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S_4-symmetry of $6j$-symbols and Frobenius–Schur indicators in rigid monoidal C^*-categories

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Abstract

We show that a left-rigid monoidal C^*-category with irreducible monoidal unit is also a sovereign and spherical category. Defining a Frobenius–Schur type indicator we obtain selection rules for the fusion coefficients of irreducible objects. As a main result we prove S_4-invariance of $6j$-symbols in such a category.

1 Introduction

The quantities that are known as $6j$-*symbols* or F-*coefficients* appear in various disguises in mathematics and physics, for instance as: recoupling coefficients in the theory of groups [1–3] and quantum groups [4], as (partially gauge-fixed) fusing matrices [5–7] in conformal field theory, as the Boltzmann weights for triangulations of three-manifolds [8–11] giving rise to topological lattice field theories [11–13], as expansion coefficients of exchange algebra relations in the algebraic theory of superselection sectors [14], as the components of the 3-cocycle in Ocneanu's non-Abelian cohomology [15], as structure constants in the theory of 3-algebras [16], as specific endomorphisms of a von Neumann factor M in the theory of subfactors [17], and as components of the coassociator φ of a rational Hopf algebra [18–20]. More generally, the F-coefficients can be described as the projections of the associativity constraint φ of a semisimple, rigid, monoidal category onto irreducible objects hence they map a pair of basic intertwiner spaces to another pair. The F-coefficients depend on the choice of basis (gauge choices) in the basic intertwiner spaces (spaces of 3-point functions).

An important property of the F-coefficients is their 'tetrahedral symmetry', or more precisely, the fact that the gauge freedom that is present in their definition can be fixed in such a way that they are invariant under a set of transformations which form the permutation group S_4. This symmetry property is for example needed in three-dimensional lattice theories in order for the partition function that is defined in terms of the F-coefficients to be independent of the triangulation so that the theory is topological (see e.g. [21, 9]). Other places where this symmetry plays an important rôle are the derivation of identities for 'higher order fusion coefficients' [22] and the construction of solutions to the 'big pentagon equation' [23]. Also, in pursuing a project for finding (possibly with the aid of computers) explicit solutions of the Moore–Seiberg polynomial equations for 'small' fusion rings the assumption of tetrahedral symmetry allows for the substantial increase in the number of accessible fusion rings. (In conformal field theory the explicit computation of fusing matrices is e.g. required for the calculation of the operator product coefficients.)

Let us remark that in conformal field theory it seems to be common lore that the F-coefficients possess S_4-invariance (see e.g. [7, 24]); but as a matter of fact no complete and detailed proof has ever been published. In the theory of subfactors it was realized that the S_4-symmetry follows from Frobenius reciprocity of intertwiners between bimodules [17], and in algebraic field theory the same aspects of the symmetry are implicit in the results of [14] (see in particular the appendix of that paper). The Frobenius maps between basic intertwiners generate the group S_3, and the coherent choice of bases in the orbits of this S_3 group lead to S_4-symmetric F-coefficients. In [9] it was shown that in order to have S_4-symmetry one does not need a braiding – it is sufficient that the category be spherical; however, in that paper the possibility of non-trivial Frobenius–Schur (FS) indicators [25–27] was not considered.

In this paper we present a proof that in a left-rigid monoidal C^*-category the F-coefficients possess S_4-symmetry. We would like to emphasize two fine points. The first is that, in general, due to the possibility of having non-trivial FS indicators, the Frobenius maps provide only a $\mathbb{Z}_2$-projective representation of S_3; but nevertheless the corresponding signs cancel in the transformation of the F-coefficients so that the F's are truly S_4-invariant. The second point is the following. The Frobenius transformations of order 3 map the basic intertwiners space $(p \times p, \hat{p})$ into itself; hence its eigenvalues are third roots of unity (here $\hat{p}$ is the conjugate of the irreducible object p). When considering F's that involve such intertwiner spaces, then in order to verify S_4-invariance we must calculate the different transforms of F in different bases of the space $(p \times p, \hat{p})$. If, instead, we use only a single basis in this space, then the transforms of F will possibly carry factors of third roots of unity (this is illustrated on an example in the appendix).

Let us also briefly mention a possible application of our results to the quest of explicitly solving the polynomial equations for the braiding and fusing matrices, which constitutes e.g. a part of the problem of classifying all rational conformal field theories. Namely, one expects that when the category is modular, then the trace of the braiding matrices $R^{pp,q}$ can be expressed completely in terms of the modular data, i.e. of the fusion rules, the modular S matrix and balancing phases (in the case of doubles of finite groups this was proved in [27]). Assuming that this expectation is indeed correct, it follows that the Frobenius–Schur indicators as well as the multiplicities of the above-mentioned third roots of unity do not constitute independent data, but are already determined uniquely by the modular data. In particular, one can immediately write down all the F-coefficients for which at least one line is colored by the unit object as well as the R-coefficients. In fact, there are further special F-coefficients that can easily be solved for. Finally, our results concerning the S_4-symmetry allow to reduce the number of unknowns in the polynomial equations drastically, namely for generic F-coefficients by a factor of 24. [1]

2 Rigid monoidal C^*-categories with irreducible monoidal unit

Our starting point is a left-rigid monoidal C^*-category $(\mathcal{C}; (\varepsilon, \times, \{\lambda, \rho, \varphi\}); (\widehat{\ }, \{e, c\}); {}^*)$ with the restriction that the monoidal unit ε is irreducible. Here $\times$ is the monoidal product and λ_a, ρ_a, $\varphi_{a,b,c}$ with $a, b, c \in \mathrm{Obj}\,\mathcal{C}$ are natural isomorphisms

$$\begin{aligned} &\lambda_a\colon\ a \to a \times \varepsilon\,, \qquad \rho_a\colon\ a \to \varepsilon \times a\,, \\ &\varphi_{a,b,c}\colon\ a \times (b \times c) \to (a \times b) \times c \end{aligned} \tag{2.1}$$

that satisfy the triangle identity

$$\varphi_{a,\varepsilon,c} = (\lambda_a \times 1_c)\,(1_a \times \rho_c^{-1}), \tag{2.2}$$

and the pentagon identity

$$\varphi_{a\times b,c,d}\,\varphi_{a,b,c\times d} = (\varphi_{a,b,c} \times 1_d)\,\varphi_{a,b\times c,d}\,(1_a \times \varphi_{b,c,d})\,. \tag{2.3}$$

[1] Further simplifications come from the action of simple currents on the F-coefficients, which should be derivable in a way similar to the S_4-action. Moreover, one would expect the pentagon equation itself, which can be regarded as the boundary of a 4-simplex, to possess S_5-symmetry, just like the tetrahedra (F-coefficients) – the boundaries of 3-simplices – possess S_4-symmetry.

The C^*-property requires the arrows (the intertwiners)

$$(a,b) \equiv \mathrm{Hom}\,(a,b) := \{T\colon\ a \to b\} \tag{2.4}$$

between two objects a, b to form a complex Banach space. The * is an involutive monoidal contravariant functor acting as identity on the objects and antilinearly on the intertwiner spaces. The norm of the intertwiners satisfies the C^*-property, $\|T^*T\| = \|T\|^2$, and the natural equivalences $\{\lambda, \rho, \varphi\}$ are isometries. Irreducibility of ε implies that $(\varepsilon, \varepsilon) = \mathbb{C} \cdot 1_\varepsilon$.

The left conjugation $L\colon\ \mathcal{C} \to \mathcal{C}$ is an antimonoidal contravariant (linear) functor, which is the identity on the monoidal unit. The functor L is built up from a left rigidity structure $(\widehat{\ }, \{e, c\})$, where $\widehat{\ }$ is the left conjugation on objects: $\hat{a} \equiv L(a)$. The left evaluation and coevaluation maps $e_a\colon\ \hat{a} \times a \to \varepsilon$ and $c_a\colon\ \varepsilon \to a \times \hat{a}$, $a \in \mathrm{Obj}\,\mathcal{C}$ satisfy the left rigidity equations

$$\lambda_a^*\,(1_a \times e_a)\,\varphi_{a,\hat{a},a}^*\,(c_a \times 1_a)\,\rho_a = 1_a\,, \qquad \rho_{\hat{a}}^*\,(e_a \times 1_{\hat{a}})\,\varphi_{\hat{a},a,\hat{a}}\,(1_{\hat{a}} \times c_a)\,\lambda_{\hat{a}} = 1_{\hat{a}} \tag{2.5}$$

and lead to the definition

$$(a,b) \ni T \mapsto L(T) := \rho_{\hat{a}}^*\,(e_b \times 1_{\hat{a}})\,((1_{\hat{b}} \times T) \times 1_{\hat{a}})\,\varphi_{\hat{b},a,\hat{a}}\,(1_{\hat{b}} \times c_a)\,\lambda_{\hat{b}} \in (\hat{b}, \hat{a}) \tag{2.6}$$

of the conjugated intertwiners $L(T)$. Note that two left conjugations L_1 and L_2 that arise from left rigidity structures are always related by canonical natural equivalences $\mu\colon\ L_1 \to L_2$ given by

$$\begin{aligned} \mu_a &= \rho_{a^2}^*\,(e_a^1 \times 1_{a^2})\,\varphi_{a^1,a,a^2}\,(1_{a^1} \times c_a^2)\,\lambda_{a^1} \in (a^1, a^2)\,, \\ \mu_a^{-1} &= \rho_{a^1}^*\,(e_a^2 \times 1_{a^1})\,\varphi_{a^2,a,a^1}\,(1_{a^2} \times c_a^1)\,\lambda_{a^2} \in (a^2, a^1)\,, \end{aligned} \qquad \text{for every } a \in \mathrm{Obj}\,\mathcal{C}\,, \tag{2.7}$$

where $a^i \equiv L_i(a)$ for $i = 1, 2$.

