Two-dimensional rational solitons and their blow-up via the Moutard transformation*

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1 Introduction

This article deals with applications of the Moutard transformation [15] which is a two-dimensional version of the well-known in solitonics Darboux transformation to some problems of the spectral theory of two-dimensional operators and of $(2+1)$-dimensional nonlinear evolution equations.

In particular, the main results consist in the explicit construction of

- *two-dimensional Schrödinger operators*

$$H = -\Delta + u = -(\partial_x^2 + \partial_y^2) + u(x,y)$$

  with fast decaying smooth rational potentials such that their L_2-kernels are nontrivial and, moreover, contain at least two-dimensional subspaces spanned by rational eigenfunctions (see Theorems 2 and 3);

- *blow-up of solutions to the Novikov–Veselov (NV) equation, which is a two-dimensional generalization of the Korteweg–de Vries (KdV) equation, with fast decaying rational Cauchy data (see Theorem 4).*

The first construction was already announced and briefly sketched in [19]. For operators with such fast decaying potentials there exists a nice spectral theory [8, 16]. We note that for one-dimensional Schrödinger operators

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with fast decaying potentials the existence of square-summable eigenfunctions at zero energy level is impossible (see, for instance [8]) and for higher-dimensional operators (i.e., for dimensions greater or equal than 5) one may easily construct such examples for which the kernel contains a smoothing of the Green function of Δ. However for two-dimensional operators these are the first known examples with nontrivial L_2-kernel.

The Novikov–Veselov equation has the form

$$\begin{aligned} U_t = \partial^3 U + \bar{\partial}^3 U + 3\partial(VU) + 3\bar{\partial}(\bar{V}U) = 0, \\ \bar{\partial} V = \partial U, \end{aligned} \tag{1}$$

and it is the first in the hierarchy of equations which have the form

$$H_t = HA + BH,$$

where A and B are differential operators. Here and everywhere below we use the standard derivatives $\partial = \partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ w.r.t. $z = x + i\,y$. These equations were introduced in [17] as equations which preserve the zero-level spectrum of H. The principal part of the n-th equation from the NV hierarchy takes the form

$$U_t = \partial^{2n+1} U + \bar{\partial}^{2n+1} U + \dots$$

where $\dots$ stays for terms of lower order.

It follows from the inverse scattering method that solutions to the KdV equation with analytical fast decaying Cauchy data do not blow up and the well-posedness of the Cauchy problem for this equation is established in many functional spaces (see [13] and references therein). However as we show this is not true for its natural two-dimensional generalization.

The examples of blow-up solutions were obtained not by using the inverse scattering method which is not well-developed in this situation. Indeed, the inverse spectral problem for a two-dimensional Schrödinger operator at a fixed energy level was first posed in [6] and has been studied for positive energy levels [10] or levels below the ground state [11]. We think that the study of this problem on the zero energy level will help to understand both phenomena which we discuss in this article.

We also have to remark that our potentials obtained by iterations of the Moutard transformation may be considered as two-dimensional generalizations of one-dimensional rational solitons obtained in the same way using the Darboux transformation (see §2.2). However in the one-dimensional case these potentials are always singular.

Finally we remark that our potentials are constructed by iterations of the Moutard transformation from the constant potential and all such potentials are integrable on the zero energy level in the sense that all solutions to the equation $H\psi = 0$ may be explicitly constructed via quadratures from linear combinations of harmonic functions. We give the details of this construction in Section 5.

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2 Darboux and Moutard transformations

2.1 The Darboux transformation

Let

$$H = -\frac{d^2}{dx^2} + u(x)$$

be a one-dimensional Schrödinger operator and let ω satisfy the equation

$$H\omega = 0.$$

The function ω determines a factorization of H:

$$H = A^\top A, \quad A = -\frac{d}{dx} + v, \quad A^\top = \frac{d}{dx} + v, \quad v = \frac{\omega_x}{\omega}. \tag{2}$$

Indeed we have

$$A^\top A = \left(\frac{d}{dx} + v\right)\left(-\frac{d}{dx} + v\right) = -\frac{d^2}{dx^2} + v^2 + v_x$$

and the equation

$$v_x + v^2 = u$$

is equivalent to $H\omega = 0$. If v is real-valued we have $A^* = A^\top$.

The Darboux transformation of H [5] is the swapping of $A^\top$ and A:

$$H = A^\top A \to \widetilde{H} = AA^\top,$$

or in terms of u:

$$u = v^2 + v_x \to \widetilde{u} = v^2 - v_x.$$

It is easy to check the following:

Proposition 1 *If φ satisfies the equation $H\varphi = E\varphi$ with $E = \mathrm{const}$ then $\widetilde{\varphi} = A\varphi$ satisfies the equation $\widetilde{H}\widetilde{\varphi} = E\widetilde{\varphi}$.*

Remark 1. In general the Darboux transformation is defined for any solution to the equation $H\omega = c\omega$ with $c = \mathrm{const}$. In this case it reduces to the transformation of $H' = H - c$ for which $H'\omega = 0$.

2.2 One-dimensional solitons via the Darboux transformation

Let $u = 0$ and $\omega = \omega_1 = x = \tau_1$. Then

$$v = \frac{1}{x}, \quad v_x = -\frac{1}{x^2}, \quad u_1 = \widetilde{u} = \frac{2}{x^2}.$$

The function

$$\psi(P, x) = \left(1 - \frac{1}{i\sqrt{E}x}\right) e^{i\sqrt{E}x}$$

is meromorphic in $P = (E, \lambda)$ on the Riemann surface $\Gamma = \{\lambda^2 = E\}$ and for every E its branches give a basis of solutions to the equation

$$H_1\psi = \left(-\frac{d^2}{dx^2} + u_1\right)\psi = E\psi$$

(we normalize ψ by the condition $\psi \approx e^{i\sqrt{E}x}$ as $E \to \infty$). Now we may apply the Darboux transformation defined by $\omega_2 = x^2 + \frac{\tau_2}{\omega_1}$ and obtain another potential. In particular, for every n the potential

$$U_n = \frac{n(n+1)}{x^2},$$

is obtained after n iterations. The orbit of U_n under the Korteweg–de Vries hierarchy is the n-dimensional family M_n of potentials.

