On the set-theoretic complete intersection problem for monomial curves in A^n and P^n

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Abstract

In this paper we deal with the problem of the expression of monomial curves in the affine or projective n-dimensional space as set-theoretic complete intersections. We develop two techniques for finding monomial curves which are set-theoretic complete intersections. Using these two techniques we are able to generalize all previous known results and give infinitely many examples of monomial curves which are set-theoretic complete intersections in an affine or projective n-dimensional space, for any n.

1 Introduction

Let K be a field of characteristic zero and $m_1 < m_2 < \cdots < m_n$ be positive integers, the g.c.d. of which equals 1. By an *affine monomial curve* $C = C(m_1, m_2, \cdots, m_n)$ we mean a curve with generic zero $(t^{m_1}, t^{m_2}, \cdots, t^{m_n})$ in the affine n-dimensional space A^n, over the field K. By a *projective monomial curve* we mean a curve with generic zero

$$(u^{m_n}, u^{m_n - m_1} v^{m_1}, \cdots, v^{m_n})$$

in the projective n-dimensional space P^n, over the field K.

An ideal $\mathbf{I}$ in a Noetherian ring R is called a *set-theoretic complete intersection* (s.t.c.i., for short), if there are $s = height(\mathbf{I})$ elements $f_1, f_2, \cdots, f_s \in \mathbf{I}$, such that $Rad(\mathbf{I}) = Rad(f_1, f_2, \cdots, f_s)$. In particular, a curve C in A^n or in P^n, is called a *set-theoretic complete intersection* if its defining ideal $\mathbf{I}(C)$ is generated by $n-1$ elements up to radical.

The general problem of whether all monomial curves are set-theoretic complete intersections is still open. There are nevertheless some partial results in this direction.

It is well known that:

(i) all monomial curves in A^3 are s.t.c.i. (see [1], [11], [21]),

(ii) in A^4, if the numerical semigroup $< m_1, m_2, m_3, m_4 >$ is symmetric, then the monomial curve $C(m_1, m_2, m_3, m_4)$ is a s.t.c.i. (see [2], [7]),

(iii) in A^n, if $n-1$ numbers among $m_1, m_2, \cdots, m_n$ form an arithmetic sequence, then $C(m_1, m_2, \cdots, m_n)$ is a s.t.c.i. (see [15]).

Correspondingly, in the projective space case, it is known that:

(iv) the arithmetically Cohen-Macaulay curves are s.t.c.i. in P^3 (see [16], [17], [18]) and

(v) the rational normal curves are s.t.c.i. in P^n, for any n (see [17], [22]).

The purpose of this paper is to develop two techniques, which are applied to prove that a large number of monomial curves in A^n or P^n, are s.t.c.i.. The first technique starts with a projective monomial curve which is a s.t.c.i. and produces infinitely many monomial curves that are s.t.c.i. in affine or projective space of one dimension higher. This technique has an inductive power. The second technique starts from an affine curve which is a s.t.c.i. and produces infinitely many affine monomial curves in the same space, which are set-theoretic complete intersections. Using this technique we can generalize all previous known results in the affine space and the results that we are getting using the first technique. Note also that, in both techniques, we get explicitly the defining equations of the curves, provided that we know the defining equations of the first curve.

The first technique will be developed in sections 2 and 3, and the second in section 4.

2 The main Theorem

In this section we associate to each monomial curve in A^n a projective monomial curve in P^{n-1}. The monomial curve $C = C(m_1, m_2, \cdots, m_n)$ is associated to the projective monomial curve $P(C)$ with generic zero

$$(u^{(m_n-m_1)^*}, \cdots, u^{(m_n-m_i)^*}v^{(m_i-m_1)^*}, \cdots, v^{(m_n-m_1)^*})$$

of P^{n-1}, where $(m_i - m_j)^*$ are the numbers $(m_i - m_j)$ divided by their g.c.d.. From now on we shall denote this g.c.d. by g.

Theorem 2.1. *Let C be a monomial curve $C(m_1, m_2, \cdots, m_n)$ in A^n, such that:*
a) $P(C)$ is set theoretic complete intersection and
b) a power of the binomial $(u^{m_n} - v^{m_1})$ belongs to the ring

$$K[u^{(m_n-m_1)^*}, \cdots, u^{(m_n-m_i)^*}v^{(m_i-m_1)^*}, \cdots, v^{(m_n-m_1)^*}].$$

Then:
(i) C is set theoretic complete intersection,
(ii) The projective closure $\bar{C}$ of C is set theoretic complete intersection, and
(iii) The affine counterpart of C, $C(m_n, m_n - m_1, \cdots, m_n - m_{n-1})$, is set theoretic complete intersection.

