TIGHT SURFACES AND REGULAR HOMOTOPY

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1) INTRODUCTION

If $f : M^2 \longrightarrow \mathbf{R}^3$ is an immersion of a compact surface M^2 then

$$(1) \qquad \tau(f) = \frac{1}{2\pi} \int_{M^2} |K| \, dA$$

is called the total absolute curvature of f. Here K denotes the Gaussian curvature of f.

In the theory of total absolute curvature there is a basic inequality, essentially due to Chern and Lashof [4]:

$$(2) \qquad \tau(f) \geq \beta(M^2) \quad ,$$

where β denotes the sum of the $\mathbf{Z}_2$-Betti numbers.

If equality holds in (2) then the immersion f is called tight. Tight surfaces are characterized by many interesting properties. See [3] for a good introduction to this topic.

Nearly all compact surfaces admit tight immersions into $\mathbf{R}^3$. There are only the following exceptions:

1) There is no tight immersion into $\mathbf{R}^3$ of the projective plane and the Klein bottle [8].
2) It is still unknown whether there exists a tight surface in $\mathbf{R}^3$ of Euler charakteristic -1.

The estimate (2) involves only an assumption on the intrinsic topological type of M^2 . One might ask whether it is possible to improve (2) if one is given further topological information about the immersion f . For example one can restrict attention to embeddings and ask for lower bounds of $\tau(f)$ in a given isotopy class of embeddings. A typical result in this direction is the following

<u>THEOREM</u>: a) [12,13] If an embedding $f : M^2 \longrightarrow \mathbb{R}^3$ is knotted (i.e. not isotopic to a standard embedding) then

(3) $$\tau(f) \geq \beta(M^2) + 4 \quad .$$

b) [11] If the genus g of M^2 is greater than two, then there are knotted embeddings $f : M^2 \longrightarrow \mathbb{R}^3$ for which equality holds in (3). For $g \leq 2$ there are no such embeddings.

Further details can be found in [11].

If one does not restrict f to be an embedding it is natural to ask for the infimum of $\tau(f)$ within a given <u>regular homotopy class</u>† of immersions $f : M^2 \longrightarrow \mathbb{R}^3$. In some sense the regular homotopy classes are the connected components of the space of immersions. Thus we have the following problem, which was formulated by Kuiper in 1960 [9]:

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†A regular homotopy is a smooth homotopy that is an immersion at each stage

a) In each regular homotopy class of immersions $f : M^2 \longrightarrow \mathbb{R}^3$ determine the infimum of $\tau(f)$.

b) Determine those regular homotopy classes in which this infimum is attained.

In this paper we will solve part a) of the problem completely:

THEOREM 1: In each regular homotopy class F of immersions $f : M^2 \longrightarrow \mathbb{R}^3$ we have

$$\inf_{f \in F} \tau(f) = \beta(M^2) . \tag{4}$$

A complete solution of part b) of the above problem seems to be very difficult. Such a solution would involve for example an answer to the question whether there exists in $\mathbb{R}^3$ a tight surface of Euler characteristic -1 . At least we are able to show that the infimum is attained in all but a finite number of regular homotopy classes:

THEOREM 2: If the Euler characteristik of M^2 is less than -9 then every immersion $f : M^2 \longrightarrow \mathbb{R}^3$ is regularly homotopic to a tight immersion.

On the other hand there are immersions $f : M^2 \longrightarrow \mathbb{R}^3$ which are not regularly homotopic to a tight immersion. For example this is always the case if M^2 is the projective plane or the Klein bottle. Beyond this we can show for example:

<u>THEOREM 3</u>: **Every tight immersion of the torus T^2 is regularly homotopic to a standard embedding.**

Thus in the nonstandard regular homotopy class of immersed tori [14] there are no tight surfaces.

Here is an outline of the paper: Theorem 1 is proved in section 2. Section 3 contains the proof of Theorem 3 and some further relations between tightness and regular homotopy. In section 4 we construct some explicit examples of tight surfaces and prove Theorem 2.

