Birationally rigid iterated Fano double covers

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Abstract

Iterating the procedure of making a double cover over a given variety, we construct large families of smooth higher-dimensional Fano varieties of index 1. These varieties can be realized as complete intersections in various weighted projective spaces. A generic variety in these families is proved to be birationally superrigid; in particular, it admits no non-trivial structures of a fibration into rationally connected (or uniruled) varieties, it is non-rational and its groups of birational and biregular self-maps coincide.

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Introduction

0.1 Birationally rigid varieties

Birational rigidity is one of the most striking phenomena of higher-dimensional algebraic geometry. Speaking informally, birational rigidity means that certain algebraic varieties, on which there are no non-zero global regular differential forms (that is, rationally connected varieties), behave as if they were of general type. Starting from the pioneer paper of V.A.Iskovskikh and Yu.I.Manin on the three-dimensional quartic [IM] of 1970 it has been gradually understood that in higher dimensions birationally rigid varieties and fibrations not only do not form an exceptional exotic class, but on the contrary, are quite typical. Sarkisov's theorem [S1,S2] means that in a sense "almost all" conic bundles of dimension three and higher are birationally rigid. In [P1,P7,P8,P10,P12] it was proved that Fano hypersurfaces and more generally certain complete intersections of index 1 are birationally rigid. In [P6,P9] it was proved that "almost all" del Pezzo fibrations and fibrations into Fano hypersurfaces over $\mathbb{P}^1$ are birationally rigid. Nowadays it is clear that, on the contrary, non-rigid Fano varieties and fibrations are less typical.

In the present paper we use the more traditional definition of birational rigidity. We work over the field $\mathbb{C}$ of complex numbers. A smooth variety X of dimension ≥ 3 is said to be *birationally superrigid*, if for each birational map $\chi\colon X - - \to X'$ onto a variety X' of the same dimension, smooth in codimension one, and each linear system Σ' on X', free in codimension 1 (that is, $\operatorname{codim} \operatorname{Bs} \Sigma' \geq 2$), the inequality

$$c(\Sigma, X) \leq c(\Sigma', X') \tag{1}$$

holds, where $\Sigma = (\chi^{-1})_* \Sigma'$ is the proper inverse image of Σ' on X with respect to χ, and $c(\Sigma, X) = c(D, X)$ stands for the *threshold of canonical adjunction*

$$c(D, X) = \sup\{b/a | b, a \in \mathbb{Z}_+ \setminus \{0\}, |aD + bK_X| \neq \emptyset\}$$

$D \in \Sigma$, and similarly for Σ', X'.

A smooth variety X of dimension ≥ 3 is said to be *birationally rigid*, if for each X', χ and Σ' as above there exists a birational *self*-map

$$\chi^* = \chi^*_{(X',\chi,\Sigma')} \in \operatorname{Bir} X,$$

depending on the triple (X', χ, Σ'), such that

$$c(\Sigma^*, X) \leq c(\Sigma', X'),$$

where Σ^* is the strict transform of Σ with respect to $(\chi^*)^{-1}$, or, equivalently, the strict transform of Σ with respect to the composition

$$\chi \circ \chi^*\colon X - - \to X'.$$

The difference between the rigid and superrigid cases is not too big. Roughly speaking, superrigidity is rigidity combined with the property that the groups of birational and biregular maps coincide (indeed, if this is the case, then twisting by a birational=biregular self-map χ^* does not change the threshold of canonical adjunction, hence rigidity implies the inequality (1), that is, superrigidity).

The most natural object for the rigidity theory is formed by Fano varieties. Assume that X is a smooth Fano variety of dimension ≥ 3 with $\operatorname{Pic} X = \mathbb{Z}K_X$. The important geometric properties of birationally rigid and superrigid varieties from this class (the properties that motivate the very choice of the word "rigidity") are summarized in the following

Proposition 0.1. *Assume that X is rigid. Then:*

(i) *X can not be fibered into uniruled varieties by a non-trivial rational map,*

(ii) *if $\chi: X - - \to X'$ is a birational map onto a Fano variety X' with $\mathbb{Q}$-factorial terminal singularities such that* $\operatorname{Pic} X' \otimes \mathbb{Q} = \mathbb{Q}K_{X'}$, *then X is (biregularly) isomorphic to X' (although χ itself is not necessarily an isomorphism),*

(iii) *X is non-rational.*

Assume moreover that X is superrigid. Then any birational map onto a Fano variety X' with $\mathbb{Q}$-factorial terminal singularities such that $\operatorname{Pic} X' \otimes \mathbb{Q} = \mathbb{Q}K_{X'}$ *is a (biregular) isomorphism. In particular, the groups of birational and biregular self-maps coincide:*

$$\operatorname{Bir} X = \operatorname{Aut} X.$$

These implications of birational rigidity are well known. For convenience of the reader, we give a (very easy) proof of Proposition 0.1 below in Sec. 1.5.

Note that Corti and Reid [C2,CR] take the properties (i) and (ii) (property (iii) is an immediate implication of (i)) as a *definition* of birational rigidity.

Although in this paper we study only Fano varieties, to make the picture complete, let us give the definition of birational rigidity for Fano fibrations, too. The relative version is completely analogous to the absolute one (see the definition above), with the only difference: the group of birational self-maps is replaced by the group of *fiber-wise* birational self-maps.

Assume that X/S is a rationally connected fibration:

$$\begin{array}{ccl} & X & \\ \pi & \downarrow & \text{generic fiber } F_\eta \\ & S & \end{array}$$

The fiber of general position is assumed to be rationally connected, so that X itself is uniruled and the thresholds of canonical adjunction are finite. We define the group of *proper* birational self-maps of the fibration X/S, setting

$$\operatorname{Bir}(X/S) = \operatorname{Bir} F_\eta,$$

that is, $\operatorname{Bir}(X/S) \subset \operatorname{Bir} X$ is the subgroup consisting of all maps $\chi: X - - \to X$ such that χ transforms each fiber into itself. Now the fibration is said to be *birationally rigid* (as a fibration!), if for any variety X' of the same dimension, smooth in codimension one, any birational map $\chi\colon X - - \to X'$ and any linear system Σ' on X' there exists a self-map

$$\chi^* \in \mathrm{Bir}(X/S)$$

such that for the strict transform Σ^* of the linear system Σ' with respect to the composition $(\chi \circ \chi^*)^{-1}$ the inequality

$$c(\Sigma^*, X) \le c(\Sigma', X')$$

is satisfied.

Proposition 0.2. *Assume that X/S is a Fano fibration with X, S smooth such that*

$$\mathrm{Pic}\, X = \mathbb{Z}K_X \oplus \pi^* \mathrm{Pic}\, S$$

and for any effective class $D = mK_X + \pi^* T$ the class NT is effective on S for some $N \ge 1$. Assume furthermore that X/S is birationally rigid. Then for any rationally connected fibration X'/S' and any birational map

$$\chi\colon X - - \to X'$$

(provided that such maps exist) there is a rational dominant map

$$\alpha\colon S - - \to S'$$

making the following diagram commutative:

$$\begin{array}{ccccc} & X & \stackrel{\chi}{- - \to} & X' & \\ \pi & \downarrow & & \downarrow & \pi' \\ & S & \stackrel{\alpha}{- - \to} & S'. & \end{array}$$

0.2 The known results and natural conjectures

At the moment birational superrigidity is proved for the following classes of higher-dimensional Fano varieties:

- smooth hypersurfaces $V_M \subset \mathbb{P}^M$ of degree M, $M \ge 4$ (for $M = 4$ it is the classical theorem of V.A.Iskovskikh and Yu.I.Manin [IM], the case $M = 5$ was proved in [P1], for generic hypersurfaces of degree $M \ge 6$ superrigidity was proved in [P7], finally, for arbitrary smooth hypersurfaces it was proved in [P12]);

- generic (in the sense of Zariski topology) Fano complete intersections

$$\begin{array}{ccc} V = & V_{d_1\cdot d_2\cdot\ldots\cdot d_k} & \subset \mathbb{P}^{M+k} \\ & \| & \\ & F_1\cap F_2\cap\ldots\cap F_k, & \end{array}$$

  where $\deg F_i = d_i$, $d_1+\ldots+d_k = M+k$, $d_i \geq 2$ and the inequality $2k < M$ holds [P10];

- generic (in the sense of Zariski topology) Fano double covers

$$\sigma\colon V \to Q = Q_m \subset \mathbb{P}^{M+1},$$

  where $M \geq 4$, Q is a hypersurface of degree $m \geq 3$, σ is ramified over the divisor

$$W = W_{m\cdot 2l} = W^* \cap Q,$$

  where $W^* = W^*_{2l} \subset \mathbb{P}^{M+1}$ is a hypersurface of degree $2l$, $m+l = M+1$ [P8]. For $m = 1,2$ birational superrigidity was earlier proved in [P2] for smooth double spaces and double quadrics without the assumption of them being generic;

- the known examples [P3,P11] show that mild singularities do not change the picture: the varieties remain rigid, and usually even superrigid.

Conjecture 1 (the absolute case). *A smooth Fano variety V of dimension $\dim V \geq 5$ with the Picard group $\operatorname{Pic} V = \mathbb{Z}K_V$ is birationally superrigid.*

The assumption of V being smooth seems to be unnecessarily strong.

Conjecture 2 (the absolute case). *A Fano variety V of dimension $\dim V \geq 4$ with factorial terminal singularities and the Picard group $\operatorname{Pic} V = \mathbb{Z}K_V$ is birationally rigid.*

One can make Conjecture 2 stronger in its turn, replacing the factorial singularities by the $\mathbb{Q}$-factorial ones and modifying the condition on the Picard group in an appropriate way. However, Fano varieties with terminal quotient singularities have not yet been studied in higher (≥ 4) dimensions from the viewpoint of their birational rigidity. Since there are no completely studied examples it seems it is yet too early to formulate general conjectures.

To complete the picture, we give a relative version of the conjectures on birational rigidity.

Let $V \to S$ be a fibration into Fano varieties satisfying the following standard conditions:

(i) V is smooth, $\dim V \geq 4$,

(ii) $\dim S \geq 1$, the anticanonical class $-K_V$ is relatively ample,

(iii) $\operatorname{Pic} V = \mathbb{Z}K_V \oplus \pi^* \operatorname{Pic} S$.

(In dimension $\dim V = 3$ this means that V/S is a Mori fiber space.)

Conjecture 3 (the relative case). *If the fibration V/S is sufficiently twisted over the base S, then it is birationally superrigid.*

If $\dim S = 1$, that is, $S = \mathbb{P}^1$, then the twistedness of the fibration V/S is characterized by the properties of the numerical class

$$K_V^2 \in A^2(V),$$

more precisely, in the known cases it is sufficient to assume that the following "K_V^2-condition" is satisfied: K_V^2 does not lie in the interior of the cone of the effective cycles of codimension two [P6,P9]. However, this K_V^2-condition can be somewhat weakened, see [G1,G2,Sob1,Sob2].

In the general case the "twistedness" of the fibration V/S can be imagined in the following way. Let the fiber of general position F_s, $s \in S$, belong to a fixed family of Fano varieties $\mathcal{F}$. For simplicity we assume that for a general member $F \in \mathcal{F}$ the anticanonical class $-K_V$ is very ample and determines an embedding

$$F \hookrightarrow \mathbb{P}^N.$$

Let $\mathcal{H} = \mathrm{Hilb}(F)$ be the Hilbert scheme of *embedded* varieties of the family $\mathcal{F}$. Then the fibrations V/S with the general fiber in $\mathcal{F}$ correspond to the maps

$$S \to \mathcal{H}.$$

The degree of the image of S with respect to a fixed class on $\mathcal{H}$ can be considered as a characteristic of twistedness of V/S. However, for particular fibrations the twistedness condition takes a simple form and is easy to check, see [P6,P9].

0.3 Iterated Fano double covers

The aim of the present paper is to prove birational superrigidity of a large class of higher-dimensional Fano varieties, generalizing Fano double hypersurfaces [P8]. The varieties under consideration can be realized as complete intersections in weighted projective spaces. Let us give their construction.

The ground field is assumed to be the field of complex numbers $\mathbb{C}$. The symbol $\mathbb{P}$ stands for the projective space $\mathbb{P}^{M+k}$, where $M \geq 4$, $2k \leq M-1$. Let us choose a system of homogeneous coordinates on $\mathbb{P}$, say $(x_0 : x_1 : \ldots : x_{M+k})$. Consider a set of homogeneous polynomials

$$f_1, \ldots, f_k, g_1, \ldots, g_m$$

in the variables (x_*) of degrees, respectively,

$$d_1, \ldots, d_k, 2l_1, \ldots, 2l_m,$$

where $m \geq 1$, $l_i \geq 2$ and the following equality holds:

$$\sum_{i=1}^{k} d_i + \sum_{i=1}^{m} l_i = M + k.$$

Set

$$Q(f_*) = Q(f_1, \ldots, f_k) = \bigcap_{i=1}^{k} \{f_i = 0\}$$

to be the complete intersection of hypersurfaces $F_i = \{f_i = 0\}$ in $\mathbb{P}$. Set also

$$W_i = \{g_i = 0\} \subset \mathbb{P}.$$

Define a sequence of double covers

$$\sigma_i \colon \mathbb{P}^{(i)} \to \mathbb{P}^{(i-1)},$$

where $\mathbb{P}^{(0)} = \mathbb{P}$, $i = 1, \ldots, m$, in the following way. The cover σ_1 is branched over W_1. Assume that $\sigma_1, \ldots, \sigma_i$ are already constructed. For an arbitrary $j \in \{0, \ldots, i-1\}$ set

$$\sigma_{i,j} = \sigma_{j+1} \circ \ldots \circ \sigma_i \colon \mathbb{P}^{(i)} \to \mathbb{P}^{(j)}.$$

In particular, $\sigma_i = \sigma_{i,i-1}$. Obviously $\sigma_{i,j}$ is a finite morphism of degree 2^{i-j}. Now the double cover

$$\sigma_{i+1} \colon \mathbb{P}^{(i+1)} \to \mathbb{P}^{(i)}$$

is determined by the branch divisor

$$\widetilde{W}_{i+1} = \sigma_{i,0}^{-1}(W_{i+1}) \subset \mathbb{P}^{(i)}.$$

As a result we get a sequence of double covers

$$\mathbb{P}^{(m)} \stackrel{\sigma_m}{\to} \mathbb{P}^{(m-1)} \stackrel{\sigma_{m-1}}{\to} \ldots \stackrel{\sigma_1}{\to} \mathbb{P}^{(0)} = \mathbb{P}.$$

Finally, set

$$\sigma = \sigma_{m,0} \colon \mathbb{P}^{(m)} \to \mathbb{P}.$$

It is a finite morphism of degree 2^m.