Proof. (i) From condition (a) we deduce that there exist homogeneous polynomials $F_1, F_2, \cdots, F_{n-2}$, such that

$$\mathbf{I}(P(C)) = rad(F_1, F_2, \cdots, F_{n-2}).$$

Let f be the ring homomorphism from $K[X_1, X_2, \cdots, X_n]$ to $K[u, v]$, given by $f(X_i) = u^{(m_n-m_i)^*}v^{(m_i-m_1)^*}$, $i = 1, \cdots, n$.
The kernel of f is the defining ideal $\mathbf{I}(P(C))$ of the projective monomial curve $P(C)$. According to condition (b), there exists at least one polynomial F_{n-1}, such that $f(F_{n-1}) = (u^{m_n} - v^{m_1})^r$ for some r.

We claim that C is the s.t.c.i. of $F_1, F_2, \cdots, F_{n-1}$.

We shall prove the claim by considering a common zero of $F_1, F_2, \cdots, F_{n-1}$, say $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ in $\bar{K}^n$, where $\bar{K}$ denotes the algebraic closure of K. This $\mathbf{x}$ is also a common zero of $F_1, \cdots, F_{n-2}$. Therefore it can be written as

$$\mathbf{x} = (u^{(m_n-m_1)^*}, \cdots, u^{(m_n-m_i)^*}v^{(m_i-m_1)^*}, \cdots, v^{(m_n-m_1)^*})$$

for some u,v in $\bar{K}$. Furthermore, since $\mathbf{x}$ is a zero of F_{n-1} too, it is $f(F_{n-1}(\mathbf{x})) = (u^{m_n} - v^{m_1})^r = 0$. Thus u and v satisfy the relation $u^{m_n} = v^{m_1}$. Setting $u = t^{m_1}$, we get $v = \omega t^{m_n}$, where ω is a m_1-root of unity. The g.c.d. of g and m_1 is 1, which means that there exist k and l, such that k is positive and $kg + lm_1 = 1$. Replacing the values of u and v in $\mathbf{x}$, we get $\mathbf{x} = (T^{m_1}, T^{m_2}, \cdots, T^{m_n})$, where $T = \omega^k t^{(m_n - m_1)^*}$.

(ii) We claim that the projective closure

$$\tilde{C} = (u^{m_n}, u^{m_n - m_1} v^{m_1}, \cdots, u^{m_n - m_{n-1}} v^{m_{n-1}}, v^{m_n})$$

of C is the s.t.c.i. of $F_1, F_2, \cdots, F_{n-1}^h$, where F^h is equal to

$$X_0^{degF} F(X_1/X_0, \cdots, X_n/X_0).$$

Note that $F_1, \cdots, F_{n-2}$ are already homogeneous.

To prove the claim we only need to show that, for $X_0 = 0$, the above polynomials have only one common zero, namely the point $(0, 0, \cdots, 0, 1)$, because we know from (i) that C is the s.t.c.i. of $F_1, F_2, \cdots, F_{n-1}$.

The highest degree term in F_{n-1} is $X_1^{m_n}$. Therefore, by setting $X_0 = 0$ in F_{n-1}^h, we get $X_1 = 0$. But the common zeros of $F_1, \cdots, F_{n-2}$ are in the form

$$(u^{(m_n - m_1)^*}, \cdots, u^{(m_n - m_i)^*} v^{(m_i - m_1)^*}, \cdots, v^{(m_n - m_1)^*})$$

and thus, if $X_1 = 0$, $u = 0$ and $X_2 = X_3 = \cdots = X_{n-1} = 0$.
(iii) It follows from (ii).

Example 2.2 In [9] S. Eliahou proves that the curve (t^4, t^6, t^7, t^9) is s.t.c.i. in A_C^4, by using a refinement of Cowsik's Lemma for monomial curves, which says that: if the symbolic Rees algebra of a monomial curve C is finitely generated, then C is globally a s.t.c.i. (see [5], [8]).

