Mixed motives and algebraic cycles II

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The idea of motives goes back to A. Grothendieck (cf. [Kl 1, 2] [Ma]). As he worked on the Weil conjecture on the zeta function of a variety X over a finite field k, he noticed the importance of studying the action of the Frobenius map on the ℓ-adic etale cohomology $H^i(X \otimes_k \bar{k}, \mathbb{Q}_\ell)$; to prove the conjecture, one must show the characteristic polynomial of the Frobenius is a $\mathbb{Q}$-polynomial independent of ℓ, and its eigenvalues have expected absolute values. He postulated certain conjectures on algebraic cycle classes, called *the standard conjectures*, and observed that they imply the Weil conjecture.

The standard conjectures are concerned not only with ℓ-adic cohomology, but with any other Weil cohomology (such as Betti (singular) or de Rham cohomology) as well, and the consequences are further-reaching. They indicate the existence of the theory of (pure Grothendieck) motives : there is a semi-simple abelian category of Grothendieck motives $\mathcal{M}(k)$ (where k is any ground field) satisfying the properties:

(1) To a smooth projective variety X over k, its *motive* $h(X)$, an object of $\mathcal{M}(k)$, is associated; $h(X)$ is contravariantly functorial in X.

(2) Any Weil cohomology H^* (which is a contravariant functor from the category of smooth projective varieties to the category of graded vector spaces) factors through $\mathcal{M}(k)$. The resulting functor (called the realization functor) from $\mathcal{M}(k)$ to (*graded vector sp.*) is exact and faithful.

Morally, the vector space $H^*(X)$ together with its functorial behavior is ruled by the motive $h(X)$. The functor h is the universal Weil cohomology taking values in a semi-simple abelian category.

In view of the discovery of mixed Hodge structures by Deligne – the theory of weights for Betti cohomology [De 2], one may naturally suspect the existence of the category of mixed motives – motives with weights – which contains the "motives" of quasi-projective singular varieties. This abelian category is no more semi-simple, which makes it all the more interesting, because extension groups between objects are non-trivial. The significance of such a category was made clear by the conjecture that these extension groups should be related to Chow groups and higher K-groups of varieties, [Be], [De]; see also [Ja].

The purpose of this paper is to construct the category $\mathcal{D}(k)$ of mixed motives (over a field k) and study some properties. Such a construction was made possible by incorporating the theory of higher Chow groups due to S. Bloch. We perceive a mixed motive as a *diagram* built up of smooth projective varieties and correspondences (in the sense of higher Chow groups) between them.

Part of this paper appeared in [Ha 1] and part III is in preparation where we discuss conjectural properties of the category $\mathcal{D}(k)$. The paper [Ha 2] treats the material in §6 more fully. The references at the end are limited; for a more comprehensive bibliography related to mixed motives, the reader may consult [Ja], for example.

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For other approaches to categorical constructions, see [Le 2] and [Vo]. In our conception, higher correspondences are built into the very definition of motives, while in the other constructions higher Chow groups are revealed later in the theories.

§1 – §3 are preliminary materials. We discuss some results on cycle complexes in §1. A large part is concerned with the notion of distinguished subcomplexes of a cycle complex; on many occasions it is necessary to restrict oneself to a subcomplex on which certain operations are defined. We need to define such distinguished subcomplexes, and show the "moving lemma": a distinguished subcomplex is quasi-isomorphic to the whole cycle complex. The results are stated in (1.1)-(1.7). The detailed proofs take the rest of the section; the reader may skip them in the first reading.

In §2, we recall the construction of the category of pure motives by Grothendieck. The reason of describing the construction in a way slightly different from the literature – introducing the category of formal symbols – is because it will also appear as an intermediate category to construct $\mathcal{D}(k)$.

As a generalization of the notion of double complexes, we introduce C-complexes in §3 in an abstract form. Similar types of objects appear in the definition of mixed motives in §4 and in their cohomology realizations in §5.

We define the category $\mathcal{D}(k)$ in §4. Remark both the objects and the morphisms will be quite explicitly given. We also study the structures of $\mathcal{D}(k)$.

(4.10) **Theorem.** *The triangulated category $\mathcal{D}(k)$ has the following properties:*

(1) $\mathcal{D}(k)$ has dual, tensor product, inner Hom, the unit object $\underline{\mathbb{Q}}$, and the Tate objects $\underline{\mathbb{Q}}(r)$;

(2) ($ch(k) = 0$) The contravariant functor $h : (Smooth\,Proj./k) \to \mathcal{D}(k)$ can be extended to a functor from the category of quasi-projective varieties; to each quasi-projective X there is associated its (cohomological) motive $h(X)$, an object of $\mathcal{D}(k)$;

(3) If X is smooth and projective, one has

$$\mathrm{Hom}_{\mathcal{D}(k)}(\underline{\mathbb{Q}}, h(X)(r)[2r-m]) = K_m(X)^{(r)}_{\mathbb{Q}} \ .$$

Here the right hand side is an Adams-graded piece of the K-group of X.

(4) If H^* denotes one of the etale, Betti or algebraic de Rham cohomology, there is the corresponding realization functor $H^* : \mathcal{D}(k) \to Vect\,(\Lambda)$; here $Vect\,(\Lambda)$ is the category of Λ-vector spaces, and Λ is the coefficient field of the cohomology theory.

In §5, the cohomology realization functors are constructed as stated in (4).

§6 is an exposition of (2) and more. In particular, for a quasi-projective variety over k of characteristic zero, we define its higher Chow cohomology/homology groups with/without compact support.

The conjectural properties of the category $\mathcal{D}(k)$ are discussed in Part III, particularly the existence of the expected t-structure. Assuming some "standard conjectures" (slightly more than the conjectures by Grothendieck, Murre and Soulé-Beilinson) it can be proven: There is a unique t-structure on $\mathcal{D}(k)$ such that any one of singular, etale, or de Rham cohomology functor $\mathbb{R}\Gamma : \mathcal{D}(k) \to D(\Lambda)$ (where $D(\Lambda)$ is the derived category of complexes of Λ-vector spaces) is a t-exact functor with respect to the t-structure and the natural t-structure on $D(\Lambda)$. The t-structure would then provide an abelian subcategory of $\mathcal{D}(k)$ as its heart; this is the candidate of the abelian category of mixed motives.

The following displays categories and functors appearing in this paper.($\hookrightarrow$ indicates a fully faithful functor.)

$$\begin{array}{ccccc} & & \mathcal{D}_{finite}(k) & & \\ & \nearrow & & \searrow & \\ Symb_{finite}(k) & \hookrightarrow & CH\mathcal{M}(k) & \hookrightarrow & \mathcal{D}(k) \\ \downarrow & & \downarrow & & \downarrow \\ Symb_{inf}(k) & \hookrightarrow & CH\mathcal{M}_{inf}(k) & \hookrightarrow & \mathcal{D}_{inf}(k) \end{array}$$

There is the "cohomological motive" functor

$$h : (Smooth\, Proj/k)^{opp} \to \mathcal{D}_{finite}(k) \ .$$

This factors through $Symb_{finite}(k)$. If $ch(k) = 0$, h extends to $h : (Quasi - Proj/k)^{opp} \to \mathcal{D}_{finite}(k)$.

(See Part III) Under certain conjectures, an abelian subcategory (consisting of cohomological degree 0 objects) $\mathcal{D}(k)^{\deg 0} \hookrightarrow \mathcal{D}(k)$ exists, and one has a *non-commutative* square

$$\begin{array}{ccc} CH\mathcal{M}(k) & \hookrightarrow & \mathcal{D}(k) \\ \downarrow cano & & \uparrow \\ \mathcal{M}(k) & \hookrightarrow & \mathcal{D}(k)^{\deg 0} \end{array}$$

§1. Cycle complexes and higher Chow groups.

In this paper k is an arbitrary ground field, and one considers separated schemes of finite type (we will simply say schemes) over k. A variety is a reduced, possibly reducible scheme over k.

The references for cycle complexes are [Bl 1,2,3] [Bl-Le] (the last one is yet to appear). The exposition below will be self-contained except for the citation of (1.1) and (1.8). For generalities on algebraic cycles and intersection theory, see [Fu]. Among the various versions, we employ the alternating cubical cycle complex [Bl 3], since we need the graded commutativity for the internal product.

Let $\square^n = \mathbb{A}^n_k$ with coordinates $(x_1, \cdots, x_n)$. Faces of $\square^n$ are intersections of codimension one faces, and the latter are divisors of the form $\square^{n-1}_{i,a} = \{x_i = a\}$ where $a = 0$ or 1. A face of dimension m is canonically isomorphic to $\square^m$. Let Σ_n be the permutation group of the set $\{1, \cdots, n\}$ acting on $\square^n$ by permutations of the coordinates.

Let X be an equi-dimensional variety (or a scheme) over a field k. Let $\mathcal{Z}^r(X, n)$ be the $\mathbb{Q}$-vector space of $\mathbb{Q}$-cycles z of codimension r on $X \times \square^n$ such that

(i) each irreducible component of z meets each face $X \times \square^m$ properly, and

(ii) z is alternating with respect to Σ_n, namely, for any $\sigma \in \Sigma_n$, $\sigma(z) = sgn(\sigma)z$.

The inclusions of codimension one faces $\delta_{i,a} : \square^{n-1}_{i,a} \hookrightarrow \square^n$ induce the map

$$\partial = \sum (-1)^{i+a} \delta^*_{i,a} : \mathcal{Z}^r(X,n) \to \mathcal{Z}^r(X,n-1)$$

(namely, ∂ sends alternating cycles to alternating ones) and $(\mathcal{Z}^r(X,\cdot), \partial)$ is a homology complex. We call this the *cycle complex* (of codimension r) of X. By definition the (rational) higher Chow groups are the homology groups of this complex:

$$CH^r(X,n) = H_n \mathcal{Z}^r(X,\cdot) \ .$$

Note $CH^r(X,0) = CH^r(X)$, the rational Chow group of X.

In [Bl-Le], considering $\mathbb{Z}$- cycles in $X \times \square^n$ which are "normalized", in place of $\mathbb{Q}$-cycles, integral higher Chow group is defined, and denoted by $CH^r(X,n)$. In this paper we only consider rational cycle complexes and rational Chow groups. Rational Chow groups are isomorphic to Adams graded pieces of K-groups of X.

(1.1) **Theorem** ([Bl 2] [Le 1]). *There is a canonical isomorphism*

$$CH^r(X,n) \cong K'_n(X)^{(r)}_{\mathbb{Q}}$$

where $K'_n(X)$ is the K-group of the exact category of coherent sheaves on X, $K'_n(X)_{\mathbb{Q}} = K'_n(X) \otimes \mathbb{Q}$, and $K'_n(X)^{(r)}_{\mathbb{Q}}$ is the r-th Adams eigenspace.

For the rest of the section, all varieties are equi-dimensional. A proper map of varieties $f : X \to Y$ induces a map of cycle complexes $f_* : \mathcal{Z}^r(X,\cdot) \to \mathcal{Z}^{r+\dim Y - \dim X}(Y,\cdot)$. If $f : X \to Y$ is a flat map, there is induced a map $f^* : \mathcal{Z}^r(Y,\cdot) \to \mathcal{Z}^r(X,\cdot)$. If Y is smooth, there is a quasi-isomorphic subcomplex $\mathcal{Z}^r(Y,\cdot)' \subset \mathcal{Z}^r(Y,\cdot)$ on which $f^* : \mathcal{Z}^r(Y,\cdot)' \to \mathcal{Z}^r(X,\cdot)$ is defined.

For more general results, valid only for smooth varieties, consider the following operations. Let X, Y, Z be smooth projective varieties.

The map induced by a correspondence. For an element $f \in \mathcal{Z}^s(X \times Y, \ell)$, the partially defined map (of graded vector spaces)

$$f_* : \mathcal{Z}^r(X,\cdot) - - \to \mathcal{Z}^{r+s}(Y, \cdot + \ell), \qquad f_*(z) = p_{Y*}(f \cdot (z \times Y)) \ .$$

Here $f \cdot (z \times Y)$ denotes the alternation of the intersection theoretic $f \cdot (z \times Y)$ with respect the action of $\Sigma_{n+\ell}$. One has the formula

$$(\partial f)_* = \partial \circ f_* - (-1)^{\ell-1} f_* \circ \partial$$

whenever all the terms are defined.

Transpose. The involution

$$\mathcal{Z}^r(X \times Y, \cdot) \xrightarrow{\sim} \mathcal{Z}^r(Y \times X, \cdot), \quad z \mapsto {}^t z := s_* z$$

where $s : X \times Y \to Y \times X$ is the isomorphism which switches the factors.

External product. The map

$$\mathcal{Z}^r(X,\cdot) \otimes \mathcal{Z}^s(Y,\cdot) \to \mathcal{Z}^{r+s}(X \times Y,\cdot)$$

which sends $z \otimes w$ to $z \times w$. (More precisely, if $z \in \mathcal{Z}^r(X,n)$ and $w \in \mathcal{Z}^s(Y,m)$ $z \times w$ is a cycle on $X \times \square^n \times Y \times \square^m$. The isomorphism which switches the factors $s : X \times \square^n \times Y \times \square^m \xrightarrow{\sim} X \times Y \times \square^n \times \square^m$ gives the cycle $s_*(z \times w)$ on $X \times Y \times \square^{n+m}$; take then its alternation with respect to Σ_{n+m}. Denote this by $z \times w$ by abuse of notation.) This is associative and a map of complexes, namely satisfies Leibniz formula

$$\partial(z \times w) = (\partial z) \times w + (-1)^{\deg z}\, z \times (\partial w)\ .$$

The transpose is graded anti-commutative:

$${}^t(z \times w) = (-1)^{\deg z\ \deg w}\ {}^t w \times {}^t z\ .$$

Internal product. The partially defined map

$$\mathcal{Z}^r(X,\cdot) \otimes \mathcal{Z}^s(X,\cdot) \dashrightarrow \mathcal{Z}^{r+s}(X,\cdot)$$

which sends $z \otimes w$ to $z \cdot w :=$ the alternation of $(z \times w) \cdot \Delta_X$. This is associative, graded commutative and satisfies Leibniz formula. The graded commutativity reads: $w \cdot z = (-1)^{\deg w\ \deg z} z \cdot w$.

Composition of correspondences. The partially defined map

$$\mathcal{Z}^r(X \times Y,\cdot) \otimes \mathcal{Z}^s(Y \times Z,\cdot) \dashrightarrow \mathcal{Z}^{r+s-\dim Y}(X \times Z,\cdot)$$

which sends $u \otimes v \in \mathcal{Z}^r(X \times Y, n) \otimes \mathcal{Z}^s(Y \times Z, m)$ to $v \circ u :=$ the alternation of $p_{XZ\,*}[(X \times v \times \square^n) \cdot (u \times Z \times \square^m)]$. (Here again, $u \times Z \times \square^m$, for example, is a cycle on $X \times Y \times \square^n \times Z \times \square^m$ and one regards it as a cycle on $X \times Y \times Z \times \square^{n+m}$. For the rest of the paper, we will use similar abuse of notation without further explanation.) This is associative and satisfies Leibniz formula. One also has

$${}^t(u \circ v) = (-1)^{\deg u\ \deg v}\ {}^t v \circ {}^t u\ .$$

Relations among the above operations. If $f : X \to Y$ is a map of smooth projective varieties, and $\Gamma_f \subset Y \times X$ its graph, one has

$$(\Gamma_f)_* = f^* \text{ and } ({}^t\Gamma_f)_* = f_*\ .$$

If $f \in \mathcal{Z}(X \times Y, \ell)$ and $g \in \mathcal{Z}(Y \times Z, m)$, then

$$g_* f_* = (g \circ f)_*\ .$$

If $u \in \mathcal{Z}(X \times Y, \ell)$ and $v \in \mathcal{Z}(Y \times Z, m)$, then

$$v \circ u = (\Delta_X \times v)_* u$$

where $\Delta_X \times v \in \mathcal{Z}((X \times Y) \times (X \times Z), m)$. Similarly, $v \circ u = ({}^t u \times \Delta_Z)_* v$.

Remark. Some of the above operations are defined for smooth quasi-projective varieties. For example, for the map f_* be defined, one only needs X smooth projective and Y smooth quasi-projective; external and internal products are defined for smooth quasi-projective X. For the rest of the section also, one may often work under weaker assumptions on varieties. Yet we will not be concerned with that, and assume varieties X, Y, etc. are smooth projective (unless otherwise stated); it will suffice for our purposes.

Two subvarieties V, W on a equi-dimensional variety X *intersect properly* if $\dim(V \cap W) \leq \dim V + \dim W - \dim X$. We define that two cycles α, β on X intersect properly if each component of α and each component of β intersect properly. We will denote proper intersection by $\alpha \quad \not{P} \quad \beta$.

For a set $I = \{(a, i)\} \subset \{0, 1\} \times [m]$, such that for each i, there is at most one a such that $(a, i) \in I$, there is the corresponding face $\square_I^{m-|I|} = \{x_i = a \quad \text{for} \quad (a, i) \in I\} \subset \square^m$ and the boundary map $\partial_I : \mathcal{Z}(X, m) \to \mathcal{Z}(X, m - |I|)$.

Let X be a smooth projective variety. Let Y be a smooth projective variety and $W = \{W_i\}$ a finite set where W_i is an algebraic cycle on $X \times Y \times \square^{\ell_i}$ meeting faces properly. Let $\mathcal{Z}_W^r(X, \cdot)$ be the subcomplex of $\mathcal{Z}^r(X, \cdot)$ defined as follows: $z \in \mathcal{Z}^r(X, n)$ is in $\mathcal{Z}_W^r(X, n)$ iff for each W_i

$$\begin{aligned} &z \times Y \times \square^{\ell_i} \quad \not{P} \quad W_i \times \square^n \quad \text{in} \quad X \times Y \times \square^{n+\ell_i} \quad \text{and} \\ &(z \times Y \times \square^{\ell_i}) \cdot (W_i \times \square^n) \quad \not{P} \quad \text{any face of } X \times Y \times \square^{n+\ell_i} \end{aligned}$$

where $(z \times Y \times \square^{\ell_i}) \cdot (W_i \times \square^n)$ is the cycle theoretic intersection, not its alternation. When there is no confusion, we will simply put "$(z \times Y \times \square^{\ell_i}) \cdot (W_i \times \square^n)$ is defined and $\in \mathcal{Z}(X \times Y, n + \ell_i)$." The condition is equivalent to:

$$\begin{aligned} &\text{For any faces} \quad \square_J \subset \square^{\ell_i} \quad \text{and} \quad \square^{n'} \subset \square^n, \\ &z \times Y \times \square^{\ell_i} \quad \not{P} \quad \partial_J W_i \times \square^{n'} \quad \text{in} \quad X \times Y \times \square^{n+\ell_i}\ . \end{aligned}$$

The equivalence follows from the following lemma (readily verified and often to be used).

(1.2) **Lemma.** Let A, B and C be algebraic cycles on a quasi-projective variety X such that $A \quad \not{P} \quad B$, $B \quad \not{P} \quad C$, and $A \quad \not{P} \quad C$. Then

$$A \cdot C \quad \not{P} \quad B \Leftrightarrow A \cdot B \quad \not{P} \quad C\ .$$

Remark. In the definition of $\mathcal{Z}_W(X, \cdot)$ one might be tempted to require only the alternation of the cycle theoretic intersection meet properly the faces. The stronger requirement will turn out to be more appropriate: for example Proposition (1.5) holds because "$g \circ f$ is defined and $\in \mathcal{Z}(X \times Z)$".

A subcomplex of the form $\mathcal{Z}_W^r(X, \cdot)$ is called a *distinguished subcomplex.* We will use the notation $\mathcal{Z}^r(X, \cdot)'$ if it is not necessary to specify W. Note that the intersection of a finite collection of distinguished subcomplexes is again distinguished. The proof of the following is in (1.12).