The collections of homogeneous polynomials $(f_*; g_*)$ are parametrized by the space

$$\mathcal{H} = \prod_{i=1}^{k} \left[H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d_i)) \setminus \{0\} \right] \times \prod_{i=1}^{m} \left[H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2l_i)) \setminus \{0\} \right].$$

For a general collection $(f_*; g_*) \in \mathcal{H}$ all the varieties

$$Q = Q(f_*), \quad \mathbb{P}^{(i)}, \quad \widetilde{W}_i, \quad \widetilde{W}_i \cap \sigma_{i-1,0}^{-1}(Q(f_*)),$$

are obviously smooth. Set

$$V = V(f_*; g_*) = \sigma^{-1}(Q) \subset \mathbb{P}^{(m)}.$$

This is a smooth subvariety of codimension k in $\mathbb{P}^{(m)}$. Obviously

$$V = \bigcap_{i=1}^{k} \sigma^{-1}(F_i)$$

is a smooth complete intersection in $\mathbb{P}^{(m)}$. It is easy to see that

$$K_V = \left[-(M+k+1) + \sum_{i=1}^{k} d_i + \sum_{i=1}^{m} l_i \right] \sigma^* H$$

$$\|$$

$$-\sigma^* H,$$

where $H \in \operatorname{Pic} \mathbb{P}$ is the class of a hyperplane. By the Lefschetz theorem we get $\operatorname{Pic} V = \mathbb{Z}\sigma^* H$, so that V is a smooth Fano variety of index 1 and dimension M.

Here is the main result of the present paper.

Theorem. *There is a non-empty open Zariski subset $U \subset \mathcal{H}$ such that for any collection of polynomials $(f_*; g_*) \in U$ the variety $V = V(f_*; g_*)$ is a (smooth) birationally superrigid variety.*

The open subset U is defined below by explicit *regularity conditions.* There is no doubt that *any smooth* variety $V = V(f_*; g_*)$ is birationally superrigid. However, both the classical technique of hypertangent divisors, which is used in this paper, and the new technique based on the connectedness principle of Shokurov and Kollár, are not strong enough to prove this fact.

0.4 The structure of the paper

The paper is organized in the following way. Section 1 contains the "general theory" of the method of maximal singularities, which for convenience of the reader is given here with the necessary details. In Sec. 1.1 we give a criterion of birational superrigidity, reducing the proof of birational superrigidity to checking certain explicit conditions for subvarieties of codimension two. In Sec. 1.2 we give the first proof of the main claim of this criterion. This proof makes use of the technique of counting multiplicities. Sec. 1.3 contains another proof, which makes use of Corti's idea [C2] of reduction to a non-log-canonical singularity of a linear system on a smooth surface. The corresponding two-dimensional fact is proved by means of an elementary technique. Actually, it is a simple implication of one fact which was proved already in [P3]. In Sec. 1.4 the proof of the basic criterion of Sec. 1.1 is completed. Finally, in Sec. 1.5 we give an elementary proof of Proposition 0.1, describing geometry of birationally rigid varieties.

Section 2 is of more technical character. Its aim is to describe the regularity condition that defines the open set $U \subset \mathcal{H}$ in the main theorem and to prove that this set is non-empty. In order to do it, we use the method developed in [P10]. However, in the present case this method should be modified. In this section we also give certain convenient coordinate presentations, which are used later.

Section 3 contains the heart of the proof — there we check the criterion of Section 1.1 for the regular iterated double covers. In Sec. 3.1 the technique of estimating the multiplicity of a subvariety at a given point is developed. It is based on the concept of a hypertangent divisor. The method is presented in a general form. After that in Sec. 3.2 we construct a family of hypertangent divisors for an arbitrary point $x \in V$ of the iterated double cover V. Sec. 3.3 studies some properties of hypertangent divisors. These properties follow from the regularity condition. It is here that this condition is made use of. Finally, in Sec. 3.4 – 3.5 we complete the proof of the main theorem.

0.5 Historical remarks and acknowledgements

Not pretending to be complete in any sense, we just point out the principal landmarks in the development of the theory of birational rigidity. The very phenomenon of rigidity was guessed by Fano [F1-F3] when he tried to extend Noether's method [N] to dimension three. However, the techniques of algebraic geometry of his time were not strong enough (intersection theory, sheaves and cohomology, resolution of singularities either were non-existent at all or just made its first steps) to enable him to obtain complete results. The first outlines of the theory of birational rigidity can be seen in the papers of Yu.I.Manin of 60ies on surfaces over non-closed fields, see, for instance, [M1-M3], where, using B.Segre's earlier results, it is proved what is now formulated as birational superrigidity of del Pezzo surfaces of degree 1 with the Picard group $\mathbb{Z}$. In [M2,M3] a graph is associated to a finite sequence of blow ups. Its combinatorial invariants are very important for the classical technique of counting multiplicities.

The decisive step was made by V.A.Iskovskikh and Yu.I.Manin in [IM] where birational superrigidity of a smooth three-dimensional quartic was established (the authors prove the coincidence of birational and biregular isomorphisms of three-dimensional quartics, however the arguments of the paper do not need any modification to produce birational superrigidity, so that essentially it is superrigidity that was proved there). After that the technique that was developed in [IM] was applied in [I1] to a few families of Fano 3-folds, which resulted in proving their birational rigidity.

The next step was made by V.G.Sarkisov in [S1,S2]. Starting with V.A.Iskovskikh's results on the surfaces with a pencil of rational curves, V.G.Sarkisov proved birational rigidity of conic bundles with a "sufficiently big" discriminant divisor. The concept of "birational rigidity" was in 1980 yet non-existent, so in [S1,S2] the main result is formulated as uniqueness of the conic bundle structure on a given variety. The breakthrough that was made in [S1,S2] was the more impressing that embraced conic bundles in arbitrary dimension.

In a few years after Sarkisov's theorem the first attempts were made to extend the three-dimensional technique of V.A.Iskovskikh and Yu.I.Manin to the field of higher-dimensional Fano varieties [P1,P2]. Besides, birational geometry of a three-dimensional quartic with a non-degenerate double point was described [P3]. However, the technical side of this work was getting more and more complicated. The methods needed to be improved.

In [P4] and especially [P5] (the latter paper was written in 1995 and has been distributed among the experts since that time, although published only in 2000) the classical technique of the method of maximal singularities was essentially simplified and clarified, which in particular made it possible to prove birational rigidity of del Pezzo fibrations over $\mathbb{P}^1$ [P6] — that is, the only class of three-dimensional Mori fiber spaces, which stubbornly refused all attempts to study its birational geometry for about 15 years (there were quite a few well-studied examples of Fano varieties, starting from the quartic, and for the conic bundles there was Sarkisov's theorem).

The further development of the theory went in two directions. Already in the late 80ies V.G.Sarkisov suggested a general program of factorization of birational maps between three-dimensional Mori fiber spaces into a composition of elementary links [S3]. M.Reid did a lot of work to popularize Sarkisov's ideas among the experts in Mori theory [R]. In [C1] Corti gave a complete proof of the main claim of Sarkisov's program and thus brought the construction of this theory to an end. Combining the classical methods with the Sarkisov program, it was proved in [CPR] that three-dimensional $\mathbb{Q}$-Fano hypersurfaces of index 1 in weighted projective spaces are birationally rigid. Generators of their groups of birational self-maps were described. Besides, in [C2] Corti suggested to use the connectedness principle of Shokurov and Kollár (based, in its turn, on the Kawamata-Viehweg vanishing theorem) in the investigation of maximal singularities. This idea turned out to be very fruitful and has already been used several times, both in dimension three [CM] and in arbitrary dimension [P12]. This technique was discussed and further developed in [Ch,ChPk], see also [I2,I3].

The classical technique was developing parallel to the ideas coming from the Mori theory and the log minimal model program. In [P7] the construction of hypertangent divisors was introduced. This construction proved extremely fruitful and made it possible to prove birational rigidity of generic Fano varieties and Fano fibrations for several important families [P8-P11]. This construction makes the basis of the present paper, either.

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1 The method of maximal singularities

1.1 A criterion of birational superrigidity

We will prove the main theorem checking the following convenient sufficient condition of birational superrigidity.

Proposition 1.1. *Let X be a smooth Fano variety with* $\operatorname{Pic} X = \mathbb{Z}K_X$. *Assume that for any irreducible subvariety $Y \subset X$ of codimension two the following two properties are satisfied:*

(i) $\operatorname{mult}_Y \Sigma \leq n$ *for any linear system $\Sigma \subset |-nK_X|$ without fixed components*;

(ii) *the inequality*

$$\operatorname{mult}_x Y \leq \frac{4}{\deg X} \deg Y \tag{2}$$

holds for any point $x \in Y$, where

$$\deg X = (-K_X)^{\dim X}, \quad \deg Y = (Y \cdot (-K_X)^{\dim Y})$$

and $\operatorname{mult}_Y \Sigma$ *means multiplicity of a general divisor $D \in \Sigma$ along Y.*

Then the variety X is birationally superrigid.

It is in this way that birational superrigidity was proved for the majority of known classes of Fano varieties [P7-P11].

Proof of Proposition 1.1. For convenience of the reader we give it here with all the significant details. For the other details and comments see [P5,P8]. Assume that X is not superrigid. Then, by the definition of superrigidity, we get a birational map $\chi\colon X - - \to X'$ and a pair of linear systems Σ, Σ', transformed one into another by χ, such that inequality (1) is not true. The next step in the arguments is given by

Proposition 1.2. *There exists a geometric discrete valuation ν on X such that the Noether-Fano inequality*

$$\nu(\Sigma) > n \cdot \operatorname{discrepancy}(\nu) \tag{3}$$

holds, where $n \in \mathbb{Z}_+$ is defined by the inclusion $\Sigma \subset |-nK_X|$ and $\nu(\Sigma) = \nu(D)$ for a general divisor $D \in \Sigma$. The discrete valuations ν, satisfying (3), are called maximal singularities of the linear system Σ.

Recall [C2,CPR,P5,P8] that a discerete valuation is said to be *geometric*, if it is realizable by a prime Weil divisor on some model of the field of rational function $\mathbb{C}(X)$.

Let $B \subset X$ be the centre of ν on X. If $\operatorname{codim}_X B = 2$, then it is easy to see that $\operatorname{mult}_B \Sigma > n$, which contradicts (i). Therefore $\operatorname{codim}_X B \geq 3$. Here we come to the crucial point of the proof.

Let $D_1, D_2 \in \Sigma$ be general divisors. They have no common components and therefore their intersection is of codimension 2. Denote by $(D_1 \circ D_2)$ the effective cycle of their scheme-theoretic intersection. Obviously,

$$\deg(D_1 \circ D_2) = n^2 \deg X. \tag{4}$$

Now the crucial fact is given by

Proposition 1.3. *The following inequality holds*

$$\mathrm{mult}_B(D_1 \circ D_2) > 4n^2. \tag{5}$$

Proof of the proposition is given below. We give here both known arguments: the classical one, based on the technique of counting multiplicities, and the recent argument of Corti, based on his idea of using the connectedness principle of Shokurov and Kollár.

Now, comparing (4) and (5), we find an irreducible subvariety $Y \subset X$ of codimension 2 (a component of the effective cycle $(D_1 \circ D_2)$) such that

$$\mathrm{mult}_B Y > \frac{4}{\deg X} \deg Y,$$

which contradicts the assumption (ii) of Proposition 1.1.

Therefore, out initial assumption is false and X is superrigid. Q.E.D. for Proposition 1.1.

1.2 The first proof: counting multiplicities

First of all, we take a resolution of the discrete valuation ν. Consider the sequence of blow ups

$$\begin{array}{rccc} \varphi_{i,i-1}: & X_i & \to & X_{i-1} \\ & \bigcup & & \bigcup \\ & E_i & \to & B_{i-1} \end{array}$$

$i \geq 1$, where $X_0 = X, \varphi_{i,i-1}$ blows up the cycle $B_{i-1} = Z(X_{i-1}, \nu)$ of codimension ≥ 2, $E_i = \varphi_{i,i-1}^{-1}(B_{i-1}) \subset X_i$. Set also for $i > j$

$$\varphi_{i,j} = \varphi_{j+1,j} \circ \ldots \circ \varphi_{i,i-1} : X_i \to X_j,$$

$$\varphi_{i,i} = \mathrm{id}_{X_i}.$$

For any cycle $(\ldots)$ we denote its proper inverse image on X_i by adding the upper index i: $(\ldots)^i$.

Remark. (i) Note that $\varphi_{i,j}(B_i) = B_j$ for $i \geq j$.
(ii) Note also that although all the X's are possibly singular, $B_i \not\subset \mathrm{Sing}\, X_i$ for all i.

For some $K\in\mathbb{Z}_+$ the divisor $E_K\subset X_K$ realizes the discrete valuation ν.

Let us define the structure of an oriented graph on the set of exceptional divisors or, equivalently, on the set of indices $\{1,\dots,K\}$. We draw an arrow

$$i\to j$$

if $i>j$ and $B_{i-1}\subset E_j^{i-1}$. Set p_{ij} to be the number of paths from i to j if $i\neq j$, and $p_{ii}=1$. Finally, set $p_i=p_{Ki}$ for all $i=1,\dots,K$.