2) ALMOST TIGHT SURFACES

In this section we will construct in each regular homotopy class of immersions $f : M^2 \longrightarrow \mathbb{R}^3$ (M^2 a compact surface) an "almost tight" immersion. "Almost tight" means that $\tau(f)$ comes arbitrarily close to $\tau(M^2)$. The main idea is contained in the following lemma, that is a special case of a more general result [7,10,15]:

<u>LEMMA</u>: Let $f : M^2 \longrightarrow \mathbb{R}^3$ be an immersion, e_1, e_2, e_3 an orthonormal basis of $\mathbb{R}^3$ such that the height function $x \longmapsto \langle e_3, f(x)\rangle$ on M^2 has exactly k critical points, all of them nondegenerate. Then for the immersions $f_\lambda : M^2 \longrightarrow \mathbb{R}^3$, $\lambda \in (0,1]$,

$$(5) \qquad f_\lambda(x) = \lambda(\langle e_1, f(x)\rangle e_1 + \langle e_2, f(x)\rangle e_2) + \langle e_3, f(x)\rangle e_3$$

we have $\lim_{\lambda\to 0} \tau(f_\lambda) = k$.

By this lemma it is sufficient to find in each regular homotopy class an immersion $f : M^2 \longrightarrow \mathbf{R}^3$ such that the height function $h : M^2 \longrightarrow \mathbf{R}$, $h(x) = \langle e_3, f(x)\rangle$ has exactly $\beta(M^2)$ critical points, all nondegenerate.

The required property of f depends only on the "immersed surface" $[f]$ corresponding to the immersion f (see [14] for a definition). Thus it suffices to construct in each regular homotopy class of <u>immersed surfaces</u> an example of the desired type. We construct such surfaces M by describing the intersections of M with planes orthogonal to e_3 (i.e. the level curves of the height function h). The critical points of h can be seen from the behaviour of the level curves: In a maximum of h there appears in the moving plane a small convex curve. In a minimum such a curve shrinks to a point and then disappears. In the neighborhood of a regular value of h the level curve only performs a regular homotopy. Finally near a nondegenerate critical value of index 1 the level curve behaves as indicated in figure 1.

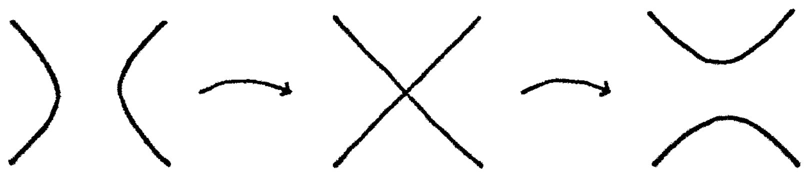
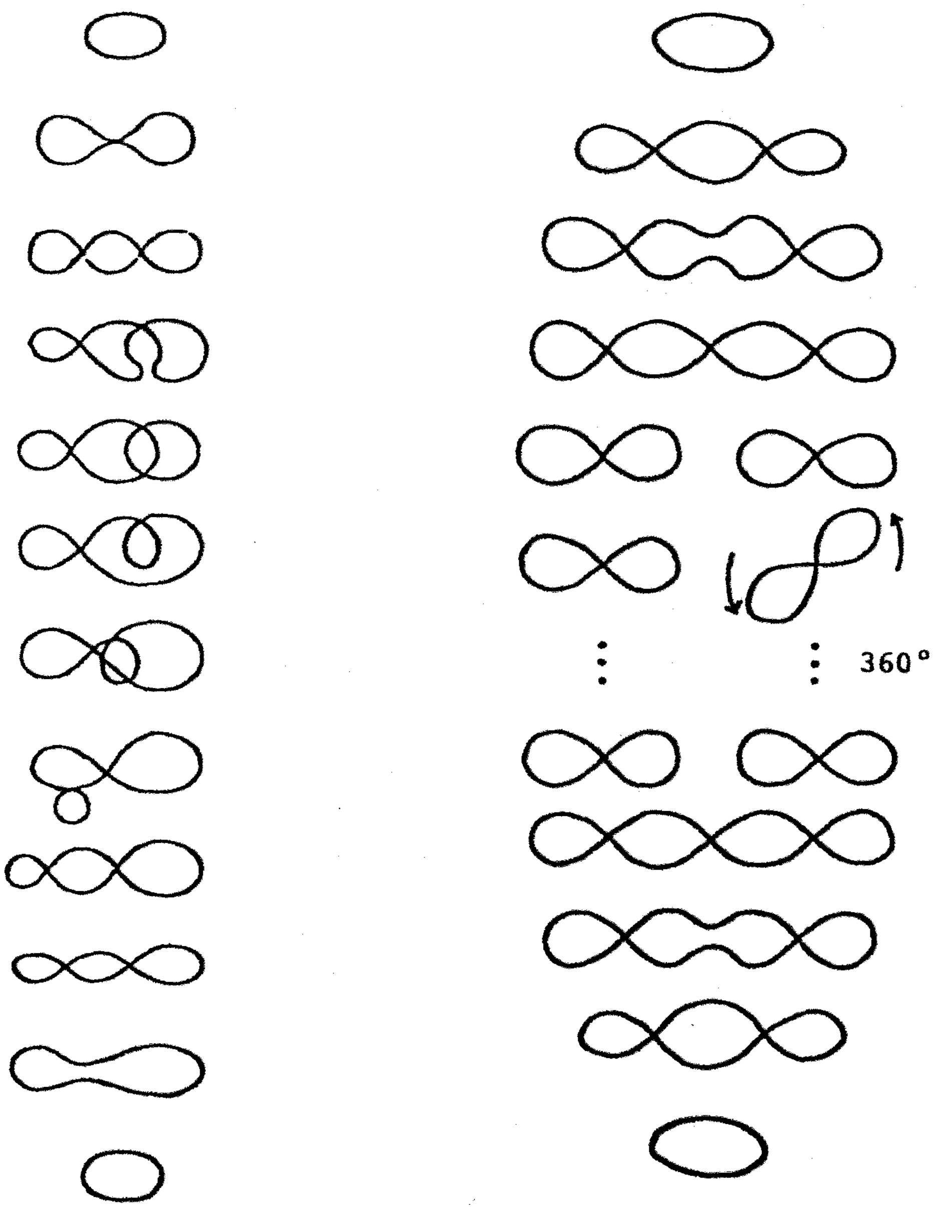