(1.3) **Proposition.** *For a smooth projective variety X the inclusion of a distinguished subcomplex $\mathcal{Z}^r(X,\cdot)' \to \mathcal{Z}^r(X,\cdot)$ is a quasi-isomorphism.*

(1.4) **Proposition.** *(1) Let $f \in \mathcal{Z}^s(X\times Y,\ell)$. Define the subcomplex $\mathcal{Z}^r_{f_*}(X,\cdot) \subset \mathcal{Z}^r(X,\cdot)$ by: $z \in \mathcal{Z}^r(X,n)$ is in $\mathcal{Z}^r_{f_*}(X,n)$ iff*

$$\begin{array}{rcll} z \times Y \times \square^\ell & \not R & f \times \square^n & \text{in} \quad X \times Y \times \square^{n+\ell}, \quad \text{and} \\ (f \times \square^n)\cdot(z \times Y \times \square^\ell) & \not R & & \text{any face of} \quad X \times Y \times \square^{n+\ell}\ . \end{array}$$

(in other words f_*z is defined and $\in \mathcal{Z}(X\times Y, n+\ell)$). Then $\mathcal{Z}^r_{f_*}(X,\cdot)$ is a distinguished subcomplex and one has $f_* : \mathcal{Z}^r_{f_*}(X,\cdot) \to \mathcal{Z}^{r+s}(Y,\cdot+\ell)$.

(2) Let $w \in \mathcal{Z}^s(X,\ell)$. Define the subcomplex $\mathcal{Z}^r_{w\cdot}(X,\cdot) \subset \mathcal{Z}^r(X,\cdot)$ by: $z \in \mathcal{Z}^r(X,n)$ is in $\mathcal{Z}^r_{w\cdot}(X,n)$ iff

$$\begin{array}{rcll} z \times \square^\ell & \not R & w \times \square^n & \text{in} \quad X \times \square^{n+\ell}, \quad \text{and} \\ (w \times \square^n)\cdot(z \times \square^\ell) & \not R & & \text{any face of} \quad X \times \square^{n+\ell}\ . \end{array}$$

Then $\mathcal{Z}^r_{w\cdot}(X,\cdot)$ is distinguished and one has $w\cdot : \mathcal{Z}^r_{w\cdot}(X,\cdot) \to \mathcal{Z}^{r+s}(X,\cdot+\ell)$.

(3) Let $v \in \mathcal{Z}^s(Y\times Z,\ell)$. Define the subcomplex $\mathcal{Z}^r_{v\circ}(X\times Y,\cdot) \subset \mathcal{Z}^r(X\times Y,\cdot)$ by: $u \in \mathcal{Z}^r(X\times Y,n)$ is in $\mathcal{Z}^r_{v\circ}(X\times Y,n)$ iff

$$\begin{array}{rcll} u \times Z \times \square^\ell & \not R & X \times v \times \square^n & \text{and} \\ (X \times v \times \square^n)\cdot(u \times Z \times \square^\ell) & \not R & & \text{any face of} \quad X \times Y \times Z \times \square^{n+\ell}\ . \end{array}$$

Then the subcomplex is distinguished and one has $v\circ : \mathcal{Z}^r_{v\circ}(X\times Y,\cdot) \to \mathcal{Z}^{r+s-\dim Y}(X\times Z,\cdot+\ell)$.

One has a similar statement for $w \in \mathcal{Z}^s(Z\times X,\ell)$ and the map $\circ w$.

Proof. (1) Clearly, $\mathcal{Z}^r_{f_*}(X,\cdot) = \mathcal{Z}^r_{\{f\}}(X,\cdot)$.

(2) Let $\delta : X \to X\times X$ be the diagonal map. Then $\mathcal{Z}^r_{w\cdot}(X,\cdot)$ is the distinguished subcomplex with respect to the cycle $\delta_*(w)$ on $X\times X\times\square^\ell$.

(3) $\mathcal{Z}^r_{v\circ}(X\times Y,\cdot)$ is the distinguished subcomplex with respect to the cycle $X\times v$ on $X\times Y\times Z\times\square^\ell$.

The following amplification holds. It will be used in §4.

(1.5) **Proposition.** *(1) Let $f \in \mathcal{Z}(X\times Y,\ell)$, $g \in \mathcal{Z}(Y\times Z,m)$ such that $g\circ f$ is defined and $\in \mathcal{Z}(X\times Z,\ell+m)$. Then there are distinguished subcomplexes $\mathcal{Z}(X,\cdot)'$ and $\mathcal{Z}(Y,\cdot)'$ such that the three maps*

$$f_* : \mathcal{Z}(X,\cdot)' \to \mathcal{Z}(Y,\cdot+\ell)', \qquad g_* : \mathcal{Z}(Y,\cdot)' \to \mathcal{Z}(Z,\cdot+m), \ \text{and}$$
$$(g\circ f)_* : \mathcal{Z}(X,\cdot)' \to \mathcal{Z}(Z,\cdot+\ell+m)$$

are all defined and $(g{\circ}f)_* = g_*{\circ}f_*$.

(2) Let $v \in \mathcal{Z}(Y\times Z,\ell)$, $w \in \mathcal{Z}(Z\times W, m)$ such that $w{\circ}v$ is defined and $\in \mathcal{Z}(Y\times W,\ell+m)$. Then there are distinguished subcomplexes $\mathcal{Z}(X\times Y,\cdot)'$ and $\mathcal{Z}(Y\times Z,\cdot)'$ such that the three maps

$$v\circ : \mathcal{Z}(X\times Y,\cdot)' \to \mathcal{Z}(X\times Z,\cdot+\ell)', \qquad w\circ : \mathcal{Z}(X\times Z,\cdot)' \to \mathcal{Z}(X\times W,\cdot+m), \text{ and}$$
$$(w{\circ}v)\circ : \mathcal{Z}(X\times Y,\cdot)' \to \mathcal{Z}(X\times W,\cdot+\ell+m)$$

are all defined and $(w\circ)(v\circ) = (w{\circ}v)\circ$. (There is a similar statement for compositions "from the right".)

Remark. These can be generalized to the case where a finite sequence of composable correspondences are given. For example let X, Y and Z be smooth projective varieties and $f \in \mathcal{Z}(X\times Y,\ell)$, $g\in\mathcal{Z}(Y\times Z,m)$ and $h\in\mathcal{Z}(Z\times W,n)$. We assume: $g{\circ}f$ is defined and $\in\mathcal{Z}(X\times Z,\ell+m)$, $h{\circ}g$ is defined and $\in \mathcal{Z}(Y\times W, m+n)$, and $h{\circ}g{\circ}f$ is defined and $\in\mathcal{Z}(X\times W,\ell+m+n)$. The last condition is defined as follows:

$(f\times Z\times W\times\square^{m+n})\cdot(X\times g\times W\times\square^{\ell+n}) \quad \pitchfork \quad X\times Y\times h\times\square^{\ell+m}$ and the intersection meeting properly the faces of $X\times Y\times Z\times W\times\square^{\ell+m+n}$.

then there are distinguished subcomplexes $\mathcal{Z}(X,\cdot)'$, $\mathcal{Z}(Y,\cdot)'$ and $\mathcal{Z}(Z,\cdot)'$ such that the maps

$$\mathcal{Z}(X,\cdot)' \xrightarrow{f_*} \mathcal{Z}(Y,\cdot+\ell)' \xrightarrow{g_*} \mathcal{Z}(Z,\cdot+\ell+m)' \xrightarrow{h_*} \mathcal{Z}(W,\cdot+\ell+m+n)$$

are all defined. Similarly for (2). (The proof below is for the case of two correspondences; the general case is similar.)

Proof. (1) Since $g{\circ}f$ is defined,

$$f\times Z\times\square^m \quad\pitchfork\quad X\times g\times\square^\ell \text{ in } X\times Y\times Z\times\square^{\ell+m}\ .$$

Take the distinguished subcomplex $\mathcal{Z}(X,\cdot)'\subset\mathcal{Z}(X,\cdot)$ with respect to the set

$$\{f\times Z,\quad X\times g,\quad (f\times Z\times\square^m)\cdot(X\times g\times\square^\ell)\}$$

of cycles on $X\times Y\times Z\times\square^?$ (the third one is the intersection theoretic one, not its alteration) and the distinguished subcomplex $\mathcal{Z}(Y,\cdot)'$ with respect to $\{g\}$ on $Y\times Z\times\square^m$. We claim these satisfy the required conditions.

For $z\in\mathcal{Z}(X,n)'$, we need to show

$$(*)\qquad p_{Y*}[(f\times\square^n)\cdot(z\times Y\times\square^\ell)]\times Z\times\square^m \quad\pitchfork\quad g\times\square^{n+\ell} \text{ in } Y\times Z\times\square^{n+m+\ell}$$

and more generally

$$(**)\quad p_{Y*}[(f\times\square^n)\cdot(z\times Y\times\square^\ell)]\times Z\times\square^m \quad\pitchfork\quad \partial_K g\times\square_J\times\square^{n'} \text{ in } \quad Y\times Z\times\square^{n+m+\ell}\ .$$

for faces $\square_J\subset\square^\ell$, $\square_K\subset\square^m$ and $\square^{n'}\subset\square^n$.

¿From

$$(z\times Y\times Z\times\square^{\ell+m})\quad\pitchfork\quad(f\times Z\times\square^{n+m})\cdot(X\times g\times\square^{n+\ell})$$

using Lemma (1.2) we have

$$(z \times Y \times Z \times \square^{\ell+m}) \cdot (f \times Z \times \square^{n+m}) \quad \not R \quad X \times g \times \square^{n+\ell} \ .$$

By the lemma below (easily verified) applied to the projection $p : X \times Y \times Z \times \square^{n+m+\ell} \to Y \times Z \times \square^{n+m+\ell}$ we obtain (*). The proof of (**) is similar. (2) is proved in a similar manner.

Lemma. Let W be a quasi-projective variety, X an equi-dimensional projective variety and $p : X \times W \to W$ the projection. For algebraic cycles u on $X \times W$ and w on W,

$$u \quad \not R \quad X \times w \quad \Rightarrow \quad p_* u \quad \not R \quad w \ .$$

We now consider the tensor product of cycle complexes

$$\mathcal{Z}^r(X, \cdot) \otimes \mathcal{Z}^s(Y, \cdot)$$

where X, Y are smooth projective. Let $W = \{W_i\}$ be a finite set such that for a smooth projective Z, W_i are cycles on $X \times Y \times Z \times \square^{\ell_i}$ meeting faces properly. Define the subcomplex

$$\left[\mathcal{Z}^r(X, \cdot) \otimes \mathcal{Z}^s(Y, \cdot)\right]_W$$

by the condition: An element $z \otimes w \in \mathcal{Z}^r(X, n) \otimes \mathcal{Z}^s(Y, m)$ is in the subcomplex iff

$$\begin{array}{l} z \times w \times Z \times \square^{\ell_i} \quad \not R \quad W_i \times \square^{n+m} \quad \text{in} \quad X \times Y \times Z \times \square^{n+m+\ell_i} \ \text{and} \\ \text{the intersection} \quad \not R \quad \text{any face of} \quad X \times Y \times Z \times \square^{n+m+\ell_i} \ . \end{array}$$

A subcomplex of this form is called a *distinguished subcomplex.* The intersection of a finite number of distinguished subcomplexes is distinguished. We again postpone the proof of the following until (1.13).

(1.6) **Proposition.** *The inclusion of a distinguished complex into $\mathcal{Z}^r(X, \cdot) \otimes \mathcal{Z}^s(Y, \cdot)$ is a quasi-isomorphism.*

(1.7) **Proposition.** *(1) Let $\mathcal{Z}^{r+s}(X \times Y, \cdot)'$ be a distinguished subcomplex. Define the subcomplex $\left[\mathcal{Z}^r(X, \cdot) \otimes \mathcal{Z}^s(Y, \cdot)\right]'$ by the condition: an element $z \otimes w$ is in the subcomplex iff $z \times w \in \mathcal{Z}^{r+s}(X \times Y, \cdot)'$. Then the subcomplex is distinguished.*

(2) Let $\mathcal{Z}^{r+s}(X, \cdot)'$ be a distinguished subcomplex. Define the subcomplex $\left[\mathcal{Z}^r(X, \cdot) \otimes \mathcal{Z}^s(X, \cdot)\right]'$ by the condition: an element $z \otimes w$ is in the subcomplex iff $z \cdot w$ is defined and $\in \mathcal{Z}^{r+s}(X, \cdot)'$. Then the subcomplex is distinguished.

(3) Let $\mathcal{Z}^{r+s}(X \times Z, \cdot)'$ be a distinguished subcomplex. Define the subcomplex $\left[\mathcal{Z}^r(X \times Y, \cdot) \otimes \mathcal{Z}^s(Y \times Z, \cdot)\right]'$ by the condition: an element $u \otimes v$ is in the subcomplex iff $v \circ u$ is defined and $\in \mathcal{Z}^{r+s}(X \times Z, \cdot)'$. Then the subcomplex is distinguished.

Remark. In (2), in particular, the subcomplex (of the pairable cycles) $\left[\mathcal{Z}^r(X, \cdot) \otimes \mathcal{Z}^s(X, \cdot)\right]^{pairable}$, consisting of $z \times w$ for which $z \cdot w$ is defined, is distinguished. Similarly for the composition (3).

Proof. (1) There is a smooth variety Z and cycles W_i on $X \times Y \times Z \times \square^{\ell_i}$ such that $\mathcal{Z}^{r+s}(X \times Y, \cdot)' = \mathcal{Z}^{r+s}_W(X \times Y, \cdot)$. Then the subcomplex in consideration is equal to $\left[\mathcal{Z}^r(X, \cdot) \otimes \mathcal{Z}^s(Y, \cdot)\right]_W$.

(2) Let W_i are on $X \times Y \times \square^{\ell_i}$ such that $\mathcal{Z}^{r+s}(X, \cdot)' = \mathcal{Z}^{r+s}_W(X, \cdot)$. Then the subcomplex $\left[\mathcal{Z}^r(X, \cdot) \otimes \mathcal{Z}^s(X, \cdot)\right]'$ is equal to the distinguished subcomplex with respect to the collection $\{\delta_* W_i\}$ on $X \times X \times Y \times \square^{\ell_i}$ ($\delta : X \times Y \to X \times X \times Y$ is the diagonal).

(3) Similar to (1), (2) and left to the reader.

The rest of §1 is concerned with the proof of (1.3) and (1.6). It will be an adaptation of [Bl 1]. For example, (1.3) is proven there in the case $Y = pt$ and $\ell_i = 0$. We first show (1.3) in the special case $X = \mathbb{P}^N$. The following lemma is from *loc. cit.*

(1.8) **Lemma.** *Let G be a connected algebraic group acting on an equi-dimensional variety X. Let A, B be equi-dimensional closed subsets of X. Assume that the map induced from the action $G \times A \to X$ is surjective and equi-dimensional (i.e. all fibers are equi-dimensional of the same dimension d). Then:*

(1) There is an non-empty open subset $U \subset G$ such that for any $g \in U(\bar{k})$, $g(A)$ Ꝑ B in X.

(2) Let K be a function field over k and $g \in G(K)$ be K-point which is k-generic. Then $g(A_K)$ Ꝑ B_K in X_K. [(2) follows from (1).]

We will apply this in the following situation. Let $G = SL_{N+1}$ act on $X = \mathbb{P}^N_k$ in the natural way. Let $A, B \subset X$ be any equi-dimensional closed sets. Since G acts on X transitively, the map $G \times A \to X$ is surjective and equi-dimensional. So if $g \in G(K)$ is k-generic, then gA Ꝑ B in X_K.

Let $p : U \to W$ be a quasi-projective map between (equi-dimensional) quasi-projective varieties. Let w be a subvariety of W and assume the scheme $p^{-1}w$ has the expected dimension, i.e., $p^{-1}w$ is equi-dimensional and *cod* $(p^{-1}(w) \subset U) =$ *cod* $(w \subset W)$. Then the *pull-back as a cycle* can be defined:

$$p^* w := [p^{-1} w] \ .$$

([ ] indicates the cycle associated to a scheme.) This definition extends by linearity to cycles w on W.

If u is an algebraic cycle on U, let $|u|$ be the support of u and

$$p(u)_r := \{x \in W \mid \dim(|u| \cap p^{-1}(x)\,) \geq r.\ \} \quad (r = 0, 1, \cdots)$$

This is a locally closed set. Let $\overline{p(u)_r}$ be the closure of $p(u)_r$ and

$$p_\sharp(u) = \{\overline{p(u)_r}\} \ .$$

Note $p_\sharp(u)$ is a collection of closed subsets rather than a cycle; it may not be empty even if $p_*(u) = 0$ as a cycle.

(1.9) **Proposition.** *Let $p : U \to W$ be a quasi-projective map between quasi-projective varieties, u a cycle on U, and w a cycle on W. Suppose $p^*(w)$ is defined. We then have*

$$p_\sharp(u) \quad \not R \quad w \quad \textit{in} \quad W \quad \Rightarrow \quad u \quad \not R \quad p^* w \quad \textit{in} \quad U.$$

(The proof is left for the reader. Hint for the proof: Let

$$p(u)^0_r := \{x \in W \mid \dim(|u| \cap p^{-1}(x)\,) = r.\ \}\ .$$

Then it is locally closed and if $w \quad \not R \quad p_\sharp(u)$, then $w \quad \not R \quad p(u)^0_r$. Evaluate $\dim w \cap p(u)^0_r$, and then $\dim(p^* w \cap u) \cap p^{-1}(p(u)^0_r)$ and hence $\dim(p^* w \cap u)$.)

(1.10) **Proposition.** *Let $W = \{W_i\}$, with W_i cycles on $\mathbb{P}^N \times Y \times \square^{\ell_i}$ meeting faces properly. Then the inclusion $\mathcal{Z}^r_W(\mathbb{P}^N, \cdot) \subset \mathcal{Z}^r(\mathbb{P}^N, \cdot)$ is a quasi-isomorphism.*

Proof. Let $X = \mathbb{P}^N_k$, and K be a purely transcendental function field over k. Let $\pi : X_K \to X$ be the canonical morphism, and

$$\pi^* : \mathcal{Z}^r(X, \cdot) \to \mathcal{Z}^r(X_K, \cdot)$$

be the pull-back by π. Since it induces $\pi^* : \mathcal{Z}^r_W(X, \cdot) \to \mathcal{Z}^r_{W_K}(X_K, \cdot)$, there is the induced map of complexes

$$\pi^* : \mathcal{Z}^r(X, \cdot)/\mathcal{Z}^r_W(X, \cdot) \to \mathcal{Z}^r(X_K, \cdot)/\mathcal{Z}^r_{W_K}(X_K, \cdot)\ .$$

If we show π^* is injective and induces zero on homology, the acyclicity of $\mathcal{Z}^r(X, \cdot)/\mathcal{Z}^r_W(X, \cdot)$ will follow.

The injectivity is verified as follows. First assume k is infinite and $K = k(X_1, \cdots X_n)$. If $z \in \mathcal{Z}^r(X, \cdot)$ satisfies

$$\pi^* z = z_1 + \partial w, \quad z_1 \in \mathcal{Z}^r_W(X_K, \cdot), \quad \text{and} \quad w \in \mathcal{Z}^r(X_K, \cdot + 1)$$

then for an non-empty open set $U \subset \mathbb{A}^n_k$ and $x \in U(k)$, the specializations satisfy $z_1(x) \in \mathcal{Z}^r_W(X, \cdot)$, $w(x) \in \mathcal{Z}^r(X, \cdot + 1)$ and

$$z = z_1(x) + \partial w(x)\ .$$

In general, note if k'/k is a finite field extension, one has maps

$$\mathcal{Z}^r(X_k, \cdot) \xrightarrow{\pi^*} \mathcal{Z}^r(X_{k'}, \cdot) \xrightarrow{\pi_*} \mathcal{Z}^r(X_k, \cdot)$$

and $\pi_* \pi^* = [k' : k]$. Consider an infinite tower of field extensions of p-power degrees (for two different primes p) and apply the above argument.

We have to show π^* induces zero on homology. Take a purely transcendental function field K over k, and a map $\psi : \mathbb{A}^1_K \to G_K$ such that $\psi(0)$ is the identity of $G(K)$ and for any $t \in \mathbb{A}^1(K)$, $t \neq 0$, $\psi(t)$ is k-generic. Let $pr_1 : X \times \mathbb{A}^1 \to X$ be the projection and let

$$\phi : X \times \mathbb{A}^1 \to X \times \mathbb{A}^1$$

be the isomorphism over K given by $(x,a) \mapsto (\psi(a).x, a)$. Let G act on $X \times \mathbb{A}^1$ by $g.(x,a) = (g.x, a)$. Set

$$\rho^* = \phi^* \, pr_1^* \, \pi^* : \mathcal{Z}^r(X_k, \cdot) \to \mathcal{Z}^r(X_K, \cdot + 1) \ .$$

Claim. ρ^* sends $\mathcal{Z}^r_W(X_k, \cdot)$ into $\mathcal{Z}^r_W(X_K, \cdot + 1)$.