The C^*-property allows us to define a right rigidity structure within $\mathcal{C}$:

$$(R(a), e_a^R, c_a^R) := (L(a), c_a^{L*}, e_a^{L*}) \equiv (\hat{a}, c_a^*, e_a^*) \quad \text{for} \ \ a \in \mathrm{Obj}\,\mathcal{C}\,, \tag{2.8}$$

leading to a right conjugate functor $R\colon\ \mathcal{C} \to \mathcal{C}$ similarly to (2.6):

$$\begin{aligned} (a,b) \ni T \mapsto R(T) &:= \lambda_{R(a)}^*\,(1_{R(a)} \times e_b^R)\,(1_{R(a)} \times (T \times 1_{R(b)}))\,\varphi_{R(a),a,R(b)}^*\,(c_a^R \times 1_{R(b)})\,\rho_{R(b)} \\ &\equiv \lambda_{\hat{a}}^*\,(1_{\hat{a}} \times c_b^*)\,(1_{\hat{a}} \times (T \times 1_{\hat{b}}))\,\varphi_{\hat{a},a,\hat{b}}^*\,(e_a^* \times 1_{\hat{b}})\rho_{\hat{b}} \in (\hat{b}, \hat{a})\,. \end{aligned} \tag{2.9}$$

Since the right and left conjugated objects are identical, $R(a) = L(a) \equiv \hat{a}$ for all $a \in \mathrm{Obj}\,\mathcal{C}$, the canonical natural isomorphisms

$$\{\kappa_a\colon\ R(L(a)) \to a\}\,, \qquad \{\tilde{\kappa}_a\colon\ L(R(a)) \to a\}\,, \tag{2.10}$$

are given by

$$\begin{aligned} \kappa_a &= \lambda_a^*\,(1_a \times e_{L(a)}^R)\,\varphi_{a,L(a),R(L(a))}^*\,(c_a^L \times 1_{R(L(a))})\,\rho_{R(L(a))} \equiv \lambda_a^*\,(1_a \times c_{\hat{a}}^*)\,\varphi_{a,\hat{a},\hat{\hat{a}}}^*\,(c_a \times 1_{\hat{\hat{a}}})\,\rho_{\hat{\hat{a}}}\,, \\ \tilde{\kappa}_a &= \rho_a^*\,(e_{R(a)}^L \times 1_a)\,\varphi_{L(R(a)),R(a),a}\,(1_{L(R(a))} \times c_a^R)\,\lambda_{L(R(a))} \equiv \rho_a^*\,(e_{\hat{a}} \times 1_a)\,\varphi_{\hat{\hat{a}},\hat{a},a}\,(1_{\hat{\hat{a}}} \times e_a^*)\,\lambda_{\hat{\hat{a}}}\,, \end{aligned} \tag{2.11}$$

which satisfy

$$\begin{aligned} \kappa_a^{-1} &= \lambda_{\hat{a}}^* \,(1_{\hat{a}} \times e_a)\, \varphi_{\hat{\hat{a}},\hat{a},a}^* \,(e_{\hat{a}}^* \times 1_a)\, \rho_a = \tilde{\kappa}_a^* \,, \\ \tilde{\kappa}_a^{-1} &= \rho_{\hat{a}}^* \,(c_a^* \times 1_{\hat{a}})\, \varphi_{a,\hat{a},\hat{\hat{a}}} \,(1_a \times c_{\hat{a}})\, \lambda_a = \kappa_a^* \,. \end{aligned} \tag{2.12}$$

One can define (positive) left and right inverses for $a \in \mathrm{Obj}\,\mathcal{C}$ [28]:

$$\Phi_a^L\colon\ (a \times b, a \times c) \to (b, c)\,, \qquad \Phi_a^R\colon\ (b \times a, c \times a) \to (b, c)\,, \quad \text{for } b, c \in \mathrm{Obj}\,\mathcal{C} \tag{2.13}$$

using evaluation and coevaluation maps. In the special case $b = c = \varepsilon$ these maps

$$\begin{aligned} (a,a) \ni T \mapsto \Phi_a^L(T) &:= e_a\,(1_{\hat{a}} \times T)\, e_a^* \in (\varepsilon, \varepsilon) = \mathbb{C} \cdot 1_\varepsilon \\ (a,a) \ni T \mapsto \Phi_a^R(T) &:= c_a^*\,(T \times 1_{\hat{a}})\, c_a \in (\varepsilon, \varepsilon) = \mathbb{C} \cdot 1_\varepsilon \end{aligned} \tag{2.14}$$

are faithful positive linear functionals. Since they are bounded from below [28],

$$\begin{aligned} (a,a) \ni T^*T &\le \rho_a^*\,(c_a^* c_a \times 1_a)\rho_a \cdot \lambda_a^*\,(1_a \times \Phi_a^L(T^*T))\,\lambda_a\,, \\ (a,a) \ni T^*T &\le \lambda_a^*\,(1_a \times e_a e_a^*)\lambda_a \cdot \rho_a^*\,(\Phi_a^R(T^*T) \times 1_a)\,\rho_a\,, \end{aligned} \qquad \text{for } T \in (a,b)\,, \tag{2.15}$$

it follows that $\mathrm{End}\,a \equiv (a,a)$ is finite-dimensional for every $a \in \mathrm{Obj}\,\mathcal{C}$; therefore $\mathcal{C}$ is semisimple. We denote the subset of irreducible objects by $\mathcal{I} \subset \mathrm{Obj}\,\mathcal{C}$.

Semisimplicity allows us to construct the so-called *standard* rigidity intertwiners [28], leading to a left conjugation functor L_s among the equivalent ones (cf. (2.7)) that obeys $L_s = R_s$ by (2.8) and (2.9). This conjugation is achieved by the following procedure.

In the case of an irreducible object $p \in \mathcal{I}$ one uses the scalar freedom $e_p \mapsto z_p e_p,\ c_p \mapsto z_p^{-1} c_p$ with $z_p \in \mathbb{C} \setminus \{0\}$ to set

$$e_p e_p^* = d_p\, 1_\varepsilon = c_p^* c_p \qquad \text{for every } p \in \mathcal{I}\,, \tag{2.16}$$

where d_a is the *quantum* or *statistics dimension* of an arbitrary object $a \in \mathcal{C}$, defined to be

$$d_a := \|e_a\|\, \|c_a\|\,. \tag{2.17}$$

Then for an arbitrary object a one chooses two orthogonal and complete sets of partial isometries

$$\begin{aligned} V_a^{p\alpha}\colon &\quad p \to a\,, \\ W_a^{p\alpha}\colon &\quad \hat{p} \to \hat{a}\,, \end{aligned} \qquad \text{for } p \in \mathcal{I},\ \alpha = 1, \ldots, m_a^p\,, \tag{2.18}$$

satisfying

$$V_a^{p\alpha *} V_a^{p'\alpha'} = \delta_{pp'} \delta_{\alpha\alpha'} 1_p\,, \qquad W_a^{p\alpha *} W_a^{p'\alpha'} = \delta_{pp'} \delta_{\alpha\alpha'} 1_{\hat{p}}\,, \tag{2.19}$$

and

$$\sum_{p,\alpha} V_a^{p\alpha} V_a^{p\alpha *} = 1_a\,, \qquad \sum_{p,\alpha} W_a^{p\alpha} W_a^{p\alpha *} = 1_{\hat{a}}\,, \tag{2.20}$$

to define

$$e_a := \sum_{p,\alpha} e_p (W_a^{p\alpha *} \times V_a^{p\alpha *}) \in (\hat{a} \times a, \varepsilon)\,, \qquad c_a := \sum_{p,\alpha} (V_a^{p\alpha} \times W_a^{p\alpha}) c_p \in (\varepsilon, a \times \hat{a})\,. \tag{2.21}$$

Then $(\widehat{\ }, \{e_a, c_a\})$ becomes a left rigidity structure, i.e. it satisfies (2.5) due to naturality of $\{\lambda, \rho, \varphi\}$ and (2.19), (2.20). The rigidity intertwiners (e_a, c_a) of an object a satisfying (2.16) and (2.21) are called *standard*.

Now the corresponding right rigidity structure (cf. (2.8) and (2.9)) $(\widehat{\ }, \{c_a^*, e_a^*\})$ leads to a right conjugation functor R that is identical to the left conjugation L. As a matter of fact, by an explicit calculation one obtains

$$(b,a) \ni T \mapsto R(T) = L(T) = \sum_{p,\alpha,\beta} W_b^{p\beta} t_p^{\alpha\beta} 1_{\hat{p}} W_a^{p\alpha *} \in (\hat{a}, \hat{b}), \tag{2.22}$$

with

$$t_p^{\alpha\beta} 1_p := V_a^{p\alpha *} T V_b^{p\beta}, \quad t_p^{\alpha\beta} \in \mathbb{C}\,. \tag{2.23}$$

Therefore in case of standard conjugation we use the notation $\hat{T} \equiv R(T) = L(T)$ for the conjugated intertwiners.