We have [1, 2]:

1. there is a general recursion procedure found by Adler and Moser [1] for deriving the polynomials

$$\theta_n(\tau_1, \dots, \tau_n)$$

such that

(a) θ_n is a polynomial of degree $\frac{n(n+1)}{2}$ in $x = \tau_1$. In particular, we have

$$\theta_1 = x = \tau_1,$$

$$\theta_2 = x^3 + \tau_2,$$

$$\theta_3 = x^6 + 5\tau_2 x^3 + \tau_3 x - 5\tau_2^2;$$

(b) the function

$$u_n(x) = -2\frac{d^2}{dx^2}\log\theta_n(\tau_1 + x, \tau_2, \dots, \tau_n)$$

is the potential of H obtained after n iterations of the Darboux transformation (which starts at $u_0 = 0$). Here $\tau_k, k = 2, \dots,$ are free scalar parameters (integration constants) which appear one by one at every step of the iteration;

(c) for $\varphi_n = \frac{\theta_{n+1}}{\theta_n}$ we have

$$\left(-\frac{d^2}{dx^2} + u_n\right)\varphi_n = 0;$$

2. M_n is an n-dimensional family parameterized by $\tau \in \mathbb{C}^n$ and consists of the potentials $u_n(\tau_1 + x, \tau_2, \dots, \tau_n)$;

3. there are birational transformations $\tau \to t$ of the form

$$t_k = a_k\tau_k + g_k(\tau_1, \dots, \tau_{k-1}), \quad a_k \neq 0,$$

such that

$$\frac{\partial u_n}{\partial t_k} = X_k(u_n)$$

is the k-th KdV flow;

4. the zeroes $x_1, \dots, x_{n(n+1)/2}$ of the polynomials θ_n are evolved by the KdV-flows as some integrable Hamiltonian systems and, for instance, for the original KdV equation

$$u_t = 3uu_x - \frac{1}{2}u_{xxx} = X_2(u)$$

their dynamics is described by the Calogero-Moser system.

2.3 The Moutard transformation

Let H be a two-dimensional potential Schrödinger operator and let ω be a solution to the equation

$$H\omega = (-\Delta + u)\omega = 0.$$

Then *the (elliptic) Moutard transformation* of H (cf. [15]) is defined as

$$\widetilde{H} = -\Delta + u - 2\Delta\log\omega = -\Delta - u + 2\frac{\omega_x^2 + \omega_y^2}{\omega^2}.$$

Proposition 2 *If φ satisfies the equation $H\varphi = 0$, then the function θ defined from the system*

$$(\omega\theta)_x = -\omega^2 \left(\frac{\varphi}{\omega}\right)_y, \quad (\omega\theta)_y = \omega^2 \left(\frac{\varphi}{\omega}\right)_x \tag{3}$$

satisfies $\widetilde{H}\theta = 0$.

Obviously, if θ satisfies (3) then

$$\theta + \frac{C}{\omega}, \quad C = \text{const}, \tag{4}$$

satisfies (3) for any constant C.

We shall use the following notation for the Moutard transformation:

$$M_\omega(u) = \widetilde{u} = u - 2\Delta \log \omega, \quad M_\omega(\varphi) = \{\theta + \frac{C}{\omega},\ C \in \mathbb{C}\}.$$

In the one-dimensional limit the Moutard transformation reduces to the Darboux transformation. Indeed, let $u = u(x)$ depend on x only and $\omega = f(x)e^{\sqrt{c}y}$. Then f satisfies the one-dimensional Schrödinger equation

$$H_0 f = \left(-\frac{d^2}{dx^2} + u\right) f = cf$$

and the Moutard transformation reduces to the Darboux transformation of H_0 defined by f:

$$H = H_0 - \frac{\partial^2}{\partial y^2} \quad \longrightarrow \quad \widetilde{H} = \widetilde{H_0} - \frac{\partial^2}{\partial y^2}.$$

If $g = g(x)$ satisfies $H_0 g = Eg$, then $H\varphi = 0$ with $\varphi = e^{\sqrt{E}y}g(x)$. We derive from (3) that $\theta = e^{\sqrt{E}y}h(x)$ satisfies $\widetilde{H}\theta = 0$ if $h = \frac{1}{\sqrt{c}+\sqrt{E}}\left(\frac{d}{dx} - \frac{f_x}{f}\right) g$, i.e. h is a multiple of the Darboux transform of g: $h = -\frac{1}{\sqrt{c}+\sqrt{E}} Ag$ where $H_0 - c = A^\top A$ is the factorization of $H_0 - c$ defined by f. The inverse Darboux transformation is given by $g = \frac{1}{\sqrt{c}-\sqrt{E}}\left(\frac{d}{dx} + \frac{f_x}{f}\right) h$.

Remark 2. There is another two-dimensional generalization of the Darboux transformation called the Laplace transformation. It is defined for a more general operator, i.e. for the Schrödinger operator with an electromagnetic field:

$$H = 4(\bar{\partial} + \beta)(-\partial + \alpha) + u,$$

and has the form

$$H \to \widetilde{H} = 4u(-\partial + \alpha)u^{-1}(\bar{\partial} + \beta) + u.$$

So if $H\varphi = 0$, then $\tilde{\varphi} = (-\partial + \alpha)\varphi$ satisfies the equation $\widetilde{H}\widetilde{\varphi} = 0$. In the one-dimensional limit it also reduces to the Darboux transformation. Its relation to integrable systems was recently studied in [18].

