ALMOST UNIVERSAL QUADRATIC FORMS: AN EFFECTIVE SOLUTION OF A PROBLEM OF RAMANUJAN

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1. Introduction

The object of this paper is to prove several results giving an *effective* method for deciding whether a positive definite integral quaternary quadratic form is *almost universal*, that is, whether it represents all large positive integers. In this way we obtain an effective and definitive solution to a problem first addressed and investigated by Ramanujan 90 years ago (cf. [11]).

The following set

$$\Sigma = \{1, 2, 3, 5, 6, 7, 10, 14\}$$

will play an important role in our investigations. Observe that the 8 elements of Σ are the smallest possible positive integers representing all 8 different square classes of $\mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$.

Theorem 1.1. *For any positive definite integral quaternary quadratic form f, the following conditions are equivalent:*

(i) *f is almost universal.*

(ii) *f is locally universal and, either*

(a) *f is p-isotropic for every prime number p,*

or

(b) *f is 2-anisotropic and represents every element of the set*

$$16\Sigma = \{16, 32, 48, 80, 96, 112, 160, 224\},$$

or

(c) *f is equivalent to one of the following four forms*

$$x^2 + 2y^2 + 5z^2 + 10t^2,\ x^2 + 2y^2 + 3z^2 + 5t^2 + 2yz,$$
$$x^2 + y^2 + 3z^2 + 3t^2,\ x^2 + 2y^2 + 4z^2 + 7t^2 + 2yz.$$

We shall see later on that the set 16Σ in Theorem 1.1 is optimal in the sense that removing any of its elements would lead to a false statement (cf. Section 6).

Unless stated otherwise, by an "integral form" we shall always mean a positive definite quadratic form having integer matrix (this is sometime called "classically integral"). An integral form f is called *locally universal* if f is *p-universal* for every prime p, that is, if f represents every p-adic integer over the ring $\mathbb{Z}_p$ of p-adic integers. If f represents zero nontrivially over $\mathbb{Z}_p$, we say that f is *p-isotropic*, otherwise f is called *p-anisotropic*. Effective criteria for p-universality are known and easy to verify (cf. [9]). Simple effective criteria for p-isotropy, involving the discriminant and the Hasse-Minkowski invariant of a form, are also well known (cf. [13]). Therefore all conditions characterizing almost universality, stated in Theorem 1.1, are effective.

Theorem 1.2. *An almost universal integral quadratic form is p-anisotropic for at most one prime p. The four forms listed in Theorem 1.1* (c) *are the only ones* (*up to equivalence*) *which are p-anisotropic for some $p > 2$.*

If follows that the properties (a),(b) and (c) in Theorem 1.1 occur disjointly. The four forms listed in (c) are all *universal*, that is, each represents all positive integers. The first two are 5-anisotropic, while the next two are, respectively, 3- and 7-anisotropic.

The part of Theorem 1.1 concerning the 2-anisotropic case can be replaced by an even more precise description of almost universal 2-anisotropic forms.

First define an invariant $\beta(f)$ of an integral quaternary 2-anisotropic, 2-universal quadratic form f: $\beta(f)$ is the smallest integer $b \geq 1$ such that always, if

$$f(x) \equiv 0 \pmod{2^{b+2}}$$

then $x = 2y$ for some y in $\mathbb{Z}^4$. In Section 2 we shall see that $\beta(f)$ is one of the numbers 1, 2, 3 or 4, and that any of these numbers is the $\beta(f)$ for some form f.

Theorem 1.3. *Given an integral* 2*-anisotropic quaternary quadratic form f, the following conditions are equivalent:*

(i) *f is almost universal.*

(ii) *f is* 2*-universal and represents every number in the set $2^{\beta(f)}\Sigma$.*

We shall see later on that $\beta(f) = 1$ if and only if the discriminant $d(f)$ of f is odd (cf. Corollary 2.3).

Corollary 1.4. *Let f be an integral* 2*-anisotropic quaternary quadratic form having an odd discriminant. Then f is almost universal if and only if f represents every number in the set*

$$2\Sigma = \{2, 4, 6, 10, 12, 14, 20, 28\}.$$

In view of the facts reported above, we shall focus our attention mainly on 2-anisotropic almost universal forms. We are able to describe completely and explicitly all genera of such forms. Before formulating this description we need some preparation.

Given a genus Γ containing a form f, let Γ_p be the class of equivalence over $\mathbb{Z}_p$ of $\mathbb{Z}_p$-forms represented by $f_p = f \otimes \mathbb{Z}_p$. Clearly, the sequence $\{\Gamma_p\}_{p \in P}$, where P is the set of all primes, completely determines Γ, and conversely. By abuse of notation, we shall identify Γ and the sequence $\{\Gamma_p\}_{p \in P}$, $\Gamma = \{\Gamma_p\}_{p \in P}$.

Next we shall define three important sets $\mathcal{A}$, $\mathcal{M}$ and $\mathcal{N}$ of quaternary forms.

Let $\mathcal{A}$ be the set of all (up to equivalence over $\mathbb{Z}_2$) quaternary $\mathbb{Z}_2$-forms which are 2-universal and 2-anisotropic. This set has precisely 8 elements enumerated in Proposition 2.1.

Let $\mathcal{M}$ be the set of all (up to equivalence over $\mathbb{Z}$) integral quaternary quadratic forms which are even, 2-anisotropic, represent all even positive integers, and have the discriminant of the type $4(8m + 1)$ for some integer

$m \geq 0$. We shall explicitly enumerate all elements of $\mathcal{M}$ (cf. Section 3 and Table 4 in Section 6). In particular, we shall prove that $\mathcal{M}$ contains exactly 79 forms.

Finally, let $\mathcal{N}$ be a subfamily of $\mathcal{M}$ of all forms representing different genera. The family $\mathcal{N}$ contains exactly 65 elements; they are listed in Section 6.

All these three sets $\mathcal{A}$, $\mathcal{M}$ and $\mathcal{N}$ play the crucial rule in the proofs of the results of this paper. In particular, $\mathcal{A}$ and $\mathcal{N}$ are needed to describe all genera containing some almost universal 2-anisotropic quadratic forms.

Theorem 1.5. *Let $\Gamma = \{\Gamma_p\}_{p \in P}$ be a genus of an integral quaternary quadratic form. Then the following conditions are equivalent:*

(i) *Γ contains an integral almost universal 2-anisotropic quadratic form.*
(ii) *There are forms (h, g) in $\mathcal{A} \times \mathcal{N}$, such that $\Gamma_2 = h$ and $\Gamma_p = g_p$ for all primes $p > 2$.*

Clearly, Theorem 1.5 provides all necessary information allowing to decide effectively whether a genus contains an almost universal 2-anisotropic form.

Corollary 1.6. *There are precisely 520 genera containing some almost universal 2-anisotropic quadratic form. Denoting by $\mathcal{D}$ the set of discriminants of these genera, we have*

$$\mathcal{D} = \{4^s d(g) \mid s = -1, 0, 1, 2, \text{ and } g \in \mathcal{N}\}.$$

All integers in $\mathcal{D}$ can be read off from Table 4; for $g \in \mathcal{N}$ one has $d(g) = 4v$, v odd. The largest discriminant in $\mathcal{D}$ is $4^3 19^2 = 23104$, represented by several nonequivalent 2-anisotropic almost universal forms, all belonging to the same genus.

An almost universal form which is p-anisotropic for some prime p is called *exceptional*. Corollary 1.6 implies, in particular, the finitude of the set of all nonequivalent exceptional almost universal forms, a fact first established in [10]. The number of all genera of nonexceptional almost universal forms is, of course, infinite.

The problem of finding an effective characterization of all almost universal positive definite forms was open only for the quaternary forms. No positive definite quadratic form in less than 4 variables is almost universal. In more than 4 variables, almost universal forms are precisely those that are locally universal (this is an immediate consequence of the Tartakowsky theorem (cf. [2], p. 204)).

Of course, some parts of Theorem 1.1 are known. We shall now make a few brief historical remarks. A great number of special cases of the diagonal form $ax^2 + by^2 + cz^2 + dt^2 = \langle a, b, c, d \rangle$ have been investigated by Lagrange, Liouville, Eisenstein, Smith, Dickson, and others (cf. [4], vol.3 ch.X). The first systematic search for *diagonal* almost universal quaternaries was initiated by Ramanujan in 1916 in his influential paper [11]: he determined explicitly all diagonal almost universal forms of the type $\langle a, a, a, d \rangle$. In the same paper he also wrote down all 54 diagonal universal quaternaries. Ramanujan's article was the starting point for a huge amount of work on the subject by numerous authors.

