Bounds on degrees of projective schemes

Bernd Sturmfels *
Ngô Viêt Trung **
Wolfgang Vogel ***

*
Cornell University
Department of Mathematics
Ithaca, NY 14853
USA

**
Institute of Mathematics
P.O.Box 631
Bò Hô, Hanoi
Vietnam

***
Massey University
Department of Mathematics
Palmerston North
New Zealand

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn
Germany

MPI / 93-70

Author for correspondences: B. Sturmfels
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BOUNDS ON DEGREES OF PROJECTIVE SCHEMES

Bernd Sturmfels
Cornell University, Dept. of Mathematics, Ithaca, NY 14853, U.S.A.
e-mail: bernd@math.cornell.edu

Ngô Viêt Trung
Institute of Mathematics, P.O. Box 631, Bò Hô, Hanoi, Vietnam

Wolfgang Vogel
Massey University, Department of Mathematics, Palmerston North, New Zealand
e-mail: W.Vogel@massey.ac.nz

1. Introduction

Computational complexity issues in algebraic geometry necessitate two refinements in the notion of *degree* of a projective scheme, namely, the *geometric degree* and the *arithmetic degree* (cf. [Har], [B-M], [Kol]). Given a homogeneous ideal I in $S := k[x_1, \ldots, x_n]$, k any field, a fundamental problem is to bound these refined degrees in terms of the generators of I.

The main results in this paper give bounds for the arithmetic degree of a monomial ideal (Theorem 3.1) and for the geometric degree of an arbitrary ideal (Theorem 4.3). One novelty of independent interest is a combinatorial construction (called *standard pairs*) for these degrees in the case of monomial ideals. In §5 we present applications to the effective division problem and to extensions of Bezout's Theorem. In §2 we discuss two theorems which relate the arithmetic degree to the Nullstellensatz and to Gröbner bases. These two are essentially due to Kollár [Kol] and Hartshorne [Har], while our contribution lies in providing new, self-contained proofs.

In this section we recall the basic definitions. Let $I = q_0 \cap q_1 \cap \ldots \cap q_t$ be a primary decomposition of I, with associated primes $p_i := \sqrt{q_i}$, defining (irreducible) varieties $Z_i := V(p_i)$ in P_k^{n-1} for $i \geq 0$, while $p_0 = (x_1, \ldots, x_n)$ and thus $Z_0 := V(p_0) = \emptyset$. Suppose that $q_1, \ldots, q_s$ are the isolated components of I, so that $V(I) = Z_1 \cup \ldots \cup Z_s$ is set-theoretically the minimal decomposition of $V(I)$ into varieties. Let $mult(q_i)$ be the classical length-multiplicity (see, e.g. [Gro]), that is, $mult(q_i)$ is the length l of a maximal strictly increasing chain of p_i-primary ideals:

$$q_i \ = \ J_l \ \subset \ J_{l-1} \ \subset \ \cdots \ \ \subset \ J_1 \ = \ p_i.$$

Equivalently, $mult(q_i)$ is the length of the local ring $(S/I)_{p_i}$.

The usual geometric degree of Z_i is the cardinality of $Z_i \cap L$ for almost all linear subspaces L of complementary dimension. It is denoted by $deg\ Z_i$. The *degree* of I is

$$deg(I) \ := \ \sum_{\substack{i \text{ such that} \\ dim\, Z_i = dim(I)}} mult(q_i) \cdot deg\ Z_i.$$

This is also the normalized leading coefficient of the Hilbert polynomial of S/I. By contrast, the *geometric degree* of I is defined by taking the sum over all isolated components:

$$geom\text{-}deg(I) \ := \ \sum_{i=1}^{s} mult(q_i) \cdot deg\ Z_i,$$

This degree is at the heart of the so-called refined Bezout's Theorem (see e.g. [Ful], [Vo]).

The two notions of degree defined so far ignore the embedded components q_i, $i \in \{0, s+1, \ldots, t\}$. To measure their contributions to I, we need to recall the notion of length multiplicity relative to I (see e.g. [B-M, §3], [E-H, p. 61]). Given any homogeneous prime ideal p in S, we consider the ideal $J \ := \ \cup_{j>0} (I : p^j)$. This is the intersection of the primary components of I with associated primes not containing p. We define $mult_I(p)$ to be the length ℓ of a maximal strictly increasing chain of ideals

$$(1.1) \qquad I \ = \ J_\ell \ \subset \ J_{\ell-1} \ \subset \ \cdots \ \subset \ J_2 \ \subset \ J_1 \ \subset \ J,$$

where each J_k equals $q \ \cap \ J$ for some p-primary ideal q. Equivalent definitions are:

(1.2) $mult_I(p)$ is the length of the module JS_p/IS_p;

(1.3) $mult_I(p)$ is the length of the largest ideal of finite length in the ring S_p/IS_p.

We have $mult_I(p) > 0$ if and only if p is an associated prime of I. If q_i is an isolated component of I, then $mult_I(p_i) = mult(q_i)$. The *arithmetic degree* of I is now defined as

$$(1.4) \qquad arith\text{-}deg(I) \ := \ \sum mult_I(p) \cdot deg(p),$$

where p runs over all homogeneous primes in S. (Note: this includes the irrelevant ideal.)

The following simple example may serve as an illustration. Let $n = 3$ and

$$(1.5) \qquad I \ = \ (x^2y, x^2z, xy^2, xyz^2) \ = \ (x^2, y^2, z^2) \cap (x) \cap (y, z) \cap (x^2, y).$$

Then $deg(I) = 1$, $geom\text{-}deg(I) = 2$, and $arith\text{-}deg(I) = 5$. The contributions of the two embedded components are $mult_I\big((x,y)\big) = 1$ and $mult_I\big((x,y,z)\big) = 2$. That the irrelevant ideal (x,y,z) has multiplicity 2 in I can be seen from the maximal sequence

$$J \;=\; \cup_j\big(I:(x,y,z)^j\,\big) \;=\; (x^2z, xy) \quad \supset \quad J \cap (x^2,y^2,z) \quad \supset \quad J \cap (x^2,y^2,z^2) \;=\; I.$$

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2. On the arithmetic degree

The importance of the concept of arithmetic degree for computational problems in algebraic geometry is highlighted by the following version of the Nullstellensatz.