Note that the claim implies that ρ^* induces a map

$$\mathcal{Z}^r(X_k, \cdot)/\mathcal{Z}^r_W(X_k, \cdot) \to \mathcal{Z}^r(X_K, \cdot + 1)/\mathcal{Z}^r_W(X_K, \cdot + 1)$$

which provides a homotopy

$$\pi^* \simeq \psi(1)^* : \mathcal{Z}^r(X_k, \cdot)/\mathcal{Z}^r_W(X_k, \cdot) \to \mathcal{Z}^r(X_K, \cdot)/\mathcal{Z}^r_W(X_K, \cdot)$$

and $\psi(1)^*$ is zero by Lemma (1.8) (2).

To prove the claim, we must show, for $z \in \mathcal{Z}_W(X_k, n)$ and faces $F \subset \mathbb{A}^1 \times \square^n$, $\square_J \subset \square^\ell$,

$$(*) \qquad \phi^*(z \times Y \times \mathbb{A}^1 \times \square^\ell) \quad \not R \quad (\partial_J W \times F) \quad \text{in} \quad X \times Y \times \mathbb{A}^1 \times \square^{n+\ell} \ .$$

(Here and below, we write W and ℓ for W_i and ℓ_i. ϕ is being used for the automorphism

$$\phi \times id_{Y \times \square^{n+\ell}} : X \times Y \times \mathbb{A}^1 \times \square^{n+\ell} \to X \times Y \times \mathbb{A}^1 \times \square^{n+\ell} \ .$$

Below are used similar abbreviations for $\psi(t)$.) If $F = \{0\} \times \square^{n'}$ with $\square^{n'} \subset \square^n$, this is clear since $z \in \mathcal{Z}(X, n)$.

Suppose $F = \{1\} \times \square^{n'}$. The condition $(*)$ is equivalent to

$$\psi(1)^*(z \times Y \times \square^\ell) \quad \not R \quad (\partial_J W \times \square^{n'}) \qquad \text{in} \quad X \times Y \times \square^n \times \square^\ell$$

or

$$z \times Y \times \square^\ell \quad \not R \quad \psi(1)_*(\partial_J W \times \square^{n'}) \qquad \text{in} \quad X \times Y \times \square^n \times \square^\ell \ .$$

By Proposition (1.9) applied to the projection $p : X \times Y \times \square^{n+\ell} \to X \times \square^n$, this follows from

$$z \quad \not R \quad p_\sharp \psi(1)_*(\partial_J W \times \square^{n'}) \quad \text{in} \quad X \times \square^n \ .$$

Since p is compatible with $\psi(1)$,

$$p_\sharp \psi(1)_*(\partial_J W \times \square^{n'}) = \psi(1)_* p_\sharp(\partial_J W \times \square^{n'})$$

so the above is equivalent to

$$z \quad \not R \quad \psi(1)_* p_\sharp(\partial_J W \times \square^{n'}) \quad \text{in} \quad X \times \square^\ell \ .$$

This holds by Proposition (1.8), (2).

We now consider the least obvious case $F = \mathbb{A}^1 \times \square^{n'}$ and show

$$\phi^*(z \times Y \times \mathbb{A}^1 \times \square^\ell) \quad \not R \quad \partial_J W \times \mathbb{A}^1 \times \square^{n'} \quad \text{in} \quad X \times Y \times \mathbb{A}^1 \times \square^{n+\ell}$$

or equivalently

$$z \times Y \times \mathbb{A}^1 \times \square^\ell \quad \pitchfork \quad \phi_*(\partial_J W \times \mathbb{A}^1 \times \square^{n'}) \quad \text{in} \quad X \times Y \times \mathbb{A}^1 \times \square^{n+\ell} \ .$$

Since both cycles are equi-dimensional and surjective over $\mathbb{A}^1$, this will follow from: for each $t \in \mathbb{A}^1(k)$

$$z \times Y \times \{t\} \times \square^\ell \quad \pitchfork \quad \psi(t)_*(\partial_J W \times \square^{n'}) \quad \text{in} \quad X \times Y \times \square^{n+\ell} \ .$$

If $t = 0$, it evidently holds. Assume $t \neq 0$. We have only to show ($p : X \times Y \times \square^{n+\ell} \to X \times \square^n$ is the projection)

$$z \quad \pitchfork \quad p_\sharp \psi(t)_*(\partial_J W \times \square^{n'}) = \psi(t)_* p_\sharp(\partial_J W \times \square^{n'}) \quad \text{in} \quad X \times \square^n$$

which is true by Proposition (1.8)(2).

Let X be a smooth projective variety. Define, for a cycle $z = \sum a_i\, z_i$ and a subvariety $Y \subset X$,

$$e(z) = \operatorname*{Max}_i(\dim(Y \cap z_i) - \dim z_i - \dim Y + \dim X) \ .$$

Note $e(z) \geq 0$ and $e(z) = 0$ iff $z \ \pitchfork\ Y$. We recall the moving lemma in [Ro] to the effect that "coning operation" on cycles improves the intersection properties.

Take a closed embedding $j : X \hookrightarrow \mathbb{P}^N$ with $N > \dim X$. Let F be the function field of the Grassmannian variety $Grass(N, N - \dim X - 1)$, and $L \subset \mathbb{P}^N_F$ be the restriction of the universal family of planes.

For a cycle z on X, let $C_L(z)$ be its cone with center L; it is a cycle on $\mathbb{P}^N$ defined over F meeting X properly. Also let $\sigma_L(z) = z - C_L(z) \cap X$; it is a cycle on X defined over F.

(1.11) **Lemma.** *In this situation, one has*

$$e(\sigma_L(z)\,) \leq Max\,(e(z) - 1, 0) \ .$$

Corollary. *There is a sequence of finitely generated purely transcendental extensions $k = K_0 \subset K_1 \subset \cdots \subset K_r = K$ and subspaces $L_i \subset \mathbb{P}^N_{K_i}$ such that: for any k-cycle z on X and any subvariety $Y \subset X$, one has*

$$\sigma_{L_r} \cdots \sigma_{L_1}(z) \quad \pitchfork \quad Y \quad \textit{in} \quad X.$$

(1.12) **Proof of Proposition (1.3).** Keeping the above notation let

$$X_{K_r} \xrightarrow{\pi_r} X_{K_{r-1}} \to \cdots \to X_1 \xrightarrow{\pi_1} X$$

be the projections. Consider the maps

$$\pi_i^* : \mathcal{Z}^r(X_{K_{i-1}}, \cdot)/\mathcal{Z}^r_W(X_{K_{i-1}}, \cdot) \to \mathcal{Z}^r(X_{K_i}, \cdot)/\mathcal{Z}^r_W(X_{K_i}, \cdot) \ .$$

By a specialization argument as in (1.10), this map is injective on homology. We will show that $\pi_r^*\circ\cdots\circ\pi_1^*$ induces zero on homology, so $\mathcal{Z}^r(X,\cdot)/\mathcal{Z}^r_W(X,\cdot)$ is acyclic.

If $z\in\mathcal{Z}^r(X,\cdot)$ (resp. $\in\mathcal{Z}^r_W(X,\cdot)$) then $C_{L_1}(z)\in\mathcal{Z}^r_{\{X\}}(\mathbb{P}^N,\cdot)$ (resp. $\in\mathcal{Z}^r_{\{W,X\}}(\mathbb{P}^N,\cdot)$) so one has the first map in the following:

$$\mathcal{Z}^r(X,\cdot)/\mathcal{Z}^r_W(X,\cdot)\stackrel{C_{L_1}}{\to}\mathcal{Z}^r_{\{X\}}(\mathbb{P}^N_{K_1},\cdot)/\mathcal{Z}^r_{\{X,W\}}(\mathbb{P}^N_{K_1},\cdot)\stackrel{j^*}{\to}\mathcal{Z}^r(X_{K_1},\cdot)/\mathcal{Z}^r_W(X_{K_1},\cdot)\ .$$

The map j^* is the restriction by j. By the previous proposition, the complex in the middle is acyclic, so the composition $j^*C_{L_1}$ induces zero on homology. The same holds for $j^*C_{L_i}$. So

$$(\pi_r^*-j^*C_{L_r})\cdots(\pi_1^*-j^*C_{L_1})=\sigma_{L_r}\cdots\sigma_{L_1}:\mathcal{Z}^r(X,\cdot)/\mathcal{Z}^r_W(X,\cdot)\to\mathcal{Z}^r(X_K,\cdot)/\mathcal{Z}^r_W(X_K,\cdot)$$

and π^* induce the same map on homology. By the above corollary the former is the zero map of complexes, hence π^* induces zero on homology. We are done.

(1.13) **Proof of Proposition (1.6).** As for (1.3), we first consider the special case $X=Y=\mathbb{P}^N_k$. We will take finitely generated purely transcendental extensions K/k and E/K, and consider the map of complexes

$$\pi_X^*\otimes\pi_Y^*:\mathcal{Z}^r(X_k,\cdot)\otimes\mathcal{Z}^s(Y_k,\cdot)\to\mathcal{Z}^r(X_K,\cdot)\otimes\mathcal{Z}^s(Y_E,\cdot)$$

where $\pi_X:X_K\to X_k$ and $\pi_Y:Y_E\to Y_k$. Denote the first complex by $C(X_k,Y_k)$, and its distinguished subcomplex with respect to W by $C(X_k,Y_k)_W$. Similarly, the second complex is denoted by $C(X_K,Y_E)$ and let $C(X_K,Y_E)_W$ be the subcomplex consisting of elements whose images in $C(X_E,Y_E)$ are in $C(X_E,Y_E)_W$. Since $\pi_X^*\otimes\pi_Y^*$ sends $C(X_k,Y_k)_W$ into $C(X_K,Y_E)_W$, the above map passes to a map of quotient complexes

$$\pi_X^*\otimes\pi_Y^*:C(X_k,Y_k)/C(X_k,Y_k)_W\to C(X_K,Y_E)/C(X_K,Y_E)_W\ .$$

We will show this map is injective and zero on homology, whence the acyclicity of $C(X_k,Y_k)/C(X_k,Y_k)_W$. The injectivity is again verified by a specialization argument.

Let $G=SL_{N+1}$, and take K/k and $\psi_1:\mathbb{A}^1_K\to G_K$ as in before. Take a further extension E/K and $\psi_2:\mathbb{A}^1_E\to G_E$ with similar properties. Let $g_1=\psi_1(1)$, $g_2=\psi_2(1)$. Define

$$\rho_1^*:\mathcal{Z}^r(X_k,\cdot)\to\mathcal{Z}^r(X_K,\cdot+1)\text{ and }\rho_2^*:\mathcal{Z}^s(Y_k,\cdot)\to\mathcal{Z}^s(Y_E,\cdot+1)$$

as in the proof of (1.10).

We claim the degree one map

$$\rho_1^*\otimes id+g_1^*\otimes\rho_2^*:C(X_k,Y_k)\to C(X_K,Y_E)$$

sends $C(X_k,Y_k)_W$ into $C(X_K,Y_E)_W$. Then it provides homotopy

$$\pi_X^*\otimes\pi_Y^*\simeq g_1^*\otimes g_2^*:C(X_k,Y_k)/C(X_k,Y_k)_W\to C(X_K,Y_E)/C(X_K,Y_E)_W\ .$$

Since $g_1^*\otimes g_2^*$ sends $C(X_k,Y_k)$ into $C(X_K,Y_E)_W$ by (1.8), (2) applied to $SL_{N+1}\times SL_{N+1}$ acting on $X\times Y$ it is the zero map of quotient complexes.

For the proof of the claim, let $A \subset X \times \square^n$, $B \subset Y \times \square^m$ be irreducible subvarieties meeting faces properly such that

$$A \times B \times Z \times \square^\ell \quad \pitchfork \quad \partial_J W \times \square^{n+m} \quad \text{in} \quad X \times Y \times Z \times \square^{n+m+\ell} \ .$$

We have to show

$$\begin{aligned} &\rho_1^* A \times B \times Z \times \square^\ell \quad \pitchfork \quad \partial_J W \times \mathbb{A}^1 \times \square^{n+m} \quad \text{and} \\ &g_1^* A \times \rho_2^* B \times Z \times \square^\ell \quad \pitchfork \quad \partial_J W \times \mathbb{A}^1 \times \square^{n+m} \quad \text{in} \quad X \times Y \times Z \times \mathbb{A}^1 \times \square^{n+m+\ell} \ . \end{aligned}$$

The first of the two will follow from Lemma (1.9) and the fact ($p : X \times Y \times Z \times \mathbb{A}^1 \times \square^{n+m+\ell} \to X \times \mathbb{A}^1 \times \square^n$ is the projection)

$$\begin{aligned} \rho_1^* A \quad \pitchfork \quad & p_\sharp\big((X \times B \times Z \times \mathbb{A}^1 \times \square^{n+\ell}) \cdot (\partial_J W \times \mathbb{A}^1 \times \square^{n+m})\big) \\ &= p_\sharp\big((X \times B \times Z \times \square^{n+\ell}) \cdot (\partial_J W \times \square^{n+m})\big) \times \mathbb{A}^1 \end{aligned}$$

which in turn is verified "fiberwise over $\mathbb{A}^1$" as in (1.10). The proof of the second is similar; since $g_1^* A$ is defined over K, the argument goes through over a further extension E/K. This is the reason we needed to take E.

In the general case, let $\mathcal{Z}(X, \cdot)'$, $\mathcal{Z}(Y, \cdot)'$ be distinguished subcomplexes and consider the commutative square

$$\begin{array}{ccc} \mathcal{Z}(X)' \otimes \mathcal{Z}(Y)' & \to & \mathcal{Z}(X) \otimes \mathcal{Z}(Y) \\ \uparrow & & \uparrow \\ [\mathcal{Z}(X)' \otimes \mathcal{Z}(Y)']_W & \to & [\mathcal{Z}(X) \otimes \mathcal{Z}(Y)]_W \end{array}$$

where $[\mathcal{Z}(X)' \otimes \mathcal{Z}(Y)']_W := [\mathcal{Z}(X) \otimes \mathcal{Z}(Y)]_W \cap \mathcal{Z}(X)' \otimes \mathcal{Z}(Y)'$. The upper horizontal map is a quasi-isomorphism.

Claim 1. The lower horizontal map is a quasi-isomorphism (for any choice of distinguished subcomplexes).

It is enough to show $[\mathcal{Z}(X)' \otimes \mathcal{Z}(Y)']_W \to [\mathcal{Z}(X) \otimes \mathcal{Z}(Y)']_W$ is a quasi-isomorphism. This is shown as in (1.10) if $X = \mathbb{P}^N$, and as in (1.12) in general.

Since the two horizontal maps are quasi-isomorphisms, one of the vertical maps is a quasi-isomorphism iff so is the other. In particular, if $X = Y = \mathbb{P}^N$, we know both maps are quasi-isomorphisms. The proof is completed by:

Claim 2. For a choice of a distinguished subcomplex $\mathcal{Z}(Y)'$, the inclusion

$$[\mathcal{Z}(X) \otimes \mathcal{Z}(Y)']_W \to \mathcal{Z}(X) \otimes \mathcal{Z}(Y)'$$

is a quasi-isomorphism.

Take $\mathcal{Z}(Y)' = \mathcal{Z}_{p_{2\sharp} W}(Y)$ where $p_2 : X \times Y \times \square^\ell \to Y \times \square^\ell$. For a purely transcendental extention K/k we will show the map

$$\pi_X^* \otimes id : \frac{\mathcal{Z}(X) \otimes \mathcal{Z}(Y)'}{[\mathcal{Z}(X) \otimes \mathcal{Z}(Y)']_W} \to \frac{\mathcal{Z}(X_K) \otimes \mathcal{Z}(Y)'}{[\mathcal{Z}(X_K) \otimes \mathcal{Z}(Y)']_W}$$

is injective and zero on homology. The injectivity is again seen by a specialization argument. Take a closed embedding $j : X \hookrightarrow \mathbb{P}^N$ with $N > \dim X$. Consider the maps

$$\frac{\mathcal{Z}(X)\otimes\mathcal{Z}(Y)'}{[\mathcal{Z}(X)\otimes\mathcal{Z}(Y)']_W} \xrightarrow{C_L\otimes id} \frac{\mathcal{Z}_X(\mathbb{P}^N_K)\otimes\mathcal{Z}(Y)'}{[\mathcal{Z}_X(\mathbb{P}^N_K)\otimes\mathcal{Z}(Y)']_W} \xrightarrow{j^*\otimes id} \frac{\mathcal{Z}(X_K)\otimes\mathcal{Z}(Y)'}{[\mathcal{Z}(X_K)\otimes\mathcal{Z}(Y)']_W} .$$

If $Y = \mathbb{P}^N$, the complex in the middle is acyclic, so $j^*{\circ}C_L \otimes id$ induces zero on homology. On the other hand, one can show: if $z \otimes w \in \mathcal{Z}(X)\otimes\mathcal{Z}(Y)'$, then $\sigma_{L_r}\cdots\sigma_{L_1}(z)\otimes w \in [\mathcal{Z}(X_K)\otimes\mathcal{Z}(Y)']_W$ (notation as in (1.12)). So $\pi^*\otimes id$ induces zero on homology. If Y is general, since one now knows the acyclicy of the middle complex in the above sequence, one can repeat the above argument.

§2. Grothendieck's construction.

In [Ma], the construction of the category of pure motives is presented. One starts with the correspondence category of smooth projective varieties, takes its pseudo-abelianization, and then add Tate twists.

In this section we construct the same category as follows: take the correspondence category, add Tate twists and then do the pseudo-abelianization. The motivation of the introduction of the correspondence category with Tate twists comes from the construction of $\mathcal{D}(k)$.

For an equi-dimensional smooth projective variety X over k, let $C^r(X)$ be the $\mathbb{Q}$-vector space of $\mathbb{Q}$-algebraic cycles of codimension r on X modulo an adequate equivalence relation. (We will be concerned only with rational, homological or numerical equivalence.) An element of $C^*(X\times Y)$ is called a C-correspondence from X to Y. One has the operations of transpose, internal product, external product, the induced map by an element of $C^*(X\times Y)$ and composition. The composition of correspondences is the map

$$C^{\dim X+a}(X\times Y)\otimes C^{\dim Y+b}(Y\times Z)\to C^{\dim X+a+b}(X\times Z)$$

which sends $u\otimes v$ to

$$v{\circ}u = (p_{XZ})_*(p_{XY}^*u\cdot p_{YZ}^*v) .$$

Here p_{XY}, for example, is the projection $X\times Y\times Z\to X\times Y$.

(2.1) The additive category of *finite C-symbols $C\,Symb_{finite}(k)$* is defined as follows. The objects are of the form

$$\bigoplus_{\alpha\in A}(X_\alpha, r_\alpha)$$

(called *(finite) C-symbols*, or simply *symbols*) where A is a finite set, X_α is an *irreducible* smooth projective variety over k and $r_\alpha\in\mathbb{Z}$. (More formally, a symbol is a map from a finite set A into the product of the set of irreducible smooth projective varieties and the set $\mathbb{Z}$.) The homomorphism group is the $\mathbb{Q}$-vector space

$$\mathrm{Hom}_{CSymb_{finite}(k)}(\oplus(X_\alpha,r_\alpha),\oplus(Y_\beta,s_\beta)) = \bigoplus_{\alpha,\beta} C^{\dim X_\alpha+s_\beta-r_\alpha}(X_\alpha\times Y_\beta) .$$

The composition of morphisms is induced by the composition of correspondences.

The category is additive, with the direct sum

$$\Big(\bigoplus_\alpha (X_\alpha, r_\alpha)\Big) \oplus \Big(\bigoplus_\beta (Y_\beta, s_\beta)\Big) = \bigoplus_\gamma (Z_\gamma, t_\gamma)$$

where γ varies in $A \coprod B$, and $Z_\gamma = X_\alpha$ or Y_β, $t_\gamma = r_\alpha$ or s_β. There is also the tensor product defined as:

$$\Big(\bigoplus_\alpha (X_\alpha, r_\alpha)\Big) \otimes \Big(\bigoplus_\beta (Y_\beta, s_\beta)\Big) = \bigoplus_{(\alpha,\beta)} (X_\alpha \times Y_\beta, r_\alpha + s_\beta) \ .$$

The axioms of tensor product [MacL] is verified.

