Moduli of Abelian Surfaces with a $(1,p^2)$ Polarisation

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MODULI OF ABELIAN SURFACES WITH A $(1, p^2)$ POLARISATION

V.A. Gritsenko & G.K. Sankaran

The moduli space of abelian surfaces with a polarisation of type $(1, p^2)$ for p a prime was studied by O'Grady in [O'G], where it is shown that a compactification of this moduli space is of general type if $p \geq 17$. We shall show that in fact this is true if $p \geq 11$. Our methods overlap with those of [O'G], but are in some important ways different. We borrow notation freely from that paper when discussing the geometry of the moduli space.

1. Methods

Let $\mathcal{A}_2(p)$ denote the moduli space of abelian surfaces over $\mathbb{C}$ with a polarisation of type $(1, p^2)$. We denote by $\bar{\mathcal{A}}_2(p)$ the toroidal compactification of $\mathcal{A}_2(p)$ and by $\hat{\mathcal{A}}_2(p)$ a partial desingularisation of $\bar{\mathcal{A}}_2(p)$ having only canonical singularities.

To show that $\hat{\mathcal{A}}_2(p)$ is of general type (that is, roughly, that the pluricanonical bundles have many sections), one chooses an $\mathcal{E} \in \operatorname{Pic} \hat{\mathcal{A}}_2(p) \otimes \mathbb{Q}$ such that $n\mathcal{E}$ (which is a bundle if n is sufficiently divisible) is not too far from the pluricanonical bundle $nK_{\hat{\mathcal{A}}_2(p)}$, and such that the space of sections $H^0(n\mathcal{E})$ can be calculated or at least estimated by some method. Then by knowing about the geometry of $\hat{\mathcal{A}}_2(p)$ one can estimate the plurigenera, because the difference between $nK_{\hat{\mathcal{A}}_2(p)}$ and $\mathcal{E}$ is known. In [O'G] the bundle used to play the rôle of $n\mathcal{E}$ arises from pulling back powers of the Hodge bundle on the moduli space $\overline{\mathfrak{M}}_2$ of semi-stable genus 2 curves via the map $\hat{\mathcal{A}}_2(p) \to \overline{\mathfrak{M}}_2$ constructed there. Here, by contrast, we consider $\mathcal{A}_2(p)$ as a Siegel modular variety, i.e., as a quotient of the Siegel upper half-space by the paramodular group Γ_{p^2} (an arithmetic subgroup of $\mathrm{Sp}(4,\mathbb{Q})$), and obtain a suitable bundle $n\mathcal{E}$ by considering cusp forms of weight $3n$ for Γ_{p^2}. A similar procedure is adopted in [HS] for a different Siegel modular variety. But here we do not use all cusp forms of weight $3n$. Instead, we use the cusp form F_2 of weight 2 for Γ_{p^2}, constructed by the first author in [G], and we consider modular forms of weight $3n$ of the form $F_n F_2^n$, where F_n is a modular form of weight n. The bundle that results has fewer sections than the one arising from all cusp forms but is much closer to $nK_{\tilde{\mathcal{A}}_2(p)}$ and this turns out to give a better bound on the plurigenera for small p. Hulek and the first author have applied this idea to the situation of [HS] and the improvement that results is described in [GH].

2. Modular forms

If t is a positive integer, the paramodular group is defined to be the arithmetic subgroup of $\mathrm{Sp}(4,\mathbb{Q})$

$$\Gamma_t = \left\{ \gamma \in \mathrm{Sp}(4,\mathbb{Q}) \mid \gamma \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t\mathbb{Z} \\ t\mathbb{Z} & \mathbb{Z} & t\mathbb{Z} & t\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t\mathbb{Z} \\ \mathbb{Z} & \frac{1}{t}\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}.$$

It acts on the Siegel upper half-plane

$$\mathbb{H}_2 = \left\{ Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in M_{2\times 2}(\mathbb{C}) \mid Z = {}^T Z,\ \mathrm{Im}\, Z > 0 \right\}$$

by fractional linear transformations, i.e.,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \longmapsto (AZ+B)(CZ+D)^{-1}.$$

