# Polylogarithms and the Bloch-Kato conjecture on special values of L-functions

by Guido Kings
Arbeitstagung 14.6.2001

In this talk we give an overview of the state of the Bloch-Kato conjecture on special values of L-functions and explain the role of the polylogarithm sheaf in the proofs.

## 1 What is the Bloch-Kato conjecture?

Let $F$ be a number field, $\mathcal{O}_F$ its ring of integers. The Dedekind zeta function is defined by the Euler product

$$\zeta_F(s) := \prod_{\mathfrak{p}} \frac{1}{1 - \frac{1}{N\mathfrak{p}^s}}$$

for $Re(s) > 1$. This function has a meromorphic continuation to $\mathbb{C}$ and satisfies a functional equation. In the case, where $F = \mathbb{Q}$ we get the Riemann zeta function. An theorem due to Euler gives now:

$$\zeta_{\mathbb{Q}}(1-n) = -\frac{B_n}{n}$$

for $n \geq 1$, where $B_n$ is the $n$-th Bernoulli number. We ask:

*What is the arithmetic meaning of these numbers?.*

A prototype answer is provided by the *class number formula*:

$$\zeta_F(0)^* = -\operatorname{vol}(H/r_\infty(\mathcal{O}_F^*)) \frac{\#Cl(\mathcal{O}_F)}{\#\mu(F)}.$$

Here $\zeta_F(0)^*$ denotes the leading term of the Taylor series at 0, $Cl(\mathcal{O}_F)$ is the ideal class group and $\mu(F)$ is the group of roots of unity in $F$. Moreover $r_\infty$ is the Dirichlet regulator map

$$\begin{aligned} r_\infty : \mathcal{O}_F^* &\longrightarrow \left( \bigoplus_{\sigma: F \to \mathbb{C}} \mathbb{R} \right)^+ \\ u &\longmapsto (\log|\sigma(u)|)_\sigma . \end{aligned}$$

Here $+$ means invariants under complex conjugation.

*The Bloch-Kato conjecture generalizes this formula to the L-values at all integers of the L-functions of all varieties over number fields.*

Consider the motivic cohomology of $F$, which we take for definiteness to be the $2n-1$ Quillen K-group and its $n$-th Adams eigenspace, thus $H^1_{\mathcal{M}}(F, \mathbb{Q}(n)) := (K_{2n-1}(\mathcal{O}_F) \otimes \mathbb{Q})^n$. One has the Beilinson regulator

$$r_\infty : H^1_{\mathcal{M}}(F, \mathbb{Q}(n)) \to \left( \bigoplus_{\sigma: F \to \mathbb{C}} (2\pi i)^{n-1} \mathbb{R} \right)^+ =: H_B^+$$

and the Soulé regulator

$$r_p : H^1_{\mathcal{M}}(F, \mathbb{Q}(n)) \to H^1(G_{F,p}, \mathbb{Z}_p(n)) \otimes \mathbb{Q}_p$$

into Galois cohomology. The group $G_{F,p}$ is the Galois group of the maximal unramified outside of $p$ extension of $F$. One denotes $\mathbb{Z}_p(1) := \varprojlim_r \mu_{p^r}$ and $\mathbb{Z}_p(n) := \mathbb{Z}_p(1)^{\otimes n}$.

**Conjecture 1.0.1. (Bloch-Kato conjecture for $F$)** *Let $n > 1$ and choose $\Omega \subset H^1_{\mathcal{M}}(F, \mathbb{Q}(n))$ a $\mathbb{Z}$-lattice, then*

**BK(F,1-n)** $\zeta_F(1-n)^* = \pm \operatorname{vol}\left(H_B^+ / r_\infty(\Omega)\right) \prod_p \frac{\# H^2(G_{F,p}, \mathbb{Z}_p(n))}{\#\left(H^1(G_{F,p}, \mathbb{Z}_p(n))/r_p(\Omega \otimes \mathbb{Z}_p)\right)}$.

*Here $\zeta_F(1-n)^*$ denotes again the leading term in the Taylor series expansion.*

**Remark:**

1. The conjecture is independent of $\Omega$
2. There exists a version for all $n \in \mathbb{Z}$ and for Artin motives
3. In the case described above the conjecture is already due to Lichtenbaum
4. A similar formula is conjectured for the L-functions of all varieties

## 2 What is known?

The following is a nearly comprehensive list of known results:

**Theorem 2.0.2. (Bloch-Kato '89)** *$BK(\mathbb{Q}, n)$ is true for all $n \in \mathbb{Z}$.*

**Theorem 2.0.3. (Wiles '90)** *$BK(F, 1-n)$ is true for totally real fields $F$ and $n \geq 0$ even.*

**Theorem 2.0.4. (Huber-Kings '00, Burns-Greither '00 for $n \leq 0$)** *$BK(F, n)$ is true for all* abelian *field extensions $F/\mathbb{Q}$ (more generally for* abelian *Artin motives) and all $n \in \mathbb{Z}$.*

For higher dimensional varieties one has:

**Theorem 2.0.5. (Kings '99)** *Let $E/K$ be a CM elliptic curve of conductor $\mathfrak{f}$ over an imaginary quadratic field. Fix $k \geq 0$ and a prime $p \nmid 6\mathfrak{f}\bar{\mathfrak{f}}$ such that $H^2(G_{K,p\mathfrak{f}}, T_pE(k+1)$ is finite. Then there is a sublattice $\Omega \subset H^2_{\mathcal{M}}(E, \mathbb{Q}(k+2))$ such that the p valuation of*

$$\frac{L(E/K, -k)^*}{\mathrm{vol}\,(H^1(E(\mathbb{C}), (2\pi i)^{k+1}\mathbb{R})^+/r_\infty(\Omega))}$$