Figure 1

Figure 2a, which is from the original paper [1] of Boy, shows the level curves of a height function with three nondegenerate critical points on a right-handed Boy surface B (concerning immersed surfaces we adopt the terminology of [14]). In figure 2b we indicate a height function on an immersed torus of type T with four critical points.



a) b)

Figure 2

This torus is essentially the same as the one pictured in figure 1 of [14]. In the part of figure 2b indicated only by dots the right hand lemniscate rotates by 360°.

We obtain a height function of the required kind on a left-handed Boy surface $\overline{B}$ by considering the mirror image of figure 2a. Nearly all height functions on a tight torus S meet our conditions. Thus we have found a surface with the desired properties in all four "basic" homotopy classes S,T,B and $\overline{B}$.

By Theorem 4 of [14] every regular homotopy class of immersed surfaces in $\mathbb{R}^3$ can be represented by a connected sum of several copies of S,T,B and $\overline{B}$. Now it is easy to construct such connected sums together with the desired height functions by "stacking" several diagrams as those in figure 3. This completes the proof of Theorem 1.

3) REGULAR HOMOTOPY CLASSES OF TIGHT SURFACES

On every tight surface in $\mathbb{R}^3$ which is not homeomorphic to S^2 there is a homologically nontrivial untwisted annulus (we use the terminology of [14], section 4). This fact was already used in [8] to show that the projective plane $\mathbb{R}P^2$ does not admit a tight immersion into $\mathbb{R}^3$. Suppose now we are given a regular homotopy class of immersions $f : M^2 \longrightarrow \mathbb{R}^3$ $(M^2 \not\approx S^2)$ such that for the corresponding quadratic form q_f [14] we have

$$(6) \qquad q_f(x)=0 \rightarrow x=0$$

Then this regular homotopy class cannot contain tight immersions. An easy consequence of this is the following extended version of Theorem 3:

THEOREM 3': a) Every tight torus in $\mathbb{R}^3$ is regularly homotopic to a standard torus S.

b) There are no tight surfaces in $\mathbb{R}^3$ of type $B \# B \# B$ or $\bar{B} \# \bar{B} \# \bar{B}$.