The quotient $\Gamma_t \backslash \mathbb{H}_2$ is a coarse moduli space for abelian surfaces over $\mathbb{C}$ with a polarisation of type $(1,t)$. The group Γ_t is conjugate to a subgroup of $\mathrm{Sp}(4,\mathbb{Z})$ only if t is a perfect square: in particular, if p is a prime and $t = p^2$, then Γ_{p^2} is conjugate to $\Gamma'_{p^2} < \mathrm{Sp}(4,\mathbb{Z})$, where

$$\Gamma'_{p^2} = \left\{ \gamma \in \mathrm{Sp}(4,\mathbb{Z}) \mid \gamma \in \begin{pmatrix} \mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}.$$

We denote $\Gamma_{p^2}\backslash\mathbb{H}_2$ by $\mathcal{A}_2(p)$. This moduli space is a finite covering of the rational variety of abelian surfaces with principal polarisation. Let $\bar{\mathcal{A}}_2(p)$ be the toroidal compactification and $\hat{\mathcal{A}}_2(p)$ the canonical partial resolution described in [O'G]. If we can show that $h^0(nK_{\hat{\mathcal{A}}_2(p)}) \sim n^3$ we shall have shown that $\hat{\mathcal{A}}_2(p)$ is of general type. For that we shall use special examples of modular forms of weight 2 with respect to Γ_{p^2}, which we now describe. We first define the so-called Jacobi forms (see [EZ]).

Definition: A holomorphic function $\varphi(\tau, z) : \mathbb{H}_1 \times \mathbb{C} \to \mathbb{C}$ is called a Jacobi form of index $m \in \mathbb{N}$ and weight k if for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and any $\mu, \lambda \in \mathbb{Z}$ it satisfies the two functional equations

$$\varphi(\tau,\, z) = (c\tau + d)^{-k} \exp\Big(-\frac{2\pi i c m z^2}{c\tau + d}\Big)\varphi\Big(\frac{a\tau + b}{c\tau + d},\, \frac{z}{c\tau + d}\Big),$$

$$\varphi(\tau, z) = \exp\big(2\pi i m(\lambda^2\tau + 2\lambda z)\big)\varphi(\tau, z + \lambda\tau + \mu)$$

and it has a Fourier expansion of the type

$$\varphi(\tau,\, z) = \sum_{n=0}^{\infty} \sum_{\substack{l \in \mathbb{Z} \\ 4nm \geq l^2}} f(n, l) \exp\big(2\pi i(n\tau + lz)\big).$$

We call the function $\varphi(\tau, z)$ a Jacobi cusp form if we have the strict inequality $4nm > l^2$ in the last summation. We shall denote the space of all Jacobi cusp forms of index m and weight k by $\mathfrak{S}^J_{k,m}$.

The next theorem was proved in [G, Theorem 3].

Proposition 2.1. *Let $\varphi(\tau, z)$ be a Jacobi cusp form of weight k and index t with the Fourier expansion*

$$\varphi(\tau, z) = \sum_{n\in\mathbb{N}} \sum_{\substack{l\in\mathbf{Z} \\ 4nt>l^2}} f(n,l) \exp\big(2\pi i(n\tau + lz)\big).$$

Then the following function

$$F_\varphi(\tau, z, \tau') = \sum_{n,\, m\in\mathbb{N}} \sum_{\substack{l\in\mathbf{Z} \\ 4tmn>l^2}} \sum_{a|(n,l,m)} a^{k-1} f\Big(\frac{nm}{a^2}, \frac{l}{a}\Big) \exp\Big(2\pi i\Big(n\tau + \frac{lz}{t} + \frac{m\tau'}{t}\Big)\Big).$$

is a nontrivial cusp form of weight k with respect to the group Γ_t.