Theorem 2.1. *Let I be a homogeneous ideal of S. Then* $\sqrt{I}^{\,arith\text{-}deg(I)} \subseteq I$.

This theorem had been stated as a conjecture in an early draft of this paper. J. Kollár kindly informed us that a proof can be gotten from his approach describing $mult_I(p)$ in [Kol, Remark 1.12]. Subsequently, we found the following direct proof. It starts out with an easy splitting property of the arithmetic degree.

Lemma 2.2. *Let I be a homogeneous ideal of S and P a maximal associated prime of I. Let J be the intersection of the primary components of I with associated primes different from P, in any primary decomposition of I. Then*

$$arith\text{-}deg(I) \quad = \quad arith\text{-}deg(J) + mult_I(P) \cdot deg(P).$$

Proof: In view of the maximality hypothesis, no associated prime of I or J strictly contains P. In other words, we have $mult_I(p) = mult_J(p) = 0$ for any homogeneous prime ideal p in S which strictly contains P. Since $P^rJ \subseteq I \subseteq J$ for sufficiently large integers $r > 0$, we have $IS_p = JS_p$ for any prime p not containing P. By (1.3), this implies $mult_I(p) = mult_J(p)$ for any prime p other than P. Finally, since P is not an associated prime of J, we have $mult_J(P) = 0$. Evaluating the sum (1.4) for both I and J, we obtain the desired result. ∎

We note that Lemma 2.2 is false if P is not a maximal associated prime of I. (In this case the ideal J and its arithmetic degree depend on the chosen primary decomposition.) For instance, taking $P = (x, y)$ in (1.5), we find $\textit{arith-deg}(J) = \textit{arith-deg}(I) = 5$, where

$$J \quad = \quad (x^2, y^2, z^2) \cap (x) \cap (y, z) \quad = \quad (x^2y, x^2z, xy^2, xz^2). \tag{2.1}$$

Proof of Theorem 2.1: Let P and J be as in Lemma 2.2. Then we may split

$$\begin{aligned} \sqrt{I}^{\,\textit{arith-deg}(I)} &= \sqrt{I}^{\,\textit{arith-deg}(J)} \cdot \sqrt{I}^{\,\textit{mult}_I(P)\textit{deg}(P)} \\ &\subseteq \sqrt{J}^{\,\textit{arith-deg}(J)} \cdot P^{\textit{mult}_I(P)}. \end{aligned}$$

By induction on the number of the associated primes we may assume that

$$\sqrt{J}^{\,\textit{arith-deg}(J)} \subseteq J.$$

It remains to show that $JP^{\textit{mult}_I(P)} \subseteq I$. This can be done locally at every associated prime p of I. For $p \neq P$, the inclusion is immediate because $JS_p = IS_p$. For $p = P$ we assume, on the contrary, that $JP^{\textit{mult}_I(P)}S_P \neq IS_P$. Then $(JP^j + I)S_P \neq (JP^{j+1} + I)S_P$ for $j = 0, \ldots, \textit{mult}_I(P)$. For, otherwise we would have

$$JP^{\textit{mult}_I(P)}S_P \;\subseteq\; (JP^j + I)S_P \;=\; \bigcap_{r \geq j} (JP^r + I)S_P = IS_P,$$

by Krull's Intersection Theorem. We get a strictly increasing chain of $\textit{mult}_I(P) + 2$ ideals

$$IS_P \;\subset\; (JP^{\textit{mult}_I(P)} + I)S_P \;\subset \ldots \subset\; (JP^2 + I)S_P \;\subset\; (JP + I)S_P \;\subset\; JS_P,$$

which gives a contradiction to the definition (1.2) of $\textit{mult}_I(P)$. ∎

We next establish a connection to Gröbner bases theory by showing that the arithmetic degree of a homogeneous ideal is bounded above by that of any initial ideal, and, moreover this inequality holds for the contributions in each dimension. Theorem 2.3 is a special case of a more general result due to R. Hartshorne. Indeed, in [Har, Theorem 2.10] it is shown that $\textit{arith-deg}_r(\,\cdot\,)$ is upper-semicontinuous with respect to flat families of projective schemes, and a well-known result of Gröbner basis theory states that, for any term order, the initial ideal $in(I)$ is a flat deformation of I (see e.g. [B-M], [Eis]). We write $\textit{arith-deg}_r(I)$ for the subsum over all terms in (1.4) where p has (affine) dimension r, for $r = 0, 1, \ldots, n$.

Theorem 2.3. *Fix any term order on S, and let $I \subset S$ be any homogeneous ideal. Then*

$$arith\text{-}deg_r(I) \;\leq\; arith\text{-}deg_r(in(I)) \qquad \text{for all } r = 0, 1, \ldots, n.$$

The following self-contained proof uses a different characterization of the arithmetic degree, which is purely enumerative and, in fact, serves as the definition in [Har]. Let I be a homogeneous ideal in S. For each integer $r = 0, 1, \ldots, n$ we consider

$$I_{\geq r} \quad := \quad \{ f \in S \,:\, dim\,(I : f) \,<\, r \}. \tag{2.2}$$

Using elementary properties of ideal quotients (as in e.g. [Gro]), it can be verified that the set $I_{\geq r}$ is an ideal in S, containing I, and that it equals the intersection of all primary components of dimension $\geq r$ in any primary decomposition of I. We also find that the Hilbert polynomial of the graded S-module $I_{\geq r}/I$ has the form

$$h_{I_{\geq r}/I}(d) \quad = \quad \frac{arith\text{-}deg_r(I)}{(r-1)!} \cdot d^{r-1} \;+\; O(d^{r-2}). \tag{2.3}$$

Proof of Theorem 2.3: We claim the following inclusion of monomial ideals:

$$in(I_{\geq r}) \quad \subseteq \quad \big(in(I)\big)_{\geq r}. \tag{2.4}$$

Indeed, suppose $m \in in(I_{\geq r})$. Then $m = in(f)$ for some $f \in I_{\geq r}$, and therefore $r > dim\,(I : f) \doteq dim\; in(I : f)$. Since $in(I : f)$ is contained in $(in(I) : in(f)) = (in(I) : m)$, we conclude $dim\,(in(I) : m) \,<\, r$, and therefore $m \in \big(in(I)\big)_{\geq r}$.