One can also prove [28] that if (e_a, c_a) and (e'_a, c'_a) are both standard, then

$$e'_a = e_a(1_{\hat{a}} \times U_a), \qquad c'_a = (U_a^* \times 1_{\hat{a}})c_a, \tag{2.24}$$

where $U_a \in (a,a)$ is unitary. Moreover,

$$\begin{aligned}
e_{(a,b)} &:= e_b\,(1_{\hat{b}} \times \rho_b)\,(1_{\hat{b}} \times (e_a \times 1_b))\,(1_{\hat{b}} \times \varphi_{\hat{a},a,b})\,\varphi^*_{\hat{b},\hat{a},a\times b} &&\in ((\hat{b} \times \hat{a}) \times (a \times b), \varepsilon) \\
c_{(a,b)} &:= \varphi^*_{a\times b,\hat{b},\hat{a}}\,(\varphi_{a,b,\hat{b}} \times 1_{\hat{a}})\,((1_a \times c_b) \times 1_{\hat{a}})\,(\lambda_a \times 1_{\hat{a}})\,c_a &&\in (\varepsilon, (a \times b) \times (\hat{b} \times \hat{a}))
\end{aligned} \tag{2.25}$$

are standard if (e_a, c_a) and (e_b, c_b) are standard. Thus in case of standard conjugation, $L = R$, the natural equivalence $\{\alpha_{a,b}\colon \widehat{a\times b} \to \hat{b} \times \hat{a}\}$ that expresses the antimonoidality of the conjugation functor is given by

$$\begin{aligned}
\alpha_{a,b} &= \rho^*_{\hat{b}\times\hat{a}}(e_{a\times b} \times 1_{\hat{b}\times\hat{a}})\,\varphi_{\widehat{a\times b},a\times b,\hat{b}\times\hat{a}}(1_{\widehat{a\times b}} \times c_{(a,b)})\,\lambda_{\widehat{a\times b}} &&\in (\widehat{a\times b}, \hat{b} \times \hat{a})\,, \\
\alpha^{-1}_{a,b} &= \rho^*_{\widehat{a\times b}}(e_{(a,b)} \times 1_{\widehat{a\times b}})\,\varphi_{\hat{b}\times\hat{a},a\times b,\widehat{a\times b}}\,(1_{\hat{b}\times\hat{a}} \times c_{a\times b})\,\lambda_{\hat{b}\times\hat{a}} &&\in (\hat{b} \times \hat{a}, \widehat{a\times b})\,.
\end{aligned} \tag{2.26}$$

3 Sovereignty, traces and sphericity

We note that in case of standard rigidity intertwiners the choice $\psi_a := 1_{\hat{a}}$ for every $a \in \mathrm{Obj}\,\mathcal{C}$ leads to a natural equivalence $\psi\colon R \to L$ and makes $\mathcal{C}$ a *sovereign* category [29]. Indeed, $\psi_\varepsilon = 1_\varepsilon$ and monoidality of ψ is clear, one has only to show the commutativity of the sovereignty diagram, which reads in this case as $\kappa_a = \tilde{\kappa}_a$. Since κ and $\tilde{\kappa}$ are natural equivalences it is enough to show this equality for $p \in \mathcal{I}$. But then one has

$$\kappa_p^* \kappa_p = x_p 1_{\hat{\hat{p}}} \quad \text{and} \quad \tilde{\kappa}_p^* \tilde{\kappa}_p = \tilde{x}_p 1_{\hat{\hat{p}}} \quad \text{for all } \ p \in \mathcal{I}\,, \tag{3.1}$$

with $x_p, \tilde{x}_p \in \mathbb{R}_+$ and $x_p = \tilde{x}_p^{-1}$ due to (2.12). Using (2.16) and rigidity one obtains

$$\begin{aligned}
x_p d_{\hat{p}}\, 1_\varepsilon &= e_{\hat{p}}\,(x_p 1_{\hat{\hat{p}}} \times 1_{\hat{p}})\,e^*_{\hat{p}} = e_{\hat{p}}\,(\kappa_p^*\kappa_p \times 1_{\hat{p}})\,e^*_{\hat{p}} = c_p^* c_p = d_p\, 1_\varepsilon\,, \\
\tilde{x}_p d_{\hat{p}}\, 1_\varepsilon &= c^*_{\hat{p}}\,(1_{\hat{p}} \times \tilde{x}_p 1_{\hat{\hat{p}}})\,c_{\hat{p}} = c^*_{\hat{p}}\,(1_{\hat{p}} \times \tilde{\kappa}_p^*\tilde{\kappa}_p)\,c_{\hat{p}} = e_p e_p^* = d_p\, 1_\varepsilon\,,
\end{aligned} \tag{3.2}$$

which imply $x_p = \tilde{x}_p$, hence $x_p = \tilde{x}_p = 1$ and $d_{\hat{p}} = d_p$. Then (3.1) and (2.12) lead to the equalities $\tilde{\kappa}_p = \kappa_p$, $p \in \mathcal{I}$, of isometries. Owing to naturality the maps $\tilde{\kappa}_a \equiv \kappa_a$: $\hat{\hat{a}} \to a$, $a \in \mathrm{Obj}\,\mathcal{C}$, are isometries.

Having reached a standard conjugation, $L = R$, and sovereignty, let us consider the conjugates of standard rigidity intertwiners. Using (2.26) one obtains

$$\alpha_{\hat{a},a}\hat{e}_a = c_{\hat{a}} \in (\varepsilon, \hat{a} \times \hat{\hat{a}})\,, \qquad \hat{c}_a \alpha^{-1}_{a,\hat{a}} = e_{\hat{a}} \in (\hat{\hat{a}} \times \hat{a}, \varepsilon)\,, \tag{3.3}$$

and the rigidity intertwiners of conjugated objects are related to the original ones as

$$e_{\hat{a}} = c_a^* \,(\kappa_a \times 1_{\hat{a}})\,, \qquad c_{\hat{a}} = (1_{\hat{a}} \times \kappa_a^{-1})\, e_a^*\,, \tag{3.4}$$

as is seen by using (2.11) and (2.12). From (3.4) it follows that further specification of the rigidity intertwiners, e.g. identification of the (e_a, c_a) and $(c^*_{\hat{a}}, e^*_{\hat{a}})$ rigidity pairs, may be achieved only in case of involutive conjugation, which will be discussed in Chapter 5.

In a sovereign monoidal category one can define left and right traces $\mathrm{tr}_a^{L/R}$: $(a,a) \to (\varepsilon,\varepsilon)$ for each $a \in \mathrm{Obj}\,\mathcal{C}$:

$$\begin{aligned} \mathrm{tr}_a^L(T) &:= e_a^L\,(\psi_a \times T)\, c_a^R\,, \\ \mathrm{tr}_a^R(T) &:= e_a^R\,(T \times \psi_a)\, c_a^L\,, \end{aligned} \qquad \text{for} \ \ \psi_a\colon\ R(a) \to L(a)\,, \tag{3.5}$$

obeying the property

$$\mathrm{tr}_a^{R/L}(TS) = \mathrm{tr}_b^{R/L}(ST) \qquad \text{for all} \ \ T \in (b,a),\ S \in (a,b)\,. \tag{3.6}$$

In the case of standard conjugation in a C^*-category, together with the previously chosen sovereignty maps, $\{\psi_a = 1_{\hat{a}}\}$, these left (right) traces become identical to left (right) inverses (2.14), hence they are positive traces. Moreover, due to standardness they are equal:

$$\mathrm{tr}_a^L(T) = \mathrm{tr}_a^R(T) =: \mathrm{tr}_a(T) \qquad \text{for all} \ \ T \in (a,a)\,; \tag{3.7}$$

hence $\mathcal{C}$ is also a *spherical* category [9].

4 Frobenius–Schur indicators

For any finite group G one defines the Frobenius–Schur element σ of the group algebra $\mathbb{C}G$ as in [25], i.e. (up to normalization)

$$\sigma := \frac{1}{|G|} \sum_{g \in G} g^2\,. \tag{4.1}$$

The Frobenius–Schur element is a central selfadjoint element of $\mathbb{C}G$; its central decomposition reads

$$\sigma = \sum_{r \in \mathcal{I}} \frac{\nu_r}{d_r}\, e_r \qquad \text{with} \qquad \nu_r = \begin{cases} \ \ 0 & \text{for } r \neq \hat{r}\,, \\ \pm 1 & \text{for } r = \hat{r}\,, \end{cases} \tag{4.2}$$

where e_r, $r \in \mathcal{I}$, are the minimal central projectors and $d_r = \mathrm{tr}_r(e_r)$ is the (integral) dimension of the corresponding simple ideal of the group algebra $\mathbb{C}G$. The three-valent indicator ν_r that is defined by (4.2) is the *Frobenius–Schur indicator*, which is zero on non-selfconjugate simple ideals while on selfconjugate simple ideals the $+(-)$ sign indicates (pseudo)reality.

There is an extension of σ for C^*-Hopf [30] or weak C^*-Hopf algebras [23], which is obtained using the Haar integral $h \in H$ of the (weak) C^*-Hopf algebra H:

$$\sigma := \frac{1}{\epsilon(\mathbf{1})} h^{(1)}\, h^{(2)}\,; \tag{4.3}$$

here we use Sweedler notation for the coproduct, i.e. $h^{(1)} \otimes h^{(2)} \equiv \Delta(h)$, and $\mathbf{1}$ and ϵ are the unit and the counit of H, respectively. Since weak C^*-Hopf algebras contain C^*-Hopf algebras as special cases, we prove the property (4.2) for σ as defined in (4.3) only for the former case.

There is a unique positive $g \in H$ [31] that implements the square of the antipode, i.e. $gxg^{-1} = S^2(x)$ for all $x \in H$, and satisfies the normalization condition $\mathrm{tr}\, D_r(g) = \mathrm{tr}\, D_r(g^{-1}) =: \tau_r \in \mathbb{R}_+$. Then $S(g) = g^{-1}$ holds, and if the unit representation is irreducible then $\tau_r = d_r/\epsilon(\mathbf{1})$. Using the properties [32]

$$h^2 = h^* = S(h) = h\,, \qquad h^{(1)} \otimes x h^{(2)} y = S(x) h^{(1)} S^{-1}(y) \otimes h^{(2)} \quad \text{for all } x, y \in H \tag{4.4}$$

and

$$h^{(2)} \otimes h^{(1)} = h^{(1)} \otimes g h^{(2)} g\,, \qquad S(h^{(1)}) \otimes h^{(2)} = \sum_{r\in\mathcal{I}} \frac{1}{\tau_r} \sum_{i,j} e_r^{ij} g^{-1/2} \otimes g^{-1/2} e_r^{ji} \tag{4.5}$$

of the Haar integral $h \in H$, one obtains

$$\sigma^* = \frac{1}{\epsilon(\mathbf{1})}\, h^{(2)} h^{(1)} = \frac{1}{\epsilon(\mathbf{1})}\, h^{(1)} g h^{(2)} g = \frac{1}{\epsilon(\mathbf{1})}\, h^{(1)} S^{-1}(g) g h^{(2)} = \sigma \tag{4.6}$$