This result gives us, for example, a differential 3-form on a smooth model of $\Gamma_t\backslash\mathbb{H}_2$. As a corollary we have irrationality of the moduli space of abelian surfaces with polarisation $(1,t)$ for $t \geq 13$ and $t \neq 14, 15, 16, 20, 24, 30, 36$ (see [G] for details). To construct pluricanonical forms on $\hat{\mathcal{A}}_2(p)$ we use

Corollary 2.2. *If $p \geq 11$ is a prime then there exists a nontrivial cusp form F_2 of weight 2 for Γ_{p^2}.*

Proof: The dimension of the space of Jacobi cusp forms of weight k and index t is well known (see [EZ] and [SZ]). In our particular case

$$\dim_{\mathbb{C}} \mathfrak{S}^J_{2,p^2} = \sum_{j=1}^{p^2} \{1+j\}_6 - \left\lfloor \frac{j^2}{4t} \right\rfloor$$

with

$$\{m\}_6 = \begin{cases} \lfloor \frac{m}{6} \rfloor & \text{if } m \neq 1 \bmod 6, \\ \lfloor \frac{m}{6} \rfloor - 1 & \text{if } m = 1 \bmod 6. \end{cases}$$

This formula shows that $\mathfrak{S}^J_{2,p^2}$ is nontrivial for $p \geq 11$. ∎

It is not known whether such a cusp form F_2 exists for Γ_{p^2} if $p \leq 7$.

Proposition 2.3. *The space* $\mathcal{M}_n^*(\Gamma_{p^2})$ *of cusp forms of weight* n *for* Γ_{p^2} *satisfies*

$$\dim \mathcal{M}_n^*(\Gamma_{p^2}) = \frac{p^2(p^2+1)}{8640} n^3 + O(n^2)$$

for any prime p.

Proof: Γ_{p^2} is conjugate to a subgroup of $\mathrm{Sp}(4, \mathbb{Z})$ so we may as well work with Γ'_{p^2}. Exactly as in [HS] (cf. also [T]) we obtain, for l large

$$\dim \mathcal{M}_n^*(\Gamma(l)) \sim \frac{n^3}{8640}[\Gamma(1) : \Gamma(l)]$$

and if $p|l$ then $\Gamma(l) \subseteq \Gamma'_{p^2}$ and the cusp forms for Γ'_{p^2} are the $\Gamma'_{p^2}/\Gamma(l)$-invariant cusp forms for $\Gamma(l)$. Using the Atiyah-Bott fixed point theorem, as in [T], we obtain

$$\dim \mathcal{M}_n^*(\Gamma'_{p^2}) \sim \frac{2}{[\Gamma'_{p^2} : \Gamma(l)]} \dim \mathcal{M}_n^*(\Gamma(l))$$

since there is a contribution from $\gamma = -I$ as well as from $\gamma = I$. But

$$\begin{aligned}
\frac{2}{[\Gamma'_{p^2} : \Gamma(l)]} \dim \mathcal{M}_n^*(\Gamma(l)) &\sim \frac{2}{[\Gamma'_{p^2} : \Gamma(l)]} \cdot \frac{n^3}{8640}[\Gamma(1) : \Gamma(l)] \\
&= \frac{1}{[\Gamma'_{p^2}/\pm I : \Gamma(l)]} \cdot \frac{n^3}{8640}[\Gamma(1) : \Gamma(l)] \\
&= \frac{n^3}{8640}[\Gamma(1) : \Gamma'_{p^2}/\pm I] \\
&= \frac{n^3}{8640} \deg(\pi : \mathcal{A}_2(p) \to \mathcal{A}_2) \\
&= \frac{p^2(p^2+1)}{8640} n^3
\end{aligned}$$