Using Theorem 2.1, we see that the curve (t^4, t^6, t^7, t^9) is associated to the curve $(u^5, u^3v^2, u^2v^3, v^5)$ and the binomial $(u^9 - v^4)$. The curve is arithmetically Cohen-Macaulay and it is therefore the s.t.c.i. of $X_2^3 - X_1X_3^2$ and

$$X_3^5 - 3X_2^2X_3^2X_4 + 3X_1X_2X_3X_4^2 - X_1^2X_4^3.$$

On the other hand, $(u^9 - v^4)^5$ belongs to $K[u^5, u^3v^2, u^2v^3, v^5]$. Therefore, the Eliahou's curve is the s.t.c.i. of $X_2^3 - X_1X_3^2$, $X_3^5 - 3X_2^2X_3^2X_4 + 3X_1X_2X_3X_4^2 - X_1^2X_4^3$ and

$$X_1^9 - 5X_1^6X_2^2 + 10X_1^4X_2X_3^2 - 10X_1^2X_3^4 + 5X_2^3X_4^2 - X_4^4.$$

From Theorem 2.1 we conclude also that the projective closure of the Eliahou's curve, $(u^9, u^5v^4, u^3v^6, u^2v^7, v^9)$, and the affine counterpart, (t^2, t^3, t^5, t^9), are s.t.c.i..

Example 2.3. Let C be the monomial curve (t, t^3, t^4). C is the s.t.c.i. of $X_2 - X_1^3$ and $X_3 - X_1^4$. By the method introduced in this section, C is associated with the projective monomial curve (u^3, uv^2, v^3) and the binomial $(u^4 - v)$. No power of $(u^4 - v)$ belongs to $K[u^3, uv^2, v^3]$, since 1 does not belong to the semigroup generated by 2 and 3. Hence, by the use of Theorem 2.1, we can not prove that (t, t^3, t^4) is s.t.c.i.. But that was not unexpected, since the projective closure of C is (u^4, u^3v, uv^3, v^4), which is not a bihomogeneous s.t.c.i. (see [19], [20]).

Example 2.4 In the case, in which the characteristic p of our underlying field K is positive, all affine (see [6]) and projective (see [4], [10], [13], [17]) monomial curves are s.t.c.i.. We can use Theorem 2.1 to give an easy proof of these two results.
Choosing the power of the binomial in the condition (b) of Theorem 2.1 to be $p^k(m_n - m_1)^*$, where k is big enough, we see that condition (b) remains always true.

In the projective plane all (monomial) curves are s.t.c.i.. Therefore, Theorem 2.1 automatically proves that all monomial curves are s.t.c.i. in A^3 and P^3 and, of course, inductively, that all monomial curves are s.t.c.i. in A^n and P^n.

3 The first technique

In this section we describe the first technique which is based on Theorem 2.1 and produces infinitely many examples of monomial curves, which are set-theoretic complete intersections in A^n or P^n, for every n. Let us first introduce some terminology that will help us to give a better expression of the condition (b) in Theorem 2.1.

Let N denote the set of nonnegative integers and e_i denote the tuple $((m_n - m_i)^*, (m_i - m_1)^*)$. Let S be the semigroup in N^2 generated by the set $\{e_i | i = 1, \cdots, n\}$, S_1 the numerical semigroup generated by the set $\{(m_i - m_1)^* | i = 1, \cdots, n\}$, S_n the numerical semigroup generated by the set $\{(m_n - m_i)^* | i = 1, \cdots, n\}$, $W^l := \{w_i^l = ((l-i)m_n, im_1) | i = 1, \cdots, n\}$,

$$H := \{(\alpha, \beta) \in N^2 | \alpha + \beta \equiv 0 mod (m_n - m_1)^*\},$$

and $S' := \{e \in H | e + p_1 e_1 \in S$ and $e + p_n e_n \in S$ for some $p_1, p_n \in N\}$. If $x \in N^2$, we write $x = ([x]_1, [x]_2)$. For an element $e \in H$, the number $\delta(e) = ([e]_1 + [e]_2)/(m_n - m_1)^*$ will be called *degree* of e . For an element $s \in S_1$, the number $\delta_{S_1}(s) = min\{\sum_{i=1}^n \alpha_i | s = \sum_{i=1}^n \alpha_i (m_i - m_1)^*\}$ will be called *degree* of s with respect to S_1. Respectively one defines the degree of an element in S_n.