In 1926 Kloosterman published a monumental work [6] in which he applied very complicated and delicate analytic methods (of the Hardy-Littlewood style) to prove, for the *diagonal* quaternaries, what in modern language is the implication (ii)(a) $\Rightarrow$ (i) in Theorem 1.1. Kloosterman first proved an asymptotic formula for the number $r_f(n)$ of representations of a positive integer n by a form f. He then narrowed the problem to forms whose coefficients satisfy certain conditions. It was later recognized that these are the conditions for f to be locally universal. Assuming further that f is (in p-adic terms) p-isotropic for every prime p, he showed that $r_f(n) > 0$ for all large n, proving the implication (ii)(a) $\Rightarrow$ (i). Being aware that p-isotropy for all p is not a necessary condition for almost universality, Kloosterman also provided an almost complete list of the exceptional *diagonal* forms. More precisely, his list is complete, save that he was not able to decide whether the four forms (namely, $\langle 1, 2, 11, 38\rangle$ and three other) are almost universal. Twenty years later Pall completed the answer by showing that these four forms in Kloosterman's paper indeed represent all large integers. The total number of the exceptional diagonal quaternaries is 199 (cf. [9]). Thus by 1946 the Ramanujan problem of characterizing all diagonal almost universal forms was essentially solved.

In a series of three papers [9], [10], [12] published in 1946, Ross and Pall extended some results of Kloosterman to arbitrary, not necessarily diagonal forms; in particular in [10], they proved the implication (ii)(a) $\Rightarrow$ (i) in the general case, and the finitude of the set of nonequivalent exceptional quaternaries. In [12], Ross found all 3 almost universal forms which are p-anisotropic for some $p > 2$ and have an odd discriminant.

However, no effective method was known that would allow to decide almost universality of arbitrary locally universal 2-anisotropic forms.

Even, surprisingly, the real progress on the question of enumerating all universal forms did not come until very recently, when Conway and Schneeberger proved in 1993 their remarkable "The 15 Theorem". They showed that: an integral positive definite quadratic form is universal if and only if it represents all positive integers up to 15. Bhargava found a simpler proof of "The 15 Theorem" and enumerated all 204 universal quaternary forms (cf. [1], [3]). The statements of our Theorems 1.1 and 1.3 were inspired by reading the remarkable Bhargava's paper. Recently Hanke provided a list of all integer-valued universal quaternaries (cf. [5]).

The results of this paper allow, at least in principle, to enumerate explicitly all exceptional almost universal forms.

The paper is organized as follows. In Section 2, we prove an important technical result saying that every 2-universal and 2-anisotropic integral quaternary quadratic form f contains a "subform" g which, over $\mathbb{Z}_2$, is equivalent to

$$2^{\beta(f)-1}(2x^2 + 2xy + 2y^2 + 4z^2 + 4zt + 4t^2).$$

This result is better expressed in the language of lattices which we shall use in the proofs (cf. Theorem 2.4). In Section 3 we study various properties of the family $\mathcal{M}$. Theorems 2.4 and 3.1 are the main technical ingredients needed in the proof of the results stated above. Section 4 contains the proof of Theorem 1.3, and Section 5 the proofs of Theorems 1.1, 1.2 and 1.5. Section 6 contains Table 4 enumerating all 79 lattices of family $\mathcal{M}$ and the list of all 65 lattices of family $\mathcal{N}$.

2. Universal and anisotropic $\mathbb{Z}_2$-lattices

In this and the next section we shall study various 2-adic properties of the quaternary $\mathbb{Z}$-lattices.

As is well known, there is a natural bijection between classes of integral quadratic forms and lattices having integral inner products: the Gram matrix of a quadratic form f can be regarded as the matrix of the corresponding lattice L_f. Although the results in the Introduction were stated in the language of forms, it is more convenient to use the language of lattices in the proofs. Therefore we shall oscillate freely between these two languages. A $\mathbb{Z}$-lattice (by definition having an integral inner product) is called universal, almost universal, p-universal, etc., if the corresponding form is universal, almost universal, p-universal, etc. We shall also work with $\mathbb{Z}_p$-lattices, always equipped with $\mathbb{Z}_p$-valued inner product.

Given a $\mathbb{Z}$-lattice L we shall denote by $L_p = L \otimes \mathbb{Z}_p$ the lattice L regarded as a lattice over $\mathbb{Z}_p$. Given two lattices L and M over a ring R, we shall use the notation $L \simeq M$ to indicate that they are isometric over R. Often we shall make no distinction between a lattice L and its Gram matrix, and we shall use the same symbol to denote both objects. In particular, throughout this paper we shall always denote by Λ the $\mathbb{Z}$-lattice corresponding to the matrix

$$\Lambda = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The lattice $\Lambda \perp 2\Lambda$ will play an important role in our investigation. This lattice corresponds to the form $2x^2 + 2xy + 2y^2 + 4z^2 + 4zt + 4t^2$.

All $\mathbb{Z}$-lattices are assumed to be positive definite.

The family $\mathcal{A}$ defined in the Introduction is explicitly described in the following proposition.

Proposition 2.1. *A quaternary $\mathbb{Z}_2$-lattice H is 2-universal and 2-anisotropic if and only if H is isometric to one of the following diagonal lattices*

$$H_1 = \langle 1,1,1,1\rangle, \quad H_2 = \langle 1,1,2,2\rangle, \quad H_3 = \langle 1,1,1,4\rangle, \quad H_4 = \langle 3,3,3,12\rangle,$$
$$H_5 = \langle 1,2,2,4\rangle, \quad H_6 = \langle 1,1,2,8\rangle, \quad H_7 = \langle 3,3,6,24\rangle, \quad H_8 = \langle 1,2,4,8\rangle.$$

Proof. Any 2-universal $\mathbb{Z}_2$-lattice H is diagonalizable, $H \simeq \langle a_1\rangle \perp \cdots \perp \langle a_4\rangle$, $a_i \in \mathbb{Z}_2$ (cf. [10], Lemma 2.3). Moreover, one can take $a_i = 2^{m_i} v_i$, where v_i are among the numbers $1, 3, 5$ or 7 and where several restrictions are imposed on the m_i, including $0 = m_1 \le m_2 \le m_3 \le m_4 \le 3$ (cf. [9], Lemma 1). Taking this into account and removing redundant isometric cases, we obtain a short list of all 2-universal $\mathbb{Z}_2$-lattices. Further elimination of those which are 2-isotropic leaves a list of 8 lattices which are isometric over $\mathbb{Z}_2$ to 8 members of the family $\mathcal{A}$ enumerated in the proposition. □

As mentioned earlier, we shall use the notation that often makes no distinction between a lattice and its Gram matrix. More precisely, if $e_1, e_2, \ldots, e_n$ is a basis of an R-lattice L, and M is an $n \times n$ matrix with entries in a ring R, then the notation

$$L = Re_1 + \cdots + Re_n = M$$

signifies that $M = (e_i \cdot e_j)_{i,j}$ is the Gram matrix of L with respect to this basis. For elements a and b in L, we denote by $a \cdot b$ their inner product and we use the notation $L(a) = a \cdot a$. Recall that given a 2-universal and 2-anisotropic $\mathbb{Z}$- (or $\mathbb{Z}_2$-) lattice L, the invariant $\beta(L)$ is defined as follows: $\beta(L)$ is the smallest integer $b \geq 1$, such that every x in L satisfying $L(x) \equiv 0 \pmod{2^{b+2}}$ is of the form $x = 2y$ for some y in L.

Proposition 2.2. *Let $K_1, \ldots, K_8$ be the $\mathbb{Z}$-lattices defined below by their Gram matrices (with respect to a basis $e_1, \ldots, e_4$ of K_i):*

$$K_1 = \Lambda \perp \langle 1, 3 \rangle,\ K_2 = \langle 1, 3, 2, 6 \rangle,\ K_3 = \Lambda \perp \langle 3, 4 \rangle,\ K_4 = \Lambda \perp \langle 1, 12 \rangle,$$
$$K_5 = \langle 2, 6, 1, 12 \rangle,\ K_6 = \langle 1, 3, 6, 8 \rangle,\ K_7 = \langle 1, 3, 2, 24 \rangle,\ K_8 = \langle 1, 12, 6, 8 \rangle.$$

Define $s_1 = 1$, $s_2 = 2$, $s_3 = s_4 = s_5 = 4$, $s_6 = s_7 = s_8 = 8$, and let, for $i = 1, 2, \ldots, 8$,

$$E(K_i, 2s_i) = \{x \in K_i \mid K_i(x) \equiv 0 \pmod{2s_i}\}.$$

Then

(i) *One has $(K_i)_2 \simeq H_i$, where H_i is a $\mathbb{Z}_2$-lattice of family $\mathcal{A}$ listed in Proposition 2.1.*

(ii) *For each $i = 1, 2, \ldots, 8$, one can find vectors $f_1, \ldots, f_4$ in K_i, where $f_k = \sum_{j=1}^4 \alpha_{kj} e_j$, with α_{kj} in $\mathbb{Z}$ depending on K_i, such that the determinant of $(\alpha_{kj})_{k,j}$ is a power of 2, and*

$$E(K_i, 2s_i) = \mathbb{Z}f_1 + \cdots + \mathbb{Z}f_4 = s_i(\Lambda \perp 2\Lambda). \tag{1}$$

In particular, $E(K_i, 2s_i)$ is a sublattice of K_i representing all positive integers divisible by $2s_i$.