and

$$\begin{aligned} \sigma \cdot y \;&= \frac{1}{\epsilon(\mathbf{1})}\, h^{(1')} h^{(1)} h^{(2')} h^{(2)} \cdot y = \frac{1}{\epsilon(\mathbf{1})}\, S(h^{(1)}) h^{(1')} h^{(2')} h^{(2)} \cdot y \\ &= \frac{1}{\epsilon(\mathbf{1})}\, S(h^{(1)} S^{-1}(y)) h^{(1')} h^{(2')} h^{(2)} = y \cdot \sigma \end{aligned} \tag{4.7}$$

for all $y \in H$, i.e. σ is a central selfadjoint element of H. From the second identity in (4.5) it follows that σ vanishes on non-selfconjugate ideals, hence $\nu_r = 0$ for $r \neq \hat{r}$. For a selfconjugate ideal, $e_r = e_{\hat{r}}$, one obtains

$$\begin{aligned} \epsilon(\mathbf{1}) \cdot e_r \sigma \;&= \frac{1}{\tau_r} \sum_{i,j} S^{-1}(e_r^{ij} g^{-1/2}) g^{-1/2} e_r^{ji} \\ &= \frac{1}{\tau_r} \sum_{i,j} g^{1/2} S^{-1}(e_r^{ij}) g^{-1/2} e_r^{ji} = \frac{1}{\tau_r} \sum_{i,j} S_0(e_r^{ij}) e_r^{ji}\,, \end{aligned} \tag{4.8}$$

where $S_0 := \mathrm{Ad}_{g^{1/2}} \circ S^{-1}$ is the involutive antipode: $S_0^2 = id_H$, $S_0 \circ {}^* = {}^* \circ S_0$. Since $e_r = e_{\hat{r}}$ for matrix units, it follows that $S_0(e_r^{ij}) = v_r e_r^{ji} v_r^*$ with $v_r \in e_r H$ unitary. Moreover,

$$e_r^{ij} = S_0^2(e_r^{ij}) = S_0(v_r^*) v_r e_r^{ij} v_r^* S_0(v_r) \tag{4.9}$$

implies that $S_0(v_r^*)v_r$ is a central unitary element in $e_r H$, hence $S_0(v_r^*) = \pm v_r^* =: \nu_r v_r^*$ for $r = \hat{r}$ due to the involutivity of S_0. But then

$$\begin{aligned}\sigma &= \sum_{\substack{r\in\mathcal{I}\\ r=\hat{r}}} \frac{1}{\epsilon(\mathbf{1})\tau_r} \sum_{i,j} S_0(e_r^{ij}) e_r^{ji} = \sum_{\substack{r\in\mathcal{I}\\ r=\hat{r}}} \frac{1}{d_r} \sum_{i,j} v_r e_r^{ji} v_r^* e_r^{ji} \\ &= \sum_{\substack{r\in\mathcal{I}\\ r=\hat{r}}} \frac{1}{d_r} v_r v_r^{*T} = \sum_{\substack{r\in\mathcal{I}\\ r=\hat{r}}} \frac{1}{d_r} v_r v_r^* S_0(v_r^*) v_r = \sum_{r\in\mathcal{I}} \frac{\nu_r}{d_r} e_r \,,\end{aligned} \tag{4.10}$$

proving (4.2).

There exists a purely categorical definition of the Frobenius–Schur indicator. Let $\mathcal{C}$ be a category as in section 2 and p an irreducible object. If p is selfconjugate, then there exists an invertible intertwiner $J_p\colon\ p \to \hat{p}$. Let us define the self-intertwiner

$$\nu_p := J_p^{-1} \hat{J}_p \kappa_p^{-1} \in (p,p)\,, \tag{4.11}$$

which is independent of the choice of J_p due to irreducibility of p and linearity of the conjugation. Hence ν_p is an isometry, because an isometric J_p can be chosen. In order to prove that even $\nu_p = \pm 1_p$ holds, one first computes that

$$\Phi_p^R(\nu_p) := c_p^* \,(\nu_p \times 1_{\hat{p}})\, c_p = c_p^* \,(J_p^{-1} \times J_p)\, e_p^*\,, \tag{4.12}$$

using (2.6) and (2.12). However, the rigidity intertwiners

$$(e'_p, c'_p) := (c_p^*(J_p^{-1} \times J_p), (J_p^{-1} \times J_p)e_p^*) \in ((\hat{p}\times p, \varepsilon), (\varepsilon, p \times \hat{p})) \tag{4.13}$$

are standard, hence owing to (2.24) we have

$$e'_p = e_p u_p \quad \text{and} \quad c'_p = u_p^* c_p \quad \text{for some } \; u_p \in \mathbb{C} \; \text{ with } \; |u_p| = 1\,. \tag{4.14}$$

Therefore

$$u_p d_p = u_p e_p e_p^* = e'_p e_p^* = \Phi_p^R(\nu_p) = c_p^* c'_p = u_p^* c_p^* c_p = u_p^* d_p\,, \tag{4.15}$$

which proves that $u_p = \pm 1$ and $\nu_p = \pm 1_p$.

Defining ν_p as the zero intertwiner for non-selfconjugate irreducible objects p, one can obtain a natural map ν between the identity functors.

Since the representation category of a (pure weak) C^*-Hopf algebra is a rigid monoidal C^*-category with irreducible unit object, the consistency of the Hopf algebraic and the categorical definitions of the FS indicator requires that $\nu_p = d_p D_p(\sigma)$ for $p \in \mathcal{I}$, where $D_p\colon\ H \to \mathrm{End}\, V_p$ is an irreducible representation of H. This is indeed the case: One can use the natural equivalences λ, ρ, the standard rigidity intertwiners that were given in [23], and the involutive conjugation on the objects $D \mapsto \hat{D} := D^T \circ S_0$ to arrive at $\kappa_D = 1_D$. Then for an irreducible selfconjugate object D_p the choice $J_p^{\hat{\alpha}\alpha} := D_p^{\hat{\alpha}\alpha}(S_0(v_p^*))$ for the matrix elements of the map $J_p\colon\ V_p \to \hat{V}_p$ leads to the desired result.

5 Involutive conjugation

The existence of the natural isomorphism $\{\kappa_a\colon\ \hat{\hat{a}} \to a\}$ suggests that a convenient choice $\hat{\hat{a}} = a$ can be achieved. This requires, however, a further assumption on the category $\mathcal{C}$; namely, the cardinalities of the sets of objects in any two equivalence classes that are related by conjugation must be equal. This quite harmless assumption will always hold after adjoining, if necessary, new objects to $\mathcal{C}$, and this procedure does not change the category within equivalence.

Therefore from now on we assume that the object map $a \mapsto \hat{a}$ of our conjugation functor is involutive, $\hat{\hat{a}}=a$ for all $a \in \mathrm{Obj}\,\mathcal{C}$, and that $\hat{a}=a$ whenever $\hat{a} \simeq a$. It follows that every choice of standard rigidity intertwiners leads to a conjugation functor $L \equiv R$ that is involutive on the arrows as well. As a matter of fact, for each $T\in(a,b)$ we have $T^{LL} = T^{LR} = \kappa_b^{-1} T \kappa_a = T$.

Due to the involutivity of the conjugation, for irreducible objects p the natural isomorphism $\kappa_p\colon\ \hat{\hat{p}} = p \to p$ becomes $\kappa_p = \chi_p 1_p$ with $\chi_p \in \mathbb{C}$ and $|\chi_p| = 1$. The question arises whether the scalar χ_p that appears here has something to do with the Frobenius–Schur indicator or it can be “gauged” away. The relations (3.4) now read as

$$\begin{aligned} e_{\hat{p}} &= c_p^*\,(\kappa_p \times 1_{\hat{p}}) = \chi_p\, c_p^*\,, \\ c_{\hat{p}} &= (1_{\hat{p}} \times \kappa_p^{-1})\, e_p^* = \chi_p^{-1} e_p^* \equiv \bar{\chi}_p\, e_p^*\,. \end{aligned} \tag{5.1}$$

Inserting $p=\hat{q}$ in the first equation, we obtain $\chi_{\hat{q}} e^*_{\hat{\hat{q}}} = c_{\hat{q}} = \bar{\chi}_q e_q^*$, implying that $\chi_{\hat{q}} = \bar{\chi}_q \equiv \chi_q^{-1}$. For selfconjugate irreducibles $q=\hat{q}$ we obtain $\chi_q = \pm 1$. In this case by choosing the map $J_q\colon\ q \to \hat{q}$ in the definition (4.12) of the FS indicator to be 1_q, we obtain

$$\nu_q = J^{-1} \hat{J} \kappa_q^{-1} = 1_q 1_q (\chi_q^{-1} 1_q) = \chi_q^{-1} 1_q \qquad \text{for all } \ q\in\mathcal{I} \ \text{ with } \ q=\hat{q}\,. \tag{5.2}$$

If $p \neq \hat{p}$ for $p\in\mathcal{I}$, then we can employ the freedom (2.24) so as to change the rigidity intertwiners of one of the irreducible objects of the pair $\{p,\hat{p}\}$, let us say of $\hat{p}$:

$$e'_{\hat{p}} := \chi_p^{-1} e_{\hat{p}}\,, \quad c'_{\hat{p}} := \chi_p c_{\hat{p}}\,, \qquad e'_p := e_p, \quad c'_p := c_p\,. \tag{5.3}$$

Then $e'_{\hat{p}} = c'^*_p$ and $c'_{\hat{p}} = e'^*_p$, i.e. the coefficients χ that are present in (5.1) are gauged away for $p\in\mathcal{I}$, $p\neq\hat{p}$.