Now let Σ^j be the strict transform of the linear system Σ on V_j. Set

$$\nu_j=\operatorname{mult}_{B_{j-1}}\Sigma^{j-1}.$$

Obviously, the multiplicity of the linear system Σ with respect to the valuation E_j is equal to

$$\nu_{E_j}(\Sigma)=\sum_{i=1}^{j}p_{ji}\nu_i.$$

Setting $\delta_i=\operatorname{codim}B_{i-1}-1$, we get the well-known expression for the discrepancy

$$a(X,\nu_{E_j})=\sum_{i=1}^{j}p_{ji}\delta_i.$$

The Noether-Fano inequality takes the form

$$\sum_{i=1}^{K}p_i\nu_i>\sum_{i=1}^{K}p_i\delta_i n.$$

Now let us consider the following general situation. Let $B\subset X, B\not\subset\operatorname{Sing}X$ be an irreducible cycle of codimension ≥ 2, $\sigma_B\colon X(B)\to X$ be its blowing up, $E(B)=\sigma_B^{-1}(B)$ the exceptional divisor. Let

$$Z=\sum m_iZ_i,\quad Z_i\subset E(B)$$

be a k-cycle, $k\geq\dim B$. We define the *degree* of Z as

$$\deg Z=\sum_i m_i\deg\left(Z_i\bigcap\sigma_B^{-1}(b)\right),$$

where $b\in B$ is a generic point, $\sigma_B^{-1}(b)\cong\mathbb{P}^{\operatorname{codim}B-1}$ and the right-hand side degree is the ordinary degree in the projective space.

Note that $\deg Z_i=0$ if and only if $\sigma_B(Z_i)$ is a proper closed subset of B.

Our computations will be based upon the following statement.

Let D and Q be two different prime Weil divisors on X, D^B and Q^B be their proper inverse images on $X(B)$.

Lemma 1.1. (i) *Assume that* $\operatorname{codim}B\geq 3$. *Then*

$$D^B\circ Q^B=(D\circ Q)^B+Z,$$

where $\circ$ *stands for the cycle of the scheme-theoretic intersection,* $\operatorname{Supp} Z \subset E(B)$*, and*

$$\operatorname{mult}_B(D \circ Q) = (\operatorname{mult}_B D)(\operatorname{mult}_B Q) + \deg Z.$$

(ii) *Assume that* $\operatorname{codim} B = 2$*. Then*

$$D^B \circ Q^B = Z + Z_1,$$

where $\operatorname{Supp} Z \subset E(B), \operatorname{Supp} \sigma_B(Z_1)$ *does not contain* B*, and*

$$D \circ Q = [(\operatorname{mult}_B D)(\operatorname{mult}_B Q) + \deg Z]\, B + (\sigma_B)_* Z_1.$$

Proof follows easily from the standard intersection theory [Ful].

Now let us come back to our discrete valuation ν.

We divide the resolution $\varphi_{i,i-1} : X_i \to X_{i-1}$ into the *lower part*, $i = 1, \ldots, L \leq K$, when $\operatorname{codim} B_{i-1} \geq 3$, and the *upper part*, $i = L+1, \ldots, K$, when $\operatorname{codim} B_{i-1} = 2$. It may occur that $L = K$ and the upper part is empty.

Let $D_1, D_2 \in \Sigma$ be two different general divisors. We define a sequence of codimension 2 cycles on X_i's setting

$$\begin{array}{l} D_1 \circ D_2 = Z_0, \\ D_1^1 \circ D_2^2 = Z_0^1 + Z_1, \\ \vdots \\ D_1^i \circ D_2^i = (D_1^{i-1} \circ D_2^{i-1})^i + Z_i, \\ \vdots \end{array}$$

where $Z_i \subset E_i$. Thus for any $i \leq L$ we get

$$D_1^i \circ D_2^i = Z_0^i + Z_1^i + \ldots + Z_{i-1}^i + Z_i.$$

For any $j > i, j \leq L$ set

$$m_{i,j} = \operatorname{mult}_{B_{j-1}}(Z_i^{j-1})$$

(the multiplicity of an irreducible cycle along a smaller cycle is understood in the usual sense; for an arbitrary cycle we extend the multiplicity by linearity).

Set $d_i = \deg Z_i$.

We get the following system of equalities:

$$\begin{array}{l} \nu_1^2 + d_1 = m_{0,1}, \\ \nu_2^2 + d_2 = m_{0,2} + m_{1,2}, \\ \vdots \\ \nu_i^2 + d_i = m_{0,i} + \ldots + m_{i-1,i}, \\ \vdots \\ \nu_L^2 + d_L = m_{0,L} + \ldots + m_{L-1,L}. \end{array}$$

Now

$$d_L \geq \sum_{i=L+1}^{K} \nu_i^2 \deg(\varphi_{i-1,L})_* B_{i-1} \geq \sum_{i=L+1}^{K} \nu_i^2.$$

Definition. A function $a : \{1, \ldots, L\} \to \mathbb{R}_+$ is said to be *compatible with the graph structure*, if

$$a(i) \geq \sum_{j \to i} a(j)$$

for any $i = 1, \ldots, L$.

We will actually use only one compatible function, namely $a(i) = p_{Ki} = p_i$.

Proposition 6. *Let $a(\cdot)$ be any compatible function. Then*

$$\sum_{i=1}^{L} a(i) m_{0,i} \geq \sum_{i=1}^{L} a(i) \nu_i^2 + a(L) \sum_{i=L+1}^{K} \nu_i^2.$$

Proof. Multiply the i-th equality by $a(i)$ and put them all together: in the right-hand part for any $i \geq 1$ we get the expression

$$\sum_{j \geq i+1} a(j) m_{i,j}$$

In the left-hand part for any $i \geq 1$ we get

$$a(i) d_i.$$

Lemma 1.2. *If $m_{i,j} > 0$, then $i \to j$.*

Proof. If $m_{i,j} > 0$, then some component of Z_i^{j-1} contains B_{j-1}. But $Z_i^{j-1} \subset E_i^{j-1}$. Q.E.D.

Besides, we can compare the multiplicities m_{ij} with the corresponding degrees.

Lemma 1.3. *For any $i \geq 1, j \leq L$ we have*

$$m_{i,j} \leq d_i.$$

Proof. The cycles B_a are non-singular at their generic points. But since

$$\varphi_{a,b} : B_a \to B_b$$

is surjective, we can count multiplicities at generic points. Now the multiplicities are non-increasing with respect to blowing up of a non-singular cycle, so we are reduced to the obvious case of a hypersurface in a projective space. Q.E.D.

Consequently, we have the following estimate:

$$\sum_{j\geq i+1} a(j)m_{i,j} = \sum_{\substack{j\geq i+1\\ m_{i,j}\neq 0}} a(j)m_{i,j} \leq d_i \sum_{j\to i} a(j) \leq a(i)d_i.$$

So we can throw away all the $m_{i,*}, i\geq 1$, from the right-hand part, and all the $d_i, i\geq 1$, from the left-hand part, replacing $=$ by $\leq$. Q.E.D.

Corollary 1.1. *Set* $m = m_{0,1} = \mathrm{mult}_B(D_1 \circ D_2), D_i \in \Sigma$. *Then*

$$m\left(\sum_{i=1}^{L} a(i)\right) \geq \sum_{i=1}^{L} a(i)\nu_i^2 + a(L)\sum_{i=L+1}^{K} \nu_i^2.$$

Corollary 2. *The following inequality holds:*

$$m\left(\sum_{i=1}^{L} p_i\right) \geq \sum_{i=1}^{K} p_i\nu_i^2.$$

Proof: for $i\geq L+1$ obviously $p_i \leq p_L$. Q.E.D.

Taking into account the Noether-Fano inequality for ν, we see that the right-hand part here is strictly greater than the value of the quadratic form $\sum_{i=1}^{K} p_i\nu_i^2$ at the point

$$\nu_1 = \ldots = \nu_K = \frac{\sum_{i=1}^{K} p_i\delta_i n}{\sum_{i=1}^{K} p_i}.$$

Now set

$$\Sigma_l = \sum_{\delta_j\geq 2} p_j,$$

$$\Sigma_u = \sum_{\delta_j=1} p_j.$$

Then

$$\mathrm{mult}_x Z > \frac{(2\Sigma_l + \Sigma_u)^2}{\Sigma_l(\Sigma_l+\Sigma_u)} n^2.$$

Now easy computations show that the right-hand side is not smaller than $4n^2$. Q.E.D. for Proposition 2.

1.3 The second proof: the connectedness principle of Shokurov and Kollár

Here we give an alternative proof of Proposition 2, suggested by Corti [C2]. Our version is slightly different from that of [C2], we reduce the proof to one simple fact about oriented graphs which was proved originally in [P3].

It is sufficient to consider the case when $B = o$ is a point. Let $S \ni o$ be a general germ of a smooth surface on the variety X. Obiovuly, $\Lambda = \Sigma|_S$ is a germ of a linear system of curves on S with the point o as an isolated base point. Since the pair

$$(X, \frac{1}{n}\Sigma)$$

is not canonical at the point o (it is a reformulation of the Noether-Fano inequality (3)), according to the inversion of adjunction (which is a direct consequence of the connectedness principle of Shokurov and Kollár [K,Sh]), the pair

$$(S, \frac{1}{n}\Lambda)$$

is not *log*-canonical at the point o. In other words, for a certain birational morphism

$$\varphi: \widetilde{S} \to S$$

of smooth surfaces there exists a prime divisor $E \subset \widetilde{S}$, satisfying the log-version of the Noether-Fano inequality

$$\nu_E(\Lambda) > n(a(E) + 1), \tag{6}$$

where $a(\cdot)$ is the discrepancy, $\nu_E(\cdot)$ is the multiplicity of a general divisor of the system at E. Let $D_1, D_2 \in \Lambda$ be generic curves,

$$Z = (D_1 \circ D_2)$$

be a zero-dimensional subscheme on S. One may assume that it is supported at the point o.

Proposition 1.4. *The following inequality holds*

$$\mathrm{mult}_o\, Z (= \deg Z) > 4n^2.$$

Since our considerations are local, $\mathrm{mult}_o\, Z = \deg Z$ is just the degree of the zero-dimensional scheme Z. Since $S \ni o$ is a general germ of a surface, we obtain immediately the claim of Proposition 1.3.

Proof. We give an elementary argument based on explicit computations. The original argument of Corti see in [C2]. Let

$$\begin{array}{cccc} & S_i & \supset & E_i \\ \varphi_{i,i-1} & \downarrow & & \downarrow \\ & S_{i-1} & \ni & x_{i-1} \end{array}$$

be the resolution of the discrete valuation ν_E, $i = 1, \ldots, N$, $x_0 = o$, $x_1, \ldots, x_{N-1}$ points on $S_1, \ldots, S_{N-1}$, respectively, where

$$x_i \in E_i \quad \text{and} \quad \nu_{E_N} = \nu_E.$$

Set Γ to be the graph of this resolution:

$$\{1, \ldots, N\}$$

is the set of vertices, and the vertices i and j, $i > j$, are connected by an oriented edge (notation:

$$i \to j$$

always implies that $i > j$), if and only if

$$x_{i-1} \in E_j^{i-1},$$

where E_j^{i-1} is the strict transform of the exceptional line E_j on S_{i-1}. Set also

$$p_j = (\text{the number of paths from } N \text{ to } j)$$

for $j \leq N-1$, $p_N = 1$. Set

$$\nu_i = \mathrm{mult}_{x_{i-1}} \Lambda^{i-1},$$

where Λ^{i-1} is the strict transform of the linear system Λ on S_{i-1}. It is easy to see that in terms of the resolution the log-inequality (6) takes the form

$$\sum_{i=1}^{N} p_i \nu_i > n \left(\sum_{i=1}^{N} p_i + 1 \right).$$

Besides, the following estimate is true

$$\mathrm{mult}_o Z \geq \sum_{i=1}^{N} \nu_i^2.$$

Lemma 1.4. *The following inequality holds*

$$\mathrm{mult}_o Z > \frac{\left(\sum\limits_{i=1}^{N} p_i + 1 \right)^2}{\sum\limits_{i=1}^{N} p_i^2} n^2.$$

Proof is obtained by elementary computations: the minimum of the quadratic form

$$\sum_{i=1}^{N} s_i^2$$

under the restriction

$$\sum_{i=1}^{N} p_i s_i = c \tag{7}$$

is attained at $s_i = p_i a$, where the common constant a is found from (7). Q.E.D. for the lemma.

In view of this lemma Proposition 1.4 is an implication of the following fact.

Lemma 1.5. *The following inequality holds*

$$\left(\sum_{i=1}^{N} p_i + 1\right)^2 \geq 4\sum_{i=1}^{N} p_i^2. \tag{8}$$

Proof. Note first of all that in (8) the equality can be attained, for instance when $N = 1$. Assume that $N \geq 2$. Set

$$\{2, \ldots, k \leq N\} = \{i \,|\, i \to 1\}.$$

By the definition of the integers p_i we get the equality

$$p_1 = \sum_{i \to 1} p_i = \sum_{i=2}^{k} p_i.$$

Consequently, (8) can be rewritten as

$$\left(2p_1 + \sum_{i=k+1}^{N} p_i + 1\right)^2 \geq 4\sum_{i=1}^{N} p_i^2$$

or, after an easy transformation,

$$4\left(\sum_{i=2}^{k} p_i\right)\left(\sum_{i=k+1}^{N} p_i + 1\right) + \left(\sum_{i=k+1}^{N} p_i + 1\right)^2 \geq 4\sum_{i=2}^{k} p_i^2 + 4\sum_{i=k+1}^{N} p_i^2.$$

It is easy to see that if $k = N$, then the subgraph of the graph Γ with the vertices $\{2, \ldots, N\}$ is of the form

$$2 \leftarrow 3 \leftarrow \ldots \leftarrow N$$

(since on any surface S_i the curve

$$\bigcup_{j \leq i} E_j^i$$

is by smoothness a normal crossing divisor). Hence $p_2 = \ldots = p_N = 1$ and the inequality (8) holds in an obvious way. So let us assume that $N \geq k+1$. Arguing by induction on the number of vertices of the graph Γ we may assume that the inequality

$$\left(\sum_{i=k+1}^{N} p_i + 1\right)^2 \geq 4\sum_{i=k+1}^{N} p_i^2$$

is true. Therefore, it is enough to show that the following estimate is true:

$$\left(\sum_{i=2}^{k} p_i\right)\left(\sum_{i=k+1}^{N} p_i + 1\right) \geq \sum_{i=2}^{k} p_i^2. \tag{9}$$

If $k = 2$, then by construction we get

$$p_2 \leq \sum_{i=3}^{N} p_i,$$

which immediately implies (9). If $k \geq 3$, then the subgraph of the graph Γ with the vertices $\{2, \ldots, \}$ is a chain:

$$2 \leftarrow 3 \leftarrow \ldots \leftarrow k.$$

Since $k \to (k-1)$ and $k \to 1$, the vertices k and i, $i \in \{2, \ldots, k-2\}$, are not joined by an arrow (oriented edge). Consequently,

$$j \not\to i$$

for $j \geq k+1$, $i \in \{2, \ldots, k-2\}$. Thus each path from the vertex N to the vertex $i \in \{2, \ldots, k-2\}$ must go through the vertex $k-1$. Therefore

$$p_2 = \ldots = p_{k-1} = p_k + \sum_{\substack{i \to k-1 \\ i \geq k+1}} p_i. \tag{10}$$

Lemma 1.6. *For any $i \in \{1, \ldots, N\}$ the following inequality holds:*

$$p_i \leq \sum_{j \geq i+2} p_j + 1 \tag{11}$$

(if the set $\{j \geq i+2\}$ is empty, the sum is assumed to be equal to zero).

