AN EXPLICIT FORMULA FOR LOCAL DENSITIES OF QUADRATIC FORMS

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0. Introduction.

Let S and T be two positive definite integral matrices. It is an ancient but still very challenging problem to determine how many times S can represent T, i.e., the number of integral matrices X with ${}^tXSX = T$. However, Siegel proved in his celebrated paper ([Si]) that certain weighted averages of these numbers over the genus of S can be expressed as the Euler product of pure local data—confluent hypergeometric functions for $p = \infty$ and local densities $\alpha_p(T, S)$ for $p < \infty$ (see (1.5) for definition). This reduced a difficult global problem to pure local problems. Siegel himself extended this result, as did Weil and most recently Kudla and Rallis. They show that theta integrals equal the central value of Eisenstein series (see [We] or [KR] for example). When the central value of the Eisenstein series is forced to be zero by its functional equation, Kudla asserted and proved ([Ku1]) in a very interesting special case that the central derivative of the Eisenstein series is deeply connected with height pairing on Shimura varieties. One part of his proof depends on the explicit formula of local density $\alpha_p(T, S)$ for rank $T = 1$ or 2, which was only known in the 'unramified case', i.e., p does not divide $2\det S$, ([Si], [Ki]), until very recently. Bruce Meyers ([Me]) extended the local density formula to the case that pS^{-1} is p-integral and $p \neq 2$ (rank $T \leq 2$). Local density was also used in Gross and Keating's work on the interesection number of modular correspondence ([GK]). The purpose of this work is to give a formula for $\alpha_p(T, S)$ for arbitrary S and rank $T \leq 2$, with one exception rank $T = 2 = p$. Our method is quite different from others in this subject. Instead of using reduction and functional equation to reduce the problem to a simpler one and then computing the simpler one, we relate the local density to a (Whittaker) integral and then compute the integral directly. The results may be used to complete Kudla's work in the special case mentioned above. The local Whittaker integrals we computed are local factors of the Fourier Coefficients of Siegel Eisenstein series on $\mathrm{Sp}(n)$ and are of importance in their own right. It should be mentioned that functional equation in the general case is very complicated and is not relating S to itself. The reduction formula is also absent in general and is only true after a certain stage (see Theorem 3.3). We should also mention that the case where S is hyperbolic, i.e., $S = \frac{1}{2}\begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}$, has been extensively studied as related to the classical Siegel Eisenstein series. Most recently, Katsurada gave a beautiful formula in this case with both T and p arbitrary. We refer to [Kat] and its bibliography for further reference in this matter. On the other hand, the confluent hypergeometric functions, the local problem at $p = \infty$, have been extensively studied by Shimura ([Sh]).

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In section 1, we review the relation between local densities and local Whittaker functions and set up basic notations. In section 2, we record formulas for a couple of simple Gauss type integrals which will be used in sections 3 and 7. In sections 3 ($p \neq 2$) and 4 ($p = 2$), we obtain the local density formula when T is a nonzero number, and derive some of its consequences. In particular, we generalize Siegel's reduction formula and give an explicit formula for local Whittaker integrals in this case, which is also true for $T = 0$. In the appendix after section 4, we work out a special example which includes a unexpected simple relation between local Whittaker functions coming from split and nonsplit quaternion algebras. In section 5, we prove an integral transformation formula which will translate the local Whittaker integral problem to a problem on Gauss integral on $\mathrm{GL}_2(\mathbb{Z}_p)$. In section 6, we give a complete solution to the Gauss integral on $\mathrm{GL}_2(\mathbb{Z}_p)$. Results in both sections 5 and 6 should have independent interests. It is also interesting to note that the Gauss integral just mentioned is related to rational points of certain elliptic curves over the finite field $\mathbb{F}_p$ in some case (Proposition 6.5). In section 7, we obtain an explicit formula for $\alpha_p(T, S)$ in the case rank $T = 2$, $p \neq 2$ and arbitrary S.

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1. Whittaker functionals and local densities.

In this section, we recall some basic facts on Whittaker functionals and local density and their relations and set up notations for this paper. We mainly follow Kudla ([Ku1, Appendix]).

Let (V, q) be a nondegerate quadratic space over $\mathbb{Q}_p$ of dimension m with quadratic form $q(x) = \frac{1}{2}(x, x)$. Let

$$G = \begin{cases} \mathrm{Sp}(n)(\mathbb{Q}_p) & \text{if } m = \text{ even}, \\ \mathrm{Mp}(n)_{\mathbb{Q}_p} & \text{if } m = \text{ odd}. \end{cases}$$

Here $\mathrm{Mp}(n)_{\mathbb{Q}_p}$ is the metaplectic cover of $\mathrm{Sp}(n)(\mathbb{Q}_p)$. We choose and fix an additive character of $\mathbb{Q}_p$ as follows

$$\psi(x) = e^{-2\pi i \lambda(x)}$$

with $\lambda : \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathbb{Q}/\mathbb{Z}$. Then the reductive dual pair $(O(V), \mathrm{Sp}(n))$ gives rise to a Weil representation $\omega = \omega_{\psi, V}$ of G on $S(V^n)$. We refer [Ku2] for explicit formula and refer ([Ho], [MVW]) for basic facts on reductive dual pair and Weil representation. Let

$$\chi_V(a) = (a, (-1)^{\frac{m(m-1)}{2}} \det V))_p$$

be the quadratic character of $\mathbb{Q}_p^*$ associated to V, where $(x, y)_p$ is the Hilbert symbol. Let $P = NM$ be the Siegel parabolic subgroup of G where

$$N = \{n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : {}^t b = b\}$$

is the unipotent radical of P and

$$M = \{m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} : a \in \mathrm{GL}_n(\mathbb{Q}_p)\}$$

is a Levi factor of P. Then ([Ku2])

$$\begin{aligned} (1.1) \qquad \omega(m(a))\phi(x) &= \chi_{V,\psi}(a)|a|^{m/2}\phi(xa) \\ \omega(n(b))\phi(x) &= \psi(\frac{1}{2}\ \mathrm{tr}(b(x,x)))\phi(x) \\ \omega(w)\phi(x) &= \gamma(V)\int_{V^n}\phi(y)\psi(-\ \mathrm{tr}(x,y))d_V y \end{aligned}$$

where

$$\chi_{V,\chi}(a) = \begin{cases} \chi_V(\det a) & \text{if } m = \text{ even,} \\ \chi_V(a)\gamma(\det a, \frac{1}{2}\psi)^{-1} & \text{if } m = \text{ odd} \end{cases}$$

is a character of $\mathrm{GL}_n(\mathbb{Q}_p)$ (and of P), $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$$\gamma(V) = \epsilon(V)\gamma(\frac{1}{2}\psi)^m\gamma(\det V, \frac{1}{2}\psi).$$

Here $\epsilon(V)$ is the Hasse invariant of V, $\gamma(\psi)$ and $\gamma(a,\psi)$ are local Weil indices ([We], [Rao]). The Haar measure $d_V y$ on V is chosen to be self-dual with respect to the bilinear character $\psi((x,y))$, $x, y \in V$. The Haar measure on V^n is the product measure.

Let $I_n(s,\chi_V) = \mathrm{Ind}_P^G \chi_V|\,|^s$ be the degenerated principal series. Then the map

$$S(V^n) \longrightarrow I_n(s_0,\chi_V), \quad \phi \mapsto \{g \mapsto \omega(g)\phi(0)\}$$

is G-intertwining, where $s_0 = \frac{m-n-1}{2}$ and $S(V^n)$ is the space of locally constant functions on V^n with compact support. Given $\phi \in S(V^n)$, we associate a standard section $\Phi \in I_n(s,\chi_V)$ via

$$\Phi(g,s) = \omega(g)\phi(0)|a(g)|^s,$$

where $a(g) = \det a$ if $g = n(b)m(a)k$ with $b \in \mathrm{Sym}_n(\mathbb{Q}_p)$, $a \in \mathrm{GL}_n(\mathbb{Q}_p)$ and $k \in \mathrm{Sp}(n)(\mathbb{Z}_p)$. We remark that $a(g)$ is not well-defined but $|a(g)|$ is. Given $T \in \mathrm{Sym}_n(\mathbb{Q}_p)$, one defines the T-th Whittaker functional of ϕ (or Φ) by

$$W_T(g,s,\phi) = \int_{\mathrm{Sym}_n(\mathbb{Q}_p)}\Phi(wn(b)g,s)\psi(-\ \mathrm{tr}Tb)db.$$

Here db is the Haar measure on $\mathrm{Sym}_n(\mathbb{Q}_p)$ so that $\mathrm{meas}(\mathrm{Sym}_n(\mathbb{Z}_p)) = 1$. One may assume $g = 1$ by replacing ϕ with $\omega(g)\phi$. Easy calculation gives

$$(1.3) \quad W_T(1,s_0,\phi) = \gamma(V)\int_{\mathrm{Sym}_n(\mathbb{Q}_p)}\int_{V^n}\phi(x)\psi(\frac{1}{2}\ \mathrm{tr}b(x,x))\psi(-\ \mathrm{tr}Tb)\,d_V x\,db.$$

¿From now on, we assume that T is nonsingular, i.e., $\det T \neq 0$. Define the Weighted local density for a datum (T,V,ϕ) by

$$\alpha_p(T,V,\phi) = \lim_{t\longrightarrow\infty}\int_{\substack{x\in V^n \\ \frac{1}{2}(x,x)\equiv T \bmod p^t}}\phi(x)d_V x.$$

Then Kudla proved in [Ku1] that

$$(1.4) \qquad W_T(1,s_0,\phi) = \gamma(V)\alpha_p(T,V,\phi).$$

The following basic properties of weighted local densities follow easily from definition and (1.1).

Lemma 1.1. *Write $\phi^a(x) = \phi(xa)$ for $a \in GL_n(\mathbb{Q}_p)$. Then*

(1) $\alpha_p(T, V, \omega(n(b))\phi) = \psi(\mathrm{tr} Tb)\alpha_p(T, V, \phi)$.

(2) $\alpha_p(T, V, \omega(m(a))\phi) = \chi_{V,\psi}(a)|a|^{m/2}\alpha_p(T, V, \phi)$.

(3) *For $a \in \mathbb{Q}_p^*$, one has*

$$\alpha(aT, aV, \phi) = |a|^{-\frac{n(n+1)}{2}}\alpha_p(T, V, \phi).$$

Here aV means the same space V with quadratic form aq.

Two symmetric matrices T and T' over $\mathbb{Q}_p$ of degree n are called $\mathbb{Z}_p$-equivalent if there is a matrix $a \in \mathrm{GL}_n(\mathbb{Z}_p)$ such that ${}^taTa = T'$. We often write $T[a]$ for ${}^taTa$.

Lemma 1.2. *For $a \in GL_n(\mathbb{Z}_p)$, one has*

$$\alpha_p(T[a], V, \phi) = \alpha_p(T, V, \phi^a)$$

In particular, $\alpha_p(T, V, \phi)$ is only dependent on the $\mathbb{Z}_p$-equivalence class of T when ϕ is right $GL_n(\mathbb{Z}_p)$-invariant.

Now turn to the classical local density. Given two integral nondegenerated symmetric matrices S and T over $\mathbb{Z}_p$ of degree m and n respectively. We assume $m > n \geq 1$. Then the local density of representing T by S is defined by Siegel ([Si]) as follows:

$$\alpha_p(T, S) = \lim_{t \longrightarrow \infty} (p^t)^{\frac{n(n+1)}{2} - mn} A_t(T, S), \tag{1.5}$$

where

$$A_t(T, S) = \#\{X \in M_{m,n}(\mathbb{Z}_p/p^t) : S[X] \equiv T \bmod p^t\}.$$

For a nondegenerate symmetric matrix S of degree m, we associate a quadratic space $V = \mathbb{Q}_p^m$ of columns of length m with quadratic form $q(x) = \frac{1}{2}(x, x) = {}^txSx$. Let $L = \mathbb{Z}_p^m$ be its standard lattice. We will use a different Haar measure on V instead of the self-dual one d_Vx in this paper. On $\mathbb{Q}_p$, we fix the Haar measure so that $\mathrm{meas}(\mathbb{Z}_p) = 1$. On V, we use the product measure dx. The difference between the product measure dx and the self-dual measure d_Vx is a factor $|\det 2S|^{\frac{1}{2}}$, that is

$$\int_V \phi(x) d_Vx = |\det 2S|^{\frac{1}{2}} \int_V \phi(x) dx, \quad \phi \in S(V). \tag{1.6}$$

Set

$$W(T, S) = \int_{\mathrm{Sym}_n(\mathbb{Q}_p)} \int_{L^n} \psi(\mathrm{tr} bq(x))\psi(-\mathrm{tr} Tb)\, dx\, db. \tag{1.7}$$

Lemma 1.3. *With notation as above, one has*

$$\alpha_p(T, S) = W(T, S) = |\det S|^{\frac{-n}{2}}\gamma(V)^{-1}W_T(1, s_0, \; char(L)^n),$$

where $s_0 = \frac{m-n-1}{2}$.

Proof. The last equality follows from (1.3), (1.6) and (1.7). So one has by (1.4) for $t >> 0$,

$$\begin{aligned}
W(T,S) &= p^{t\frac{n(n+1)}{2}} \int_{\substack{X\in L^n \\ \frac{1}{2}(X,X)\equiv T \bmod p^t}} dX \\
&= p^{t\frac{n(n+1)}{2}} \sum_{\substack{X\in (L/p^tL)^n \\ \frac{1}{2}(X,X)\equiv T \bmod p^t}} \int_{X+p^tL^n} dX \\
&= p^{t\frac{n(n+1)}{2}} \sum_{\substack{X\in (L/p^tL)^n \\ \frac{1}{2}(X,X)\equiv T \bmod p^t}} p^{tmn} \operatorname{meas}(L,dx)^n \\
&= (p^t)^{\frac{n(n+1)}{2}-mn} \sum_{\substack{X\in (L/p^tL)^n \\ S[X]\equiv T \bmod p^t}} 1
\end{aligned}$$

since $\operatorname{meas}(L,dx)=1$. This finishes the proof.

To compute the Whittaker function $W_T(1,s,(\operatorname{char}L)^n)$, one can use Kudla's trick ([Ku1, Appendix]) as follows.

Lemma 1.4. *For any integer* $r\ge 0$, *let* $S_r = S \perp \frac{1}{2}\begin{pmatrix} 0 & I_r \\ I_r & 0\end{pmatrix}$. *Then*

$$W_T(1, r+s_0, (\operatorname{char}L)^n) = |\det 2S|^{\frac{n}{2}}\gamma(V)\alpha_p(T,S_r).$$

It is known that $W_T(1,s,\phi)$ is a rational function of $X=q^{-s}$. So Lemma 1.4 gives a formula for $W_T(1,s,(\operatorname{char}L)^n)$ once we know a formula for $\alpha_p(T,S)$.

By definition, $W(T,S)$ is only dependent on the $\mathbb{Z}_p$-equivalence class of S. By theory of quadratic forms, we may assume, for $p\ne 2$, that S is $\mathbb{Z}_p$-equivalent to

$$\text{(1.8)}\qquad \operatorname{diag}(\epsilon_1 p^{l_1},\cdots,\epsilon_m p^{l_m}),\quad \epsilon_i\in\mathbb{Z}_p^*, l_1\le l_2\le\cdots\le l_m\in\mathbb{Z}.$$

Notice that l_m is the smallest integer such that $p^{l_m}S^{-1}$ is half-integral. We end this section with some more notation needed later. Denote for any $b,c\in\mathbb{Z}$.

$$\text{(1.9)}\qquad \begin{aligned} L(b) &= \{1\le i\le m:\, b+l_i<0\},\quad l(b)=\#L(b), \\ L(b,c) &= \{i\in L(b):\, b+l_i\equiv k \bmod 2\},\quad l(b,c)=\#L(b,c).\end{aligned}$$

Notice that $l(b)=l(b,0)+l(b,1)$ and that $l(b)=m$ when $b+l_m<0$. Moreover, if $b+l_m<0$, $L(b,c)$ is only dependent on the parities of b and c. Denote

$$\text{(1.10)}\qquad \delta_p = \begin{cases} 1 & \text{if } p\equiv 1 \bmod 4, \\ \sqrt{-1} & \text{if } p\equiv -1 \bmod 4.\end{cases}$$

Finally, we write $(\frac{\alpha}{p})$ for the Hilbert symbol $(\alpha,p)_p$ for $\alpha\in\mathbb{Z}_p$ in this paper.

2. Some integrals.

In this section, we compute the following four integrals needed later.

$$I(t) = \int_{\mathbb{Z}_p} \psi(tx^2) dx, \quad t \in \mathbb{Q}_p^*, \tag{2.1}$$

$$I^*(t) = \int_{\mathbb{Z}_p^*} \psi(tx^2) dx, \quad t \in \mathbb{Q}_p^*, \tag{2.2}$$

$$J(\beta p^b) = \int_{\mathbb{Z}_p^m} \psi(\beta p^b q(x)) dx, \quad \beta \in \mathbb{Z}_p^*, b \in \mathbb{Z}, \tag{2.3}$$

$$I(t, \chi) = \int_{\mathbb{Z}_p^*} \chi(x)\psi(tx) dx, \quad t \in \mathbb{Q}_p^*. \tag{2.4}$$

Here $q(x)$ the quadratic form associated to S as in section 1, and χ is a character of $\mathbb{Q}_p^*$. Here the Haar measure on $\mathbb{Q}_p^*$ is the restriction of the Haar measure on $\mathbb{Q}_p$. We assume $p \neq 2$ in this section.

Lemma 2.1. *Let $t = \alpha p^a \in \mathbb{Q}_p^*$ with $\alpha \in \mathbb{Z}_p^*$ and $a \in \mathbb{Z}$.*

(1) *Let $\gamma_a(\alpha) = 1$ if a is even, and $= \delta_p(\frac{-\alpha}{p})$ if a is odd. Then*

$$I(t) = \begin{cases} 1 & \text{if } a \geq 0, \\ p^{\frac{a}{2}}\gamma_a(\alpha) & \text{if } a < 0. \end{cases}$$

(2)

$$I^*(t) = I(t) - p^{-1}I(p^2 t) = \begin{cases} 1 - p^{-1} & \text{if } a \geq 0, \\ p^{-\frac{1}{2}}\gamma_1(\alpha) - p^{-1} & \text{if } a = -1, \\ 0 & \text{if } a \leq -2. \end{cases}$$

In particular, $I(t) = p^{-1}I(p^2t)$ when $\mathrm{ord}_p t \leq -2$.

