\alpha)\right). \tag{5.8}$$

### 5.4 Symplectic reduction.

Now consider the lift of $\mathcal{G}$ to the global canonical transformations of $T^*\mathcal{R}$. In addition to (5.3),(5.4),(5.5) we have the following action of $\mathcal{G}$

$$\Phi \to f^{-1}k\partial_x f + f^{-1}\Phi f,\quad k\to k, \tag{5.9}$$

$$p_\alpha \to f_\alpha^{-1}p_\alpha f_\alpha + r_\alpha f_\alpha^{-1}\partial_x f_\alpha,\quad r_\alpha \to r_\alpha\,. \tag{5.10}$$

This transformation leads to the moment map from the phase space to the Lie coalgebra of the gauge group $\mu:\ T^*\mathcal{R}\to \text{Lie}^*(\mathcal{G})$. It takes the form

$$\mu = \left(\bar{\partial}\Phi - kp_x\bar{A} + [\bar{A},\Phi] + \sum_{\alpha=1}^{n} p_\alpha\delta(z_\alpha),\ \bar{\partial}k + \sum_{\alpha=1}^{n} r_\alpha\delta(z_\alpha)\right). \tag{5.11}$$

We assume that $\mu = (0,0)$. Therefore, we have the two holomorpheity conditions

$$\bar{\partial}\Phi - kp_x\bar{A} + [\bar{A},\Phi] + \sum_{\alpha=1}^{n} p_\alpha\delta(z_\alpha) = 0\,, \tag{5.12}$$

$$\bar{\partial}k + \sum_{\alpha=1}^{n} r_\alpha\delta(z_\alpha) = 0\,. \tag{5.13}$$

The constraint equation (5.13) means that the $k$-component of the Higgs field is a holomorphic one-form on $\Sigma$ with first order poles at the marked points.

Let us fix a gauge

$$\bar{L} = f^{-1}\bar{\partial} f + f^{-1}\bar{A}f. \tag{5.14}$$

The same gauge action transform $\Phi$ as

$$L = kf^{-1}\partial_x f + f^{-1}\Phi f. \tag{5.15}$$

We preserve the same notations $g_\alpha, p_\alpha$ for the gauge transformed variables. The moment constraint equation (5.12) has the same form in terms of $\bar{L}$ and $L$

$$\bar{\partial} L - k\partial_x \bar{L} + [\bar{L}, L] + \sum_{\alpha=1}^{n} p_\alpha \delta(z_\alpha) = 0\,. \tag{5.16}$$

Solutions of this equation along with (5.13) define the reduced phase space

$$T^*\mathcal{R}//\mathcal{G} \sim T^*\mathcal{N}\,.$$

The symplectic form (5.7) on $T^*\mathcal{N}$ becomes

$$\omega = < \delta L | \delta \bar{L} > + \int_{\Sigma_n} \delta k \delta \lambda + \sum_{\alpha=1}^{n} \omega_\alpha\,. \tag{5.17}$$

### 5.5 Coadjoint orbits.

Consider in detail the symplectic form $\omega$ (5.8) on $T^*\hat{L}(G) \sim \{(p,r);(g,c)\}$. We omit the subscript $_\alpha$ below. The following canonical transformation of $\omega$ by $(f,s) \in \hat{L}(G)$, where $s$ is a central element,

$$g \to fg,\ p \to p,\ r \to r,\ c \to sc,\quad f \in L(G)\,, \tag{5.18}$$

has not been used so far. The symplectic reduction with respect to this transformation leads to the coadjoint orbits of $\hat{L}(\mathrm{GL}(N,\mathbb{C}))$. In fact, the moment map

$$\mu : T^*\hat{L}(G) \to \mathrm{Lie}^*(\hat{L}(\mathrm{GL}(N,\mathbb{C})))$$

takes the form

$$\mu = (-gpg^{-1} + r\partial_x g g^{-1}, r).$$

Let us fix the moment $\mu = (p^{(0)}, r^{(0)})$. The result of the symplectic reduction of $T^*\hat{L}(G)$ is the coadjoint orbit

$$\mathcal{O}(p^{(0)}, r^{(0)}) = (p = -g^{-1}p^{(0)}g - r^{(0)}g^{-1}\partial_x g,\ r^{(0)}) = \mu^{-1}\left(T^*\hat{L}(\mathrm{SL}(N,\mathbb{C}))\right)/G_0,$$

where $G_0$ is the subgroup of $\hat{L}(\mathrm{GL}(N,\mathbb{C}))$ that preserves $\mu$

$$G^0 = \{g \in L(\mathrm{GL}(N,\mathbb{C}),\ s \text{ is arbitrary}) \mid p^{(0)} = -g^{-1}p^{(0)}g + r^{(0)}g^{-1}\partial_x g\}.$$

The symplectic form (5.8) being pushed forward on $\mathcal{O}$ takes the form

$$\omega = \oint \mathrm{tr}(D(pg^{-1})Dg) + \frac{r^{(0)}}{2}\oint \mathrm{tr}\left(g^{-1}DgD(g^{-1}\partial_x g)\right). \tag{5.19}$$

In what follows we will consider the collection of the orbits $\mathcal{O}_\alpha(p^{(0)}_\alpha, r^{(0}_\alpha)$ at the marked points instead of the cotangent bundles $T^*\hat{L}(G_\alpha)$. In this way we modify the notion of the cotangent bundle $T^*\mathcal{R}$ (5.6)

$$T^*\mathcal{R} = \left\{(\bar{A},\Phi),(\lambda,k),\mathcal{O}_1(p^{(0)}_1, r^{(0}_1),\ldots,\mathcal{O}_n(p^{(0)}_n, r^{(0)}_n)\right\}\,. \tag{5.20}$$

### 5.6 Conservation laws I

.

The Higgs field $\Phi$ is transformed as a connection with respect to the circles $S^1$ (5.9). If the central charge $k \neq 0$, the standard Hitchin integrals (2.23) cease to be gauge invariant. Invariant integrals are generated by the traces of the monodromies of the Higgs field $\Phi$. The generating function of Hamiltonians is given by

$$H(z) = \mathrm{tr}\left( P \exp \frac{1}{k} \oint_{S_1} \Phi \right) , \tag{5.21}$$

where $z$ is a local coordinate of an arbitrary point. At a marked point, $\Phi$ has a first order pole and

$$H(z) = \sum_{j=-N}^{+\infty} H_j z^j . \tag{5.22}$$

Since $H(z)$ is gauge invariant one can replace $\Phi$ by $L$ in (5.21)

$$H(z) = \mathrm{tr}\left( P \exp \frac{1}{k} \oint_{S_1} L \right) . \tag{5.23}$$

### 5.7 Equations of motion.

Consider the equations of motion on the "upstairs" space $T^*\mathcal{R}$ (5.20). They are derived by means of the symplectic form $\omega$ (5.7), where $\omega_\alpha$ is replaced by (5.19), and the Hamiltonians (5.21), (5.22). Let $t_j$ be a time variable corresponding to the Hamiltonian $H_j$. Taking into account that $H_j$ is a functional depending on the Higgs field and the central charge $k$ only, we arrive to the following free system

$$\partial_j \Phi = 0, \tag{5.24}$$

$$\partial_j \bar{A} = \frac{\delta H_j}{\delta \Phi} , \tag{5.25}$$

$$\partial_j k = 0, \quad \partial_j \lambda = \frac{\delta H_j}{\delta k}, \quad \partial_j p_\alpha = 0 . \tag{5.26}$$