Whenever we have an inclusion of homogeneous ideals $I \subseteq I'$ in S, then their quotient I'/I is isomorphic as a graded vector space to $in(I')/in(I)$. To see this, it suffices to note that the canonical monomial basis for $in(I')/in(I)$ lifts to a basis for I'/I. Applying this observation to $I' = I_{\geq r}$, and using (2.4), we obtain the inclusion of graded vector spaces:

$$I_{\geq r}/I \quad \simeq \quad in(I_{\geq r})/in(I) \quad \hookrightarrow \quad \big(in(I)\big)_{\geq r}/\,in(I). \tag{2.5}$$

The leading term of the Hilbert polynomial of the right hand side in (2.5) exceeds that of the left hand side. By (2.3), this proves Theorem 2.3. ∎

Theorem 2.3 shows that in bounding the arithmetic degree it makes sense to concentrate on the case of a monomial ideal. This is what we will do in the next section.

Example 2.4. Let $n = 4$ and let I be the prime ideal of the monomial curve given parametrically by $(s^7 : s^5t^2 : s^2t^5 : t^7)$. Clearly, $deg(I) = geom\text{-}deg(I) = arith\text{-}deg(I) = 7$. Using the methods in [Tho, §3], we find that I has the universal Gröbner basis

$$\big\{x_1^5x_4^2 - x_2^7,\; x_1^4x_3x_4 - x_2^6,\; x_1^3x_3^2 - x_2^5,\; x_1^2x_4^5 - x_3^7,\; x_1x_2x_4^4 - x_3^6,$$
$$x_1x_4 - x_2x_3,\; x_1^2x_3^3 - x_2^4x_4,\; x_1x_3^4 - x_2^3x_4^2,\; x_2^2x_4^3 - x_3^5\big\},$$

and that there are precisely 14 distinct initial ideals. All of them fail to be square-free, which implies that the inequalities $arith\text{-}deg(in(I)) > 7$ are strict. The gap can vary widely: the arithmetic degrees of the 14 initial monomial ideals are between 9 and 18.

3. The arithmetic degree of a monomial ideal

The objective of this section is to prove the following lower and upper bounds.

Theorem 3.1. *Let I be a proper monomial ideal in $S = k[x_1, \ldots, x_n]$ with minimal set of monomial generators $\{m_1, m_2, \ldots, m_s\}$, and let $e := dim(I) + s - n$. Then*

$$max\,\{deg(m_i) : i = 1, \ldots, s\} \quad \leq \quad arith\text{-}deg(I) \quad \leq \quad \prod_{i=1}^{s} deg(m_i) \;-\; e.$$

The upper bound in Theorem 3.1 is false for a general homogeneous ideal which is not generated by monomials, even if we delete the excess dimension term e. We do not know whether the lower bound generalizes to arbitrary homogeneous ideals.

Example 3.2. (see also [Kol, p. 966]) Let I be the ideal generated by the forms

$$f := x_1 x_4 - x_2 x_3,\; g := x_1^{b-a} x_3^a - x_2^b,\; h := x_3^b - x_2^a x_4^{b-a},$$

for any integers $b > a > 0$. It follows from [S-V, p. 171 and Prop. 1.9, p. 162] that

$$geom\text{-}deg(I) = deg(I) = a + b \quad \text{and} \quad arith\text{-}deg(I) = a + b + \binom{b-a+1}{3}.$$

For $b \gg a$, the arithmetic degree of I exceeds $deg(f) \cdot deg(g) \cdot deg(h) \;=\; 2b^2$. ∎

The key idea in proving Theorem 3.1 is to give a combinatorial rule for the length-multiplicity $mult_I(\cdot)$ of a monomial ideal I. Each associated prime of I has the form $P_Z := (x_i : x_i \in X \setminus Z)$, where Z is a subset of $X = \{x_1, \ldots, x_n\}$. Let $\mathcal{M}$ denote the set of all monomials in $k[X]$. There is a natural map $\mathcal{M} \to 2^X$, $m \mapsto supp(m)$, which takes each monomial to its set of variables. A pair (m, Z) in $\mathcal{M} \times 2^X$ is called *admissible* if $Z \cap supp(m) = \emptyset$. We define a partial order "$\leq$" on the set of all admissible pairs by

$$(3.1) \qquad (m, Z) \;\leq\; (m', Z') \;\Longleftrightarrow\; m \text{ divides } m' \quad \text{and} \quad supp(m'/m) \,\cup\, Z' \;\subseteq\; Z.$$

The condition (3.1) is equivalent to the inclusion of graded vector spaces $m' \cdot k[Z'] \subseteq m \cdot k[Z]$. An admissible pair (m, Z) is called *standard* (with respect to the monomial ideal I) if $m \cdot k[Z] \cap I \;=\; \emptyset$, and (m, Z) is minimal with this property in the partial order (3.1). Let $std(I)$ denote the number of all standard pairs of a monomial ideal I.