(iii) *One has $s_i = 2^{\beta(K_i)-1}$, and s_i is the smallest power of 2 satisfying (1) above.*

Proof. (i) Checking that the $\mathbb{Z}_2$-lattices $(K_i)_2$ and H_i are isometric is an easy exercise (cf. [2], Chapter 8).

(ii) For each $i = 1, \ldots, 8$ we shall first indicate explicitly the vectors $f_1, \ldots, f_4$ as linear combinations of the e_i.

K_i	f_1	f_2	f_3	f_4	K_i	f_1	f_2	f_3	f_4
K_1	e_1	e_2	$2e_3$	e_3+e_4	K_5	$2e_1$	e_1+e_2	$4e_3$	$2e_3+e_4$
K_2	$2e_1$	e_1+e_2	$2e_3$	e_3+e_4	K_6	$4e_1$	$2e_1+2e_2$	$2e_3+e_4$	$2e_4$
K_3	$2e_1$	$2e_2$	$2e_3+e_4$	$2e_4$	K_7	$4e_1$	$2e_1+2e_2$	$4e_3$	$2e_3+e_4$
K_4	$2e_1$	$2e_2$	$4e_3$	$2e_3+e_4$	K_8	$4e_1$	$2e_1+e_2$	$2e_3+e_4$	$2e_4$

Looking at the coefficients α_{kj} in $f_k = \sum \alpha_{kj} e_j$, one sees immediately that $\det(\alpha_{kj})$ is a power of 2.

Denote

$$V_i = \mathbb{Z}f_1 + \cdots + \mathbb{Z}f_4.$$

Checking that $V_i = s_i(\Lambda \perp 2\Lambda)$ amounts to the computation of the products $f_k \cdot f_j$, which is trivial. To check that $V_i = E(K_i, 2s_i)$, we argue as follows. Since we know already that $V_i = s_i(\Lambda \perp 2\Lambda)$, and the lattice $\Lambda \perp 2\Lambda$ represents precisely all even positive integers, it follows that $V_i \subset E(K_i, 2s_i)$.

To prove the inclusion in the opposite direction, we go "backward". We take $x = a_1e_1 + \cdots + a_4e_4$ in K_i, a_j in $\mathbb{Z}$, satisfying $K_i(x) \equiv 0 \pmod{2s_i}$.

Then, by examining the restrictions imposed on the coefficients a_j by this congruence, we prove that x can be presented as a $\mathbb{Z}$-linear combination of the f_k. We shall illustrate this procedure on two examples, say, K_1 and K_8. The remaining six cases can be checked easily, following the identical pattern.

Assume therefore that for $x = a_1e_1 + \cdots + a_4e_4$ in K_1, $a_j \in \mathbb{Z}$, one has $K_1(x) \equiv 0 \pmod 2$. It follows that

$$K_1(x) = 2(a_1^2 + a_1a_2 + a_2^2) + a_3^2 + 3a_4^2 \equiv 0 \pmod 2,$$

which can occur only if $a_3 = a_4 + 2v$ for some v in $\mathbb{Z}$. In turn, this implies

$$x = a_1e_1 + a_2e_2 + 2ve_3 + a_4(e_3 + e_4) = a_1f_1 + a_2f_2 + vf_3 + a_4f_4,$$

showing that x is in V_1.

The case K_8 is similar: let $x = a_1e_1 + \cdots + a_4e_4$ in K_8 satisfy $K_8(x) \equiv 0 \pmod{16}$. It follows that

$$K_8(x) = a_1^2 + 12a_2^2 + 6a_3^2 + 8a_4^2 \equiv 0 \pmod{16}.$$

Necessarily, a_1 must be even, say, $a_1 = 2t_1$ for some t_1 in $\mathbb{Z}$. Then

$$2t_1^2 + 6a_2^2 + 3a_3^2 + 4a_4^2 \equiv 0 \pmod 8.$$

It follows that $a_3 = 2t_3$ for some t_3 in $\mathbb{Z}$ and therefore

$$t_1^2 + 3a_2^2 + 6t_3^2 + 2a_4^2 \equiv 0 \pmod 4.$$

But then $t_1 = a_2 + 2v$ for some v in $\mathbb{Z}$ and thus

$$2(a_2^2 + a_2v + v^2) + 3t_3^2 + a_4^2 \equiv 0 \pmod 2.$$

The last congruence implies $a_4 = t_3 + 2u$ for some u in $\mathbb{Z}$. One has therefore

$$x = 2t_1e_1 + a_2e_2 + 2t_3e_3 + a_4e_4,$$

and substituting for t_1 and a_4 the values computed above, one gets

$$x = v_1f_1 + a_2f_2 + t_3f_3 + uf_4,$$

proving that x is in V_8 and ending the proof of (ii).

(iii) First observe that for each $i = 1, \ldots, 8$, there is a primitive element x_i in K_i such that $K_i(x_i) = 4s_i$: indeed, it suffices to take $x_i = a_ie_3+e_4$, where $a_1 = a_2 = 1$ and $a_3 = \cdots = a_8 = 2$. Then it follows from the definition of the invariant $\beta(K_i)$, that $\beta(K_i) \geq l_i + 1$, where $s_i = 2^{l_i}$.

On the other hand, (1) above implies easily that

$$\{x \in K_i \mid K_i(x) \equiv 0 \pmod{8s_i}\} = \mathbb{Z}(2f_1) + \cdots + \mathbb{Z}(2f_4) = 4s_i(\Lambda \perp 2\Lambda).$$

Hence, if $K_i(x) \equiv 0 \pmod{8s_i}$, then $x = 2y$ for some y in K_i, implying $\beta(K_i) \leq l_i + 1$. Therefore $\beta(K_i) = l_i + 1$, as claimed. This argument also shows that s_i is the smallest power of 2 satisfying (1). □

Corollary 2.3. (i) *If L is a 2-universal and 2-anisotropic quaternary $\mathbb{Z}$-lattice, then $\beta(L) = \beta(L_2)$.*

(ii) *If H_i is one of the eight $\mathbb{Z}_2$-lattices of family $\mathcal{A}$ enumerated in Proposition 2.1, then*

$$\beta(H_1) = 1, \quad \beta(H_2) = 2, \quad \beta(H_3) = \beta(H_4) = \beta(H_5) = 3,$$

$$\beta(H_6) = \beta(H_7) = \beta(H_8) = 4.$$

(iii) *For a 2-anisotropic quaternary $\mathbb{Z}$-lattice L one has $\beta(L) = 1$ if and only if its discriminant $d(L)$ is odd.*

Proof. (i) is obvious, and (ii) follows from (i) and Proposition 2.2. To show (iii) observe that a 2-anisotropic quaternary $\mathbb{Z}$-lattice L has $d(L)$ odd if and only if L_2 is isometric over $\mathbb{Z}_2$ to H_1. Then (iii) follows from (ii). □

In general, an arbitrary 2-universal and 2-anisotropic quaternary $\mathbb{Z}$-lattice L does not contain a sublattice $2^{\beta(L)-1}(\Lambda \perp 2\Lambda)$, as it was the case for 8 special lattices in Proposition 2.2. However, the $\mathbb{Z}_2$-lattice L_2 does contain $2^{\beta(L)-1}(\Lambda \perp 2\Lambda)_2$, as we shall see in the next theorem. This result is crucial for our investigations.

Theorem 2.4. *Let L be a 2-universal and 2-anisotropic quaternary $\mathbb{Z}$-lattice, and let $s = 2^{\beta(L)-1}$. Then*

(i) *The set*

$$E = E(L, 2s) = \{x \in L \mid L(x) \equiv 0 \pmod{2s}\}$$

is a sublattice of L (representing all integers divisible by $2s$ which are represented by L).

(ii) $E_2 \simeq s(\Lambda \perp 2\Lambda)_2$.

(iii) $E_q \simeq L_q$ *for every odd prime q.*

Proof. (i) Among the 8 lattices $K_1, \ldots, K_8$ listed in Proposition 2.2 choose the one, denoted by M, for which $L_2 \simeq M_2$. The existence of M follows from Proposition 2.1 and 2.2 (i). By Corollary 2.3, $\beta(L) = \beta(M)$.

Let $e_1, \ldots, e_4$ be a basis of M with the properties described in Proposition 2.2. Choose a basis $e'_1, \ldots, e'_4$ of L such that

$$M(x) \equiv L(\phi(x)) \pmod{2^{18}}, \qquad (*)$$

for all x in M, where $\phi : M \to L$ is a $\mathbb{Z}$-linear bijection defined by the requirement $\phi(e_i) = e'_i$. The existence of such a basis $e'_1, \ldots, e'_4$ is an immediate consequence of the fact that $L_2 \simeq M_2$. (Such a basis exists, of course, for every power of 2; for our purpose we need a sufficiently larger power).

The congruence $(*)$ above implies that

$$\phi(E(M, 2s)) = E(L, 2s) = E.$$

Since ϕ is a $\mathbb{Z}$-linear bijection and, by Proposition 2.2, $E(M, 2s)$ is a sublattice of M having a basis $f_1, \ldots, f_4$, $f_k = \sum a_{kj} e_j$, it follows that E is a sublattice of L with a basis $f'_1, \ldots, f'_4$, $f'_k = \sum a_{kj} e'_j$. This shows (i). Observe that the coefficients α_{kj} in f_k and f'_k are identical.