Finally we define $\epsilon(m_1) := max\{\delta_{S_1}(rm_1) | 1 \leq r \leq (m_n - m_1)^*\}$and $\epsilon(m_n) := max\{\delta_{S_n}(rm_n) | 1 \leq r \leq (m_n - m_1)^*\}$.

Lemma 3.1. [3], [12] *Let $e \in S'$ and $e \neq (0,0)$. Then the following three conditions are equivalent:*

(i) $\quad e \in S$.

(ii) $\quad \delta(e) \geq \delta_{S_1}([e]_2)$,

(iii) $\quad \delta(e) \geq \delta_{S_n}([e]_1)$.

Proposition 3.2. *If $l = h(m_n - m_1)^*$, then $W^l \subset H$.*

Proof. If $1 \leq i \leq n$, then $w_i^l = ((l-i)m_n, im_1)$ and $(l-i)m_n + im_1 = lm_n - i(m_n - m_1) = (hm_n - ig)(m_n - m_1)^*$. Therefore $W^l \subset H$.

Proposition 3.3. *The condition (b) of Theorem 2.1 and the following two conditions are equivalent.*

(b_1) $\quad W^l \subset S$ *for some* l

(b_2) $\quad m_1 \in S_1$ *and* $m_n \in S_n$.

Proof. Rephrasing condition (b) in terms of the previous terminology we get condition (b_1). So, in fact, we only have to prove that conditions (b_1),(b_2) are equivalent.

Suppose that (b_1) is true. Then $((l-1)m_n, m_1)$ is an element of S and $m_1 \in S_1$. Similarly we get $m_n \in S_n$.

Suppose that (b_2) is true. Set $l = h(m_n - m_1)^*$, where

$$h \geq max\{2\epsilon(m_1)/m_n, 2\epsilon(m_n)/m_1\}.$$

According to proposition 3.2, w_i^l belongs to H. Since $m_1 \in S_1$and $m_n \in S_n$, there exist a_j, b_j, such that $m_1 = \sum_{j=1}^{n} a_j(m_j - m_1)^*$ and

$$m_n = \sum_{j=1}^{n} b_j(m_n - m_j)^*,$$

which means that w_i^l can be written as

$$w_i^l = z_i e_1 + \sum_{j=1}^{n} a_j e_j = y_i e_n + \sum_{j=1}^{n} b_j e_j.$$

If one of z_i, y_i is nonnegative, then $w_i^l \in S$ and therefore $w_i^l \in S'$. If both of them are negative, then $w_i^l - z_i e_1 = \sum_{j=1}^{n} a_j e_j \in S$,$w_i^l - y_i e_n = \sum_{j=1}^{n} b_j e_j \in S$ and $-z_i, -y_i \in N$. Thus $w_i^l \in S'$.

We have to distinguish between two cases. First consider the case when $l/2 \geq i$ and write $i = q_i(m_n - m_1)^* + r_i$, where $0 \leq r_i \leq (m_n - m_1)^*$.

We have $\delta(w_i^l) = hm_n - ig \geq hm_n - h(m_n - m_1)^* g/2 = hm_n/2 + hm_1/2 \geq \epsilon(m_1) + q_i m_1 \geq \delta_{S_1}(r_i m_1) + q_i m_1 = \delta_{S_1}(im_1) = \delta_{S_1}([w_i^l]_2)$. Hence, from the Lemma 3.1, we conclude that $w_i^l \in S$.

The second case, i.e. when $l/2 \leq i$, can be treated similarly.

Combining Theorem 2.1 and the Proposition 3.3 we get the following Theorem:

Theorem 3.4. *If $C = (u^d, u^{a_1}v^{b_1}, \cdots, v^d)$ is a projective monomial curve in P^n, which is set-theoretic complete intersection, then*
(i) for any k, l, such that $k \in < b_1, \cdots, b_{n-1}, d >$ and $k+ld \in < a_1, \cdots, a_{n-1}, d >$, the affine monomial curves $(t^k, t^{k+lb_1}, \cdots, t^{k+ld})$, $(t^{la_{n-1}}, \cdots, t^{la_1}, t^{ld}, t^{k+ld})$ as well as their projective closure

$$(u^{k+ld}, u^{ld}v^k, u^{la_1}v^{k+lb_1}, \cdots, u^{la_{n-1}}v^{k+lb_{n-1}}, v^{k+ld})$$

are set-theoretic complete intersections, and
(ii) for any k, l, such that $k \in < a_1, \cdots, a_{n-1}, d >$ and $k+ld \in < b_1, \cdots, b_{n-1}, d >$, the affine monomial curves $(t^k, t^{k+la_{n-1}}, \cdots, t^{k+ld})$, $(t^{lb_1}, \cdots, t^{lb_{n-1}}, t^{ld}, t^{k+ld})$ as well as their projective closure

$$(u^{k+ld}, u^{ld}v^k, u^{lb_{n-1}}v^{k+la_{n-1}}, \cdots, u^{lb_1}v^{k+la_1}, v^{k+ld})$$

are set-theoretic complete intersections.

Starting from any given example of a projective monomial curve, which is s.t.c.i., and using Theorem 3.4, we can get infinitely many examples of affine and projective monomial curves, which are s.t.c.i. in the affine or projective space ofone dimension higher. For instance, one could consider as starting point of this procedure any monomial curve in P^2 or any arithmetically Cohen-Macaulay monomial curve in P^3.

Example 3.5. In this example we are going to start from the arithmetically Cohen-Macaulay monomial curve (u^7, u^6v, u^4v^3, v^7) in P^3 and use Theorem 3.4 to deduce that certain monomial curves in A^4 or P^4 are s.t.c.i.. This curve is the s.t.c.i. of $X_2^3 - X_1^2X_3$ and $X_3^7 - 3X_2^2X_3^4X_4 + 3X_1^2X_2X_3^2X_4^2 - X_1^4X_4^3$, (see [16] ,[17] ,[18]). Theorem 3.4 says that for any k, l, such that $k \in < 1, 3, 7 >$ and $k + 7l \in < 4, 6, 7 >$, the affine monomial curve $(t^k, t^{k+l}, t^{k+3l}, t^{k+7l})$ and its projective closure

$$(u^{k+7l}, u^{7l}v^k, u^{6l}v^{k+l}, u^{4l}v^{k+3l}, v^{k+7l})$$

are s.t.c.i..
The only values of k and l, for which this does not happen, are $k = 2$ and $l = 1$. These lead to the curve (t^2, t^3, t^5, t^9). But even this curve as well as its projective closure $(u^9, u^7v^2, u^6v^3, u^4v^5, v^9)$, are s.t.c.i., according to our previous Example 2.2.

Summarising our results, we see that all affine monomial curves of the form $(t^k, t^{k+l}, t^{k+3l}, t^{k+7l})$ and all projective monomial curves of the form

$$(u^{k+7l}, u^{7l}v^k, u^{6l}v^{k+l}, u^{4l}v^{k+3l}, v^{k+7l})$$

are indeed s.t.c.i..
We should also mention, that for all, up to the last one, of the curves of the above type, two of the three defining equations are exactly the same.

Example 3.6. In this example we prove that all normal curves are s.t.c.i. (see [17] ,[22]). We are going to prove it by induction. For P^2 the proof is obvious. Suppose we know that $(u^n, u^{n-1}v, \cdots, v^n)$ is s.t.c.i.. Then, applying Theorem 3.4 to this curve, we conclude that $(u^{n+1}, u^nv, \cdots, uv^n, v^{n+1})$ is s.t.c.i., since both 1 and $n + 1$ belong to the semigroup $< 1, 2, \cdots, n >$.

4 The second technique

In this section we first prove a Lemma concerning the arithmetical ranks of certain types of monomial curves. Based on this Lemma, we develop a second technique by means of which one can find infinitely many examples of set-theoretic complete intersection monomial curves in A^n, just starting from a given one.
Consider an affine monomial curve $C = C(m_1, \cdots, m_n)$ and the monomial curve $C_{(a,i)} = C(am_1, \cdots, am_{i-1}, m_i, am_{i+1}, \cdots, am_n)$, where $g.c.d.(a, m_i) = 1$. Note that, in this section, we do not assume necessarily that $m_1 < m_2 < \cdots < m_n$ holds.

If f is an element of $K[X_1, X_2, \cdots, X_n]$, we denote by $f_{(a,i)}$ the polynomial $f(X_1, X_2, \cdots, X_i^a, \cdots, X_n)$.