Lemma 3.3. *Every monomial ideal I in S satisfies arith-deg(I) = std(I). More precisely, for fixed $Z \in 2^X$, $mult_I(P_Z)$ equals the number of standard pairs of the form $(\,\cdot\,, Z)$.*

Proof: Our monomial ideal has the following decomposition into irreducible ideals:

$$I \;=\; \bigcap \,(x_i^{deg_{x_i}(m)+1} : \; x_i \in X \backslash Z), \tag{3.2}$$

where the intersection is over all standard pairs (m, Z). Split this intersection with respect to the different sets Z and apply the definition of $mult_I(\cdot)$ given in the introduction. ∎

Note that any monomial ideal can be recovered from its standard pairs using (3.2). For an example consider the ideal I in (1.5). Its standard pairs are

$$(xy, \{\,\}),\; (xyz, \{\,\}),\; (1, \{y, z\}),\; (1, \{x\}),\; (x, \{z\}). \tag{3.3}$$

If we replace $(x, \{z\})$ by $(xz, \{\,\})$ in (3.3) then we get precisely the standard pairs of the ideal J in (2.1). We need three more lemmas for the proof of Theorem 3.1.

Lemma 3.4.
(a) For a principal monomial ideal $I = (m)$, we have std(I) = deg(m).
(b) For any two monomial ideals I_1 and I_2, we have $std(I_1 + I_2) \;\leq\; std(I_1) \cdot std(I_2)$.

Proof:
(a): The standard pairs are $(x^i, X \setminus \{x\})$ where $x \in supp(m)$ and $0 \leq i < deg_x(m)$.
(b): For each standard pair (m_1, Z_1) of I_1 and each standard pair (m_2, Z_2) of I_2, determine their supremum in the partial order (3.1). In the resulting poset of $\leq std(I_1) \cdot std(I_2)$ pairs select the minimal elements. They are precisely the standard pairs for $I_1 + I_2$. ∎

Lemma 3.5. *Let $I \subset k[x_1, \ldots, x_n]$ be any monomial ideal, and $P = (x_1, \ldots, x_d)$. Then P is an isolated prime of I if and only if $(1, \{x_{d+1}, \ldots, x_n\})$ is a standard pair of I.*

Proof: The inclusion $I \subseteq P$ is equivalent to the condition $k[x_{d+1}, \ldots, x_n] \cap I \;=\; \{0\}$. This means there exists a standard pair (m, Z) smaller or equal to $(1, \{x_{d+1}, \ldots, x_n\})$ in the partial order "$\leq$". In this case we must have $m = 1$ and $\{x_{d+1}, \ldots, x_n\} \subseteq Z$.

We conclude that I is contained in P if and only if there exists a subset Z of variables such that $(1, Z)$ is standard and $\{x_{d+1}, \ldots, x_n\} \subseteq Z$. If this inclusion is proper, then the ideal $(x_i \,:\, x_i \in X \setminus Z)$ contains I and is properly contained in P, so that P is not a minimal prime of I. On the other hand, if $\{x_{d+1}, \ldots, x_n\} = Z$ then no other ideal of the form $(x_i \,:\, x_i \in X \setminus Z')$, $Z' \supset Z$, contains I. ∎

Lemma 3.5 implies that the geometric degree *geom-deg*(I) equals the number of standard pairs (m, Z) for which $(1, Z)$ is also standard.

Lemma 3.6. *Let I be a monomial ideal, let m be a monomial of degree at least 2 and suppose that $dim(I) = dim(I + (m))$. Then $std(I + (m)) \leq std(I) \cdot deg(m) - 1$.*

Proof: Since $dim(I) = dim(I + (m))$, the monomial m is contained in a minimal prime P of I. We may assume $P = (x_1, \ldots, x_d)$, $m = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$, and $i_1 \geq 1$. By Lemma 3.5, $\Pi := (1, \{x_{d+1}, \ldots, x_n\})$ is a standard pair of I. We need to show that two of the crosswise suprema formed in the proof of Lemma 3.4 (b) are comparable in the poset (3.1), which means one of them is discarded when passing to minimal elements.

Case 1: $d_1 \geq 2$. Then $(1, \{x_2, \ldots, x_n\})$ and $(x_1, \{x_2, \ldots, x_n\})$ are standard pairs of (m), giving the same supremum:

$$\Pi \vee (1, \{x_2, \ldots, x_n\}) \;=\; \Pi \vee (x_1, \{x_2, \ldots, x_n\}) \;=\; (1, \{x_{d+1}, \ldots, x_n\}).$$

Case 2: $d_1 = 1$ *and* $d_2 \geq 1$. Then $(1, \{x_2, \ldots, x_n\})$ and $(1, \{x_1, x_3, \ldots, x_n\})$ are standard pairs of (m), giving comparable suprema:

$$\Pi \vee (1, \{x_2, \ldots, x_n\}) \;>\; (1, \{x_{max\{3,d+1\}}, \ldots, x_n\}) \;=\; \Pi \vee (1, \{x_1, x_3, \ldots, x_n\}).$$

Case 3: $d_1 = 1$ *and* $d_n \geq 1$. Then $(1, \{x_2, \ldots, x_n\})$ and $(1, \{x_1, x_2, \ldots, x_{n-1}\})$ are standard pairs of (m), giving comparable suprema:

$$\Pi \vee (1, \{x_2, \ldots, x_n\}) \;>\; \Pi \vee (1, \{x_1, \ldots, x_{n-1}\}) \;=\; (1, \{x_{d+1}, \ldots, x_{n-1}\}).$$

This exhausts all cases up to relabeling. ■

Proof of Theorem 3.1. We first remove all variables which appear in the set $\{m_1, \ldots, m_s\}$. This does not alter any of the three expressions in the two claimed inequalities. So, we may assume $deg(m_i) \geq 2$ for $i = 1, \ldots, s$. Starting with Lemma 3.4 (a), we apply Lemma 3.4 (b) and Lemma 3.6 iteratively to $I = (m_1, \ldots, m_{i-1})$ and $m = m_i$, for $i = 2, \ldots, s$. The excess dimension e equals the number of indices i for which the dimension hypothesis of Lemma 3.6 is satisfied. This proves the asserted upper bound for *arith-deg*(I).