3 Soliton potentials via the Moutard transformation

In [19] we gave simple examples of fast decaying rational potentials of the two-dimensional Schrödinger operator which have a degenerate L_2-kernel. These examples are constructed using the Moutard transformation as follows.

Main construction. Let

$$H_0 = -\Delta = -\Delta + u_0$$

be an operator with a potential $u_0 = 0$ and let ω_1 and ω_2 satisfy the equations

$$H_0\omega_1 = H_0\omega_2 = 0.$$

We take the Moutard transformations M_{ω_1} and M_{ω_2} defined by ω_1 and ω_2 and obtain the operators

$$H_1 = -\Delta + u_1, \quad H_2 = -\Delta + u_2$$

where $u_1 = M_{\omega_1}(u_0), u_2 = M_{\omega_2}(u_0)$. By the construction, we have

$$H_1 M_{\omega_1}(\omega_2) = 0, \quad H_2 M_{\omega_2}(\omega_1) = 0.$$

Let us choose some function

$$\theta_1 \in M_{\omega_1}(\omega_2)$$

and put

$$\theta_2 = -\frac{\omega_1}{\omega_2}\theta_1 \in M_{\omega_2}(\omega_1).$$

These functions define the Moutard transformations of H_1 and H_2 and we obtain the operators H_{12} and H_{21} with the potentials

$$u_{12} = M_{\theta_1}(u_1), \quad u_{21} = M_{\theta_2}(u_2).$$

The following classical lemma is checked by a straightforward computation which we omit.

Lemma 1 *1. $u_{12} = u_{21} = u$, i.e. the diagram*

$$\begin{array}{cccccc} & u_0 & \xrightarrow{\omega_1} & u_1 & & \\ \omega_2 & \downarrow & & \downarrow & \theta_1 & , \\ & u_2 & \xrightarrow{\theta_2} & u_{12} = u_{21} & & \end{array}$$

where $\theta_1 \in M_{\omega_1}(\omega_2)$, $\theta_2 = -\frac{\omega_1}{\omega_2}\theta_1 \in M_{\omega_2}(\omega_1)$, is commutative;

2. For $\psi_1 = \frac{1}{\theta_1}$ and $\psi_2 = \frac{1}{\theta_2}$ we have

$$H\psi_1 = H\psi_2 = 0$$

where $H = -\Delta + u$.

We note that in this construction we have a free scalar parameter C (see (4)) for the choice of $\theta_1 \in M_{\omega_1}(\omega_2)$. This parameter can be used in some cases to build a non-singular potential u and functions ψ_1 and ψ_2.

Now let us apply this construction to the situation when

$$u_0 = 0.$$

The desired formula for the potential u is given by the following theorem which is derived by simple and straightforward computations which we omit.

Theorem 1 *Let*

$$\omega_1 = p_1(z) + \overline{p_1(z)}, \quad \omega_2 = p_2(z) + \overline{p_2(z)},$$

where p_1 and p_2 are holomorphic functions of z. Let us consider the Moutard transformation of the operator $H_0 = -\Delta$ defined by ω_1. Then the corresponding transformation (3) of ω_2 has the form

$$\theta_1 = \frac{i}{p_1 + \bar{p}_1}\left((p_1\bar{p}_2 - p_2\bar{p}_1) + \int \left((p_1' p_2 - p_1 p_2')dz + (\bar{p}_1\bar{p}_2' - \bar{p}_1'\bar{p}_2)d\bar{z}\right)\right).$$

Therefore the second iteration defined by θ_1 gives us the operator $H = -\Delta + u$ with

$$u = -2\Delta\log\omega_1 - 2\Delta\log\theta_1 = -2\Delta\log(\omega_1\theta_1) =$$

$$-2\Delta\log i\left((p_1\bar{p}_2 - p_2\bar{p}_1) + \int \left((p_1' p_2 - p_1 p_2')dz + (\bar{p}_1\bar{p}_2' - \bar{p}_1'\bar{p}_2)d\bar{z}\right)\right). \quad (5)$$

The free scalar parameter which we mentioned above appears as the integration constant in (5).

Let us apply this theorem to obtain some interesting examples of operators. We consider hereafter only the cases when ω_1 and ω_2 are real-valued harmonic polynomials.

EXAMPLE 1. Let

$$\omega_1 = x + 2(x^2 - y^2) + xy, \quad \omega_2 = x + y + \frac{3}{2}(x^2 - y^2) + 5xy,$$

which in terms of complex polynomials is written as

$$p_1(z) = \left(1 - \frac{i}{4}\right) z^2 + \frac{z}{2}, \quad p_2(z) = \frac{1}{4}(3 - 5i)z^2 + \frac{1-i}{2} z. \tag{6}$$

For some appropriate constant C in θ_1 we obtain

$$\begin{aligned} u = -\frac{5120(1 + 8x + 2y + 17x^2 + 17y^2)}{(160 + 4x^2 + 4y^2 + 16x^3 + 4x^2y + 16xy^2 + 4y^3 + 17(x^2+y^2)^2)^2} = \\ -\frac{5120|1 + (4-i)z|^2}{(160 + |z|^2|2 + (4-i)z|^2)^2} \end{aligned} \tag{7}$$

and

$$\begin{aligned} \psi_1 &= \frac{x + 2x^2 + xy - 2y^2}{160 + 4x^2 + 4y^2 + 16x^3 + 4x^2y + 16xy^2 + 4y^3 + 17(x^2+y^2)^2}, \\ \psi_2 &= \frac{2x + 2y + 3x^2 + 10xy - 3y^2}{160 + 4x^2 + 4y^2 + 16x^3 + 4x^2y + 16xy^2 + 4y^3 + 17(x^2+y^2)^2} \end{aligned} \tag{8}$$

(here we simplify the expressions for ψ_1 and ψ_2 by multiplying them by some constant). On page 21 we put the graphs of this potential and the solutions ψ_1, ψ_2.