After the symplectic reduction we are led to the fields $\bar{L}$ (5.14) and $L$ (5.15). For simplicity, we keep the same notation for the coadjoint orbits variables $p_\alpha$, so they are transformed as in (5.10). Substituting (5.15) in (5.24) we obtain the Zakharov-Shabat equation

$$\partial_j L - k\partial_x M_j + [M_j, L] = 0 , \quad (\partial_j = \partial_{t_j}) , \tag{5.27}$$

where $M_j = \partial_j f f^{-1}$. The operator $M_j$ can be restored partly from the second equation (5.25)

$$\bar{\partial} M_j - \partial_j \bar{L} + [M_j, \bar{L}] = \frac{\delta H_j}{\delta L} . \tag{5.28}$$

The last two equations along with the moment constraint equation (5.16) are the consistency conditions for the linear system

$$(k\partial_x + L)\Psi = 0 , \tag{5.29}$$

$$(\bar{\partial} + \bar{L})\Psi = 0 , \tag{5.30}$$

$$(\partial_j + M_j)\Psi = 0 . \tag{5.31}$$

## 5.8 Conservation laws II

.

The matrix equation (5.29) allows to write down the conservation laws. Its generic solutions can be represented in the form

$$\Psi(x) = (I + R)\exp\frac{1}{k}\int_0^x S dx' , \tag{5.32}$$

where $I$ is the identity matrix, $R$ is an off-diagonal periodic matrix and $S = \mathrm{diag}(S^1, \dots, S^N)$ is a diagonal matrix. The equation (5.29) means that $L$ can be gauge transformed to the diagonal form

$$(I + R)S = k\partial_x(I + R) + L(I + R). \tag{5.33}$$

Consider this equation in neighborhood of a point on $\Sigma_n$ with a local coordinate $z$. Assume, for simplicity, that it is a pole of the Lax operator and $k$ is a constant. In particular, it follows from (5.13) that $r^0_\alpha = 0$ and the coadjoint orbits have the form $\mathcal{O}_\alpha = \{p_\alpha = -gp^0_\alpha g^{-1}\}$. Then substitute into (5.33) the series expansions

$$L(z) = L_{-1}z^{-1} + L_0 + L_1 z + \dots, \quad (L_{-1} = \mathrm{res}L = p),$$
$$S(z) = S_{-1}z^{-1} + S_0 + S_1 z + \dots,$$
$$(I + R)(z) = h + R_1 z + R_2 z^2 + \dots, \quad (\mathrm{diag}(R_m) \equiv 0).$$

It follows from (5.32) that the diagonal matrix elements $S^m_j$ are the densities of the conservation laws

$$\log H_{j,l} = \oint S^l_j dx.$$

We present a recurrence procedure to define the diagonal matrices $S_j$. On the first step we find that

$$S_{-1} = h^{-1}L_{-1}h = h^{-1}ph, \quad p = L_{-1} = \mathrm{Res}\ L_{z=0}\,. \tag{5.34}$$

In other words the diagonal matrix $S_{-1}$ determines the orbit located at the point $z = 0$.

In the general case we get the following equation

$$S_k + [h^{-1}R_{k+1}, S_{-1}] = h^{-1}k\partial_x R_k + h^{-1}\sum_{l=0}^{k} L_l R_{k-l} - R_l S_{k-l}\,.$$

Separating the diagonal and the off-diagonal parts allows us to express $S_k$ and $R_k$ in terms of the lower coefficients

$$S_k = \left(h^{-1}k\partial_x R_k + h^{-1}\sum_{l=0}^{k-1} L_l R_{k-l} + h^{-1}L_k h R_l S_{k-l}\right)_{diag} , \tag{5.35}$$

$$[h^{-1}R_k, S_{-1}] = \left(h^{-1}k\partial_x R_{k-1} + h^{-1}L_{k-1}hh^{-1}\sum_{l=0}^{k-2} L_l R_{k-l} - h^{-1}\sum_{l=1}^{k-1} R_l S_{k-l}\right)_{nondiag} . \tag{5.36}$$

In particular,

$$S_0 = h^{-1}k\partial_x h\,, \tag{5.37}$$

$$S_1 = \left(h^{-1}k\partial_x R_1 + h^{-1}L_1 h - h^{-1}R_1 S_0\right)_{diag} , \tag{5.38}$$

where $R_1$ is defined from the equation

$$R_1 S_{-1} - pR_1 = k\partial_x h\,. \tag{5.39}$$

### 5.9 Hamiltonians in $\mathrm{SL}(2,\mathbb{C})$ case.

Let us perform the gauge transformation

$$f^{-1}Lf + kf^{-1}\partial_x f = L' , \tag{5.40}$$

with $f$ defined as follows:

$$f = \begin{pmatrix} \sqrt{L_{12}} & 0 \\ -\frac{L_{11}}{\sqrt{L_{12}}} - k\frac{\partial_x\sqrt{L_{12}}}{L_{12}} & \frac{1}{\sqrt{L_{12}}} \end{pmatrix} . \tag{5.41}$$

Then the Lax matrix $L$ is transformed into

$$L' = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix} , \tag{5.42}$$

where

$$T = L_{21}L_{12} + L_{11}^2 + k\frac{L_{11}\partial_x L_{12}}{L_{12}} - k\partial_x L_{11} - \frac{1}{2}k^2\frac{\partial_x^2 L_{12}}{L_{12}} + \frac{3}{4}k^2\frac{(\partial_x L_{12})^2}{L_{12}^2} . \tag{5.43}$$

The linear problem

$$\left\{ \begin{array}{l} (xk\partial_x + L')\psi = 0 , \\ (\partial_j + M_j')\psi = 0 , \end{array} \right. \tag{5.44}$$

where $\psi$ is the Bloch wave function $\psi = exp\{-i\oint\chi\}$, leads to the Riccati equation:

$$ik\partial_x\chi - \chi^2 + T = 0. \tag{5.45}$$

The decomposition of $\chi(z)$ provides densities of the conservation laws (see [26]):

$$\chi = \sum_{k=-1}^{\infty} z^k\chi_k , \tag{5.46}$$

$$H_k \sim \oint dx\chi_{k-1} . \tag{5.47}$$

The values of $\chi_k$ can be found from (5.45) using the expression (5.43) for $T(z) = \sum\limits_{k=-2}^{\infty} z^k T_k$ in a neighborhood of zero. For $k = -2,\ -1$ and $0$ we have:

$$\left\{ \begin{array}{l} \chi_{-1} = \sqrt{T_{-2}} = \sqrt{h} , \\ 2\sqrt{h}\chi_0 = T_{-1} + ik\partial_x\chi_{-1} = T_{-1} , \\ 2\sqrt{h}\chi_1 = T_0 + ik\partial_x\chi - \chi_0^2 . \end{array} \right. \tag{5.48}$$

In Subsections 7.2, 7.3 below, explicit formulae for $T_k$ are used for the computation of the Hamiltonians for the elliptic 2d Calogero-Moser and the elliptic Gaudin models.