For the lower bound we may assume that $deg(m_1) \geq deg(m_i)$ for all i, and $m_1 = x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r}$, where $0 < r \leq n$ and $a_1, \ldots, a_r > 0$. Consider the following sequence of $deg(m_1) = \sum a_j$ *standard monomials* (i.e. monomials not in I), arranged in r groups:

$$(3.4) \qquad \begin{array}{c} x_1^{i_1} x_2^{a_2} \cdots x_r^{a_r},\; i_1 = 0, 1, \ldots, a_1 - 1, \qquad x_1^{a_1} x_2^{i_2} \cdots x_r^{a_r},\; i_2 = 0, 1, \ldots, a_2 - 1, \\ \ldots \quad \ldots, \qquad x_1^{a_1} x_2^{a_2} \cdots x_{r-1}^{a_{r-1}} x_r^{i_r},\; i_r = 0, 1, \ldots, a_r - 1. \end{array}$$

For each monomial m' in (3.4) there exists a standard pair (m, Z) which is a *cover* of m' in the sense that $m' \in m \cdot K[Z]$, or, equivalently, m divides m' and $supp(m'/m) \subseteq Z$.

In view of Lemma 3.3, it suffices to show that the covers (m, Z) of any two monomials in the list (3.4) are distinct. We proceed by contradiction and assume there exist two standard monomials m', m'' in (3.4) which have the same cover (m, Z).

Case 1: m' and m'' are from the same group, say $m' = x_1^{i_1} x_2^{a_2} \cdots x_r^{a_r}$ and $m'' = x_1^{j_1} x_2^{a_2} \cdots x_r^{a_r}$, $i_1 < j_1$. Then $\{x_1\} = supp(m''/m') \subseteq supp(m''/m) \subseteq Z$. Writing $U := \{x_2, \ldots, x_r\} \setminus Z$, this implies $m = \prod_{i \in U} x_i^{a_i}$. Therefore $m_1 \in m \cdot k[Z]$, which is a contradiction to the fact that $m_1 = x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r}$ lies in I.

Case 2: m' and m'' are from different groups, say $m' = x_1^{i_1} x_2^{a_2} \cdots x_r^{a_r}$ and $m'' = x_1^{a_1} x_2^{i_2} \cdots x_r^{a_r}$. Then

$$\begin{aligned} \{x_1, x_2\} &= supp(m''/gcd(m', m'')) \cup supp(m'/gcd(m', m'')) \\ &\subseteq supp(m''/m) \cup supp(m'/m) \subseteq Z. \end{aligned}$$

Writing $U := \{x_3, \ldots, x_r\} \setminus Z$, this implies $m = \prod_{i \in U} x_i^{a_i}$, and hence $m_1 \in m \cdot k[Z]$, in contradiction to $m_1 = x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r} \in I$. This completes the proof of Theorem 3.1. ∎

4. On the geometric degree

The geometric degree behaves quite differently from the arithmetic degree. For instance, while the arithmetic degree goes up under Gröbner basis computations (Theorem 2.3), it turns out that the geometric degree goes in the opposite direction.

Proposition 4.1. *Let I be a homogeneous ideal in $k[x_1, \ldots, x_n]$ and $in(I)$ its initial ideal with respect to any term order. Then*

$$geom\text{-}deg(I) \geq geom\text{-}deg(in(I)). \tag{4.1}$$

Proof: First suppose that I is pure d-dimensional, that is, each isolated prime of I has dimension d. Then $in(I)$ is pure d-dimensional as well (see e.g. [K-S]). Since the degree is preserved under taking initial ideals, and since degree and geometric degree coincide for pure ideals, we have $geom\text{-}deg(I) = deg(I) = deg(in(I)) = geom\text{-}deg(in(I))$.

Suppose now that I has dimension d but is *not* pure. Write $I = J \cap K$, where J is pure d-dimensional and each isolated prime of K has dimension $\leq d-1$ and is isolated in I as well. We have $deg(I) = deg(J)$ and $geom\text{-}deg(I) = deg(J) + geom\text{-}deg(K)$. If $in(I)$ is pure, then we are done since $geom\text{-}deg(in(I)) = deg(in(I)) = deg(I) \leq geom\text{-}deg(I)$. If $in(I)$ is *not* pure, then write $in(I) = J' \cap K'$ where J' is a pure d-dimensional monomial ideal and each isolated prime of K' has dimension $\leq d-1$ and is isolated in $in(I)$. By induction on the dimension we have $geom\text{-}deg(in(K)) \leq geom\text{-}deg(K)$, and therefore it suffices to show $geom\text{-}deg(K') \leq geom\text{-}deg(in(K))$. This follows from our construction because each isolated prime of K' is isolated and of the same multiplicity in $in(K)$. ∎

Example 4.2. Let I' be any ideal which has an isolated component of dimension less than $dim(I')$, and let I be obtained from I' by a generic linear change of coordinates. For the lexicographic term order (coordinate projection) we get the sharp inequality in (4.1):

$$\text{geom-deg}(in(I)) \;=\; deg(in(I)) \;=\; deg(I) \;<\; \text{geom-deg}(I). \tag{4.2}$$

The main goal of this section, however, is to prove the following Bezout-type result.

Theorem 4.3. *Let $I \subset J$ be homogeneous ideals in $S = k[x_1, \ldots, x_n]$ such that S_p/IS_p is Cohen-Macaulay for every isolated prime P of J. Let $f_1, \ldots, f_m$ be forms in J such that $J = I + (f_1, \ldots, f_m)$. Set $d_i := deg(f_i)$ and suppose that $d_1 \geq d_2 \geq \cdots \geq d_m$. Then*

$$geom\text{-}deg(J) \;\leq\; d_1 d_2 \cdots d_t \cdot geom\text{-}deg(I), \tag{4.3}$$

where $t := \max\{ht(P/I) \mid P \text{ is an isolated prime of } J\}$.