To summarize: after enlarging $\mathcal{C}$ appropriately, there exists a *standard conjugation functor* that is involutive both on the objects and on the arrows and has the property that for $p\in\mathcal{I}$

$$\begin{aligned} e_{\hat{p}} &= \chi_p\, c_p^*\,, \\ c_{\hat{p}} &= \chi_p^{-1} e_p^*\,, \end{aligned} \qquad \text{with} \quad \chi_p = \begin{cases} 1 & \text{for } \ p\neq\hat{p}\,, \\ \nu_p & \text{for } \ p=\hat{p}\,. \end{cases} \tag{5.4}$$

6 Frobenius maps on basic intertwiner spaces

Let us call the space $(p, q\times r)$, for $p,q,r\in\mathcal{I}$, a *basic intertwiner space*. It is a Hilbert space with scalar product determined by

$$(t_1,t_2)\,1_p := t_1^* t_2 \qquad \text{for } \ t_1,t_2\in(p,q\times r)\,. \tag{6.1}$$

We define two antilinear maps

$$x: \ (p, q \times r) \to (r, \hat{q} \times p\,,) \qquad y: \ (p, q \times r) \to (q, p \times \hat{r}) \tag{6.2}$$

by

$$x(t) := (1_{\hat{q}} \times t^*)\, \varphi^*_{\hat{q},q,r}\, (e^*_q \times 1_r)\, \rho_r \sqrt{\frac{d_r}{d_p}}\,, \quad y(t) := (t^* \times 1_{\hat{r}})\, \varphi^*_{q,r,\hat{r}}\, (1_q \times c_r)\, \lambda_r \sqrt{\frac{d_q}{d_p}} \tag{6.3}$$

and call them the Frobenius maps. These maps are antilinear isometries, i.e. for the scalar product the formula

$$(x(t_1), x(t_2)) = (t_2, t_1) = (y(t_1), y(t_2)) \tag{6.4}$$

holds due to the trace property (3.6). They generate a $\mathbb{Z}_2$-graded antilinear and projective action of S_3 on the basic intertwiner spaces. A $\mathbb{Z}_2$-graded antilinear action of a $\mathbb{Z}_2$-graded group means linear (antilinear) action for even (odd) elements of the group. A permutation group is naturally $\mathbb{Z}_2$-graded by distinguishing even and odd permutations. Moreover, the Frobenius maps generate only a projective action since using (5.4) one proves that

$$x^2 = \chi_q \cdot id_{(p,q\times r)}\,, \qquad y^2 = \chi_r \cdot id_{(p,q\times r)} \tag{6.5}$$

as well as

$$xyx = yxy\colon \ \ (p, q \times r) \to (\hat{p}, \hat{r} \times \hat{q})\,, \tag{6.6}$$

which constitutes just a projective version of the defining S_3-relations for the transpositions: $\sigma_{12} \leftrightarrow x$, $\sigma_{23} \leftrightarrow y$. In a generic case the Frobenius maps lead to an orbit of six different intertwiner spaces

$$\begin{aligned}
\mathrm{Im}(id) \ &= (p, q \times r)\,, \qquad & \mathrm{Im}(xy) &= (\hat{r}, \hat{p} \times q)\,,\\
\mathrm{Im}(x) \ &= (r, \hat{q} \times p)\,, \qquad & \mathrm{Im}(yx) &= (\hat{q}, r \times \hat{p})\,,\\
\mathrm{Im}(y) \ &= (q, p \times \hat{r}), \qquad & \mathrm{Im}(yxy) &= (\hat{p}, \hat{r} \times \hat{q})\,.
\end{aligned} \tag{6.7}$$

It is easy to give the action of the Frobenius maps on canonical basic intertwiners $\rho_p \in (p, \varepsilon \times p)$ and $\lambda_p \in (p, p \times \varepsilon)$, namely,

$$x(\rho_p) = \rho_p\,, \qquad y(\rho_p) = \frac{1}{\sqrt{d_p}}\, c_p\,, \qquad xy(\rho_p) = \lambda_{\hat{p}} \tag{6.8}$$

and

$$x(\lambda_p) = \frac{1}{\sqrt{d_p}}\, e^*_p\,, \qquad y(\lambda_p) = \lambda_p\,, \qquad yx(\lambda_p) = \rho_{\hat{p}}\,. \tag{6.9}$$

Therefore they are not generic but length three orbits of S_3 with a possible projective $\mathbb{Z}_2$-extension by χ_p.

There are other cases, too, where the orbits are not generic. Then we have a representation of the corresponding non-trivial stabilizer subgroup of S_3 on a basic intertwiner space. In the list below we present all the possible irreducible representations of the stabilizer subgroups in the non-generic cases. In two cases we obtain certain restrictions on fusion coefficients.

1. $q = \hat{q},\ r = p \neq q:$ orbit $= \{(p, p \times q), (q, \hat{p} \times p), (\hat{p}, q \times \hat{p})\}$.

Therefore there is a $\mathbb{Z}_2$-graded antilinear (projective if $\chi_q = -1$) representation of $\mathbb{Z}_2$ on $(p, p \times q)$ generated by y. But an antilinear unitary is always of order 2, therefore this intertwiner space is the null space if $\chi_q = -1$, i.e. the corresponding fusion coefficient is $N^p_{pq} = 0$. If $\chi_q = 1$ we have a $\mathbb{Z}_2$-graded antilinear representation of $\mathbb{Z}_2$, and such irreducible representations are unique up to unitary equivalence.

2. $q = r = \hat{p} \neq p:$ orbit $= \{(p, \hat{p} \times \hat{p}), (\hat{p}, p \times p)\}$.

Therefore there is a linear representation of $\mathbb{Z}_3$ on $(p, \hat{p} \times \hat{p})$, generated by xy. One can use an orthonormal basis in $(p, \hat{p} \times \hat{p})$ in which the basis elements carry one of the three possible (one-dimensional) irreducible representations of $\mathbb{Z}_3$. (A simple example is given in the Appendix.)

3. $q = r = p = \hat{p}:$ orbit $= \{(p, p \times p)\}$.

Therefore there is a $\mathbb{Z}_2$-graded antilinear (projective if $\chi_p = -1$) representation of S_3 on $(p, p \times p)$. Similarly to the first case we have $N^p_{pp} = 0$ if $\chi_p = -1$. If $\chi_p = 1$ we get a $\mathbb{Z}_2$-graded antilinear proper representation of S_3. Such irreducible representations are one-dimensional and can be labelled by the three possible irreducible representations of $\mathbb{Z}_3$. One can use a basis in $(p, p \times p)$ where basis elements carry these irreducible representations.

7 S_4-transformed simplicial maps of a tetrahedron

Let us consider the following Hilbert space containing certain four-fold tensor products of basic intertwiner spaces as orthogonal subspaces:

$$\begin{aligned} \mathcal{H} \equiv \bigoplus_{\mathbf{s}} \mathcal{H}_{\mathbf{s}} \ &:= \bigoplus_{p,q,r,t,u,v \in \mathcal{I}} (p \times q, u) \otimes (u \times r, t) \otimes (v, q \times r) \otimes (t, p \times v) \\ &\equiv \bigoplus_{p,q,r,t,u,v \in \mathcal{I}} (u, p \times q)^* \otimes (t, u \times r)^* \otimes (v, q \times r) \otimes (t, p \times v)\,, \end{aligned} \tag{7.1}$$

where we put $(u, p \times q)^* := \{t^* \,|\, t \in (u, p \times q)\}$. Vectors in subspaces $\mathcal{H}_{\mathbf{s}}$ of $\mathcal{H}$ that correspond to a four-fold tensor product of basic intertwiner spaces can be labelled by the two-dimensional simplicial complex of the boundary of a tetrahedron $\triangle^3$. More precisely, there are simplicial maps

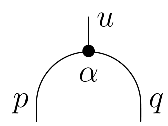
$$\nu = (\nu^0, \nu^1, \nu^2):\ \partial\triangle^3 \to \mathcal{H}_{\mathbf{s}} \tag{7.2}$$

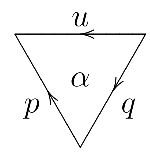
as follows. ν^0 maps all vertices into one point thus the vertices of the tetrahedron remain unlabelled. (Nontrivial ν^0 is needed only in 2-categories which is out of the scope of the present paper.) ν^1 associates to each oriented edge of $\triangle^3$ an irreducible object from the set of representants $\mathcal{I}$. Thus the edges become labelled with $\mathcal{I}$:

$$\begin{aligned} \nu^1(12) &:= p\,, & \nu^1(14) &:= t\,, \\ \nu^1(23) &:= q\,, & \nu^1(13) &:= u\,, \\ \nu^1(34) &:= r \quad , & \nu^1(24) &:= v\,. \end{aligned} \tag{7.3}$$

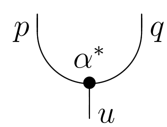
Edges with opposite orientation carry the conjugate objects, e.g., $\nu^1(21) = \hat{p}$.

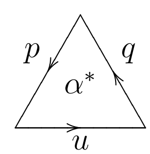
To every element of a basic intertwiner space one can associate an oriented face by a right hand rule:[2]

$(u, p \times q) \ni \alpha =$ 

$\longleftrightarrow$ 

$= \nu^2(123)_+$

$(u, p \times q)^* \ni \alpha^* =$ 

$\longleftrightarrow$ 

$= \nu^2(123)_-$

[2] In the vertex pictures, all arrows point downwards; correspondingly we can safely suppress them.