Let R be the polynomial ring $K[X_1, \cdots, X_n]$, S the graded polynomial ring $K[X_0, X_1, \cdots, X_n]$ with the usual grading, R_C the polynomial ring R with the σ-grading: $\sigma - deg(X_j) = m_j$, and S_C the ring S with two gradings, namely: the usual grading and the σ-grading: $\sigma - deg(X_0) = 0, \sigma - deg(X_i) = m_i$.

If I is an ideal of R (respectively of S), we will write $rad(I)$ for the radical of I. The arithmetical rank of I, written $ara(I)$, is the smallest integer s for which there exist elements (resp. homogeneous elements) $f_1, f_2, \cdots, f_s$ in I, such that $rad(I) = rad(f_1, f_2, \cdots, f_s)$.

If I is an ideal of R_C (respectively of S_C), then the homogeneous arithmetical rank of I, written $arah(I)$, is the smallest integer s for which there exist σ-homogeneous elements (resp. bihomogeneous elements) $f_1, f_2, \cdots, f_s$ in I, such that $rad(I) = rad(f_1, f_2, \cdots, f_s)$.

If I is the ideal of a monomial curve, we shall use the notation $ara(C)$ instead of $ara(\mathbf{I}(C))$ and $arah(C)$ instead of $arah(\mathbf{I}(C))$.

Lemma 4.1. *Let C be a monomial curve in the affine n-space. The arithmetical ranks of C and $C_{(a,i)}$ are related by the following inequalities:*
(i) $ara(C) \leq arah(C)$
(ii) $ara(C_{(a,i)}) \leq ara(C)$
(iii) $ara(C) \leq arah(C_{(a,i)})$

Proof. (i) It follows from the definitions.
(ii) Suppose that $ara(C) = s$, i.e. that there exist $f_1, f_2, \cdots, f_s$, such that

$\mathbf{I}(C) = rad(f_1, f_2, \cdots, f_s)$. Then we claim that

$$\mathbf{I}(C_{(a,i)}) = rad(f_{1(a,i)}, \cdots, f_{s(a,i)}).$$

Let h be an element of $\mathbf{I}(C_{(a,i)}) \subset R_{C(a,i)}$. This element h can be written in the form $h = \tilde{h}_0 + X_i\tilde{h}_1 + \cdots + X_i^{a-1}\tilde{h}_{a-1}$, in which the exponents of the variable X_i in the polynomials $\tilde{h}_j$ are multiples of a. Note that the σ-homogeneous terms in $X_i^j\tilde{h}_j$ have different σ-degrees from those in $X_i^t\tilde{h}_t$, when $j \neq t$, because the σ-degree of $\tilde{h}_j$ is a multiple of a, for every j. Therefore, each $X_i^j\tilde{h}_j$ belongs to $\mathbf{I}(C_{(a,i)})$. Note that $\mathbf{I}(C_{(a,i)})$ is a prime ideal and $X_i \notin \mathbf{I}(C_{(a,i)})$. Thus, each $\tilde{h}_j$ belongs to $\mathbf{I}(C_{(a,i)})$ and, for each j, there exists an h_j in $\mathbf{I}(C)$, such that $\tilde{h}_j = h_{j(a,i)}$. Now h_j belongs to $rad(f_1, f_2, \cdots, f_s)$. This means, that each $\tilde{h}_j$ belongs to $rad(f_{(a,i)1}, \cdots, f_{(a,i)s})$. Hence the last inclusion is also valid for the whole h.
(iii) Suppose that $arah(C_{(a,i)}) = s$, i.e. that there exist $\tilde{g}_1, \cdots, \tilde{g}_s$, such that $\mathbf{I}(C_{(a,i)}) = rad(\tilde{g}_1, \cdots, \tilde{g}_s)$. Since $\tilde{g}_1, \cdots, \tilde{g}_s$ are σ-homogeneous and none of them has X_i as a factor, there are elements $g_1, \cdots, g_s$ of $\mathbf{I}(C)$, such that $g_{j(a,i)} = \tilde{g}_j$.