(3) *For any $\beta \in Z_p^*$, one has*

$$\int_{\beta + p\mathbb{Z}_p} \psi(tx^2) dx = p^{-1}\psi(t\beta^2)\ \mathrm{char}(p^{-1}\mathbb{Z}_p)(t).$$

Proof. (1). It is obvious for $a \geq 0$. Assume $a < 0$. Let $k = [\frac{-a+1}{2}]$ be the integral part of $\frac{-a+1}{2}$. Then

$$\begin{aligned} I(t) &= \sum_{y \in \mathbb{Z}_p/p^k} \int_{y + p^k\mathbb{Z}_p} \psi(tx^2) dx \\ &= \sum_{y \in \mathbb{Z}_p/p^k} p^{-k}\psi(ty^2) \int_{\mathbb{Z}_p} \psi(2\alpha p^{a+k} yx) dx \\ &= p^{-k} \sum_{y \in \mathbb{Z}_p/p^k} \psi(ty^2)\ \mathrm{char}(p^{-a-k})(y) \end{aligned}$$

When a is even, $a = -2k$ and $-a-k = k$. So the only nonzero term in last integral is when $y = 0$. Therefore $I(t) = p^{-k} = p^{a/2}$. When a is odd, $a = -2k+1$ and $-a-k = k-1$. So

$$\begin{aligned} I(t) &= p^{-k} \sum_{y \in \mathbb{Z}_p/p} \psi(tp^{2k-2}y^2) \\ &= p^{-k} \sum_{y \in \mathbb{Z}/p} e^{\frac{2\pi i \alpha y^2}{p}} \\ &= p^{-k} \sqrt{p} \delta_p (\frac{-\alpha}{p}) \end{aligned}$$

as expected, the final step is a well-known Gauss sum formula . Claim (2) follows from (1) easily. To prove (3), one first notices that

$$\int_{\beta + p\mathbb{Z}_p} \psi(tx^2) dx = p^{-1} \psi(t\beta^2) \int_{\mathbb{Z}_p} \psi(2\beta tpx) \psi(tp^2x^2) dx.$$

Write $t = \alpha p^a$ with $\alpha \in \mathbb{Z}_p^*$ and $a \in \mathbb{Z}$. When $a \ge -1$,

$$\int_{\mathbb{Z}_p} \psi(2\beta tpx) \psi(tp^2x^2) dx = \int_{\mathbb{Z}_p} dx = 1$$

as expected. When $a < -1$, we have to prove that the integral vanishes. Substitute x by $x + p^{-a-2}$, one has

$$\int_{\mathbb{Z}_p} \psi(2\beta tpx) \psi(tp^2x^2) dx = \psi(2\alpha\beta p^{-1}) \int_{\mathbb{Z}_p} \psi(2\beta tpx) \psi(tp^2x^2) dx.$$

Since $\psi(2\alpha\beta p^{-1}) \ne 1$, one sees that the integral has to be zero.

As for integral (2.3), notice that $q(x) = \sum \epsilon_i p^{l_i} x_i^2$ by (1.8). So Lemma 2.1 implies

Lemma 2.2. *With notation as above, one has*

$$J(\beta p^b) = p^{\frac{l(b)b}{2} + \frac{1}{2}\sum_{i \in L(b)} l_i} (-(\frac{\beta}{p})\delta_p)^{l(b,1)} \prod_{i \in L(b,1)} (\frac{\epsilon_i}{p}).$$

In particular, $J(\beta p^b) = 1$ *when* $b \ge 0$.

Corollary 2.3. *Notation as above.*

(1) $J(\beta p^b) = (\frac{\beta}{p})^{l(b,1)} J(p^b)$.

(2) *If* $b + l_m < 0$ *then*

$$J(p^b) = p^{\frac{1}{2}(mb + \sum l_i)} (-\delta_p)^{l(b,1)} \prod_{i \in L(b,1)} (\frac{\epsilon_i}{p}).$$

Lemma 2.4. *Let $t = \alpha p^a$ with $\alpha \in \mathbb{Z}_p^*$ and $a \in \mathbb{Z}$, and let χ be a character of $\mathbb{Q}_p$ of conductor $n(\chi)$. Then*

$$I(t, \chi) = \begin{cases} 1 - p^{-1} & \text{if } a \geq 0 = n(\chi), \\ p^a \chi(-\alpha) G(\chi) & \text{if } n(\chi) = -a, \\ 0 & \text{otherwise.} \end{cases}$$

Here $G(\chi)$ is the Gauss sum of χ. In particular, one has

$$I(t, (\frac{}{p})) = p^{-\frac{1}{2}} \delta_p(\frac{-\alpha}{p}) \ \mathrm{char}(p^{-1}\mathbb{Z}_p^*)(t).$$

Proof. We may assume that $a \leq 0$. First assume $n(\chi) = -a$. Notice that χ can be viewed as a character of $(\mathbb{Z}/p^{n(\chi)})^*$. So

$$\begin{aligned} I(t, \chi) &= \sum_{y \in (\mathbb{Z}/p^{-a})} \int_{y + p^{-a}\mathbb{Z}_p} \chi(x) \psi(tx) \, dx \\ &= p^a \sum_{y \in (\mathbb{Z}/p^{-a})^*} \chi(y) \psi(\alpha p^a y) \\ &= p^a \sum_{y \in (\mathbb{Z}/p^{-a})^*} \chi(y) e^{2\pi i \alpha y p^a} \\ &= p^a \chi(-\alpha) G(\chi) \end{aligned}$$

as desired. Next we assume $n(\chi) > -a$. In this case, one has

$$\begin{aligned} I(t, \chi) &= \sum_{y \in (\mathbb{Z}/p^{-a})^*} \int_{y + p^{-a}\mathbb{Z}_p} \chi(x) \psi(tx) \, dx \\ &= \sum_{y \in (\mathbb{Z}/p^{-a})^*} \psi(\alpha p^a y) \chi(y) \int_{1 + p^{-a}\mathbb{Z}_p} \chi(x) dx \\ &= 0. \end{aligned}$$

The case $n(\chi) < -a$ is similar.

3. The case $n = 1$ and $p \neq 2$.

In this section, we assume that $n = 1$ and $p \neq 2$ so that S is $\mathbb{Z}_p$-equivalent to $\mathrm{diag}(\epsilon_1 p^{l_1}, \cdots, \epsilon_m p^{l_m})$ with $\epsilon_i \in \mathbb{Z}_p^*$ and $l_1 \leq l_2 \cdots \leq l_m$. Set $\epsilon_S = \prod \epsilon_i \in \mathbb{Z}_p^*/\mathbb{Z}_p^{*2}$. For any positive integer k, define

$$v_k = (\frac{-1}{p})^{[\frac{l(-k,1)}{2}]} \prod_{i \in L(-k,1)} (\frac{\epsilon_i}{p}). \tag{3.1}$$

Here we recall $l(-k, c) = \#L(-k, c)$ for $c = 0$ or 1, and

$$L(-k, c) = \{1 \leq i \leq m : \ l_i - k < 0 \text{ and } l_i - k \equiv c \bmod 2\}.$$

Denote

$$d(k) = k + \frac{1}{2} \sum_{l_i < k} (l_i - k). \tag{3.2}$$

Theorem 3.1. *Let the notation be as above. Let $t = \alpha p^a$ with $\alpha \in \mathbb{Z}_p^*$ and $a \in \mathbb{Z}$. Assume that $p \neq 2$. Then $\alpha_p(t, S) = 0$ unless $a \geq l_1$. When $a \geq l_1$ one has*

$$\alpha_p(t, S) = p^{l_1} + (1 - p^{-1}) \sum_{\substack{l_1 < k \leq a \\ l(-k,1) \text{ even}}} v_k p^{d(k)} + v_{a+1} f(t) p^{d(a+1)}$$

where

$$f(t) = \begin{cases} -\frac{1}{p} & \text{if } l(-a-1, 1) \text{ is even,} \\ (\frac{\alpha}{p}) \frac{1}{\sqrt{p}} & \text{if } l(-a-1, 1) \text{ is odd.} \end{cases}$$

Proof. First notice that one has has by Lemma 1.1(3)

$$\alpha_p(t, S) = p^l \alpha_p(p^{-l} t, p^{-l} S).$$

It is not hard to see from this fact that it suffices to prove the formula for $l_1 = 0$. In this case, it is obvious that $\alpha_p(t, S) = 0$ when $a < 0$. Assume now $a \geq 0$. By Lemmas 1.3 and 2.2, one has

$$\begin{aligned} \alpha_p(t, S) &= W(t, S) \\ &= \int_{\mathbb{Q}_p} J(b) \psi(-tb) db \\ &= 1 + \sum_{k>0} p^k \int_{\mathbb{Z}_p^*} J(\beta p^{-k}) \psi(-\alpha \beta p^{a-k}) d\beta \\ &= 1 + \sum_{\substack{k>0 \\ l(-k,1) \text{ even}}} p^{d(k)} \delta_p^{l(-k,1)} \prod_{i \in L(-k,1)} (\frac{\epsilon_i}{p}) \int_{\mathbb{Z}_p^*} \psi(-\alpha \beta p^{a-k}) d\beta \\ &\quad + \sum_{\substack{k>0 \\ l(-k,1) \text{ odd}}} p^{d(k)} (-\delta_p)^{l(-k,1)} \prod_{i \in L(-k,1)} (\frac{\epsilon_i}{p}) \int_{\mathbb{Z}_p^*} (\frac{\beta}{p}) \psi(-\alpha \beta p^{a-k}) d\beta \end{aligned}$$

Here δ_p is the number defined by (1.10). Notice that $\delta_p^2 = (\frac{-1}{p})$ and thus

$$v_k = \begin{cases} \delta_p^{l(-k,1)} \prod_{i \in L(-k,1)} (\frac{\epsilon_i}{p}) & \text{if } l(-k,1) \text{ is even,} \\ -\delta_p^{l(-k,1)+1} \prod_{i \in L(-k,1)} (\frac{\epsilon_i}{p}) & \text{if } l(-k,1) \text{ is odd.} \end{cases}$$

So one has by Lemma 2.4

$$\begin{aligned} \alpha_p(t, S) = 1 &+ \sum_{\substack{k>0 \\ l(-k,1) \text{ even}}} v_k p^{d(k)} (\operatorname{char}(p^k \mathbb{Z}_p) - \frac{1}{p} \operatorname{char}(p^{k-1} \mathbb{Z}_p))(t) \\ &+ \sum_{\substack{k>0 \\ l(-k,1) \text{ odd}}} v_k p^{d(k)} (\frac{\alpha}{p}) p^{-\frac{1}{2}} \operatorname{char}(p^{k-1} \mathbb{Z}_p^*)(t) \\ &= 1 + (1 - \frac{1}{p}) \sum_{\substack{k>0 \\ l(-k,1) \text{ even}}} v_k p^{d(k)} + v_{a+1} f(t) p^{d(a+1)}. \end{aligned}$$

When all $l_i = 0$, we recover a nice formula of Siegel ([Si])

Corollary 3.2. *(Siegel) Assume $p \neq 2$ and that S is unimodular and $m > 1$. Assume $t = \alpha p^a$ with $a \in \mathbb{Z}$ and $\alpha \in \mathbb{Z}_p^*$. Then $\alpha_p(t, S) = 0$ unless t is integral, i.e., $a \geq 0$. In this case, let*

$$\nu = \begin{cases} ((-1)^{\frac{m}{2}} \det S,\, p)_p & \text{if } m \text{ is even,} \\ ((-1)^{\frac{m-1}{2}} \alpha \det S,\, p)_p & \text{if } m \text{ is odd,} \end{cases}$$

and let

$$X = \begin{cases} p^{1-\frac{m}{2}} & \text{if } m \text{ is even,} \\ p^{2-m} & \text{if } m \text{ is odd.} \end{cases}$$

Then

$$\alpha_p(t, S) = \begin{cases} (1 - \nu p^{-1} X) \sum_{k=0}^{a} (\nu X)^k & \text{if } m \text{ is even,} \\ (1 - p^{-1} X) \sum_{k=0}^{\frac{a-1}{2}} X^k & \text{if } m \text{ and } a \text{ are odd,} \\ (1 - p^{-1} X)(\sum_{k=0}^{\frac{a}{2}-1} X^k + \frac{X^{\frac{a}{2}}}{1 - \nu p^{\frac{1-m}{2}}}) & \text{if } m \text{ is odd and } a \text{ is even.} \end{cases}$$

Proof. Since S is unimodular, S is $\mathbb{Z}_p$-equivalent to $\operatorname{diag}(\epsilon_1, \cdots, \epsilon_m)$ with $\epsilon_i \in \mathbb{Z}_p^*$. That is $l_i = 0$ for all i. So $l(-k) = m$ for all $k > 0$, and $l(-k, 1) = 0$ or m depending on whether k is even or odd.

Case 1: m is even. In this case, $v_k = \nu^k$ and $l(-k, 1)$ is always even. So

$$\begin{aligned} \alpha_p(t, S) &= 1 + (1 - \frac{1}{p}) \sum_{1 \leq k \leq a} v_k p^{k(1-\frac{m}{2})} - \frac{1}{p} v_{a+1} p^{(a+1)(1-\frac{m}{2})} \\ &= 1 + (1 - \frac{1}{p}) \sum_{1 \leq k \leq a} (\nu X)^k - \frac{1}{p} (\nu X)^{a+1} \\ &= (1 - \nu p^{-1} X) \sum_{0 \leq k \leq a} (\nu X)^k \end{aligned}$$

as expected.

Case 2: m is odd. In this case, $l(-k, 1)$ is even if and only if k is even. One also has

$$v_k = \begin{cases} 1 & \text{if } k \text{ is even,} \\ \nu(\frac{\alpha}{p}) & \text{if } k \text{ is odd.} \end{cases}$$

When a is odd, one has

$$\begin{aligned} \alpha_p(t, S) &= 1 + (1 - \frac{1}{p}) \sum_{k=1}^{\frac{a-1}{2}} p^{2k(1-\frac{m}{2})} - \frac{1}{p} p^{(a+1)(1-\frac{m}{2})} \\ &= (1 - p^{-1} X) \sum_{k=0}^{\frac{a-1}{2}} X^k \end{aligned}$$

When a is even, $a+1$ is odd.

$$\begin{aligned}
\alpha_p(t,S) &= 1+(1-\frac{1}{p})\sum_{k=1}^{\frac{a}{2}} p^{2k(1-\frac{m}{2})} + (\frac{\alpha}{p})\frac{1}{\sqrt{p}}\nu(\frac{\alpha}{p})p^{(a+1)(1-\frac{m}{2})} \\
&= \sum_{k=0}^{\frac{a}{2}} X^k - \frac{1}{p}\sum_{k=1}^{\frac{a}{2}} + \nu p^{\frac{1-m}{2}} X^{\frac{a}{2}} \\
&= (1-p^{-1}X)\sum_{k=0}^{\frac{a}{2}-1} X^k + X^{\frac{a}{2}}(1+\nu p^{\frac{1-m}{2}}) \\
&= (1-p^{-1}X)(\sum_{k=0}^{\frac{a}{2}-1} X^k + \frac{X^{\frac{a}{2}}}{1-\nu p^{\frac{1-m}{2}}})
\end{aligned}$$

as desired. Here we have used

$$1-p^{-1}X = 1-p^{1-m} = (1+\nu p^{\frac{1-m}{2}})(1-\nu p^{\frac{1-m}{2}}).$$

Theorem 3.3 (Induction formula). *Let the notation be as in Theorem* 3.1. *Then for* $\mathrm{ord}_p t \ge l_m+2$, *one has*

$$\alpha_p(tp^2,S)-\alpha_p(t,S) = p^{2-m}(\alpha_p(t,S)-\alpha_p(tp^{-2},S)).$$

Furthermore, if m is even, one has for $\mathrm{ord}_p t \ge l_m+1$

$$\alpha_p(tp,S)-\alpha_p(t,S) = \nu p^{1-\frac{m}{2}}(\alpha_p(t,S)-\alpha_p(tp^{-1},S)).$$

Here ν is the number defined in Corollary 3.2 *with* $\det S$ *replaced by* $\epsilon_S = \prod \epsilon_i$.

Proof. Let $t=\alpha p^a$ with $a \ge l_m$. First notice that $d(k+1) = d(k)+1-\frac{m}{2}$ for $k > l_m$. One has by Theorem 3.1

$$\begin{aligned}
&\alpha_p(tp^2,S)-\alpha_p(t,S) \\
&= (1-\frac{1}{p})\sum_{\substack{a+1\le k\le a+2\\ l(-k,1)\ \text{even}}} v_k p^{d(k)} + v_{a+3}f(tp^2)p^{d(a+3)} - v_{a+1}f(t)p^{d(a+1)} \\
&= p^{d(a+1)}\{(1-\frac{1}{p})\sum_{\substack{0\le k\le 1\\ l(-k-a-1,1)\ \text{even}}} v_{a+1+k}p^{k(1-\frac{m}{2})} \\
&\qquad + v_{a+3}f(\alpha p^{a+2})p^{2(1-\frac{m}{2})} - v_{a+1}f(\alpha p^a)\}.
\end{aligned}$$

Notice that v_k, $l(-k,1)$, and $f(\alpha p^{k-1})$ depend only on the parity of k when $k > l_m$. So the expression in the brace does not change when we replace a by $a+2$. Therefore

$$\alpha_p(tp^4,S)-\alpha_p(tp^2,S) = p^{2-m}(\alpha_p(tp^2,S)-\alpha_p(t,S))$$

when $a = \mathrm{ord}_p t \ge l_m$. This proves the first induction formula.