In the case C is the rational Chow group, we denote the corresponding category simply by $Symb_{finite}(k)$.

Let $(Smooth\ Proj./k)$ be the category of not necessarily connected smooth projective varieties and maps. There is a natural contravariant functor

$$h : (Smooth\, Proj./k) \to C\, Symb_{finite}(k)$$

which sends X to X itself and associates with a map $f : X \to Y$ the class of its graph $\Gamma_f \in \oplus C^{\dim Y_\beta}(Y_\beta \times X)$ where $Y = \coprod Y_\beta$ (irreducible components).

(2.2) An additive category $\mathcal{C}$ is said to be *pseudo-abelian* if for any object X in $\mathcal{C}$ and a projector $P \in \mathrm{End}(X)$ (i.e. $P \circ P = P$), the categorical image $\mathrm{Im}\ P$ exists. One then has

$$X = \mathrm{Im}\ P \oplus \mathrm{Im}(id_X - P) \ .$$

If $\mathcal{C}$ is an additive category, its *pseudo-abelianization* is a pseudo-abelian category $\mathcal{C}^{\tilde{}}$ together with an additive functor $F : \mathcal{C} \to \mathcal{C}^{\tilde{}}$ satisfying the condition: for an additive functor $F' : \mathcal{C} \to \mathcal{C}'$ where $\mathcal{C}'$ is pseudo-abelian, there exists a unique additive functor $G : \mathcal{C}^{\tilde{}} \to \mathcal{C}'$ such that $F' = G \circ F$.

The pseudo-abelianization exists and can be constructed as follows. The objects of $\mathcal{C}^{\tilde{}}$ are of the form

$$(X, P) \text{ where } P \in \mathrm{End}(X), P \circ P = P \ ,$$

and the homomorphism groups are

$$\mathrm{Hom}_{\mathcal{C}^{\tilde{}}}((X,P),(Y,Q)) = Q \circ \mathrm{Hom}_{\mathcal{C}}(X,Y) \circ P \ .$$

Composition of morphisms in $\mathcal{C}^{\tilde{}}$ is induced by the composition in $\mathcal{C}$ in the obvious manner. The functor $F : \mathcal{C} \to \mathcal{C}^{\tilde{}}$ is given by $X \mapsto (X, id_X)$ and the evident isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) = \mathrm{Hom}_{\mathcal{C}^{\tilde{}}}((X,id),(Y,id)) \ .$$

One can show that $F : \mathcal{C} \to \mathcal{C}^{\tilde{}}$ is the pseudo-abelianization; the object (X,P) is the image of $P : X \to X$ in $\mathcal{C}^{\tilde{}}$.

The following property of pseudo-abelinazation is easy to see. If $\mathcal{C}$ is a tensor category, then $\mathcal{C}^{\sim}$ has a unique structure of a tensor category such that F is a tensor funcor. Explicitly, the operation is given by:

$$(X, P) \otimes (Y, Q) = (X \otimes Y, P \otimes Q) \ .$$

(2.3) **Definition.** The category $C\mathcal{M}(k)$ of C-*motives* is the pseudo-abeliazation of the additive category $C\,Symb_{finite}(k)$. Explicitly, $C\mathcal{M}(k)$ has objects

$$\big(\bigoplus_{\alpha\in A}(X_\alpha, r_\alpha), P\big)$$

where A is a finite set, X_α is irreducible smooth projective, $r_\alpha \in \mathbb{Z}$, and

$$P = (P_{\alpha\,\beta}) \in \bigoplus_{\alpha\,\beta} C^{\dim X_\alpha + r_\beta - r_\alpha}(X_\alpha \times X_\beta) \text{ such that } P{\circ}P = P \ .$$

The homomorphism groups are

$$\begin{aligned}&\mathrm{Hom}(\,(\oplus(X_\alpha, r_\alpha), P), (\oplus(Y_\beta, s_\beta), Q)\,)\\ =&Q{\circ}\big(\bigoplus_{\alpha,\beta} C^{\dim X_\alpha + s_\beta - r_\alpha}(X_\alpha \times Y_\beta)\big){\circ}P \ ;\end{aligned}$$

the composition of maps are given by composition of correspondences.

The category is additive and has tensor product

$$\big(\bigoplus_{\alpha\in A}(X_\alpha, r_\alpha), P\big) \otimes \big(\bigoplus_{\beta\in B}(Y_\beta, s_\beta), Q\big) = \big(\bigoplus_{(\alpha\,\beta)}(X_\alpha \times Y_\beta\, r_\alpha + s_\beta), P \otimes Q\big) \ .$$

Let $\mathbb{P}^1$ be the projective line over k, and $\Pi^2 = \mathbb{P}^1 \times p$ be a cycle on $\mathbb{P}^1 \times \mathbb{P}^1$, where p is a k-rational point of $\mathbb{P}^1$. Denote its cycle class by the same $\Pi^2 \in C^1(\mathbb{P}^1 \times \mathbb{P}^1)$. Then it is a projector, and one has an isomorphism

$$(pt, 1) \xrightarrow{\sim} (\mathbb{P}^1, \Pi^2) \text{ in } C\mathcal{M}\,(k).$$

Hence

$$(X,) \cong X \otimes (\mathbb{P}^1, \Pi^2)^{\otimes r} \cong (X \otimes (\mathbb{P}^1)^r, \Delta_X \times \overbrace{\Pi^2 \times \cdots \times \Pi^2}^{r}) \ .$$

This shows the category $C\mathcal{M}_{inf}(k)$ allows the following description as is in the literature. The objects are of the form (X, P, r) where X is a not necessarily connected smooth projective variety with irreducible components X_α, and $P = (P_{\alpha\,\beta}) \in \oplus_i C^{\dim X_\alpha}(X_\alpha \times X_\beta)$. The homomorphism group is given by

$$\begin{aligned}&\mathrm{Hom}(\,(X, P, r), (Y, Q, s)\big)\\ &= Q{\circ}\big(\oplus_\alpha C^{\dim X_\alpha + s - r}(X_\alpha \times Y)\big){\circ}P \ ;\end{aligned}$$

the composition is induced from the composition of correspondences. The object (X, r, P) is the image of the endomorphism of (X, r) given by P.

When C^* is the rational Chow group (resp. algebraic cycles modulo numerical equivalence), the resulting category we denote by $CH\mathcal{M}(k)$ and call the category of *Chow motives* (resp. $\mathcal{M}(k)$, the category of *Grothendieck motives*). There is a canonical functor $cano : CH\mathcal{M}(k) \to \mathcal{M}(k)$. We summarize:

$$(Smooth\ Proj./k) \to Symb_{finite}(k) \hookrightarrow CH\mathcal{M}(k) \overset{cano}{\to} \mathcal{M}(k).$$

(2.4) *Infinite Grothendieck constructions.* There is an "infinite" version of construction. The reader may skip this until it is needed in §4.

We first define the additive category of *infinite* C*-symbols* $C\,Symb_{inf}(k)$; it has objects – *(infinite)* C*-symbols*:

$$\bigoplus_{\alpha \in A} (X_\alpha, r_\alpha)$$

where A is a possibly infinite set, X_α is an irreducible smooth projective variety over k and $r_\alpha \in \mathbb{Z}$. We impose that the set $\{r_\alpha\}$ is finite. The homomorphism group is the $\mathbb{Q}$-vector space

$$\mathrm{Hom}_{C\,Symb_{inf}(k)}(\oplus(X_\alpha, r_\alpha), \oplus(Y_\beta, s_\beta)) = \prod_\alpha \bigoplus_\beta C^{\dim X_\alpha + s_\beta - r_\alpha}(X_\alpha \times Y_\beta)\ ,$$

i.e., the set of elements $u_{\alpha\,\beta}$ such that for each α there are only finitely many non-zero $u_{\alpha\,\beta}$'s. The composition of maps is given by:

$$(v \circ u)_{\alpha\,\beta} = \sum_\gamma v_{\gamma\,\beta} \circ u_{\alpha\,\gamma}\ .$$

(Note this is a finite sum.)

This category is additive and has tensor product, the structures being described just as for finite symbols except that one now allows infinite formal sums. If C is the rational Chow group, we denote the category simply by $Symb_{inf}(k)$.

Then we define the category of C-motives of (possibly) infinite type, $C\mathcal{M}_{inf}(k)$ to be the psedo-abelianazation of the category $C\,Symb_{inf}(k)$. Specifically, it has objects

$$(\bigoplus_{\alpha \in A} (X_\alpha, r_\alpha), P)$$

where

(i) A is a possibly infinite set, X_α is irreducible smooth projective, $r_\alpha \in \mathbb{Z}$, such that $\{r_\alpha\}$ is finite, and

(ii) $P = (P_{\alpha\,\beta}) \in \prod_\alpha \bigoplus_\beta C^{\dim X_\alpha + s_\beta - r_\alpha}(X_\alpha \times Y_\beta)$ such that $P \circ P = P$.

The homomorphism groups are

$$\begin{aligned}&\mathrm{Hom}\big((\oplus(X_\alpha, r_\alpha), P)), (\oplus(Y_\beta, s_\beta), Q)\big)\\ &= Q \circ \big(\prod_\alpha \bigoplus_\beta C^{\dim X_\alpha + s_\beta - r_\alpha}(X_\alpha \times Y_\beta)\big) \circ P\ ;\end{aligned}$$

the composition of maps are given by composition of correspondences.

There are full embeddings of categories $C\mathcal{M}(k) \to C\mathcal{M}_{inf}(k)$, and $C\mathcal{M}_{inf}(k) \to Ind\, C\mathcal{M}(k)$, the latter the ind-category of $C\mathcal{M}(k)$.

One has the commutative square

$$\begin{array}{ccc} CH\mathcal{M}(k) & \hookrightarrow & CH\mathcal{M}_{inf}(k) \\ \downarrow & & \downarrow \\ \mathcal{M}(k) & \hookrightarrow & \mathcal{M}(k) \end{array}$$

with horizontal arrows full embeddings and vertical ones canonical functors.

§3. C-complexes.

We consider cohomological complexes of abelian groups (or more generally, complexes in an abelian category). We define the notion of C-complexes, which is one possible generalization of that of double complexes. C-complexes are used in constructing the cycle complex of a diagram in §4 and the realization of a motive in §5. For generalities on localizations, triangulated categories and derived categories, we refer to [Har], [Ve].

(3.1) **Definition.** A *C-complex* (of abelian groups) consists of

(i) Complexes $A^m = (A^{m,\bullet}, d_{A^m})$ (each of which is not necessarily bounded) for $m \in \mathbb{Z}$, such that for all but finitely many m's, $A^m = 0$; and

(ii) For $m < n$, maps of graded groups

$$F^{m,n} : A^{m,\bullet} \to A^{n,\bullet-(n-m-1)}$$

subject to the condition

$$F^{m,n}\circ(-1)^m\, d_{A^m} + (-1)^n\, d_{A^n}\circ F^{m,n} + \sum_{m<\ell<n} F^{\ell,n}\circ F^{m,\ell} = 0$$

as a map $A^{m,\bullet} \to A^{n,\bullet-n+m+2}$.

Note that the condition for $n = m+1$ amounts to $F^{m,m+1} : (A^m, d_{A^m}) \to (A^{m+1}, d_{A^{m+1}})$ being a map of complexes. Remark one may put $F^{m,m} = (-1)^m d_{A^m}$ so that the condition above and the ones below are simplified. We will, however, separate $(-1)^m d_{A^m}$ and $F^{m,n}$ $(m < n)$ in notation for the purposes in later sections.

A C-complex is denoted by $A = (A^{m,\bullet}, d_{A^m}, F^{m,n}) = (A^m, F^{m,n})$. One can associate with it a complex, the *total complex* $Tot\,(A) = (Tot\,(A)^\bullet, \mathbf{d})$, defined as:

$$Tot\,(A)^p = \bigoplus_{p\in\mathbb{Z}} A^{m,p-m}$$

(which is a finite sum), and the differential $\mathbf{d}$ is the direct sum, for m, of $(-1)^m\, d_{A^m} + F^{m,n} : A^{m,\bullet} \to \oplus_{n\geq m} A^{n,\bullet-(n-m-1)}$. The condition in (ii) is equivalent to $\mathbf{d}$ being a differential.

One can motivate the definition as follows. The condition for $n = m+1$ is equivalent to the map $F^{m,m+1} : (A^m, d_{A^m}) \to (A^{m+1}, d_{A^{m+1}})$ being a map of complexes. The composition of successive maps $F^{m+1,m+2}\circ F^{m,m+1} : (A^m, (-1)^m d_{A^m}) \to (A^{m+2}, (-1)^{m+2} d_{A^{m+2}})$

is required to be homotopic to zero (if not zero as a map of complexes), with the choice of a homotopy $-F^{m,m+2}$. In the next stage, the map

$$F^{m+1,m+3}\circ F^{m,m+1} + F^{m+2,m+3}\circ F^{m,m+2} : (A^m, (-1)^m d_{A^m}) \to (A^{m+3}, (-1)^{m+3} d_{A^{m+3}})$$

is required to be homotopic zero, with the choice of a homotopy $-F^{m,m+3}$, and so on.

(3.2) **Definition.** Let $B = (B^{m,\bullet}, d_{B^m}, G^{m,n})$ be another C-complex. A *map* of C-complexes $A \to B$ is defined to be a collection maps of graded groups, given for all pairs (m,n):

$$U^{m,n} : A^{m,\bullet} \to B^{n,\bullet-(n-m)}$$

subject to either of the equivalent conditions:

(i) If $tot\,(U^{m,n}) : Tot\,(A)^\bullet \to Tot\,(B)^\bullet$ is the direct sum of the maps $U^{m,n}$, then it is a map of complexes.

(ii) For all m, n, one has

$$(-1)^n\, d_{B^n}\circ U^{m,n} + \sum_{\ell<n} G^{\ell,n}\circ U^{m,\ell} = U^{m,n}\circ(-1)^m\, d_{A^m} + \sum_{m<\ell} U^{\ell,n}\circ F^{m,\ell}$$

as a map $A^{m,\bullet} \to B^{n,\bullet-n+m}$.

The set of maps is naturally an abelian group.

For maps of C-complexes $U = (U^{m,n}) : A \to B$ and $V = (V^{m,n}) : B \to C$, their composition $V\circ U$ is defined by

$$(V\circ U)^{m,n} = \sum_{\ell} V^{\ell,n}\circ U^{m,\ell}\ .$$

C-complexes thus form an additive category.

Let $(U^{m,n})$ and $(\bar{U}^{m,n})$ be maps $A \to B$. A *homotopy* between them is a collection of maps, given for all pairs (m,n),

$$I^{m,n} : A^{m,\bullet} \to B^{n,\bullet-(n-m+1)}$$

satisfying either of the equivalent conditions:

(i) If $tot\,(I^{m,n}) : Tot\,(A)^\bullet \to Tot\,(B)^{\bullet-1}$ is the direct sum of the maps $I^{m,n}$, it gives a homotopy between the maps of complexes $tot\,(U^{m,n})$ and $tot\,(\bar{U}^{m,n})$.

(ii) For all (m,n) one has

$$\begin{aligned} &U^{m,n} - \bar{U}^{m,n} \\ =&\sum_{\ell<n} G^{\ell,n}\circ I^{m,\ell} + (-1)^n\, d_{B^n}\circ I^{m,n} + \sum_{m<\ell} I^{\ell,n}\circ F^{m,\ell} + I^{m,n}\circ(-1)^m d_{A^m}\ . \end{aligned}$$

Two maps are *homotopy equivalent* if there is a homotopy between them. This is an equivalence relation; it is compatible with sums and compositions. One thus has the homotopy category of C-complexes: the objects are C-complexes and morphisms are homotopy equivalence classes of maps.

Denote by $C(Ab)$ the category of complexes and by $K(Ab)$ its homotopy category. Also denote by $(C-Ab)$ the category of C-complexes of abelian groups, and by $K(C-Ab)$ its homotopy category. There is a commutative square of categories and functors:

$$\begin{array}{ccc} (C-Ab) & \to & K(C-Ab) \\ \downarrow & & \downarrow \\ C(Ab) & \to & K(Ab) \end{array}$$

where the horizontal arrows are the canonical functors and the vertical arrows are Tot. The functors Tot are fully faithful, and in fact equivalences of categories.

We now show that the category $K(C-Ab)$ has the structure of a triangulated category, and that the functor Tot is an exact functor of triangulated categories.

One defines the translation functor by $(A^m, F^{m,n})[1] = \big((A[1])^m, (F[1])^{m,n}\big)$ where

$$(A[1])^m = (A^{m+1,\bullet}, -d_{A^{m+1}}), \quad \text{and} \quad (F[1])^{m,n} = -F^{m+1,n+1}\ ;$$

for a map $U : (A^m, F^{m,n}) \to (B^m, G^{m,n})$ in $K(C-Ab)$ represented by $(U^{m,n})$, $U[1] : A[1] \to B[1]$ is represented by $(U[1])^{m,n} = U^{m+1,n+1}$. Note $Tot\,(A[1]) = Tot\,(A)[1]$ functorially. (One could also define $(A^m, F^{m,n})\{1\} = \big((A\{1\})^m, (F\{1\})^{m,n}\big)$ where

$$(A\{1\})^{m,\bullet} = (A^{m,\bullet+1}, -d_{A^m}), \quad \text{and} \quad (F\{1\})^{m,n} = -F^{m+1,n+1}\ .$$

There is an evident isomorphism $A[1] \xrightarrow{\sim} A\{1\}$. To fix the idea, we will use the former notion $A[1]$ in the following.)

The cone of U, $C_U = (C_U^{m,\bullet}, d_{C_U^m}, H^{m,n})$ is defined as follows. Take $N \geq 0$ so that $U^{m,n} = 0$ for $n - m < -N$. Let

$$\begin{aligned} C_U^{m,\bullet} &= A^{m+N+1,\bullet+1} \oplus B^{m,\bullet}; \\ d_{C_U^m} &= \begin{pmatrix} -(-1)^{m+N+1} d_{A^{m+N+1}} & 0 \\ U^{m+N+1,m} & (-1)^m d_{B^m} \end{pmatrix}; \\ H^{m,n} &= \begin{pmatrix} -F^{m+N+1,n+N+1} & 0 \\ U^{m+N+1,n+1} & G^{m,n} \end{pmatrix}. \end{aligned}$$

The C_U is independent of the choice of N up to natural isomorphism, and

$$Tot\,(C_U) = C_{tot(U)}$$

where the latter is the cone of the map of complexes $tot(U) : Tot\,(A) \to Tot\,(B)$, defined as in [Har]. We have a triangle

$$A \xrightarrow{U} B \to C_U \xrightarrow{[1]}\ ,$$

which we declare to be a standard distinguished triangle; upon taking Tot it becomes the standard distinguished triangle in $K(Ab)$ as in [Har]. (We conform to the sign convention in [Har] for distinguished triangles of complexes; the convention of [Ve] is its opposite). Define the class of the *distinguished triangles* in $K(C-Ab)$ to be the ones isomorphic to the standard distinguished triangles.

One verifies that the translation functors and the distinguished triangles satisfy the axioms of triangulated category. One has proved

(3.3) **Proposition.** *There is a natural structure of a triangulated category on $K(C-Ab)$. The functor $Tot : K(C-Ab) \to K(Ab)$ is an exact fully faithful functor of triangulated categories.*

One can proceed further to discuss the derived category of $(C-Ab)$. (This will not be used in the sequel of this paper.) Consider the class of maps u in $K(C-Ab)$ such that the induced maps $tot\,(u)$ are quasi-isomorphisms. This satisfies the axioms in [Har] for the category $D(C-Ab)$ obtained by localizing the category $K(C-Ab)$ with respect to this class of maps to have the induced structure of a triangulated category. The functor Tot induces an exact functor (in fact an equivalence) $Tot : D(C-Ab) \to D(Ab)$. The following diagram summarizes the constructions in this section.

$$\begin{array}{ccccc} (C-Ab) & \to & K(C-Ab) & \to & D(C-Ab) \\ \downarrow Tot & & \downarrow Tot & & \downarrow Tot \\ C(Ab) & \to & K(Ab) & \to & D(Ab) \end{array}$$

§4. The category $\mathcal{D}(k)$.