## 6 $\hat{L}(\mathrm{SL}(N,\mathbb{C}))$-bundles over elliptic curves with marked points.

### 6.1 General case.

We apply the general construction to the $\hat{L}(\mathrm{SL}(N,\mathbb{C}))$-bundle over elliptic curve $E_\tau$ with marked points $w_\alpha,\ \alpha = 1,\dots,n$. It is a two-dimensional generalization of the elliptic Gaudin model [8]. In particular, for one marked point $z = 0$ we come to the $N$-body elliptic CM field theory.

Let us construct solutions of the moment equations (5.11), taking for simplicity at the marked points the orbits with vanishing central charges

$$\mathcal{O}_\alpha = \{p^\alpha_{ij}, \quad r_\alpha = 0\} \ .$$

For elliptic curves one can fix the central charge as $k = 1$. For the stable bundles the gauge transformation (5.3) allows to diagonalize $\bar{A}$:

$$\bar{A}_{ij} = \delta_{ij} \frac{2\pi\sqrt{-1}}{\tau - \bar{\tau}} u_i \ . \tag{6.1}$$

Then the Lax operator $L^G$ should satisfy (5.16). It takes the form:

$$\begin{array}{c} L^G_{ij} = -\frac{\delta_{ij}}{2\pi\sqrt{-1}} \left( \frac{v_i}{2} + \sum\limits_\alpha p^\alpha_{ii} \left( 2\pi\sqrt{-1}\frac{z-\bar{z}}{\tau-\bar{\tau}} + E_1(z - w_\alpha) \right) \right) - \\ -\frac{1-\delta_{ij}}{2\pi\sqrt{-1}} \sum\limits_\alpha p^\alpha_{ij} \mathbf{e}\left( \frac{z-w_\alpha-(\bar{z}-\bar{w}_\alpha)}{\tau-\bar{\tau}} u_{ij} \right) \phi(u_{ij}, z - w_\alpha), \quad (u_{ij} = u_i - u_j) \ . \end{array} \tag{6.2}$$

By the quasiperiodic gauge transform

$$f = \mathrm{diag}\{\mathbf{e}(\frac{z - \bar{z}}{\tau - \bar{\tau}} u_i)\} \tag{6.3}$$

one comes to the holomorphic quasiperiodic Lax operator

$$l^G_{ij}(z) = -\frac{\delta_{ij}}{2\pi\sqrt{-1}} \left( \frac{v_i}{2} + \sum_\alpha p^\alpha_{ii} E_1(z - w_\alpha) \right) - \frac{1 - \delta_{ij}}{2\pi\sqrt{-1}} \sum_\alpha p^\alpha_{ij} \phi(u_{ij}, z - w_\alpha) \ . \tag{6.4}$$

reducing the moment map equation to the diagonal gives the additional constraint

$$\frac{1}{2\pi\sqrt{-1}} \sum_\alpha p^\alpha_{ii} = \partial_x u_i \ . \tag{6.5}$$

## 6.2 $\hat{L}(\mathrm{SL}(2, \mathbb{C}))$-bundles over elliptic curves with marked points.

In this subsection we study 2-body elliptic Calogero field theory in details.

**The operator $L$.** According to (6.4) the holomorphic Lax operator is

$$\left\{ \begin{array}{l} l^G_{11} = -\frac{v}{4\pi\sqrt{-1}} - \sum\limits_\alpha \frac{p^\alpha_{11}}{2\pi\sqrt{-1}} E_1(z - w_\alpha) \ , \\ l^G_{12} = -\sum\limits_\alpha \frac{p^\alpha_{12}}{2\pi\sqrt{-1}} \phi(2u, z - w_\alpha) \ , \\ l^G_{21} = -\sum\limits_\alpha \frac{p^\alpha_{21}}{2\pi\sqrt{-1}} \phi(-2u, z - w_\alpha) \ , \end{array} \right. \tag{6.6}$$

with the additional constraint (6.5)

$$\frac{1}{2\pi\sqrt{-1}} \sum_\alpha p^\alpha_{11} = u_x \ . \tag{6.7}$$

We still have the freedom to fix the gauge with respect to the action of the diagonal subgroup. The corresponding moment map is (6.7).

For the one marked point $w_1 = 0$ the corresponding orbit is

$$p = 2\pi\sqrt{-1}\begin{pmatrix} u_x & -\nu \\ -\nu & -u_x \end{pmatrix}, \tag{6.8}$$

where $\nu$ =const., is the result of the gauge fixing. In this case the Lax operator is a 2d generalization of the Lax operator for the two-body CM model:

$$L_{2D}^{CM} = \begin{pmatrix} -\frac{1}{4\pi\sqrt{-1}}v - u_x E_1(z) & \nu\phi(2u, z) \\ \nu\phi(-2u, z) & \frac{1}{4\pi\sqrt{-1}}v + u_x E_1(z) \end{pmatrix}, \tag{6.9}$$

This operator is still periodic under the shift $z \to z + 1$ and

$$L_{2D}^{CM}(z + \tau) = \mathbf{e}(u) L_{2D}^{CM}(z)\mathbf{e}(-u) + \mathbf{e}(u)\partial_x \mathbf{e}(-u)\,,$$

where $\mathbf{e}(u) = \mathrm{diag}(\exp u, \exp -u)$.

**Hamiltonians for the 2d elliptic** $\mathrm{sl}(2, \mathbb{C})$ **CM model.** In this case the coefficients $T_k$ are (see (5.43)-(5.48)):

$$\begin{cases} T_{-2}^{CM} = u_x^2 + \nu^2 = h \\ T_{-1}^{CM} = 2\frac{v}{4\pi\sqrt{-1}}u_x - \frac{\nu_x}{\nu}u_x + u_{xx} \\ T_0^{CM} = -\frac{v^2}{16\pi^2} + (2u_x^2 - \nu^2)\wp(2u) - \frac{v}{4\pi\sqrt{-1}}\frac{\nu_x}{\nu} + \frac{1}{4}(\frac{\nu_x}{\nu})^2 \end{cases}, \tag{6.10}$$

where $h$ is the Casimir function, fixing the coadjoint orbit at the marked point. It can be chosen as a constant. Thus, we have

$$\nu^2 = h - u_x^2.$$

The next order Hamiltonian is quadratic

$$H_{-1}^{CM} = \oint \frac{v}{2\pi\sqrt{-1}}u_x - \frac{\nu_x}{\nu}u_x\,. \tag{6.11}$$

It can be written in the following way:

$$H_{-1}^{CM} = \oint \frac{v}{2\pi\sqrt{-1}}u_x + \frac{u_{xx}h}{\nu^2}\,. \tag{6.12}$$

Since $\{\oint dx \frac{u_{xx}}{\nu^2}, v(y)\} = 0$, the equations of motion are:

$$\begin{cases} u_t = \frac{1}{2\pi\sqrt{-1}}u_x\,, \\ v_t = \frac{1}{2\pi\sqrt{-1}}v_x\,. \end{cases} \tag{6.13}$$

Note that the L-M pair is simple in this case: $M = \frac{1}{2\pi\sqrt{-1}}L$.