Now assume that m is even. Again write $t = \alpha p^a$ with $a \geq l_m$. There are two cases. If $l(-a-1,1)$ is even, so is $l(-a-2,1) = m - l(-a-1,1)$ (thanks to $a+1 > l_m$). So

$$\begin{aligned}
&\alpha_p(tp, S) - \alpha_p(t, S) \\
&= (1-\frac{1}{p})v_{a+1}p^{d(a+1)} - \frac{1}{p}v_{a+2}p^{d(a+2)} + \frac{1}{p}v_{a+1}p^{d(a+1)} \\
&= v_{a+1}p^{d(a+1)}(1-\nu p^{-\frac{m}{2}}).
\end{aligned}$$

Here we have used the fact that when m is even

$$\frac{v_{a+1}}{v_{a+2}} = v_{a+1}v_{a+2} = \nu. \tag{3.3}$$

Because of (3.3), it is now obvious that

$$\alpha_p(tp^2, S) - \alpha_p(tp, S) = \nu p^{1-\frac{m}{2}}(\alpha_p(tp, S) - \alpha_p(t, S))$$

for $a \geq l_m$. If $l(-a-1,1)$ is odd, so is $l(-a-2,1) = m - l(-a-1,1)$. In this case Theorem 3.1 and Formula (3.3) imply

$$\begin{aligned}
\alpha_p(tp, S) - \alpha_p(t, S) &= v_{a+2}(\frac{\alpha}{p})\frac{1}{\sqrt{p}}p^{d(a+2)} - v_{a+1}(\frac{\alpha}{p})\frac{1}{\sqrt{p}}p^{d(a+1)} \\
&= -v_{a+1}(\frac{\alpha}{p})\frac{1}{\sqrt{p}}p^{d(a+1)}(1-\nu p^{1-\frac{m}{2}}).
\end{aligned}$$

So one has again by (3.3)

$$\alpha_p(tp^2, S) - \alpha_p(tp, S) = \nu p^{1-\frac{m}{2}}(\alpha_p(tp, S) - \alpha_p(t, S))$$

for $a = \mathrm{ord}_p t \geq l_m$. This proves the second induction formula.

Theorem 3.4. (1) *For any $\phi \in S(V)$, $g \in G$ and $t \neq 0$, $W_t(g, s+s_0, \phi)$ is a polynomial of $X = p^{-s}$ over $\mathbb{Q}_p$.*
(2) *Suppose $\phi =$ char(L) for a lattice L of V, let $S =$ diag$(\epsilon_1 p^{l_1}, \cdots, \epsilon_m p^{l_m})$ be the matrix of L with respect to a suitable basis with $l_1 \leq \cdots \leq l_m$ as explained in section 1. Then $W_t(1, s+s_0,$ char$(L))$ is a polynomial of X of degree max$(0,$ ord$_p t+1)$. More precisely, for $t = \alpha p^a$ with $a \geq$ min$(0, l_1)$, one has*

$$\begin{aligned}
&C^{-1}W_t(1, s+s_0, \ char(L)) \\
&= 1 + (1-p^{-1}) \sum_{0<k\leq min(l_1,a)} p^k X^k \\
&\quad + (1-p^{-1}) \sum_{\substack{min(l_1,a)<k\leq a \\ l(-k,1) \ even}} v_k p^{d(k)} X^k + v_{a+1} f(t) p^{d(a+1)} X^{a+1}
\end{aligned} \tag{3.4}$$

for $l_1 \geq 0$, and

$$\begin{aligned}
&C^{-1}W_t(1, s+s_0, \ char(L)) \\
&= p^{l_1} + (1-p^{-1}) \sum_{\substack{l_1<k\leq a \\ l(-k,1) \ even}} v_k p^{d(k)} X^{max(k,0)} + v_{a+1} f(t) p^{d(a+1)} X^{max(a+1,0)}
\end{aligned} \tag{3.5}$$

for $l_1 < 0$, where $C = |\det S|^{\frac{1}{2}}\gamma(V)$ is a constant.

Proof. Since $W_t(g, s, \phi) = W_t(1, s, \omega(g)\phi)$ and every $\phi \in S(V)$ is a linear combination of characteristic functions $\mathrm{char}(L)$ of lattices, it is enough to prove the formulas (3.4) and (3.5). For any large integer $r > 0$, one has by Lemmas 1.3 and (1.4)

$$W_t(1, r + s_0,\ \mathrm{char}(L)) = C\alpha_p(t, S_r)$$

where

$$S_r = S \perp \begin{pmatrix} 0 & \frac{1}{2}I_r \\ \frac{1}{2}I_r & 0 \end{pmatrix}.$$

Now (3.4) and (3.5) follow from Theorem 3.1 easily. For example, if $l_1 < 0$, let $l_r(-k, 1)$ and $d_r(k)$ stand for the numbers $l(-k, 1)$ and $d(k)$ for the matrix S_r respectively. Then

$$l_r(-k, 1) = \begin{cases} l(-k, 1) & \text{if } k \le 0, \\ l(-k, 1) + 2r & \text{if } k > 0, \end{cases}$$

and

$$d_r(k) = \begin{cases} d(k) & \text{if } k \le 0, \\ d(k) - kr & \text{if } k > 0. \end{cases}$$

In particular, $l_r(-k, 1)$ is even if and only if $l(-k, 1)$ is even. Also notice that v_k and $f(t)$ do not change. Applying, one obtains formula (3.5). Formula (3.4) is similar and left to the reader.

Remark 3.5. The formulas in Theorem 3.4 are also true for $t = 0$, i.e., $a = \infty$. In this case, $W_0(1, s + s_0,\ \mathrm{char} L)$ is a rational function of X and has a pole at $s = s_0$. In fact, Theorem 3.1 is also true for $t = 0$ for $m > n+1$. We will give an interesting example involving quaternion algebras in the appendix after section 4.

4. The case $n = 1$ and $p = 2$.

In this section, we consider the case $n = 1$ and $p = 2$. By [Ca, Lemma 8.4.1], a symmetric matrix (nonsingular) over $\mathbb{Q}_2$ is $\mathbb{Z}_2$ equivalent to

$$\mathrm{diag}(\epsilon_1 2^{l_1}, \cdots, \epsilon_L 2^{l_L}) \oplus \left(\oplus_{i=1}^{M} \epsilon_i' 2^{m_i} \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\right) \oplus \left(\oplus_{j=1}^{N} \epsilon_j'' 2^{n_j} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\right) \tag{4.1}$$

where $\epsilon_h, \epsilon_i', \epsilon_j'' \in \mathbb{Z}_2^*$, l_h, m_i, and n_j are all integers. Let l be the smallest integer among all l_h, m_i, and n_j. Of course, $L + 2M + 2N = m$ is the degree of S. The corresponding lattice has a basis over which the corresponding quadratic form is given by

$$q = \sum_{h=1}^{L} \epsilon_h 2^{l_h} x_i^2 + \sum_{i=1}^{M} \epsilon_i' 2^{m_i} y_{i1} y_{i2} + \sum_{j=1}^{N} \epsilon_j'' 2^{n_j} (z_{j1}^2 + z_{j1} z_{j2} + z_{j2}^2). \tag{4.2}$$

For each integer $k > 0$ and $t = \alpha 2^a$ with $\alpha \in \mathbb{Z}_2^*$ and $a \in \mathbb{Z}$, we denote

$$\begin{aligned} p(k) &= (-1)^{\sum_{n_j < k}(n_j - k)} \\ \epsilon(k) &= \prod_{h \in L(-k+1, 1)} \epsilon_h \\ \mu_k(t) &= \alpha 2^{a-k+3} - \sum_{l_h < k-1} \epsilon_h \end{aligned} \tag{4.3}$$

and

$$d(k) = k + \frac{1}{2}\sum_{l_h < k-1}(l_h - k + 1) + \sum_{m_i < k}(m_i - k) + \sum_{n_j < k}(n_j - k). \tag{4.4}$$

Here

$$L(b, c) = \{1 \le h \le L : l_h + b < 0 \,\&\, l_h + b \equiv c \bmod 2\}$$

and $l(b, c) = \#L(b, c)$ for any integers. We also define the Kronecker symbol for $x \in \mathbb{Q}_2$

$$(\frac{2}{x}) = \begin{cases} (2, x)_2 & \text{if } x \in \mathbb{Z}_2^*, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

Finally, we define $\delta(k) = 0$ or 1 depending on whether there exists $l_h = k - 1$ or not.

Theorem 4.1. *With notation as above, one has $\alpha_2(\alpha 2^a, S) = 0$ unless $a \ge l$. Furthermore, for $a \ge l$ one has*

$$\alpha_2(\alpha 2^a, S) = p^l + \sum_{\substack{l<k\le a+3 \\ l(-k+1,1) \ odd}} \delta(k)p(k)\left(\frac{2}{\mu\epsilon(k)}\right)2^{d(k)-\frac{3}{2}}$$

$$+ \sum_{\substack{l<k\le a+3 \\ l(-k+1,1) \ even}} \delta(k)p(k)\left(\frac{2}{\epsilon(k)}\right)2^{d(k)-1}\psi(\frac{\mu}{8}) \ char(4\mathbb{Z}_2))(\mu).$$

Here we write $\mu = \mu_k(t)$ to save space.

We remark that the terms in the formula with $k \le a$ are independent of a. From this it is not difficult to derive a inductive formula for $p = 2$ similar to Theorem 3.3. However, even when m is even, one does not have a reduction formula for $\alpha_2(2t, S) - \alpha_2(t, S)$ similar to that of Theorem 3.3 since there is not a good reduction formula for $\mu_k(t)$. In other words, we only have

Corollary 4.2. *Let the notation be as in Theorem 4.1. Then for $a \ge 4 + max(l_h, m_i, n_j)$, one has*

$$\alpha_2(4t, S) - \alpha_2(t, S) = 2^{2-m}\left(\alpha_2(t, S) - \alpha_2(\frac{1}{4}t, S)\right).$$

In order to prove Theorem 4.1 we need two lemmas. The first one is similar to Lemmas 2.1 and 2.4 and the proof is left to the reader.

Lemma 4.3. (1) *For $\alpha \in \mathbb{Z}_2^*$ and $a \in \mathbb{Z}$, one has*

$$\int_{\mathbb{Z}_2} \psi(\alpha 2^a x^2)dx$$

$$= \begin{cases} 1 & \textit{if } a \ge 0 \\ 0 & \textit{if } a = -1 \\ 2^{\frac{a+1}{2}}\psi(\frac{\alpha}{8})(\frac{2}{\alpha})^{a+1} & \textit{if } a < -1. \end{cases}$$

(2) *Let χ be a character of $\mathbb{Q}_2^*$ with conductor $n(\chi)$. Then for α and a as above, one has*

$$\int_{\mathbb{Z}_2^*} \chi(x)\psi(\alpha 2^a x)dx$$
$$= \begin{cases} \frac{1}{2} & \text{if } n(\chi) = 0 \,\&\, a \geq 0 \\ -\frac{1}{2} & \text{if } n(\chi) = 0 \,\&\, a = -1 \\ 2^a \chi(-\alpha) G(\chi) & \text{if } n(\chi) = -a \\ 0 & \text{otherwise.} \end{cases}$$

In particular, one has

$$\int_{\mathbb{Z}_2^*} (\frac{2}{x})\psi(\alpha 2^a x)dx = \frac{1}{2\sqrt{2}}(\frac{2}{-\alpha})\delta_{-a,3}.$$

Here $\delta_{m,n}$ is the usual Kronecker symbol.

Lemma 4.4. *For $t = \alpha 2^a$ with $\alpha \in \mathbb{Z}_2^*$ and $a \in \mathbb{Z}$, one has*

$$\int_{\mathbb{Z}_2^2} \psi(ty_1y_2)dy_1\, dy_2 = min(1, 2^a) \tag{4.6}$$

and

$$\int_{\mathbb{Z}_2^2} \psi(t(z_1^2 + z_1z_2 + z_2^2)dz_1dz_2 \tag{4.7}$$
$$= \begin{cases} 1 & \text{if } a \geq 0 \\ 2^a(-1)^a & \text{if } a < 0. \end{cases}$$

Proof. Only (4.7) needs some verification for $a < 0$. Set $k = [\frac{-a+1}{2}]$. Then

$$\int_{\mathbb{Z}_2} \psi(t(z_1z_2 + z_2^2))dz_2$$
$$= \sum_{x \in \mathbb{Z}/2^k} 2^{-k}\psi(t(z_1x + x^2)) \int_{\mathbb{Z}_2} \psi(\alpha 2^{a+k}(z_1 + 2x)z_2)dz_2$$
$$= 2^{-k} \sum_{x \in \mathbb{Z}/2^k} \psi(t(z_1x + x^2))\ \mathrm{char}(-2x + 2^{-a-k}\mathbb{Z}_2)(z_1).$$

So

$$\int_{\mathbb{Z}_2^2} \psi(t(z_1^2 + z_1z_2 + z_2^2))dz_1dz_2$$
$$= 2^{-k} \sum_{x \in \mathbb{Z}/2^k} \int_{-2x+2^{-a-k}\mathbb{Z}_2} \psi(t(z_1^2 + z_1x + x^2))dz_1$$
$$= 2^a \sum_{x \in \mathbb{Z}/2^k} \psi(3tx^2) \int_{\mathbb{Z}_2} \psi(-3x\alpha 2^{-k}z_1)\psi(\alpha 2^{-2k-a}z_1^2)dz_1.$$

When $a=-2k$ is even, the last integral is

$$\int_{\mathbb{Z}_2} \psi(-3x\alpha 2^{-k} z_1) dz_1 = \ \mathrm{char}(2^k\mathbb{Z}_2)(x).$$

This proves the even case.
When $a=-2k+1$ is odd, the last integral is

$$\begin{aligned}
&\int_{\mathbb{Z}_2} \psi(-3x\alpha 2^{-k} z_1)\psi(\frac{\alpha}{2}z_1^2) dz_1 \\
&= \int_{2\mathbb{Z}_2} \psi(-3x\alpha 2^{-k} z_1) dz_1 - \int_{1+2\mathbb{Z}_2} \psi(-x\alpha 2^{-k} z_1) dz_1 \\
&= \frac{1}{2}(1-\psi(-3x\alpha 2^{-k}))\ \mathrm{char}(2^{k-1}\mathbb{Z}_2)(x) \\
&= \ \mathrm{char}(2^{k-1}\mathbb{Z}_2^*)(x).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{\mathbb{Z}_2^2} \psi(t(z_1^2+z_1z_2+z_2^2)) dz_1 dz_2 \\
&= 2^a\psi(3\alpha 2^a 2^{2k-2}) \\
&= 2^a\psi(\frac{3\alpha}{2}) = -2^a.
\end{aligned}$$

Proof of Theorem 4.1 Now the proof of Theorem 4.1 becomes a formality. Just as in the proof of Theorem 3.1, we may assume that $l=0$. Obviously, $\alpha_2(t,S)=0$ for $a<0$. Assume $a\geq 0$. By Lemma 1.3, one has for $t=\alpha 2^a$

$$\begin{aligned}
&\alpha_2(t,S)) \\
&= \int_{\mathbb{Q}_2}\int_{\mathbb{Z}_2^m} \psi(bq(x)) dx \psi(-tb) db \\
&= 1+\sum_{k=1}^{\infty} 2^k \int_{\mathbb{Z}_2^*} J_1(\beta 2^{-k}) J_2(\beta 2^{-k}) J_3(\beta 2^{-k}) \psi(-\alpha\beta 2^{a-k}) d\beta
\end{aligned}$$

where

$$\begin{aligned}
J_1(b) &= \prod_{h=1}^{L}\int_{\mathbb{Z}_2} \psi(\epsilon_h 2^{l_h} b x_h^2) dx \\
J_2(b) &= \prod_{i=1}^{M}\int_{\mathbb{Z}_2^2} \psi(\epsilon_i 2^{m_i} b y_{i1} y_{i2}) dy_{i1} dy_{i2} \\
J_3(b) &= \prod_{j=1}^{N}\int_{\mathbb{Z}_2^2} \psi(\epsilon_j 2^{n_j} b (z_{j1}^2 + z_{j1}z_{j2} + z_{j2}^2)) dz_{j1} dz_{j2}.
\end{aligned}$$

By Lemma 4.3, one has

$$J_1(\beta 2^{-k}) = \delta(k)(\frac{2}{\epsilon(k)})2^{\sum_{l_h<k-1}\frac{l_h-k+1}{2}} \prod_{l_h<k-1} \psi(\frac{\epsilon_h\beta}{8})(\frac{2}{\alpha})^{l(-k+1,1)}.$$

By Lemma 4.4, one has

$$J_2(\beta 2^{-k}) = 2^{\sum_{m_i<k} m_i - k}$$

and

$$J_3(\beta 2^{-k}) = p(k) 2^{\sum_{n_j<k} n_j - k}.$$

Therefore

$$\begin{aligned}&\alpha_2(\alpha 2^a, S)\\ &= 1 + \sum_{k=1}^{\infty} \delta(k)p(k)(\frac{2}{\epsilon(k)})2^{d(k)} \int_{\mathbb{Z}_2^*} (\frac{2}{\alpha})^{l(-k+1,1)} \psi(\frac{-\mu_k(t)}{8}\beta)d\beta.\end{aligned}$$

The last integral is always zero for $k > a+3$ since $\mu_k(t) \notin \mathbb{Z}_2$ in this case.

When $l(-k+1,1)$ is odd, the last integral is equal to $2^{-\frac{3}{2}}(\frac{2}{\mu_k(t)})$ by Lemma 4.3 and our notation (4.5). When $l(-k+1,1)$ is even, the last integral is equal to

$$\begin{aligned}\int_{\mathbb{Z}_2^*} \psi(\frac{-\mu_k(t)}{8}\beta)d\beta &= \frac{1}{2}\psi(-\frac{\mu_k(t)}{8}) \int_{\mathbb{Z}_2} \psi(\frac{-\mu_k(t)}{4}\beta)d\beta\\ &= \frac{1}{2}\psi(\frac{\mu_k(t)}{8}) \ \mathrm{char}(4\mathbb{Z}_2)(\mu).\end{aligned}$$

Now Theorem 4.1 follows easily.

Just as Theorem 3.4, the following theorem follows easily from Theorem 4.1.

Theorem 4.5. *Let L be a lattice of V with matrix S given by (4.1) (with respect to a suitable basis). Let $t = \alpha 2^a$. Then $W_t(1, s+s_0, L) = 0$ unless $a \geq \ min(0,l)$. Set $C = |\det 2S|^{\frac{1}{2}}\gamma(V)$ and $X = 2^{-s}$.*

(1) *When $l \geq 0$, one has*

$$\begin{aligned}&C^{-1}W_t(1, s+s_0,\ char(L))\\ &= 1 + \sum_{\substack{l<k\leq a+3\\ l(-k+1,1)\ odd}} \delta(k)p(k)\left(\frac{2}{\mu\epsilon(k)}\right)2^{d(k)-\frac{3}{2}}X^k\\ &+ \sum_{\substack{0<k\leq a+3\\ l(-k+1,1)\ even}} \delta(k)p(k)\left(\frac{2}{\epsilon(k)}\right)2^{d(k)-1}\psi(\frac{\mu}{8})\ char(4\mathbb{Z}_2))(\mu)X^k.\end{aligned}$$

where $\mu = \mu_k(t)$. Moreover, the k-th term with $0 < k \leq \min(l,a)$ is just $2^{k-1}X^k$.