Remarks: (a) The inequality (4.3) fails to hold if the degrees of the generators are not sorted decreasingly: For instance, if $I = (0)$, $J = (x^2y, xy^2, x^m, y^m)$ then $deg(J) = 2m > deg(x^2y) \cdot deg(xy^2) = 9$, which leads to a violation of (4.3) for $m \geq 5$.
(b) Example 3.2 shows that Theorem 4.3 does not hold for the arithmetic degree.
(c) The case where S/I is a Cohen-Macaulay ring was already considered by P. Philipon [Ph, Prop. 3.3] and D. Brownawell [Br2]. They proved that $geom\text{-}deg(J) \leq d_1^t \cdot deg(I)$.

Before embarking on the proof of Theorem 4.3 we derive two lemmas.

Lemma 4.4. *Let I be a homogeneous ideal and f a form in S such that S_P/IS_P is Cohen–Macaulay for all isolated associated prime ideals P of (I, f). Then*

$$geom\text{-}deg(I, f) \;\leq\; deg(f) \cdot geom\text{-}deg(I).$$

Proof: Let I_1 resp. I_2 denote the intersection of all isolated primary components of I whose associated prime ideals contain resp. do not contain f. Then

$$geom\text{-}deg(I) \;=\; geom\text{-}deg(I_1) + geom\text{-}deg(I_2).$$

Similarly, let J_1 and J_2 be the intersection of all isolated primary components of (I,f) whose associated prime ideals are resp. are not associated isolated prime ideals of I. Then

$$geom\text{-}deg(I,f) \;=\; geom\text{-}deg(J_1) + geom\text{-}deg(J_2).$$

It suffices to prove the two inequalities $geom\text{-}deg(J_\nu) \le deg(f)\cdot geom\text{-}deg(I_\nu)$ for $\nu = 1,2$. The set of isolated prime ideals P of I which contain f is exactly the set of prime ideals P which are isolated prime ideals of both I and (I,f). Hence

$$\begin{aligned} geom\text{-}deg(I_1) &= \sum \ell(S_P/IS_P)\cdot deg(P) \\ &\geq \sum \ell(S_P/(I,f)S_P)\cdot deg(P) \;=\; geom\text{-}deg(J_1). \end{aligned}$$

Let U_i denote the intersection of all i–dimensional isolated primary components of I_2 ($U_i = S$ in the absence of such components). Then $geom\text{-}deg(I_2) = \sum_i deg(U_i)$. Since U_i is an unmixed ideal and f is relatively prime to U_i, all isolated primes of (U_i,f) have the same dimension and $geom\text{-}deg(U_i,f) = deg(U_i,f) = deg(f)\cdot deg(U_i)$. Therefore,

$$\sum_i geom\text{-}deg(U_i,f) \;=\; deg(f)\cdot geom\text{-}deg(I_2).$$

The proof will be completed if we can show that

$$geom\text{-}deg(J_2) \;\le\; \sum_i deg(U_i,f).$$

For this we only need to show that every isolated primary component U of J_2 is also an isolated primary component of some ideal (U_i,f). Let P be the radical of U. By the definition of J_2, the prime P is not an isolated prime of I. Hence any isolated prime ideal of I contained in P does not contain f. Since S_P/IS_P is a Cohen–Macaulay ring, IS_P is an unmixed ideal. Hence all associated prime ideals of IS_P come from isolated prime ideals of I which have the same dimension $dim S/P + 1$ and which do not contain f. So we can conclude that $IS_P = U_iS_P$ for $i = dim S/P + 1$. Since $US_P = (I,f)S_P = (U_i,f)S_P$, the ideal U must be an isolated primary component of (U_i,f), as required. ■

The Cohen–Macaulay condition of Lemma 4.4 cannot be replaced by the condition that $f \notin P$ for any minimal prime P of I. For example, let $I = (x_1^2, x_1x_2)$ and $f = x_2$. Then $(I,f) = (x_1^2, x_2)$ and $geom\text{-}deg(I,f) = deg(I,f) = 2 > 1 = geom\text{-}deg(I)$.

For the proof of Theorem 4.3 we also need the following lemma which shows that J can be approximated by a sequence of extensions as in Lemma 4.4.

Lemma 4.5. *Let I and $J = (I, f_1, \ldots, f_m)$ and the integer t be as in Theorem 4.3. Suppose that k is an infinite field and that $f_1, \ldots, f_m$ define a minimal k-basis for J/I. Then there exist forms $g_1, \ldots, g_t$ in J with $deg(g_i) = d_i = deg(f_i)$ which satisfy:*

(i) The ring $S_P/(I, g_1, \ldots, g_{i-1})S_P$ is Cohen–Macaulay for any isolated prime ideal P of $(I, g_1, \ldots, g_i)$, $i = 1, \ldots, t$.

(ii) Every isolated prime ideal of J is also an isolated prime ideal of $(I, g_1, \ldots, g_t)$.

Proof: We may assume that $t > 0$. For any integer $d > 0$ let J_d denote the ideal generated by I and the homogeneous elements of degree d of J. Put $d = d_1$. Since any homogeneous minimal basis for J/I contains at least an element of degree d, J_d is not contained in the ideal $I + MJ$, where M denotes the irrelevant ideal $(x_1, \ldots, x_n)$. Moreover, since J_d has the same radical as J, any prime $Q \not\supseteq J$ of S does not contain J_d. Thus, $(I + MJ) \cap J_d$ and $Q \cap J_d$ are proper subideals of J_d. Since k is infinite, we can find a form $g_1 \in J_d$, $g_1 \notin I + MJ$, with $deg(g_1) = d$ such that $g_1 \notin Q$ for any associated prime $Q \not\supseteq J$ of I.

We will show that S_P/IS_P is Cohen–Macaulay for any isolated prime P of (I, g_1). If P is also an isolated prime of I, then $dim S_P/IS_P = 0$ and we are done. If P is not an isolated prime of I, we may assume that $P \not\supseteq J$. For, if $P \supseteq J$, then P must be an isolated prime of J because $(I, g_1) \subseteq J$, and in this case, S_P/IS_P is Cohen–Macaulay by the assumption of Theorem 4.3. If $P \not\supseteq J$, by the choice of g_1, the form g_1 does not belong to any associated prime of I contained in P. Hence g_1 is a non-zero-divisor of S_P/IS_P. Since $dim S_P/IS_P = 1$, S_P/IS_P is a Cohen–Macaulay ring.