The first nontrivial Hamiltonian $H_0$ is quadratic in the momenta field $v$. It is a two-dimensional generalization of the quadratic CM Hamiltonian

$$H_0^{CM} = \oint dx 2\sqrt{h}\chi_1 = \oint dx(T_0 - \frac{1}{4h}T_{-1}^2)\,. \tag{6.14}$$

A direct evaluation yields:

$$T_0^{CM} - \frac{1}{4h}(T_{-1}^{CM})^2 = -\frac{v^2}{16\pi^2}(1 - \frac{u_x^2}{h}) + (3u_x^2 - h)\wp(2u) - \frac{u_{xx}^2}{4\nu^2}\,. \tag{6.15}$$

The equations of motion produced by $H_0^{CM}$ are:

$$u_t = -\frac{v}{8\pi^2}(1 - \frac{u_x^2}{h})\,, \tag{6.16}$$

$$v_t = \frac{1}{8\pi^2 h}\partial_x(v^2 u_x) - 2(3u_x^2 - h)\wp'(2u) + 6\partial_x(u_x\wp(2u)) + \frac{1}{2}\partial_x(\frac{u_{xxx}\nu - \nu_x u_{xx}}{\nu^3})\,.$$

It is reduced to the two-body elliptic CM system for the $x$-independent fields.

**The L-M pair for the 2d elliptic** $\mathrm{sl}(2,\mathbb{C})$ **CM model.** The equations of motion (6.16) produced by the quadratic Hamiltonian $H_0^{CM}$ can be represented in a form of the Zakharov-Shabat equation with the $L$ matrix defined by (6.9) and the $M$ matrix given as follows:

$$\begin{cases} M_{11} = -u_t E_1(z) - \frac{1}{4\pi\sqrt{-1}}\left(\frac{1}{8\pi^2 h}v^2 u_x + 6u_x\wp(2u) + \frac{u_{xxx}\nu - \nu_x u_{xx}}{2\nu^3}\right) + \\ +\frac{u_x}{2\pi\sqrt{-1}}(E_2(2u) - E_2(z))\,, \\ M_{12} = -\frac{\nu}{2\pi\sqrt{-1}}\phi'(2u,z) + \left(\frac{\nu}{2\pi\sqrt{-1}}E_1(z) + \frac{vu_x\nu}{8\pi^2 h} - \frac{1}{4\pi\sqrt{-1}}\frac{u_{xx}}{\nu}\right)\phi(2u,z)\,, \\ M_{21} = -\frac{\nu}{2\pi\sqrt{-1}}\phi'(-2u,z) + \left(\frac{\nu}{2\pi\sqrt{-1}}E_1(z) + \frac{vu_x\nu}{8\pi^2 h} + \frac{1}{4\pi\sqrt{-1}}\frac{u_{xx}}{\nu}\right)\phi(-2u,z)\,. \end{cases} \tag{6.17}$$

See Appendix C for details of the proof. This construction completes the description of right vertical arrow in Fig.1

**2d CM - LL correspondence** . The upper modification that produces the map of the elliptic CM system into the elliptic rotator (4.9), (4.10) works in the two-dimensional case as well.

The two-dimensional extension of the $\mathrm{SL}(2,\mathbb{C})$-elliptic rotator is the Landau-Lifshitz (LL) equation

$$\partial_t \mathbf{S} = \frac{1}{2}[\mathbf{S}, J(\mathbf{S})] + \frac{1}{2}[\mathbf{S}, \partial_{xx}\mathbf{S}]. \tag{6.18}$$

This equation can be fitted in the Zakharov-Shabat form [28]. The Lax operator $L^{LL}$ has the same form as for the $\mathrm{SL}(2,\mathbb{C})$ elliptic rotator $L^{rot}$ (4.14). For $\mathrm{sl}(2,\mathbb{C})$ the basis of the sigma matrices coincides with the basis of the sin-algebra and $L^{LL}$ takes the form

$$L = \sum_a u_a(z) S_a \sigma_a\,,$$

$$u_1 = \varphi\begin{bmatrix} 0 \\ 1 \end{bmatrix}(z)\,,\ u_2 = \varphi\begin{bmatrix} 1 \\ 1 \end{bmatrix}(z)\,,\ u_3 = \varphi\begin{bmatrix} 1 \\ 0 \end{bmatrix}(z)\,.$$

The $M^{LL}$ operator has a very simple extension

$$M^{LL} = M^{rot} + \sum_a u_a(z)\mathrm{tr}(\sigma_a[\mathbf{S}, \partial_x\mathbf{S}])\sigma_a\,.$$

It is easy to check that the Zakharov-Shabat equation leads to (6.18) if

$$\sum_a S_a^2 = 1\,.$$

Thereby be have defined the right vertical arrow in Fig.1.

Now the upper modification acts as

$$L^{LL} = \Xi\partial_x\Xi^{-1} + \Xi L_{2D}^{CM}\Xi^{-1}\,,$$

where $L_{2D}^{CM}$ is defined by (6.9). The gauge transformation $\Xi$ has the same quasi-periodicity properties as in the one-dimensional situation. Therefore, it is defined by (4.30) and (4.23) for $N=2$. It means that (4.32) allows to pass from the $CM$ fields $v(x,t), u(x,t)$ and the constant $\nu$ to the LL fields $\mathbf{S}=(S_1,S_2,S_3)$ with the orbit fixing condition

$$\sum_a S_a^2 = -\frac{1}{2\pi^2}(u_x^2+\nu^2) = 1\,.$$

It completes the description of the diagram on Fig.1.

**Limit to the Sinh-G equation.** It is known that the LL model is universal; it contains as a special limit the Sinh-Gordon and the Nonlinear Schrödinger models [3]. In this way they can be derived within the 2d CM system. The scaling limit in the CM model is a combination of the trigonometric limit $Im\tau\to\infty$ with shifts of coordinates: $u=U+\frac{1}{2}Im\tau$ and renormalization of the coupling constant $\nu=\bar{\nu}e^{\frac{1}{2}Im\tau}$ [27]. This procedure applied to the 2d elliptic CM Hamiltonian yields the sinh-Gordon system:

$$H_{SG} = -\frac{v^2}{16\pi^2} - \bar{\nu}^2(e^{2U}+e^{-2U}) + \frac{U_x^2}{4}\,. \tag{6.19}$$

The equations of motion are:

$$\left\{\begin{array}{l} U_t = -\frac{v}{8\pi^2}\,, \\ v_t = 2\bar{\nu}^2(e^{2U}-e^{-2U}) + \frac{1}{2}U_{xx}\,. \end{array}\right. \tag{6.20}$$

The L-M pair is:

$$L^{SG} = \begin{pmatrix} -\frac{v}{4\pi\sqrt{-1}} - \frac{1}{2}U_x & \bar{\nu}(1-e^{2U}Z) \\ \bar{\nu}(\frac{1}{Z}-e^{-2U}) & \frac{v}{4\pi\sqrt{-1}} + \frac{1}{2}U_x \end{pmatrix}\,. \tag{6.21}$$

$$M^{SG} = \begin{pmatrix} -\frac{U_t}{2} - \frac{1}{8\pi\sqrt{-1}}U_x & \frac{\bar{\nu}}{4\pi\sqrt{-1}}(1+e^{2U}Z) \\ \frac{\bar{\nu}}{4\pi\sqrt{-1}}(e^{-2U}+\frac{1}{Z}) & \frac{U_t}{2} + \frac{1}{8\pi\sqrt{-1}}U_x \end{pmatrix}\,. \tag{6.22}$$