(2) *When $l < 0$, one has*

$$\begin{aligned}&C^{-1}W_t(1, s+s_0,\ char(L))\\ &= p^l + \sum_{\substack{l<k\leq a+3, k\neq 0\\ l(-k+1,1)\ odd}} \delta(k)p(k)\left(\frac{2}{\mu\epsilon(k)}\right)2^{d(k)-\frac{3}{2}}X^{max(0,k)}\\ &+ \sum_{\substack{l<k\leq a+3, k\neq 0\\ l(-k+1,1)\ even}} \delta(k)p(k)\left(\frac{2}{\epsilon(k)}\right)2^{d(k)-1}\psi(\frac{\mu}{8})\ char(4\mathbb{Z}_2))(\mu)X^{max(0,k)}\end{aligned}$$

Here $\mu = \mu_k(t)$ as above.

(3) *For any $\phi \in S(V)$, $g \in G$ and $t \neq 0$, $W_t(g, s+s_0, \phi)$ is a polynomial of $X = 2^{-s}$ over $\mathbb{Q}_2$.*

We remark that the formulas in Theorem 4.5 is also true for $t = 0$, i.e., $a = \infty$. In this case, $W_0(1, s+s_0, \ \mathrm{char}(L))$ is a rational function of X instead of a polynomial. One has $\mu_k(0) = \epsilon_2(k)$ for all k.

Appendix: An example.

We work out an interesting example in this appendix for future application. Let $p < \infty$ be any prime number. Let B be a quaternion algebra over $\mathbb{Q}_p$, and let R be a maximal order of B. Let $\kappa \in \mathbb{Q}_p^*$. Let $V = V_\kappa = \{x \in B : \ \mathrm{tr} x = 0\}$ be with quadratic form $q(x) = -\kappa N(x)$, and let $L = R \cap V$ be a lattice of V. We use superscript 's' or 'r' to refer to the split or ramified case. For example, B^s stands for the split quaternion algebra over $\mathbb{Q}_p$. Denote $l = \ \mathrm{ord}_p \kappa$ and assume $l = 0$ or 1. $t \in \mathbb{Q}_p^*$, we write $t = \alpha p^{a+l}$ with $\alpha \in \mathbb{Z}_p^*$ and $a \in \mathbb{Z}$. Put

$$C = C(\kappa) = p^{-\frac{3}{2}l - \frac{1}{2}\delta_{p,2}} (\kappa, -1)_p \gamma(-\kappa, \frac{1}{2}\psi)\gamma(\frac{1}{2}\psi)^3, \tag{A1}$$

and

$$v(a) = v(t, \kappa) = \begin{cases} (\alpha\kappa p^{-l}, \delta_{p,2} - p)_p & \text{if } a \text{ is even,} \\ -1 & \text{if } a \text{ is odd.} \end{cases} \tag{A2}$$

Here $\delta_{m,n}$ is the usual Kronecker symbol. Notice that $\alpha\kappa p^{-1}$ is just the 'unit' part of $t\kappa$. We also define

$$\begin{aligned} \delta_8(x) &= \psi(\frac{x}{8}) \ \mathrm{char}(4\mathbb{Z}_2)(x) \\ &= \begin{cases} 1 & \text{if } x \equiv 0 \bmod 8, \\ -1 & \text{if } x \equiv 4 \bmod 8, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{A3}$$

Finally, we associate a polynomial $F(x) = F_{t,\kappa}(x)$ to the pair (t, κ) as follows. When $p = 2$ and $a = \ \mathrm{ord}_p t - \ \mathrm{ord}_p k$ is even,

$$F(x) = \frac{1}{2} \sum_{k=1}^{\frac{a}{2}} 2^{-k} x^{2k} + v(a) 2^{-\frac{a}{2}-2} x^{a+2} + \delta_8(\alpha - \kappa p^{-l}) 2^{-\frac{a}{2}-2} x^{a+3}, \tag{A4}$$

and otherwise,

$$F(x) = (1 - p^{-1}) \sum_{k=1}^{[\frac{a}{2}]} p^{-k} x^{2k} + v(a) p^{-[\frac{a+1}{2}]-1} X^{a+1}. \tag{A5}$$

Then calculation using Theorems 3.4 and 4.5 will give the following theorem.

Theorem *A*1. *Let p be any prime number and let the notation be as above. Set $X = p^{-s}$. Then $W_t(1, s + \frac{1}{2},\ charL) = 0$ unless $t \in \mathbb{Z}_p$. Moreover,*

(1) *When $a \ge 0$, one has*

$$C^{-1}W_t(1, s+\frac{1}{2},\ charL) = \begin{cases} 1 + p^l X^l((1-p^{-1})l + F(X)) & \text{if } B = B^s, \\ -p^{-1} + p^l X^l(-p^{-1}(1-p^{-1})l + F(X)) & \text{if } B = B^r. \end{cases}$$

(2) *When $l = 1$ and $a = -1$ (so $t \in \mathbb{Z}_p^*$), one has*

$$C^{-1}W_t(1, s+\frac{1}{2},\ charL) = (1-X)\begin{cases} 1 & \text{if } B = B^s, \\ -\frac{1}{p} & \text{if } B = B^r. \end{cases}$$

Corollary *A*2. *With notation as in Theorem 4.6, one has for $a \ge 0$*

$$W_t(1, s+\frac{1}{2},\ charL^s) - W_t(1, s+\frac{1}{2},\ charL^r) = C(1-p^{-1}) + C(1-p^{-2})lp^l X^l.$$

In particular, when $l = 0$, i.e., $\kappa \in \mathbb{Z}_p^*$, the difference is a constant.

As mentioned before, the results here are also true for $t = 0$ with $a = \infty$, i.e., with $F(x)$ replaced by a simpler rational function $\frac{p^{-1}(1-p^{-1})x^2}{1-p^{-1}x^2}$. More precisely, we have

Corollary *A*3. *With notation as above, one has*

$$W_0(1, s+\frac{1}{2},\ charL) = \begin{cases} \frac{1+lp^l(1-p^{-1})X^l - p^{-1}X^2 + p^{l-1}(1-p^{-1})(1-l)X^{2+l}}{1-p^{-1}X^2} & \text{if } B = B^s, \\ \frac{-1}{p}\frac{1+lp^l(1-p^{-1})X^l - p^{-1}X^2 - p^l(1-p^{-1})(1+lp^{-1})X^{2+l}}{1-p^{-1}X^2} & \text{if } B = B^r. \end{cases}$$

We divide the proof of Theorem *A*1 into two parts and four cases.

Part 1 We first assume that $p \ne 2$. We will verify the case $B = B^r$ in detail and leave the case $B = B^s$ to the reader where the matrix of L^s is $\mathbb{Z}_p$-equivalent to $\kappa\ \mathrm{diag}(1, 1, -1)$. We use the notation in section 3. When B is a division quaternion algebra, the matrix of L is $\mathbb{Z}_p$-equivalent to $S = \kappa\ \mathrm{diag}(\beta, p, -\beta p)$ where $\beta \in \mathbb{Z}_p^*$ is a non-square modulo p. So we have $l_1 = l$, $l_2 = l_3 = l + 1$,

$$l(-k-l, 1) = \begin{cases} 1 & \text{if } k > 0 \text{ is odd}, \\ 2 & \text{if } k > 0 \text{ is even}, \end{cases}$$

and

$$v_{k+l} = \begin{cases} -(\kappa p^{-1}, p)_p & \text{if } k > 0 \text{ is odd}, \\ -1 & \text{if } k > 0 \text{ is even}. \end{cases}$$

Moreover, one has $d(k+l) = -\frac{1}{2}k + l + 1$ for $k > 0$, and

$$f(\alpha p^{a+l}) = \begin{cases} -\frac{1}{p} & \text{if } a \ge 0 \text{ is odd}, \\ (\frac{\alpha}{p})\frac{1}{\sqrt{p}} & \text{if } a \ge 0 \text{ is even}. \end{cases}$$

Finally, the constant in Theorem 3.4 is $C_L = -p^{-1}C(\kappa)$ in this case. Now all we need to do is to plug all the data to the formula in Theorem 3.4. The case $l=1$ and $a=-1$ is obvious. When $a \ge 0$ is odd, one has

$$\begin{aligned}
&-pC^{-1}W_t(1, s+\frac{1}{2},\ \mathrm{char}L^r)\\
&= 1 + (1-p^{-1})lp^lX^l + (1-p^{-1})\sum_{0<k\le a,\ \text{even}} -p^{-\frac{1}{2}k+l+1}X^k\\
&\qquad + \frac{1}{p}p^{-\frac{1}{2}(a+1)+l+1}X^{a+1+l}\\
&= 1 + p^{l+1}X^l(lp^{-1}(1-p^{-1}) - F(X))
\end{aligned}$$

Multiplying both sides by $-p^{-1}$, one gets the desired formula. When $a \ge 0$ is even, one has

$$\begin{aligned}
&-pC^{-1}W_t(1, s+\frac{1}{2},\ \mathrm{char}L^r)\\
&= 1 + (1-p^{-1})lp^lX^l + (1-p^{-1})\sum_{0<k\le a,\ \text{even}} -p^{-\frac{1}{2}k+l+1}X^k\\
&\qquad - v(a)\frac{1}{\sqrt{p}}d^{-\frac{a+1}{2}+l+1}X^{a+1+l}\\
&= 1 + p^{l+1}X^l(lp^{-1}(1-p^{-1}) - F(X))
\end{aligned}$$

Multiplying both sides by $-p^{-1}$, one gets the desired formula again.

Part 2 Now we assume that $p=2$. First we change the definition of $F(x)$ a little and rephrase the theorem in a slightly better way, taking advantage of the fact $1-\frac{1}{2}=\frac{1}{2}$. Let

$$F_1(x) = \sum_{k=1}^{[\frac{a}{2}]} 2^{-k}x^{2k} - 2^{-\frac{a+1}{2}}x^{a+1} \tag{E2}$$

when $a>0$ is odd, and

$$F_1(x) = \sum_{k=1}^{[\frac{a}{2}]} 2^{-k}x^{2k} + \left(\frac{-1}{\alpha\kappa p^{-l}}\right)2^{-\frac{a+2}{2}}x^{a+2} + \delta_8(\alpha-\kappa')2^{-\frac{a+2}{2}}X^{a+3} \tag{E3}$$

when $a \ge 0$ is even. Then $F_1(x) = 2F(x)$.

Theorem *A*4. *$W_t(1, s+\frac{1}{2},\ charL) = 0$ unless $t \in \mathbb{Z}_2$. Put $X = 2^{-s}$.*

(1) *When $l=1$ and $a=-1$ (so $t \in \mathbb{Z}_2^*$), one has*

$$C^{-1}W_t(1, s+\frac{1}{2},\ charL) = (1-X)\begin{cases} 1 & \text{if } B = B^s,\\ -\frac{1}{2} & \text{if } B = B^r.\end{cases}$$

(2) *When $a \ge 0$, one has*

$$\begin{aligned}
&C^{-1}W_t(1, s+\frac{1}{2},\ charL)\\
&= \begin{cases} 1 + 2^{l-1}X^l(l + F_1(X)) & \text{if } B = B^s,\\ -\frac{1}{2} + 2^{l-1}X^l(-\frac{1}{2}l + F_1(X)) & \text{if } B = B^r.\end{cases}
\end{aligned}$$

Proof. First we assume that $B = B^s$ is split. We may identify B with $M_2(\mathbb{Q}_2)$ in such a way that $R = M_2(\mathbb{Z}_2)$. So for $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix}$, one has $q(x) = \kappa(x_1^2 + x_2x_3)$. In the notation of section 4, we have $L = M = 1$, $l_1 = m_1 = l$, and $N = 0$. Now the case $l = 1$ and $a = -1$ can be verified without any difficulty from Theorem 4.5. We assume $a \ge 0$ from now on. Write $\kappa' = \kappa p^{-l}$. Then $\delta(k+l) = 0$ for $k = 1$ and $= 1$ otherwise. One has also $p(k+l) = 1$. Furthermore

$$l(-k-l+1, 1) = \begin{cases} 1 & \text{if } k > 0 \text{ is even}, \\ 0 & \text{if } k > 0 \text{ is odd}, \end{cases}$$

$$\epsilon_1(k+l) = \begin{cases} \kappa' & \text{if } k > 0 \text{ is even}, \\ 1 & \text{if } k > 0 \text{ is odd}, \end{cases}$$

and $\mu_{k+l}(t) = \alpha p^{a-k+3} - \kappa'$ for $k > 0$. So $l(-k-l+1, 1)$ is odd if and only if $k > 0$ is even. When $k > 0$ is even, we compute ($\mu = \mu_{k+l}(t)$ for short)

$$\left(\frac{2}{\mu\epsilon_1(k+l)}\right) = \begin{cases} 1 & \text{if } 0 < k \le a, \\ -1 & \text{if } k = a+1, \\ (\frac{-1}{\alpha\kappa'}) & \text{if } k = a+2, \\ 0 & \text{if } k = a+3. \end{cases}$$

Here we have used the fact that $(\frac{2}{4x-1}) = -1$ and $(\frac{2}{2x-1}) = (\frac{-1}{x})$ for any $x \in \mathbb{Z}_2^*$, where $(\frac{-1}{x}) = (-1, x)_2$. Moreover,

$$\mathrm{char}(4\mathbb{Z}_2)(\mu) = \begin{cases} 0 & \text{if } 0 < k \le a+2, \\ \mathrm{char}(4\mathbb{Z}_2)(\alpha - \kappa') & \text{if } k = a+3. \end{cases}$$

Finally, we have $d(k+l) = -\frac{1}{2}k + l + \frac{1}{2}$ for $k > 0$, and that the constant $C_L = |\det 2S|^{\frac{1}{2}}\gamma(V)$ in Theorem 4.5 is equal to $C(\kappa)$ defined by $(A1)$. Now all we need to do it to plug in all the data to the formula in Theorem 4.5, with a replaced by $a + l$. When $a > 0$ is odd, $a + 3$ is even, one has

$$\begin{aligned} &C^{-1}W_t(1, s+\frac{1}{2}, \ \mathrm{char}L) \\ &= 1 + 2^{l-1}X^l\delta_{l,1} + \sum_{\substack{0<k\le a \\ \text{even}}} 2^{-\frac{1}{2}+k-1}X^{k+l} - 2^{-\frac{a+1}{2}+l-1}X^{a+1+l} \\ &= 1 + 2^{l-1}X^l(l + F(X)). \end{aligned}$$

Here we have used the trivial fact that $\delta_{l,1} = l$ for $l = 0$ or 1. When $a \ge 0$ is even, $a + 3$ is odd. One has in this case

$$\begin{aligned} &C^{-1}W_t(1, s+\frac{1}{2}, \ \mathrm{char}L) \\ &= 1 + 2^{l-1}X^l l + \sum_{\substack{0<k\le a \\ \text{even}}} 2^{-\frac{1}{2}k+l-1}X^{k+l} + (\frac{-1}{\alpha\kappa'})2^{-\frac{a+2}{2}+l-1}X^{a+2+l} \\ &\quad + \psi(\alpha - \kappa')\ \mathrm{char}(4\mathbb{Z}_2)(\alpha - \kappa')2^{-\frac{a+2}{2}+l-1}X^{a+3+l} \\ &= 1 + 2^{l-1}X^l(l + F_1(X)). \end{aligned}$$

This proves the split case.

Finally we assume that $B = B^r$ is a division quaternion algebra over $\mathbb{Q}_2$. The following easy lemma is well known and we include a proof here for completeness..

Lemma *A*5. *There is a $\mathbb{Z}_2$ basis $\{v_1, v_2, v_3\}$ of L with respect to which the quadratic form q is given by*

$$q(\sum x_i v_i) = \kappa(-3x_1^2 + 2(x_2^2 + x_2x_3 + x_3^2)). \tag{A6}$$

Proof. We may assume that $\kappa = 1$, i.e., $q(x) = -N(x)$ is the negative reduced norm. Since every quadratic extension of $\mathbb{Q}_2$ splits in B, there is $\alpha \in B$ such that $\alpha^2 = -3$. There exists $\beta \in B^*$ such that $\beta^2 \in \mathbb{Q}_2$ and $\beta k = \bar{k}\beta$ for every $k \in \mathbb{Q}_2(\alpha)$. Since $\mathbb{Q}_2(\alpha)$ is unramified over $\mathbb{Q}_2$, every units in $\mathbb{Z}_2$ is a norm from $\mathbb{Q}_2(\alpha)$. So replacing β by βk for some $k \in \mathbb{Q}_2(\alpha)$ if necessary, we may assume $\beta^2 = 2$. Set $\gamma = \alpha\beta$. The condition $\beta\alpha = -\alpha\beta$ implies that $\gamma^2 = 6$, $\alpha\gamma = -\gamma\alpha$, and $\beta\gamma = -\gamma\beta$. Put $v_1 = \alpha$, $v_2 = \frac{1}{2}(\beta + \gamma)$, and $v_3 = \frac{1}{2}(\beta - \gamma)$. We claim that $\{v_1, v_2, v_3\}$ is what we sought. First $v_1^2 = -3$ and $v_2^2 = v_3^2 = 2$ imply $v_i \in L$. Direct computation shows that the quadratic form $q(x)$ has the form $(A6)$ with respect to $\{v_1, v_2, v_3\}$. It remains to check that v_i generate L as a $\mathbb{Z}_2$ module. Given $x \in L$, we write $x = \sum x_i v_i$ with $x_i \in \mathbb{Q}_2$, we need to prove $x_i \in \mathbb{Z}_2$. Taking norm, one finds

$$-Nx = -3x_1^2 + 2(x_2^2 + x_2x_3 + x_3^2) \in \mathbb{Z}_2.$$

It is easy to see that $\operatorname{ord}_2 x_2 = \operatorname{ord}_2 x_3$ if either of the two is not in $\mathbb{Z}_2$. Assume $m = \operatorname{ord}_2 x_2 = \operatorname{ord}_2 x_3$, and write $x_i = 2^m y_i$ for some $y_i \in \mathbb{Z}_2^*$, $i = 2, 3$. So

$$-Nx = -3x_1^2 + 2^{2m+1}(y_2^2 + y_2y_3 + y_3^2) \in \mathbb{Z}_2.$$

Notice that $y_2^2 + y_2y_3 + y_3^2$ is a unit. So one must have that $x_1 \in \mathbb{Z}_2$ and $2m+1 \geq 0$, i.e., $m \geq 0$. This proves $x_i \in \mathbb{Z}_2$.