We claim that $\mathbf{I}(C) = rad(g_1, \cdots, g_s)$. Let g be a σ-homogeneous element of $\mathbf{I}(C)$. Then $g_{(a,i)}$ belongs to $\mathbf{I}(C_{(a,i)}) = rad(\tilde{g}_1, \cdots, \tilde{g}_s)$. This means, that there exist an m, such that $(g_{(a,i)})^m = A_1\tilde{g}_1 + \cdots + A_s\tilde{g}_s$. On the other hand, $(g_{(a,i)})^m, \tilde{g}_1, \cdots, \tilde{g}_s$ are all σ-homogeneous. Therefore, by forgetting the terms T occuring in A_j, for which $\sigma - degree(T) + \sigma - degree(\tilde{g}_j) \neq \sigma - degree(g_{(a,i)})^m$, the above equality still holds and the new A_j turn out to be σ-homogeneous. Note firstly that the $\sigma - degree(g_{(a,i)})$ is a multiple of a, since it is the image of a σ-homogeneous element of $\mathbf{I}(C)$, and secondly that the σ-degrees of $\tilde{g}_1, \cdots, \tilde{g}_s$ are multiples of a, since they are σ-homogeneous and none of them has X_i as factor. Hence, there exist $B_1, \cdots, B_s$, such that $g^m = B_1g_1 + \cdots + B_sg_s$.

The proof follows from the fact that every element in $\mathbf{I}(C)$ is a sum of σ-homogeneous elements.

For more informations about the ideals of C and $C_{(a,i)}$ we refer to [14].

Remark 4.2. We do not know any example of an affine or projective monomial curve, for which $ara(C)$ is different from $arah(C)$. We nevertheless know some monomial curves in P^3, like the *Macaulay curve* (u^4, u^3v, uv^3, v^4), which have $arah(C) = 3$ (not σ-homogeneous s.t.c.i.), but it is still uknown if $ara(C)$ is exactly 2 or exactly 3, (see [20]).

Theorem 4.3. *Let C be a monomial curve in the affine n-dimensional space.*

(i) If C is a set theoretic complete intersection, then $C_{(a,i)}$ is a set theoretic complete intersection, for every a, i, such that $1 \leq i \leq n$ and $g.c.d.(a, m_i) = 1$.

(ii) If $C_{(a,i)}$ is a σ-homogeneous set theoretic complete intersection, then C is a set theoretic complete intersection.

Proof. The proof follows from Lemma 4.1 and the generalized Krull's Principal Ideal Theorem.

Definition 4.4 *A monomial curve $C(m_1, m_2, \cdots, m_n)$ is called* minimal *if*

$$g.c.d.(m_1, m_2, \cdots, \hat{m}_i, \cdots, m_n) = 1$$

for any $1 \leq i \leq n$.

By a series of applications of operations $T_{(a,i)} : C \to C_{(a,i)}$, we can obtain any monomial curve starting from a minimal one. We shall say that two monomial curves belong to *the same class*, if both can be obtained from the same minimal curve.

Theorem 4.5. *If a monomial curve C is σ-homogeneous set theoretic complete intersection, then every monomial curve within the class of C is also set-theoretic complete intersection.*

Proof. The proof follows immediately from Theorem 4.3.

Example 4.6. Theorem 4.5 generalizes all known results in the affine space (see Introduction), as well as the results we get from sections 2 and 3. And this is due to the fact, that all the cases which are known to be s.t.c.i., are also σ-homogeneous s.t.c.i. For instance, D. Patil proves in [15], that any affine monomial curve $C(m_1, m_2, \cdots, m_n)$, for which $n - 1$ numbers among $m_1, m_2, \cdots, m_n$ form an arithmetic sequence, is s.t.c.i. Using the Theorem 4.5 we can generalize this result:
Every monomial curve within the class of a monomial curve for which $n - 1$ numbers among $m_1, \cdots, m_n$ form an arithmetic sequence, is s.t.c.i..

Example 4.7. One of the simplest open cases for the set theoretic complete intersection problem for affine monomial curves, was the curve (t^5, t^6, t^8, t^9). This curve is associated to the projective monomial curve (u^4, u^3v, uv^3, v^4), which is not arithmetically Cohen-Macaulay and so it is unknown if it is s.t.c.i. or not. Therefore, none of the Theorems of the sections 2 and 3 can be applied to conclude that (t^5, t^6, t^8, t^9) is s.t.c.i..