by [O'G, Lemma 2.1]. ∎

3. Pluricanonical forms and extension to the boundary

Choose a cusp form F_2 of weight 2 for Γ_{p^2}, $p \geq 11$; we can do this in view of Corollary 2.2. Suppose F_n is a modular form for Γ_{p^2} of weight n: then $\Phi = F_n F_2^n$ is a cusp form of weight $3n$. Let $\omega = d\tau \wedge dz \wedge d\tau'$ be the standard 3-form on $\mathbb{H}_2$. The form $\Phi\omega^{\otimes n}$ is invariant under Γ_{p^2} and therefore descends to give a pluricanonical form on $\mathcal{A}_2(p)$ except at the branch locus of $\mathbb{H}_2 \to \mathcal{A}_2(p)$. If Φ were a general element of $\mathcal{M}_{3n}^*(\Gamma_{p^2})$ we should expect this form to have logarithmic poles at the boundary $\bar{\mathcal{A}}_2(p)\setminus\mathcal{A}_2(p)$, but because the cusp form we have chosen is special these poles do not occur. That is because Φ vanishes to high order (at least order n) at the cusps.

Proposition 3.1. *The differential $3n$-form coming from $\Phi\omega^{\otimes n}$ extends over the generic point of each codimension 1 boundary component of $\bar{\mathcal{A}}_2(p)$.*

Proof: According to [SC, Chapter IV, Theorem 1] (see also [HS, Proposition 1.1]), we need to check that in the Fourier-Jacobi expansion

$$\Phi(Z) = \sum_{m\geq 0} \theta^D_{m,\Phi}(\tau_D, z_D)\exp\{2\pi i m\tau'_D\}$$

near the boundary component D, the coefficients $\theta^D_{m,\Phi}$ vanish for $m < n$. But we can write the expansion of $\Phi(Z)$ as a product of expansions of $F_2(Z)$ and $F_n(Z)$: we have

$$F_2(Z) = \sum_{m>0} \theta^D_{m,F_2}(\tau_D, z_D)\exp\{2\pi i m\tau'_D\}$$

(with $\theta^D_{0,F_2}(\tau_D, z_D) \equiv 0$ as F_2 is a cusp form), and similarly for F_n. Hence

$$\theta^D_{m,\Phi} = \sum_{m_0+\cdots+m_n=m} \theta^D_{m_0,F_n}\prod_{i=1}^{n}\theta^D_{m_i,F_2}$$

which is zero if $m < n$ as then $m_i = 0$ for some $i \geq 1$. ■

$\bar{\mathcal{A}}_2(p)$ is smooth in codimension 1, but the quotient map $\mathbb{H}_2 \to \mathcal{A}_2(p)$ is branched along two divisors (Humbert surfaces). These are the divisors whose closures in $\bar{\mathcal{A}}_2(p)$ are denoted $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_2$ in [O'G]. At a point of $\mathbb{H}_2$ lying over a general point of $\widetilde{\Delta}_1$ or $\widetilde{\Delta}_2$ the isotropy group in Γ_{p^2} is $\mathbb{Z}/2$ and it acts by a reflection, so $\bar{\mathcal{A}}_2(p)$ is smooth at a general point of $\widetilde{\Delta}_1$ or $\widetilde{\Delta}_2$.

Corollary 3.2. *If n is sufficiently divisible then $n(K_{\bar{\mathcal{A}}_2(p)} + \frac{1}{2}\widetilde{\Delta}_1 + \frac{1}{2}\widetilde{\Delta}_2)$ is a bundle and if $\Phi = F_n F_2^n$ is a cusp form of the type described above then $\Phi\omega^{\otimes n}$ determines an element of $H^0(\bar{\mathcal{A}}_2(p); n(K_{\bar{\mathcal{A}}_2(p)} + \frac{1}{2}\widetilde{\Delta}_1 + \frac{1}{2}\widetilde{\Delta}_2))$*

Proof: $\bar{\mathcal{A}}_2(p)$ has only quotient singularities, which are, in particular, $\mathbb{Q}$-Gorenstein. From the description of the action of Γ_{p^2} above a general point of $\widetilde{\Delta}_1$ or $\widetilde{\Delta}_2$ it is clear that $\Phi\omega^{\otimes n}$ acquires poles of order $\frac{1}{2}n$ along $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_2$.