Now turn back to the proof of Theorem *A*4. Let $\{v_1, v_2, v_3\}$ be a basis of L given by Lemma *A*5. In the notation of section 4, one has $L = N = 1$, $l_1 = l$, $n_1 = l + 1$, and $M = 0$. We again verify the case $a \geq 0$ and leave the other case to the reader. First, as in the split case, $\delta(k + l) = 0$ for $k = 1$ and $= 1$ otherwise. One also has

$$l(-k - l + 1, 1) = \begin{cases} 1 & \text{if } k > 0 \text{ is even,} \\ 0 & \text{if } k > 0 \text{ is odd,} \end{cases}$$

is odd if and only if $k > 0$ is even. However,

$$\epsilon_1(k + l) = \begin{cases} -3\kappa' & \text{if } k > 0 \text{ is even,} \\ 1 & \text{if } k > 0 \text{ is odd,} \end{cases}$$

and $\mu_{k+l}(t) = \alpha p^{a-k+3} + 3\kappa'$ for $k > 0$. When $k > 0$ is even, we have as in the split case

$$\left(\frac{2}{\mu\epsilon_1(k + l)}\right) = \begin{cases} 1 & \text{if } 0 < k \leq a, \\ -1 & \text{if } k = a + 1, \\ \left(\frac{-1}{\alpha\kappa'}\right) & \text{if } k = a + 2, \\ 0 & \text{if } k = a + 3, \end{cases}$$

and

$$\operatorname{char}(4\mathbb{Z}_2)(\mu) = \begin{cases} 0 & \text{if } 0 < k \leq a + 2, \\ \operatorname{char}(4\mathbb{Z}_2)(\alpha + 3\kappa') & \text{if } k = a + 3. \end{cases}$$

Moreover, we have $p(k+l) = (-1)^{k-1}$, and $d(k+l) = -\frac{1}{2}k + l + \frac{3}{2}$ for $k > 0$. Finally the constant $C_L = -\frac{1}{2}C(\kappa)$. Now one can obtain the desired formula by doing the same calculation as in the split case, plus the following identity:

$$\delta_8(\alpha + 3\kappa') = -\delta_8(\alpha - \kappa').$$

We leave the detail to the reader.

5. A proposition.

The purpose of this section is to prove a result needed for the main calculation in section 7. We fix an element $u \in \mathbb{Z}_p^*$ such that $(\frac{u}{p}) = -1$. Let $\mathcal{H} = \mathrm{Sym}_2(\mathbb{Q}_p) \cap \mathrm{GL}_2(\mathbb{Q}_p)$ be the set of nonsingular symmetric matrices over $\mathbb{Q}_p$, and let $\mathrm{GL}_2(\mathbb{Z}_p)$ acts on $\mathcal{H}$ via $g.X = {}^t gXg$. By a well-known theorem ([Ca, Theorem 8.3.1]), the following is a complete representative set of the $\mathrm{GL}_2(\mathbb{Z}_p)$-orbits in $\mathcal{H}$

$$p^b \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad p^b \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}, \quad \text{and} \begin{pmatrix} u_1 p^{b_1} & 0 \\ 0 & u_2 p^{b_2} \end{pmatrix} \tag{5.1}$$

with $b_1 < b_2$ and $b \in \mathbb{Z}$ and $u_i = 1$ or u. Recall also that every $\mathrm{GL}_2(\mathbb{Z}_p)$-orbit is open in $\mathcal{H}$ since two quadratic forms over $\mathbb{Q}_p$ sufficiently close p-adically are $\mathrm{GL}_2(\mathbb{Z}_p)$-equivalent ([Ca, chapter 8]).

As before, we choose on $\mathbb{Q}_p$ the Haar measure dx so that $\mathrm{meas}(\mathbb{Z}_p) = 1$. On symmetric matrices, we choose the Haar measure $dx = dx_1 dx_2 dx_3$ for $x = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}$. On $\mathrm{GL}_2(\mathbb{Q}_p)$ we choose the Haar measure $dx = dx_1 dx_2 dx_3 dx_4$. In particular, the measure of $\mathrm{GL}_2(\mathbb{Z}_p)$ is $(1-p^{-1})^2(1+p^{-1})$ instead of 1.

Proposition 5.1. *Let f be a locally constant bounded function on $Sym_2(\mathbb{Q}_p)$ such that $f \in L^1(Sym_2(\mathbb{Q}_p)$. Then*

$$\begin{aligned}
&\int_{Sym_2(\mathbb{Q}_p)} f(x)dx \\
&= \sum_{b\in\mathbb{Z}} p^{-3b} \left(\frac{1}{2(1-(\frac{-1}{p})p^{-1})} \int_{GL_2(\mathbb{Z}_p)} f(p^b\, {}^t xx)dx \right. \\
&\qquad \left. + \frac{1}{2(1+(\frac{-1}{p})p^{-1})} \int_{GL_2(\mathbb{Z}_p)} f(p^b\, {}^t x \ diag(1,u)x)dx \right) \\
&\quad + \frac{1}{4} \sum_{b_1<b_2, u_i=1 \ or \ u} p^{-2b_1-b_2} \int_{GL_2(\mathbb{Z}_p)} f({}^t x \ diag(u_1p^{b_1}, u_2p^{b_2})x)dx
\end{aligned}$$

Proof. Since $\mathrm{Sym}_2(\mathbb{Q}_p) - \mathcal{H}$ is a closed subvariety of codimension 1, one has

$$\begin{aligned}
\int_{\mathrm{Sym}_2(\mathbb{Q}_p)} f(x)dx &= \int_{\mathcal{H}} f(x)dx \\
&= \sum \int_{\mathbb{O}} f(x)dx.
\end{aligned} \tag{5.2}$$

Here the sum runs over all $\mathrm{GL}_2(\mathbb{Z}_p)$-orbits $\mathbb{O}$ of $\mathcal{H}$. For each matrix $\mathrm{diag}(t_1,t_2)$, let $\mathbb{O}(t_1,t_2)$ be the $\mathrm{GL}_2(\mathbb{Z}_p)$-orbit of $\mathrm{diag}(t_1,t_2)$, and let $O(t_1,t_2)$ be the stabilizer of $\mathrm{diag}(t_1,t_2)$ in $\mathrm{GL}_2(\mathbb{Z}_p)$. Let dh be the Haar measure on $O(t_1,t_2)$ with total measure 1. Since the Haar measure on $\mathcal{H}$ is $\mathrm{GL}_2(\mathbb{Z}_p)$ invariant, there is a constant $C = C(t_1,t_2)$ such that for every locally constant function ϕ in $\mathrm{GL}_2(\mathbb{Z}_p)$, one has

$$C \int_{\mathrm{GL}_2(\mathbb{Z}_p)} f(x)dx = \int_{\mathbb{O}(t_1,t_2)} \int_{O(t_1,t_2)} f(hx)dhdx. \tag{5.3}$$

Take $\phi =$ char$(1+p^n M_2(\mathbb{Z}_p))$ for sufficiently large integer n. Then the left hand side of (5.3) is Cp^{-4n}, and the right hand side is equal to

$$\mathrm{meas}(O(t_1,t_2)\cap(1+p^n M_2(\mathbb{Z}_p)))\cdot\ \mathrm{meas}(\mathbb{O}_n(t_1,t_2)).$$

Here $\mathbb{O}_n(t_1,t_2)$ is the $(1+p^n M_2(\mathbb{Z}_p))$-orbit of $\mathrm{diag}(t_1,t_2)$. Since

$$\begin{aligned}&{}^t(1+p^n\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix})\ \mathrm{diag}(t_1,t_2)(1+p^n\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix})\\ &=\begin{pmatrix}t_1+2t_1p^nx_1 & t_1p^nx_2+t_2p^nx_3\\ t_1p^nx_2+t_2p^nx_3 & t_2+2t_2p^nx_4\end{pmatrix}p^{2n}\begin{pmatrix}* & *\\ * & *\end{pmatrix},\end{aligned}$$

one has

$$\mathrm{meas}(\mathbb{O}_n(t_1,t_2))=p^{-3n-2b_1-b_2}.$$

On the other hand,

$$\mathrm{meas}(O(t_1,t_2)\cap(1+p^nM_2(\mathbb{Z}_p)))=\frac{1}{[O(t_1,t_2):O(t_1,t_2)\cap(1+p^nM_2(\mathbb{Z}_p))]}.$$

Applying the lemma below, one has then

$$C(t_1,t_2)=p^{-2b_1-b_2}\mu(t_1,t_2) \tag{5.4}$$

where

$$\mu(t_1,t_2)=\begin{cases}\dfrac{1}{4} & \text{if } b_1<b_2,\\[2ex] \dfrac{1}{2(1-(\frac{-u_1u_2}{p})p^{-1})} & \text{if } b_1=b_2.\end{cases} \tag{5.5}$$

($t_i=u_ip^{b_i}$). Applying (5.3) – (5.5) to (5.2), one proves the proposition.

Lemma 5.2. *One has*

$$[O(t_1,t_2):O(t_1,t_2)\cap(1+p^nM_2(\mathbb{Z}_p))]=\mu(t_1,t_2)^{-1}p^n.$$

Proof. Direct computation gives

$$O(t_1,t_2)=SO(t_1,t_2)\propto\{\pm1\}$$

where

$$\begin{aligned}SO(t_1,t_2)&=\{\begin{pmatrix}x_1 & x_2\\ -\frac{t_1}{t_2}x_2 & x_1\end{pmatrix}:x_i\in\mathbb{Z}_p,\frac{t_1}{t_2}x_2\in\mathbb{Z}_p,x_1^2+\frac{t_1}{t_2}x_2^2=1\}\\ &\cong\{(x_1,x_2)\in\mathbb{Z}_p^2:x\in\frac{t_2}{t_1}\mathbb{Z}_p,x_1^2+\frac{t_1}{t_2}x_2^2=1\}\end{aligned}$$

and

$$\begin{aligned}&O(t_1,t_2)\cap(1+p^nM_2(\mathbb{Z}_p))\\ &\cong\{(x_1,x_2)\in\mathbb{Z}_p^2:x_1\equiv1 \bmod p^n, x_2\equiv 0 \bmod p^{n+b_2-b_1},x_1^2+\frac{t_1}{t_2}x_2^2=1\}.\end{aligned}$$

When $b_1 < b_2$, the map $(x_1, x_2) \mapsto x_2$ is a two-to-one map from $SO(t_1, t_2)$ onto $p^{b_2-b_1}\mathbb{Z}_p$ and a one-to-one correspondence from $O(t_1, t_2) \cap (1 + p^n M_2(\mathbb{Z}_p))$ onto $p^{n+b_2-b_1}\mathbb{Z}_p$. This proves the case $b_1 < b_2$.

When $b_1 = b_2$, $\frac{t_1}{t_2} = \frac{u_1}{u_2} \in \mathbb{Z}_p^*$. Let $\alpha = \sqrt{-\frac{u_1}{u_2}}$. If $\alpha \in \mathbb{Z}_p^*$, i.e., $(\frac{-u_1 u_2}{p}) = 1$, then $(x_1, x_2) \mapsto x_1 + \alpha x_2$ is an isomorphism between $SO(t_1, t_2)$ to $\mathbb{Z}_p^*$ and maps $O(t_1, t_2) \cap (1 + p^n M_2(\mathbb{Z}_p))$ onto $1 + p^n\mathbb{Z}_p$. This implies

$$[SO(t_1, t_2) : O(t_1, t_2) \cap (1 + p^n M_2(\mathbb{Z}_p))] = [\mathbb{Z}_p^* : 1 + p^n\mathbb{Z}_p] = p^n(1 - p^{-1}).$$

If $\alpha \notin \mathbb{Z}_p^*$, i.e., $(\frac{-u_1 u_2}{p}) = -1$, then $K = \mathbb{Q}_p(\alpha)$ is a quadratic extension. Let K^1 be the norm-1 subgroup of K and let

$$K^1(n) = \{z \in K^1 : z \equiv 1 \bmod p^n\}.$$

Then the map $(x_1, x_2) \mapsto x_1 + \alpha x_2$ gives an isomorphism from $SO(t_1, t_2)$ onto K^1 and from $O(t_1, t_2) \cap (1 + p^n M_2(\mathbb{Z}_p))$ onto $K^1(n)$. So in this case, one has

$$[SO(t_1, t_2) : O(t_1, t_2) \cap (1 + p^n M_2(\mathbb{Z}_p))] = [K^1 : K^1(n)] = p^n(1 + p^{-1}).$$

This finishes the proof of the lemma.

6. Gauss integrals over GL_2.

For any matrix $M = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix} \in M_2(\mathbb{Q}_p)$, we compute the following Gauss integral in this section.

$$I^*(t_1, t_2, t_3, t_4) = I(M) = \int_{\mathrm{GL}_2(\mathbb{Z}_p)} \psi(\sum t_i x_i^2) dx_1 dx_2 dx_3 dx_4. \tag{6.1}$$

Here dx_i is the Haar measure on $\mathbb{Q}_p$ so that $\mathrm{meas}(\mathbb{Z}_p) = 1$. So

$$\mathrm{meas}(\mathrm{GL}_2(\mathbb{Z}_p)) = (1 + p^{-1})(1 - p^{-1})^2.$$

The integral is a generalization of the integral (2.2) to GL_2, and will be needed in next section. It also seems to be of independent interest. We will use either of the two notations whenever convenient. The following lemma is obvious from the definition.

Lemma 6.1. (1) *The value of the integral $I^*(t_1, t_2, t_3, t_4)$ does not change under any of the following operations: switching t_1 with t_4, switching t_2 with t_3, or switching the pair (t_1, t_4) with the pair (t_2, t_3).*

(2) *For any $\alpha \in \mathbb{Z}_p^*$, one has*

$$I(\alpha^2 t_1, \alpha^2 t_2, t_3, t_4) = I(t_1, t_2, t_3, t_4).$$

In other words, multiplying a column or a row of M by a square does not change the value of $I(M)$.

Lemma 6.2. (1)

$$\begin{aligned}
&I^*(t_1,t_2,t_3,t_4)\\
&= I^*(t_1)I^*(t_4)I^*(t_2)I(t_3) + I^*(t_1)I(t_4)I^*(t_2)I^*(t_3) + p^{-1}I^*(t_1)I^*(t_4)I(p^2t_2)I(t_3)\\
&+ p^{-1}I(p^2t_1)I(t_4)I^*(t_2)I^*(t_3) - I^*(t_1)I^*(t_4)I^*(t_2)I^*(t_3)\\
&- p^{-4}\ \mathit{char}((p^{-1}\mathbb{Z}_p)^4)(t_1,t_2,t_3,t_4) \sum_{x_i\in\mathbb{F}_p^*, x_1x_4=x_2x_3} \psi(\sum t_1x_i^2).
\end{aligned}$$

Here $\mathbb{F}_p$ is the finite field of p elements, $I(t)$ and $I^*(t)$ are integrals given by (2.1) *and* (2.2)*.*

(2) *For $\alpha_i \in \mathbb{Z}_p$, one has*

$$\begin{aligned}
&I^*(\alpha_1p^{-1},\alpha_2p^{-1},\alpha_3p^{-1},\alpha_4p^{-1})\\
&= p^{-4} \sum_{X\in\ GL_2(\mathbb{F}_p)} e_p(-\sum_{i=1}^4(\alpha_ix_i^2)).
\end{aligned}$$

Here $e_p(x) = e^{\frac{2\pi ix}{p}}$, and $X = \begin{pmatrix} x_1 & x_2\\ x_3 & x_4\end{pmatrix}$.

Proof. We first remark that for $x \in \mathbb{Z}_p$ or $x \in \mathbb{F}_p$, we choose a rational integer $n \equiv x \bmod p$, then $e_p(x) = e_p(n)$ is independent of the choice of n and is what we mean in (2). By the definition of ψ, one has $\psi(xp^{-1}) = e_p(-x)$ for $x \in \mathbb{Z}_p$. Now (2) is obvious.

To prove (1), first notice that $\mathrm{GL}_2(\mathbb{Z}_p) \subset \mathbb{Z}_p^4$ is the disjoint union of the subsets (coordinates are in the order (x_1,x_4,x_2,x_3))

$$p\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_p^*\times\mathbb{Z}_p^*,\quad \mathbb{Z}_p^*\times p\mathbb{Z}_p\times\mathbb{Z}_p^*\times\mathbb{Z}_p^*,\quad \mathbb{Z}_p^*\times\mathbb{Z}_p^*\times p\mathbb{Z}_p\times\mathbb{Z}_p,$$

and

$$\{(x_1,x_4,x_2,x_3)\in\mathbb{Z}_p^{*3}\times\mathbb{Z}_p : x_3 \not\equiv x_2^{-1}x_1x_4 \bmod p\}.$$

So

$$\begin{aligned}
&I^*(t_1,t_2,t_3,t_4)\\
&= p^{-1}I(p^2t_1)I(t_4)I^*(t_2)I^*(t_3) + p^{-1}I^*(t_1)I(p^2t_4)I^*(t_2)I^*(t_3)\\
&+ p^{-1}I^*(t_1)I^*(t_4)I(p^2t_2)I(t_3) + I^*(t_1)I^*(t_4)I^*(t_2)I(t_3)\\
&- \int_{\mathbb{Z}_p^{*3}} \psi(t_1x_1^2+t_2x_2^2+t_4x_4^2)dx_1dx_2d_4 \int_{x_3\equiv x_2^{-1}x_1x_4 \bmod p} \psi(t_3x_3^2)dx_3.
\end{aligned}$$

Since $p^{-1}I(p^2t_4) = I(t_4) - I^*(t_4)$, it suffices to show that

$$\begin{aligned}
&\int_{\mathbb{Z}_p^{*3}} \psi(t_1x_1^2+t_2x_2^2+t_4x_4^2)dx_1dx_2d_4 \int_{x_3\equiv x_2^{-1}x_1x_4 \bmod p} \psi(t_3x_3^2)dx_3\\
&= p^{-4}\ \mathrm{char}((p^{-1}\mathbb{Z}_p)^4)(t_1,t_2,t_3,t_4) \sum_{x_i\in\mathbb{F}_p^*, x_1x_4=x_2x_3} \psi(\sum t_1x_i^2).
\end{aligned} \tag{6.2}$$

By Lemma 2.1, the left hand side of (6.2) is

$$p^{-1}\ \mathrm{char}(p^{-1}\mathbb{Z}_p)(t_3)\int_{\mathbb{Z}_p^{*3}}\psi(t_1x_1^2)\psi(t_2x_2^2)\psi(t_4x_4^2)\psi(t_3x_2^{-2}x_1^2x_4^2)dx_1dx_2dx_4.$$

This is zero unless $t_1, t_4 \in p^{-1}\mathbb{Z}_p$ since it contains a factor $I^*(t_1+t_3x_2^{-1}x_2^2)$ resp. $I^*(t_4+t_3x_2^{-2}x_1^2)$. The same proof as in Lemma 2.1(3) would make the integral zero unless $t_2 \in p^{-1}\mathbb{Z}_p$ (under the condition $t_3 \in p^{-1}\mathbb{Z}_p$). On the other hand, if $t_1, ..., t_4 \in p^{-1}\mathbb{Z}_p$, then the integral equals

$$\begin{aligned}
&p^{-1}\sum_{x_1,x_2,x_4\in\mathbb{F}_p^*}\psi(t_1x_1^2)\psi(t_2x_2^2)\psi(t_4x_4^2)\psi(t_3x_2^{-2}x_1^2x_4^2)\int_{(1+p\mathbb{Z}_p)^3}dy\\
&=p^{-4}\sum_{x_i\in\mathbb{F}_p^*, x_1x_4=x_2x_3}\psi(\sum t_ix_i^2).
\end{aligned}$$

This finishes the proof of Lemma 6.2.