(ii) The congruence $(*)$ above implies also

$$E(M, 2s) \equiv E \pmod{2^{17}}. \qquad (**)$$

By Proposition 2.2, we know that $E(M, 2s) = s(\Lambda \perp 2\Lambda)$. In particular, $d(E(M, 2s)) = 9 \cdot 2^{4\beta(L)-2}$, and $4\beta(L) - 2 \leq 14$. We can therefore apply Lemma 5.1 in [2] p.123, to deduce from $(**)$ that

$$E_2 \simeq (E(M, 2s))_2 \simeq s(\Lambda \perp 2\Lambda)_2,$$

which shows (ii).

(iii) Finally, since by Proposition 2.2 (iii) the determinant $\det(\alpha_{kj})$ is a power of 2, and thus a unit in $\mathbb{Z}_q$ for $q > 2$, it follows that $f'_1, \ldots, f'_4$ is a basis of L_q. Hence $E_q \simeq L_q$ for every odd prime q. This shows (iii) and ends the proof of Theorem 2.4. $\square$

We end this section with a result needed in the proof of Theorems 1.1 and 1.5.

Corollary 2.5. *Let L be a 2-universal and 2-anisotropic quaternary $\mathbb{Z}$-lattice and let*

$$F(L,4s) = \{x \in L \mid L(x) \equiv 0 \pmod{4s}\},$$

where $s = 2^{\beta(L)-1}$. Then $F(L,4s)$ is a sublattice of L, $(F(L,4s))_q \simeq L_q$ for every odd prime q, and

$$(F(L,4s))_2 \simeq 2s(\Lambda \perp 2\Lambda)_2.$$

Proof. Let M be one of the eight $\mathbb{Z}$-lattices of Proposition 2.2 for which $L_2 \simeq M_2$. Keeping the notation of Proposition 2.2, one checks easily that

$$F(M,4s) = \mathbb{Z}(2f_1) + \mathbb{Z}(2f_2) + \mathbb{Z}(f_3) + \mathbb{Z}(f_4) = 2s(2\Lambda \perp \Lambda).$$

(Observe that in the last equality, the natural notation $\Lambda \perp 2\Lambda$ is reversed). Then arguing as in the proof of Theorem 2.4, one shows the corollary. $\square$

3. Escalators

This section is entirely devoted to the proof of the following crucial result:

Theorem 3.1. *Let M be a quaternary $\mathbb{Z}$-lattice such that $M_2 \simeq (\Lambda \perp 2\Lambda)_2$. If M represents every number of the set*

$$2\Sigma = \{2, 4, 6, 10, 12, 14, 20, 28\},$$

then M represents every even positive integer. Moreover, M is q-universal and q-isotropic for every odd prime q.

Before giving the proof we need some preparation. We shall call a quaternary $\mathbb{Z}$-lattice L *admissible* if $L_2 \simeq (\Lambda \perp 2\Lambda)_2$. Observe that L is admissible if and only if L is even, 2-anisotropic and has the discriminant $d(L)$ of the form $4(8m+1)$, for some integer $m \geq 0$.

Important in the proof of Theorem 3.1 is the notion of "2-anisotropic escalator lattice" (here we adapt the terminology and the methods used in Bhargava's paper [1] to suit our purpose).

Given a $\mathbb{Z}$-lattice L of dimension n, let us denote by $\mu_n(L)$ its n-th successive minimum, and by $\delta(L)$ the smallest even positive integer *not* represented by L (if exists). We call $\delta(L)$ the *defect* of L. Clearly, $\delta(L)$ always exists if $n \leq 3$.

A 2-*anisotropic escalation* (or, briefly *an escalation*) of an even, 2-anisotropic n-dimensional lattice L is, by definition, any $n+1$ dimensional 2-anisotropic lattice L' which is generated by L and a vector whose norm is an even integer e satisfying

$$\mu_n(L) \leq e \leq \delta(L).$$

Moreover, we add an extra requirement if $\dim(L') = 4$. Namely, the 4-dimensional escalation should have the discriminant $d(L') = 4(8m+1)$, for some integer $m \geq 0$. In particular, each 4-dimensional escalation is an admissible lattice.

A (2-anisotropic) *escalator* lattice is a lattice which can be obtained as the result of a sequence of successive escalations of the zero-dimensional lattice.

In the next subsection we shall study all escalators of dimension ≤ 3. In particular, we shall prove that all 3-dimensional escalators are p-isotropic for every odd prime p. This will imply immediately that all 4-dimensional escalators are also p-isotropic for every odd prime p.

In the second subsection we shall study all 4-dimensional escalators. We shall prove that, up to isometry, there are exactly 81 of them. Two do not represent the number 20. The remaining 79 represent all even positive integers; in particular, they are necessarily p-universal for every odd prime p. These 79 $\mathbb{Z}$-lattices will form the family $\mathcal{M}$ defined in the Introduction.

All these will prove Theorem 3.1.

3.1. **Escalators of dimension ≤ 3.** The unique escalation of zero-dimensional lattice is the lattice generated by a single vector of norm 2. This lattice corresponds to the form $2x^2$ (or, to the matrix $[2]$), and its defect is equal to 4. Hence an escalation of $L = [2]$ has the Minkowski-reduced Gram matrix of the form

$$\begin{bmatrix} 2 & a \\ a & e \end{bmatrix}$$

where e is even, $2 = \mu_1(L) \leq e \leq \delta(L) = 4$, and $a = 0, \pm 1$. The choice with $a = -1$ leads to an isometric lattice with $a = 1$, and the choice with $e = 4, a = 1$ leads to a 2-isotropic lattice. Therefore we obtain exactly 3 nonisometric binary escalators, namely those having reduced Gram matrices

$$B_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Their defects $\delta(B_i)$ are, respectively, $6, 4$ and 10. Of course, the number in the lower right corner of B_i indicates the second successive minimum of the lattice B_i. As usual, we identify a lattice with its Gram matrix.

If we escalate each of these binary escalators in the same manner, we find that we obtain exactly 14 nonisometric ternary escalators, namely those having reduced Gram matrices

$$T_1 = \begin{bmatrix} 2&0&0 \\ 0&2&0 \\ 0&0&2 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 2&0&1 \\ 0&2&1 \\ 1&1&2 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 2&0&0 \\ 0&2&0 \\ 0&0&4 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 2&0&1 \\ 0&2&1 \\ 1&1&6 \end{bmatrix}, \quad T_5 = \begin{bmatrix} 2&1&0 \\ 1&2&0 \\ 0&0&4 \end{bmatrix},$$

$$T_6 = \begin{bmatrix} 2&0&0 \\ 0&4&0 \\ 0&0&4 \end{bmatrix}, \quad T_7 = \begin{bmatrix} 2&0&0 \\ 0&4&2 \\ 0&2&4 \end{bmatrix}, \quad T_8 = \begin{bmatrix} 2&0&0 \\ 0&4&0 \\ 0&0&6 \end{bmatrix}, \quad T_9 = \begin{bmatrix} 2&0&1 \\ 0&4&0 \\ 1&0&6 \end{bmatrix}, T_{10} = \begin{bmatrix} 2&0&1 \\ 0&4&2 \\ 1&2&6 \end{bmatrix},$$

$$T_{11} = \begin{bmatrix} 2&0&0 \\ 0&4&0 \\ 0&0&8 \end{bmatrix}, T_{12} = \begin{bmatrix} 2&0&0 \\ 0&4&2 \\ 0&2&10 \end{bmatrix}, T_{13} = \begin{bmatrix} 2&0&1 \\ 0&4&0 \\ 1&0&10 \end{bmatrix}, T_{14} = \begin{bmatrix} 2&0&1 \\ 0&4&2 \\ 1&2&10 \end{bmatrix}.$$

Table 1.

Again, the third successive minimum $\mu_3(T_i)$ is the number in the right lower corner of the matrix T_i. The defects $\delta(T_i)$ and the discriminants $d(T_i)$ are listed in the Table 2 below.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\delta(T_n)$	14	28	28	12	20	14	10	20	20	28	28	14	20	20
$d(T_n)$	2^3	2^2	2^4	$2^2 5$	$2^2 3$	2^5	$2^3 3$	$2^4 3$	$2^2 11$	$2^2 3^2$	2^6	$2^3 3^2$	$2^2 19$	$2^2 17$

Table 2.

Let us quickly indicate how these 14 ternary escalators were obtained. Starting with binary escalator B_1, we obtain precisely 4 nonisometric ternary escalators $T_1, \ldots, T_4$. Indeed, each such an escalator T has a reduced Gram matrix of the form

$$T = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{B_1}} & a \\ & & b \\ \hline a & b & e \end{array}\right],$$

where $B_1 = (b_{ij})$, $2|a| \le b_{11} = 2$, $2|b| \le b_{22} = 2$ and e is even, satisfying $2 = \mu_2(B_1) \le e \le \delta(B_1) = 6$. Disregarding choices of a and b leading to isometric lattices, we get 3 possible choices for (a, b), namely $(0,0), (0,1)$ and $(1,1)$. With three possible choices for $e = 2, 4$ or 6, it gives 9 candidates for an escalator of B_1. Among these 9 candidates only 4 are 2-anisotropic: they are $T_1, \ldots, T_4$ listed above.