We will define the three triangulated categories $\mathcal{D}(k)$, $\mathcal{D}_{finite}(k)$, and $\mathcal{D}_{inf}(k)$. The category $\mathcal{D}(k)$ is called the triangulated category of *mixed motives.* We first construct the category $\mathcal{D}_{finite}(k)$ of *finite diagrams* and the larger category $\mathcal{D}_{inf}(k)$ of *(possibly) infinite diagrams.* Then we construct $\mathcal{D}(k)$ as an intermediate triangulated category:

$$\mathcal{D}_{finite}(k) \hookrightarrow \mathcal{D}(k) \hookrightarrow \mathcal{D}_{inf}(k) \ .$$

(4.1) In this section, $X, Y, \cdots$ will denote smooth projective varieties over k. We consider formal symbols of the form

$$\bigoplus_{\alpha \in I} (X_\alpha, r_\alpha)$$

where X_α is an *irreducible* smooth projective variety, I an index set (possibly infinite) and $r_\alpha \in \mathbb{Z}$; we impose the condition that the set $\{r_\alpha\}$ is finite. We call such a symbol infinite (resp. finite) if I is infinite (resp. finite). We write 0 for the corresponding symbol when I is an empty set. If X is a not necessarily connected smooth projective variety with irreducible components X_α, we define $(X, r) = \oplus(X_\alpha, r)$.

Formal symbols were also considered in §2 as objects of the category $Symb_{inf}(k)$; in this section they are to be the building blocks of objects in a larger category $\mathcal{D}_{inf}(k)$.

Define the dual of a finite symbol by:

$$\big(\oplus(X_\alpha, r_\alpha)\big)^\vee = \oplus(X_\alpha, \dim X_\alpha - r_\alpha) \ .$$

Tensor product of two (possibly infinite) symbols by

$$(\oplus(X_\alpha, r_\alpha)) \otimes (\oplus(X'_{\alpha'}, r_{\alpha'})\,) = \oplus(X_\alpha \times X'_{\alpha'}, r_\alpha + r'_{\alpha'}) \ ;$$

and, when the first object is finite, inner Hom by

$$\underline{Hom}(\oplus(X_\alpha, r_\alpha), \oplus(X'_{\alpha'}, r_{\alpha'})) = (\oplus(X_\alpha, r_\alpha))^\vee \otimes (\oplus(X'_{\alpha'}, r_{\alpha'})) \ .$$

Define the *cycle complex* of a formal symbol by

$$\mathcal{Z}^0(\oplus(X_\alpha, r_\alpha), \cdot) = \oplus \mathcal{Z}^{r_\alpha}(X_\alpha, \cdot) \ .$$

This is a homological complex, but often regarded as a cohomological complex in the obvious manner. (If $C_\bullet$ is a homological complex, $C^\bullet = C_{-\bullet}$ is the corresponding cohomological complex.)

Also define the *function complex* of a pair of formal symbols by (note the cohomological indexing)

$$\mathcal{Z}^0(\underline{Hom}(\oplus(X_\alpha, r_\alpha), \oplus(X_\beta, r_\beta)\,)\,)^\bullet := \prod_\alpha \bigoplus_\beta \mathcal{Z}^0(\ \underline{Hom}((X_\alpha, r_\alpha), (X_\beta, r_\beta)), -\bullet) \ .$$

It consists of elements $(f_{\alpha\beta})$ satisfying the condition that for each α, there are only a finite number of β's such that $f_{\alpha\beta}$ are non-zero. An element of this complex may be called a *higher correspondence* from $\oplus(X_\alpha, r_\alpha)$ to $\oplus(X_\beta, r_\beta)$. If $\oplus_\alpha$ is a finite direct sum, the function complex coincides with the cycle complex of $\underline{Hom}(\oplus(X_\alpha, r_\alpha), \oplus(X_\beta, r_\beta)\,)$; in general, $\underline{Hom}(-,-)$ is only a notation.

There are several operations related to the cycle complexes and function complexes of formal symbols. One may for instance consider the composition of correspondences as follows. There is a partially defined map

$$\mathcal{Z}^0\big(\underline{Hom}(\oplus(X_\alpha, r_\alpha), \oplus(X_\gamma, r_\gamma)\,)\big)^\bullet \otimes \mathcal{Z}^0\big(\underline{Hom}(\oplus(X_\gamma, r_\gamma), \oplus(X_\beta, r_\beta)\,)\big)^\bullet$$

$$--\to \mathcal{Z}^0\big(\underline{Hom}(\,\oplus(X_\alpha, r_\alpha), \oplus(X_\beta, r_\beta)\,)\big)^\bullet$$

given by the composition of correspondences in §1, extended by linearity: $f \otimes g$ is sent to $g \circ f$ where

$$(g \circ f)_{\alpha\beta} = \sum_\gamma g_{\gamma\beta} \circ f_{\alpha\gamma} \quad \text{(finite sum)}.$$

We list below similar operations.

Operations on cycles complexes and function complexes; case of formal symbols. Let $K, L, \cdots$ be formal symbols.

The map induced by a correspondence. For an element $u \in \mathcal{Z}^0(\underline{Hom}(K,L)\,)^{-\ell}$, the partially defined map (of graded vector spaces)

$$u_* : \mathcal{Z}^0(K, \cdot) --\to \mathcal{Z}^0(L, \cdot + \ell), \ .$$

One has $(\partial u)_* = \partial \circ u_* - (-1)^{\ell-1} u_* \circ \partial$.

Transpose. The map

$$\mathcal{Z}^0\big(\underline{Hom}(K,L)\,\big)^\bullet \to \mathcal{Z}^0(\underline{Hom}(L^\vee, K^\vee)\,)^\bullet \ , \quad u \mapsto {}^t u \ .$$

Composition of correspondences. The partially defined map

$$\mathcal{Z}^0(\underline{Hom}(K,L))^\bullet \otimes \mathcal{Z}^0(\underline{Hom}(L,M))^\bullet --\to \mathcal{Z}^0(\underline{Hom}(K,M))^\bullet , \quad u\otimes v \mapsto v{\circ}u \ .$$

This is associative and satisfies Leibniz formula; it satisfies

$$^t(u{\circ}v) = (-1)^{\deg u \ \deg v} \ {}^tv{\circ}{}^tu \ .$$

External product of cycles. The map

$$\times : \ \mathcal{Z}^0(K,\cdot)\otimes \mathcal{Z}^0(L,\cdot) \to \mathcal{Z}^0(K\otimes L,\cdot) \ , z\times w \mapsto z\times w \ .$$

It is associative, satisfies Leibniz formula, and one has

$$^t(z\times w) = (-1)^{\deg z \ \deg w} \ {}^tw \times {}^tz \ .$$

Tensor product of correspondences. The map

$$\otimes : \ \mathcal{Z}^0(\underline{Hom}(K,L))^\bullet \otimes \mathcal{Z}^0(\underline{Hom}(M,N))^\bullet \to \mathcal{Z}^0(\underline{Hom}(K\otimes M, L\otimes N))^\bullet \ ,$$

which sends $u\otimes v \in \mathcal{Z}^0(\underline{Hom}(K,L),\ell)\otimes \mathcal{Z}^0(\underline{Hom}(M,N),m)$ to $u\otimes v \in \mathcal{Z}^0(\underline{Hom}(K\otimes M, L\otimes N),\ell+m)$. It is associative, satisfies Leibniz formula, and one has

$$^t(u\otimes v) = (-1)^{\deg u \ \deg v} \ {}^tv\otimes {}^tu \ .$$

Relations among the operations.

$$v_*{\circ}u_* = (v{\circ}u)_* \ .$$

$$v{\circ}u = (1\otimes v)_*u = (-1)^{\deg u \ \deg v}({}^tu\otimes 1)_*v \quad \text{(note the sign).}$$

$$(u{\circ}u')\otimes v = (u\otimes v){\circ}(u'\otimes v); \quad \text{similarly for } v \ .$$

(4.2) **Definition.** An object of $\mathcal{D}_{inf}(k)$ (resp. $\mathcal{D}_{finite}(k)$) is a set of data – called an *infinite diagram* (resp. *finite diagram*)

$$K = (K^m) = (K^m; f^{m,n})$$

where

(i) For each integer m, $K^m = \oplus_{\alpha\in I(m)}(X_\alpha, r_\alpha)$, a possibly infinite formal symbol (resp. finite symbol). We require: For all but finitely many m's, $K^m = 0$.

(ii) For (m,n) with $m<n$, given $f^{m,n} = (f^{m,n}_{\alpha\beta}) \in \mathcal{Z}^0(\underline{Hom}(K^m,K^n))^{-n+m+1}$, which are subject to the conditions:

For $f^{m_k,m_{k+1}} \in \mathcal{Z}^0(\underline{Hom}(K^{m_k},K^{m_{k+1}}))^{-m_{k+1}+m_k+1}$ $(k=1,2,\cdots,r)$ one has

$$f^{m_r,m_{r+1}}{\circ}f^{m_r,m_{r-1}}{\circ}\cdots{\circ}f^{m_1,m_2} \text{ is defined and } \in \mathcal{Z}^0(\underline{Hom}(K^{m_1},K^{m_{r+1}}))^{-m_{r+1}+m_1+r} \ ;$$

(see Remark to (1.5) for the definition).

For $m < n$, one has

$$(-1)^n \, \partial f^{m,n} + \sum_{m<\ell<n} f^{\ell,n} \circ f^{m,\ell} = 0 \ .$$

Note that the condition includes $\partial f^{m,m+1} = 0$ (vacuous).

The object with all $K^m = 0$ is denoted by 0. The object with $K^0 = pt$, $K^m = 0$ for $m \neq 0$ is denoted by $\underline{\mathbb{Q}}$.

For an integer r, let the *Tate objects* be

$$\underline{\mathbb{Q}}(r) := (pt, r)[-2r]$$

namely (pt, r) placed in degree $2r$. ($\underline{\mathbb{Q}}(0) = \underline{\mathbb{Q}}$.)

We extend the definitions of cycle complexes and function complexes to diagrams. First, to an object K of $\mathcal{D}_{inf}(k)$ we associate its cycle complex $\mathcal{Z}^0(K, \cdot)$.

To construct $\mathcal{Z}^0(K, \cdot)$, take a distinguished subcomplex $\mathcal{Z}^0(K^m_\alpha, \cdot)' \subset \mathcal{Z}^0(K^m_\alpha, \cdot)$ for each $\alpha \in I(m)$ so that the maps

$$(f^{m,n}_{\alpha\beta})_* : \mathcal{Z}^0(K^m_\alpha, \cdot)' \to \mathcal{Z}^0(K^n_\beta, \cdot + n - m - 1)'$$

are all defined. This is possible by Remark to (1.5) and since for each β there are only a finite number of non-zero $f^{m,n}_{\alpha\beta}$'s. Let $\mathcal{Z}^0(K^m, \cdot)' = \oplus \mathcal{Z}^0(K^m_\alpha, \cdot)'$. We then have

$$f^{m,n}_* : \mathcal{Z}^0(K^m, \cdot)' \to \mathcal{Z}^0(K^n, \cdot + n - m - 1)' \ .$$

The condition (ii) of Definition (4.2) and the formulas after (1.1) imply

$$(-1)^n \partial \circ f^{m,n}_* + (-1)^m f^{m,n}_* \circ \partial + \sum f^{\ell,n}_* \circ f^{m,\ell}_* = 0 \ ,$$

namely $(\mathcal{Z}^0(K^m, \cdot), f^{m,n}_*)$ is a C-complex. We define $\mathcal{Z}^0(K, \cdot)$ to be its total complex $Tot\,(\mathcal{Z}^0(K^m, \cdot), f^{m,n}_*)$, i.e. the complex $(\mathbb{K}, d)$ with

$$\mathbb{K}^i = \bigoplus_j \mathcal{Z}^0(K^j, j - i)' \ ,$$

and

$$d^i = \sum_j \big((-1)^j \partial_j + \sum_{j<\ell} f^{j,\ell}_* \big) \ .$$

The result is unique up to canonical quasi-isomorphism: Denote temporarily by $\mathcal{Z}^0(K, \cdot)'$ the total complex above, and by $\mathcal{Z}^0(K, \cdot)''$ the total complex coming from another choice of distinguished subcomplexes. Then the two inclusions

$$\mathcal{Z}^0(K, \cdot)' \leftarrow \mathcal{Z}^0(K, \cdot)' \cap \mathcal{Z}^0(K, \cdot)'' \to \mathcal{Z}^0(K, \cdot)''$$

are both quasi-isomorphisms.

Let $(K;f)$ and $(L;g)$ be objects in $\mathcal{D}_{inf}(k)$. The *function complex* $Hom(K,L)^\bullet$ is a cochain complex of abelian groups defined as follows. (If $K=(pt,0)$, it coincides with the cycle complex of L.) By an argument similar to the one above, one can take distinguished subcomplexes

$$\mathcal{Z}^0(\underline{Hom}(K^m_\alpha, L^{m'}_\beta))'^\bullet \subset \mathcal{Z}^0(\underline{Hom}(K^m_\alpha, L^{m'}_\beta))^\bullet$$

so that if

$$\mathcal{Z}^0(\underline{Hom}(K^m, L^{m'}))'^\bullet := \prod_\alpha \bigoplus_\beta \mathcal{Z}^0(\underline{Hom}(K^m_\alpha, L^{m'}_\beta))'^\bullet$$

the maps

$$(-)\circ f^{n,m} : \mathcal{Z}^0(\underline{Hom}(K^m, L^{m'}))'^\bullet \to \mathcal{Z}^0(\underline{Hom}(K^n, L^{m'}))'^{\bullet-n+m+1}$$

are defined; similarly for $g^{m',n'}\circ(-)$. In the following we omit the $'$ which indicates subcomplex for simplicity. The cohomological complex to be defined has the group of N-cochains

$$Hom(K,L)^N = \bigoplus_{-m+m'+p=N} \mathcal{Z}^0(\underline{Hom}(K^m, L^{m'}))^p$$

The differential of this complex, which we denote by D, is the sum of the three types of maps:

$$(-1)^{p+m'+n+1}(\circ f^{n,m}) : \mathcal{Z}^0(\underline{Hom}(K^m, L^{m'}))^p \to \mathcal{Z}^0(\underline{Hom}(K^n, L^{m'}))^{p-n+m+1} ,$$

$$(-1)^{m'+n'}(g^{n',m'}\circ) : \mathcal{Z}^0(\underline{Hom}(K^m, L^{m'})^p \to \mathcal{Z}^0(\underline{Hom}(K^m, L^{n'}))^{p-n'+m'+1} ,$$

and

$$(-1)^{m'}\partial : \mathcal{Z}^0(\underline{Hom}(K^m, L^{m'}))^\bullet \to \mathcal{Z}^0(\underline{Hom}(K^m, L^{m'}))^{\bullet+1} .$$

One verifies $D\circ D=0$. The complex is defined uniquely up to canonical quasi-isomorphism (as for $\mathcal{Z}^0(K,\cdot)$). An element of this complex is a higher correspondence from K to L.

If K happens to be finite, so that $\underline{Hom}(K,L)$ exists as an object (to be defined in (4.5)), its cycle complex $\mathcal{Z}^0(\underline{Hom}(K,L),\cdot)$ and the complex defined above are the same up to canonical quasi-isomorphism.

As for formal symbols one may define operations on the cycle and function complexes of diagrams; we first need to consider the composition of correspondences. Given three objects K, L and M, the composition is the partially defined map

$$Hom(K,L)^\bullet \otimes Hom(L,M)^\bullet \dashrightarrow Hom(K,M)^\bullet$$
$$u\otimes v \mapsto v\circ u; \qquad (v\circ u)^{m,n} = \sum_{\ell\in\mathbb{Z}} v^{\ell,n}\circ u^{m,\ell} .$$

It is associative and satisfies the Leibniz formula

$$D(v\circ u) = Dv\circ u + (-1)^{\deg v} v\circ Du ,$$

where $\deg v$ is the total degree of v in the cohomological complex.

(4.3) **Proposition.** *There is a quasi-isomorphic subcomplex, unique up to canonical quasi-isomorphism, of the tensor product*

$$Hom(K,L)^\bullet \otimes Hom(L,M)^\bullet$$

on which the composition $u\otimes v \mapsto v\circ u \in Hom(K,M)^\bullet$ is defined.

Proof. In each double complex

$$C^{m,\ell,n} := \mathcal{Z}^0(\underline{Hom}(K^m,L^\ell)\,)^\bullet \otimes \mathcal{Z}^0(\underline{Hom}(L^\ell,M^n)\,)^\bullet$$

one can take a distinguished subcomplex $(C^{m,\ell,n})'$ on which the composition map into $\mathcal{Z}^0(\underline{Hom}(K^m,M^n)\,)^\bullet$ is defined, (1.7) (3). Then those subcomplexes $(C^{m,\ell,n})'$ form a quasi-isomorphic subcomplex C' of the double complex considered in the statement.

By definition

$$\mathrm{Hom}_{\mathcal{D}_{inf}(k)}(K,L) = H^0\mathcal{Z}^0(\underline{Hom}(K,L)\,)^\bullet \ .$$

The composition of morphisms is induced from the composition of the function complexes. A morphism $u:K\to L$ is represented by $u^{m,n}\in\mathcal{Z}^0(\underline{Hom}(K^m,L^n),n-m)'$ (for a choice of a collection of distinguished subcomplexes) subject to the condition

$$(-1)^n\partial u^{m,n} - \sum(-1)^{m+\ell}\,u^{\ell,n}\circ f^{m,\ell} + \sum(-1)^{\ell+n}\,g^{\ell,n}\circ u^{m,\ell} = 0 \ .$$

It defines the zero morphism if there exist $U^{m,n}\in\mathcal{Z}^0(\underline{Hom}(K^m,L^n),n-m+1)'$ such that

$$u^{m,n} = (-1)^n\partial U^{m,n} + \sum(-1)^{m+\ell}\,U^{\ell,n}\circ f^{m,\ell} + \sum(-1)^{\ell+n}\,g^{\ell,n}\circ U^{m,\ell} \ .$$

(4.4) **Proposition.** *(1) The category $\mathcal{D}_{inf}(k)$ is an additive category.*

(2) The association $K\mapsto\mathcal{Z}^0(K,\cdot)$ is a functor $\mathcal{D}_{inf}(k)\to D(\mathbb{Q})$, the latter the unbounded derived category of $\mathbb{Q}$-vector spaces.

(3) There is a fully faithful functor $Symb_{inf}(k)\to\mathcal{D}_{inf}(k)$ such that the composition

$$(\textit{Smooth Proj./k})^{\,opp} \to Symb_{inf}(k)\to\mathcal{D}_{inf}(k)$$

(which will also be denoted h) assigns to a smooth projective variety X the diagram $(X,0)[0]$ (i.e. $(X,0)$ placed in degree 0).

All the statements are valid if $\mathcal{D}_{inf}(k)$, $Symb_{inf}(k)$, are replaced with $\mathcal{D}_{finite}(k)$ and $Symb_{finite}(k)$, respectively.

Proof. (1) is easily seen. For (2), let $(u^{m,n})$ represent a morphism $u:K\to L$; then $(u_*^{m,n})$ satisfy the condition

$$(-1)^n\partial\circ u_*^{m,n} + \sum_\ell g_*^{\ell,n}\circ u_*^{m,\ell} = (-1)^m u_*^{m,n}\circ\partial + \sum_\ell u_*^{\ell,n}\circ f_*^{m,\ell} \ .$$

So $(u_*^{m,n})$ represents a map of C-complexes

$$(\mathcal{Z}^0(K^m,\cdot),f_*^{m,n}) \to (\mathcal{Z}^0(L^m,\cdot),g_*^{m,n})$$

and induces a map

$$tot\,(u_*^{m,n}) : \mathcal{Z}^0(K,\cdot) \to \mathcal{Z}^0(L,\cdot)\ .$$

If $(U^{m,n})$ gives $(u^{m,n}) \simeq 0$, then $tot\,(U_*^{m,n})$ gives $tot\,(u_*^{m,n}) \simeq 0$.

(3) The functor sends an object $\oplus(X_\alpha, r_\alpha)$ to $\oplus(X_\alpha, r_\alpha)$ itself placed in degree 0 (i.e. the object K such that $K^0 = \oplus(X_\alpha, r_\alpha)$, $K^m = 0$ for $m \neq 0$). The corresponding homomorphism groups are identical.