The two possible orientations tell us whether we are in a basic intertwiner space $(u, p\times q)$ or in its adjoint $(u, p\times q)^* \equiv (p\times q, u)$. Now the images of the faces of the tetrahedron are defined to be elements of basic intertwiner spaces:

$$\begin{aligned}
\nu^2(123) &:= \alpha^u_{pq} \in (u, p\times q) \equiv (\nu^1(13), \nu^1(12)\times\nu^1(23))\,,\\
\nu^2(134) &:= \beta^t_{ur} \in (t, u\times r) \equiv (\nu^1(14), \nu^1(13)\times\nu^1(34))\,,\\
\nu^2(234) &:= \gamma^v_{qr} \in (v, q\times r) \equiv (\nu^1(24), \nu^1(23)\times\nu^1(34))\,,\\
\nu^2(124) &:= \delta^t_{pv} \in (t, p\times v) \equiv (\nu^1(14), \nu^1(12)\times\nu^1(24))\,.
\end{aligned} \tag{7.4}$$

Permutation of the index set $\{1,2,3,4\}$ of the points of the tetrahedron defines, using the Frobenius maps x, y, an S_4-action on the maps ν: $\partial\triangle^3 \to \mathcal{H}_{\mathbf{s}}$ as follows:

$$(\nu_\tau)^0 := \nu^0 \circ \tau\,, \qquad (\nu_\tau)^1(ij) := \nu^1(\tau(i),\tau(j)) \qquad \text{for all } \tau\in S_4 \text{ and } i,j\in\{1,2,3,4\}\,. \tag{7.5}$$

The definition of the transforms of ν^2 uses the Frobenius maps. Let $\tau_{12}, \tau_{23}, \tau_{34}$ denote the generating transpositions of S_4. Then

$$\begin{aligned}
(\nu_{\tau_{12}})^2(123) &:= x(\nu^2(123))\,, & (\nu_{\tau_{12}})^2(234) &:= \nu^2(134)\,,\\
(\nu_{\tau_{12}})^2(134) &:= \nu^2(234)\,, & (\nu_{\tau_{12}})^2(124) &:= x(\nu^2(124))\,;
\end{aligned} \tag{7.6}$$

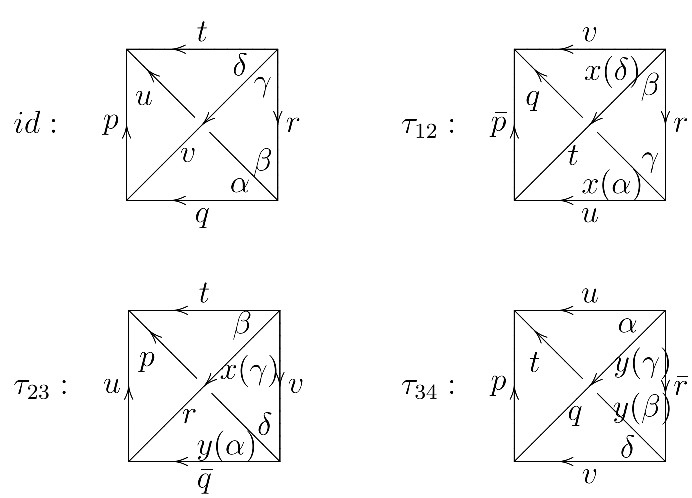
in other words, introducing the notation

$$\alpha^*\otimes\beta^*\otimes\gamma\otimes\delta = \nu^2(123)^*\otimes\nu^2(134)^*\otimes\nu^2(234)\otimes\nu^2(124) \in \mathcal{H}_{\mathbf{s}}\,, \tag{7.7}$$

one has the induced transformations

$$\begin{aligned}
\tau_{12}:&\quad \alpha^*\otimes\beta^*\otimes\gamma\otimes\delta \to x(\alpha)^*\otimes\gamma^*\otimes\beta\otimes x(\delta)\,,\\
\tau_{23}:&\quad \alpha^*\otimes\beta^*\otimes\gamma\otimes\delta \to y(\alpha)^*\otimes\delta^*\otimes x(\gamma)\otimes\beta\,,\\
\tau_{34}:&\quad \alpha^*\otimes\beta^*\otimes\gamma\otimes\delta \to \delta^*\otimes y(\beta)^*\otimes y(\gamma)\otimes\alpha
\end{aligned} \tag{7.8}$$

on the ν^2-images. The images of the original and the transformed maps $\nu, \nu_{\tau_{12}}, \nu_{\tau_{23}}, \nu_{\tau_{34}}$ can be pictured as



Extending the maps in (7.8) antilinearly, one obtains an action of S_4 on the Hilbert space $\mathcal{H}$ in (7.1). The important property of this S_4 action is the following:

Lemma. *The above defined action of the transpositions $\tau_{12}, \tau_{23}, \tau_{34}$ induces a $\mathbb{Z}_2$-graded antilinear representation of S_4 on $\mathcal{H}$.*

Proof. One has to check only the defining relations of S_4:

$$\tau_{12}^2 = \tau_{23}^2 = \tau_{34}^2 = id\,, \qquad \tau_{12}\tau_{23}\tau_{12} = \tau_{23}\tau_{12}\tau_{23}\,, \qquad \tau_{23}\tau_{34}\tau_{23} = \tau_{34}\tau_{23}\tau_{34}\,. \tag{7.9}$$

Indeed one gets

$$\begin{aligned}
\tau_{12}^2(\alpha^* \otimes \beta^* \otimes \gamma \otimes \delta) &= x^2(\alpha)^* \otimes \beta^* \otimes \gamma \otimes x^2(\delta) = \chi_p^2 \cdot \alpha^* \otimes \beta^* \otimes \gamma \otimes \delta = \alpha^* \otimes \beta^* \otimes \gamma \otimes \delta\,,\\
\tau_{23}^2(\alpha^* \otimes \beta^* \otimes \gamma \otimes \delta) &= y^2(\alpha)^* \otimes \beta^* \otimes x^2(\gamma) \otimes \delta = \chi_q^2 \cdot \alpha^* \otimes \beta^* \otimes \gamma \otimes \delta = \alpha^* \otimes \beta^* \otimes \gamma \otimes \delta\,,\\
\tau_{34}^2(\alpha^* \otimes \beta^* \otimes \gamma \otimes \delta) &= \alpha^* \otimes y^2(\beta)^* \otimes y^2(\gamma) \otimes \delta = \chi_r^2 \cdot \alpha^* \otimes \beta^* \otimes \gamma \otimes \delta = \alpha^* \otimes \beta^* \otimes \gamma \otimes \delta\,,
\end{aligned} \tag{7.10}$$

while

$$\begin{aligned}
\tau_{12}\tau_{23}\tau_{12}(\alpha^* \otimes \beta^* \otimes \gamma \otimes \delta) &= xyx(\alpha)^* \otimes x(\beta)^* \otimes x(\delta) \otimes x(\gamma)\\
&= yxy(\alpha)^* \otimes x(\beta)^* \otimes x(\delta) \otimes x(\gamma) = \tau_{23}\tau_{12}\tau_{23}(\alpha^* \otimes \beta^* \otimes \gamma \otimes \delta)\,,\\
\tau_{23}\tau_{34}\tau_{23}(\alpha^* \otimes \beta^* \otimes \gamma \otimes \delta) &= y(\beta)^* \otimes y(\alpha)^* \otimes xyx(\gamma) \otimes y(\delta)\\
&= y(\beta)^* \otimes y(\alpha)^* \otimes yxy(\gamma) \otimes y(\delta) = \tau_{34}\tau_{23}\tau_{34}(\alpha^* \otimes \beta^* \otimes \gamma \otimes \delta)\,. \blacksquare
\end{aligned} \tag{7.11}$$

Notice that although the Frobenius maps lead to a ($\mathbb{Z}_2$-graded antilinear) *projective* representation of S_3 on each basic intertwiner space, due to the fact that $\partial\triangle^3$ is a closed orientable surface without boundary, the action of S_4 on $\mathcal{H}$ is a *proper* (i.e., non-projective) representation. As a matter of fact, according to (7.10), the Frobenius transformations x^2 and y^2 that lead to the signs χ are always coming in pairs.

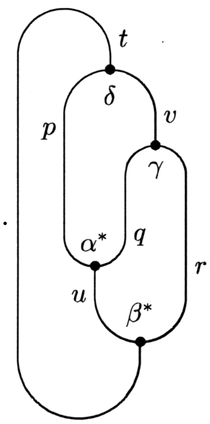
8 An S_4-invariant linear functional on $\mathcal{H}$

Let us define a linear functional Φ: $\mathcal{H} \to \mathbb{C}$ on the Hilbert space $\mathcal{H}$ (7.1) by

$$\begin{aligned}
&\Phi(\alpha_{pq}^{u*} \otimes \beta_{ur}^{t*} \otimes \gamma_{qr}^{v} \otimes \delta_{pv}^{t})\\
&\quad := \sqrt{\tfrac{d_p d_q d_r}{d_t}} \cdot e_t(1_{\hat{t}} \times \beta_{ur}^{t*})(1_{\hat{t}} \times (\alpha_{pq}^{u*}))(1_{\hat{t}} \times \varphi_{p,q,r})(1_{\hat{t}} \times (1_p \times \gamma_{qr}^{v}))(1_{\hat{t}} \times \delta_{pv}^{t})e_t^*\,.
\end{aligned} \tag{8.1}$$

Pictorially, the value is:

$$\Phi(\alpha^{u*}_{pq} \otimes \beta^{t*}_{ur} \otimes \gamma^{v}_{qr} \otimes \delta^{t}_{pv}) = \sqrt{\tfrac{d_p d_q d_r}{d_t}} \cdot$$



(8.2)

(In order not to overburden the picture, we do not indicate the maps λ, ρ, φ; due to coherence and naturality they can be put back unambiguously.)