To obtain a more precise formula for $I^*(t_1, t_2, t_3, t_4)$, write $t_i = \alpha_i p^{a_i}$ with $\alpha_i \in \mathbb{Z}_p^*$ and $a_i \in \mathbb{Z}$. We may and will assume that $a_1 \le a_2, a_4$, and $a_2 \le a_3$ without loss of generality in the rest of this section (see Lemma 6.1).

Corollary 6.3. *With notation and assumption as above, one has*

$$I^*(t_1,t_2,t_3,t_4)=\begin{cases}(1-p^{-1})^2(1+p^{-1}) & \text{if } a_1\ge 0,\\ I(t_1)I(t_4)I^*(t_2)I^*(t_3) & \text{if } a_1\le -2.\end{cases}$$

In particular, if $a_1, a_2 \le -2$, then $I^*(t_1, t_2, t_3, t_4) = 0$.

Proof. When $a_1 \ge 0$, one has $a_i \ge 0$ for $1 \le i \le 4$. So

$$\begin{aligned}
I^*(t_1,t_2,t_3,t_4)&=\int_{\mathrm{GL}_2(\mathbb{Z}_p)}dx_1dx_2dx_3dx_4\\
&=p^{-4}\#\ \mathrm{GL}_2(\mathbb{F}_p)\\
&=(1-p^{-1})^2(1+p^{-1}).
\end{aligned}$$

On the other hand, if $a_1 \le -2$ then $I^*(t_1) = 0$ by Lemma 2.1. So Lemma 6.2 implies

$$I^*(t_1,t_2,t_3,t_4)=p^{-1}I(p^2t_1)I(t_4)I^*(t_2)I^*(t_3)=I(t_1)I(t_4)I^*(t_2)I^*(t_3).$$

The case $a_1 = -1$ is a little bit more complicated and more interesting. We first assume at least one of a_i, say a_4 is nonnegative. Then (assuming $a_4 \ge 0$)

$$\begin{aligned}
&\sum_{x_i\in\mathbb{F}_p^*, x_1x_4=x_2x_3}\psi(\sum t_ix_i^2)\\
&=(\sum_{x_1\in\mathbb{F}_p^*}\psi(t_1x_1^2))(\sum_{x_2\in\mathbb{F}_p^*}\psi(t_2x_2^2))(\sum_{x_3\in\mathbb{F}_p^*}\psi(t_3x_3^2))\\
&=p^3I^*(t_1)I^*(t_2)I^*(t_3)
\end{aligned}$$

Also notice that $I(t_4) = 1$ and $I^*(t_4) = 1 - p^{-1}$ in this case. Now Lemmas 6.2 and 2.1 give

Lemma 6.4. *Let the notation be as above. Assume that $a_1 = -1$ and $a_4 \ge 0$. Then*

$$I^*(t_1, t_2, t_3, t_4) = (1 - p^{-1}) I^*(t_1) I(t_2) I(t_3) + p^{-1} I^*(t_2) I^*(t_3).$$

More precisely, one has

$$\begin{aligned} &I^*(t_1, t_2, t_3, t_4) \\ &= \begin{cases} (1 - p^{-1}) \left(\frac{\delta_p}{\sqrt{p}} (\frac{-\alpha_1}{p}) - \frac{1}{p^2} \right) & \text{if } a_2, a_3 \ge 0, \\ p^{-1}(1 - p^{-1}) \left((\frac{-\alpha_1 \alpha_2}{p}) - \frac{1}{p} \right) & \text{if } a_2 = -1, a_3 \ge 0, \\ p^{-3}(1 + (\frac{-\alpha_2\alpha_3}{p})) + \frac{\delta_p}{p\sqrt{p}} (\frac{\alpha_1\alpha_2\alpha_3}{p}) & \\ \quad - \frac{\delta_p}{p^2\sqrt{p}} \left((\frac{\alpha_1\alpha_2\alpha_3}{p}) + (\frac{-\alpha_2}{p}) + (\frac{-\alpha_3}{p}) \right) & \text{if } a_2 = a_3 = -1. \end{cases} \end{aligned}$$

Finally, it comes to the case where $a_1 = a_2 = a_3 = a_4 = -1$. By Lemma 6.2, it reduces to compute the following Gauss sum

$$S(M) = \sum_{X \in \mathrm{GL}_2(\mathbb{F}_p)} e_p(\sum \alpha_i x_i^2) \tag{6.3}$$

where $M = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in M_2(\mathbb{F}_p)$, $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, and $e_p(x) = e^{\frac{2\pi i x}{p}}$. We also write $S(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ for $S(M)$ when it is more convenient. One has

$$I^*(\alpha_1 p^{-1}, \alpha_2 p^{-1}, \alpha_3 p^{-1}, \alpha_4 p^{-1}) = p^{-4} S(-\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4) \tag{6.4}$$

for $\alpha_i \in \mathbb{Z}_p$. Here we also use α_i for its image in $\mathbb{F}_p$. When some of $\alpha_i = 0$, the sum (or the integral) is given by Lemma 6.4. The following lemma was proved with the help of Don Zagier. I thank him for his help and for allowing me to publish it here.

Lemma 6.5. *Let $M = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$ with $\alpha_i \in \mathbb{F}_p^*$.*

(1) *When* $\det M = 0$, *i.e*, $M = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta\alpha_1 & \beta\alpha_2 \end{pmatrix}$ *with some $\beta \in \mathbb{F}_p^*$, one has*

$$S(M) = -p^2((\frac{-\beta}{p}) + (\frac{-\alpha_1\alpha_2}{p}))(1 - (\frac{-\alpha_1\alpha_2}{p})p^{-1}).$$

(2) *When* $\det M \neq 0$, *let $\lambda = \frac{\alpha_2\alpha_3}{\alpha_1\alpha_4}$, and let*

$$E_\lambda : y^2 = x(x-1)(x-\lambda)$$

be the corresponding elliptic curve over $\mathbb{F}_p$. Let

$$a_p(\lambda) = a_p(E_\lambda) = \#E_\lambda(\mathbb{F}_p) - p - 1.$$

be the trace of the Frobenious on E_λ. Then

$$\begin{aligned} S(M) = &(\frac{\prod \alpha_i}{p}) p^2 - (\frac{\alpha_1\alpha_4}{p}) p a_p(\lambda) + p \\ &- p\sqrt{p}\delta_p(\frac{\det M}{p}) \left((\frac{\alpha_1}{p}) + (\frac{-\alpha_2}{p}) + (\frac{-\alpha_3}{p}) + (\frac{\alpha_4}{p}) \right). \end{aligned}$$

Here δ_p is the number defined by (1.10), i.e., it is 1 or i depending on whether $p \equiv 1 \mod 4$ or $p \equiv -1 \mod 4$.

Proof. By the classical Gauss sum formula, one has

$$\begin{aligned} S(M) &= \sum_{X \in M_2(\mathbb{F}_p)} e_p(\sum \alpha_i x_i^2) - \sum_{\det X=0} e_p(\sum \alpha_i x_i^2) \\ &= (\frac{\prod \alpha_i}{p})p^2 - \sum_{\det X=0} e_p(\sum \alpha_i x_i^2) \end{aligned}$$

Since $\det X = 0$ means that either $X = \begin{pmatrix} a & b \\ \mu a & \mu b \end{pmatrix}$ with $a, b, \mu \in \mathbb{F}_p$ or $X = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$ with $X = 0$ being counted $p+1$ times, one has

$$\begin{aligned} \sum_{\det X=0} e_p(\sum \alpha_i x_i^2) = &\sum_\mu \sum_a e_p((\alpha_1 + \alpha_3\mu^2)a^2) \sum_b e_p((\alpha_2 + \alpha_4\mu^2)b^2) \\ &+ \sum_{c,d} e_p(\alpha_3 c^2 + \alpha_4 d^2) - p. \end{aligned}$$

Applying the classical Gauss sum formula to the second term, one obtains

$$\begin{aligned} S(M) = &(\frac{\prod \alpha_i}{p})p^2 + p - p(\frac{-\alpha_3\alpha_4}{p}) \\ &- \sum_\mu \sum_a e_p((\alpha_1 + \alpha_3\mu^2)a^2) \sum_b e_p((\alpha_2 + \alpha_4\mu^2)b^2) \end{aligned} \tag{6.5}$$

Notice that

$$\sum_a e_p((\alpha_1+\alpha_3\mu^2)a^2) = \begin{cases} p & \text{if } (\frac{-\alpha_1\alpha_3}{p}) = 1 \,\&\, \mu = \pm\sqrt{-\frac{\alpha_1}{\alpha_3}}, \\ \sqrt{p}\delta_p(\frac{\alpha_1+\alpha_3\mu^2}{p}) & \text{otherwise.} \end{cases} \tag{6.6}$$

Case 1 When $\det M = 0$, $\frac{-\alpha_1}{\alpha_3} = \frac{-\alpha_2}{\alpha_4}$. In this case, the roots $\mu = \pm\sqrt{\frac{-\alpha_1}{\alpha_3}}$ and $\mu = \pm\sqrt{\frac{-\alpha_2}{\alpha_4}}$ are the same if they exist in $\mathbb{F}_p$. Let $\beta = \frac{\alpha_3}{\alpha_1}$, then one has by (6.6)

$$\begin{aligned} &\sum_\mu \sum_a e_p((\alpha_1 + \alpha_3\mu^2)a^2) \sum_b e_p((\alpha_2 + \alpha_4\mu^2)b^2) \\ &= \sum_\mu p(\frac{-1}{p})(\frac{\alpha_1 + \alpha_3\mu^2}{p})(\frac{\alpha_2 + \alpha_4\mu^2}{p}) + (1 + (\frac{-\beta}{p}))p^2 \\ &= (\frac{-\alpha_1\alpha_2}{p})p \sum_\mu (\frac{1 + \beta\mu^2}{p})^2 + (1 + (\frac{-\beta}{p}))p^2 \\ &= (\frac{-\alpha_1\alpha_2}{p})p(p - 1 - (\frac{-\beta}{p})) + (1 + (\frac{-\beta}{p}))p^2 \\ &= (1 + (\frac{-\beta}{p}) + (\frac{-\alpha_1\alpha_2}{p}))p^2 - (\frac{-\alpha_1\alpha_2}{p})(1 + (\frac{-\beta}{p}))p. \end{aligned}$$

Plugging this into (6.5), one proves (1).

Case 2 When $\det M \neq 0$, the roots $\mu = \pm\sqrt{\frac{-\alpha_1}{\alpha_3}}$ and $\mu = \pm\sqrt{\frac{-\alpha_2}{\alpha_4}}$ are different if they exist in $\mathbb{F}_p$. So

$$\begin{aligned}
&\sum_\mu \sum_a e_p((\alpha_1+\alpha_3\mu^2)a^2)\sum_b e_p((\alpha_2+\alpha_4\mu^2)b^2)\\
&=\sum_\mu p(\frac{-1}{p})(\frac{\alpha_1+\alpha_3\mu^2}{p})(\frac{\alpha_2+\alpha_4\mu^2}{p})\\
&\quad+(1+(\frac{-\alpha_1\alpha_3}{p}))p\sum_b e_p((\alpha_2-\frac{\alpha_1\alpha_4}{\alpha_3})b^2)\\
&\quad+(1+(\frac{-\alpha_2\alpha_4}{p}))p\sum_a e_p((\alpha_1-\frac{\alpha_2\alpha_3}{\alpha_4})a^2)\\
&=(\frac{-\alpha_3\alpha_4}{p})p\sum_{x\in\mathbb{F}_p}(\frac{x+\alpha_1\alpha_3^{-1}}{p})(\frac{x+\alpha_2\alpha_4^{-1}}{p})(1+(\frac{x}{p}))\\
&\quad+((\frac{\alpha_1}{p})+(\frac{-\alpha_3}{p}))(\frac{\det M}{p})p\sqrt{p}\delta_p+((\frac{-\alpha_2}{p})+(\frac{\alpha_4}{p}))(\frac{\det M}{p})p\sqrt{p}\delta_p.
\end{aligned}$$

So

$$\begin{aligned}
&\sum_\mu \sum_a e_p((\alpha_1+\alpha_3\mu^2)a^2)\sum_b e_p((\alpha_2+\alpha_4\mu^2)b^2)\\
&=(\frac{-\alpha_3\alpha_4}{p})p\sum_x(\frac{(x+\alpha_1\alpha_3^{-1})(x+\alpha_2\alpha_4^{-1})}{p})\\
&\quad+(\frac{-\alpha_3\alpha_4}{p})p\sum_x(\frac{x(x+\alpha_1\alpha_3^{-1})(x+\alpha_2\alpha_4^{-1})}{p})\\
&\quad+p\sqrt{p}\delta_p(\frac{\det M}{p})((\frac{\alpha_1}{p})+(\frac{-\alpha_2}{p})+(\frac{-\alpha_3}{p})+(\frac{\alpha_4}{p})).
\end{aligned}$$

Given a polynomial $f(x)\in\mathbb{F}_p[x]$, let C_f^a be the affine curve defined by $y^2=f(x)$. Since, for a fixed $x\in\mathbb{F}_p$, the equation $y^2=f(x)$ has $1+(\frac{f(x)}{p})$ solutions in $\mathbb{F}_p$, one has

$$\sum_{x\in\mathbb{F}_p}(\frac{f(x)}{p})=\#C_f^a(\mathbb{F}_p)-p. \tag{6.7}$$

It is easy to see from this that

$$\sum_{x\in\mathbb{F}_p}(\frac{x^2+ax+b}{p})=\begin{cases}p-1 & \text{if } a^2-4b=0,\\ -1 & \text{if } a^2-4b\neq 0.\end{cases} \tag{6.8}$$

In particular, one has ($\det M\neq 0$)

$$\sum_x(\frac{(x+\alpha_1\alpha_3^{-1})(x+\alpha_2\alpha_4^{-1})}{p})=-1.$$

Next, let E_M be the elliptic curve defined by

$$E_M : y^2 = x(x + \alpha_1\alpha_3^{-1})(x + \alpha_2\alpha_4^{-1}).$$

Notice that E_M has one extra point (infinity) to its affine points. So (6.7) gives

$$(\frac{x(x + \alpha_1\alpha_3^{-1})(x + \alpha_2\alpha_4^{-1})}{p}) = a_p(E_M).$$

So

$$\begin{aligned}
&\sum_\mu \sum_a e_p((\alpha_1 + \alpha_3\mu^2)a^2) \sum_b e_p((\alpha_2 + \alpha_4\mu^2)b^2) \\
&= -(\frac{-\alpha_3\alpha_4}{p})p + (\frac{-\alpha_3\alpha_4}{p})pa_p(E_M) \\
&\quad + p\sqrt{p}\delta_p(\frac{\det M}{p})((\frac{\alpha_1}{p}) + (\frac{-\alpha_2}{p}) + (\frac{-\alpha_3}{p}) + (\frac{\alpha_4}{p})).
\end{aligned}$$

Plugging this into (6.5) and comparing the result with the desired formula in (2) leads us to verify

$$a_p(E_M) = (\frac{-\alpha_1\alpha_3}{p})a_p(\lambda). \tag{6.9}$$

Notice that the map $x \mapsto -\alpha_1\alpha_3^{-1}x$ and $y \mapsto (-\alpha_1\alpha_3^{-1})^2 y$ gives a bijection between $E'(\mathbb{F}_p)$ and $E_M(\mathbb{F}_p)$, where

$$E' : -\alpha_1\alpha_3^{-1}y^2 = x(x-1)(x-\lambda)$$

is the quadratic twist of E_λ by $-\alpha_1\alpha_3^{-1}$ and $\lambda = \frac{\alpha_2\alpha_3}{\alpha_1\alpha_4}$. So (6.9) follows from the general fact that

$$a_p(E^d) = (\frac{d}{p})a_p(E) \tag{6.10}$$

for any elliptic curve E over $\mathbb{F}_p$ and any quadratic twist E^d with $d \in \mathbb{F}_p^*$. Indeed, if d is a square, E^d is isomorphic to E over $\mathbb{F}_p$ and $(\frac{d}{p}) = 1$. If d is not a square, one has $(\frac{d}{p}) = -1$. On the other hand, we may assume that E is defined by an equation $y^2 = f(x)$ with a nonsigular cubic polynomial $f(x)$ over $\mathbb{F}_p$ $(p \neq 2)$. Then E^d is given by $dy^2 = f(x)$. So (6.7) implies

$$a_p(E^d) + a_p(E) = \sum_x (\frac{f(x)}{p}) + \sum_x (\frac{df(x)}{p}) = 0.$$

Corollary 6.6**.** (1) *For $\alpha_i, \beta \in \mathbb{Z}_p^*$, one has*

$$\begin{aligned}
&I^*(\alpha_1p^{-1}, \alpha_2p^{-1}, \alpha_1\beta p^{-1}, \alpha_2\beta p^{-1}) \\
&= -p^{-2}((\frac{-\beta}{p}) + (\frac{-\alpha_1\alpha_2}{p}))(1 - (\frac{-\alpha_1\alpha_2}{p})p^{-1}).
\end{aligned}$$