*and of*

$$\frac{\# H^2(G_{K,p\mathfrak{f}}, T_pE(k+1))}{\#\,(H^1(G_{K,p\mathfrak{f}}, T_pE(k+1))/r_p(\Omega \otimes \mathbb{Z}_p))}$$

*coincide.*

The proofs have two essential steps:

1. Construct $\Omega \subset H^1_{\mathcal{M}}(F, \mathbb{Q}(n))$ and compute $r_\infty(\Omega)$ and $r_p(\Omega)$. Moreover, relate $r_\infty(\Omega)$ to L-values. If this is possible, choose an $\Omega$ such that the value of the zeta function is exactly $\zeta_F(1-n)^* = \pm \mathrm{vol}\left(H_B^+/r_\infty(\Omega)\right)$. Then it remains to show:

$$\# H^2(G_{F,p}, \mathbb{Z}_p(n)) = \#\left(H^1(G_{F,p}, \mathbb{Z}_p(n))/r_p(\Omega \otimes \mathbb{Z}_p)\right).$$

   One should see this as an Euler characteristic type formula.

2. use Iwasawa theory to reduce to the case $n = 1$. More specifically consider

$$\varprojlim{}_r H^1(G_{F(\mu_{p^r}),p}, \mathbb{Z}_p(n)).$$

   This is up to isomorphism independent of $n$ and $H^1(G_{F,p}, \mathbb{Z}_p(n))$ is essentially a quotient of it. Thus, one can reduce to the case $n = 1$ provided one knows:

$$r_p(\Omega \otimes \mathbb{Z}_p) \textit{ is compatible with "twist".}$$

To show these two points one needs polylogarithms:

# 3 What is the polylogarithm sheaf?

We sketch this beautiful idea of Beilinson very roughly: Let $X = \mathbb{G}_m(\mathbb{C})$ or $= E(\mathbb{C})$. Fix a point $x \in X \setminus \{1\}$. Denote by $\mathbb{Q}[\pi_1(X \setminus x, 1)]^\wedge$ the completion of the group algebra of $\pi_1(X \setminus x, 1)$ with respect to the augmentation ideal. Similarly we consider $\mathbb{Q}[\pi_1(X, 1)]^\wedge$. These are Hopf algebras and we have a map

$$\mathbb{Q}[\pi_1(X \setminus x, 1)]^\wedge \to \mathbb{Q}[\pi_1(X, 1)]^\wedge.$$

This induces a map of the Lie algebras $L_x \to \mathcal{H}$ of primitive elements $L_x$ resp. $\mathcal{H}$ in $\mathbb{Q}[\pi_1(X \setminus x, 1)]^\wedge$ resp. $\mathbb{Q}[\pi_1(X, 1)]^\wedge$. Let $K_x := \ker(L_x \to \mathcal{H})$ and define:

**Definition 3.0.6.** *The polylog Lie algebra at $x$ is*

$$\mathcal{P}ol_x := \frac{L_x}{[K_x, K_x]}.$$

Note that when $x$ varies, these form a sheaf of Lie algebras on $X \setminus x$. There is also a Hodge and an etale version of $\mathcal{P}ol_x$.

The Lie algebra $\mathcal{P}ol_x$ sits in an exact sequence

$$0 \to K_x/[K_x, K_x] \to \mathcal{P}ol_x \to \mathcal{H} \to 0.$$

It turns out that $\mathcal{H} = H_1(X, \mathbb{Q})$ and we define the logarithm Lie algebra to be $\mathcal{L}og_x(1) := K_x/[K_x, K_x]$.

**Theorem 3.0.7.** *If $x \in X$ is a torsion point, then*

$$\mathcal{L}og_x(1) \cong \prod_{k \geq 0} \operatorname{Sym}^k \mathcal{H}(1).$$

In particular one gets for $X = \mathbb{G}_m(\mathbb{C})$ two extension classes (note $\mathcal{H} = \mathbb{Q}(1)$):

$$\mathcal{P}ol_{x,\infty} \in \operatorname{Ext}^1_{\mathbb{R}-Hodge-str}(\mathbb{R}(1), \mathbb{R}(k+1))$$
$$\mathcal{P}ol_{x,l} \in \operatorname{Ext}^1_{\operatorname{Gal}(\overline{K}/K)}(\mathbb{Q}_l(1), \mathbb{Q}_l(k+1)).$$

The properties of these classes are summarized in the next theorem:

**Theorem 3.0.8.** *1. $X = \mathbb{G}_m$:*

- *(a) (Huber-Wildeshaus): $\mathcal{P}ol_{x,\infty}$ and $\mathcal{P}ol_{x,l}$ are in the image of the K-theory regulator.*
- *(b) (Beilinson-Deligne): $\mathcal{P}ol_{x,\infty}$ is given by $Li_k(x) = \sum_{n \geq 1} \frac{x^n}{n^k}$ for $k > 1$.*
- *(c) (Deligne): $\mathcal{P}ol_{x,l}$ is given by "twists" of cyclotomic units $1 - x$.*

*2. $X = E$:*

- *(a) (Beilinson-Levin): $\mathcal{P}ol_{x,\infty}$ and $\mathcal{P}ol_{x,l}$ are in the image of the K-theory regulator.*
- *(b) (Beilinson-Levin): $\mathcal{P}ol_{x,\infty}$ is given by Eisenstein-Kronecker series of weight $k + 2$.*
- *(c) (Kings): $\mathcal{P}ol_{x,l}$ is given by "twists" of elliptic units.*

This theorem gives the desired construction of K-theory elements, the relation to L-values and the relation to Iwasawa theory.

# References


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