4. Obstructions from elliptic fixed points.

We take a canonical partial resolution $\varphi : \hat{\mathcal{A}}_2(p) \to \bar{\mathcal{A}}_2(p)$, as in [O'G], adopting also the notations of [O'G, Definition 3.8] for (Weil) divisors on $\hat{\mathcal{A}}_2(p)$.

Proposition 4.1. *If n is sufficiently divisible then $\Phi\omega^{\otimes n}$ determines an element of*

$$H^0(\hat{\mathcal{A}}_2(p); n(K_{\hat{\mathcal{A}}_2(p)} + \frac{1}{2}E_1' + \frac{1}{2}E_1'' + \frac{1}{2}\hat{\Delta}_1 + \frac{1}{2}\hat{\Delta}_2 + (1 - \frac{2}{p})E_2 - \frac{1}{4}E' - \frac{1}{4}E'')).$$

Proof: By Corollary 3.2., above, we have a section of $\varphi^*\big(n(K_{\bar{\mathcal{A}}_2(p)} + \frac{1}{2}\widetilde{\Delta}_1 + \frac{1}{2}\widetilde{\Delta}_2)\big)$, and the formulae for $\varphi^* K_{\bar{\mathcal{A}}_2(p)}$, $\varphi^*\widetilde{\Delta}_1$ and $\varphi^*\widetilde{\Delta}_2$ given in [O'G] provide the required expression. ■

We assume henceforth that n is sufficiently divisible, so that everything we have written so far is a bundle (in fact it is enough that $24p|n$). For the rest of this section we assume that $p \geq 5$, as in [O'G], but we shall need $p \geq 11$ in the end in order to apply Corollary 2.2.

Put $\mathcal{E} = K_{\hat{\mathcal{A}}_2(p)} + \frac{1}{2}E_1' + \frac{1}{2}E_1'' + \frac{1}{2}\hat{\Delta}_1 + \frac{1}{2}\hat{\Delta}_2 + (1-\frac{2}{p})E_2 - \frac{1}{4}E' - \frac{1}{4}E''$. We want to make use of O'Grady's calculations (and to avoid either resolving the singularities of $\hat{\mathcal{A}}_2(p)$ or using adjunction and Riemann-Roch on singular varieties), so we express $\mathcal{E}$ in terms of the pullback of the Hodge bundle λ on $\overline{\mathfrak{M}}_2$ via the map $\pi\varphi : \hat{\mathcal{A}}_2(p) \to \overline{\mathfrak{M}}_2$.

Lemma 4.2. *In* $\operatorname{Pic}\hat{\mathcal{A}}_2(p) \otimes \mathbb{Q}$ *we have*

$$\mathcal{E} = 3\varphi^*\pi^*(\lambda) - \frac{1}{p}\varphi^*\pi^*(\Delta_1) - \frac{p-1}{p}\hat{\Delta}_0 - \frac{p-1}{p}\hat{\hat{\Delta}}_0. \qquad (\star)$$

Proof: See [O'G, Theorem 3.1]. ∎

We have

$$\begin{aligned} h^0(nK_{\hat{\mathcal{A}}_2(p)}) &\geq h^0(nK_{\hat{\mathcal{A}}_2(p)} - \frac{n}{4}E' - \frac{n}{4}E'') \\ &= h^0\big(n\mathcal{E} - \frac{n}{2}E_1' - \frac{n}{2}E_1'' - \frac{n}{2}\hat{\Delta}_1 - \frac{n}{2}\hat{\Delta}_2 - n(1-\frac{2}{p})E_2\big) \end{aligned}$$

so we want to estimate the five obstructions coming from E_1', E_1'', $\hat{\Delta}_1$, $\hat{\Delta}_2$ and E_2. Our $\mathcal{E}$ plays the rôle of $\varphi^*\pi^*(\alpha_p\lambda)$ in [O'G]: by comparison, we have replaced $\alpha_p = 3 - \frac{10}{p}$ by 3, which makes some of the obstructing sheaves more positive, but we also have vanishing to order $\frac{p-1}{p}$ along the boundary components $\hat{\Delta}_0$ and $\hat{\hat{\Delta}}_0$, which makes them more negative. So much more negative, in fact, that we have the following result.