We obtain further restrictions on the regular homotopy class of a tight surface if we make assumptions on the number of top-cycles. For the definition of a top-cycle we refer to the paper [2] of Cecil and Ryan, where also the following theorem is proved:

THEOREM [2]: Let M^2 be a compact surface of Euler characteristic χ, $f : M^2 \longrightarrow \mathbb{R}^3$ a tight immersion, $\alpha(f)$ the number of top-cycles of f. Then

a) $2 \leq \alpha(f) \leq 2-\chi$.

b) If M^2 is nonorientable then

$$\alpha(f) \leq 1-\chi .$$

Cecil and Ryan give examples of tight surfaces with the maximal number of top-cycles ($2-\chi$ if M^2 is orientable, $1-\chi$ otherwise) for all even values of χ. It is unknown whether there exist tight surfaces with odd χ having $1-\chi$ top-cycles. It is shown in [2] that this problem is equivalent to the question whether there is a tight surface with $\chi = -1$. By part b) of the following theorem such surfaces can exist only in certain regular homotopy classes:

<u>THEOREM 4</u>: Let M be a tight immersed surface in $\mathbf{R}^3$ of Euler characteristic χ, $\alpha(M)$ the number of top-cycles of M. Then

a) If $\alpha(M) = 2-\chi$ then M is regularly homotopic to the standard surface $S \# \ldots \# S$.

b) If χ is odd and $\alpha(f) = 1-\chi$ then M is regularly homotopic to $B \# S \# \ldots \# S$ or to $\overline{B} \# S \# \ldots \# S$.

<u>PROOF</u>: In the proof of Theorem 1 in [2] it is shown that under the assumptions of a) M can be decomposed as $M = U \cup V_1 \cup \ldots \cup V_k$, where

(i) $U = K-(D_1 \cup \widetilde{D}_1 \cup \ldots \cup D_k \cup \widetilde{D}_k)$. Here K is the boundary of the convex hull of M and $D_i, \widetilde{D}_i \subset K$ are plane convex disks.

(ii) For $i = 1,\ldots,k$ V_i is an immersed annulus and $\partial V_i = \partial D_i \cup \partial \widetilde{D}_i$.

It is obvious that such a surface is cobordant to an immersed sphere [14], and therefore regularly homotopic to $S \# \ldots \# S$. This proves a).

In the proof of Theorem 2 in [2] it is shown that under the conditions of b) M admits a decomposition $M = U \cup V_1 \cup \ldots \cup V_k$, where $U, V_1, \ldots, V_k$ are defined as above, with one exception: V_1 is not homeomorphic to an annulus but to the projective plane with two holes. Obviously such an immersed surface is cobordant to B or to $\overline{B}$.

4) TIGHT SURFACES IN PRESCRIBED REGULAR HOMOTOPY CLASSES

In [14] it is shown that the immersed surfaces $[f]$ in $\mathbb{R}^3$ of intrinsic type M^2 are classified up to regular homotopy by the Arf-invariant $\sigma_f \in \mathbb{Z}_8$, where σ_f is subject only to the following conditions:

(i) $\sigma_f \equiv \chi(M^2) \mod 2$

(ii) $\sigma_f \equiv 0 \mod 4$ if M^2 is orientable

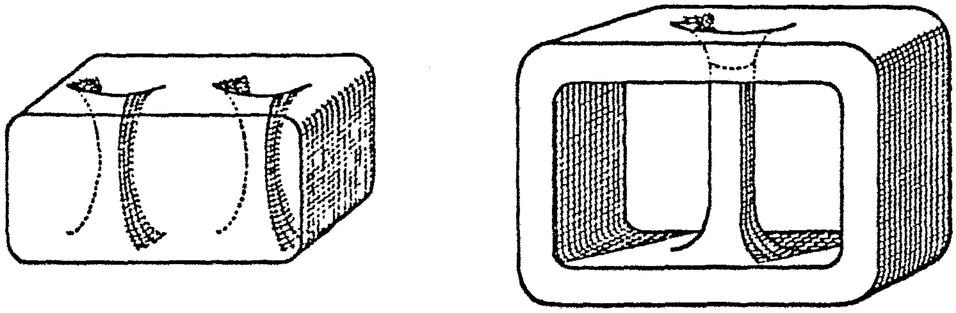
(iii) $|\sigma_f| \leq 2-\chi(M^2)$

Here for $\xi \in \mathbb{Z}_8$ we have defined

(7) $$|\xi| = \inf \{ |x| \mid x \in \mathbb{Z},\ x \equiv \xi \mod 8 \} .$$

(iii) is a real restriction only in case M^2 is the projective plane or the Klein bottle.