(4.5) *Duals, tensor products, and inner Hom's.* One can define the dual $K^\vee$ and tensor product $K \otimes L$ as follows. For K finite, define the dual $K^\vee = ((K^\vee)^m, (f^\vee)^{m,n})$ by $(K^\vee)^{-m} = (K^m)^\vee = \oplus_{\alpha \in I(m)} (K^m_\alpha)^\vee$, and, for $m < n$, $\alpha \in I(m), \beta \in I(n)$,

$$\begin{aligned}(f^\vee)^{-n,-m} = (-1)^{mn+m+1}\ {}^t f^{m,n} \in &\mathcal{Z}^0\big(\underline{Hom}(K^{-n\vee}, K^{-m\vee})\big)^{n-m-1} \\ = &\mathcal{Z}^0\big(\underline{Hom}(K^m, K^n)\big)^{n-m-1}\ .\end{aligned}$$

The association $K \mapsto K^\vee$ is functorial. In fact, define an isomorphism of complexes

$$\alpha(K,L) : \mathrm{Hom}(K,L)^\bullet \to \mathrm{Hom}(L^\vee, K^\vee)^\bullet$$

by

$$\begin{aligned}\mathcal{Z}^0(\underline{Hom}(K^m, L^{m'})\,)^p &\to \mathcal{Z}^0(\underline{Hom}((L^\vee)^{-m'}, (K^\vee)^{-m'})\,)^p \\ u &\mapsto u^\vee := (-1)^{mm'+mp+m'p+m+1}\ {}^t u\ .\end{aligned}$$

(Note for $f^{m,n} \in \mathcal{Z}^0(\underline{Hom}(K^m, K^n)\,)^{-n+m+1}$, $(f^\vee)^{-n,-m}$ coincides with $(f^{m,n})^\vee$.) One verifies the transitivity: if $u \in \mathrm{Hom}(K,L)^0$ and $v \in \mathrm{Hom}(L^\vee, K^\vee)^0$, then $(v \circ u)^\vee = u^\vee \circ v^\vee$. So the $\alpha\,(K,L)$ induces an isomorphism $\mathrm{Hom}\,(K,L) \xrightarrow{\sim} \mathrm{Hom}\,(L^\vee, K^\vee)$, also satisfying the transitivity. Hence the association $K \to K^\vee$ is a functor.

There is an isomorphism $K \xrightarrow{\sim} K^{\vee\vee}$ given by $(-1)^m$ on $K^m \to (K^{\vee\vee})^m = K^m$. It is shown that the composition

$$\begin{aligned}\alpha(L^\vee, K^\vee) \circ \alpha(K,L) : &\mathrm{Hom}(K,L)^\bullet \to \mathrm{Hom}(L^\vee, K^\vee)^\bullet \\ &\to \mathrm{Hom}(K^{\vee\vee}, L^{\vee\vee})^\bullet \xrightarrow{\sim} \mathrm{Hom}(K,L)^\bullet\end{aligned}$$

is the identity. Hence $K \to K^{\vee\vee}$ is canonically isomorphic to the identity functor.

The tensor product $(K; f) \otimes (L; g)$ is the object $(M; h)$ where

$$M^{\bar m} = \bigoplus_{(\alpha,\beta) \in I(m) \times I(m'), m+m' = \bar m} K^m_\alpha \otimes L^{m'}_\beta\ , \quad h^{\bar m, \bar n} = (h^{\bar m, \bar n}_{(\alpha,\beta),(\alpha',\beta')});$$

$$h^{m+m', n+m'}_{(\alpha,\beta),(\gamma,\beta)} = (-1)^{m'(n-m-1)}\quad f^{m,n}_{\alpha,\gamma} \otimes id_{L^{m'}_\beta}\ ,$$

$$h^{m+m', m+n'}_{(\alpha,\beta),(\alpha,\gamma)} = (-1)^m id_{K^m_\alpha} \otimes g^{m',n'}_{\beta,\gamma}\ ,$$

and $h^{\bar m, \bar n}_{(\alpha,\beta),(\alpha',\beta')} = 0$ otherwise.

For K of finite type, define the inner Hom by

$$\underline{Hom}(K,L) = K^\vee \otimes L\ .$$

Operations on cycles complexes and function complexes; case of diagrams. Let $K, L, \cdots$ be objects in $\mathcal{D}_{inf}(k)$. The differences from the case of formal symbols is: one has the dual in place of the transpose; there are no signs in some formulas involving the duals.

The map induced by a correspondence. For an element $u \in Hom(K,L)^{-\ell}$, the partially defined map (of graded vector spaces)

$$u_* : \mathcal{Z}^0(K, \cdot) - - \to \mathcal{Z}^0(L, \cdot + \ell), \ .$$

One has $(\partial u)_* = \partial \circ u_* - (-1)^{\ell-1} u_* \circ \partial$.

Dual. The map (defined above)

$$^\vee : \ Hom(K,L)^\bullet \to Hom(L^\vee, K^\vee)^\bullet \ , \quad u \mapsto u^\vee \ .$$

Composition of correspondences. The partially defined map

$$Hom(K,L)^\bullet \otimes Hom(L,M)^\bullet - - \to Hom(K,M)^\bullet, \quad u \otimes v \mapsto v \circ u \ .$$

This is associative and satisfies Leibniz formula; (if K, L are symbols) it satisfies

$$(u \circ v)^\vee = v^\vee \circ u^\vee \ ;$$

unlike the case of formal symbols (where transposes appeared), there is no sign.

External product of cycles. The map

$$\tilde{\times} : \ \mathcal{Z}^0(K, \cdot) \otimes \mathcal{Z}^0(L, \cdot) \to \mathcal{Z}^0(K \otimes L, \cdot)$$

which sends $(z, w) \in \mathcal{Z}^0(K^m, p) \otimes \mathcal{Z}^0(L^n, q)$ to $z \tilde{\times} w := (-1)^{np} z \times w$. It is associative, satisfies Leibniz formula, and one has

$$(z \tilde{\times} w)^\vee = w^\vee \tilde{\times} z^\vee \ .$$

Tensor product of correspondences. The map

$$\tilde{\otimes} : \ Hom(K,L)^\bullet \otimes Hom(M,N)^\bullet \to Hom(K \otimes M, L \otimes N)^\bullet$$

which sends $(u, v) \in \mathcal{Z}^0(Hom(K^m, L^{m'}), p) \otimes \mathcal{Z}^0(Hom(M^n, N^{n'}), q)$ to $u \tilde{\otimes} v$ $:= (-1)^{n'p + mq + mn' + nm'} \ u \otimes v$. It is associative, satisfies Leibniz formula, and one has

$$(u \tilde{\otimes} v)^\vee = v^\vee \tilde{\otimes} u^\vee \ .$$

Relations among the operations.

$$v_* \circ u_* = (v \circ u)_* \ .$$

$$v \circ u = (1 \tilde{\otimes} v)_* u = (u^\vee \tilde{\otimes} 1)_* v \ .$$

$$(u \circ u') \tilde{\otimes} v = (u \tilde{\otimes} v) \circ (u' \tilde{\otimes} v); \quad \text{similarly for } v \ .$$

$$(u\tilde{\otimes}v)_* = u_* \otimes v_* \ .$$

The above operations on complexes induce maps on cohomology. For example one has

$$\tilde{\otimes} : \mathrm{Hom}(K,L) \otimes \mathrm{Hom}(M,N) \to \mathrm{Hom}(K \otimes M, L \otimes N) \ .$$

The same relations among the operations hold.

So the association $(K,L) \mapsto K \otimes L$ is a bifunctor. The category $\mathcal{D}_{inf}(k)$ with the tensor product satisfies the axioms of monoidal category, [MacL].

As a special case, one has the functor from $\mathcal{D}_{inf}(k)$ to itself, called the r-th Tate twist

$$K \mapsto K(r) := K \otimes \underline{\mathbb{Q}}(r);$$

concretely, $K(r) = (K(r)^m, f(r)^{m,n})$ where $(K(r))^m = K^{m+2r} \otimes (pt, r)$ and $(f(r))^{m,n} = f^{m+2r,n+2r} \otimes id_{(pt,r)}$.

Remark. For a smooth projective variety X, $h(X)(r)$ is $(X,r)[-2r]$, namely "(X,r) placed in degree $2r$". On the other hand, $h(\,(X,r)\,) = (X,r)[0]$. In other words, the Tate twists are not compatible with the functor h.

We now describe the structure of a triangulated category on $\mathcal{D}_{inf}(k)$. Define the translation functor $K \mapsto K[1]$ in the category $\mathcal{D}_{inf}(k)$ as follows. On objects $(K[1])^m = K^{m+1}$ and $(f[1])^{m,n} = -f^{m+1,n+1}$. For a morphism $u : K \to L$ represented by $(u^{m,n})$, $u[1] : K[1] \to L[1]$ is to be represented by $(u[1])^{m,n} = u^{m+1,n+1}$.

Define the cone of a morphism $u : K \to L$ to be the object $C_u = (C; h)$ where

$$C^m = K^{m+1} \oplus L^m$$

and

$$h^{m,n} = \begin{pmatrix} -f^{m+1,n+1} & 0 \\ u^{m+1,n} & g^{m,n} \end{pmatrix} \ .$$

There are canonical morphisms $L \to C_u$ given by

$$\begin{pmatrix} 0 \\ id \end{pmatrix} : L^m \to K^{m+1} \oplus L^m$$

and $C_u \to K[1]$ given by

$$(\,0 \quad id\,) : K^{m+1} \oplus L^m \to K^{m+1}$$

so that we have a triangle

$$K \to L \to C_u \stackrel{[1]}{\to} \ .$$

Such a triangle is called a standard distinguished triangle. We declare the distinguished triangles in the category $\mathcal{D}_{inf}(k)$ to be the ones isomorphic to the standard ones. The category $\mathcal{D}_{finite}(k)$ also has parallel structures. One shows

(4.6) **Theorem.** *The category $\mathcal{D}_{inf}(k)$ (resp. $\mathcal{D}_{finite}(k)$) equipped with the translation functor and the distinguished triangles is a triangulated category.*

Let K be an object of $\mathcal{D}_{inf}(k)$. The object $\bigoplus_{\alpha\geq 1} K_\alpha$ (K_α is a copy of K and α varies over natural numbers) in an obvious manner. It satisfies, for any object L of $\mathcal{D}_{inf}(k)$, $\mathrm{Hom}(\oplus K_\alpha, L) = \prod \mathrm{Hom}(K_\alpha, L)$ and $\mathrm{Hom}(L, \oplus K_\alpha) = \oplus \mathrm{Hom}(L, K_\alpha)$; namely it is a direct sum and a direct product. (To see these note for example $Hom(\oplus K_\alpha, L)^\bullet = \prod Hom(K_\alpha, L)^\bullet$.)

For an object K in $\mathcal{D}_{inf}(k)$ and a projector $p : K \to K$, let $[K,p]$ be the cone of the morphism

$$u = (u_{\alpha,\beta}) : \bigoplus_{\alpha\geq 1} K_\alpha \to \bigoplus_{\alpha\geq 1} K_\alpha$$

where K_α is a copy of K and $u_{\alpha,\alpha} = id$, $u_{\alpha,\alpha+1} = -p$, and $u_{\alpha,\beta} = 0$ if otherwise. The " first direct summand" morphism $K \to \oplus K_\alpha$ followed by the canonical morphism $\oplus K_\alpha \to [K,p]$ gives a morphism $K \to [K,p]$. It induces canonical functorial maps

$$\mathrm{Hom}(L, K) \to \mathrm{Hom}(L, [K,p])$$

and

$$\mathrm{Hom}([K,p], L) \to \mathrm{Hom}(K, L)\ .$$

(4.7) **Proposition.** *The above maps induce isomorphisms*

$$\mathrm{Hom}(L, [K,p]\,) \cong p{\circ}\,\mathrm{Hom}(L, K)\ ;$$

$$\mathrm{Hom}([K,p], L) \cong \mathrm{Hom}(K, L){\circ}p\ .$$

There is a canonical isomorphism $K = [K,p] \oplus [K, 1-p]$ *according to which* $p = \left(\begin{smallmatrix} id & 0 \\ 0 & 0 \end{smallmatrix}\right)$*; in particular* $[K,p]$ *is the image of the projector* p.

Proof. One has a long exact sequence

$$\to \mathrm{Hom}(\oplus K_\alpha, L[i-1]) \to \mathrm{Hom}(\,[K,p], L[i]) \to \mathrm{Hom}(\oplus K_\alpha, L[i]) \stackrel{\circ u}{\to} \mathrm{Hom}(\oplus K_\alpha, L[i]) \to\ .$$

Note that $\circ u$ is surjective with kernel $\mathrm{Hom}(K, L[i]){\circ}p$, more precisely, the following diagram is commutative:

$$\begin{array}{ccc} Ker\,(\circ u) & \hookrightarrow & \mathrm{Hom}(\oplus K_\alpha, L[i]) \\ \downarrow\cong & & \downarrow \text{1st projection} \\ \mathrm{Hom}(K, L[i]){\circ}p & \hookrightarrow & \mathrm{Hom}(K, L[i]) \end{array}$$

One hence obtains $\mathrm{Hom}([K,p], L) = \mathrm{Hom}(K, L){\circ}p$.

As for $\mathrm{Hom}(L, [K,p]\,)$ one argues similarly, now using the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(L[i], \oplus K_\alpha) & \twoheadrightarrow & Cok\,(u\circ) \\ \uparrow \text{1st summand} & & \uparrow\cong \\ \mathrm{Hom}(L[i], K) & \twoheadrightarrow & p{\circ}\,\mathrm{Hom}(L[i], K)\ . \end{array}$$

(4.8) **Definition.** Let $\mathcal{D}(k)$ be the full subcategory of $\mathcal{D}_{inf}(k)$ consisting of objects of the form $[K,p]$ with K an object of $\mathcal{D}_{finite}(k)$ and p a projector.

This is a triangulated subcategory: if $u_{11} : [K,p] \to [L,q]$ is a morphism, its cone is a direct summand of the cone of the morphism $u = \begin{pmatrix} u_{11} & 0 \\ 0 & 0 \end{pmatrix} : K \to L$, the matrix being with respect to $K = [K,p] \oplus [K, 1-p]$ and $L = [L,q] \oplus [L, 1-q]$.

The category is closed under the operations of duals, tensor products, and inner Hom's:

$$[K,p]^\vee = [K^\vee, p^\vee] \qquad \text{and}$$

$$[K,p] \otimes [L,q] = [K \otimes L, p \otimes q] \ .$$

The category of Chow motives is fully embedded in $\mathcal{D}(k)$: in fact let the functor be:

$$(X,r,P) \mapsto [(X,r)[0], P]$$

where the right hand side is the image of the projector $P : (X,r)[0] \to (X,r)[0]$. The homomorphism groups between such objects are identical. Hence we have

(4.9) **Theorem.** *There are fully faithful functors $CH\mathcal{M}(k) \to \mathcal{D}(k)$ and $CH\mathcal{M}_{inf}(k) \to \mathcal{D}_{inf}(k)$.*

The following summarizes the main properties of $\mathcal{D}(k)$ which can be shown without assuming any conjectures. The properties (2) (4) will be discussed in subsequent sections. (3) is clear from the definitions and Theorem (1.1).

(4.10) **Theorem.** *The triangulated category $\mathcal{D}(k)$ has the following properties:*

(1) $\mathcal{D}(k)$ has dual, tensor product, inner Hom, the unit object $\underline{\mathbb{Q}}$, and the Tate objects $\underline{\mathbb{Q}}(r)$;

(2) ($ch(k) = 0$) The contravariant functor $h : (Smooth\,Proj./k) \to \mathcal{D}(k)$ can be extended to a functor from the category of quasi-projective varieties; to each quasi-projective X there is associated its (cohomological) motive $h(X)$, an object of $\mathcal{D}(k)$;

(3) If X is smooth and projective, one has

$$\mathrm{Hom}_{\mathcal{D}(k)}(\underline{\mathbb{Q}}, h(X)(r)[2r-m]) = K_m(X)^{(r)}_{\mathbb{Q}} \ .$$

Here the right side is an Adams-graded piece of the K-group of X.

(4) If H^* denotes one of the etale, singular or algebraic de Rham cohomology, there is the corresponding realization functor $H^* : \mathcal{D}(k) \to Vect\,(\Lambda)$; here $Vect\,(\Lambda)$ is the category of Λ-vector spaces, and Λ is the coefficient field of the cohomology theory.

(4.11) **Definition.** For an object K of $\mathcal{D}_{inf}(k)$, define

$$\mathcal{Z}^r(K,\cdot) := \mathcal{Z}^0(K \otimes (pt, r), \cdot)$$

and the *higher Chow group (or motivic cohomology)* of K by

$$CH^r(K,n) := H_n \mathcal{Z}^r(K,\cdot) \ .$$

Note $CH^r(K,n) = \mathrm{Hom}_{\mathcal{D}_{inf}(k)}(\underline{\mathbb{Q}}, K(r)[2r-n])$.

§5. Cohomology realizations of motives.

The purpose of this section is to construct the realization functors. We benefited from a discussion with S. Bloch and [Bl 4]. We consider Betti, etale and de Rham cohomogies. Let Λ be the coefficient ring:

(1) Betti: $H^* = H^*_B$. $\quad k \subset \mathbb{C}$ and $\Lambda = \mathbb{Q}$.

(2) Etale: $H^* = H^*_{et}$. $\quad k$ is any field, and $\Lambda = \mathbb{Q}_\ell$ where ℓ is a prime integer not dividing char (k).

(3) de Rham: $H^* = H^*_{DR}$. $\quad$ and $\Lambda = k$.

Let $(\Lambda - vect)$ be the category of Λ-vector spaces, and $D(\Lambda - vect)$ the derived category of complexes of Λ-vector spaces with unbounded cohomology. In each case (1)-(3), there is the cohomology functor $\mathbb{R}\Gamma : (Smooth\, Proj./k)^{opp} \to D(\Lambda - vect)$, and the induced functor $H^* : (Smooth\, Proj./k)^{opp} \to (\Lambda - vect)$.

(5.1) **Theorem.** *Let H^* denote one of H^*_B, H^*_{et} or H^*_{DR}. There is an exact functor of triangulated categories*

$$\mathbb{R}\Gamma : \mathcal{D}_{inf}(k) \to D\,(\Lambda - vect)$$

the composition of which with $h : (Smooth\, Proj./k)^{opp} \to \mathcal{D}_{inf}(k)$ coincides with the cohomology functor $X \mapsto \mathbb{R}\Gamma(X)$. We have also the induced functor

$$H^* : \mathcal{D}_{inf}(k) \to (\Lambda - vect)\ .$$

Remark. The functors restrict to

$$\mathbb{R}\Gamma : \mathcal{D}(k) \to D^b_f\,(\Lambda - vect)$$

(where D^b_f is the subcategory with bounded and finite dimensional cohomology) and

$$H^* : \mathcal{D}(k) \to (finite\, dim.\, \Lambda - vect)\ .$$

We consider smooth projective varieties X, Y, etc. in the following. For a sheaf $\mathcal{F}$ on X (with respect to a Grothendieck topology), let $\mathcal{F} \to C^\bullet(\mathcal{F})$ be its Godement resolution. This is functorial for $\mathcal{F}$. Recall the following properties:

(1) If $f : X \to Y$ is a map, then there is a canonical map $f^*C^\bullet(\mathcal{F}) \to C^\bullet(f^*\mathcal{F})$ which is a quasi-isomorphism.

(2) For two sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y, there is a canonical map $p^*_X C^\bullet(\mathcal{F}) \otimes p^*_Y C^\bullet(\mathcal{G}) \to C^\bullet(p^*_X\mathcal{F} \otimes p^*_Y\mathcal{G})$ which is a quasi-isomorphism. If $X = Y$ there is a canonical map $C^\bullet(\mathcal{F}) \otimes C^\bullet(\mathcal{G}) \to C^\bullet(\mathcal{F} \otimes \mathcal{G})$ which is a quasi-isomorphism.

One specializes to the case of constant sheaves. We let Λ_X be:

(1) Betti : $\Lambda_X = \mathbb{Q}_X$, the constant sheaf on $X(\mathbb{C})$ in the classical topology.

(2) Etale : $\Lambda_X =$ "$\varprojlim$"$(\mathbb{Z}/\ell^\nu\mathbb{Z})_X \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, the constant $\mathbb{Q}_\ell$-sheaf in the etale topology.