Theorem 2 ([19]) *The potential u given by (7) is smooth, rational, and decays like $1/r^6$ for $r \to \infty$.*

The functions ψ_1 and ψ_2 given by (8) are smooth, rational, decay like $1/r^2$ for $r \to \infty$ and span a two-dimensional space in the kernel of the operator $L = -\Delta + u : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$.

Here and in the next example we put $r = \sqrt{x^2+y^2}$.

EXAMPLE 2. Let us take for ω_1 and ω_2 two harmonic polynomials of the third order:

$$\omega_1 = x + \frac{x^2 - y^2 - 3xy}{5} + 2(-x^3 - 3x^2y + 3xy^2 + y^3),$$

$$\omega_2 = x + y + \frac{x^2 - y^2}{2} - \frac{xy}{5} - 4(3x^2y - y^3)$$

which in terms of complex polynomials take the form

$$p_1(z) = (i-1)z^3 + \left(\frac{1}{10} + \frac{3}{20}i\right) z^2 + \frac{z}{2}, \ p_2(z) = 2iz^3 + \left(\frac{1}{4} + \frac{i}{20}\right) z^2 + \frac{1-i}{2}z. \quad (9)$$

Then the potential u and the functions ψ_1 and ψ_2 take the form

$$u = \frac{F_0(x,y)}{G(x,y)^2}, \quad \psi_1 = \frac{F_1(x,y)}{G(x,y)}, \quad \psi_2 = \frac{F_2(x,y)}{G(x,y)} \quad (10)$$

with

$$F_0(x,y) = -1280000(25+20x-287x^2+60x^3+1800x^4-30y-600xy-300x^2y+$$

$$313y^2 + 60xy^2 + 3600x^2y^2 - 300y^3 + 1800y^4),$$

$$G(x,y) = 40000 + 100x^2 + 40x^3 - 387x^4 + 40x^5 + 800x^6 - 60x^2y-$$

$$800x^3y - 200x^4y + 100y^2 + 40xy^2 + 26x^2y^2 + 80x^3y^2 + 2400x^4y^2 - 60y^3-$$

$$800xy^3 - 400x^2y^3 + 413y^4 + 40xy^4 + 2400x^2y^4 - 200y^5 + 800y^6,$$

$$F_1(x,y) = -10x - 2x^2 + 20x^3 + 6xy + 60x^2y + 2y^2 - 60xy^2 - 20y^3,$$

$$F_2(x,y) = -10x - 5x^2 - 10y + 2xy + 120x^2y + 5y^2 - 40y^3$$

(as in the previous example we simplify the expressions for ψ_1 and ψ_2 multiplying them by some constant). On pages 21–22 we put the graphs of this potential u and the solutions ψ_1, ψ_2.

Theorem 3 ([19]) *The potential u given by (10) is smooth, rational, and decays like $1/r^8$ for $r \to \infty$.*

The functions ψ_1 and ψ_2 given by (10) are smooth, rational, decay like $1/r^3$ for $r \to \infty$ and span a two-dimensional space in the kernel of the operator $L = -\Delta + u : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$.

Remark 3. We guess that for every $N > 0$ by applying this construction to other harmonic polynomials one can construct smooth rational potentials u and the eigenfunctions ψ_1 and ψ_2 decaying faster than $\frac{1}{r^N}$.

4 Soliton equations

4.1 Explicit solutions to the Novikov–Veselov equation

For simplicity, in this section we renormalize the Schrödinger operator as follows

$$H = \partial\bar{\partial} + U = \frac{1}{4}\Delta - \frac{u}{4} \tag{11}$$

with the standard $\partial = \partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$.

The Moutard transformation defined by a function ω takes the form

$$U \to U + 2\partial\bar{\partial}\log\omega$$

and the transformation of eigenfunctions is given by the same formulas (3) which are rewritten in terms of a complex coordinate as

$$\left(\bar{\partial} + \frac{\omega_{\bar{z}}}{\omega}\right)\theta = i\left(\bar{\partial} - \frac{\omega_{\bar{z}}}{\omega}\right)\varphi, \quad \left(\partial + \frac{\omega_z}{\omega}\right)\theta = -i\left(\partial - \frac{\omega_z}{\omega}\right)\varphi,$$

or

$$\varphi \to \theta = \frac{i}{\omega}\int(\varphi\partial\omega - \omega\partial\varphi)dz - (\varphi\bar{\partial}\omega - \omega\bar{\partial}\varphi)d\bar{z}.$$

Let us assume that the function φ depends also on the time t and satisfies the equations

$$\begin{gathered} H\varphi = 0, \\ \partial_t\varphi = (\partial^3 + \bar{\partial}^3 + 3V\partial + 3V^*\partial)\varphi \end{gathered} \tag{12}$$

where

$$\bar{\partial}V = \partial U, \quad \partial V^* = \bar{\partial}U.$$

Then by straightforward computations we derive

Proposition 3 (The extended Moutard transformation) [1] *The system (12) is invariant under the extended Moutard transformation*

$$\begin{gathered} \varphi \to \theta = \frac{i}{\omega}\int(\varphi\partial\omega - \omega\partial\varphi)dz - (\varphi\bar{\partial}\omega - \omega\bar{\partial}\varphi)d\bar{z} + \\ [\varphi\partial^3\omega - \omega\partial^3\varphi + \omega\bar{\partial}^3\varphi - \varphi\bar{\partial}^3\omega + 2(\partial^2\varphi\partial\omega - \partial\varphi\partial^2\omega) - \\ 2(\bar{\partial}^2\varphi\bar{\partial}\omega - \bar{\partial}\varphi\bar{\partial}^2\omega) + 3V(\varphi\partial\omega - \omega\partial\varphi) + 3V^*(\omega\bar{\partial}\varphi - \varphi\bar{\partial}\omega)]dt, \\ U \to U + 2\partial\bar{\partial}\log\omega, \quad V \to V + 2\partial^2\log\omega, \quad V^* \to V^* + 2\bar{\partial}^2\log\omega. \end{gathered} \tag{13}$$

[1] This theorem was first formulated in [14]. However in [14] the third line in the formula for the Moutard transformation $\varphi \to \theta$ was omitted and here we make this correction. Due to this fact the formula (6.2.5) in [14] does not give a solution of the Novikov-Veselov equation. Recently we learned that such a correction was already done in [12]. See also [3] for another approach to the extended Moutard transformations.