Fig. 3 (which is taken from [9]) shows two tight surfaces with trivial Arf-invariant $\sigma_f = 0$.



a) b)

Figure 3

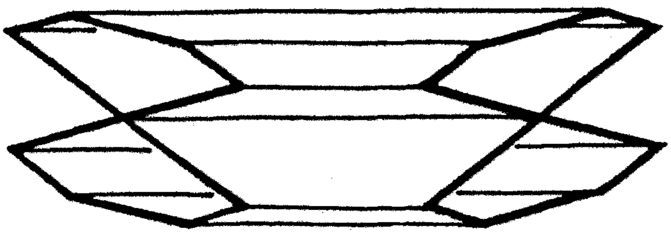
By adding further handles to these surfaces (preserving tightness) one easily obtains

<u>THEOREM 5</u>: Let M^2 be a compact surface, not homeomorphic to the Klein bottle. Then every immersion $f : M^2 \longrightarrow \mathbb{R}^3$ with $\sigma_f = 0$ is regularly homotopic to a tight immersion.

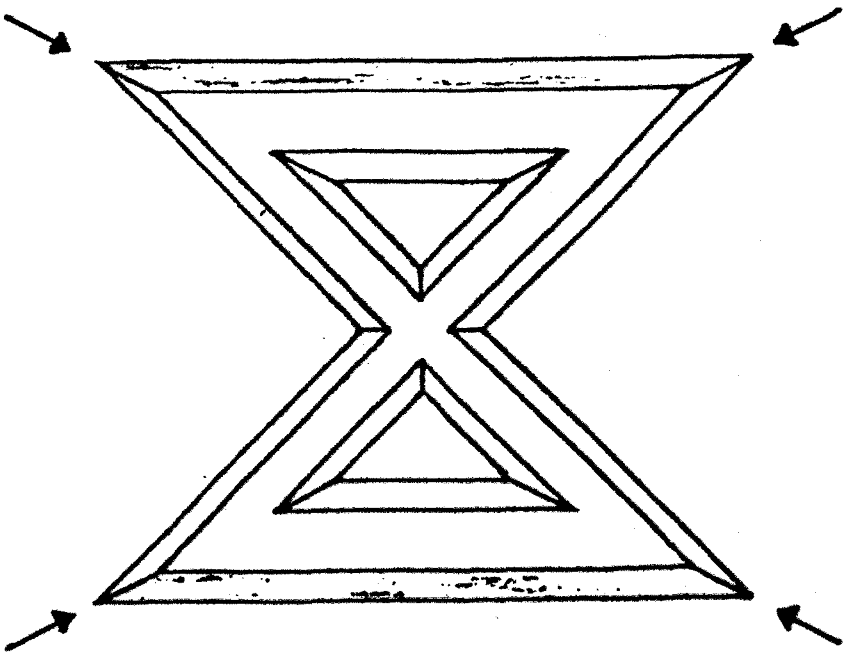
Since the Klein bottle does not admit tight immersions into $\mathbb{R}^3$ [8] part b) of Kuipers problem stated in the introduction is solved for regular homotopy classes with trivial Arf-invariant.

We now construct a tight orientable surface in $\mathbb{R}^3$ of genus 4 with Arf-invariant $\sigma_f = 4$. This construction is based on a polyhedral surface due to T. Banchoff. We only have modified here this surface slightly in order to be able to apply the smoothing procedure developed in [6].

The starting point for the construction is the polyhedral immersed torus pictured in figure 4 b. This torus is built with the help of four figure eight-shaped polygons that lie in the planes indicated in figure 4b by arrows



a)



b)

Figure 4

(the planes are orthogonal to the drawing plane). These four polygonal figure eights are then connected by four polyhedral cylinders as indicated in figure 4b. Figure 4a shows the upper one of the two horizontal cylinders in a slightly different view. The two boundary curves of the cylinder in Figure 4a are two of the four mentioned polygonal figure eights.