Similarly, starting with B_2 we get two nonisometric escalators, namely T_2 (already obtained from B_1, and therefore disregarded) and a new T_5.

Finally, proceeding in the same manner we obtain 9 new ternary escalators $T_6, \ldots, T_{14}$ from B_3.

Schematically, we have the following tower of escalators of dimension ≤ 3.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & {[2]} \arrow[dl] \arrow[d] \arrow[dr] & & \\
 & B_1 \arrow[dl] \arrow[d] & B_2 \arrow[d] & B_3 \arrow[d] \arrow[dr] & \\
 T_1 & \ldots, T_4 & T_5 & T_6, \ldots & T_{14}
\end{tikzcd}$$

Observe that the set 2Σ coincides with the set of all defects $\delta(L)$ of the escalator latices L of dimension ≤ 3.

To end this subsection let us record the following result.

Proposition 3.2. *Each of the* 14 *ternary escalators T_i enumerated above is p-isotropic for every odd prime p.*

Proof. It suffices to consider only primes dividing the discriminant $d(T_i)$. Each $d(T_i)$ has at most one odd prime factor (cf. Table 2). Since the total number of primes q at which any positive definite ternary lattice is q-anisotropic is odd, and since each T_i is 2-anisotropic, the proposition follows. $\square$

3.2. **Escalators of dimension** 4. The next step is to determine all 4-dimensional escalators. Surprisingly, their number is relatively small, namely 81.

Let us indicate how they are obtained. Each such an escalator G is obtained from some ternary escalator, say T_k. A Gram matrix of G is among the matrices of the form

$$G = \left[\begin{array}{ccc|c} & & & a \\ & T_k & & b \\ & & & c \\ \hline a & b & c & e \end{array}\right], \tag{$*$}$$

where e is an even integer satisfying $\mu_3(T_k) \le e \le \delta(T_k)$, and where a, b, c are integers satisfying $2|a| \le t_{11}$, $2|b| \le t_{22}$, $2|c| \le t_{33}$, $T_k = (t_{ij})$. Moreover, G satisfies the condition $d(G) = 4(8m+1)$ for some $m \ge 0$; in particular $d(G)$ is a square in $\mathbb{Q}_2$. We investigate all lattices corresponding to these matrices. Some of them might be isometric. We choose one in each isometry class. Then we remove all which are 2-isotropic (they are exactly those G which have the Hasse-Minkowski invariant $\epsilon_2(G) = 1$, cf. [12] p.37). At the end we obtain a full list of 81 nonisometric 4-dimensional escalators.

In some cases, namely for $k = 3, 6, 8$ and 11, the ternary escalator T_k produces no quaternary escalator G: the reason is simple, each lattice G corresponding to the matrix $(*)$ above has then the discriminant $d(G)$ not of the form $4(8m+1)$. Let us see it for T_6. It is trivial to check that in this case G would have $d(G) = 8(4e - 2a^2 - b^2 - c^2)$, clearly not of the required type. Similar arguments work for $k = 3, 8$ and 11.

In the next table, we enumerate for each ternary escalator T_k, $k = 1, \dots, 14$, all quaternary escalators $G_{i_1}, \dots, G_{i_m}$ obtained from T_k. In order to avoid repetitions, if G is obtained from T_k and T_l, with $k < l$, we shall list G as obtained only from T_k.

An explicit list of all 4 dimensional escalators $G_1, \dots, G_{79}, E_1, E_2$, together with their discriminants, is given in Table 4 in Section 6.

The escalators E_1 and E_2 are special, they do not represent the number 20 belonging to the set 2Σ. They are therefore irrelevant to our purposes. We shall see later that each of the remaining escalators $G_1, \dots, G_{79}$ represents all even positive integers.

T_1	T_2	T_3	T_4	T_5	T_6
$G_1, \dots, G_4$	$G_5, \dots, G_7$	$\emptyset$	$G_8, \dots, G_{10}$	$G_{11}, \dots, G_{15}$	$\emptyset$
T_7	T_8	T_9	T_{10}	T_{11}	T_{12}
$G_{16}, \dots, G_{19}$	$\emptyset$	$G_{20}, \dots, G_{31}$	$G_{32}, \dots, G_{45}$	$\emptyset$	$G_{46}, \dots, G_{55}$
T_{13}	T_{14}				
$G_{56}, \dots, G_{69}, E_1$	$G_{70}, \dots, G_{79}, E_2$				

Table 3.

Example. Now we shall show on a concrete example how, starting with a 3-dimensional escalator T_1, we can obtain all 4-dimensional escalators $G_1, \dots, G_4$ containing T_1. Each such an escalator G has the Gram matrix of the form

$$G = \left[\begin{array}{ccc|c} & T_1 & & \begin{smallmatrix} a \\ b \\ c \end{smallmatrix} \\ \hline a & b & c & e \end{array}\right]$$

for some choice $a, b, c = 0, \pm 1$, and some e even, $2 = \mu_3(T_1) \leq e \leq \delta(T_1) = 14$. One sees immediately that if some of the coefficients a, b or c are equal to -1, then they can be replaced by 1, without changing the isometry class of G. Also, since $d(G) = 4(2e - (a^2 + b^2 + c^2))$ and since $d(G)$ must be of the form $4(8m + 1)$, it follows that

$$2e - (a^2 + b^2 + c^2) \equiv 1 \pmod 8.$$

This can occur only if e is not divisible by 4 and $a^2 + b^2 + c^2 = 3$. In other words, one has necessarily $a = b = c = 1$ and $e = 2, 6, 10$ or 14. The corresponding $\mathbb{Z}$-lattices are all 2-anisotropic and mutually nonisometric. They are precisely all 4-dimensional escalators $G_1, \ldots, G_4$ extending T_1. □

Proceeding in the same manner with the remaining ternary escalators $T_2, \ldots, T_{14}$, we obtain all 81 nonisometric 4-dimensional escalators $G_1, \ldots, G_{79}$, E_1, E_2.

Proposition 3.3. *Every admissible quaternary $\mathbb{Z}$-lattice M representing every number of the set 2Σ is isometric to some 4-dimensional escalator G_i.*

Proof. Let $e_1, \ldots, e_4$ be a basis of M such that $M(e_1), \ldots, M(e_4)$ are the successive minima of M. The existence of such a basis, which is not obvious, follows from Corollary 6.2.3, p. 195 in [7] (observe that such a basis does not exist in general for lattices of dimension greater that 4).

Since the set 2Σ coincides with the set of all defects of n-dimensional escalators, $n \leq 3$, it follows that taking

$$L_i = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_i, \quad i = 1, \ldots, 4,$$

one obtains a tower of escalators

$$[2] = L_1 \subset L_2 \subset L_3 \subset L_4 = M.$$

The proposition follows. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let M be an admissible $\mathbb{Z}$-lattice representing all elements of the set 2Σ. By Proposition 3.3, M is isometric to one of the 79 escalators $G_1, \ldots, G_{79}$ constructed earlier in this section and listed in Table 4. It follows from Proposition 3.2 that each of them is q-isotropic for every odd prime q. The proof of Theorem 3.1 will be completed if we can show that each G_i represents all even integers (in particular, this would also imply q-universality of G_i for every odd q). This can be seen as follows. Each G_i is an even $\mathbb{Z}$-lattice. It follows that the off-diagonal coefficients of the Gram matrix of the lattice $K_i = G_i^{1/2}$, obtained from G_i by scaling 1/2, are half integers. The quadratic forms corresponding to K_i's are integer-valued (in general, not integer-matrix). A list of all integer-valued universal quaternary quadratic forms is known (cf. [5]), and checking against this list, we see that each K_i is universal. It follows that G_i itself represents all even positive integers. □

Since each escalator G_i, $i = 1, \dots, 79$, is an admissible $\mathbb{Z}$-lattice, we get immediately the following explicit description of the family $\mathcal{M}$ of all admissible $\mathbb{Z}$-lattices representing all even positive integers.

Corollary 3.4. *Up to isometry, there are exactly* 79 *admissible quaternary $\mathbb{Z}$-lattices representing all even positive integers. They form the family*

$$\mathcal{M} = \{G_1, \dots, G_{79}\}$$

consisting of all 4*-dimensional escalators G_i representing all even positive integers. Each G_i is q-universal and q-isotropic for every odd prime q.*

4. Proof of Theorem 1.3

We shall need the following 4-dimensional version of the Tartakowsky theorem (for an analytic proof see [10], for an arithmetic one [2] p.235-249).

Theorem 4.1. *Let L be a quaternary $\mathbb{Z}$-lattice. For each prime q (if any) such that L is q-anisotropic, fix a bound to the power of q which may divide a positive integer n represented locally (i.e., over every $\mathbb{Z}_p$) by L. Then L represents every such n sufficiently large.*

Recall that the invariant $\beta(L)$ of a 2-universal and 2-anisotropic quaternary $\mathbb{Z}$-lattice L depends only on L_2, $\beta(L) = \beta(L_2)$, and that $\beta(L)$ takes its values in the set $\{1, 2, 3, 4\}$ (cf. Corollary 2.3).