If ω is a real-valued function, then the latter transformations preserve the property $V^* = \overline{V}$.

The compatibility condition for the system (12) is the system

$$U_t = \partial^3 U + \bar{\partial}^3 U + 3\partial(VU) + 3\bar{\partial}(V^*U) = 0,$$

$$\bar{\partial}V = \partial U, \quad \partial V^* = \bar{\partial}U$$

which for

$$V^* = \overline{V}$$

reduces to the Novikov–Veselov equation. In the sequel, we consider only the cases when the above equality holds.

Let us consider the flow on the space of functions $p(z,t)$ holomorphic in z:

$$\frac{\partial p}{\partial t} = \frac{\partial^3 p}{\partial z}. \tag{14}$$

If $p_1(z,t)$ and $p_2(z,t)$ satisfy (14) then the functions $\omega = p_1 + \bar{p}_1$ and $\varphi = p_2 + \bar{p}_2$ satisfy the system (12) with $U = V = 0$. The extended Moutard transformation differs from the original one by the dt-term:

$$\theta = \frac{i}{\omega}\int \Big(\Psi_1(\omega,\varphi)dz + \Psi_2(\omega,\varphi)d\bar{z} + \Theta(\omega,\varphi)dt\Big).$$

Let us substitute $\omega = p_1 + \bar{p}_1$, $\varphi = p_2 + \bar{p}_2$ into Ψ_s. By (14) and $V = V^* = 0$ the integrand is a closed form which implies

$$\frac{\partial \Psi_1(p_1+\bar{p}_1, p_2+\bar{p}_2)}{\partial t} = \frac{\partial \Theta(p_1+\bar{p}_1, p_2+\bar{p}_2)}{\partial z},$$

$$\frac{\partial \Psi_2(p_1+\bar{p}_1, p_2+\bar{p}_2)}{\partial t} = \frac{\partial \Theta(p_1+\bar{p}_1, p_2+\bar{p}_2)}{\partial \bar{z}}.$$

Hence Θ_z and $\Theta_{\bar{z}}$ are defined by Ψ_s and we have the following analog of Theorem 1:

Proposition 4 *Let*

$$\omega_1 = p_1(z,t) + \overline{p_1(z,t)}, \quad \omega_2 = p_2(z,t) + \overline{p_2(z,t)},$$

where p_1 and p_2 are holomorphic functions of z which satisfy (14). Let us consider the extended Moutard transformation of the operator $H_0 = \partial\bar{\partial}$ defined by ω_1, and let θ_1 be the image of ω_2 under this transformation. Let

$H = \partial\bar\partial + U$ is obtained by the iteration of the Moutard transformation defined by θ_1. Then

$$U(z,\bar z,t) = 2\partial\bar\partial \log i\Big((p_1\bar{p_2} - p_2\bar{p_1}) + \int \big((p_1'p_2 - p_1p_2')dz + (\bar{p_1}\bar{p_2}' - \bar{p_1}'\bar{p_2})d\bar z\big) +$$
$$+ \int (p_1'''p_2 - p_1p_2''' + 2(p_1'p_2'' - p_1''p_2') + \bar{p_1}\bar{p_2}''' - \bar{p_1}'''\bar{p_2} + 2(\bar{p_1}''\bar{p_2}' - \bar{p_1}'\bar{p_2}''))dt\Big). \tag{15}$$

Corollary 1 *Let $p_1(z,t)$ and $p_2(z,t)$ be holomorphic functions in z which satisfy the equation (14).*

Then the substitution of them into (15) gives a solution to the Novikov–Veselov equation, which is rational in $z, \bar z$, and t.

Remark 4. It is clear that the analogs of (13) may be derived for all equations from the Novikov–Veselov hierarchy and hence the explicit solutions to them, in particular, of the form (13) may be constructed.

4.2 σ-flows

Let us denote by $\mathcal{H}_N$ the space of all harmonic polynomials $p(z) + \overline{p(z)}$ of the form

$$p(z) = \sigma_0 z^N + \sigma_1 z^{N-1} + \dots + \sigma_{N-1}z + \sigma_N$$

and denote by $\mathcal{H}_N^0$ the subspace $\sigma_0 = 1$.

The flow (14) generates on $\mathcal{H}_N$ a linear flow

$$\dot\sigma_k = (N-k+3)(N-k+2)(N-k+1)\sigma_{k-3}, \quad k = 0,\dots,N. \tag{16}$$

Since σ_0 is constant along the flow we restrict these equations onto $\mathcal{H}_N^0$ and obtain a particular example of the linear system on $\sigma_1,\dots,\sigma_N$:

$$\dot\sigma = A + B\sigma \tag{17}$$

which generates a dynamical system on the n-th symmetric product $S^n\mathbb{C}$ of $\mathbb{C}$. Indeed, $\sigma_1,\dots,\sigma_n$ are the elementary symmetric polynomials in the roots $z_1,\dots,z_n$ of $p(z)$:

$$\sigma_1(z_1,\dots,z_n) = -(z_1+\dots+z_n),\dots,\sigma_n(z_1,\dots,z_n) = (-1)^n z_1\dots z_n$$

and the integrable (even linear) evolution of σ induces a dynamical system on $S^n\mathbb{C}$. We call such a dynamical system on $S^n\mathbb{C}$ a σ-system.