The action of $\tau \in S_4$ on Φ is the one induced by the action on $\mathcal{H}$,

$$(\tau\Phi)(\alpha^* \otimes \beta^* \otimes \gamma \otimes \delta) := \overline{\Phi(\tau^{-1}(\alpha^* \otimes \beta^* \otimes \gamma \otimes \delta))}^{\deg \tau}, \tag{8.3}$$

where the notation "overline to the power of $\deg \tau$" means complex conjugation for odd S_4-elements and the identity operation for even elements. This transformation property is required by the linearity of Φ and the antilinearity of S_4 on $\mathcal{H}$ to be compatible:

$$(\tau\Phi)(\lambda X) = \overline{\Phi(\tau^{-1}(\lambda X))}^{\deg \tau} = \overline{\bar{\lambda}^{\deg \tau} \Phi(\tau^{-1}(X))}^{\deg \tau} = \lambda \cdot \overline{\Phi(\tau^{-1}(X))}^{\deg \tau} \tag{8.4}$$

for $X \in \mathcal{H}$. Hence the functional Φ is S_4-invariant if

$$(\tau\Phi)(X) := \overline{\Phi(\tau^{-1}(X))}^{\deg \tau} = \Phi(X) \tag{8.5}$$

for all $\tau \in S_4$, or equivalently

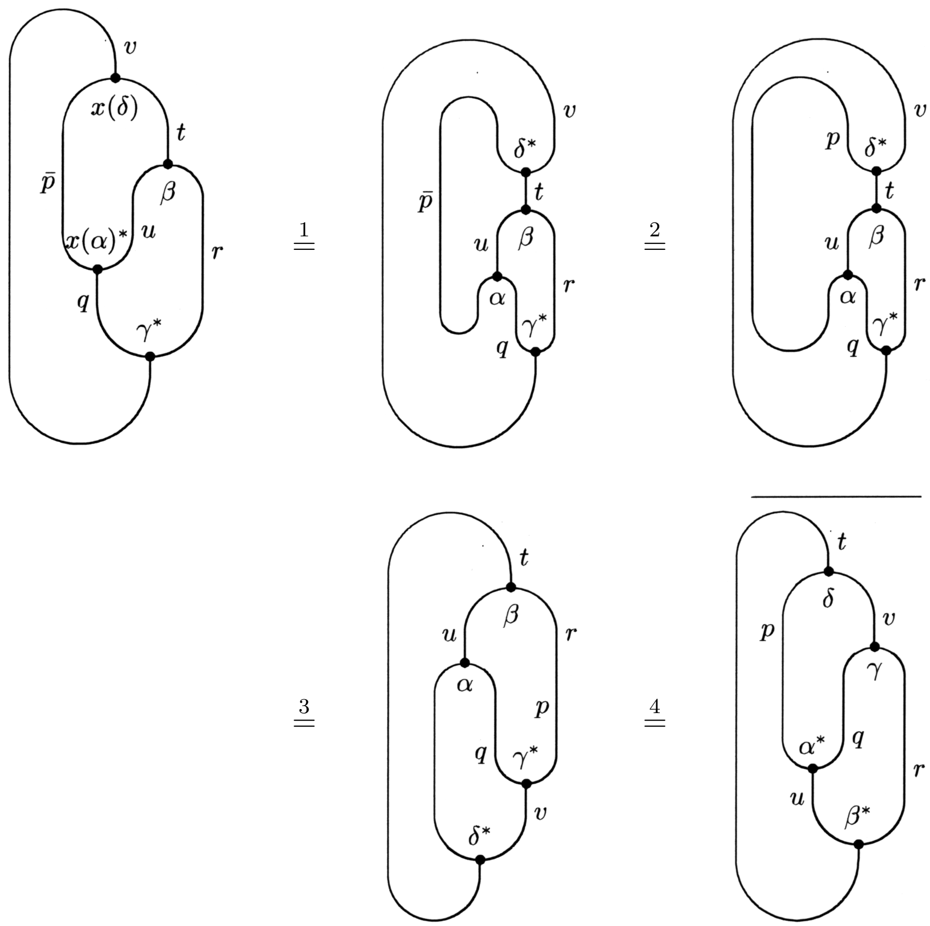
$$\Phi(\tau^{-1}(X)) = \overline{\Phi(X)}^{\deg \tau} \tag{8.6}$$

for all $\tau \in S_4$.

Proposition. *The functional Φ is constant on S_4-orbits.*

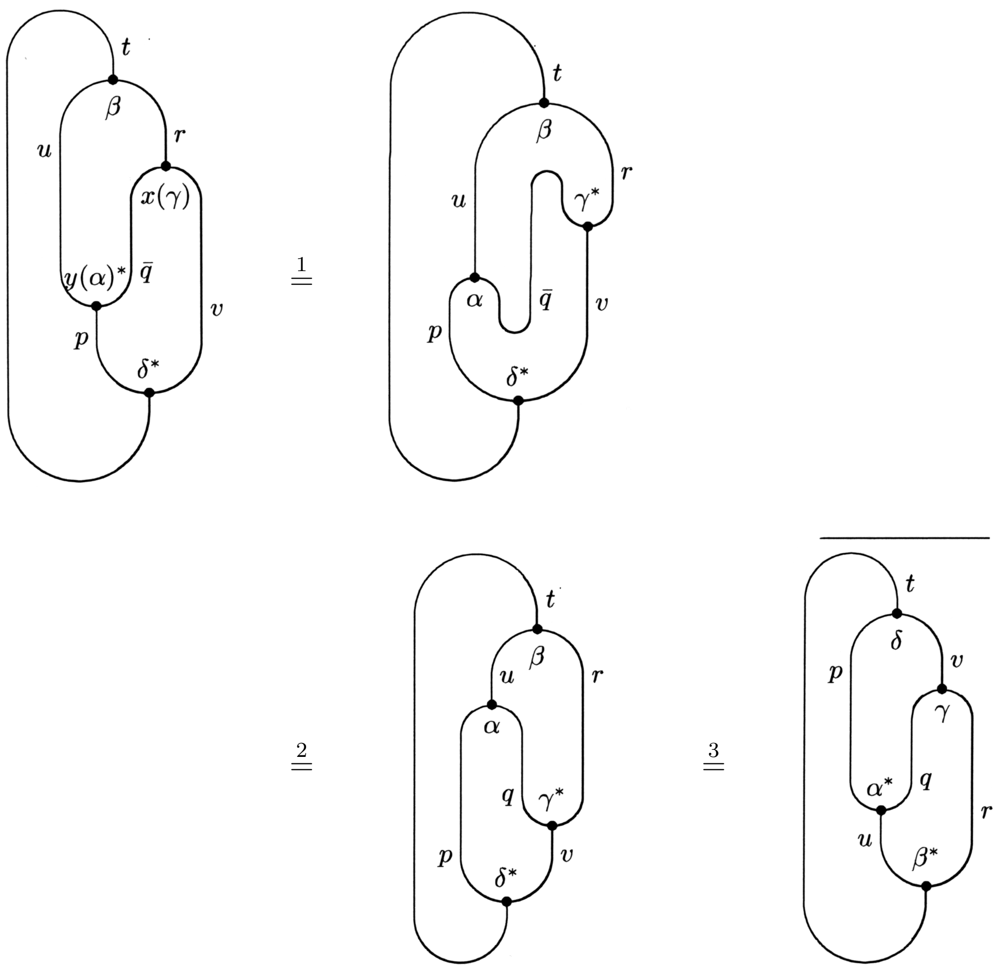
Proof. It is sufficient to show this property for the generators of S_4. Below there is a diagrammatic proof where we used the definitions of the Frobenius maps, the trace property, rigidity, the property $\mathrm{End}\,\varepsilon = \mathbb{C} \cdot 1_\varepsilon$ and sphericity in the last series of pictures.

Invariance with respect to τ_{12} is shown by the following chain of equalities:



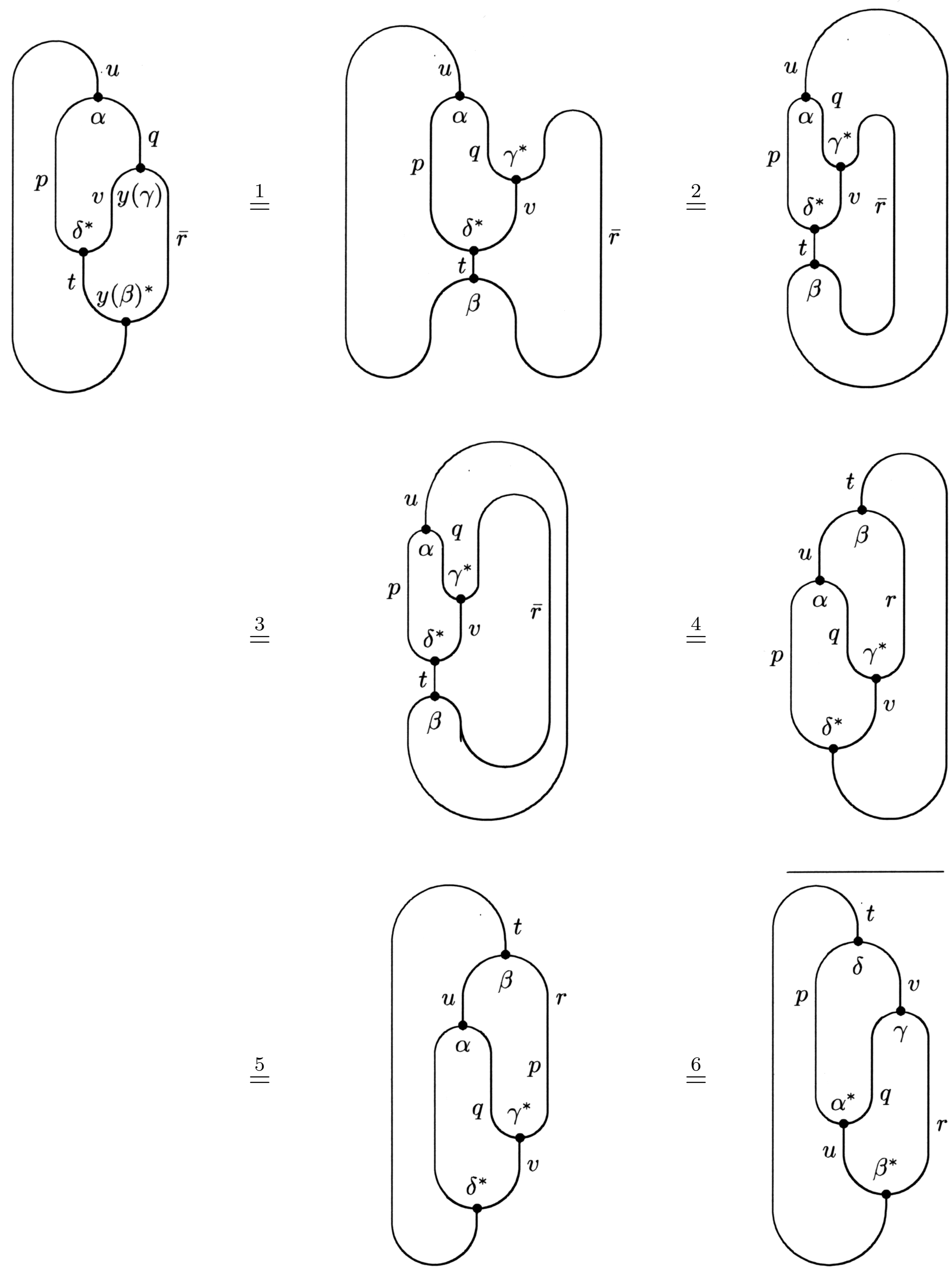
Here we have used the definition of the Frobenius map x in the first equality, the monoidality of the functor of taking duals in the second, used the trace property in the third, and the definition of '$*$' in the fourth equality.