(3) de Rham: $\Lambda_X = \Omega^\bullet_X$ in the Zariski topology.

Let $\Lambda_X \to C^\bullet(\Lambda_X)$ be the Godement resolution. (More precisely, in the etale case, one lets

$$C^\bullet(\Lambda_X) := \text{“}\varprojlim\text{”}\, C^\bullet(\mathbb{Z}/\ell^\nu\mathbb{Z})_X \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell;$$

and in the de Rham case, $C^\bullet(\Lambda_X) = C^\bullet(\Omega^\bullet_X)$ is a double complex of sheaves.)

For a morphism $f : X \to Y$, there is induced a canonical quasi-isomorphism $f^* : f^* C^\bullet(\Lambda_Y) \to C^\bullet(\Lambda_X)$. There are canonical quasi-isomorphisms $p_X^* C^\bullet(\Lambda_X) \otimes p_Y^* C^\bullet(\Lambda_Y) \to C^\bullet(\Lambda_{X\times Y})$ and $C^\bullet(\Lambda_X) \otimes C^\bullet(\Lambda_X) \to C^\bullet(\Lambda_X)$.

For a closed subset $Z \subset X$ let Γ_Z be the functor of sections supported on Z. One has a subcomplex

$$\Gamma_Z(X, C^\bullet(\Lambda_X)\,) \subset \Gamma(X, C^\bullet(\Lambda_X)\,)\ .$$

For this one has the following purity theorem (cf. [Mi]).

(5.2) **Theorem.** *If Z is of pure codimension r the complex $\Gamma_Z(X, C^\bullet(\Lambda_X)\,)$ is acyclic in degrees $< 2r$.*

For a formal symbol (X, r), and $a, b \in \mathbb{Z}$ with $b \le 0$, consider the sheaf

$$C^{a,b}(X, r) := C^{a+2r}(\Lambda_{X\times\square^{-b}})$$

on the variety $X \times \square^{-b}$. Let

$$\Gamma C^{a,b}\big(X, r\big) = \Gamma\big(X \times \square^{-b}, C^{a+2r}\left(\Lambda_{X\times\square^{-b}}\right)\big)^{alt}$$

(or zero if $b > 0$). Here *alt* indicates the alternating part under the action of Σ_{-b} on $\square^{-b}$. The Λ-vector spaces $\Gamma C^{a,b}\big(X, r\big)$ form a double complex with the differentials given as follows. The differential d of degree $(1, 0)$ is the one induced by the differential of the Godement resolution and

$$\partial : \Gamma C^{a,b}\big(X, r\big) \to \Gamma C^{a,b+1}\big(X, r\big)$$

is induced from the face maps $X \times \square^{-b-1} \to X \times \square^{-b}$.

Given $Z \in \mathcal{Z}^r(X, n)$ define a subcomplex

$$\begin{aligned}\Gamma_{|Z|} C^{\bullet,\bullet}\big(X, r\big) &\subset \Gamma C^{\bullet,\ge -n}\big(X, r\big)\\ &\subset \Gamma C^{\bullet,\bullet}\big(X, r\big)\end{aligned}$$

where

$$\Gamma_{|Z|} C^{a,b}\big(X, r\big) = \Gamma_{\cup_\delta |\delta^* Z|} C(X \times \square^{-b}, C^{a+2r}\Lambda_{X\times\square^{-b}})^{alt}$$

(δ varies over the face maps $\delta : \square^{-b} \to \square^n$).

We will use the notation $\Gamma C\big(X, r\big)$ (or with supports) for the associated simple complex whose differential is $D = d + (-1)^b \partial$. As long as the cohomology is concerned one may replace $\Gamma C^\bullet\big(X, r\big)$ by $\Gamma C^{\bullet\bullet}\big(X, r\big)$ while $\Gamma_{|Z|} C^{\bullet,b}\big(X, r\big)$ has a purity property:

(5.3) **Proposition.** *(1) Let $Z \in \mathcal{Z}^r(X, n)$. For each b the complex $\Gamma_{|Z|} C^{\bullet,b}\big(X, r\big)$ is acyclic in degrees < 0.*

(2) The map of complexes induced by the augmentation $\Gamma C^\bullet\big(X, r\big) \to \Gamma C^{\bullet\bullet}\big(X, r\big)$ is a quasi-isomorphism.

Proof. (1) This follows from the purity theorem.

(2) The augmented complex

$$0 \to \Gamma C^{\bullet}(\Lambda_X) \to \Gamma C^{\bullet}(\Lambda_{X\times\square^0})^{alt} \to \Gamma C^{\bullet}(\Lambda_{X\times\square^1})^{alt} \to \cdots$$

has a contracting homotopy, whence the claim.

The complexes $\Gamma C^{\bullet\bullet}\big(X,r\big)$ have external and internal products, defined as follows. The canonical map

$$p_X^* C^{a+2r}\Lambda_{X\times\square^{-b}} \otimes p_Y^* C^{b+2s}\Lambda_{Y\times\square^{-c}} \to C^{a+b+2r+2s}\Lambda_{X\times Y\times\square^{-b-c}}$$

induces the external product

$$\Gamma C^{\bullet\bullet}(X,r)\otimes\Gamma C^{\bullet\bullet}(Y,s) \to \Gamma C^{\bullet\bullet}(X\times Y, r+s), \quad F\otimes G \mapsto F\times G\ .$$

This is a map of complexes, i.e. the Leibniz rule holds: $D(F\times G) = (DF)\times G + (-1)^{\deg F} F\times (DG)$.

Similarly, the internal product on the Godement resolution induces the internal product

$$\Gamma C^{\bullet\bullet}(X,r)\otimes\Gamma C^{\bullet\bullet}(X,s) \to \Gamma C^{\bullet\bullet}(X, r+s), \quad F\otimes G \mapsto F\cdot G$$

satisfying

$$D(F\cdot G) = DF\cdot G + (-1)^{\deg F} F\cdot DG\ .$$

Define the composition map

$$\begin{aligned}&\Gamma C^{\bullet\bullet}\big(X\times Y,r\big)\otimes\Gamma C^{\bullet\bullet}\big(Y\times Z,s\big) \to \Gamma C^{\bullet\bullet}\big(X\times Z, r+s-\dim Y\big),\\ &\qquad F\otimes G \mapsto G\circ F\end{aligned}$$

as the composite of

$$\begin{aligned}&\Gamma C^{\bullet\bullet}\big(X\times Y,r\big)\otimes\Gamma C^{\bullet\bullet}\big(Y\times Z,s\big)\\ &\xrightarrow{p_{XY}^*\otimes p_{YZ}^*}\Gamma C^{\bullet\bullet}\big(X\times Y\times Z,r\big)\otimes\Gamma C^{\bullet\bullet}\big(X\times Y\times Z,s\big)\\ &\xrightarrow{int.\,product}\Gamma C^{\bullet\bullet}\big(X\times Y\times Z,r+s\big)\\ &\xrightarrow{p_{XZ\,*}}\Gamma C^{\bullet\bullet}\big(X\times Z,r+s-\dim Y\big)\ .\end{aligned}$$

One verifies the property:

$$D(G\circ F) = DG\circ F + (-1)^{\deg G} G\circ DF\ .$$

An element $F\in\Gamma C^{t,s}\big(X\times Y,u\big) = \Gamma C^{t+2u,s}\big(X\times Y\big)$ induces a map of graded groups

$$\begin{aligned}&F_*:\Gamma C^{\bullet\bullet}\big(X,r\big) = \Gamma C^{\bullet+2r,\bullet}\big(X\big) \xrightarrow{p_X^*} \Gamma C^{\bullet+2r,\bullet}(X\times Y) \xrightarrow{F\cdot} \Gamma C^{\bullet+2r+t+2u,\bullet+s}(X\times Y)\\ &\xrightarrow{p_{Y*}} \Gamma C^{\bullet+2r+t+2u-2\dim X,\bullet+s}(Y) = \Gamma C^{\bullet+t,\bullet+s}\big(Y, r+u-\dim X\big)\ .\end{aligned}$$

One has the property:
$$(DF)_* = D\circ F_* - (-1)^{\deg F} F_*\circ D$$

The external product, the composition map, and the induced map carry over to formal symbols by linearity. If $K = \oplus(X_\alpha, r_\alpha)$ and $L = \oplus(Y_\beta, s_\beta)$ let $\Gamma C^{\bullet\bullet}(K) = \oplus C^{\bullet\bullet}(X_\alpha, r_\alpha)$ and
$$\Gamma C^{\bullet\bullet}\big(\underline{Hom}(K,L)\big) = \prod_\alpha \bigoplus_\beta C^{\bullet\bullet}\underline{Hom}\big((X_\alpha, r_\alpha), (Y_\beta, s_\beta)\big) \ .$$

(5.4) **Proposition.** For symbols K, L and M, we have

(1) the external product
$$\Gamma C^{\bullet\bullet}(K) \otimes \Gamma C^{\bullet\bullet}(L) \to \Gamma C^{\bullet\bullet}(K\otimes L)$$
satisfying the Leibniz rule.

(2) the composition map
$$\Gamma C^{\bullet\bullet}\big(\underline{Hom}(K,L)\big) \otimes \Gamma C^{\bullet\bullet}\big(\underline{Hom}(L,M)\big) \to \Gamma C^{\bullet\bullet}\big(\underline{Hom}(K,M)\big) \ .$$
satisfying the Leibniz rule.

(3) An element $F \in \Gamma C^{s,t}(\underline{Hom}(K,L))$ induces a map
$$F_* : \Gamma C^{\bullet\bullet}(K) \to \Gamma C^{\bullet+s,\bullet+t}(L) \ .$$
One has $(DF)_* = D\circ F_* - (-1)^{\deg F} F_*\circ D$.

We would like to define similar operations for supported subcomplexes. To do it for the composition (which is what we will need later) suppose given smooth projective varieties X, Y, Z and cycles
$$f \in \mathcal{Z}^0(\underline{Hom}(\,(X,r),(Y,s)\,),a), \quad g \in \mathcal{Z}^0(\underline{Hom}(\,(Y,s),(Z,t)\,),b) \ .$$
We have already defined
$$\Gamma_{|f|}C^{\bullet\bullet}\underline{Hom}\big((X,r),(Y,s)\big) \quad \text{and} \quad \Gamma_{|g|}C^{\bullet\bullet}\underline{Hom}\big((Y,s),(Z,t)\big) \ .$$
If we define the subcomplex
$$\Gamma_{|g\circ f|}C^{\bullet\bullet}\underline{Hom}\big((X,r),(Z,t)\big) \subset \Gamma C^{\bullet\bullet}\underline{Hom}\big((X,r),(Z,t)\big)$$
by the support condition (let $\ell = \dim X + s - r + a$, $m = \dim Y + t - r + b$ so f is a cycle on $X \times Y \times \square^\ell$ and g on $Y \times Z \times \square^m$)
$$\bigcup_{\delta:\square^{-b}\to\square^{\ell+m}} |\delta^* p_{XZ\,*}(f \times Z \times \square^m \cdot X \times g \times \square^\ell)| \ .$$
(the notation is a bit of abuse; the condition is stronger than what the cycle $g\circ f$ imposes) then there is a map
$$\Gamma_{|f|}C^{\bullet\bullet}\underline{Hom}\big((X,r),(Y,s)\big) \otimes \Gamma_{|g|}C^{\bullet\bullet}\underline{Hom}\big((Y,s),(Z,t)\big) \to \Gamma_{|g\circ f|}C^{\bullet\bullet}\underline{Hom}\big((X,r),(Z,t)\big) \ .$$

More generally given a finite sequence of varieties $X_0, X_1, \cdots, X_N$ and cycles $f^{i,i+1} \in \mathcal{Z}^0\big(\underline{Hom}\big((X_i, r_i), (X_{i+i}, r_{i+1})\big), a_i\big)$ such that $f^{N-1,N}\circ\cdots\circ f^{0,1}$ is defined, one may define a subcomplex

$$\Gamma_{|f^{N-1,N}\circ\cdots\circ f^{0,1}|}C^{\bullet\bullet}\underline{Hom}\big((X_0, r_0), (X_N, r_N)\big)$$

which is the target of a map from

$$\Gamma_{|f^{0,1}|}C^{\bullet\bullet}\underline{Hom}\big((X_0, r_0), (X_1, r_1)\big) \otimes \cdots \otimes \Gamma_{|f^{N-1,N}|}C^{\bullet\bullet}\underline{Hom}\big((X_{N-1}, r_{N-1}), (X_N, r_N)\big) \ .$$

We apply this construction to a diagram. Specifically, let

$$\Gamma_{|f|}C^{\bullet\bullet}\underline{Hom}(K^m, K^n) \subset \Gamma C^{\bullet\bullet}\underline{Hom}(K^m, K^n)$$

be subcomplexes where $|f|$ indicates the support condition (in the above sense) with respect to all possible compositions:

$$|f| := \cup |f^{i_{r-1},i_r}\circ\cdots\circ f^{i_1,i_2}|, \quad m = i_1 < i_2 < \cdots < i_r = n \ .$$

There are maps

$$\Gamma_{|f|}C^{\bullet\bullet}\underline{Hom}(K^m, K^n) \otimes \Gamma_{|f|}C^{\bullet\bullet}\underline{Hom}(K^n, K^\ell) \to \Gamma_{|f|}C^{\bullet\bullet}\underline{Hom}(K^m, K^\ell) \ .$$

We recall facts on cycle classes. If f is an algebraic cycle of codimension r, there is associated its cycle class

$$cl(f) \in H^{2r}\Gamma_{|f|}\big(C^\bullet(\Lambda_X)\big)$$

as well as its image $cl(f) \in H^{2r}(X)$. One has the following properties.

(1) If $\varphi : X' \to X$ is map of smooth projective varieties, and f is a cycle on X, one has $cl(\varphi_* f) = \varphi_* cl(f)$. If f is a cycle on X, then $cl(\varphi^* f) = \varphi^* cl(f)$.

(2) If f, g are cycles on X, Y, respectively, then $cl(f \times g) = cl(f) \times cl(g)$.

(3) If u, v are correspondences from X to Y, and from Y to Z, respectively, then $cl(v\circ u) = cl(v)\circ cl(u)$.

The idea of constructing the cohomology realization is the following. Take a diagram $K = (K^m, f^{m,n})$, consider the complexes

$$\Gamma C(K^m)$$

for all m. On has cycle classes $cl(f^{m,n}) \in H^0\Gamma C^{\bullet,-(n-m-1)}\underline{Hom}(K^m, K^n)$, so one may take their representatives $\in \Gamma C^{0,-(n-m-1)}\underline{Hom}(K^m, K^n)$. We will extend these to elements $F^{m,n} \in \Gamma C^{-(n-m-1)}\underline{Hom}(K^m, K^n)$ that induce graded maps

$$F^{m,n}_* : \Gamma C(K^m) \to \Gamma C(K^n)[-(n-m-1)] \ .$$

If $\{F^{m,n}\}$ may be so chosen that $(\Gamma C(K^m), F^{m,n}_*)$ is a C-complex then one may take its total complex. We can achieve this by taking elements $F^{m,n} \in \Gamma_{|f|}C^{-(n-m-1)}\underline{Hom}(K^m, K^n)$ satisfying a condition similar to the one for $\{f^{m,n}\}$.

One may verify the following lemma, which will be used in creating cochains with prescribed coboundaries in a double complex.

(5.5) **Lemma.** *Let $(A^{\bullet\bullet}; d, \partial)$ be a double complex, with differentials d, ∂ of degree $(1,0)$, $(0,1)$, respectively; let $D = d + (-1)^b\,\partial$ on $A^{a,b}$. Suppose for each $b \in \mathbb{Z}$ the complex $(A^{\bullet,b}, d)$ is acyclic in degrees < 0. Also, there is a large enough integer N such that $A^{a,b} = 0$ for $b \geq N$.*

(1) Suppose given $e = (e^{a,-a+1}) \in \oplus_{a\leq 0} A^{a,-a+1}$ and $c^{0,0} \in A^{0,0}$ such that:

$$D(e) = 0, \quad d(c^{0,0}) = 0, \quad \text{and} \quad [e^{0,1}]_d = [\partial(c^{0,0})]_d \quad \text{in} \quad H^0(A^{\bullet,1}) \ .$$

Then there exists $c = (c^{a,-a}) \in \oplus_{a\leq 0} A^{a,-a}$ having the given $c^{0,0}$ as a component and $D(c) = e$.

(2) If $c = (c^{a,-a}) \in \oplus_{a\leq 0} A^{a,-a}$ satisfies $D(c) = 0$ and $[c^{0,0}]_d = 0$, there is $f = (f^{a,-a-1}) \in \oplus_{a\leq -1} A^{a,-a-1}$ such that $D(f) = c$.

If $c = (c^{a,-a}) \in \oplus_{a\leq -1} A^{a,-a}$ satisfies $D(c) = 0$, there is $f = (f^{a,-a-1})$ $\in \oplus_{a\leq -2} A^{a,-a-1}$ such that $D(f) = c$.

(5.6) **Proposition.** *(1) Let $K = (K^\bullet; f^{\bullet,\bullet})$ be an object of $\mathcal{D}_{inf}(k)$. Then there exist, for $m < n$, elements*

$$F^{m,n} \in \bigoplus_{a+b=-(n-m-1),\, a\leq 0} \Gamma_{|f|} C^{a,b}(\underline{Hom}(K^m, K^n))$$

such that its $(0, -n+m+1)$-component ${}^0F^{m,n} \in \Gamma_{|f|} C^{0,-(n-m-1)}\big(\underline{Hom}(K^m, K^n)\big)$ satisfies

$$[{}^0F^{m,n}]_d = cl(f^{m,n}) \in H^0\Gamma_{|f|} C^{\bullet,-(n-m-1)}\big(\underline{Hom}(K^m, K^n)\big)$$

and one has the relation

$$(-1)^n D(F^{m,n}) + \sum_{m<\ell<n} F^{\ell,n} \circ F^{m,\ell} = 0$$

in $\bigoplus_{a+b=-(n-m-2),\, a\leq 0} \Gamma_{|f|} C^{a,b}(\underline{Hom}(K^m, K^n))$. We call such $(F^{m,n})$ a representative of $\big(cl(f^{m,n})\big)$.

(2) If $(\bar{F}^{m,n})$ is another representative of $\big(cl(f^{m,n})\big)$, there exist, for $m < n$, elements

$$I^{m,n} \in \bigoplus_{a+b=-(n-m),\, a\leq -1} \Gamma_{|f|} C^{a,b}(\underline{Hom}(K^m, K^n))$$

such that

$$(-1)^n D(I^{m,n}) - (-1)^{m+n} F^{m,n} + (-1)^{m+n} \bar{F}^{m,n} - \sum_{m<\ell<n} (-1)^{m+\ell} I^{\ell,n} \circ F^{m,\ell} + \sum_{m<\ell<n} (-1)^{\ell+n} \bar{F}^{\ell,n} \circ I^{m,\ell} = 0 \ .$$

Moreover, if $({}'I^{m,n})$ *also satisfies the same condition as* $(I^{m,n})$*, then there exist*

$$J^{m,n} \in \bigoplus_{a+b=-n+m-1, a\leq -2} \Gamma_{|f|}C^{a,b}\big(\underline{Hom}(K^m, K^n)\big) \quad (m < n)$$

such that

$$I^{m,n} -' I^{m,n} = (-1)^n D(J^{m,n}) + \sum (-1)^{\ell+n} J^{\ell,n} \circ F^{m,\ell} + \sum (-1)^{\ell+n} \bar{F}^{\ell,n} \circ J^{m,\ell} \ .$$

Proof. (1) We will abbreviate the double complex $\Gamma_{|f|}C^{\bullet\bullet}\underline{Hom}(K^m, K^n)$ by $\Gamma_{|f|}C^{\bullet\bullet}(m,n)$. The term $\Gamma_{|f|}C^{a,b}(m,n)$ is zero for $b > 0$ and the complex $(\Gamma_{|f|}C^{\bullet,b}(m,n), d)$ is acyclic for in degree < 0.