In contrast to the Calogero–Moser systems, σ-systems do not describe the evolution of singularities of a solution to the NV equation (as the Calogero–Moser flow does for the KdV equation), i.e. the evolution of particle-type solutions [7, 2] (see also §2.2). In fact,

- *the Calogero–Moser systems and σ-systems are different in many respects.*

For example, let us consider the simplest flow induced by (14) on $\mathcal{H}_3$:

$$\dot{\sigma}_1 = 0, \quad \dot{\sigma}_2 = 0, \quad \dot{\sigma}_3 = 6.$$

The general solution is

$$\sigma_1 = a_1, \quad \sigma_2 = a_2, \quad \sigma_3 = a_3 + 6t$$

However for a solution $(x_1(t), \dots, x_n(t))$ to the Calogero–Moser system the k-th elementary symmetric polynomial $\sigma_k(x_1, \dots, x_n)$ is a polynomial in t of degree k and especially any function $x_k(t)$ is algebraic (see Corollary 3 on p. 118 in [2]).

So we may present examples when the solutions to the Calogero–Moser system and to a σ-system are related by a reparameterization of the time variable. For instance, such is the equilateral triangular solution to the Calogero–Moser system which corresponds to the solution $-2\frac{d^2}{dx^2}\left[(x - x_1(t))(x - x_2(t))(x - x_3(t))\right]$ of the KdV equation:

$$x_k(t) = \varepsilon^k \sqrt[3]{\varepsilon} t, \quad \varepsilon = e^{\frac{2\pi i}{3}}, \quad k = 1, 2, 3;$$

and a solution to the σ-system corresponding to $p(z,t) = z^3 + 6t$:

$$z_k(t) = \varepsilon^k \sqrt[3]{6t}, \quad \varepsilon = e^{\frac{2\pi i}{3}}, \quad k = 1, 2, 3.$$

By Corollary 1, we have

- Given two solutions $p_1 \in \mathcal{H}^0_M$ and $p_2 \in \mathcal{H}^0_N$ of (16), by a substitution of $e^{i\lambda_1} p_1$ and $e^{i\lambda_2} p_2$, where λ_1 and λ_2 are real-valued constants, into (15) we obtain solutions of the Novikov–Veselov equation.[2] Therefore to each pair of solutions to (16) there corresponds an $(S^1 \times S^1)$-family of solutions to the Novikov–Veselov equation.

We think that due to their natural appearance and not only because of their relation to the Novikov–Veselov equation which we did establish, σ-systems deserve a special investigation.

[2]If we multiply, for instance, p_1 by a constant $\mu \in \mathbb{R}$ then $\omega = p_1 + \bar{p_1}$ is multiplied by μ and, since the Moutard transformation depends on ω taken up to a multiple, this does not change the transformation.

4.3 On the blow-up of solutions

For simplicity let us write the formula (15) as

$$U = 2\partial\bar{\partial}\log\Phi(p_1,p_2).$$

(see the renomalization formula (11)). Due to the integration in (5) the function $\Phi(p_1,p_2)$ is defined up to a constant which we take real-valued to obtain a real-valued potential $U(z,\bar{z})$. By Proposition 3, the potential $V=\bar{V}^*$ equals

$$V = 2\partial^2\log\Phi(p_1,p_2).$$

It is easy to check the following:

Corollary 2 *The potential (7) (see Theorem 2) is a stationary solution of the Novikov–Veselov equation.*

On the other hand one can obtain solutions of the Novikov-Veselov equations which blow up in finite time:

Theorem 4 *The solution $U(z,\bar{z},t)$ of the Novikov–Veselov equation obtained form the polynomials $p_1 = i\,z^2$, $p_2 = z^2+(1+i)z$ using Proposition 4 and the integration constant $C=-20$ in (13) has the form*

$$U = \frac{H_1}{H_2},$$

with $H_1 = -12\Big(24tx^2+12tx+24ty^2+12ty+x^5-3x^4y+2x^4-2x^3y^2-4x^3y-2x^2y^3-60x^2-3xy^4-4xy^3-30x+y^5+2y^4-60y^2-30y\Big)$,

$$H_2 = (3x^4+4x^3+6x^2y^2+3y^4+4y^3+30-12t)^2.$$

This solution decays at infinity as r^{-3} and obviously blows up for some $t>0$.

5 The cubic superposition formula

In this Section we extend the algebraic superposition formula of L. Bianchi [4] for three initial solutions ω_1, ω_2, ω_3 of the Moutard equation to the case of the Novikov-Veselov equation. In fact, already the extended Moutard transformation (13) should be considered as an "extended superposition formula" which produces a new solution θ starting with two solutions ω, φ of the linear problem (12) with some initial potentials U, V, V^*. In contrast to the well-known algebraic superposition formula for two solutions of the Sine-Gordon equation, (13) is not algebraic and requires a quadrature. This is typical for $(2+1)$-dimensional Bäcklund transformations. In this case an algebraic superposition formula exists for three initial solutions (cf. [9] for a detailed discussion of this phenomenon).