Note that figure 4b shows in some sense a "plaster model" of the surface, that is we did not attempt to indicate in the middle of figure 4b the self-intersections.

As a second step we now displace the four faces indicated in figure 4b by shading slightly towards the interior (keeping them parallel to the original faces). In other words along these faces we "grind off" a thin layer of the mentioned plaster model. During that process each of the eight vertices at maximal distance from the center of the surface splits into two vertices of valence three (figure 5).

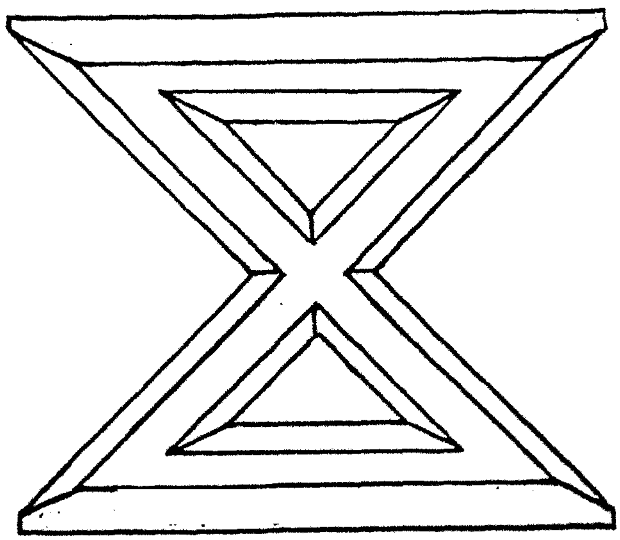


Figure 5

The four displaced faces are then cut out and the remaining surface is "inserted" into a suitable convex polyhedron (into which four corresponding holes have been cut). It is not necessary to describe here the latter convex surface in detail. All strict local support planes of the constructed surface of genus four are strict global support planes. Hence this polyhedral surface is tight [6].

All non-convex vertices of the above surface are either 3-valent or standard saddle vertices [6]. Thus by [6] this surface can be approximated by a tight C^∞-surface. It is easy to see that this smooth surface is of type $T \# S \# S \# S$ and therefore has Arf-invariant $\sigma_f = 4$.

It is now easy to attach to this surface further handles without violating the tightness or changing the Arf-invariant. Thus for orientable surfaces we have a sharper version of Theorem 2:

<u>THEOREM 6</u>: If M^2 is an orientable surface of genus $g \geq 4$ then every immersion $f : M^2 \longrightarrow \mathbb{R}^3$ is regularly homotopic to a tight immersion.

Kuiper described in [9] a tight surface with Euler characteristic -3 . In [6] one can find a more symmetric version of such a surface. The regular homotopy type of both surfaces is that of a Boy surface with two handles, hence the Arf-invariant σ_f is ± 1 . By adding further handles to such a surface we obtain for all odd numbers $\chi \leq -3$ tight surfaces with Euler characteristic χ and $\sigma_f = \pm 1$.

By gluing two of the polyhedral surfaces with $\chi = -3$ described in [6] together we obtain a tight polyhedral surface with $\chi = -8$. The smoothing procedure of [6] applies to this surface and the resulting smooth surface has $\sigma_f = \pm 2$. By adding handles we obtain tight surfaces with $\sigma_f = \pm 2$ for any even Euler characteristic $\chi \leq -8$.

Tight nonorientable surfaces with $\sigma_f = 4$ can be constructed for any even Euler characteristic $\chi \leq -8$ by adding nonorientable handles of the kind indicated in figure 3b to the orientable surface with $\sigma_f = 4$ constructed above.

Finally we obtain a tight surface with $\sigma_f = \pm 3$ and $\chi = -11$ by considering a tight connected sum of a surface with $\sigma = 4$ and $\chi = -6$ and another surface with $\sigma = \pm 1$ and $\chi = -3$. Adding handles then yields tight surface with $\sigma_f = \pm 3$ for any odd $\chi \leq -11$. This completes the proof of Theorem 2.

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