Also recall that $\Sigma = \{1, 2, 3, 5, 6, 7, 10, 14\}$.

Although Theorems 1.1, 1.2, 1.3 and 1.5 were formulated in the language of quadratic forms, we shall give their proofs in the language of lattices. A form f in the statement of a theorem is replaced by the corresponding lattice $L_f = L$ in its proof.

Proof of Theorem 1.3. Let L be a 2-anisotropic quaternary $\mathbb{Z}$-lattice.

(i) $\Longrightarrow$ (ii). Assume that L is almost universal and let us show that L represents every positive integer n divisible by $2^{\beta(L)}$. Since L is almost universal, L represents $4^k n$ for some $k \geq 1$, say, $L(x) = 4^k n$. Since $4^k n \equiv 0 \pmod{2^{\beta(L)+2}}$, it follows that $x = 2y$ for some y in L. Hence $4^{k-1} n = L(y)$ is represented by L. Arguing by induction, we deduce that n itself is represented by L.

(ii) $\Longrightarrow$ (i). Assume now that L is 2-universal and representing all 8 elements of the set $2^{\beta(L)}\Sigma$. By Theorem 2.4, the set

$$E = \{x \in L \mid L(x) \equiv 0 \pmod{2^{\beta(L)}}\}$$

is a sublattice of L, $E_2 \simeq s(\Lambda \perp 2\Lambda)_2$, where $s = 2^{\beta(L)-1}$, and $E_q \simeq L_q$ for every odd prime q. Since L represents all elements of $2^{\beta(L)}\Sigma$, clearly, E also represents all these elements. The scale of E is $s\mathbb{Z}$. It follows that the $\mathbb{Z}$-lattice $G = E^{1/s}$, obtained from E by scaling $1/s$, represents all elements of 2Σ and $G_2 \simeq (\Lambda \perp 2\Lambda)_2$.

Since by Theorem 3.1, G represents all even positive integers, E necessarily represents all positive integers divisible by $2^{\beta(L)}$. Also, by Theorem 3.1, G is q-universal and q-isotropic for every odd prime q. The scaling $1/s$ is a unit in $\mathbb{Z}_q$ for $q > 2$. Therefore the lattice E itself is q-universal and q-isotropic for $q > 2$.

It follows that L is locally universal, q-isotropic for every odd prime q, and representing all positive integers divisible by $2^{\beta(L)}$. We can now apply Theorem 4.1 to deduce that L represents all large integers of the form $2^m v$, where $0 \le m < \beta(L)$, and v is odd. This shows that L is almost universal. □

Let us record the following fact showed in the proof of Theorem 1.3.

Corollary 4.2. *Every almost universal* 2*-anisotropic* $\mathbb{Z}$*-lattice* L *is* q*-isotropic for every odd prime* q *and represents every positive integer divisible by* $2^{\beta(L)}$.

5. Proofs of Theorems 1.1, 1.2 and 1.5

Lemma 5.1. *Let* L *be a quaternary* $\mathbb{Z}$*-lattice which is* q*-universal and* q*-anisotropic for some odd prime* q*. If* $L(x) \equiv 0 \pmod{q^2}$*, then* $x = qy$ *for some* $y \in L$.

Proof. First recall a known and easy fact that a quaternary $\mathbb{Z}_q$-lattice is q-universal and q-anisotropic if and only if it is isometric to a diagonal lattice of the type

$$\langle 1, b, q, qb \rangle,$$

where $-b$ is a nonsquare unit in $\mathbb{Z}_q$ (cf. [9]). Without loss of generality we can assume that L_q is of such a type.

If $L_q(v) \equiv 0 \pmod{q^2}$, for $v = (v_1, \ldots, v_4), v_i \in \mathbb{Z}_q$, then

$$v_1^2 + bv_2^2 \equiv 0 \pmod{q}.$$

If v_1 and v_2 were both units in $\mathbb{Z}_q$, then $-b$ would be a square unit in $\mathbb{Z}_q$. Thus v_1 and v_2 are in $q\mathbb{Z}_q$. But then $q(v_3^2 + bv_4^2) \equiv 0 \pmod{q^2}$, which again implies that v_3 and v_4 are in $q\mathbb{Z}_q$. The lemma follows. □

Corollary 5.2. *If* L *is an almost universal* $\mathbb{Z}$*-lattice which is* q*-anisotropic for some odd prime* q*, then* L *is universal.*

Proof. Let n be an arbitrary positive integer. Then nq^{2k} is represented by L for some $k \ge 1$, say, $L(x) = nq^{2k}$. By Lemma 5.1, $x = qy$ for some y in L, and therefore $nq^{2(k-1)} = L(y)$ is represented by L. Arguing by induction, one sees that n is represented by L. □

Proof of Theorem 1.1 (i) $\Longrightarrow$ (ii). Assume that L is an almost universal quaternary $\mathbb{Z}$-lattice. Then clearly L is locally universal. Assume further that L is p-anisotropic for some prime p.

If p is odd, then L is universal by Corollary 5.2. Checking the list of all 204 universal quaternaries (cf. [1]), we see easily that there are only four of them which are p-anisotropic for some prime $p > 2$. These are precisely the four lattices enumerated in Theorem 1.1 (c).

If $p = 2$, then by Corollary 4.2, L represents every positive integer divisible by $2^{\beta(L)}$, in particular, every element of 16Σ (recall that $\beta(L) \le 4$).

(ii) $\Longrightarrow$ (i). Let L be a locally universal quaternary $\mathbb{Z}$-lattice. We shall show that each property (a), (b) or (c) in Theorem 1.1 (ii) implies that L is almost universal.

If L is p-isotropic for every prime p, then L is almost universal by Theorem 4.1.

If L is one of the four universal lattices listed in (c), then there is nothing further to prove.

Assume therefore that L is 2-anisotropic and represents every number in the set 16Σ. If $\beta(L) = 4$ then, by Theorem 1.3 already proved, L is almost universal. If $\beta(L) = 1$ or 2, then the fact that L represents 16Σ implies that L represents 4Σ. So again, if $\beta(L) = 2$ then by applying Theorem 1.3, we deduce that L is almost universal.
Finally, let $\beta(L) = 1$ or 3 and let $s = 2^{\beta(L)-1}$. By Corollary 2.5, the set

$$F = F(L, 4s) = \{x \in L \mid L(x) \equiv 0 \pmod{4s}\}$$

is a sublattice of L satisfying $F_2 \simeq 2s(\Lambda \perp 2\Lambda)_2$ and $F_q \simeq L_q$ for every odd prime q. We just have seen that L, and thus F, represents every number in 4Σ, if $\beta(L) = 1$. By assumption, L represents every number in 16Σ. Therefore the same holds true for F, if $\beta(L) = 3$. The scale of the lattice F is $2s\mathbb{Z}$. It follows that the $\mathbb{Z}$-lattice $G = F^{1/2s}$, obtained from F by scaling $1/2s$, represents every element in 2Σ, and $G_2 \simeq (\Lambda \perp 2\Lambda)_2$. By Theorem 3.1, G represents every positive even integer. Hence F, and therefore L, represents all positive integers divisible by $4s = 2^{\beta(L)+1}$. Also by Theorem 3.1, G_q, and therefore $L_q \simeq F_q$, is q-isotropic for every odd prime q.

In conclusion, L is locally universal, q-isotropic for every odd prime q, and represents all positive integers divisible by $2^{\beta(L)+1}$. Hence, by Theorem 4.1, L is almost universal also when $\beta(L) = 1$ or 3. □

Proof of Theorem 1.2. Let L be an almost universal $\mathbb{Z}$-lattice which is p-anisotropic for some prime p. If p is odd, we have seen in the proof of Theorem 1.1 that L is one of four lattices listed in Theorem 1.1 (c). Each of them is q-isotropic for every prime $q \neq p$. If $p = 2$, then by Corollary 4.2, L is q-isotropic for every $q > 2$. The theorem follows. □

Proof of Theorem 1.5. Let $\Gamma = \{\Gamma_p\}_{p \in P}$ be a genus of an almost universal 2-anisotropic quaternary $\mathbb{Z}$-lattice L. Clearly, $\Gamma_2 = L_2$ is in $\mathcal{A}$. We shall show that $\Gamma_q = G_q$ for some lattice G in family $\mathcal{N}$.