### 6.3 Hamiltonians for the 2d elliptic Gaudin model.

Using (B.28) we obtain the Hamiltonian:

$$H_{-1,a}^{G} = 2\frac{v}{4\pi\sqrt{-1}}\frac{p_{11}^a}{2\pi\sqrt{-1}} + 2\sum_b \frac{p_{11}^a}{2\pi\sqrt{-1}}\frac{p_{11}^b}{2\pi\sqrt{-1}}E_1(z_a-z_b) - \tag{6.23}$$

$$-\sum_{a\neq b}\frac{p_{12}^a p_{21}^b}{(2\pi\sqrt{-1})^2}\phi(2u,z_b-z_a) + \sum_{a\neq b}\frac{p_{12}^b p_{21}^a}{(2\pi\sqrt{-1})^2}\phi(2u,z_a-z_b) - \frac{p_{11}^a}{2\pi\sqrt{-1}}\frac{\partial_x p_{12}^a}{p_{12}}\,.$$

The last term makes the above Hamiltonian different from the one-dimensional version.

Let us consider the sl$(2,\mathbb{C})$ case with two marked points on the elliptic curve.

We will use the following notations:

$$\left\{\begin{array}{ll} p_{11}^1 = 2\pi\sqrt{-1}\gamma_1, & p_{11}^2 = 2\pi\sqrt{-1}\gamma_2\,, \\ p_{12}^1 = -2\pi\sqrt{-1}\nu_+, & p_{21}^1 = -2\pi\sqrt{-1}\nu_-\,, \\ p_{12}^2 = -2\pi\sqrt{-1}\mu_+, & p_{21}^2 = -2\pi\sqrt{-1}\mu_-\,. \end{array}\right. \tag{6.24}$$

The $L$ matrix is:

$$\begin{cases} l_{11}^G = -\frac{v}{4\pi\sqrt{-1}} - \gamma_1 E_1(z-z_1) - \gamma_2 E_1(z-z_2)\,, \\ l_{12}^G = \nu\phi(2u, z-z_1) + \mu_+\phi(2u, z-z_2)\,, \\ l_{21}^G = \nu\phi(-2u, z-z_1) + \mu_-\phi(-2u, z-z_2)\,. \end{cases} \tag{6.25}$$

The solution exists if

$$\gamma_1 + \gamma_2 = u_x\,. \tag{6.26}$$

The gauge fixing condition is chosen to be

$$\nu_+ = \nu_- = \nu. \tag{6.27}$$

We fix the Casimir elements $h_1 = \gamma_1^2 + \nu^2$ and $h_2 = \gamma_2^2 + \mu_+\mu_-$ to be constants: $h_1,\ h_2 \in \mathbb{C}$.

On the reduced phase space there are two independent fields besides $u$ and $v$. Let them be for example $\nu$ and $\mu_+$, then

$$\begin{cases} \gamma_1 = \sqrt{h_1 - \nu^2}\,, \\ \gamma_2 = u_x - \sqrt{h_1 - \nu^2}\,, \\ \mu_- = \frac{1}{\mu_+}(h_2 - (u_x - \sqrt{h_1-\nu^2})^2)\,. \end{cases} \tag{6.28}$$

However we are going to use all kinds of variables in order to make the formulae more transparent. The non-trivial brackets on the reduced phase space are:

$$\begin{gathered} \{v(x), u(y)\} = \delta(x-y)\,, \quad \{v(x), \gamma_1(y)\} = -\delta'(x-y)\,, \quad \{v(x), \nu(y)\} = \tfrac{\gamma_1}{\nu}\delta'(x-y)\,, \\ \{\mu_+(x), \gamma_1(y)\} = -\tfrac{1}{2\pi\sqrt{-1}}\mu_+\delta(x-y)\,, \quad \{\mu_+(x), \mu_-(y)\} = -2\tfrac{1}{2\pi\sqrt{-1}}\gamma_2\delta(x-y)\,, \\ \{\mu_+(x), \gamma_2(y)\} = \tfrac{1}{2\pi\sqrt{-1}}\mu_+\delta(x-y)\,, \quad \{\mu_+(x), \nu(y)\} = \tfrac{1}{2\pi\sqrt{-1}}\tfrac{\gamma_1}{\nu}\mu_+\delta(x-y)\,, \\ \{\nu(x), \mu_-(y)\} = \tfrac{1}{2\pi\sqrt{-1}}\tfrac{\gamma_1}{\nu}\mu_-\delta(x-y)\,. \end{gathered} \tag{6.29}$$

The Hamiltonian is:

$$\begin{aligned} H_{-1}^G = \oint dx \Big(2\gamma_1\tfrac{v}{4\pi\sqrt{-1}} - \gamma_1\tfrac{\nu_x}{\nu} + \partial_x\gamma_1 + \nu\mu_+\phi(2u, z_1-z_2) - \nu\mu_-\phi(2u, z_2-z_1) + \\ -2\gamma_1\gamma_2 E_1(z_1-z_2)\Big)\,. \end{aligned} \tag{6.30}$$

The equations of motion are:

$$\begin{cases} \partial_t u(x) = \frac{1}{2\pi\sqrt{-1}}\gamma_1(x)\,, \\ \partial_t v(x) = \frac{1}{2\pi\sqrt{-1}}v_x - \partial_x(\frac{\gamma_1\mu_+}{\nu}\phi(2u, z_1-z_2)) + \partial_x(\frac{\gamma_1\mu_-}{\nu}\phi(2u, z_2-z_1)) - \\ -2\nu\mu_+\phi'(2u, z_1-z_2) + 2\nu\mu_-\phi'(2u, z_2-z_1) - \partial_x\left(\frac{\gamma_1\partial_x\gamma_1}{\nu^2}\right)\,, \\ \partial_t\nu = -\frac{1}{2\pi\sqrt{-1}}\partial_x\left(\frac{\gamma_1^2}{\nu}\right) + \frac{\gamma_1}{2\pi\sqrt{-1}\nu}(\mu_+\phi(2u, z_1-z_2) - \mu_-\phi(2u, z_2-z_1))\,, \\ \partial_t\mu_+ = \frac{1}{2\pi\sqrt{-1}}(2\frac{v}{4\pi\sqrt{-1}}\mu_+ 2\mu_+(\gamma_2-\gamma_1)E_1(z_1-z_2)) - \frac{2\nu\gamma_2}{2\pi\sqrt{-1}}\phi(2u, z_2-z_1) - \\ -\frac{\gamma_1\mu_+}{2\pi\sqrt{-1}\nu}(\mu_+\phi(2u, z_1-z_2) - \mu_-\phi(2u, z_2-z_1))\,. \end{cases} \tag{6.31}$$