Namely, as one may check, the following statement, obtained by L. Bianchi [4] for the case of the Moutard equation $\varphi_{xy} = u(x,y)\varphi$, is also valid in our case:

Theorem 5 *If ω_1, ω_2, ω_3 are three solution of (12) where $U = U_0$, $V = V_0$, $V^* = V_0^*$ is some given solution of the Novikov-Veselov equation (1), and θ_1, θ_2 (transformed solutions of (12) with new $U_1 = U_0 - 2\partial\bar{\partial}(\ln\omega_1)$, $U_2 = U_0 - 2\partial\bar{\partial}(\ln\omega_2)$ and respectively transformed V_i, V_i^*—another set of solutions of the Novikov-Veselov equation) are obtained from ω_3 via (13) then there exists a unique solution θ' of the 4th linear system (12) with the potentials $U = U_{12}$, $V = V_{12}$, $V^* = V_{12}^*$ such that θ' is connected to θ_1, θ_2 with the extended Moutard transformations (13). This θ' is expressible with an algebraic formula*

$$\theta' - \omega_3 = \frac{\omega_1\omega_2}{\lambda}(\theta_2 - \theta_1), \qquad \lambda = \omega_1\omega_1' = -\omega_2\omega_2', \tag{18}$$

where ω_1', ω_2' are obtained from ω_1, ω_2 according to (13).

It is useful to explain this statement and the "extended superposition formula" (13) by commutative rhombic and cubic diagrams ("Bianchi superposition cube") shown on Fig. 1.

The rhombic diagram in the left part of Fig. 1 shows two solutions ω_1 and ω_2 of (12) with the initial potential U_0 and the respective V_0, V_0^* (which form a solution of the Novikov-Veselov equation (1)). Those ω_i produce two new potentials U_1, U_2 (and the respective V_i, V_i^*)— two new solutions of the Novikov-Veselov equation (this is *step one*). The formula (13) then gives us (a set of) θ_1—a solution of (12) with the potentials U_1, V_1, V_1^* as well as $\theta_2 = -\frac{\omega_1}{\omega_2}\theta_1$—a solution of (12) with the potentials U_2, V_2, V_2^*. Using either θ_1 or θ_2 we can find the potentials $U_{12} = U_1 + 2\partial\bar{\partial}\log\theta_1 = U_2 + 2\partial\bar{\partial}\log\theta_2$, $V_{12} = V_1 + 2\partial^2\log\theta_1 = V_2 + 2\partial^2\log\theta_2$, $V_{12}^* = V_1^* + 2\bar{\partial}^2\log\theta_1 = V_2^* + 2\bar{\partial}^2\log\theta_2$ shown as the last fourth circle of the rhombic diagram (this is *step two*). As we see, this second step requires a quadrature and produces a free integration constant in the final potentials U_{12}, V_{12}, V_{12}^*.

If we have done those two steps for each of the three pairs $\{\omega_1, \omega_2\}$, $\{\omega_1, \omega_3\}$, $\{\omega_2, \omega_3\}$ shown on the right cubic diagram (and we have accumulated also three additional constants of integration), the Bianchi algebraic

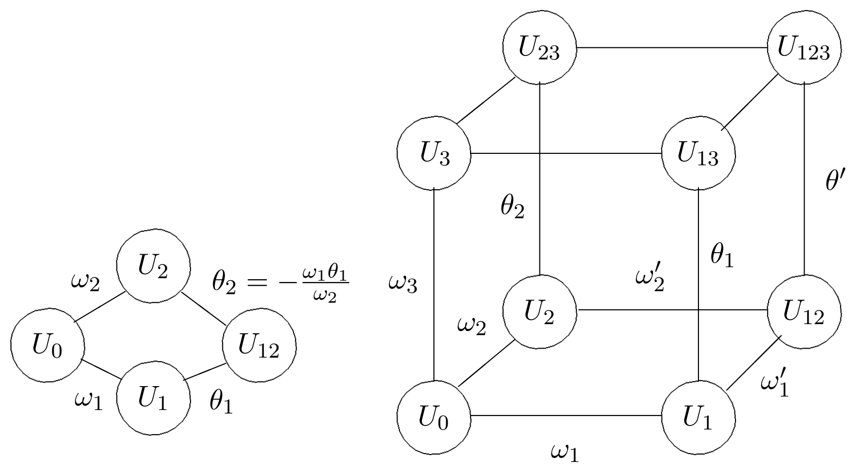

Figure 1: Rhombic and cubic superposition formulas for $(2+1)$-dimensional integrable systems

superposition formula (18) gives a solution θ' of (12) for the set U_{12}, V_{12}, V_{12}^* of the potentials obtained on step two.

The formula (18) was proved in [4] for the case of the Moutard equation $\omega_{xy} = U(x,y)\omega$. A straightforward computation shows that it preserves the dynamics of the potentials defined by (1), as Theorem 5 states.

Using (12) and (18) one can see that

- *if we start with $U_0 = V_0 = V^* = 0$ and take arbitrary harmonic polynomials as the initial solutions ω_i, $i = 1, \ldots, N$, then the potentials obtained on the first two steps are rational functions of (x, y, t). All subsequent potentials obtained on the next steps will be rational solutions of the Novikov-Veselov equation as well.*

- *If we know some large set $\{\psi\}$ of solutions of the Schrödinger equation $(\partial\bar{\partial} + U_0)\psi = 0$ for the initial potential U_0, then using Theorem 5 we can construct the corresponding solutions of $(\partial\bar{\partial} + U_{12})\tilde{\psi} = 0$ using quadratures (on step two) if we simply set $\omega_3 = \psi$ in the superposition formulas (12) and (18).*

- *This proves that we obtain potentials U_{12} integrable on zero energy level if we start from some integrable U_0, for example $U_0 = 0$.*

6 A remark on periodic and quasiperiodic solitons

The similar construction can be applied to the case when the initial functions ω_1 and ω_2 satisfy the Schrödinger equation with the constant potential $u = -k^2$:

$$(-\Delta - k^2)\omega_1 = (-\Delta - k^2)\omega_2 = 0.$$

It appears that the second iteration may give an integrable potential which is periodic and smooth. Let us expose one such an example.