Theorem 4.3. *All the obstructions vanish: that is, every section of $n\mathcal{E}$ gives a section of $nK_{\tilde{\mathcal{A}}_2(p)}$ if $p \geq 5$ and n is sufficiently divisible.*

Proof: We prove this in five steps, taking each obstruction separately. Only E_1' and E_1'' require much attention.

1. The obstruction from E_1'.

We need (cf. [O'G], p. 146) to estimate $h^0([n\mathcal{E} - iE_1']|_{E_1'})$ for $0 \leq i \leq \frac{n}{2} - 1$. Using $(\star)$ we get

$$h^0([n\mathcal{E} - iE_1']|_{E_1'}) =$$

$$h^0\Big(3n[\varphi^*\pi^*(\lambda)]|_{E_1'} - \frac{n}{p}[\varphi^*\pi^*(\Delta_1)]|_{E_1'} - n\frac{p-1}{p}\hat{\Delta}_0|_{E_1'} - n\frac{p-1}{p}\hat{\hat{\Delta}}_0|_{E_1'} - iE_1'|_{E_1'}\Big)$$
$$\leq h^0\Big(3n[\varphi^*\pi^*(\lambda)]|_{E_1'} - n\frac{p-1}{p}\hat{\Delta}_0|_{E_1'} - iE_1'|_{E_1'}\Big).$$

Replacing α_p by 3 in [O'G, Corollary 4.1 et seq.] gives

$$3n[\varphi^*\pi^*(\lambda)]|_{E_1'} - iE_1'|_{E_1'} = 3i\Sigma + (\frac{n}{4} + 4i)L - 3iG.$$

As a set, $\hat{\Delta}_0 \cap E_1'$ consists of the fibres of $\varphi_2 : E_1 \to \widetilde{\Gamma}$ over the points where $\widetilde{\Gamma}$ meets the boundary. These fibres are smooth, so $\hat{\Delta}_0|_{E_1'}$ is a positive integer multiple of the general fibre L. Hence

$$h^0\Big(3n[\varphi^*\pi^*(\lambda)]|_{E_1'} - n\frac{p-1}{p}\hat{\Delta}_0|_{E_1'} - iE_1'|_{E_1'}\Big) \leq h^0(3i\Sigma + (\frac{n}{4} + 4i - n\frac{p-1}{p})L - 3iG)$$
$$\leq h^0(3i\Sigma + ((\frac{1}{p} - \frac{3}{4})n + 4i)L).$$

But

$$\Sigma \cdot (3i\Sigma + ((\frac{1}{p} - \frac{1}{4})n + 4i)L) = (\frac{1}{p} - \frac{3}{4})n + i$$
$$\leq (\frac{1}{p} - \frac{1}{4})n - 1$$

which is negative if $p \geq 5$, so there are no sections and no obstructions.

2. The obstruction from E_1''.

The calculation here is very similar: this time it is $\hat{\hat{\Delta}}_0$ that plays a part. We need to estimate $h^0([n\mathcal{E} - \frac{n}{2}E_1' - iE_1'']|_{E_1''})$ for $0 \leq i \leq \frac{n}{2} - 1$, and $(\star)$ and [O'G, Corollary 4.2] together give

$$\begin{aligned} h^0([n\mathcal{E} - \frac{n}{2}E_1' - iE_1'']|_{E_1''}) &\leq h^0\Big(3n[\varphi^*\pi^*(\lambda)]|_{E_1''} - \frac{n}{2}E_1'|_{E_1''} - n\frac{p-1}{p}\hat{\hat{\Delta}}_0|_{E_1''} - iE_1''|_{E_1''}\Big) \\ &\leq h^0\Big(3i\Sigma + (4i - \frac{n}{4})F - n\frac{p-1}{p}\hat{\hat{\Delta}}_0|_{E_1''}\Big) \end{aligned}$$