(2) *For $\alpha_i \in \mathbb{Z}_p^*$ with $\alpha_1\alpha_4 \neq \alpha_2\alpha_3$, one has*

$$\begin{aligned}
&I^*(\alpha_1p^{-1}, \alpha_2p^{-1}, \alpha_3p^{-1}, \alpha_4p^{-1}) \\
&= (\frac{\prod \alpha_i}{p})p^{-2} + p^{-3} - (\frac{\alpha_1\alpha_4}{p})p^{-3}a_p(\lambda) \\
&\quad - p^{-3}\sqrt{p}\delta_p(\frac{\alpha_1\alpha_4 - \alpha_2\alpha_3}{p})\left((\frac{-\alpha_1}{p}) + (\frac{\alpha_2}{p}) + (\frac{\alpha_3}{p}) + (\frac{-\alpha_4}{p})\right).
\end{aligned}$$

7. The case $n = 2$ and $p \neq 2$.

In the section, we will obtainan explicit formula of local density for $n = 2$ and $p \neq 2$.. We first recall some notations for the convenience of the reader. Let S be a nondegenerate symmetric matrix over $\mathbb{Q}_p$ of rank $m > 2$. Then S is $\mathbb{Z}_p$ equivalent to $\operatorname{diag}(\epsilon_1 p^{l_1}, \cdots, \epsilon_m p^{l_m})$ with $\epsilon_i \in \mathbb{Z}_p^*$ and $l_1 \le l_2 \cdots \le l_m$. Denote, for integers b and c,

$$(7.1)\quad L(b, c) = \{1 \le i \le m : b + l_i < 0 \text{ and } b + l_i \equiv k \bmod 2\}, \quad l(b, c) = \# L(b, c).$$

Let T be a 2×2 nondegenerate symmetric matrix over $\mathbb{Q}_p$, $\mathbb{Z}_p$-equivalent to $\operatorname{diag}(\alpha_1 p^{a_1}, \alpha_2 p^{a_2})$ with $\alpha_i \in \mathbb{Z}_p^*$, and $a_1 \le a_2$. When both S and T are $\mathbb{Z}_p$-integral, the local density $\alpha_p(T, S)$ is defined as

$$\alpha_p(T, S) = \lim_{t \longrightarrow \infty} (p^t)^{3-2m} A_t(T, S), \tag{7.2}$$

where

$$A_t(T, S) = \#\{X \in M_{m,2}(\mathbb{Z}_p/p^t) : S[X] \equiv T \bmod p^t\}.$$

In general, we associate S to a $\mathbb{Z}_p$-lattice $L = \mathbb{Z}_p^m$ with the quadratic form $q(x) = \frac{1}{2}(x, x) = {}^t x S x$. Define

$$W(T, S) = \int_{\operatorname{Sym}_2(\mathbb{Q}_p)} \int_{L^2} \psi(\operatorname{tr} b q(x)) \psi(-\operatorname{tr} T b)\, dx\, db. \tag{7.3}$$

where ψ is a canonical additive character of $\mathbb{Q}_p$ given by $\psi(x) = e^{-2\pi i x'}$ where x' is a rational number with $x' - x \in \mathbb{Z}_p$. Then Lemma 1.3 gives

$$\alpha_p(T, S) = W(T, S) \tag{7.4}$$

when S and T are integral. We use this equation as the definition of $\alpha_p(T, S)$ when either S or T is not integral. It is known (Lemma 1.2) that $\alpha_p(T, S)$ depends only on the $\mathbb{Z}_p$-equivalence classes of S and T. For this reason, we sometime write $\alpha_p(\alpha_1 p^{a_1}, \alpha_2 p^{a_2}, S)$ for $\alpha_p(T, S)$. As in section 5, we fix a unit $u \in \mathbb{Z}_p^*$ with $(\frac{u}{p}) = -1$.

Denote

$$d(k) = k + \frac{1}{2} \sum_{l_i < k} (l_i - k), \tag{7.5}$$

$$v_k = (\frac{-1}{p})^{[\frac{l(-k,1)}{2}]} \prod_{i \in L(-k,1)} (\frac{\epsilon_i}{p})$$

as before, and

$$\delta^{\pm}(k) = \frac{1 \pm (-1)^{l(-k,1)}}{2}. \tag{7.6}$$

Note that $\delta^+(k)$ (resp. $\delta^-(k)$) is 1 or 0 depending on whether $l(-k, 1)$ is even (resp. odd) or odd (resp. even). Define

$$f_1(\alpha p^a) = \begin{cases} -\frac{1}{p} & \text{if } l(-1-a, 1) \text{ is even,} \\ (\frac{\alpha}{p}) \frac{1}{\sqrt{p}} & \text{if } l(-1-a, 1) \text{ is odd,} \end{cases}$$

$$f_2(\alpha p^a) = \begin{cases} -\frac{1}{p} & \text{if } l(-1-a, 1) \text{ is odd,} \\ (\frac{-\alpha}{p}) \frac{1}{\sqrt{p}} & \text{if } l(-1-a, 1) \text{ is even.} \end{cases} \tag{7.7}$$

We also define

$$g(k)=\begin{cases}\delta^+(k) & \text{if } a_1-k \text{ is even,}\\ (\frac{\alpha_1}{p})\delta^-(k) & \text{if } a_1-k \text{ is odd.}\end{cases} \tag{7.8}$$

Theorem 7.1. *Assume $a_1 = a_2 = a$ and $\alpha_i \in \mathbb{Z}_p^*$. Then the local density $\alpha_p(\alpha_1 p^a, \alpha_2 p^a, S) = 0$ unless $a \ge l_1$. When $a \ge l_1$, one has*

$$\begin{aligned}
&\alpha_p(\alpha_1 p^a, \alpha_2 p^a, S)\\
&= p^{3l_1} + (1-p^{-2})p^{l_1} \sum_{\substack{0<k\le a\\ l(-k,1)\ \ even}} v_k p^{k+d(k)}\\
&\quad + \delta^+(1+a)v_{1+a}p^{l_1}((\frac{-\alpha_1\alpha_2}{p}) - p^{-1})p^{a+d(1+a)}\\
&\quad + (1-p^{-1}) \sum_{0<k\le a} (\delta^+(k) + p^{-1}\delta^-(k))p^{k+2d(k)}\\
&\quad + (1-p^{-1})^2(1+p^{-1}) \sum_{\substack{0<k_2<k_1\le a\\ l(-k_i,1)\ \ even}} v_{k_1}v_{k_2}p^{k_1+d(k_1)+d(k_2)}\\
&\quad + \delta^+(1+a)v_{1+a}((\frac{-\alpha_1\alpha_2}{p}) - p^{-1})(1-p^{-1}) \sum_{\substack{0<k\le a\\ l(-k,1)\ \ even}} v_k p^{a+d(1+a)+d(k)}\\
&\quad - ((\frac{-\alpha_1\alpha_2}{p})\delta^+(1+a) + \delta^-(1+a))p^{a-1+2d(1+a)}.
\end{aligned}$$

Theorem 7.2. *Assume $a_1 < a_2$ and $\alpha_i \in \mathbb{Z}_p^*$. Then $\alpha_p(\alpha_1 p^{a_1}, \alpha_2 p^{a_2}, S) = 0$ unless $a_1 \ge l_1$. When $a_1 \ge l_1$, one has*

$$
\begin{aligned}
&\alpha_p(\alpha_1 p^{a_1}, \alpha_2 p^{a_2}, S) \\
&= p^{3l_1} + (1-p^{-2}) p^{l_1} \sum_{l_1 < k \le a_1} v_k \delta^+(k) p^{k+d(k)} \\
&\quad + v_{1+a_1} p^{l_1} \left((\frac{\alpha_1}{p}) \frac{1}{\sqrt{p}} \delta^-(1+a_1) - p^{-2} \delta^+(1+a_1) \right) p^{1+a_1+d(1+a_1)} \\
&\quad + (1-p^{-1}) \sum_{l_1 < k \le a_1} (\delta^+(k) + p^{-1}\delta^-(k)) p^{k+2d(k)} \\
&\quad + (1-p^{-1}) \delta^-(1+a_1) p^{a_1+2d(1+a_1)} \\
&\quad + (1-p^{-1})^2 (1+p^{-1}) \sum_{l_1 < k_2 < k_1 \le a_1} v_{k_1} v_{k_2} \delta^+(k_1) \delta^+(k_2) p^{k_1+d(k_1)+d(k_2)} \\
&\quad + (1-p^{-1}) \left((\frac{\alpha_1}{p}) \frac{1}{\sqrt{p}} \delta^-(1+a_1) - p^{-2} \delta^+(1+a_1) \right) \\
&\qquad \cdot \sum_{l_1 < k \le a_1} v_{1+a_1} v_k \delta^+(k) p^{1+a_1+d(1+a_1)+d(k)} \\
&\quad + (1-p^{-1}) p^{l_1} \sum_{1+a_1 < k < 1+a_2} v_k g(k) p^{\frac{a_1+k}{2}+d(k)} \\
&\quad + (1-p^{-1})^2 \sum_{l_1 < k_2 < 1+a_1 < k_1 < 1+a_2} v_{k_1} v_{k_2} \delta^+(k_2) g(k_1) p^{\frac{a_1+k}{2}+d(k_1)+d(k_2)} \\
&\quad + (1-p^{-1}) v_{1+a_1} f_1(\alpha_1 p^{a_1}) \sum_{1+a_1 < k < 1+a_2} v_k g(k) p^{\frac{a_1+k}{2}+d(1+a_1)+d(k)} \\
&\quad + v_{1+a_2} p^{\frac{1+a_1+a_2}{2}+d(1+a_2)} G(a_1, a_2).
\end{aligned}
$$

Here

$$
\begin{aligned}
G(a_1, a_2) &= p^{l_1} f_1(\alpha_2 p^{a_2}) + v_{1+a_1} f_1(\alpha_1 p^{a_1}) f_1(\alpha_2 p^{a_2}) p^{d(1+a_1)} \\
&\quad + (1-p^{-1}) \sum_{l_1 < k \le a_1} v_k \delta^+(k) f_1(\alpha_2 p^{a_2}) p^{d(k)}
\end{aligned}
$$

when $a_1 + a_2$ is odd, and

$$
\begin{aligned}
G(a_1, a_2) &= (\frac{\alpha_1}{p}) p^{l_1} f_2(\alpha_2 p^{a_2}) + (\frac{\alpha_1}{p}) v_{1+a_1} f_1(\alpha_1 p^{a_1}) f_2(\alpha_2 p^{a_2}) p^{d(1+a_1)} \\
&\quad + (1-p^{-1}) (\frac{\alpha_1}{p}) \sum_{l_1 < k \le a_1} v_k \delta^+(k) f_2(\alpha_2 p^{a_2}) p^{d(k)}
\end{aligned}
$$

when $a_1 + a_2$ is even.

Remark 7.3. When S is unimodular, one can verify (by tedious calculation) that our formulas coincide with those of Kitaoka's ([Ki, Theorem 2]). We will give a 'shorter' version of Theorem at the end of this section.

Corollary 7.4. *Let V be a nondegerated quadratic space over $\mathbb{Q}_p$ of dimension $m > 2$. Let $\phi \in S(V^2)$ be a locally constant function on V^2 with compact support. Let T be a nonzero 2×2 symmetric matrix over $\mathbb{Q}_p$. Then the local Whittaker function $W_T(g, s+s_0, \phi)$, defined in section 1, is a polynomial of $X = p^{-s}$ for any $g \in G$, where $s_0 = \frac{m-3}{2}$ and $G =$ $Sp_2(\mathbb{Q}_p)$ or $Mp_{2,\mathbb{Q}_p}$ depending on whether m is even or odd.*

Proof. When T is non-degenerate, it follows from Theorems 7.1 and 7.2 and the proof of Theorem 3.4. When T has rank one, standard calculation reduces the Whittaker function into Whittaker functions on Sp_1 or Mp_1, where we can apply Theorem 3.4.

The rest of this section is to prove Theorems 7.1 and 7.2.

Lemma 7.5. *With notation as above, one has*

$$\alpha_p(T,S) = \sum_{k \in \mathbb{Z}} p^{3k} J(p^{-k})^2 R(k) + \sum_{k_1 > k_2} p^{2k_1+k_2} J(p^{-k_1}) J(p^{-k_2}) R(k_1, k_2). \tag{7.9}$$

Here

$$\begin{aligned} R(k) = {} & \frac{1}{2(1-(\frac{-1}{p})p^{-1})} I^*(-\alpha_1 p^{a_1-k}, -\alpha_1 p^{a_1-k}, -\alpha_2 p^{a_2-k}, -\alpha_2 p^{a_2-k}) \\ & + \frac{(-1)^{l(-k,1)}}{2(1+(\frac{-1}{p})p^{-1})} I^*(-\alpha_1 p^{a_1-k}, -\alpha_1 u p^{a_1-k}, -\alpha_2 p^{a_2-k}, -\alpha_2 u p^{a_2-k}), \end{aligned}$$

and

$$\begin{aligned} R(k_1,k_2) = {} & \frac{1}{4} \sum_{\beta_1,\beta_2 = 1 \text{ or } u} (\frac{\beta_1}{p})^{l(-k_1,1)} (\frac{\beta_2}{p})^{l(-k_2,1)} \\ & \cdot I^*(-\alpha_1\beta_1 p^{a_1-k_1}, -\alpha_1\beta_2 p^{a_1-k_2}, -\alpha_2\beta_1 p^{a_2-k_1}, -\alpha_2\beta_2 p^{a_2-k_2}). \end{aligned}$$

Proof. By Proposition 5.1 and (7.4), one has

$$\begin{aligned} \alpha_p(T,S) = {} & \sum_{b \in \mathbb{Z}} p^{3b} \left(\frac{1}{2(1-(\frac{-1}{p})p^{-1})} K(p^b, p^b) + \frac{1}{2(1+(\frac{-1}{p})p^{-1})} K(p^b, up^b) \right) \\ & + \frac{1}{4} \sum_{\substack{b_1 < b_2 \\ \beta_i = 1 \text{ or } u}} p^{-2b_1-b_2} K(\beta_1 p^{b_1}, \beta_2 p^{b_2}) \end{aligned}$$

Where

$$K(t_1,t_2) = \int_{\mathrm{GL}_2(\mathbb{Z}_p)} J(t_1,t_2) \psi(-\operatorname{tr} T\, {}^t g \operatorname{diag}(t_1,t_2) g) dg$$

and

$$J(t_1,t_2) = \int_{(\mathbb{Z}_p^m)^2} \psi(\frac{1}{2} \operatorname{tr} {}^t g \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} g \begin{pmatrix} (x,x) & (x,y) \\ (x,y) & (y,y) \end{pmatrix}) dx dy.$$

A substitution of $\begin{pmatrix} x \\ y \end{pmatrix}$ by $g\begin{pmatrix} x \\ y \end{pmatrix}$ gives

$$\begin{aligned} J(t_1, t_2) &= \int_{(\mathbb{Z}_p^m)^2} \psi(\frac{1}{2} \operatorname{tr} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} (x,x) & (x,y) \\ (x,y) & (y,y) \end{pmatrix}) \\ &= \int_{(\mathbb{Z}_p^m)^2} \psi(t_1 q(x)) \psi(t_2 q(y))\, dx\, dy \\ &= J(t_1) J(t_2). \end{aligned}$$

In particular, $J(t_1, t_2)$ does not depend on g. Direct calculation gives

$$\begin{aligned} &\int_{\mathrm{GL}_2(\mathbb{Z}_p)} \psi(-\operatorname{tr} T\, {}^t g \operatorname{diag}(t_1, t_2) g) dg \\ &= I^*(-\alpha_1 p^{a_1} t_1, -\alpha_1 p^{a_1} t_2, -\alpha_2 p^{a_2} t_1, -\alpha_2 p^{a_2} t_2). \end{aligned}$$

Finally, $J(\beta p^b) = (\frac{-1}{p})^{l(b,1)} J(p^b)$ by Corollary 2.3. Now putting pieces together, and replacing b_i by $-k_i$, one proves the lemma.

Lemma 7.6. *Let the notation be as above.*

(1) *One has*

$$J(p^{-k})^2 = p^{\sum_{l_i < k}(l_i - k)} (\frac{-1}{p})^{l(-k,1)}.$$

(2) *One has*

$$\begin{aligned} &(\frac{-1}{p})^{l(-k,1)} R(k) \\ &= \begin{cases} (1-p^{-1})(\delta^+(k) + p^{-1}\delta^-(k)) & \textit{if } k \le a_1, \\ p^{-1}(1-p^{-1})\delta^-(k) & \textit{if } k = 1 + a_1 \& a_1 < a_2, \\ -p^{-2}((\frac{-\alpha_1\alpha_2}{p})\delta^+(k) + \delta^-(k)) & \textit{if } k = 1 + a_1 \& a_1 = a_2, \\ 0 & \textit{if } k > 1 + a_1. \end{cases} \end{aligned}$$

Proof. Claim (1) follows from Lemma 2.2 directly. When $k \le a_1$, one has by Corollary 6.3

$$I^*(-\alpha_1 p^{a_1 - k}, -\beta\alpha_1 p^{a_1 - k}, -\alpha_2 p^{a_2 - k}, -\alpha_2 \beta p^{a_2 - k}) = (1 - p^{-1})^2 (1 + p^{-1}).$$

So

$$\begin{aligned} R(k) &= (1-p^{-1})^2(1+p^{-1})\{\frac{1}{2(1-(\frac{-1}{p})p^{-1})} + \frac{(-1)^{l(-k,1)}}{2(1+(\frac{-1}{p})p^{-1})}\} \\ &= (1-p^{-1})(\delta^+(k) + (\frac{-1}{p})p^{-1}\delta^-(k)). \end{aligned}$$

Notice that

$$(\frac{-1}{p})^{l(-k,1)}(x\delta^+(k) + y(\frac{-1}{p})\delta^-(k)) = x\delta^+(k) + y\delta^-(k) \tag{7.10}$$

for any number x and y. So

$$(\frac{-1}{p})^{l(-k,1)}R(k) = (1-p^{-1})(\delta^+(k) + p^{-1}\delta^-(k)).$$

When $k = 1 + a_1$ and $a_1 < a_2$, one has $a_2 - k \geq 0$. In this case, Corollary 6.4 implies

$$\begin{aligned} &I^*(-\alpha_1 p^{a_1-k}, -\beta\alpha_1 p^{a_1-k}, -\alpha_2 p^{a_2-k}, -\beta\alpha_2 p^{a_2-k}) \\ &= p^{-1}(1-p^{-1})(\frac{-\beta}{p})\{1 - (\frac{-\beta}{p})p^{-1}\}. \end{aligned}$$

Now simple calculation (using (7.10)) gives the desired formula for this case. The case where $k = 1 + a_1$ and $a_1 = a_2$ follows similarly from Corollary 6.6 and Lemma 6.1

Finally, when $k > 1 + a_1$, $a_1 + b \leq -2$. So one has by Corollary 6.3

$$I^*(-\alpha_1 p^{a_1-k}, -\beta\alpha_1 p^{a_1-k}, -\alpha_2 p^{a_2-k}, -\alpha_2\beta p^{a_2-k}) = 0.$$

So $R(k) = 0$.