Setting $\epsilon = 0$ if $\beta(L) = 1$ or 3, and $\epsilon = 1$ if $\beta(L) = 2$ or 4, define

$$K = \{x \in L \mid L(x) \equiv 0 \pmod{2^{\beta(L)+\epsilon}}\}.$$

By Theorem 2.4 (if $\epsilon = 0$) or Corollary 2.5 (if $\epsilon = 1$), K is a sublattice of L such that $K_q \simeq L_q$ by every odd prime q, and

$$K_2 \simeq v(\Lambda \perp 2\Lambda)_2$$

where $v = 2^{\beta(L)+\epsilon-1}$. By Corollary 4.2, L represents all positive integers divisible by $2^{\beta(L)}$. Hence K represents all positive integers divisible by $2^{\beta(L)+\epsilon}$. The scale of K is $v\mathbb{Z}$. The $\mathbb{Z}$-lattice $G = K^{1/v}$, obtained from K by scaling $1/v$, represents all even positive integers and $G_2 \simeq (\Lambda \perp 2\Lambda)_2$. In other words, G is isometric to a lattice of family $\mathcal{M}$ (cf. Corollary 3.4). Since the choice of ϵ ensures that v is a square unit in each $\mathbb{Z}_q$, $q > 2$, one has $G_q \simeq K_q$. It follows that $\Gamma_q \simeq L_q \simeq K_q \simeq G_q$ for all primes $q > 2$. If G is not already in $\mathcal{N}$, we can replace G by G' which is in $\mathcal{N}$ and having $G_q \simeq G'_q$ for $q > 2$. This shows that $\{\Gamma_p\}_{p \in P}$ is of the form $\Gamma_2 \simeq H$ and $\Gamma_q \simeq G_q$, $q > 2$, with (H, G) in $\mathcal{A} \times \mathcal{N}$.

Conversely, let (H, G) be any lattices in $\mathcal{A} \times \mathcal{N}$. We shall show that there is an almost universal 2-anisotropic quaternary $\mathbb{Z}$-lattice L such that $L_2 \simeq H$ and $L_q \simeq G_q$ for all odd primes q. The lattice G is in $\mathcal{M}$, thus $G_2 \simeq (\Lambda \perp 2\Lambda)_2$, G represents all even positive integers and G is q-universal for every odd prime q (cf. Corollary 3.4).

Choose a basis $e_1, \dots, e_4$ of $\Lambda \perp 2\Lambda$ and a basis $e'_1, \dots, e'_4$ of G such that

$$G = \mathbb{Z}e'_1 + \cdots + \mathbb{Z}e'_4 \equiv \Lambda \perp 2\Lambda \pmod{2^{17}}.$$

We easily check that

$$\begin{aligned}\{x \in \Lambda \perp 2\Lambda \mid (\Lambda \perp 2\Lambda)(x) \equiv 0 \pmod{16}\} \\ &= \mathbb{Z}(4e_1) + \mathbb{Z}(4e_2) + \mathbb{Z}(2e_3) + \mathbb{Z}(2e_4) \\ &= 8(2\Lambda \perp \Lambda).\end{aligned}$$

(Notice that the natural notation of $\Lambda \perp 2\Lambda$ is reversed in the last equality). Arguing as in the proof of Theorem 2.4, it follows that for

$$F = \{x \in G \mid G(x) \equiv 0 \pmod{16}\}$$

one has

$$F = \mathbb{Z}(4e'_1) + \mathbb{Z}(4e'_2) + \mathbb{Z}(2e'_3) + \mathbb{Z}(2e'_4) \equiv 8(2\Lambda \perp \Lambda) \pmod{2^{17}}.$$

The last congruence implies that

$$F_2 \simeq 8(\Lambda \perp 2\Lambda)_2.$$

Clearly, $F_q \simeq G_q$ for all odd primes q.

Let L' be a $\mathbb{Z}$-lattice such that $L'_2 \simeq H$ and $L'_q \simeq G_q \simeq F_q$ for $q > 2$. The existence of such a lattice follows from Theorem 81.14 in [8]. Moreover, since by Theorem 2.4, L'_2 contains a $\mathbb{Z}_2$-sublattice isometric to $8(\Lambda \perp 2\Lambda)_2 \simeq F_2$, we may assume that $F_2 \subset L'_2$. This, together with $F_q \simeq L'_q$ for all $q > 2$, implies that there is a $\mathbb{Z}$-lattice L in the genus of L' such that $F \subset L$ (cf. Example 102.5 in [8]). It remains to prove that L is almost universal. Since G represents all even positive integers, the lattice F represents all positive integers divisible by 16. A fortiori, L containing F also represents all such integers. By construction, L is locally universal and 2-anisotropic. Hence L is almost universal by Theorem 1.1. □

Proof of Corollary 1.6. The first part of the corollary follows directly from Theorem 1.5. Let L be an almost universal and 2-anisotropic $\mathbb{Z}$-lattice. By Theorem 1.5, $L_2 \in \mathcal{A}$ and $L_q \simeq G_q$ for some G in $\mathcal{N}$ and every odd prime q. The discriminant $d(L_2) = 4^m u$, for some $m = 0, 1, 2$, or 3 and some unit u in $\mathbb{Z}_2$ (cf. Proposition 2.1). The discriminant $d(G) = 4v$, for some odd integer v (cf. Table 4). It follows that

$$d(L) = 4^m(d(G)/4).$$

Moreover, also by Theorem 1.5, all combinations of $m = 0, 1, 2, 3$ and $G \in \mathcal{N}$ do occur. The corollary follows. □

To end this section we shall show that for each n in the set 16Σ, there exists a quaternary $\mathbb{Z}$-lattice L_n which is locally universal, 2-anisotropic, and represents every element in 16Σ except n. By Corollary 4.2, L_n is not almost universal. This will show that the set 16Σ is optimal for Theorem 1.1; the theorem fails if 16Σ is replaced by a smaller subset.

Below we shall represent a symmetric 4×4 matrix $(a_{ij})_{i,j}$ by a sequence of 10 numbers $[a_{11}, a_{22}, a_{33}, a_{44}, a_{12}, a_{13}, a_{23}, a_{14}, a_{24}, a_{34}]$.

Consider eight $\mathbb{Z}$-lattices $\{K_l \mid l \in 2\Sigma\}$, given by their Gram matrices:

$$\begin{aligned} K_2 &= [4,6,6,12,0,0,1,2,0,0], & K_4 &= [2,6,12,12,1,0,0,0,0,2], \\ K_6 &= [2,2,10,14,0,0,0,1,1,3], & K_{10} &= [2,4,4,14,0,0,2,1,0,0], \\ K_{12} &= [2,2,6,18,0,1,1,0,0,3], & K_{14} &= [2,4,10,18,0,1,2,1,2,3], \\ K_{20} &= [2,4,10,18,0,1,2,1,0,0], & K_{28} &= [2,2,2,34,1,0,1,0,1,0]. \end{aligned}$$

Each of these lattices is admissible, q-universal for every odd prime q, and K_s represents all elements of 2Σ except s.

Now, given n in 16Σ consider $K = K_s$ for $s = n/8$. Let $M = K^8$ be a $\mathbb{Z}$-lattice obtained from K by scaling 8. Since K is admissible, one has $M_2 \simeq 8(\Lambda \perp 2\Lambda)_2$. Arguing as in the proof of Theorem 1.5, we can construct a $\mathbb{Z}$-lattice M' such that M'_2 is 2-universal, 2-anisotropic, and $M \subset M'$. Moreover, any number divisible by 16 and represented by M' is also represented by M.

Since $s = n/8$ is not represented by K, it follows that M, and hence M', does not represent n. On the other hand, M represents all elements of 16Σ, except n. Also, by construction, M' is locally universal and 2-anisotropic. We can take therefore $L_n = M'$.

6. Enumeration of all lattices of the families $\mathcal{M}$ and $\mathcal{N}$

We shall first enumerate all 79 lattices of the family $\mathcal{M} = \{G_1, \ldots, G_{79}\}$. Each G_i is a 4-dimensional escalator defined by its Gram matrix of the form

$$G = \left[\begin{array}{ccc|c} & T_k & & \begin{matrix} a \\ b \\ c \end{matrix} \\ \hline a & b & c & e \end{array}\right]$$

where T_k is one of the 14 ternary escalators listed in Table 1 in Section 3. To identify a G_i it suffices to indicate its ternary component $T(G_i) = T_k$ and the elements a, b, c, e of the last column. Each discriminant $d(G_i)$ is of the form $4v_i$, for some odd integer v_i. We shall list all $d(G_i)/4$.

G_n	$T_k = T(G_n)$	a,b,c,e	$d(G_n)/4$	G_n	$T_k = T(G_n)$	a,b,c,e	$d(G_n)/4$
G_1	T_1	$1,1,1,2$	1	G_2	T_1	$1,1,1,6$	3^2
G_3	T_1	$1,1,1,10$	17	G_4	T_1	$1,1,1,14$	5^2
G_5	T_2	$0,0,1,10$	3^2	G_6	T_2	$0,0,1,18$	17
G_7	T_2	$0,0,1,26$	5^2	G_8	T_4	$1,1,1,6$	5^2
G_9	T_4	$0,0,3,10$	41	G_{10}	T_4	$0,0,1,10$	7^2
G_{11}	T_5	$0,0,2,4$	3^2	G_{12}	T_5	$0,1,2,10$	5^2
G_{13}	T_5	$0,0,2,12$	$3 \cdot 11$	G_{14}	T_5	$0,1,2,18$	7^2
G_{15}	T_5	$0,0,2,20$	$3 \cdot 19$	G_{16}	T_7	$1,0,2,6$	5^2
G_{17}	T_7	$1,0,0,6$	$3 \cdot 11$	G_{18}	T_7	$1,0,2,10$	7^2
G_{19}	T_7	$1,0,0,10$	$3 \cdot 19$	G_{20}	T_9	$0,2,3,10$	3^4

Table 4.