For $t = 1$, we have just proved (i). To see (ii) let P now be an arbitrary isolated prime of J. If $ht(P/I) = 0$, then P is an isolated prime of I. Since $I \subset (I, g_1) \subseteq J$, P must be an isolated prime of (I, g_1). If $ht(P/I) = 1$, any isolated prime of I contained in P does not contain J. The form g_1 does not belong to any such prime ideal. Hence $ht(P'/I) = 1$ for any isolated prime P' of (I, g_1) contained in P. Now, if P is not an isolated prime of (I, g_1), there exists an isolated ideal P' of (I, g_1) properly contained in P. This would imply $ht(P/I) > ht(P'/I) = 1$, a contradiction.

If $t \geq 2$, we can show similarly as above that $S_P/(I, g_1)S_P$ is Cohen–Macaulay for any isolated prime P of J. It is easily seen that $J/(I, g_1)$ can be minimally generated by $m - 1$ homogeneous elements of degree $d_2 \geq \cdots \geq d_m$ and

$$t - 1 = \max\{ht(P/(I, g_1)) \mid P \text{ is an isolated prime of } J\}.$$

Therefore, using induction, we may assume that there exist forms $g_1, \ldots, g_s$ with $deg(g_i) = d_i$ which together with g_1 satisfy the conditions (i) and (ii). The proof of Lemma 4.5 is now complete. ∎

Proof of Theorem 4.3: Clearly, the geometric degree is preserved under base field extension. So we may replace k by a field of rational functions $k(u)$ in order to get an infinite base field. Now we can apply Lemma 4.5 to find homogeneous polynomials $g_1, \ldots, g_t$ in J satisfying the conditions (i) and (ii). Using (ii) of Lemma 4.5 we get

$$\textit{geom-deg}(J) \leq \textit{geom-deg}(I, g_1, \ldots, g_t).$$

By successively applying Lemma 4.4 we obtain from (i) of Lemma 4.5 our desired bound: $\textit{geom-deg}(I, g_1, \ldots, g_t) \leq d_1 \cdots d_t \cdot \textit{geom-deg}(I)$. ∎

We close with a variant of Theorem 4.3 for the case when I is a prime ideal. In that case we may choose g_1 with $deg(g_1) = \min\ deg(f_i)$. Applying Lemma 4.5 we then find the other elements $g_2, \ldots, g_t$. Notice the change in the ordering of the degrees of $f_1, f_2, \ldots, f_m$.

Corollary 4.6. *Let I be a homogeneous prime ideal and $J \supset I$ a homogeneous ideal in S such that S_P/IS_P is Cohen–Macaulay for any minimal prime P of J. Let $f_1, \ldots, f_m$ be forms in J such that $J = (I, f_1, \ldots, f_m)$. We set $d_i := deg(f_i)$ and suppose that $d_2 \geq d_3 \geq \cdots \geq d_m \geq d_1$. Let $t = \max\{ht(P/I) \mid P$ is a minimal prime ideal of $J\}$. Then*

$$\textit{geom-deg}(J) \quad \leq \quad d_1 d_2 \cdots d_t \cdot deg(I).$$

5. On the division problem and extensions of Bezout's Theorem.

In computational algebra there is great interest in the effective Nullstellensatz and the effective division problem. For the history and the general state of the art of these problems we refer to the literature, which includes [Br1], [C-G-H], [Kol], [Amo], [B-Y], [Sch]. Our contribution in this section is an application of the results on the geometric degree in §4.

Theorem 5.1. *Let $f, f_1, \ldots, f_m$ be polynomials in $S := k[x_1, \ldots, x_n]$ such that $f \in (f_1, \ldots, f_m)$ and the homogenized ideal $I := ({}^hf_1, \ldots, {}^hf_m)$ in $R := k[x_0, x_1, \ldots, x_n]$ has no embedded components containing x_0. Put $d_i = deg(f_i)$ and suppose $d_2 \geq \ldots \geq d_m \geq d_1$. Then there exist $g_1, \ldots, g_m \in S$ such that $f = g_1 f_1 + \ldots + g_m f_m$ with*

(i) $deg(g_i f_i) \leq \max\{d_1 + \ldots + d_{n+1},\ deg(f)\}$ if $dim(R/I) = 0$.

(ii) $deg(g_i f_i) \leq d_1 d_2 \cdots d_t - \textit{geom-deg}(\overline{I}) + deg(f)$ if $dim(R/I) > 0$, where t is the maximal height of an isolated prime of I and $\overline{I}$ is the intersection of all primary components of I whose associated primes do not contain x_0.

Here the homogenization of a polynomial f is defined in the usual way as

$${}^hf \quad := \quad x_0^{deg(f)} \cdot f(x_1/x_0, \ldots, x_n/x_0).$$

Part (i) of Theorem 5.1 is an easy consequence of a classical result of Macaulay, see e.g. [Sch, Proposition 1]. Part (ii) was proved by Schiffmann [Sch] in the special case where I is a complete intersection. Here is the more general situation.

Proof of part (ii): Suppose $dim(R/I) > 0$. We have $geom\text{-}deg(I) \leq d_1 d_2 \cdots d_t$ by Corollary 4.6. Let I_0 be the intersection of all isolated primary components of I which contain x_0. Note that $arith\text{-}deg(I_0) = geom\text{-}deg(I_0)$, and hence $x_0^{geom\text{-}deg(I_0)} \in I$ by Theorem 2.1. Then

$$geom\text{-}deg(I_0) \;=\; geom\text{-}deg(I) \;-\; geom\text{-}deg(\overline{I}).$$

Hence we have $x_0^{d_1 \cdots d_t - geom\text{-}deg(\overline{I})} \cdot {}^h f \in I$, and the conclusion follows. ∎

For the Bezout version of the Nullstellensatz we obtain the following corollary.