Next come to the second sum in (7.9). First we make a renormalization of the factors involved. Set for an integer k

$$\theta(k) = \begin{cases} 1 & \text{if } l(-k,1) \text{ is even,} \\ \delta_p & \text{if } l(-k,1) \text{ is odd.} \end{cases}$$

Then the following trivial observation will be frequently used in the rest of the paper.

$$\theta(k)^{-1}(x\delta^+(k) + y\delta_p\delta^-(k)) = x\delta^+(k) + y\delta^-(k), \tag{7.11}$$

where x and y are any complex numbers and k is any integer. Denote

$$R^*(k_1,k_2) = (\theta(k_1)\theta(k_2))^{-1}R(k_1,k_2) \tag{7.12}$$

and

$$J^*(k_1,k_2) = \theta(k_1)\theta(k_2)J(p^{-k_1})J(p^{-k_2}).$$

Then

$$J^*(k_1,k_2) = v_{k_1}v_{k_2}p^{\frac{1}{2}[\sum_{l_i<k_1}(l_i-k_1)+\sum_{l_i<k_2}(l_i-k_2)]} \tag{7.13}$$

where v_k is defined by (7.5), and

$$J^*(k_1,k_2)R^*(k_1,k_2) = J(p^{-k_1})J(p^{-k_2})R(k_1,k_2). \tag{7.14}$$

The following lemma follows directly from Corollary 6.3.

Lemma 7.7. (1) *If $k_2 < k_1 \leq a_1$, then*

$$R^*(k_1,k_2) = (1-p^{-1})^2(1+p^{-1})\delta^+(k_1)\delta^+(k_2).$$

(2) *If $k_1 > 1 + a_2$ or $k_2 > 1 + a_1$, then $R^*(k_1,k_2) = 0$.*

Lemma 7.8. (1) *When $k_1 = 1 + a_1 > k_2$ and $a_1 < a_2$, one has*

$$R^*(k_1, k_2) = (1 - p^{-1})\delta^+(k_2)\left((\frac{\alpha_1}{p})\frac{1}{\sqrt{p}}\delta^-(k_1) - \frac{1}{p^2}\delta^+(k_1)\right).$$

(1′) *When $k_1 = 1 + a_1 > k_2$ and $a_1 = a_2$, one has*

$$R^*(k_1, k_2) = p^{-1}(1 - p^{-1})\left((\frac{-\alpha_1\alpha_2}{p}) - \frac{1}{p}\right)\delta^+(k_1)\delta^+(k_2).$$

(2) *When $1 + a_2 > k_1 > 1 + a_1 > k_2$, one has*

$$R^*(k_1, k_2) = (1 - p^{-1})^2 p^{\frac{a_1 - k_1}{2}}\delta^+(k_2)g(k)$$

(3) *When $1 + a_2 > b_1 > 1 + a_1 = k_2$, one has*

$$R^*(k_1, k_2) = (1 - p^{-1})p^{\frac{a_1 - k_1}{2}} f_1(\alpha_1 p^{a_1})g(k)$$

(4) *When $k_1 = 1 + a_2 > 1 + a_1 > k_2$, one has*

$$R^*(k_1, k_2) = (1 - p^{-1})p^{\frac{a_1 - a_2 - 1}{2}}\delta^+(k_2) \cdot \begin{cases} f_1(\alpha_2 p^{a_2}) & \text{if } a_1 - a_2 \text{ is odd,} \\ (\frac{\alpha_1}{p})f_2(\alpha_2 p^{a_2}) & \text{if } a_1 - a_2 \text{ is even.} \end{cases}$$

(5) *When $k_1 = 1 + a_2 > 1 + a_1 = k_2$, one has*

$$R^*(k_1, k_2) = (1 - p^{-1})p^{\frac{a_1 - a_2 - 1}{2}} f_1(\alpha_1 p^{a_1}) \cdot \begin{cases} f_1(\alpha_2 p^{a_2}) & \text{if } a_1 - a_2 \text{ is odd,} \\ (\frac{\alpha_1}{p})f_2(\alpha_2 p^{a_2}) & \text{if } a_1 - a_2 \text{ is even.} \end{cases}$$

Proof. The proof is case by case verification using results in section 6. To save space, we write

$$I(\beta_1, \beta_2) = I^*(-\alpha_1\beta_1 p^{a_1 - k_1}, -\alpha_1\alpha_2 p^{a_1 - k_2}, -\alpha_2\beta_1 p^{a_2 - k_1}, -\alpha_2\beta_2 p^{a_2 - k_2})$$

temporarily.

When $k_1 = 1 + a_1 > k_2$ and $a_1 < a_2$, one has $a_1 - k_1 = -1$, $a_1 - k_2 \geq 0$, $a_2 - k_1 \geq 0$, and $a_2 - k_2 \geq 0$. So Lemma 2.1 and Corollary 6.4 imply

$$\begin{aligned} I(\beta_1, \beta_2) &= (1 - p^{-1})I^*(-\alpha_1\beta_1 p^{-1}) + p^{-1}(1 - p^{-1})^2 \\ &= (1 - p^{-1})(\delta_p(\frac{\alpha_1\beta_1}{p})\frac{1}{\sqrt{p}} - \frac{1}{p^2}). \end{aligned}$$

Since $(\frac{u}{p}) = -1$, one has then

$$\begin{aligned} R(k_1, k_2) &= \frac{1}{4}\sum_{\beta_i = 1 \text{ or } u}(\frac{\beta_1}{p})^{l(-k_1, 1)}(\frac{\beta_2}{p})^{l(-k_2, 1)} I(\beta_1, \beta_2) \\ &= (1 - p^{-1})\delta^+(k_2)\left((\frac{\alpha_1}{p})\frac{1}{\sqrt{p}}\delta_p\delta^-(k_1) - \frac{1}{p^2}\delta^+(k_1)\right). \end{aligned}$$

Applying (7.11), one proves (1).

When $k_1 = 1 + a_1 > k_2$ and $a_1 = a_2$, one has $a_2 - k_1 = a_1 - k_1 = -1$, and $a_1 - k_2 = a_2 - k_2 \geq 0$. So Lemma 2.1 and Corollary 6.4 imply

$$\begin{aligned} I(\beta_1,\beta_2) &= (1-p^{-1})I^*(-\alpha_1\beta_1 p^{-1})I(-\alpha_2\beta_1 p^{-1}) + p^{-1}(1-p^{-1})I^*(-\alpha_2\beta_1 p^{-1}) \\ &= p^{-1}(1-p^{-1})((\frac{-\alpha_1\alpha_2}{p}) - p^{-1}) \end{aligned}$$

Here we have used $\delta^2 = (\frac{-1}{p})$. Plugging this into $R^*(b_1,b_2)$ and applying (7.11), one obtains $(1')$.

When $1+a_2 > k_1 > 1+a_1 > k_2$, one has $a_1 - k_1 \leq -2$, $a_1 - k_2 \geq 0$, $a_2 - k_1 \geq 0$, and $a_2 - k_2 \geq 0$. So Lemma 2.1 and Corollary 6.3 imply

$$\begin{aligned} I(\beta_1,\beta_2) &= (1-p^{-1})^2 I(-\alpha_1\beta_1 p^{a_1-k_1}) \\ &= (1-p^{-1})^2 p^{\frac{a_1-k_1}{2}} \begin{cases} 1 & \text{if } a_1-k_1 \text{ is even,} \\ \delta_p(\frac{\alpha_1\beta_1}{p}) & \text{if } a_1-k_1 \text{ is odd.} \end{cases} \end{aligned}$$

Plugging this into $R^*(b_1,b_2)$ and applying (7.11), one obtains (2).

When $1+a_2 > k_1 > 1+a_1 = k_2$, one has $a_1-k_1 \leq -2$, $a_1-k_2 = -1$, $a_2-k_1 \geq 0$, and $a_2 - k_2 \geq 0$. So Lemma 2.1 and Corollary 6.3 imply

$$\begin{aligned} I(\beta_1,\beta_2) &= (1-p^{-1})I(-\alpha_1\beta_1 p^{a_1-k_1})I^*(-\alpha_1\beta_2 p^{-1}) \\ &= (1-p^{-1})p^{\frac{a_1-k_1}{2}}(\delta_p(\frac{\alpha_1\beta_2}{p})\frac{1}{\sqrt{p}} - \frac{1}{p}) \\ &\quad \cdot \begin{cases} 1 & \text{if } a_1-k_1 \text{ is even,} \\ \delta_p(\frac{\alpha_1\beta_1}{p}) & \text{if } a_1-k_1 \text{ is odd.} \end{cases} \end{aligned}$$

Plugging this into $R^*(k_1,k_2)$ and applying (7.11), one obtains (3).

When $1+a_2 = k_1 > 1+a_1 > k_2$, one has $a_1-k_1 \leq -2$, $a_2-k_1 = -1$, $a_1-k_2 \geq 0$, and $a_2 - k_2 \geq 0$. So Lemma 2.1 and Corollary 6.3 imply

$$\begin{aligned} I(\beta_1,\beta_2) &= (1-p^{-1})I(-\alpha_1\beta_1 p^{a_1-k_1})I^*(-\alpha_2\beta_1 p^{-1}) \\ &= (1-p^{-1})p^{\frac{a_1-a_2-1}{2}}(\delta_p(\frac{\alpha_2\beta_1}{p})\frac{1}{\sqrt{p}} - \frac{1}{p}) \\ &\quad \cdot \begin{cases} 1 & \text{if } a_1-k_1 \text{ is even} \\ \delta_p(\frac{\alpha_1\beta_1}{p}) & \text{if } a_1-k_1 \text{ is odd} \end{cases} \\ &= (1-p^{-1})p^{\frac{a_1-a_2-1}{2}} \begin{cases} \delta_p(\frac{\alpha_2\beta_1}{p})\frac{1}{\sqrt{p}} - \frac{1}{p} & \text{if } a_1+a_2 \text{ is odd,} \\ (\frac{\alpha_1}{p})\{(\frac{-\alpha_2}{p})\frac{1}{\sqrt{p}} - \delta_p(\frac{\beta_1}{p})\frac{1}{p}\} & \text{if } a_1-a_2 \text{ is even.} \end{cases} \end{aligned}$$

Here we have used $\delta_p^2 = (\frac{-1}{p})$. Plugging this into $R^*(k_1,k_2)$ and applying (7.11), one obtains (4).

Finally, when $1+a_2 = k_1 > 1+a_1 = k_2$, one has $a_1-k_1 \leq -2$, $a_2-k_1 = -1$, $a_1 - k_2 = -1$, and $a_2 - k_2 \geq 0$. So Lemma 2.1 and Corollary 6.3 imply

$$\begin{aligned} I(\beta_1,\beta_2) &= I(-\alpha_1\beta_1 p^{a_1-k_1})I^*(-\alpha_2\beta_1 p^{-1})I^*(-\alpha_1\beta_2 p^{-1}) \\ &= p^{\frac{a_1-a_2-1}{2}}(\delta_p(\frac{\alpha_1\beta_2}{p})\frac{1}{\sqrt{p}} - \frac{1}{p}) \\ &\quad \cdot \begin{cases} \delta_p(\frac{\alpha_2\beta_1}{p})\frac{1}{\sqrt{p}} - \frac{1}{p} & \text{if } a_1+a_2 \text{ is odd} \\ (\frac{\alpha_1}{p})\{(\frac{-\alpha_2}{p})\frac{1}{\sqrt{p}} - \delta_p(\frac{\beta_1}{p})\frac{1}{p}\} & \text{if } a_1-a_2 \text{ is even.} \end{cases} \end{aligned}$$

Plugging this into $R^*(k_1,k_2)$ and applying (7.11), one obtains (5).

Proof of Theorem 7.1 Now the proof of Theorem 7.1 becomes tedious but easy calculation. We may assume $l_1 = 0$ by shifting. By Lemma 7.6, one has

(7.15)

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} p^{3k} J(p^{-k})^2 R(k) \\
&= (1-p^{-1})\sum_{k\le 0} p^{3k} + (1-p^{-1})\sum_{0<k\le a} p^{k+2d(k)}(\delta^+(k) + p^{-1}\delta^-(k) \\
&\quad - p^{-1+a+2d(1+a)}((\frac{-\alpha_1\alpha_2}{p})\delta^+(1+a) + \delta^-(1+a)) \\
&= (1-p^{-1})(1-p^{-3})^{-1} + (1-p^{-1})\sum_{0<k\le a} p^{k+2d(k)}(\delta^+(k) + p^{-1}\delta^-(k)) \\
&\quad - p^{-1+a+2d(1+a)}((\frac{-\alpha_1\alpha_2}{p})\delta^+(1+a) + \delta^-(1+a)).
\end{aligned}$$

By Lemma 7.7 and (7.13) – (7.14), one has

$$\begin{aligned}
&\sum_{k_1>k_2} p^{2k_1+k_2} J(p^{-k_1})J(p^{-k_2})R(k_1,k_2) \\
&= \sum_{k_2<k_1} v_{k_1}v_{k_2}p^{k_1+d(k_1)+d(k_2)}R^*(k_1,k_2) \\
&= \sum_{k_2<k_1\le 0} + \sum_{k_2\le 0<k_1\le a} + \sum_{k_2\le 0,\, k_1=a+1} + \sum_{0<k_2<k_1\le a} + \sum_{0<k_2\le a<k_1=a+1}
\end{aligned}$$

where every sum is on $v_{k_1}v_{k_2}p^{k_1+d(k_1)+d(k_2)}R^*(k_1,k_2)$. By Lemmas 7.7, one has

$$\begin{aligned}
&\sum_{k_2<k_1\le 0} v_{k_1}v_{k_1}v_{k_2}p^{k_1+d(k_1)+d(k_2)}R^*(k_1,k_2) \\
&= (1-p^{-1})^2(1+p^{-1})\sum_{k_2<k_1\le 0} p^{2k_1+k_2} \\
&= p^{-1}(1-p^{-1})(1+p^{-1})(1-p^{-3})^{-1}.
\end{aligned}$$

Notice that the sum of this term and the term $(1-p^{-1})(1-p^{-3})^{-1}$ in the sum just computed is one. By Lemmas 7.7, one has

$$\begin{aligned}
&\sum_{k_2\le 0<k_1\le a} v_{k_2}p^{k_1+d(k_1)+d(k_2)}R^*(k_1,k_2) \\
&= \sum_{0<k_1\le a} v_{k_1}p^{k_1+d(k_1)}\delta^+(k_1)(1-p^{-1})^2(1+p^{-1})\sum_{k_2\le 0} p^{k_2} \\
&= (1-p^{-2})\sum_{\substack{0<k\le a \\ l(-k,1)\ \text{even}}} v_k p^{k+d(k)}
\end{aligned}$$

Similar calculation using Lemmas 7.8(1′) and 7.7 gives

$$\begin{aligned}
&\sum_{k_2\le 0,\, k_1=a+1} v_{k_1}v_{k_2}p^{k_1+d(k_1)+d(k_2)}R^*(k_1,k_2) \\
&= \delta^+(1+a)v_{1+a}((\frac{-\alpha_1\alpha_2}{p}) - p^{-1})p^{a+d(1+a)},
\end{aligned}$$

and

$$\sum_{0<k_2<k_1\le a} v_{k_1}v_{k_2}p^{k_1+d(k_1)+d(k_2)}R^*(k_1,k_2)$$
$$= (1-p^{-1})^2(1+p^{-1}) \sum_{\substack{0<k_2<k_1\le a\\ l(-k_i,1)\text{ even}}} v_{k_1}v_{k_2}p^{k_1+d(k_1)+d(k_2)},$$

and

$$\sum_{0<k_2\le a<k_1=a+1} v_{k_1}v_{k_2}p^{k_1+d(k_1)+d(k_2)}R^*(k_1,k_2)$$
$$= \delta^+(1+a)v_{1+a}((\frac{-\alpha_1\alpha_2}{p})-p^{-1})(1-p^{-1}) \sum_{\substack{0<k\le a\\ l(-k,1)\text{ even}}} v_k p^{a+d(1+a)+d(k)}.$$

Putting pieces together, one proves Theorem 7.1.

The proof of Theorem 7.2 is similar. A 'shorter' version of Theorem 7.2 is

Theorem 7.2$'$**.** *Let the notation be as in Theorem* 7.2*. Then*

$$\begin{aligned}\alpha_p(\alpha_1p^{a_1},\alpha_2p^{a_2},S) = p^{3l_1} &+ (1-p^{-1})\delta^-(1+a_1)p^{a_1-1+2d(a_1+1)}\\ &+ (1-p^{-1})\sum_{l_1<k\le a_1}(\delta^+(k)+p^{-1}\delta^-(k))p^{k+2d(k)}\\ &+ (1-p^{-1})^{-1}p^{l_1}\sum_{l_1<k_1\le 1+a_2}p^{2k_1}J^*(k_1,0)R^*(k_1,0)\\ &+ \sum_{\substack{0<k_1\le 1+a_2\\ 0<k_2\le 1+a_1\\ k_2<k_1}} p^{2k_1+k_2}J^*(k_1,k_2)R^*(k_1,k_2)\end{aligned}$$

where $J^*(k_1,k_2)$ *is given by* (7.13) *and* $R^*(k_1,k_2)$ *is given by Lemma* 7.9.

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