The quadratic Hamiltonian is the direct generalization of (6.15)

$$H_0^G = \oint dx 2\sqrt{h_1}\chi_1 = \oint dx(T_0 - \frac{1}{4h_1}T_{-1}^2)\,, \tag{6.32}$$

where

$$
\begin{array}{c}
2\sqrt{h_1}\chi_1 = \\
-\frac{v^2}{16\pi^2}(1-\frac{\gamma_1^2}{h_1}) + (2u_x\gamma_1 - \nu^2)\wp(2u) - \frac{(\partial_x\gamma_1)^2}{4\nu^2} + \mu_+\mu_-(E_2(z_1-z_2) - E_2(2u)) + \\
+4\eta_1\gamma_1\gamma_2 + \nu\mu_-\phi(2u, z_2-z_1)(E_1(z_1-z_2) - E_1(2u) + E_1(2u+z_2-z_1)) - \\
-\nu\mu_+\phi(2u, z_1-z_2)(E_1(z_1-z_2) + E_1(2u) - E_1(2u+z_1-z_2)) + \gamma_2^2 E_1^2(z_1-z_2) + \\
+2\gamma_2\frac{v}{4\pi\sqrt{-1}}E_1(z_1-z_2) - \gamma_2\frac{\nu_x}{\nu}E_1(z_1-z_2) + \gamma_1\frac{\mu_+}{\nu^2}\phi(2u, z_1-z_2) - \\
-\gamma_1\phi(2u, z_1-z_2)[\frac{\partial_x\mu_+}{\nu} + 2u_x\frac{\mu_+}{\nu}(E_1(z_1-z_2+2u) - E_1(2u))] - \\
-\frac{1}{4h_1}(\nu\mu_+\phi(2u, z_1-z_2) - \nu\mu_+\phi(2u, z_2-z_1) + 2\gamma_1\gamma_2E_1(z_1-z_2))^2 - \\
-\frac{1}{2h_1}(\nu\mu_+\phi(2u, z_1-z_2) - \nu\mu_+\phi(2u, z_2-z_1) + \\
+2\gamma_1\gamma_2E_1(z_1-z_2))(2\gamma_1\frac{v}{4\pi\sqrt{-1}} - \gamma_1\frac{\nu_x}{\nu} + \partial_x\gamma_1)\,.
\end{array}
\tag{6.33}
$$

# 7 Conclusion

Here we briefly summarize the results of our analysis and discuss some unsolved related problems. The following two subjects were investigating in the paper.

(**i**) We have constructed symplectic maps between Hitchin systems related to holomorphic bundles of different degrees. It allowed us to construct the Bäcklund transformations in the Hitchin systems defined over Riemann curves with marked points. We applied the general scheme to the elliptic CM systems and constructed the symplectic map to an integrable $\mathrm{SL}(N,\mathbb{C})$ Euler-Arnold top (the elliptic $\mathrm{SL}(N,\mathbb{C})$-rotator). The open problem is to write down the explicit expressions for the spin variables in terms of the CM phase space for an arbitrary $N$ as it was done for the case $N = 2$ (4.32). It should help to construct the Bäcklund transformations for the CM systems explicitly, and more generally, to construct the generating function for them. The later can be considered as the integrable discrete time mapping [21].

(ii) We have proposed a generalization of the Hitchin approach to 2d integrable theories related to holomorphic bundles of infinite rank. The main example is the integrable two-dimensional version of the two-body elliptic CM system. The upper modification allows to define the symplectic map to the Landau-Lifshitz equation and to find, in principle, the Bäcklund transformations in the field theories.

It will be extremely interesting to find the 2d generalization of the $\mathrm{SL}(N,\mathbb{C})$-rotator for $N > 2$ (the matrix LL equation).

There is another point of view on the 2d generalizations of the Hitchin systems. One can try to define them starting from holomorphic bundles over complex surfaces, that are fibrations over Riemann curves. In this case the spectral parameter lives on the base of the fibration, while the space variable lives on the fibers. It will be interesting to analyze, for example, the known solutions of the LL equation from this point of view.

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# 9 Appendix

## 9.1 Appendix A. Sin-Algebra.

$$\mathbf{e}(z) = \exp(2\pi\sqrt{-1}z)\,, \tag{A.1}$$

$$Q = \mathrm{diag}(\mathbf{e}(1/N), \dots, \mathbf{e}(m/N), \dots, 1)\,, \tag{A.2}$$

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{A.3}$$

$$E_{mn} = \mathbf{e}(\frac{mn}{2N})Q^m\Lambda^n, \ (m = 0, \dots, N-1, \ n = 0, \dots, N-1, (\mathrm{mod} N) \ m^2 + n^2 \neq 0) \tag{A.4}$$

is the basis in $\mathrm{sl}(N, \mathbb{C})$. The commutation relations in this basis take the form

$$[E_{sk}, E_{nj}] = 2\sqrt{-1}\sin\frac{\pi}{N}(kn - sj)E_{s+n,k+j}\,, \tag{A.5}$$

$$\mathrm{tr}(E_{sk}E_{nj}) = \delta_{s,-n}\delta_{k,-j}N\,. \tag{A.6}$$

## 9.2 Appendix B. Elliptic functions

We summarize the main formulae for elliptic functions, borrowed mainly from [29] and [30]. We assume that $q = \exp 2\pi i\tau$, where $\tau$ is the modular parameter of the elliptic curve $E_\tau$.

The basic element is the theta function:

$$\vartheta(z|\tau) = q^{\frac{1}{8}}\sum_{n\in\mathbf{Z}}(-1)^n e^{\pi i(n(n+1)\tau+2nz)} = \tag{B.1}$$

$$q^{\frac{1}{8}}e^{-\frac{i\pi}{4}}(e^{i\pi z} - e^{-i\pi z})\prod_{n=1}^{\infty}(1-q^n)(1-q^n e^{2i\pi z})(1-q^n e^{-2i\pi z})\,.$$

*The Eisenstein functions*

$$E_1(z|\tau) = \partial_z \log\vartheta(z|\tau), \quad E_1(z|\tau) \sim \frac{1}{z} - 2\eta_1 z, \tag{B.2}$$

where

$$\eta_1(\tau) = \zeta(\frac{1}{2}) = \tag{B.3}$$

$$\frac{3}{\pi^2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}{}' \frac{1}{(m\tau+n)^2} = \frac{24}{2\pi i}\frac{\eta'(\tau)}{\eta(\tau)}\,,$$

where

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n>0}(1-q^n)\,.$$

is the Dedekind function.

$$E_2(z|\tau) = -\partial_z E_1(z|\tau) = \partial_z^2 \log\vartheta(z|\tau), \quad E_2(z|\tau) \sim \frac{1}{z^2} + 2\eta_1\,. \tag{B.4}$$

The next important function is

$$\phi(u,z)=\frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}\,. \tag{B.5}$$

It has a pole at $z=0$ and

$$\phi(u,z)=\frac{1}{z}+E_1(u)+\frac{z}{2}(E_1^2(u)-\wp(u))+\dots\,, \tag{B.6}$$

and

$$\phi(u,z)^{-1}\partial_u\phi(u,z)=E_1(u+z)-E_1(u)\,. \tag{B.7}$$

The following formula plays an important role in checking of the zero curvature equation:

$$\phi''(u,z)=\phi(u,z)(E_2(z)-E_1^2(z)+2E_1(z)(E_1(u+z)-E_1(u))+2E_2(u)-6\eta_1)\,. \tag{B.8}$$