Proof is obtained by decreasing induction on i. If $i = N$ or $i = N-1$, then the inequality (11) is true. Now we get

$$p_i - \sum_{j \geq i+2} p_j = \sum_{j \to i} p_j - \sum_{j \geq i+2} p_j =$$

$$= p_{i+1} - \sum_{\substack{j \geq i+2 \\ j \not\to i}} p_j.$$

Write down the set $\{j \mid j \to i\}$ as $\{i+1, \ldots, i+l\}$. If $l = 1$, then applying the induction hypothesis, we obtain (11). If $l \geq 2$, then similarly to (10) we get

$$p_{i+1} = \ldots = p_{i+l-1} = p_{i+l} + \sum_{\substack{j \to i+l-1 \\ j \geq i+l+1}} p_j.$$

Therefore

$$p_{i+1} - \sum_{\substack{j \geq i+2 \\ j \not\to i}} p_j = p_{i+l-1} - \sum_{j=i+l+1}^{N} p_j.$$

Applying the induction hypothesis to $i+l-1$, we complete the proof.

Let us come back to the proof of Lemma 1.5. We get

$$p_2 = \ldots = p_{k-1} \leq \sum_{i=k+1}^{N} p_i + 1.$$

But $p_k \leq p_{k-1}$, so that

$$\left(\sum_{i=2}^{k} p_i \right) \left(\sum_{i=k+1}^{N} p_i + 1 \right) \geq p_{k-1} \sum_{i=2}^{k} p_i \geq \sum_{i=2}^{k} p_i^2,$$

which is what we need.

The proof of Proposition 1.4 is complete.

Remark. The inequality (11) first appeared in [P3] as an auxiliary estimate. Here we have reproduced the inductive proof of this fact given in [P3] for convenience of the reader.

1.4 The Lefschetz theorem

Let us prove that the condition (i) of Proposition 2 is true for any smooth iterated double cover V. Indeed, let $\Sigma \subset |-nK_V|$ be a linear system of divisor without fixed components. Let $Y \subset \operatorname{Bs} \Sigma$ be an irreducible subvariety of codimension 2. Take two general divisors, $D_1, D_2 \in \Sigma$ and consider the algebraic cycle of their scheme-theoretic intersection:

$$Z = (D_1 \circ D_2) = aY + \sum a_i Y_i,$$

where $a_i \geq 1$, $Y_i \neq Y$ are some irreducible subvarieties of codimension 2. Obviously, $a \geq (\operatorname{mult}_Y \Sigma)^2$. Compute the degrees:

$$n^2 \deg V = \deg Z = a \deg Y + \sum a_i \deg Y_i,$$

where $\deg V = 2^m d_1 \ldots d_k$. By the Lefschetz theorem Y is numerically equivalent to mK_V^2, $m \geq 1$, so that $\deg Y = m \deg V$, whence

$$(\operatorname{mult}_Y \Sigma)^2 \leq a \leq n^2/m \leq n^2,$$

which is what we need. Q.E.D. for the condition (i).

1.5 Proof of Proposition 0.1

Part (i) of Proposition 0.1 is almost obvious. If

$$\begin{array}{cccc} \chi\colon & X & --\to & X' \\ & & & \downarrow \;\pi' \\ & & & S' \end{array}$$

is a birational map onto X', where $\dim S' \geq 1$ and fibers of π' are uniruled, take Σ' to be a pull back of a moving linear system on S'. Then $c(\Sigma', X') = 0$ and therefore by superrigidity $c(\Sigma, X) = 0$. But $\Sigma \subset |-nK_X|$ is a moving linear system, hence $c(\Sigma, X) = n \geq 1$. Contradiction.

Let us prove part (ii) of Proposition 0.1. Let $\chi\colon X - - \to X'$ be a birational map, $\varphi\colon Y \to X$ be its Hironaka resolution, so that $\psi = \chi \circ \varphi\colon Y \to X'$ is a birational morphism. The variety Y is non-singular and

$$\operatorname{Pic} Y = \mathbb{Z}\varphi^* K_X \oplus \bigoplus_{i \in I} \mathbb{Z} E_i,$$

where $\{E_i | i \in I\}$ is the set of all the φ-exceptional divisors. By assumption,

$$\operatorname{Pic} Y \otimes \mathbb{Q} = \mathbb{Q}\psi^* K_{X'} \oplus \bigoplus_{j \in J} \mathbb{Q} E'_j,$$

where $\{E'_j | j \in J\}$ is the set of all the ψ-exceptional divisors. For simplicity of notations set $K = \varphi^* K_X$, $K' = \psi^* K_{X'}$. We get

$$K_Y = K + \sum_{i \in I} a_i E_i = K' + \sum_{j \in J} a'_j E'_j, \tag{12}$$

where $a_i \in \mathbb{Z}$, $a_i \geq 1$, and $a'_j \in \mathbb{Q}$, $a'_j > 0$. Let $\Sigma' = |-mK_{X'}|$, $m \gg 0$, be a very ample system. Obviously, $c(\Sigma', X') = m$. Take $\Sigma = \chi_*^{-1}\Sigma' \subset |-nK_X|$; obviously, $c(\Sigma, X) = n$. Twisting by a suitable birational self-map, we may assume that the inequality (1) is already satisfied for χ. Hence $n \leq m$. The proper inverse image of the linear system Σ on Y coincides with the inverse image of the linear system Σ' with respect to ψ. Therefore, there exist positive integers b_i, $i \in I$, such that

$$-mK' = -nK - \sum_{i \in I} b_i E_i.$$

Dividing by m and substituting into (12), we get

$$\left(1 - \frac{n}{m}\right) K = \sum_{i \in I} \left(\frac{b_i}{m} - a_i\right) E_i + \sum_{j \in J} a'_j E'_j.$$

Since the divisors E_i are φ-exceptional and $a'_j > 0$, we get the equality $n = m$: otherwise we get a contradiction with the ampleness of $(-K_X)$. Furthermore, all the divisors E'_j turn out to be φ-exceptional and, moreover, $\{E_i | i \in I\} = \{E'_j | j \in J\}$, otherwise $\mathrm{rk}\,\mathrm{Pic}\, X' \geq 2$. Thus χ is an isomorphism in codimension one: set

$$U = X \setminus \bigcup_{i \in I} \varphi(E_i), \quad U' = X' \setminus \bigcup_{j \in J} \psi(E'_j),$$

then $\chi: U \to U'$ is an isomorphism. Therefore $\Sigma = |-nK_X|$ and χ induces an isomorphism of the linear systems Σ and Σ'. Consequently, $\chi: X \to X'$ is an isomorphism. (Strictly speaking, we have proved that for an arbitrary birational map $\chi: X - - \to X'$ there exists $\chi^* \in \mathrm{Bir}\, X$ such that $\chi \circ \chi^*$ is an isomorphism.) The rest is obvious. Proof of Proposition 0.1 is complete.

Proof of Proposition 0.2 is similar to the elementary arguments for the part (i) of Proposition 0.1 above.

2 Iterated double covers

2.1 Coordinate presentations

Let $\mathbb{P}^\sharp$ be the weighted projective space

$$\mathbb{P}(\underbrace{1, 1, \ldots, 1}_{M+k+1}, l_1, \ldots, l_m),$$

where to the weights $l_i \geq 2$ correspond the new homogeneous coordinates y_i, $i = 1, \ldots, m$. The variety $\mathbb{P}^{(m)}$ can be realized as a complete intersection of the type $2l_1 \cdot \ldots \cdot 2l_m$ in $\mathbb{P}^\sharp$:

$$\mathbb{P}^{(m)} = \bigcap_{i=1}^{m} \{y_i^2 = g_i\} \subset \mathbb{P}^\sharp.$$

In a similar way, the variety $V \subset \mathbb{P}^{(m)} \subset \mathbb{P}^\sharp$ is a complete intersection of the type $d_1 \cdot \ldots \cdot d_k \cdot 2l_1 \cdot \ldots \cdot 2l_m$.

Let $p \in Q$ be an arbitrary point, $(z_1, \ldots, z_{M+k})$ a system of affine coordinates with the origin at the point p. Set

$$f_i = g_{i,1} + \ldots + g_{i,d_i},$$

$$g_i = w_{i,0} + w_{i,1} + \ldots + w_{i,2l_i}$$

to be the Taylor decompositions of the (non-homogeneous) polynomials f_i, g_i in the coordinates z_*. Here

$$\deg q_{i,j} = j, \quad \deg w_{i,j} = j.$$

If $w_{i,0} \neq 0$, that is, $p \notin W_i$, then for the convenience of computations we assume always that $w_{i,0} = 1$.

Definition 2.1. The point $p \in Q$ is of class $e \in \{0, 1, \ldots, m\}$, if

$$e = \sharp\{i \,|\, p \in W_i\}.$$

We write the set $\mathcal{L} = \mathcal{L}(p) = \{i \mid p \in W_i\}$ as

$$\{i_1 < \ldots < i_e\}.$$

Let us define a convenient coordinate system at a point $p \in Q$ of class e. For simplicity assume that $z_i = x_i/x_0$, $i = 1, \ldots, M+k$. Set

$$u_i = y_i/x_0^{l_i}$$

for $i = 1, \ldots, m$. The set of regular functions (z_*, u_*) is a system of affine coordinates on an open affine subset $U \subset \mathbb{P}^\sharp$, $U \cong \mathbb{C}^{M+k+m}$. With respect to these coordinates the variety V is given by the system of equations

$$\left\{ \begin{array}{ll} f_i(1, z_1, \ldots, z_{M+k}) = 0, & i = 1, \ldots, k, \\ \\ u_i^2 = g_i(1, z_1, \ldots, z_{M+k}), & i = 1, \ldots, m, \end{array} \right. \tag{13}$$

and from now on we will identify the homogeneous polynomials f_i, g_i with their non-homogeneous presentations of the type $f(1, z_*)$.

If the point $p \in Q$ is of class 0, then for all $i = 1, \ldots, m$ we have $g_i(p) \neq 0$, that is, $w_{i,0} = 1$. In this case for any point $q \in \sigma^{-1}(p)$ the linear maps

$$\sigma_* \colon T_q\mathbb{P}^{(m)} \to T_p\mathbb{P},$$

$$\sigma_* \colon T_qV \to T_pQ$$

are isomorphisms, so that the σ-preimage of any system of local coordinates on $\mathbb{P}$ and Q makes a system of local coordinates on $\mathbb{P}^{(m)}$ (for instance, $(z_1, \ldots, z_{M+k})$) and V, respectively.

If the point $p \in Q$ is of class $e \geq 1$, assume that $\mathcal{L}(p) = \{1, \ldots, e\}$. In this case a natural system of local coordinates on $\mathbb{P}^{(m)}$ is given by the set of functions

$$(z_{j_1}, \ldots, z_{j_{M+k-e}}, u_1, \ldots, u_e),$$

where $z_{j_1}, \ldots, z_{j_{M+k-e}}$ make a system of local coordinates on the complete intersection $W_1 \cap \ldots \cap W_e \subset \mathbb{P}$.

Lemma 2.1. *For $i \in \{1, \ldots, e\}$ the inverse image of the hyperplane $\{w_{i,1} = 0\}$ is tangent to V at the point $q \in \sigma^{-1}(p)$. In particular, the following isomorphism holds*

$$T_qV \cong T_p(Q \cap W_1 \cap \ldots \cap W_e) \oplus \langle u_1, \ldots, u_e \rangle^*.$$

With respect to this isomorphism the tangent cone to the intersection $V \cap \{w_{i,1} = 0\}$ is given by the quadratic equation

$$u_i^2 = w_{i,2}|_{T_p(Q \cap W_1 \cap \ldots \cap W_e)}.$$

Proof. It is obvious from the system of equations (13).

2.2 The regularity condition

Let $g(z_*) = 1 + w_1 + \ldots + w_{2l}$ be a polynomial. Following [P8], consider the formal series

$$(1+t)^{1/2} = 1 + \sum_{i=1}^{\infty} \gamma_i t^i = 1 + \frac{1}{2}t - \frac{1}{8}t^2 + \ldots,$$

and make the following formal series in the variables z_*:

$$\sqrt{g} = (1 + w_1 + \ldots + w_{2l})^{1/2} = 1 + \sum_{i=1}^{\infty} \gamma_i (w_1 + \ldots + w_{2l})^i =$$

$$= 1 + \sum_{i=1}^{\infty} \Phi_i(w_1, \ldots, w_{2l}),$$

where $\Phi_i(w_1(z_*), \ldots, w_{2l}(z_*))$ are homogeneous polynomials of degree i in z_*. Obviously,

$$\Phi_i(w_*) = \frac{1}{2}w_i + (\text{polynomials in } w_1, \ldots, w_{i-1}).$$

For instance, $\Phi_1(w_*) = \frac{1}{2}w_1$. Furthermore, for $j \geq 1$ set

$$[\sqrt{g}]_j = 1 + \sum_{i=1}^{j} \Phi_i(w_*(z_*))$$

and

$$g^{(j)} = g - [\sqrt{g}]_j^2.$$

It is easy to see that the first non-zero homogeneous component of $g^{(j)}$ is of degree $j+1$. Denote it by the symbol $h_{j+1}[g]$. Obviously,

$$h_{j+1}[g] = w_{j+1} + A_j(w_1, \ldots, w_j), \tag{14}$$

where we are not interested in the particular structure of the polynomial A_j.