Corollary 5.2. *Let $f_1, \ldots, f_m$ be polynomials in S which have no common zeros. Set $d_i = deg(f_i)$ and suppose $d_2 \geq \ldots \geq d_m \geq d_1 \geq 2$. If the homogenized ideal $I := ({}^h f_1, \ldots, {}^h f_m)$ has no embedded primes which contain x_0, then there exist $g_1, \ldots, g_m$ in S such that*

$$1 \;=\; g_1 f_1 + \cdots + g_m f_m$$

with $deg(g_i f_i) \leq d_1 \cdots d_\mu$, where $\mu = \min\{m, n\}$.

Proof: Let t be the maximal height of any isolated prime ideal of I. If $t = n + 1$, then $dim(R/I) = 0$, and we can apply Theorem 5.1 (i) in order to obtain $deg(g_i f_i) \leq d_1 \cdots d_n$ (cf. [Sch]). Note that $m \geq n$ in this case. If $t < n + 1$, then $t \leq \min\{m, n\}$. In this case the conclusion follows from Theorem 5.1 (ii) with $f = 1$. ∎

Our second application concerns Bezout's theorem. The intersection theory developed in [Ful] and [Vo] provides the following refined version: Let $I_1, \ldots, I_r$ be equidimensional homogeneous ideals without embedded components in $S = k[x_1, \ldots, x_n]$ such that $S_P/I_i S_P$ is Cohen–Macaulay for all minimal primes P of $I_1 + \cdots + I_r$, and $i = 1, \ldots, r$. Then

$$geom\text{-}deg(I_1 + \cdots + I_r) \;\leq\; \prod_{i=1}^{r} deg(I_i). \tag{5.1}$$

Using Theorem 5.1 we will prove a variant without the equidimensionality hypothesis:

Theorem 5.3. *Let $I_1, \ldots, I_r$ be homogeneous ideals in S such that $S_P/I_i S_P$ is Cohen-Macaulay for all isolated primes P of $I_1 + \cdots + I_r$ and $i = 1, \ldots, r$. Then*

$$geom\text{-}deg(I_1 + \cdots + I_r) \;\leq\; \prod_{i=1}^{r} geom\text{-}deg(I_i).$$

Proof: We shall prove only the case $r = 2$ since the other cases are similar. Let I_2' be the image of I_2 under the substitutions $x_i \mapsto y_i$, where $y_1, \ldots, y_n$ are new indeterminates. We denote by I the extension ideal of $I_1 + I_2'$ in $R = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Setting $f_i = x_i - y_i$ for $i = 1, \ldots, n$, we have $R/(I, f_1, \ldots, f_n) \cong S/(I_1 + I_2)$, and hence

$$geom\text{-}deg(I_1 + I_2) = geom\text{-}deg(I, f_1, \ldots, f_n).$$

Lemma 3 of [A-H-V] implies that R_P/IR_P is Cohen–Macaulay for all minimal primes P of $(I, f_1, \ldots, f_n)$. Hence we can apply Theorem 5.1 to get:

$$geom\text{-}deg(I, f_1, \ldots, f_n) \leq geom\text{-}deg(I).$$

To complete the proof of Theorem 5.3, it suffices to show that the geometric degree is multiplicative with respect to taking joins of projective schemes:

$$geom\text{-}deg(I) = geom\text{-}deg(I_1) \cdot geom\text{-}deg(I_2) \tag{5.2}$$

Indeed, by [Vo, Lemma 2.3], every pair (Q_1, Q_2) of minimal primary components of I_1 and I_2 yields a minimal primary component $Q_1 + Q_2'$ of I in R with $deg(Q_1 + Q_2') = deg(Q_1) \cdot deg(Q_2)$, and, moreover, every minimal primary ideal of I arises in this way. ■

Corollary 5.4. *Let X, Y be locally Cohen-Macaulay subschemes of projective n-space such that $dim(X) + dim(Y) > n$. Then $geom\text{-}deg(X \cap Y) \leq geom\text{-}deg(X) \cdot geom\text{-}deg(Y)$.*

Theorem 5.3 and Corollary 5.4 are not true in general without the Cohen-Macaulay condition. Let X, Y be reduced and irreducible subvarieties of P^n meeting properly. If there is an irreducible component C of $X \cap Y$ such that $\mathcal{O}_{X,C}$ is not Cohen-Macaulay, then it follows from Bezout's Theorem (see e.g. [Vo]) that $deg(X) \cdot deg(Y) < deg(X \cap Y)$. In view of the results in [F-V], it would be nice to improve Theorem 5.3 as follows.

Problem 5.5. Let I, J be homogeneous ideals in S. We set $e := dim(I + J) + n - (dim\, I + dim\, J) \geq 0$. Assume that S/I and S/J are Cohen–Macaulay k–algebras and $I \cap J$ contains no linear forms. Is then $deg(I) \cdot deg(J) \geq geom\text{-}deg(I + J) + e$?

All inequalities for the geometric degree which we derive in this section are generally false for the arithmetic degree (cf. Example 3.2). One promising option in getting Bezout-type theorem for the arithmetic degree is to use Gröbner bases instead of ideal generators.

Corollary 5.6. *Let I be any homogeneous ideal in $S = k[x_1, \ldots, x_n]$, let $\{g_1, g_2, \ldots, g_s\}$ a Gröbner basis for I with respect to any term order, and let $e := dim(I) + s - n$. Then*

$$arith\text{-}deg(I) \leq deg(g_1) \cdot deg(g_2) \cdot \cdots \cdot deg(g_s) - e.$$

Proof: This was proved for monomial ideals in Theorem 3.1. Using Theorem 2.3, and $dim(I) = dim(in(I))$, $deg(g_i) = deg(in(g_i))$, it follows for arbitrary homogeneous ideals. ■

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