It follows from:

$$(E_1(z)+E_1(u)-E_1(z+u))^2=E_2(u)+E_2(z)+E_2(u+z)-6\eta_1\,. \tag{B.9}$$

*Relations to the Weierstrass functions*

$$\zeta(z|\tau)=E_1(z|\tau)+2\eta_1(\tau)z\,, \tag{B.10}$$

$$\wp(z|\tau)=E_2(z|\tau)-2\eta_1(\tau)\,, \tag{B.11}$$

$$\phi(u,z)=\exp(-2\eta_1 uz)\frac{\sigma(u+z)}{\sigma(u)\sigma(z)}\,, \tag{B.12}$$

$$\phi(u,z)\phi(-u,z)=\wp(z)-\wp(u)=E_2(z)-E_2(u)\,. \tag{B.13}$$

*Particular values*

$$E_1(\frac{1}{2})=0,\quad E_1(\frac{\tau}{2})=E_1(\frac{1+\tau}{2})=-\pi\sqrt{-1}\,. \tag{B.14}$$

*Series representations*

$$E_1(z|\tau)=-2\pi i(\frac{1}{2}+\sum_{n\neq 0}\frac{e^{2\pi iz}}{1-q^n})= \tag{B.15}$$

$$-2\pi i(\sum_{n<0}\frac{1}{1-q^ne^{2\pi iz}}+\sum_{n\geq 0}\frac{q^ne^{2\pi iz}}{1-q^ne^{2\pi iz}}+\frac{1}{2}),,$$

$$E_2(z|\tau)=-4\pi^2\sum_{n\in\mathbf{Z}}\frac{q^ne^{2\pi iz}}{(1-q^ne^{2\pi iz})^2}\,, \tag{B.16}$$

$$\phi(u,z)=2\pi i\sum_{n\in\mathbf{Z}}\frac{e^{-2\pi inz}}{1-q^ne^{-2\pi iu}}. \tag{B.17}$$

*Parity*

$$\vartheta(-z) = -\vartheta(z)\,, \tag{B.18}$$

$$E_1(-z) = -E_1(z)\,, \tag{B.19}$$

$$E_2(-z) = E_2(z)\,, \tag{B.20}$$

$$\phi(u,z) = \phi(z,u) = -\phi(-u,-z)\,, \tag{B.21}$$

*Quasi-periodicity*

$$\vartheta(z+1) = -\vartheta(z)\,, \quad \vartheta(z+\tau) = -q^{-\frac{1}{2}}e^{-2\pi iz}\vartheta(z)\,, \tag{B.22}$$

$$E_1(z+1) = E_1(z)\,, \quad E_1(z+\tau) = E_1(z) - 2\pi i\,, \tag{B.23}$$

$$E_2(z+1) = E_2(z)\,, \quad E_2(z+\tau) = E_2(z)\,, \tag{B.24}$$

$$\phi(u+1,z) = \phi(u,z)\,, \quad \phi(u+\tau,z) = e^{-2\pi iz}\phi(u,z)\,. \tag{B.25}$$

*Addition formula*

$$\phi(u,z)\partial_v\phi(v,z) - \phi(v,z)\partial_u\phi(u,z) = (E_2(v) - E_2(u))\phi(u+v,z)\,, \tag{B.26}$$

or

$$\phi(u,z)\partial_v\phi(v,z) - \phi(v,z)\partial_u\phi(u,z) = (\wp(v) - \wp(u))\phi(u+v,z)\,. \tag{B.27}$$

The proof of (B.26) is based on (B.6),(B.21), and (B.25). In fact, $\phi(u,z)$ satisfies more general relation which follows from the Fay three-section formula

$$\phi(u_1,z_1)\phi(u_2,z_2) - \phi(u_1+u_2,z_1)\phi(u_2,z_2-z_1) - \phi(u_1+u_2,z_2)\phi(u_1,z_1-z_2) = 0\,. \tag{B.28}$$

A particular case of this formula is

$$\phi(u_1,z)\phi(u_2,z) - \phi(u_1+u_2,z)(E_1(u_1) + E_1(u_2)) + \partial_z\phi(u_1+u_2,z) = 0\,. \tag{B.29}$$

*Integrals*

$$\int_{E_\tau} E_1(z|\tau)dzd\bar{z} = 0\,. \tag{B.30}$$

*Theta functions with characteristics.*
For $a,b \in \mathbb{Q}$ put :

$$\theta\left[\begin{array}{c} a \\ b \end{array}\right](z,\tau) = \sum_{j\in\mathbb{Z}} \mathbf{e}\left((j+a)^2\frac{\tau}{2} + (j+a)(z+b)\right)\,. \tag{B.31}$$

In particular, the function $\vartheta$ (B.1) is the theta function with a characteristic

$$\vartheta(x,\tau) = \theta\left[\begin{array}{c} 1/2 \\ 1/2 \end{array}\right](x,\tau)\,. \tag{B.32}$$

One has

$$\theta\left[\begin{array}{c}a\\b\end{array}\right](z+1,\tau)=\mathbf{e}(a)\theta\left[\begin{array}{c}a\\b\end{array}\right](z,\tau)\,, \tag{B.33}$$

$$\theta\left[\begin{array}{c}a\\b\end{array}\right](z+a'\tau,\tau)=\mathbf{e}\left(-a'^2\frac{\tau}{2}-a'(z+b)\right)\theta\left[\begin{array}{c}a+a'\\b\end{array}\right](z,\tau)\,, \tag{B.34}$$

$$\theta\left[\begin{array}{c}a+j\\b\end{array}\right](z,\tau)=\theta\left[\begin{array}{c}a\\b\end{array}\right](z,\tau)\,,\quad j\in\mathbb{Z}\,. \tag{B.35}$$

For the simplicity we denote $\theta\left[\begin{array}{c}a/2\\b/2\end{array}\right]=\theta_{ab}$.

The following identities are useful for the upper modification procedure in $\mathrm{sl}(2,\mathbb{C})$ case:

$$\begin{array}{c}\theta_{01}(x,\tau)\theta_{00}(y,\tau)+\theta_{01}(y,\tau)\theta_{00}(x,\tau)=2\theta_{01}(x+y,2\tau)\theta_{01}(x-y,2\tau)\,,\\ \theta_{01}(x,\tau)\theta_{00}(y,\tau)-\theta_{01}(y,\tau)\theta_{00}(x,\tau)=2\vartheta(x+y,2\tau)\vartheta(x-y,2\tau)\,,\\ \theta_{00}(x,\tau)\theta_{00}(y,\tau)+\theta_{01}(y,\tau)\theta_{01}(x,\tau)=2\theta_{00}(x+y,2\tau)\theta_{00}(x-y,2\tau)\,,\\ \theta_{00}(x,\tau)\theta_{00}(y,\tau)-\theta_{01}(y,\tau)\theta_{01}(x,\tau)=2\theta_{10}(x+y,2\tau)\theta_{10}(x-y,2\tau)\,.\end{array} \tag{B.36}$$