Now let us formulate the regularity condition. Set $h_{i,j} = h_j[g_i]$.

Definition 2.2. (i) A point $p \in Q$ of class $e = 0$ is regular with respect to the set $(f_*; g_*)$ if the set of homogeneous polynomials

$$\{q_{i,j} \,|\, (i,j) \in \mathcal{J}_q\} \cup \{h_{i,j} \,|\, (i,j) \in \mathcal{J}_h\}, \tag{15}$$

where

$$\mathcal{J}_q = \{(i,j) \,|\, 1 \leq i \leq k, 1 \leq j \leq d_i\},$$

$$\mathcal{J}_h = \{(i,j) \,|\, 1 \leq i \leq m, l_i + 1 \leq j \leq 2l_i, (i,j) \neq (m, 2l_m)\},$$

is regular at $p = o = (0, \ldots, 0) \in \mathbb{C}^{M+k}$, that is, makes a regular sequence in $\mathcal{O}_{p,\mathbb{P}}$ or, in other words, the set of its common zeros is one-dimensional.

(ii) A point $p\in Q$ of class $e\geq 1$ is regular with respect to the set $(f_*;g_*)$, if the set of homogeneous polynomials

$$\{q_{i,j}\,|\,(i,j)\in\mathcal{J}_q\}\cup\{w_{i,1}\,|\,1\leq i\leq e\}\cup\{h_{i,j}\,|\,(i,j)\in\mathcal{J}_e\}, \tag{16}$$

makes a regular sequence in $\mathcal{O}_{p,\mathbb{P}}$. Here for simplicity of notations we assume that

$$\mathcal{L}(p)=\{1,\dots,e\},$$

the set $\mathcal{J}_q$ was defined above and

$$\mathcal{J}_e=\{(i,j)\,|\,e+1\leq i\leq m, l_i+1\leq j\leq 2l_i\}.$$

Proposition 2.1. *For a general set $(f_*;g_*)\in\mathcal{H}$ any point $p\in Q(f_*)$ is regular.*

Proof. Let us consider, to begin with, the general problem of estimating the codimension of "incorrect" sets of polynomials. Here we follow [P10]. This problem is of an independent interest.

2.3 Non-regular sets of polynomials

Let $z_1,\dots,z_{N+1}$ be a set of variables. The symbol $\mathcal{P}_a$ stands for the space of homogeneous polynomials of degree a in the variables z_*. Set

$$\mathcal{L}=\prod_{i=1}^{l+1}\mathcal{P}_{m_i}=\{(p_1,\dots,p_{l+1})\}$$

to be the set of all $(l+1)$-uples of homogeneous polynomials in the variables z_*, $0\leq l\leq N-1$. With each $(l+1)$-uple $(p_*)\in\mathcal{L}$ we associate the projectivized set of its zeros

$$Z(p_*)=\{p_1=\ldots=p_{l+1}=0\}\subset\mathbb{P}^N=\mathbb{P}(\mathbb{C}^{N+1})=X.$$

Here we write $Z(p_*)$ and not $V(p_*)$ in order to make this notation different from our complete intersection $V=V(f_*)$, the principal object of study in this paper. Let

$$Y=\{(p_*)\in\mathcal{L}|\operatorname{codim}_X Z(p_*)\leq l\}$$

be the set of "irregular" $(l+1)$-uples. We need an estimate for the codimension of Y. The case $l=N-1$, when the "correct" dimension of $Z(p_*)$ is zero, is especially important for applications to Fano varieties. However, for technical reasons, it is more convenient to consider the general case of an arbitrary $l\in\{0,\dots,N-1\}$. Set $I=\{1,\dots,l+1\}$ and

$$\mu_j=\min_{S\subset I,\sharp S=j}\left\{\sum_{i\in S}m_i\right\},$$

$b=1,\dots,l+1$. Assume that

$$m=\mu_1=\min\{m_1,\dots,m_{l+1}\}\geq 2.$$

Proposition 2.2. *For any $l\in\{1,\dots,N-1\}$ the following estimate holds:*

$$\mathrm{codim}_{\mathcal{L}}\, Y\geq \min_{j\in\{0,\dots,l\}}\{(\mu_{j+1}-j)(N-j)+1\}. \tag{17}$$

Remark. (i) For $l=0$ we have the trivial estimate $\mathrm{codim}_{\mathcal{L}}\, Y=\dim\mathcal{P}_m$.
(ii) It is not so easy to say whether the estimate (17) is optimal or not. It is obtained below by the method which is completely different from the technique used in [P7] for a similar purpose, that is, to prove existence of regular hypersurfaces $V_M\subset\mathbb{P}$. The technique of [P7] does not work here: the resulting estimates are too weak for complete intersections; however, it is possible that a combination of the method of this paper and that of [P7] could improve (17).

Proof of Proposition 2.2. Set

$$\mathcal{L}_a=\prod_{i=1}^{a}\mathcal{P}_{m_i}=\{(p_1,\dots,p_a)\}.$$

For each irregular $(l+1)$-uple $(p_1,\dots,p_{l+1})$ fix the first (counting from the left to the right) moment when the codimension of the set of zeros $p_1=\ldots=p_a=0$ fails to take the correct value. Consider the sets

$$Y_a=\{(p_*)\in\mathcal{L}_a|\,\mathrm{codim}_X\, Z(p_1,\dots,p_a)=\mathrm{codim}_X\, Z(p_1,\dots,p_{a-1})=a-1\}.$$

Obviously,

$$Y=\coprod_{a=1}^{l+1}\left(Y_a\times\prod_{i=a+1}^{l+1}\mathcal{P}_{m_i}\right).$$

In particular,

$$\mathrm{codim}_{\mathcal{L}}\, Y=\min\{\mathrm{codim}_{\mathcal{L}_a} Y_a|1\leq a\leq l+1\}.$$

Set $I_a=\{1,\dots,a\}\subset I$ and

$$\mu_{a,j}=\min_{S\subset I_a,\sharp S=j}\left\{\sum_{i\in S}m_i\right\},$$

$j=1,\dots,a$. Obviously, $\mu_{a,j}\geq\mu_j$. Therefore, it is sufficient to prove the estimate

$$\mathrm{codim}_{\mathcal{L}_a} Y_a\geq\min_{j\in\{0,\dots,a-1\}}\{(\mu_{a,j+1}-j)(N-j)+1\} \tag{18}$$

for each $a=2,\dots,l+1$. We omit the trivial case $a=1$, because in this case $\mathrm{codim}_{\mathcal{L}_1} Y_1=\dim\mathcal{P}_{m_1}\geq\dim\mathcal{P}_m$, which is certainly higher than the right-hand side of (17), just set in (17) $j=0$.

The space $\mathcal{L}_a$, the set Y_a and the inequality (18) do not depend on l. Thus we may simplify our notations, setting $a = l+1$ and $\mu_{a,j} = \mu_j$. In other words, we prove the inequality (17) for Y_{l+1} instead of Y.

Denote Y_{l+1} by Y^*. We have reduced our original problem to a simpler task of estimating codimension of Y^* in $\mathcal{L}$, where Y^* consists of all such $(l+1)$-uples of polynomials $(p_1, \dots, p_{l+1})$ that the set $p_1 = \dots = p_l = 0$ has the correct dimension and there exists an irreducible component $B \subset Z(p_1, \dots, p_l)$, on which p_{l+1} vanishes. Let $\langle B \rangle$ be the linear span of B, and set $\mathrm{codim}\langle B \rangle = b \leq l$.

Now set $Y^*(b)$ to be the set of all those $(l+1)$-uples $(p_*) \in Y^*$, for which there exists a component $B \subset Z(p_1, \dots, p_l)$ such that

$$\mathrm{codim}_X \langle B \rangle = b, \quad p_{l+1}|_B \equiv 0.$$

Obviously,

$$Y^* = \bigcup_{b=0}^{l} Y^*(b).$$

Thus it is sufficient to prove that

$$\mathrm{codim}_{\mathcal{L}}\, Y^*(b) \geq (\mu_{b+1} - b)(N - b) + 1. \tag{19}$$

for each $b = 0, \dots, l$. Let us prove (19).

The case $\mathbf{b = 0}$. Here $\langle B \rangle = \mathbb{P}^N$ and therefore each non-zero monomial in the linear forms in $z_1, \dots, z_{N+1}$ of degree $d_{l+1} \geq d$ does not vanish on B. The space of monomials

$$\left\{ \prod_{i=1}^{m_{l+1}} (a_{i,1} z_1 + \dots + a_{i,N+1} z_{N+1}) \right\} \subset \mathcal{P}_{m_{l+1}}$$

is closed. Its dimension is equal to

$$m_{k+1} N + 1 \geq \mu_1 N + 1.$$

On the other hand, the set of polynomials $p_{l+1} \in \mathcal{P}_{m_{l+1}}$ that vanish on B is closed. These two closed sets intersect each other at zero only. Therefore the codimension of $Y(0)$ in $\mathcal{L}$ is no smaller than $\mu_1 N + 1$. This gives (19) for $b = 0$.

The case $\mathbf{b \geq 1}$. Here $\langle B \rangle = \mathbb{P}^{N-b}$. Our strategy is to reduce this case to the previous one ($b = 0$), restricting the polynomials p_i to the linear span $P = \langle B \rangle$. Although our arguments are rather simple, they are not straightforward and require some extra work.

Definition 3. Let $g_1, \dots, g_e$ be homogeneous polynomials on the projective space P, $e \leq \dim P - 1$, $\deg g_i \geq 2$ for $i = 1, \dots, e$. An irreducible subvariety $C \subset P$ such that $\langle C \rangle = P$ and $\mathrm{codim}_P\, C = e$ is called an *associated subvariety of the sequence* (g_*), if there exists a chain of irreducible subvarieties $R_j \subset P$, $j = 0, \dots, e$, satisfying the following properties:

- $R_0 = P$;
- for each $j = 0, \ldots, e-1$ the subvariety R_{j+1} is an irreducible component of the closed algebraic set
$$\{p_{j+1} = 0\} \cap R_j,$$
where $p_{j+1}|_{R_j} \not\equiv 0$, so that $\operatorname{codim}_P R_j = j$ for all j;
- $R_e = C$.

If the sequence (g_*) has an associated subvariety, this sequence is said to be *good*.

Lemma 2.2. (i) *The property of being good is an open property.*
(ii) *A good sequence (g_*) can have at most*
$$\left[\frac{1}{e+1}\prod_{j=1}^{e} \deg g_j\right]$$
associated subvarieties.

Proof is easily obtained by induction on e. For g_1 we have the condition $g_1 \not\equiv 0$, which is clearly an open one. Furthermore, at least one irreducible component of the hypersurface $g_1 = 0$ must be of degree ≥ 2, which is also an open condition. There can be at most $[\deg g_1/2]$ such components.

Assume that Lemma is true for each $e = 1, \ldots, j$, where $j \leq \dim P - 2$. Denote by G_j the open set of good sequences of length j. By (ii), for each $(g_*) \in G_j$ there exist at most
$$\left[\frac{1}{j+1}\prod_{\alpha=1}^{j} \deg g_\alpha\right]$$
associated subvarieties. The polynomial g_{j+1} should be non-zero on at least one of them, say R_j, and moreover, the intersection
$$\{g_{j+1} = 0\} \cap R_j$$
should contain an irreducible component, the linear span of which is P. Obviously, this determines an open set in
$$G_j \times H^0(P, \mathcal{O}_P(\deg g_{j+1})).$$
Each associated subvariety has codimension $j+1$ and does not lie in a hyperplane; therefore, its degree is not smaller than $j+2$. Q.E.D. for the lemma.

Now let us return to the polynomials p_* and assume that $l > b$. We claim that we can find $(l-b)$ polynomials among them — after re-numbering we may assume that they are $p_1, \ldots, p_{l-b}$ — such that the sequence
$$p_1|_P, \ldots, p_{l-b}|_P \tag{20}$$

is good and B is one of its associated subvarieties.

Proof of the claim. Assume that we have already found j polynomials — let them be $p_1, \ldots, p_j$ — such that the sequence $(p_1|_P, \ldots, p_j|_P)$ is good and one of its associated subvarieties, say R_j, contains B. If $j < l - b$, then $R_j \neq B$ and there exists a polynomial p_α, $\alpha \in \{j+1, \ldots, l\}$, such that

$$p_\alpha|_{R_j} \not\equiv 0.$$

Otherwise, $R_j \subset Z(p_1, \ldots, p_l)$ and we get a contradiction, since $\dim R_j > \dim B$. After re-numbering, we may assume that $\alpha = j+1$. Now $p_{j+1}|_B \equiv 0$, so that there exists an irreducible component R_{j+1} of the set $\{p_{j+1} = 0\} \cap R_j$, such that $R_{j+1} \supset B$. Proceeding in this way, we obtain our claim.

Now fix a projective subspace $P \subset \mathbb{P}^N$ of codimension b. Let $Y^*(P)$ be the set of all $(l+1)$-uples $(p_1, \ldots, p_{l+1}) \in Y^*$ such that there exists a component $B \subset Z(p_1, \ldots, p_l)$, whose linear span is $\langle B \rangle = P$ and $p_{l+1}|_B \equiv 0$.