(note that Lemma 4.7.(i) of [O'G] should read $\varphi^*\pi^*(\lambda)\big|_{E_1''} \cong \frac{1}{12}F$). As above, $\hat{\hat{\Delta}}_0\big|_{E_1''}$ is a positive integer multiple of F, so the obstruction becomes $h^0\big(3i\Sigma + \big(4i + (\frac{1}{p} - \frac{5}{4})n\big)F\big)$. But

$$\begin{aligned} \Sigma \cdot \big(3i\Sigma + \big(4i + (\frac{1}{p} - \frac{5}{4})n\big)F\big) &= i + (\frac{1}{p} - \frac{5}{4})n \\ &\leq (\frac{1}{p} - \frac{3}{4})n - 1, \end{aligned}$$

which is negative for all p, so again there are no sections and no obstructions.

3. The obstruction from $\hat{\Delta}_1$.

All we have to do is replace α_p by 3 in [O'G, Corollary 4.4]. The conclusion (Theorem 4.3 in [O'G]: the coefficient of $L' + L''$ should be $\frac{\alpha_p}{n} + i$) that the obstruction vanishes is unaltered.

4. The obstruction from $\hat{\Delta}_2$.

Again changing α_p to 3 makes no difference: we simply get

$$\begin{aligned} &h^0\big([n\mathcal{E} - \frac{n}{2}(E_1' + E_1'' + \hat{\Delta}_1)]|_{\hat{\Delta}_2}\big) \\ &\quad\leq h^0\Big(3n[\varphi^*\pi^*(\lambda)]|_{\hat{\Delta}_2} - \frac{n}{2}(E_1' + E_1'' + \hat{\Delta}_1)|_{\hat{\Delta}_2} - n\frac{p-1}{p}\hat{\Delta}_0 + \hat{\hat{\Delta}}_0|_{\hat{\Delta}_2} - i\hat{\Delta}_2|_{\hat{\Delta}_2}\Big) \\ &\quad= 0 \qquad \text{for all } i \geq 0. \end{aligned}$$

5. The obstruction from E_2.

This time the restriction of $\varphi^*\pi^*(\lambda)$ is trivial, so α_p does not even appear in the calculation. ∎

Theorem 4.4. *$\hat{\mathcal{A}}_2(p)$ is of general type if $p \geq 11$.*

Proof: This follows from Proposition 2.1, Proposition 2.2. amd Theorem 4.3. ∎

In fact we have shown that $h^0(nK_{\hat{\mathcal{A}}_2(p)}) \geq \frac{p^2(p^2+1)}{8640}n^3 + O(n^2)$ if $p \geq 11$. We do not really need the precise value of the leading coefficient (unless we really want an asymptotic bound on the plurigenera, but O'Grady's bound is better unless $p = 11$ or $p = 13$), because there are no obstructions to compare it with. Instead, we have an explicit pluricanonical form.

Corollary 4.5. *If $p \geq 11$ and F_n is any modular form of weight n for Γ_{p^2}, then $F_n F_2^n \omega^{\otimes n}$ gives an n-canonical form on $\hat{\mathcal{A}}_2(p)$.*

We note that the variety $\hat{\mathcal{A}}_2(p)$ is rational for $p = 2$ and unirational for $p = 3$ (see [O'G]). In [G] it was proved that $\hat{\mathcal{A}}_2(p)$ is irrational for $p = 5$ and $p = 7$. More exactly the Kodaira dimension of $\hat{\mathcal{A}}_2(p)$ is nonnegative for $p = 5$ and is positive for $p = 7$. The question of the exact value of the Kodaira dimension for these two primes is open.

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