Using the technique of this section we see that the above curve is related to $(t^9, t^{10}, t^{12}, t^{16})$. This curve is associated to the projective monomial curve (u^7, u^6v, u^4v^3, v^7) which is arithmetically Cohen-Macaulay. Moreover $(u^{16} - v^9)^7$ belongs to $K[u^7, u^6v, u^4v^3, v^7]$. Therefore $(t^9, t^{10}, t^{12}, t^{16})$ can be interpreted as s.t.c.i. of

$$X_2^3 - X_1^2X_3,$$

$$X_3^7 - 3X_2^2X_3^4X_4 + 3X_1^2X_2X_3^2X_4^2 - X_1^4X_4^3 \text{ and of}$$

$$X_1^{16} - 7X_1^{12}X_2^2X_4 + 21X_1^{10}X_2X_3X_4^2 - 35X_1^8X_3^2X_4^3 + 35X_1^6X_2X_4^5$$
$$-21X_1^4X_3X_4^6 + 7X_2^2X_3X_4^7 - X_4^9.$$

Its projective closure $(u^{16}, u^7v^9, u^6v^{10}, u^4v^{12}, v^{16})$ is also set-theoretic complete intersection, namely of $X_2^3 - X_1^2X_3$,

$$X_3^7 - 3X_2^2X_3^4X_4 + 3X_1^2X_2X_3^2X_4^2 - X_1^4X_4^3 \text{ and of}$$

$$X_1^{16} - 7X_0X_1^{12}X_2^2X_4 + 21X_0^2X_1^{10}X_2X_3X_4^2 - 35X_0^3X_1^8X_3^2X_4^3 + 35X_0^4X_1^6X_2X_4^5$$
$$-21X_0^5X_1^4X_3X_4^6 + 7X_0^6X_2^2X_3X_4^7 - X_0^7X_4^9.$$

From Theorem 4.3, we see that the original curve (t^5, t^6, t^8, t^9) is s.t.c.i.and we can get its defining polynomials by changing X_1^2 by X_4, X_2 by X_1, X_3 by X_2 and X_4 by X_3 in the polynomials defining $(t^9, t^{10}, t^{12}, t^{16})$.
Thus, we get the following polynomials:

$$X_1^3 - X_4X_2,$$

$$X_2^7 - 3X_1^2X_2^4X_3 + 3X_1X_2^2X_3^2X_4 - X_3^3X_4^2 \text{ and}$$

$$X_4^8 - 7X_1^2X_3X_4^6 + 21X_1X_2X_3^2X_4^5 - 35X_2^2X_3^3X_4^4 + 35X_1X_3^5X_4^3$$
$$-21X_2X_3^6X_4^2 + 7X_1^2X_2X_3^7 - X_3^9.$$

Note that we cannot conclude from these polynomials, that the projective closure of (t^5, t^6, t^8, t^9) is s.t.c.i..

Remark 4.8. The trick of the Example 4.7 can be applied always in A^4 but not with equal success. With the use of the following Lemma 4.9, we can relate any monomial curve in A^4 to an arithmetically Cohen-Macaulay curve in P^3 but, in general, the second condition of Theorem 2.1 will be not satisfied.

Lemma 4.9.[20] *Let C be a projective monomial curve $(u^d, u^{a_1}v^{b_1}, u^{a_2}v^{b_2}, v^d)$, where $d > a_1 > a_2$. If b_1 belongs to the numerical semigroup $< a_1^{\#}, a_1^{\#} - a_2^{\#} >$ or a_2 belongs to $< b_2^*, b_2^* - b_1^* >$, $a_1^{\#}, a_2^{\#}$ are the numbers a_1, a_2 divided by their g.c.d. and b_2^*, b_1^* are the numbers b_2, b_1 divided by their g.c.d., then C is arithmetically Cohen-Macaulay.*

Take, for example, a number l prime to m_1. Then it is $C_{(l,1)}(m_1, m_2, m_3, m_4) = C(m_1, lm_2, lm_3, lm_4)$. According to Lemma 4.9, this curve is arithmetically Cohen-Macaulay if $lm_2 - m_1 \in < (m_4 - m_2)^{\#}, (m_3 - m_2)^{\#} >$, which is always true for big l. But for this curve to be s.t.c.i., according to Theorem 2.1, we need at least $m_1 \in < lm_2 - m_1, lm_3 - m_1, lm_4 - m_1 >$, which is never true for big l.

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