We construct $F^{m,n}$ by induction on $n - m$. If $n = m+1$, one has $cl(f^{m,m+1}) \in H^0\Gamma_{|f|}C^{\bullet,0}(m, m+1)$. Take an element ${}^0F^{m,m+1} \in \Gamma_{|}f|C^{0,0}(m, m+1)$ which represents this class:

$$d\big({}^0F^{m,m+1}\big) = 0 \quad \text{and} \quad [{}^0F^{m,m+1}]_d = cl(f^{m,m+1}) \in H^0\Gamma_{|}f|C^{\bullet,0}(m, m+1) \ .$$

Let $F^{m,m+1} = ({}^0F^{m,m+1}) \in \oplus_{a\leq 0}\Gamma_{|}f|C^{a,-a}(m, m+1)$, the element having ${}^0F^{m,m+1}$ as the only component.

Assume $n \geq m+2$. The element $\sum F^{\ell,n} \circ F^{m,\ell} \in \bigoplus_{a+b=-n+m+2, a\leq 0} \Gamma_{|}f|C^{a,b}$ has its $(0, -n+m+2)$-component equal to $\sum {}^0F^{\ell,n} \circ {}^0F^{m,\ell}$, and $[\sum {}^0F^{\ell,n} \circ {}^0F^{m,\ell}]_d = \sum cl(f^{\ell,n} \circ f^{m,\ell}) = -(-1)^n \partial cl(f^{m,n})$. Take ${}^0F^{m,n} \in \Gamma_{|}f|C^{0,-n+m+1}$ such that $[{}^0F^{m,n}]_d = cl(f^{m,n})$. We apply the above lemma (1) to extend ${}^0F^{m,n}$ to $F^{m,n} \in \bigoplus_{a+b=-n+m+1, a\leq 0} \Gamma_{|}f|C^{a,b}$ such that $(-1)^n D(F^{m,n}) = -\sum F^{m,\ell} \circ F^{m,\ell}$.

(2) We construct $I^{m,n}$ by induction on $n - m$. If $n = m+1$, since $[{}^0F^{m,m+1}]_d = [{}^0\bar{F}^{m,m+1}]_d$, the lemma (2) implies that there exists $I^{m,m+1} \in \oplus_{a\leq -1}\Gamma_{|}f|C^{a,-a-1}$ such that $(-1)^{m+1}D(I^{m,m+1}) = -F^{m,m+1} + \bar{F}^{m,m+1}$. Assume $n \geq m+2$. One verifies, using the induction hypothesis the D of

$$-(-1)^{m+n}F^{m,n} + (-1)^{m+n}\bar{F}^{m,n} - \sum_{m<\ell<n} (-1)^{m+\ell} I^{\ell,n} \circ F^{m,\ell} + \sum_{m<\ell<n} (-1)^{\ell+n} \bar{F}^{\ell,n} \circ I^{m,\ell}$$

equals zero. (For instance

$$D(\sum \bar{F}^{\ell,n} \circ I^{m,\ell}) = \sum D(\bar{F}^{\ell,n}) \circ I^{m,\ell} + \sum (-1)^{\ell+n+1} \bar{F}^{\ell,n} \circ D(I^{m,\ell})$$

and $D(\bar{F}^{\ell,n})$ and $D(I^{m,\ell})$ are known by (1) and induction hypothesis.)

By the lemma (2) there exists $I^{m,n}$ such that

$$-(-1)^n D(I^{m,n}) = \bar{F}^{m,n} - F^{m,n} + \sum I^{\ell,n} \circ F^{m,\ell} + \sum \bar{F}^{\ell,n} \circ I^{m,\ell} \ .$$

The elements $(J^{m,n})$ giving the ambiguity are constructed similarly by induction using now the latter half of the lemma (2).

Now we construct the realization of an object $(K^m, f^{m,n})$. Take a representative $(F^{m,n})$ of $cl(f^{m,n})$ and let $F_*^{m,n} : \Gamma C(K^m) \to \Gamma C(K^n)$ be the map of degree $-(n-m-1)$ induced by $F^{m,n}$. One of the condition $(F^{m,n})$ satisfies implies the condition for $(\Gamma C(K^m), F_*^{m,n})$ to be a C-complex, namely

$$(-1)^n D \circ F_*^{m,n} + F_*^{m,n} \circ (-1)^m D + \sum F^{\ell,n} \circ F^{m,\ell} = 0 \ .$$

We verify that the C-complex is independent of the choice of $(F^{m,n})$. Let $(\bar{F}^{m,n})$ be another representative and $(I^{m,n})$ be as in the above proposition (2). The $I^{m,n}$ together with $I^{m,m} = id$ give a map of C-complexes $(id, I_*^{m,n}) : \big(\Gamma C(K^m), F_*^{m,n}\big) \to \big(\Gamma C(K^m), \bar{F}_*^{m,n}\big)$. This is in fact a homotopy equivalence of C-complexes, the inverse of which being given by $(id, -I_*^{m,n})$. The second part of (2) implies that the isomorphism $(id, I_*^{m,n})$ is uniquely determined up to homotopy equivalence. So one has a canonical isomorphism

$$(\Gamma C(K^m), F_*^{m,n}) \to (\Gamma C(K^m), \bar{F}_*^{m,n}) \quad \text{in} \quad K(C - Ab)$$

and a canonical isomorphism

$$Tot\,(\Gamma C(K^m), F_*^{m,n}) \to Tot\,(\Gamma C(K^m), \bar{F}_*^{m,n}) \quad \text{in} \quad K(Ab)\ .$$

If $(\bar{\bar{F}}^{m,n})$ is a third choice of a representative, then one checks that the three comparison isomorphisms among $Tot\,(\Gamma C(K^m), F_*^{m,n})$, $Tot\,(\Gamma C(K^m), \bar{F}_*^{m,n})$, and $Tot\,(\Gamma C(K^m), \bar{\bar{F}}_*^{m,n})$ are compatible. We can thus define

$$\Gamma C(K) = Tot\,\big(\Gamma C(K^m), F_*^{m,n}\big)$$

for any choice of a representative.

(5.7) **Proposition.** *Let $L = (L^\bullet; g^{\bullet\bullet})$ be another object of $\mathcal{D}_{inf}(k)$, and $G^{m,n} \in \bigoplus_{a+b=-(n-m-1), a\le 0} \Gamma_{|g|} C^{a,b}(\underline{Hom}(L^m, L^n))$ a representative of $\big(cl(g^{m,n})\big)$. Let $u : K \to L$ be a morphism in $\mathcal{D}(k)$. Let $u^{m,n} \in \mathcal{Z}^0(\underline{Hom}(K^m, L^n))^{-(n-m-1)}$ represent u, and denote by $\Gamma_{|u|} C^{a,b}\big(\underline{Hom}(K^m, L^n)\big)$ the subcomplex with the support condition with respect to all the compositions $g \circ \cdots \circ g \circ u \circ f \cdots \circ f$.*

(1) There exist

$$U^{m,n} \in \bigoplus_{a+b=-n+m, a\le 0} \Gamma_{|u|} C^{a,b}\big(\underline{Hom}(K^m, L^n)\big)$$

such that its $(0, -n+m)$-component ${}^0U^{m,n} \in \Gamma_{|u|} C^{0,-(n-m-1)}\big(\underline{Hom}(K^m, L^n)\big)$ satisfies

$$[{}^0U^{m,n}]_d = cl(u^{m,n}) \in H^0(\Gamma_{|u|} C^{\bullet,-(n-m-1)}\big(\underline{Hom}(K^m, L^n)\big)\,;$$

and one has the relation

$$(-1)^n\, D(U^{m,n}) - \sum_{m<\ell\le n} (-1)^{m+\ell} U^{\ell,n} \circ F^{m,\ell} + \sum_{m\le \ell < n} G^{\ell,n} \circ U^{m,\ell} = 0\ .$$

We call such $(U^{m,n})$ a representative of $cl(u^{m,n})$ with respect to (F) and (G).

(2) If $(\bar{U}^{m,n})$ is another representative, there exist

$$A^{m,n} \in \bigoplus_{a+b=-n+m-1, a\leq -1} \Gamma_{|u|} C^{a,b}\big(\underline{Hom}(K^m, L^n)\big) \quad (m \leq n)$$

such that

$$U^{m,n} - \bar{U}^{m,n} = (-1)^n D(A^{m,n}) + \sum (-1)^{m+\ell} A^{\ell,n} \circ G^{m,\ell} + \sum (-1)^{\ell+n} G^{\ell,n} \circ A^{m,\ell} \ .$$

Proof. The proof of (1), (2) is similar to that of the previous proposition, using now the double complex $\Gamma_{|u|} C^{\bullet\bullet}\big(\underline{Hom}(K^m, L^n)\big)$. A statement on the ambiguity of (A) may be made, but is omitted since it will not be used.

Proof of Theorem (5.1). For a morphism $u : K \to L$ with a representative $(u^{m,n})$, take a representative $(U^{m,n})$ of $(cl(u^{m,n}))$; then the maps

$$U_*^{m,n} : (\Gamma C(K^m), F_*^{m,n}) \to (\Gamma C(L^m), G_*^{m,n})$$

form a map of C-complexes. If $(\bar{U}^{m,n})$ is another representative of the same $(cl(u^{m,n}))$, take a "homotopy" $(A^{m,n})$ as in (2) above; then the maps of C-complexes $(U_*^{m,n})$ and $(\bar{U}_*^{m,n})$ are homotopic, a homotopy being provided by $(A_*^{m,n})$. Let, then

$$u_*((F^{m,n}), (G^{m,n})) := tot\,(U_*^{m,n}) : Tot\,(\Gamma C(K^m), F_*^{m,n}) \to Tot\,(\Gamma C(L^m), G_*^{m,n})$$

(well-defined as a map in the homotopy category $K(Ab)$).

The map $u_*(F,G) = u_*((F^{m,n}), (G^{m,n}))$ is functorial. Precisely speaking, let $(M^\bullet, h^{\bullet,\bullet})$ is a third object with a representative $(H^{m,n})$ of $(cl(h^{m,n}))$ and $v : L \to M$ be a morphism. Then

$$(v \circ u)_*(F,H) = v_*(G,H) \circ u_*(F,G) : Tot\,(\Gamma C(K^m), F_*^{m,n}) \to Tot\,(\Gamma C(M^m), H_*^{m,n}) \ .$$

This follows from: if $(U^{m,n})$ is a representative of $(cl(u^{m,n}))$ with respect to F, G and $(V^{m,n})$ a representative of $(cl(v^{m,n}))$ with respect to G, H, then $(\sum_\ell V^{\ell,n} \circ U^{m,\ell})$ is a representative of $(cl\,(\sum_\ell v^{\ell,n} \circ u^{m,\ell}))$ with respect to F, H.

We show that $u_*(F,G)$ is depends only on u, i.e. independent of the choice of the representatives (F), (G). If $(\bar{F})$ is another representative of $cl(u)$, and and $(\bar{G})$ of $cl(v)$, by the above functoriality, the diagram

$$\begin{array}{ccc}
Tot\,(\Gamma C(K^m), F_*) & \xrightarrow{u_*(F,G)} & Tot\,(\Gamma C(L^m), G_*) \\
\Big\downarrow {\scriptstyle id_*(F,\bar{F})} & & \Big\downarrow {\scriptstyle id_*(G,\bar{G})} \\
Tot\,(\Gamma C(K^m), \bar{F}_*) & \xrightarrow{u_*(\bar{F},\bar{G})} & Tot\,(\Gamma C(L^m), \bar{G}_*)
\end{array}$$

is commutative in $K(Ab)$.

So define

$$u_* = u_*(F,G)$$

for any choice of representatives. One has $(v \circ u)_* = v_* \circ u_*$. Therefore we have a functor

$$\mathbb{R}\Gamma : \mathcal{D}_{inf}(k) \to D(\Lambda - vect); \qquad K \mapsto Tot\,\Gamma C(K)$$

compatible with the functor h.

If $K' \to K \to K'' \xrightarrow{[1]}$ is a standard distinguished triangle in $\mathcal{D}_{inf}(k)$, there is induced a standard distinguished triangle $\Gamma C(K') \to \Gamma C(K) \to \Gamma C(K'') \xrightarrow{[1]}$ in $K(C - Ab)$ and consequently one in $K(Ab)$. So the $\mathbb{R}\Gamma$ is an exact functor.

§6. **Cohomological motives of quasi-projective varieties.**

In this section we assume that resolutions of singularities exist for varieties over k (e.g. $\mathrm{char}(k) = 0$). More details are in [Ha 2].

Recall the functor

$$h : (Smooth\,Proj./k)^{opp} \to \mathcal{D}_{finite}(k) \ .$$

For X in $(Smooth\,Proj./k)$, $h(X)$ gives both the cycle complex $\mathcal{Z}^r(X, \cdot)$ and the cohomology $\mathbb{R}\Gamma(X, \Lambda)$.

We can extend the h in two ways. The first one is the contravariant functor (called the cohomological motive functor)

$$h : (Q - Proj./k)^{opp} \to \mathcal{D}_{finite}(k) \ .$$

where $(Q - Proj./k)$ is the category of quasi-projective (possibly singular) varieties and maps between them. The second one is the contravariant functor (the cohomological motive with compact support functor)

$$h_c : (Q - Proj./k;\, proper\, maps)^{opp} \to \mathcal{D}_{finite}(k) \ .$$

where $(Q - Proj./k;\, proper\, maps)$ is the category of quasi-projective varieties and *proper* (in fact projective) maps between them.

(6.1) **Theorem.** *There is a contravariant functor*

$$h_c : (Q - Proj./k;\, proper\, maps) \to \mathcal{D}_{finite}(k)$$

satisfying the following properties.

(1) It coincides with the functor h on the subcategory of smooth projective varieties.

(2) It is covariant functorial for open immersions: for an open immersion $j : U' \to U$ there is associated a morphism $j_* : h_c(U') \to h_c(U)$; one has $(j \circ j')_* = j_* \circ j'_*$.

(3) (Localization) If $i : Z \to U$ is a closed immersion and $j : U - Z \to U$ is the open immersion of its complement, one has a distinguished triangle

$$h_c(U - Z) \xrightarrow{j_*} h_c(U) \xrightarrow{i^*} h_c(Z) \xrightarrow{[1]}$$

functorial for pairs (U, Z).

(4) There is a canonical isomorphism in the homotopy category

$$\mathcal{Z}^{-s}(h_c(U)^\vee, \cdot) = \mathcal{Z}_s(U, \cdot) \ .$$

(6.2) **Theorem.** *There is a unique contravariant functor*

$$h : (Q - Proj./k) \to \mathcal{D}_{finite}(k)$$

satisfying the following properties.

(5) It coincides with the functor h on the subcategory of smooth projective varieties.

(6) The functor h satisfies the Mayer-Vietoris property: Let

$$\begin{array}{ccc} Z' & \xrightarrow{\alpha} & U' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{\beta} & U \end{array}$$

be a Cartesian square of quasi-projective varieties such that the horizontal arrows are closed immersions, and f induces $U' - Z' \xrightarrow{\sim} U - Z$. Then there is a distinguished triangle of the form

$$h(U) \xrightarrow{(f^*, \beta^*)} h(U') \oplus h(Z) \xrightarrow{g^* - \alpha^*} h(Z') \xrightarrow{[1]}$$

functorial for such Cartesian squares.

(7) There is a canonical morphism $h_c(U) \to h(U)$. If U is projective, it is an isomorphism.

(8) (Poincaré duality) For U smooth and equi-dimensional, there is a canonical isomorphism

$$h(U) \xrightarrow{\sim} h_c(U)^\vee(-\dim U)[-2\dim U] \ .$$

Further properties of the functors h and h_c are listed below.

(9) There is the relative cohomology motive functor. A *pair of varieties* is (U, Z) where U is a quasi-projective variety and $Z \subset U$ a closed subset. A variety U may be regarded as the pair $(U, \emptyset)$. A map of of pairs $f : (U', Z') \to (U, Z)$ is a map $f : U' \to U$ such that $f^{-1}(Z) \subset Z'$. Denote by $(Pair\, of\, Q - Proj./k)$ the category of pairs of varieties and maps between them. There exists a functor

$$h : (Pair\, of\, Q - Proj./k) \to \mathcal{D}_{finite}(k) \ ; \quad (U, Z) \to h(U, Z)$$

which extends the functor h above together with distinguished triangles

$$h(U, Z) \to h(U) \to h(Z) \xrightarrow{[1]} \ .$$

(10) A *strict map* of pairs $f : (U', Z') \to (U, Z)$ is a map $f : U' \to U$ such that $f^{-1}Z = Z'$. A variety U may be regarded as the pair (U, U). There is a functor

$$h : (Pair\, of\, Q - Proj./k; strict\, maps) \to \mathcal{D}_{finite}(k) \ ; \quad (U, Z) \to h_Z(U)$$

which extends the functor h (in the sense $h_U(U) = h(U)$) together with functorial distinguished triangles

$$h_Z(U) \to h(U) \to h(U - Z) \xrightarrow{[1]} \ .$$

(11) (Purity) If U is smooth equi-dimensional, there is a canonical isomorphism

$$h_c(Z)^\vee(-\dim U)[-2\dim U] \tilde{\to} h_Z(U) \ .$$

(6.3) **Definition.** For a quasi-projective variety U, we define its *(co)homological cycle complex (with or without compact support)* as follows:

$$\begin{aligned}
\mathcal{Z}^r(U,\cdot)^* =& \mathcal{Z}^r(h(U),\cdot) \ , \quad \text{cohomological cycle complex;}\\
\mathcal{Z}^r_{cpt}(U,\cdot)^* =& \mathcal{Z}^r(h_c(U),\cdot) \ , \quad \text{compactly supported cohomological cycle complex;}\\
\mathcal{Z}_s(U,\cdot)_* =& \mathcal{Z}^{-s}(h_c(U)^\vee,\cdot) \ , \quad \text{homological cycle complex ;}\\
\mathcal{Z}^{cpt}_s(U,\cdot)_* =& \mathcal{Z}^{-s}(h(U)^\vee,\cdot) \ , \quad \text{compactly supported homological cycle complex .}
\end{aligned}$$

(These are defined as objects in $D(Ab)$. Note $\mathcal{Z}_s(U,\bullet)_* = \mathcal{Z}_s(U,\bullet)$, the cycle complex of U.

By definition, the *higher Chow cohomology/homology groups (with compact support)* are given by:

$$\begin{aligned}
CHC^r(U,n) =& H^{-n}\mathcal{Z}^r(U,\cdot)^* \ , \quad \text{Chow cohomology;}\\
CHC^r_{cpt}(U,n) =& H^{-n}\mathcal{Z}^{-s}_{cpt}(U,\cdot)^* \ , \quad \text{Chow cohomology with compact support;}\\
CH_s(U,n) =& H^{-n}\mathcal{Z}_s(U,\cdot)_* \ , \quad \text{Chow homology;}\\
CH^{cpt}_s(U,n) =& H^{-n}\mathcal{Z}^{cpt}_s(U,\cdot)_* \ , \quad \text{Chow homology with compact support .}
\end{aligned}$$

Note the Chow homology is nothing but Bloch's higher Chow group. The following notation of motivic (co)homology may be occasionally preferred:

$$\begin{aligned}
&H^i_{\mathcal{M}}(U,\mathbb{Q}(r)) = CHC^r(U,2r-i) \ ,\\
&H^i_{\mathcal{M},cpt}(U,\mathbb{Q}(r)) = CHC^r_{cpt}(U,2r-i) \ ,\\
&H^{\mathcal{M}}_i(U,\mathbb{Q}(s)) = CH_{-s}(U,2s+i) \ ,\\
&H^{\mathcal{M},cpt}_i(U,\mathbb{Q}(s)) = CH^{cpt}_{-s}(U,2s+i) \ .
\end{aligned}$$

Let $K \mapsto H^i(K,\Lambda)$ be one of the cohomology realizations in §5. The functors h and h_c give the expected cohomology:

(6.4) **Theorem.** *For a quasi-projective variety U, $H^i(h(U);\Lambda) = H^i(U,\Lambda)$, i.e. $h(U)$ calculates the cohomology of U. Similarly,*

$$\begin{aligned}
&H^i(h_c(U);\Lambda) = H^i_{cpt}(U,\Lambda) \ ;\\
&H^i(h_c(U)^\vee;\Lambda) = H^{BM}_{-i}(U,\Lambda) \ , \quad \textit{Borel-Moore homology} \ ;\\
&H^i(h(U)^\vee;\Lambda) = H^{cpt}_{-i}(U,\Lambda) \ , \quad \textit{homology (with compact support)} \ .
\end{aligned}$$

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