By Lemma 2.2, good sequences form an open set. Thus we may estimate the codimension of $Y^*(P)$ in $\mathcal{L}$, assuming that $(p_1|_P, \ldots, p_{l-b}|_P)$ make a good sequence. Let

$$B_1, \ldots, B_K$$

be all its associated subvarieties, whose linear span is P. If $(p_1, \ldots, p_{l+1}) \in Y^*(P)$, then the polynomials

$$p_{l-b+1}|_P, \ldots, p_{l+1}|_P$$

must all vanish on one of these subvarieties B_i. Now arguing as in the case $b = 0$, we get

$$N \sum_{j=l-b+1}^{l+1} \deg p_j + b + 1 \geq \mu_{b+1}(N-b) + b + 1$$

independent conditions on $p_{l-b+1}, \ldots, p_{l+1}$. Taking into account that the Grassmanian has dimension $\dim G(N+1-b, N+1) = b(N+1-b)$, we get finally

$$\begin{array}{rcl} \operatorname{codim}_{\mathcal{L}} Y^*(b) & \geq & \mu_{b+1}(N-b) + b + 1 - b(N+1-b) \\ & & \| \\ & & (\mu_{b+1} - b)(N-b) + 1, \end{array}$$

which is what we need.

In our arguments above we assumed that $l > b$. If $l = b$, then $B \subset \mathbb{P}^N$ is a line, $l = N - 1$ and the inequality (19) can be obtained by an easy dimension count: for a fixed line B the condition $p|_B \equiv 0$ for a polynomial p of degree $e \geq 1$ defines a closed algebraic set of polynomials of codimension $e + 1$ in $\mathcal{P}_e$. Therefore,

$$\operatorname{codim}_{\mathcal{L}} Y^*(N-1) \geq \sum_{i=1}^{N} (m_i + 1) - 2(N-1) = \mu_{l+1} - l + 1,$$

since $\mu_{l+1} = m_1 + \ldots + m_N$.

Q.E.D. for Proposition 2.2.

Corollary 2.1. *In the notations of Proposition 2.2 for $l \leq N-2$ the following estimate holds:*

$$\operatorname{codim}_{\mathcal{L}} Y \geq mN + 1, \tag{21}$$

whereas for $l = N-1$ the following estimate holds:

$$\operatorname{codim}_{\mathcal{L}} Y \geq \min\{mN+1, \mu_{l+1} - l + 1\}. \tag{22}$$

Proof. Obviously $\mu_j \geq jm$ for each $j = 1, \ldots, l+1$. Thus

$$(\mu_{j+1} - j)(N-j) + 1 \geq \varepsilon(j) + mN + 1,$$

where $\varepsilon(t) = -(m-1)t^2 + (Nm - N - m)t$ has two roots, $t = 0$ and $t = N - 1 - \frac{1}{m-1}$. Thus $\varepsilon(0) = 0$ and $\varepsilon(j) \geq 0$ for $j = 1, \ldots, N-2$. Therefore we may omit in (17) the values $j = 1, \ldots, l-1$. If $l \leq N-2$, we may also omit the value $j = l$. Q.E.D. for the corollary.

2.4 Start of the proof of Proposition 2.1

Let $U_{sm} \subset \mathcal{H}$ be a non-empty Zariski open subset, consisting of all collections $(f_*; g_*)$ such that:

(i) $Q = Q(f_*) \subset \mathbb{P}$ is a smooth complete intersection;

(ii) all the divisors $W_i|_Q$ are smooth and the divisor

$$(W_1 + \ldots + W_m)|_Q$$

has normal crossings. In particular, the points of class $e \geq 1$ make a smooth quasi-projective variety of codimension e (its closure is the set of points of class $\geq e$), and the very variety $V = V(f_*; g_*)$ is smooth.

Let $e \in \{0, \ldots, m\}$ be fixed. Consider the closed subset

$$Y_e = \{(x, (f_*; g_*)) \in \mathbb{P} \times U_{sm} \,|\, x \in Q(f_*) \text{ is non-regular of class } e\}.$$

Let $\pi\colon \mathbb{P} \times U_{sm} \to U_{sm}$ be the projection onto the second factor. Now Proposition 2.1 follows from

Proposition 2.3. *The closure*

$$\overline{\pi(Y_e)} \subset U_{sm}$$

is a proper closed subset for any $e \in \{0, 1, \ldots, m\}$.

Proof. We show that $\overline{\pi(Y_e)}$ has a positive codimension in U_{sm}. Set

$$Y_e(x) = Y_e \cap (\{x\} \times U_{\rm sm}) \subset \mathbb{P} \times U_{\rm sm},$$

$$I = \{(x, (f_*)) | x \in Q(f_*)\} \subset \mathbb{P} \times U_{\rm sm},$$

$$I_e = \{(x, (f_*)) | x \in Q(f_*) \text{ is of class e}\} \subset \mathbb{P} \times U_{\rm sm},$$

$$I_e(x) = I_e \cap (\{x\} \times U_{\rm sm}) \subset \mathbb{P} \times U_{\rm sm}.$$

Identifying $\{x\} \times U_{sm} \cong U_{sm}$, we think of $Y_e(x)$ and $I_e(x)$ as subsets in U_{sm}.

In the non-homogeneous presentation with respect to the system of affine coordinates $(z_1, \ldots, z_{M+k})$ the collection $(f_*; g_*)$ can be identified with the set of *homogeneous* polynomials $q_{i,j}$, $w_{i,j}$. Among the polynomials (15,16) that appear in the regularity condition there are precisely $k + e$ linear forms: these are

$$q_{1,1}, \ldots, q_{k,1}, w_{1,1}, \ldots, w_{e,1}.$$

Since $(f_*; g_*) \in U_{sm}$, these linear forms are linearly independent. Set

$$P = P(f_*; g_*) = \{v \in \mathbb{C}^{M+k} \mid q_{1,1} = \ldots = w_{e,1} = 0\},$$

$P \cong \mathbb{C}^{M-e}$. If the point x is non-regular, then the set of common zeros of the rest of polynomials in the set (15) or (16) is of a "wrong" codimension. It is easy to compute that there is not more than $M - e - 1$ polynomials of degree ≥ 2 in the list (15) or (16). Let us apply Corollary 2.1 to them.

2.5 The polynomials $h_{i,j}$ depend on each other

Note that the polynomials $h_{i,j}$ formally depend on each other, so that generally speaking we cannot apply Corollary 2.1. However, the explicit form of the polynomials $h_{i,j}$ (14) makes it possible to circumwent this obstruction. Indeed, let us assume that the polynomials

$$w_{i,1}, \ldots, w_{i,l_i}$$

for $i \geq e = 1$ are fixed.

Let us construct by induction a sequence of homogeneous polynomials

$$\xi_{i,l_i+1}, \ldots, \xi_{i,2l_i},$$

setting

$$\xi_{i,l_i+1} = -A_{l_i}(w_{i,1}, \ldots, w_{i,l_i}),$$

$$\xi_{i,j+1} = -A_j(w_{i,1}, \ldots, w_{i,l_i}, \xi_{i,l_i+1}, \ldots, \xi_{i,j}),$$

$j = l_i + 1, \ldots, 2l_i - 1$. It is obvious that the polynomials $\xi_{i,j}$ depend on $w_{i,1}, \ldots, w_{i,l_i}$ only and therefore are also fixed.

Lemma 2.3. *For any closed subset $T \subset P$ the following conditions are equivalent:*
(i) *all the polynomials $h_{i,j}$ vanish on T, $j = l_i + 1, \ldots, a \leq 2l_i$;*
(ii) *all the polynomials $w_{i,j} - \xi_{i,j}$ vanish on T, $j = l_i + 1, \ldots, a \leq 2l_i$.*

Proof. Indeed,

$$h_{i,l_i+1} \equiv w_{i,l_i+1} - \xi_{i,l_i+1},$$

so that for $a = l_i + 1$ the claim of the lemma is obvious. Now we argue by induction on $a \geq l_i + 2$. If the lemma is true for $a \leq c \leq 2l_i - 1$, and any of the conditions (i), (ii) holds for $a = c + 1$, then in any case

$$(w_{i,j} - \xi_{i,j})|_T \equiv 0$$

for $j = l_i + 1, \ldots, c$. Therefore

$$w_{i,j}|_T \equiv \xi_{i,j})|_T$$

for $j = l_i + 1, \ldots, c$, so that the conditions

$$[w_{i,c+1} + A_c(w_{i,1}, \ldots, w_{i,l_i}, w_{i,l_i+1}, \ldots, w_{i,c})]|_T \equiv 0$$

and

$$[\quad w_{i,c+1} \quad + \quad \underset{\displaystyle -\xi_{i,c+1}}{\underset{\displaystyle \|}{A_c(w_{i,1}, \ldots, w_{i,l_i}, w_{i,l_i+1}, \ldots, w_{i,c})}} \quad]|_T \equiv 0$$

are equivalent. Q.E.D. for the lemma.

Thus for any fixed set of polynomials $w_{i,1}, \ldots, w_{i,l_i}$ the polynomials $h_{i,j}$ in the regularity condition can be replaced by the polynomials $w_{i,j} - \xi_{i,j}$. However, the polynomials $w_{i,j}$, $j = l_i + 1, \ldots, 2l_i$, are arbitrary and therefore the polynomials $(w_{i,j} - \xi_{i,j})$ are also arbitrary: essentially we just shift the origin in the space of homogeneous polynomials of degree j in z_*. Thus when we estimate the codimension $\operatorname{codim}_{I(x)} Y(x)$ that comes from the regularity condition being not satisfied we may apply Corollary 2.1 as if all the polynomials $q_{i,j}$, $h_{i,j}$ were homogeneous polynomials independent of each other. Now let us, at long last, estimate this codimension.

2.6 Estimate for the codimension

Set $Y_1(x) = Y^+(x) \cup Y^\sharp(x)$, where $(f_*; g_*) \in Y^+(x)$ if and only if $l_1 \geq 3$, whereas $(f_*; g_*) \in Y^\sharp(x)$ if and only if $l_1 = 2$.

Obviously,

$$\pi(Y_e) = \bigcup_{x \in \mathbb{P}} Y_e(x). \tag{23}$$

It suffices to show that

$$\operatorname{codim}_{U_{sm}} \pi(Y_e) \geq 1. \tag{24}$$

By (23) this estimate follows immediately from the inequality

$$\mathrm{codim}_{U_{sm}} Y_e(x) \geq M + k + 1,$$

which is what we shall actually prove. Let us consider each possible case in turn.

Case $e \geq 2$**.** Here by Definition 2.2 (ii) we have

$$\sum_{i=1}^{k}(d_i - 1) + \sum_{i=e+1}^{m} l_i = M - \sum_{i=1}^{e} l_i \leq M - e - 2$$

homogeneous polynomials of degree ≥ 2 on $P \cong \mathbb{C}^{M-e}$. According to Corollary 2.1,

$$\mathrm{codim}_{I_e(x)} Y_e(x) \geq 2M - 2e - 1.$$

Since obviously

$$\mathrm{codim}_{U_{sm}} I_e(x) = k + e,$$

we get finally the estimate

$$\mathrm{codim}_{U_{sm}} Y_e(x) \geq 2M + k - e - 1.$$

Taking the union over all $x \in \mathbb{P}$, we get

$$\mathrm{codim}_{U_{sm}} \underbrace{\overline{Y_e}}_{\displaystyle \overline{\bigcup_{x\in\mathbb{P}} Y_e(x)} \,\|} \geq M - e - 1.$$

Since $e < M/2$, the inequality (24) is proved, so that $\overline{Y_e} \subset U_{sm}$ is a proper closed subset.

Case $e = 1$**.** In the $+$-subcase we proceed as above for $e \geq 2$. Let consider the $\sharp$-subcase. Here we have precisely $M - 2$ polynomials on $\mathbb{C}^{M-1}$, so that

$$\mathrm{codim}_{I(x)} Y^{\sharp}(x) \geq \min(2M - 3, \beta), \tag{25}$$

where

$$\beta = \sum_{i=1}^{k}\sum_{j=2}^{d_i}(j-1) + \sum_{i=2}^{m}\sum_{j=l_i+1}^{2l_i}(j-1) + 2$$

$$\|$$

$$1/2\left[\sum_{i=1}^{k} d_i(d_i - 1) + \sum_{i=2}^{m} l_i(3l_i - 1)\right] + 2.$$

If the minimum in (25) is attained at $2M-3$, then we proceed as in the $+$-case above. Thus it suffices to consider the case when the minimum is attained at β. Let us estimate β as a function of non-negative real varieties d_i, l_i, satisfying the constraints

$$\sum_{i=1}^{k} d_i + \sum_{i=2}^{m} l_i = M+k-2, \quad d_i \geq 2, \quad l_i \geq 2.$$

Lemma 2.4. *The following inequality holds:*

$$\beta \geq M.$$

First of all, assuming the claim of the lemma, let us complete the $\sharp$-case. We have

$$\mathrm{codim}_{I_1(x)} Y^{\sharp}(x) \geq M, \quad \mathrm{codim}_{U_{sm}} I_1(x) = k+1,$$

so that

$$\mathrm{codim}_{U_{sm}} Y^{\sharp}(x) \geq M+k+1,$$

and taking the union over all the points $x \in \mathbb{P}$ we get

$$\mathrm{codim}_{U_{sm}} \overline{Y^{\sharp}} \geq 1,$$

which is what we need.

Proof of Lemma 2.4. For convenience of computations we start with the following auxiliary claim.

Lemma 2.5. (i) *For $s_i \geq 2$, $\sum\limits_{i=1}^{c} s_i = B \geq 2c$, where $c \in \mathbb{Z}_+$ is a fixed positive integer the following inequality holds:*

$$\sum_{i=1}^{c} s_i(3s_i - 1) \geq 5B.$$

(ii) *For $s_i \geq 2$, $\sum\limits_{i=1}^{k} s_i = A \geq 2k$, where $k \in \mathbb{Z}_+$ is a fixed positive integer the following inequality holds:*

$$\sum_{i=1}^{k} s_i(s_i - 1) \geq A(\frac{A}{k} - 1).$$

Proof: elementary computations. It is easy to see that in both cases the minimum is attained at $s_1 = \ldots = s_c$ (or $= s_k$.)