$$\begin{array}{c}2\vartheta(x,2\tau)\theta_{01}(y,2\tau)=\vartheta(\frac{x+y}{2},\tau)\theta_{10}(\frac{x-y}{2},\tau)+\theta_{10}(\frac{x+y}{2},\tau)\vartheta(\frac{x-y}{2},\tau)\,,\\ 2\theta_{00}(x,2\tau)\theta_{10}(y,2\tau)=\vartheta(\frac{x+y}{2},\tau)\vartheta(\frac{x-y}{2},\tau)+\theta_{10}(\frac{x+y}{2},\tau)\theta_{10}(\frac{x-y}{2},\tau)\,,\\ 2\theta_{00}(x,2\tau)\theta_{00}(y,2\tau)=\theta_{00}(\frac{x+y}{2},\tau)\theta_{00}(\frac{x-y}{2},\tau)+\theta_{01}(\frac{x+y}{2},\tau)\theta_{01}(\frac{x-y}{2},\tau)\,,\\ 2\theta_{10}(x,2\tau)\theta_{10}(y,2\tau)=\theta_{00}(\frac{x+y}{2},\tau)\theta_{00}(\frac{x-y}{2},\tau)-\theta_{01}(\frac{x+y}{2},\tau)\theta_{01}(\frac{x-y}{2},\tau)\,.\end{array} \tag{B.37}$$

## 9.3 Appendix C: 2d $\mathrm{sl}(2,\mathbb{C})$ Calogero L-M pair

The Zakharov-Shabat equations in $\mathrm{sl}(2,\mathbb{C})$ case are:

$$\left\{\begin{array}{l}11:\partial_t L_{11}-\partial_x M_{11}=M_{21}L_{12}-M_{12}L_{21}\,,\\ 12:\partial_t L_{12}-\partial_x M_{12}=2L_{11}M_{12}-2L_{12}M_{11}\,,\\ 21:\partial_t L_{21}-\partial_x M_{21}=2M_{11}L_{21}-2L_{11}M_{21}\,.\end{array}\right. \tag{C.1}$$

Let the non diagonal terms in the M matrix be of the form:

$$\left\{\begin{array}{l}M_{12}=c(x)\Phi'(2u,z)+(f^1_{12}(x)E_1(z)+f^0_{12}(x))\Phi(2u,z)\,,\\ M_{21}=c(x)\Phi'(-2u,z)+(f^1_{21}(x)E_1(z)+f^0_{21}(x))\Phi(-2u,z)\,.\end{array}\right. \tag{C.2}$$

Then from the diagonal part of (C.1) we conclude:

$$M_{11}=-u_tE_1(z)+\alpha(x)+\triangle M_{11}\,, \tag{C.3}$$

where

$$\alpha(x)=-\frac{1}{4\pi\sqrt{-1}}\left(\frac{1}{8\pi^2h}v^2u_x+6u_x\wp(2u)+\frac{u_{xxx}\nu-\nu_xu_{xx}}{2\nu^3}\right)\,, \tag{C.4}$$

and $\triangle M_{11}$ will be defined in the following. It is supposed to be dependent on $E_2(2u)$ in order to cancel terms proportional to $E_2(2u)$ and $E_2'(2u)$ in (C.1).

Using formula (B.8) in the non diagonal part of (C.1), we arrive to some conditions equivalent to cancellations of the terms proportional to functions $\xi(2u,z)=E_1(2u+z)-E(2u)$, $E_1(z)$, $E_1^2(z)$, $E_1(z)\xi(-2u,z)$:

$$(12)\quad\left\{\begin{array}{l}E_1(z)\xi(2u,z):\quad f^1_{12}=-c\,,\\ E_1^2(z):\quad f^1_{12}=-c\,,\\ \xi(2u,z):\quad 2\nu u_t-c_x-2u_xf^0_{12}=-2c\frac{v}{4\pi\sqrt{-1}}\,,\\ E_1(z):\quad -\partial_xf^1_{12}=-2\frac{v}{4\pi\sqrt{-1}}f^1_{12}-2u_xf^0_{12}+2u_t\nu\,.\end{array}\right. \tag{C.5}$$

$$
(21)\quad \begin{cases} E_1(z)\xi(-2u,z): \quad f^1_{12} = -c\,, \\ E_1^2(z): \quad f^1_{12} = -c\,, \\ \xi(-2u,z): \quad -2\nu u_t - c_x + 2u_x f^0_{21} = 2c\frac{v}{4\pi\sqrt{-1}}\,, \\ E_1(z): \quad -\partial_x f^1_{21} = 2\frac{v}{4\pi\sqrt{-1}} f^1_{12} + 2u_x f^0_{21} - 2u_t\nu\,. \end{cases} \tag{C.6}
$$

Thus

$$
\begin{cases} f^1_{12} = f^1_{21} = -c\,, \\ 2u_x f^0_{12} = 2\nu u_t - c_x + 2c\frac{v}{4\pi\sqrt{-1}}\,, \\ 2u_x f^0_{21} = 2\nu u_t + c_x + 2c\frac{v}{4\pi\sqrt{-1}}\,. \end{cases} \tag{C.7}
$$

$$
\begin{cases} f_+ = f^0_{21} + f^0_{12} = \frac{2}{u_x}(\nu u_t + c\frac{v}{4\pi\sqrt{-1}})\,, \\ f_- = f^0_{21} - f^0_{12} = \frac{c_x}{u_x}\,. \end{cases} \tag{C.8}
$$

The remaining parts of the non diagonal equations are:

$$
\begin{cases} \nu_t + 12cu_x\eta_1 - \partial_x f^0_{12} - 2cu_x E_2(z) - 4cu_x E_2(2u) = \\ = -2\frac{v}{4\pi\sqrt{-1}} f^0_{12} - 2\nu\alpha - 2\nu\triangle M_{11}\,, \\ -\nu_t + 12cu_x\eta_1 + \partial_x f^0_{21} - 2cu_x E_2(z) - 4cu_x E_2(2u) = \\ = -2\frac{v}{4\pi\sqrt{-1}} f^0_{21} - 2\nu\alpha - 2\nu\triangle M_{11}\,. \end{cases} \tag{C.9}
$$

Subtracting the above equations we have:

$$
2\frac{u_x}{\nu}\partial_x u_t + \partial_x f_+ = -2\frac{v}{4\pi\sqrt{-1}} f_-\,. \tag{C.10}
$$

Substituting $f_+$ and $f_-$ from (C.8) into (C.10) we arrive to the equation for $c$:

$$
\frac{u_x}{\nu}\partial_x u_t + \partial_x(\frac{1}{u_x}(\nu u_t + c\frac{v}{4\pi\sqrt{-1}})) = -\frac{v}{4\pi\sqrt{-1}}\frac{c_x}{u_x}\,. \tag{C.11}
$$

Now some concrete equations of motion should be used. For $H^{CM}_{-1}$ this equation yields $c \sim \sqrt{\frac{u_x}{v}}$. However the coefficient of the proportionality appear to be equal zero. For $H^{CM}_0$ (6.15) we have $c = -\frac{\nu}{2\pi\sqrt{-1}}$.

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