G_n	$T_k = T(G_n)$	a, b, c, e	$d(G_n)/4$	G_n	$T_k = T(G_n)$	a, b, c, e	$d(G_n)/4$
G_{21}	T_9	$0, 2, 3, 18$	13^2	G_{22}	T_9	$0, 2, 2, 12$	113
G_{23}	T_9	$0, 2, 2, 20$	$3 \cdot 67$	G_{24}	T_9	$0, 2, 1, 10$	97
G_{25}	T_9	$0, 2, 1, 18$	$5 \cdot 37$	G_{26}	T_9	$0, 2, 0, 12$	11^2
G_{27}	T_9	$0, 2, 0, 20$	$11 \cdot 19$	G_{28}	T_9	$1, 2, 2, 10$	89
G_{29}	T_9	$1, 2, 2, 18$	$3 \cdot 59$	G_{30}	T_9	$1, 2, 0, 6$	7^2
G_{31}	T_9	$1, 2, 0, 14$	137	G_{32}	T_{10}	$0, 2, 3, 10$	73
G_{33}	T_{10}	$0, 2, 3, 18$	$5 \cdot 29$	G_{34}	T_{10}	$0, 2, 3, 26$	$7 \cdot 31$
G_{35}	T_{10}	$0, 2, 0, 12$	97	G_{36}	T_{10}	$0, 2, 0, 20$	13^2
G_{37}	T_{10}	$0, 2, 0, 28$	241	G_{38}	T_{10}	$0, 2, 1, 10$	3^4
G_{39}	T_{10}	$0, 2, 1, 18$	$3^2 17$	G_{40}	T_{10}	$0, 2, 1, 26$	$3^2 5^2$
G_{41}	T_{10}	$1, 0, 2, 10$	3^4	G_{42}	T_{10}	$1, 0, 2, 18$	$3^2 17$
G_{43}	T_{10}	$1, 0, 2, 26$	$3^2 5^2$	G_{44}	T_{10}	$1, 0, 0, 14$	11^2
G_{45}	T_{10}	$1, 0, 0, 22$	193	G_{46}	T_{12}	$1, 0, 5, 10$	11^2
G_{47}	T_{12}	$1, 0, 5, 14$	193	G_{48}	T_{12}	$1, 2, 3, 10$	$5 \cdot 29$
G_{49}	T_{12}	$1, 2, 3, 14$	$7 \cdot 31$	G_{50}	T_{12}	$1, 2, 1, 10$	$3^2 17$
G_{51}	T_{12}	$1, 2, 1, 14$	$3^2 5^2$	G_{52}	T_{12}	$1, 0, 3, 10$	$3^2 17$
G_{53}	T_{12}	$1, 0, 3, 14$	$3^2 5^2$	G_{54}	T_{12}	$1, 0, 1, 10$	13^2
G_{55}	T_{12}	$1, 0, 1, 14$	241	G_{56}	T_{13}	$0, 2, 5, 18$	$3 \cdot 7 \cdot 13$
G_{57}	T_{13}	$0, 2, 4, 12$	$3 \cdot 59$	G_{58}	T_{13}	$0, 2, 4, 20$	$7 \cdot 47$
G_{59}	T_{13}	$0, 2, 3, 18$	$5 \cdot 61$	G_{60}	T_{13}	$0, 2, 2, 12$	$3 \cdot 67$
G_{61}	T_{13}	$0, 2, 2, 20$	353	G_{62}	T_{13}	$0, 2, 1, 18$	$3 \cdot 107$
G_{63}	T_{13}	$0, 2, 0, 12$	$11 \cdot 19$	G_{64}	T_{13}	$0, 2, 0, 20$	19^2
G_{65}	T_{13}	$1, 2, 4, 10$	137	G_{66}	T_{13}	$1, 2, 4, 18$	17^2
G_{67}	T_{13}	$1, 2, 3, 14$	$3^2 5^2$	G_{68}	T_{13}	$1, 2, 2, 14$	233
G_{69}	T_{13}	$1, 2, 0, 10$	$7 \cdot 23$	G_{70}	T_{14}	$0, 2, 5, 18$	257
G_{71}	T_{14}	$0, 2, 4, 12$	13^2	G_{72}	T_{14}	$0, 2, 4, 20$	$5 \cdot 61$
G_{73}	T_{14}	$0, 2, 3, 18$	281	G_{74}	T_{14}	$0, 2, 0, 12$	$5 \cdot 37$
G_{75}	T_{14}	$0, 2, 0, 20$	$3 \cdot 107$	G_{76}	T_{14}	$0, 2, 1, 18$	17^2
G_{77}	T_{14}	$1, 0, 4, 18$	$3 \cdot 7 \cdot 13$	G_{78}	T_{14}	$1, 0, 3, 14$	$7 \cdot 31$
G_{79}	T_{14}	$1, 0, 2, 14$	$3^2 5^2$				

Table 4 (continued)

Two 4-dimensional escalators E_1 and E_2 which do not represent all elements of the set 2Σ are listed below (none of them represents 20).

E_i	$T(E_i)$	a,b,c,e	$d(E_i)/4$
E_1	T_{13}	$1,2,1,18$	313
E_2	T_{14}	$1,0,0,18$	$3^3 11$

Genera of 2*-anisotropic almost universal quaternary* $\mathbb{Z}$*-lattices.* We shall now enumerate all lattices in $\mathcal{M}$ which belong to the same genus: the notation $\{n_1,\dots,n_k\}$ means that $\{G_{n_1},\dots,G_{n_k}\}$ are in the same genus. We are listing only the cases where $k>1$.

$$\{3,6\},\{4,12\},\{7,16\},\{10,14,18\},\{21,54\},\{24,35\},\{31,65\},$$
$$\{34,49\},\{36,71\},\{37,55\},\{39,52\},\{42,50\},\{45,47\}.$$

We can now enumerate all 65 lattices of family $\mathcal{N}$

$$\mathcal{N}=\{G_i\in\mathcal{M}\mid 1\le i\le 79, i\notin W\},$$

where $W=\{6,12,14,16,18,35,47,49,50,52,54,55,65,71\}$.

7. Integer-valued almost universal quaternaries

In this section we shall indicate very briefly how the results described in the Introduction can be extended to the case of almost universal $\mathbb{Z}$*-valued* quaternary quadratic forms.

Theorem 1.1 is valid for the $\mathbb{Z}$-valued quaternaries if we replace the property (c) by

(c') *f is universal and p-anisotropic for some $p>2$.*

The list of all $\mathbb{Z}$-valued p-anisotropic universal quaternaries, which contains exactly 48 forms, can be obtained from the finite list of all $\mathbb{Z}$-valued universal quaternaries given in [5]. This list contains p-anisotropic universal quaternaries for every prime $p\le 37$ (contrasting with the case of $\mathbb{Z}$-matrix quaternaries, where only $p=2,3,5,7$ can occur).

The statement of Theorem 1.3 is valid without changes for the $\mathbb{Z}$-valued case. The invariant $\beta(f)$ is defined in the same way, but can sometimes be null (cf. Proposition 7.1 below).

Theorem 1.2 for the $\mathbb{Z}$-valued quaternaries also holds true and follows from a more general result concerning the almost regular quaternaries (we shall publish this result elsewhere).

The proofs of Theorems 1.1, 1.3 go along the same line as the proofs given in Sections 4 and 5 for the $\mathbb{Z}$-matrix quaternaries. Proposition 2.1 should be extended as follows.

Proposition 7.1. *A $\mathbb{Z}_2$-valued quaternary quadratic form H is 2-universal and 2-anisotropic if and only if H is equivalent over $\mathbb{Z}_2$ to one of the following $\mathbb{Z}_2$-valued forms $H_1,\dots,H_{12}$, where $H_1,\dots,H_8$ are the forms listed in Proposition 2.1 and*

$$H_9=\frac{1}{2}\Lambda\perp\Lambda,\ H_{10}=\frac{1}{2}\Lambda\perp\langle 6,8\rangle,\ H_{11}=\frac{1}{2}\Lambda\perp\langle 2,24\rangle,\ H_{12}=\frac{1}{2}\Lambda\perp\langle 2,6\rangle.$$

Moreover, $\beta(H_9)=0,\ \beta(H_{10})=2,\ \beta(H_{11})=\beta(H_{12})=4.$

We leave the proof of Proposition 7.1 to the reader. Proposition 2.2 should be extended to the cases $H_9, \ldots, H_{12}$.

Denoting by $\mathcal{A}'$ the family of 12 $\mathbb{Z}_2$-valued quaternaries given in Proposition 7.1, we obtain the $\mathbb{Z}$-valued version of Theorem 1.5 by replacing $\mathcal{A}$ with $\mathcal{A}'$. It follows that there are precisely 780 genera containing some 2-anisotropic almost universal $\mathbb{Z}$-valued quaternary quadratic form.

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