Setting

$$\sum_{i=1}^{k} d_i = A, \quad \sum_{i=2}^{m} l_i = B,$$

we obtain by Lemma 2.5 the estimate

$$\beta \geq \frac{1}{2}(A(\frac{A}{k}-1)+5B)+2.$$

Now to prove Lemma 2.4 it is sufficient to check that the inequality

$$\varepsilon(A,B)=A(\frac{A}{k}-1)+5B+4\geq 2M$$

is true under the constraints

$$1\leq k\leq \frac{M-1}{2},\quad A\geq 2k,\quad B\geq 0,\quad A+B=M+k-2.$$

Replacing B by $M+k-2-A$, let us consider the function of real variable $A\in\mathbb{R}_+$

$$\zeta(A)=A(\frac{A}{k}-6).$$

Its minimum on the interval $I=[2k,M+k-2]$ is attained either at $A=3k$ (if $2k\leq M-2$), or at $A=M+k-2$ (if $2k=M-1$). In the first case we get

$$\varepsilon(A,B)\geq 2A+5B+6=2(M+k-2)+3B+6\geq 2M+4.$$

In the second case $B=0$ and by elementary computations we get

$$\varepsilon(A,B)\geq (M+k-2)\frac{2k-1}{k}+6\geq 2M.$$

This completes the proof of Lemma 2.4.

Case $e=0$. Here to prove the estimate (24) it is sufficient to show that

$$\beta=\frac{1}{2}\left[\sum_{i=1}^{k}d_i(d_i-1)+\sum_{i=1}^{m-1}l_i(3l_i-1)+l_m(3l_m-5)\right]+2\geq M+1, \tag{26}$$

since if this is the case then the estimate

$$\operatorname{codim}_{U_{sm}} Y_0(x)\geq M+k+1$$

holds, and arguing as above we see that $\overline{Y_0}\subset U_{sm}$ is a proper closed subset.

Consider β as a function of non-negative real variables d_i, l_i, and set

$$\sum_{i=1}^{k}d_i=A,\quad \sum_{i=1}^{m-1}l_i=B.$$

Assuming A,B to be fixed, we get by Lemma 2.5 the estimate

$$\beta\geq\frac{1}{2}(A(\frac{A}{k}-1)+5B+l_m(3l_m-5))+2.$$

Thus to prove the estimate (26) it is sufficient to check that the inequality

$$A(\frac{A}{k}-1)+5B+l_m(3l_m-5)+2\geq 2M. \qquad (27)$$

First of all let us get rid of l_m. Since

$$A+B+l_m=M+k,\quad l_m\geq 2,$$

we assume A and $B+l_m$ to be fixed. Since the derivative

$$[-5t+t(3t-5)]'=6t-10$$

is positive for $t\geq 2$, the minimum of the left-hand side of (27) is attained at $l_m=2$. Therefore it is sufficient to prove the inequality

$$A(\frac{A}{k}-1)+5B+4\geq 2M$$

for $A+B=M+k-2$ and the standard constraints for A,B and k. But this has already been done when the $\sharp$-case was considered.

Proof of Proposition 2.1 is complete. For a general collection $(f_*;g_*)\in\mathcal{H}$ each point $x\in Q(f_*)$ is regular.

3 Hypertangent divisors

3.1 How to obtain a bound for the multiplicity

Let X be a smooth projective variety, $H\in\operatorname{Pic}X$ an ample class. For an irreducible subvariety $Y\subset X$ its H-degree (or, simply, degree, when it is clear what ample class is meant) is the integer

$$\deg_H Y=(Y\cdot H^{\dim Y}).$$

By linearity the H-degree is defined for any cycle (which is assumed to be equidimensional).

The symbol

$$\frac{\mathrm{mult}_x}{\deg_H}Y$$

means the ratio $(\mathrm{mult}_x Y)/\deg_H Y$, where $x\in X$ is a point. Set

$$\lambda_e(x)=\sup_{\substack{T\subset X,\\ \operatorname{codim}_X T=e}}\left\{\frac{\mathrm{mult}_x}{\deg_H}T\right\},$$

where the supremum is taken over all irreducible subvarieties of codimension $e\geq 1$.

Assume that on X there exists a set of effective divisors

$$D_i\in|a_iH|,$$

$i = 1, \ldots, N$, such that the set-theoretic intersection

$$D_1 \cap \ldots \cap D_N$$

is of codimension $N \leq \dim X$ in a neighborhood of the point x. Set

$$\mu_i = \mathrm{mult}_x D_i \geq 1.$$

Let $T \subset X$ be an irreducible subvariety of codimension $e \geq 1$, where $N \geq e+1$ and $T \ni x$.

Lemma 3.1. *There exists a subset $\mathcal{L} \subset \{1, \ldots, N\}$ of cardinality $N-e$ (after re-numbering we assume, to simplify the notations, that $\mathcal{L} = \{1, \ldots, N-e\}$) and a sequence of irreducible subvarieties T_i, $i = 0, 1, \ldots, N-e$, such that:*

(i) $\mathrm{codim}\, T_i = e + i$;

(ii) $T_0 = T$, $T_i \not\subset D_i$ *and* T_{i+1} *is an irreducible component of the effective cycle* $T_i \cap D_i$;

(iii) $T_i \ni x$ *and the following inequality holds:*

$$\frac{\mathrm{mult}_x}{\deg_H} T_i \geq \frac{\mu_i}{a_i} \cdot \frac{\mathrm{mult}_x}{\deg_H} T_{i-1} \tag{28}$$

for all $i = 1, \ldots, N-e$.

Proof. Let us prove the existence of $\mathcal{L}$ and the set of subvarieties T_i by induction on $i \in \{0, 1, \ldots, N-e\}$. For $i = 0$ we have nothing to prove.

Lemma 3.2. *There is a divisor D_i, $1 \leq i \leq N$, such that $T \not\subset D_i$.*

Proof. Assume the converse: $T \subset D_i$ for all $i = 1, \ldots, N$. Then

$$T \subset D_1 \cap \ldots \cap D_N$$

so that

$$\mathrm{codim}_x (D_1 \cap \ldots \cap D_N) \leq \mathrm{codim}\, T = e \leq N - 1$$

contrary to our assumption about the set $D_1, \ldots, D_N$. Q.E.D. for the lemma.

To simplify the notations we assume that $T \not\subset D_1$. Obviously,

$$\begin{aligned} \mathrm{mult}_x (T \circ D_1) &\geq \mathrm{mult}_x T \cdot \mathrm{mult}_x D_1, \\ \deg_H (T \circ D_1) &= a_1 \deg_H T. \end{aligned}$$

Therefore, the following estimate holds:

$$\frac{\mathrm{mult}_x}{\deg_H} (T \circ D_1) \geq \frac{\mu_1}{a_1} \cdot \frac{\mathrm{mult}_x}{\deg_H} T.$$

The inequality

$$\frac{\mathrm{mult}_x}{\deg_H} Y \geq \gamma \tag{29}$$

is certainly non-linear in Y. However, it is equivalent to the linear inequality

$$\mathrm{mult}_x Y \geq \gamma \deg_H Y.$$

Therefore if (29) holds for an effective cycle Y, then there exists a component Y^+ of this cycle, that is, an irreducible subvariety in X, such that

$$\frac{\mathrm{mult}_x}{\deg_H} Y^+ \geq \gamma$$

(since the H-degree of an irreducible subvariety is always strictly positive). So we get that there is an irreducible component T_1 of the effective cycle $(T \circ D_1)$, an irreducible subvariety of codimension $e+1$, such that

$$\frac{\mathrm{mult}_x}{\deg_H} T_1 \geq \frac{\mu_1}{a_1} \cdot \frac{\mathrm{mult}_x}{\deg_H} T,$$

which is what we need for $i = 1$.

Assume that the subset $\{1, \ldots, j\}$, $j \leq N - e - 1$, and a sequence of irreducible subvarieties $T_1, \ldots, T_j$ satisfy the conditions (i)-(iii).

Lemma 3.3. *There is a divisor D_i, $j + 1 \leq j \leq N$, such that $T_j \not\subset D_i$.*

Proof. Assume the converse: $T_j \subset D_i$ for all $i = j + 1, \ldots, N$. Then

$$T_j \subset D_{j+1} \cap \ldots \cap D_N.$$

Taking into consideration that by construction

$$T_j \subset D_1 \cap \ldots \cap D_j,$$

we get

$$T_j \subset D_1 \cap \ldots \cap D_N.$$

However $x \in T_j$ and the codimension of the subvariety T_j is equal to

$$e + j \leq N - 1,$$

which gives again (as in the proof of Lemma 3.2) a contradiction with what we assumed about the collection $D_1, \ldots, D_N$. Q.E.D. for the lemma.

After re-numbering we may assume that $T_j \not\subset D_{j+1}$. Now we argue as above:

$$\frac{\mathrm{mult}_x}{\deg_H} (T_j \circ D_{j+1}) \geq \frac{\mu_{j+1}}{a_{j+1}} \cdot \frac{\mathrm{mult}_x}{\deg_H} T_j$$

and therefore there is an irreducible component T_{j+1} of the effective cycle $(T_j \circ D_{j+1})$ that satisfies the inequality (28). Proof of Lemma 3.3 is complete.

Corollary 3.1. *The following inequality holds*

$$\lambda_e(x) \cdot \min_{\substack{\mathcal{L} \subset \{1, \ldots, N\} \\ \sharp \mathcal{L} = N - e}} \left(\prod_{i \in \mathcal{L}} \frac{\mu_i}{a_i} \right) \leq \lambda_N(x). \tag{30}$$

Proof. In the notations above

$$\lambda_N(x) \geq \frac{\mathrm{mult}_x}{\deg_H} T_N \geq \left(\prod_{i\in\mathcal{L}} \frac{\mu_i}{a_i}\right) \frac{\mathrm{mult}_x}{\deg_H} T.$$

Here $\mathcal{L} \subset \{1,\dots,N\}$ is a subset of cardinality $N-e$, which depends, generally speaking, on T. The more so,

$$\lambda_N(x) \geq \min_{\substack{\mathcal{L}\subset\{1,\dots,N\}\\ \sharp\mathcal{L}=N-e}} \left(\prod_{i\in\mathcal{L}} \frac{\mu_i}{a_i}\right) \cdot \frac{\mathrm{mult}_x}{\deg_H} T.$$

The first factor in the right-hand side does not depend on T. Since the variety $T\subset X$ is absolutely arbitrary, we get the inequality (30). Q.E.D. for the corollary.

Corollary 3.2. *Assume that the linear system $|H|$ is free and defines a (finite) morphism $\varphi_{|H|}\colon X\to\mathbb{P}^k$. Then the following estimate holds:*

$$\lambda_e(x) \leq \left(\min_{\substack{\mathcal{L}\subset\{1,\dots,N\}\\ \sharp\mathcal{L}=N-e}} \left(\prod_{i\in\mathcal{L}} \frac{\mu_i}{a_i}\right)\right)^{-1}. \tag{31}$$

Proof. For any irreducible subvariety $T\subset X$ of codimension $e\geq 1$ there exist divisors $D_i\in|H|$, $i=1,\dots,\dim T$, such that:

- $D_i \ni x$, in particular $\mathrm{mult}_x D_i \geq 1$;
- the intersection
$$T^\sharp = T\cap D_1\cap\ldots\cap D_{\dim T}$$
is zero-dimensional.

Obviously,
$$\deg T^\sharp = \deg T$$
and
$$\mathrm{mult}_x T^\sharp \geq \mathrm{mult}_x T\cdot \prod_{i=1}^{\dim T} \mathrm{mult}_x D_i \geq \mathrm{mult}_x T.$$
However, $T^\sharp$ is a zero-dimensional scheme, so that
$$\deg T^\sharp \geq \mathrm{mult}_x T^\sharp.$$
From this inequality we obtain that $\lambda_e(x)\leq 1$ for all x and e. Now applying the previous corollary we complete the proof.

3.2 Construction of hypertangent divisors

To realize the method described in Sec. 3.1 for the iterated double covers, let us first of all fix some notations. As above, we have a system of affine coordinates $(z_1, \dots, z_{M+k})$ with the origin at the point $p \in Q$. For $1 \leq i \leq k$, $1 \leq j \leq d_i - 1$ set

$$f_{i,j} = q_{i,1} + \ldots + q_{i,j},$$

$$D^{\mathbb{P}}_{i,j} = \overline{\{f_{i,j} = 0\}}$$

(the closure is taken in $\mathbb{P}$),

$$D^{Q}_{i,j} = D^{\mathbb{P}}_{i,j}|_Q,$$

$$D^{f}_{i,j} = \sigma^{-1}(D^{Q}_{i,j}).$$

Assume that the point $p \in Q$ is of class $e \geq 0$, and, moreover, if $e \geq 1$, then $p \in W_1 \cap \ldots \cap W_e$. Set

$$D^{+}_{i} = \overline{\{w_{i,1} = 0\}}|_Q,$$

$$D_i = \sigma^{-1}(D^{+}_{i}),$$

$i \in \{1, \dots, e\}$. Finally, for a point $x \in V$ such that $\sigma(x) = p$, let us define with respect to the coordinates u_i, $i \geq e+1$ (see Sec. 2.1), the following divisors:

$$D^{g}_{i,j} = \overline{\{u_i - [\sqrt{g_i}]_j = 0\}}|_V,$$

$$D^{+}_{i,j} = \sigma(D_{i,j}),$$

where $j = l_i, \dots, 2l_i - 1$. It is easy to see that

$$D^{?}_{i,j} \in |jH|, \quad D_i \in |H|$$

for any i, j, listed above, where $? \in \{f, g\}$.