Let

$$\omega_1 = \sin kx, \quad \omega_2 = \sin(ax + by), \ a^2 + b^2 = k^2,$$

and the Moutard transformation defined by ω_1 maps ω_2 into θ_1 (modulo multiples of $\frac{1}{\omega_1}$):

$$\theta_1 = \frac{b}{2\sin kx}\left(\frac{\cos(ax+by+kx)}{a+k} - \frac{\cos(ax+by-kx)}{a-k} + C\right)$$

with $C = \text{const}$ and the new potential is

$$\widetilde{u} = k^2 + 2\frac{\cos^2 kx}{\sin^2 kx}.$$

Let us iterate the Moutard transformation using θ_1 as the generating solution and obtain a new potential

$$\widetilde{\widetilde{u}} = -\widetilde{u} + 2\frac{(\theta_1)_x^2 + (\theta_1)_y^2}{\theta_1^2} = k^2 - 2\Delta\log(\omega_1\theta_1)$$

which is nonsingular for C large enough. An example of such a periodic potential $\widetilde{\widetilde{u}}$ and the corresponding solution $\psi_1 = 1/\theta_1$ for the constants $C = 3$, $a = 0$, $b = 1$, $k = 1$ is shown on Fig. 8, 9, page 22.

As we see by iterating the Moutard transformation one may construct integrable two-dimensional non-singular periodic and quasiperiodic (when a/b is not rational) potentials. The complete basis of solutions for the constructed potentials $\widetilde{\widetilde{u}} = -4U_{12}$ may be obtained by applying (18) with $\omega_3 = \exp\big(i(px+qy)\big)$, $p^2 + q^2 = k^2$. The spectral properties and the NV evolution of such potentials deserve an additional study.

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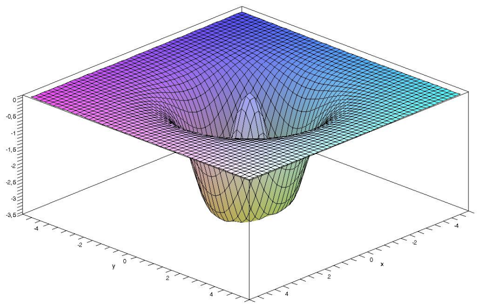

Figure 2: The potential (7).

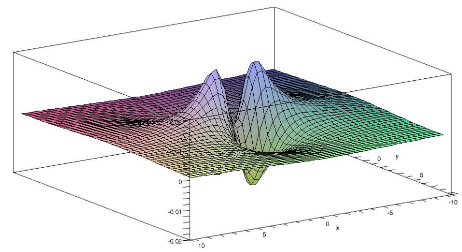

Figure 3: The solution ψ_1 in (8).

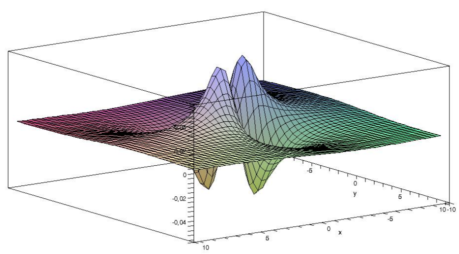

Figure 4: The solution ψ_2 in (8).

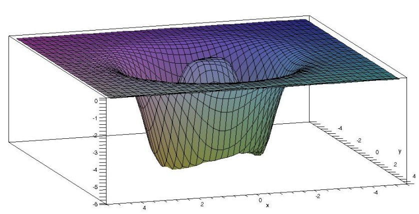

Figure 5: The potential u in (10).

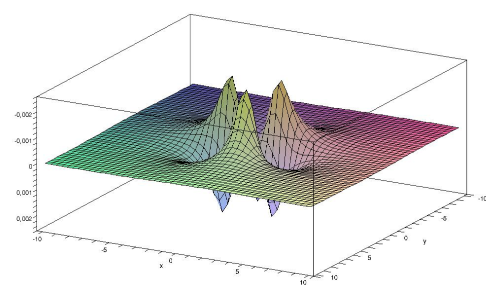

Figure 6: The solution ψ_1 in (10).

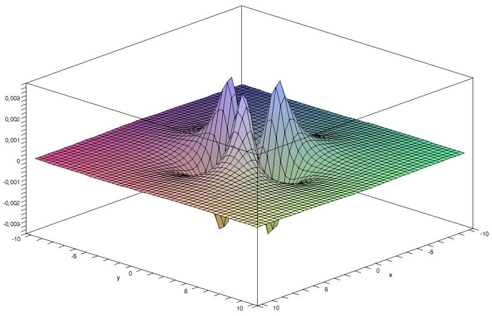

Figure 7: The solution ψ_2 in (10).

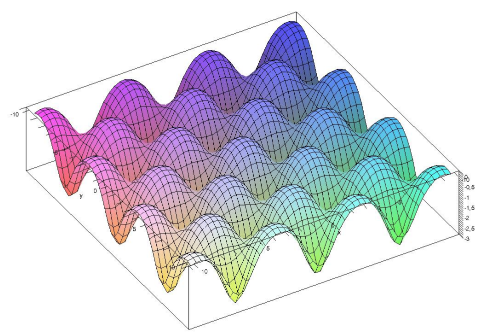

Figure 8: The periodic potential $\widetilde{\widetilde{u}}$, Section 6.

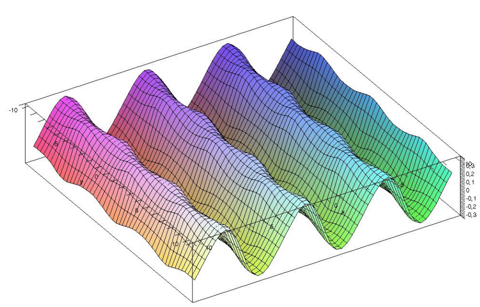

Figure 9: The solution ψ_1 for this periodic potential, Section 6.