Proof of the invariance with respect to τ_{23}:



This chain of equalities is obtained as follows. In the first equality we used the definition of the Frobenius maps x and y; in the second equality the rigidity identity is used; and in the third equality the definition of '$*$' is implemented.

Proof of the invariance with respect to τ_{34}:



In this chain of equalities we have used the definition of the Frobenius map y in the first equality; the spherical property in the second; the monoidality of the functor of taking duals in the third; used the trace property in the fourth; used the spherical property in the fifth; and used the definition of ‘ * ’ in the sixth. ∎

By definition the normalized F-coefficients are

$$\breve{F}^{(pqr)_t}_{u,v}(\alpha^*\otimes\beta^*\otimes\gamma\otimes\delta) := \Phi(\alpha^{u*}_{pq}\otimes\beta^{t*}_{ur}\otimes\gamma^{v}_{qr}\otimes\delta^{t}_{pv})\,, \tag{8.7}$$

where $\alpha, \beta, \gamma, \delta$ are elements of orthonormal bases of the corresponding basic intertwiner spaces. Their relation to the $6j$-symbols $\{F^{(pqr)_t}_{\alpha u\beta,\gamma v\delta}\}$ is given by

$$F^{(pqr)_t}_{\alpha u\beta,\gamma v\delta} = \frac{1}{\sqrt{d_p d_q d_r d_t}}\,\breve{F}^{(pqr)_t}_{u,v}(\alpha^*\otimes\beta^*\otimes\gamma\otimes\delta)\,. \tag{8.8}$$

Because of the isometry property of φ they are unitary matrices in the multi-labels $(\alpha u\beta, \gamma v\delta)$. The proposition leads to the possibility of computing $\breve{F}$ in a single point of an S_4-orbit and determine the $6j$-symbols on all the other elements of that orbit.

A Appendix

As a simple but nevertheless non-trivial illustration of the results of the main text let us consider the following three (degenerate) rational Hopf algebras [18, 19] that can be obtained as deformations of the Hopf algebra $\mathbb{C}\mathbb{Z}_3$, i.e. of the group algebra of the cyclic group $\mathbb{Z}_3$. The structural data can be summarized as follows.

$$H = \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2 \,, \qquad e_p^* = e_p^2 = e_p \;\text{ for }\; i=0,1,2\,, \tag{A.1}$$

$$\Delta(e_p) = \sum_{\substack{q,r=0 \\ q+r=p \bmod 3}}^{2} e_q \otimes e_r \,, \qquad S(e_p) = e_{-p \bmod 3}\,, \tag{A.2}$$

$$\lambda = \rho = \mathbf{1} \equiv e_0 + e_1 + e_2 = l = r \in H\,, \tag{A.3}$$

$$\varphi = \sum_{p,q,r=0}^{2} \omega_{pqr} \cdot e_p \otimes e_q \otimes e_r \in H \otimes H \otimes H\,. \tag{A.4}$$

Here $\omega_{111} = \omega_{222} = \omega_{112} = \omega_{221} = \omega_{211} = \omega_{122} =: \omega$ is a third root of unity, $\omega^3 = 1$, which parametrizes the three different rational Hopf algebras, while in all three cases one has $\omega_{pqr} = 1$ for all other combinations of indices.

The representation category $\mathbf{Rep}\,H$ is a rigid monoidal C^*-category. The irreducible representations D_p, $p = 0,1,2$ are one-dimensional and obey $D_p(e_q) = \delta_{p,q}$. The basic intertwiner spaces are one-dimensional at most, and we can choose $(p, q \times r) = \mathbb{C} \cdot 1^p_{qr}$ for the non-trivial ones, where 1^p_{qr} maps the tensor product of the chosen unit vectors into the chosen unit vector of the corresponding one-dimensional representation spaces, i.e. $1^p_{qr}(v_q \otimes v_r) = v_p$. The natural isometries connected to monoidality and the standard rigidity intertwiners are given by

$$\lambda_p = 1^p_{p0}\,, \qquad \rho_p = 1^p_{0p}\,, \qquad \varphi_{p,q,r} = (D_p \otimes D_q \otimes D_r)(\varphi)\,, \tag{A.5}$$

$$e_p = 1^{0*}_{\hat{p}p}\,, \qquad c_p = 1^0_{p\hat{p}}\,. \tag{A.6}$$

The values of χ (5.4) are all trivial: $\chi_p = 1$ for $p = 0,1,2$. Using the definitions (6.3) of the Frobenius maps, one obtains

$$x(1^p_{qr}) = \bar{\omega}_{\hat{q}qr}\, 1^r_{\hat{q}p}\,, \qquad y(1^p_{qr}) = \omega_{qr\hat{r}}\, 1^q_{p\hat{r}}\,, \tag{A.7}$$

with ω_{pqr} as in (A.4). In the case $(p, q \times r) = (1, 2 \times 2)$, this leads to

$$xy(1^1_{22}) = \omega \cdot 1^1_{22}\,, \tag{A.8}$$

which serve as examples of a degenerate orbit of type 2 in section 6 with the possible third roots of unity.

The S_4-symmetry is valid in a non-trivial way in the following sense. First note that there are five S_4-orbits of normalized F-coefficients $\breve{F}^{(pqr)_t}_{u,v}(\alpha^{u*}_{pq}\otimes\beta^{t*}_{ur}\otimes\gamma^{v}_{qr}\otimes\delta^{t}_{pv})$ that have the elements (only their pqr edges indicated):

$$\begin{aligned}&\{000\}\,,\quad \{001,200,120,012\}\,,\quad \{002,100,210,021\}\,,\\&\{010,020,102,201,121,212\}\,,\\&\{101,011,110,202,022,220,111,222,112,122,221,211\}\,,\end{aligned}\tag{A.9}$$

respectively, together with their complex conjugated quantities. One can easily compute the first normalized F-coefficient of the fifth orbit:

$$\breve{F}^{(101)_2}_{1,1}(1^{1*}_{10}\otimes 1^{2*}_{11}\otimes 1^{1}_{01}\otimes 1^{2}_{11}) = 1\,;\tag{A.10}$$

then due to the S_4-invariance one deduces that the other coefficients in the same orbit have the value 1, too. Computation of the τ_{12}-transformed quantity

$$\begin{aligned}1\ &= \tau_{12}(\breve{F}^{(101)_2}_{1,1}(1^{1*}_{10}\otimes 1^{2*}_{11}\otimes 1^{1}_{01}\otimes 1^{2}_{11})) = \overline{\breve{F}^{(211)_1}_{0,2}(x(1^{1}_{10})^*\otimes 1^{1*}_{01}\otimes 1^{2}_{11}\otimes x(1^{2}_{11}))}\\&= \overline{\breve{F}^{(211)_1}_{0,2}(\omega_{210}\bar\omega_{211}\cdot 1^{0*}_{21}\otimes 1^{1*}_{01}\otimes 1^{2}_{11}\otimes 1^{1}_{22})} = \omega\cdot\overline{\breve{F}^{(211)_1}_{0,2}(1^{0*}_{21}\otimes 1^{1*}_{01}\otimes 1^{2}_{11}\otimes 1^{1}_{22})} = \omega\cdot\bar\omega\end{aligned}\tag{A.11}$$

then shows that one may not have S_4-invariance for a fixed set of basic intertwiners in general, so it is important to take the action of the Frobenius maps into account. In this example even the following stronger statement holds: in the case of $\omega\neq 1$ there is *no* such choice for the set of orthonormal basic intertwiners, $\{b^{q\times r}_{qr} := \omega_{qr}\cdot 1^{q\times r}_{qr}\mid q,r=0,1,2\}$ with arbitrary but fixed values of $\omega_{qr}\in\mathbb{C}$, $|\omega_{qr}|=1$, that leads to a constant value of normalized F-coefficients within the chosen set of intertwiners. Indeed, the relation

$$\begin{aligned}\breve{F}^{(101)_2}_{1,1}(b^{1*}_{10}\otimes b^{2*}_{11}\otimes b^{1}_{01}\otimes b^{2}_{11})\ &= \tau_{13}(\breve{F}^{(101)_2}_{1,1}(b^{1*}_{10}\otimes b^{2*}_{11}\otimes b^{1}_{01}\otimes b^{2}_{11}))\\&= \overline{\breve{F}^{(101)_2}_{1,1}(y(b^{1}_{10})^*\otimes b^{2*}_{11}\otimes x(b^{1}_{01})\otimes b^{2}_{11})}\end{aligned}\tag{A.12}$$

implies that the common value should be ±1, but then the product of the $\tau_{12}\tau_{34}$- and $\tau_{23}\tau_{34}\tau_{12}\tau_{23}$-transformed F-coefficients leads to the contradiction

$$\begin{aligned}1\neq\ &\breve{F}^{(222)_0}_{1,1}(x(b^{2}_{11})^*\otimes y(b^{1}_{01})^*\otimes y(b^{2}_{11})\otimes x(b^{1}_{10}))\\&\cdot\breve{F}^{(111)_0}_{2,2}(xy(b^{2}_{11})^*\otimes xy(b^{1}_{01})^*\otimes xy(b^{2}_{11})\otimes yx(b^{1}_{10})) = \omega^2\,.\end{aligned}\tag{A.13}$$

We also note that for a non-trivial third root of unity, $\omega\neq 1$, there are no solutions of the hexagon equations, that is $\mathbf{Rep}\,H$ cannot be made into a braided category in those cases.

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