Lemma 3.4. (i) *For any i, j the following inequality holds:*

$$\operatorname{mult}_x D_{i,j} \geq j + 1. \tag{32}$$

(ii) *For any $i \in \{1, \dots, e\}$, where $e \geq 1$, the following inequality holds:*

$$\operatorname{mult}_x D_i \geq 2.$$

Proof. (i) To begin with, take $1 \leq i \leq k$. Obviously,

$$f_{i,j}|_Q = (-q_{i,j+1} - \ldots - q_{i,d_i})|_Q,$$

since $f_i|_Q \equiv 0$, which implies the estimate (32). Now assume that $i \geq e+1$. By the definition of the class of a point, $g_i(p) \neq 0$. In the open affine subset $U \subset \mathbb{P}^{\sharp}$, $U \cong \mathbb{C}^{M+k+m}$ with coordinates (z_*, u_*) we get

$$[u_i^2 - g_i(1, z_1, \dots, z_{M+k})]|_V \equiv 0.$$

Since obviously

$$(y_i + [\sqrt{g_i}]_j)(x) \neq 0,$$

we obtain that locally the divisor $D_{i,j}$ is given by the equation

$$g_i^{(j)}|_V = 0.$$

As we have seen above, the first non-zero homogeneous component in $g_i^{(j)}$ is $h_{j+1}[g_i]$; it is of degree $j+1$.

This completes the proof of the first part of the lemma.

(ii) Assume that $e \geq 1$. The hyperplane $\{w_{i,1} = 0\}$ is tangent to the hypersurface W_i at the point p. Hence the divisor D_i is singular at the point x. This is what we need.

3.3 The regularity condition for hypertangent divisors

To apply the techniques of Sec. 3.1 it is not enough just to know the multiplicities of hypertangent divisors at the point x. We need more precise information about the tangent cones to these divisors. Set

$$E = T_p(Q \cap W_1 \cap \ldots \cap W_e), \quad U = \langle u_1, \ldots, u_e \rangle^*.$$

As we have seen above (Lemma 2.1), the following isomorphism holds

$$T_q V \cong E \oplus U. \tag{33}$$

The subspace $E \subset T_p\mathbb{P} = \mathbb{C}^{M+k}_{(z_*)}$ is given by the system of linear equations

$$q_{1,1} = \ldots = q_{k,1} = w_{1,1} = \ldots = w_{e,1} = 0.$$

The local computations performed in the proof of the previous lemma show that by the isomorphism (33) the tangent cones to the hypertangent divisors are given by the following equations:

- to the divisors $D^f_{i,j}$ —
$$q_{i,j+1}|_E = 0 \tag{34}$$
for $1 \leq i \leq k$;
- to the divisors $D^g_{i,j}$ —
$$h_{j+1}[g_i]|_E = 0 \tag{35}$$
for $i \geq e+1$;
- to the divisors D_i —
$$u_i^2 = w_{i,2}|_E \tag{36}$$
for $1 \leq i \leq e$, if $e \geq 1$.

Indeed, the equations (34) have been obtained above in Sec. 3.2. The equations (35) and (36) follow from the local computations made in Sec. 3.2 if one takes into consideration that any linear form $L(z_*)$ that vanish on E defines an element in the *square* of the maximal ideal $\mathcal{M}_{x,V}$ of the point x in the local ring $\mathcal{O}_{x,V}$:

$$\sigma^*(L(z_*)|_Q) \in \mathcal{M}^2_{x,V}.$$

Thus if a pair of homogeneous polynomials $P^+(z_*)$, $P^-(z_*)$ of degree $a \geq 1$ coincide on E, that is, $(P^+ - P^-)|_E \equiv 0$, then

$$\sigma^*(P^+(z_*)|_Q) \equiv \sigma^*(P^-(z_*)|_Q) \operatorname{mod} \mathcal{M}^{a+1}_{x,V}.$$

From here the equations (35) and (36) follow immediately.

Lemma 3.5. *The set $\mathcal{D} = \{D^f_{i,j}, D_i, D^g_{i,j}\}$ of all hypertangent divisors satisfies the regularity condition*

$$\operatorname{codim}_x \bigcap_{D\in\mathcal{D}} D = \sharp\mathcal{D}.$$

Proof. It is sufficient to compute the codimension

$$\operatorname{codim}_{T_xV} \bigcap_{D\in\mathcal{D}} T_xD. \tag{37}$$

Since $T_xV = U \oplus E$ and the coordinates u_i come into the equations (36) only — each in its own, the codimension (37) is equal to

$$\operatorname{codim}_E\{q_{i,j+1}|_E = 0,\ h_{j+1}[g_i] = 0\}, \tag{38}$$

where the indices i, j comprise the sets

$$\{1 \leq i \leq k,\ 1 \leq j \leq d_i - 1\}$$

and

$$\{e+1 \leq i \leq m,\ l_i + 1 \leq j \leq 2l_i\},$$

respectively. Adding the equations of the hyperplane E, we see that the codimension (37) is precisely the codimension in $\mathbb{C}^{M+k}$ of the set determined by all the polynomials that come into the regularity condition (15) or (16). The claim of Lemma 3.5 follows from this fact immediately.

3.4 The Lefschetz theorem once again

Assume that the point $p = \sigma(x)$ is of class $e = 0$.

Lemma 3.6. *The set-theoretic intersection*

$$T = D_{1,1} \cap \ldots \cap D_{k,1} \subset V$$

is of codimension k, coincides with the scheme-theoretic intersection

$$T=(D_{1,1}\circ\ldots\circ D_{k,1})$$

and satisfies the equalities

$$\deg T=\deg V,\quad \mathrm{mult}_x T=2^k.$$

Proof. Let show by induction on $i=1,\ldots,k$, that the set-theoretic intersection

$$T_i=\bigcap_{j=1}^{i} D_{j,1}\subset V$$

is of codimension i, coincides with the scheme-theoretic intersection:

$$T_i=(D_{1,1}\circ\ldots\circ D_{i,1})$$

and satisfies the equalities

$$\deg T_i=\deg V,\quad \mathrm{mult}_x T_i=2^i.$$

Indeed, for $i=1$ it is true in an obvious way: the tangent cone

$$T_xD_{1,1}\subset T_xV\cong T_pQ$$

is given by the quadratic equation

$$q_{1,2}|_{\{q_{1,1}=\ldots=q_{k,1}=0\}}=0,$$

which is non-trivial by the regularity condition. To come over from i to $i+1$, let us use the Lefschetz theorem: by the regularity condition the set of common zeros of the system of equations

$$q_{1,1}=\ldots=q_{k,1}=q_{1,2}=\ldots=q_{i,2}=q_{i+1,2}=0 \tag{39}$$

is of codimension precisely $i+1$ in T_xV. Hence

$$T_xT_i\not\subset T_xD_{i+1,1}$$

and thus

$$T_i\not\subset D_{i+1,1}.$$

But $T_i\subset V$ is an irreducible subvariety of codimension i. Therefore the scheme-theoretic intersection

$$T^+_{i+1}=(T_i\circ D_{i+1,1})$$

is an effective cycle of codimension $i+1$. However,

$$\deg T_i=\deg T^+_{i+1}=\deg V$$

and $i+1 < \dim V/2$, so by the Lefschetz theorem we get that $T^+_{i+1} = T_{i+1}$ is an irreducible subvariety, the class of which generates $A^{i+1}V$. Finally, the system of equations (39) gives precisely the tangent cone $T_x T_{i+1}$ (as an effective algebraic cycle, that is, respecting the multiplicities of the components). Thus we get

$$\operatorname{mult}_x T_{i+1} = 2^{i+1},$$

which is what we need.

Definition 3.1. An irreducible subvariety $x \in Y \subset V$ of codimension 2 is said to be *correct at the point* x (where $p = \sigma(x)$ is a point of class 0), if there exists an irreducible subvariety $R \subset Y$ of codimension

$$\operatorname{codim}_V R = k+1,$$

satisfying the inequality

$$\frac{\operatorname{mult}_x}{\deg} R \geq 2^{k-1} \cdot \frac{\operatorname{mult}_x}{\deg} Y. \tag{40}$$

Lemma 3.7. *If the subvariety $Y \ni x$ is not correct at the point x, then the following estimate holds:*

$$\frac{\operatorname{mult}_x}{\deg} Y \leq \frac{4}{\deg V}.$$

Proof. Let us apply Lemma 3.1 to the irreducible subvariety Y and the set of hyperplane sections $\{D_{i,1}\}$. We obtain that there exists an irreducible subvariety $R^\sharp \subset Y$ of codimension k (with respect to V) such that

$$\frac{\operatorname{mult}_x}{\deg} R^\sharp \geq 2^{k-2} \cdot \frac{\operatorname{mult}_x}{\deg} Y.$$

Two cases are now possible. If

$$R^\sharp \neq T = D_{1,1} \cap \ldots \cap D_{k,1},$$

then there is a hyperplane section $D_{a,1}$, $1 \leq a \leq k$, which does not contain $R^\sharp$. Thus

$$\operatorname{codim}_V (R^\sharp \cap D_{a,1}) = k+1,$$

so that $(R^\sharp \circ D_{a,1})$ is an effective cycle of codimension $k+1$, satisfying the inequality

$$\frac{\operatorname{mult}_x}{\deg} (R^\sharp \circ D_{a,1}) \geq 2^{k-1} \cdot \frac{\operatorname{mult}_x}{\deg} Y.$$

Therefore, there is an irreducible component R of the effective cycle $(R^\sharp \circ D_{a,1})$, satisfying the inequality (40). Therefore, Y is a correct subvariety, which contradicts our assumption.

Thus $R^\sharp = T = D_{1,1} \cap \ldots \cap D_{k,1}$. Consequently, by Lemma 3.6 we get the inequality

$$\frac{\operatorname{mult}_x}{\deg} Y \leq 2^{2-k} \cdot \frac{\operatorname{mult}_x}{\deg} T = \frac{4}{\deg V},$$

as we have claimed. Q.E.D. for the lemma.

3.5 The points of class $e = 0$

Let us prove, at long last, the estimate (2) for a point $x \in V$ such that its image $p = \sigma(x) \in \mathbb{P}$ is of class 0. In order to do this, we apply the construction of Lemma 3.1 to an arbitrary subvariety $R \subset V$ of codimension $k+1$, $R \ni x$, and to the set of $M-1$ divisors, which we get putting together the f-collection

$$\{D^f_{i,j} \,|\, 1 \leq i \leq k, 1 \leq j \leq d_i - 1\}$$

and the g-collection

$$\{D^g_{i,j} \,|\, 1 \leq i \leq m, l_i \leq j \leq 2l_i - 1, (i,j) \neq (m, 2l_m - 1)\}.$$

Since the linear system $|H|$ is by construction free (and defines precisely the cover $\varphi_{|H|} = \sigma\colon V \to \mathbb{P}$), we get by Corollary 3.2 (inequality (31))

$$\lambda_{k+1}(x) \leq \left(\prod_{i=1}^{k}\prod_{j=2}^{d_i-1}\frac{j+1}{j}\right)^{-1} \cdot \left(\prod_{i=1}^{m}\prod_{j=l_i}^{2l_i-1}\frac{j+1}{j}\right)^{-1} \cdot \frac{2l_m}{2l_m-1} \cdot \frac{a+1}{a}, \tag{41}$$

where $a = 2$, if $\max\limits_i\{d_i\} \geq 3$, and $a = \min\limits_i\{l_i\}$ in the opposite case. Indeed, to apply the operation min in the inequality (31) means in the notations of Corollary 3.2 to delete from the product

$$\prod_{i=1}^{N}\frac{\mu_i}{a_i}$$

precisely the e highest factors. In our case these factors obviously are k twos (corresponding to the tangent hyperplane sections $D^f_{i,1}$) and the next factor $(a+1)/a$. The factor $2l_m/(2l_m - 1)$ comes into (41) simply because the divisor $D^g_{m,2l_m-1}$ is not present in our collection. Making in (41) the obvious cancellations, we obtain

$$\lambda_{k+1}(x) \leq \frac{2^k}{\deg V} \cdot \frac{2l_m}{2l_m-1} \cdot \frac{3}{2},$$

since in any case $a \geq 2$. Finally for the correct subvariety Y we get:

$$\frac{\mathrm{mult}_x}{\deg} Y \leq 2^{1-k} \cdot \frac{\mathrm{mult}_x}{\deg} R \leq$$

$$\leq \frac{4}{\deg V} \cdot \frac{3l_m}{4l_m - 2} \leq \frac{4}{\deg V},$$

which is what we need. Proof of the crucial inequality (2) for a point $x \in V$ such that the point $p = \sigma(x) \in Q$ is of class 0 is complete.

3.6 The points of class $e \geq 1$

Now assume that for the point $x \in V$ the point $p = \sigma(x) \in Q$ is of class $e \geq 1$, that is,

$$p = \sigma(x) \in Q \cap W_1 \cap \ldots \cap W_e.$$

By the regularity condition the set-theoretic intersection

$$\left(\bigcap_{i,j} D^f_{i,j}\right) \cap \left(\bigcap_i D_i\right) \cap \left(\bigcap_{i,j} D^g_{i,j}\right)$$

is in a neighborhood of the point x of the correct codimension

$$\sum_{i=1}^{k}(d_i - 1) + e + \sum_{i=e+1}^{m} l_i \geq 1$$

(the codimension is taken with respect to V). Now let us apply Corollary 3.2 and obtain an upper bound for $\lambda_2(x)$. Taking into consideration that

$$\min_{\substack{\mathcal{L}\subset\{1,\ldots,N\}\\ \sharp\mathcal{L}=N-c}} \prod_{i\in\mathcal{L}} \beta_i \times \max_{\substack{\mathcal{L}\subset\{1,\ldots,N\}\\ \sharp\mathcal{L}=c}} \prod_{i\in\mathcal{L}} \beta_i = \prod_{i=1}^{N} \beta_i,$$

we get

$$\begin{array}{rcl} \lambda_2(x) & \leq & \left(\displaystyle\prod_{i=1}^{k}\prod_{j=1}^{d_i-1} \frac{j+1}{j}\right)^{-1} \cdot 2^{-e} \cdot \left(\displaystyle\prod_{i=e+1}^{m}\prod_{j=l_i}^{2l_i-1} \frac{j+1}{j}\right)^{-1} \cdot 4 \\ & & \qquad\qquad\qquad\qquad \| \\ & & \qquad\qquad\qquad\qquad \dfrac{4}{\deg V}, \end{array}$$

which is what we need. Proof of the crucial estimate (2) (and therefore of our theorem) is complete.

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