π_1 of Smooth Points of a Log Del Pezzo Surface is Finite : II

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Introduction

A normal projective surface S over $\mathbf{C}$ is called a log del Pezzo surface if S has at most quotient singularities and $-K_S$ is ample, where K_S denotes the canonical divisor of S.
In part **I** (cf. [2]) of this paper we set out to prove the following :

Main Theorem. *The fundamental group of the space of smooth points of a log del Pezzo surface is finite.*

In this part **II**, we will complete the proof of this result. We will use the notations and results from part **I** freely. Recall from part **I** that if $\widetilde{S}$ is a minimal resolution of singularities of S, then we can find a "minimal" (-1)-curve C on $\widetilde{S}$ (cf. Lemma 3.1 and Prop. 3.6 of part **I**). In §3, §4, §5 of part **I**, we reduced to consider the cases (II-3) and (II-4) there. As remarked in the Introduction of part **I**, it suffices to consider the case (II-4) (the "2-component case"), to complete the proof of our Main Theorem. This will be done in this part **II** of our paper. As in part **I**, our proof for the case (II-4) gives quite precise information about the configuration of $C \cup D$.
After the results of parts **I** and **II** of our paper were announced in a conference in Kinosaki, Japan, A. Fujiki, R. Kobayashi and S. Lu have found another proof of our Main Theorem using differential geometric methods (cf. [1]).

Their proof of the Main Theorem is short, but it does not seem to give as precise information about the singular locus of S as our proof.

§1. The proof of the Main Theorem in the case (II-4).

In this section, we consider the case(II-4) in Remark 3.11 of part **I**. We employ the notations there. So, the (-1)-curve C meets exactly a (-2)-curve D_1 and a $(-n)$-curve D_2 with $n \geq 3$. Let Δ_i be the connected component of D containing D_i.

Our aim is to prove the following Theorem 1.1, which will imply the Main Theorem in the case (II-4).

Theorem 1.1. *Suppose that the case (II-4) in Remark 3.11 occurs. Then one of the following four cases occurs :*

(1) *Δ_i is a linear chain with D_i as a tip for $i = 1$ or 2. Hence $\pi_1(S^o)$ is finite (cf. Lemma 1.2 below).*

(2) *There are irreducible components $A_i (i = 1, \cdots, a), B_j (j = 1, \cdots, b)$ of D, there is a (-1)-curve E on $\widetilde{S}$ and there is a $\mathbf{P}^1$-fibration $\varphi : \widetilde{S} \to \mathbf{P}^1$ such that*

(2-1) *a singular fiber of φ has support equal to Supp $E + \sum_i A_i$,*

(2-2) *every irreducible component of $D - \sum_j B_j$ is contained in a singular fiber of φ, and*

(2-3) *$S_1 . \sum_j B_j \leq 2$ for a general fiber S_1 of φ.*

In particular, there is a $\mathbf{C}^*$-fibration on S^o and hence $\pi_1(S^o)$ is finite (cf. Lemma 2.2 of part **I***).*

(3) *There is a (-1)-curve E, there are two connected components $\Theta_i (i = 1, 2)$ of D, there is an irreducible component B_i in Θ_i and there is a twig $E + \widetilde{T}_i$ in $E + \Theta_i$ such that*

(3-1) *$E.D = E.(\Theta_1 + \Theta_2) = E.(B_1 + B_2) = 2$, $E.B_i = 1$ for $i = 1$ and 2. Hence $\widetilde{T}_i = 0$ if B_i is not a tip and a twig of Θ_i containing B_i otherwise, and*

(3-2) *For $i = 1$ and $j = 2$ or $i = 2$ and $j = 1$, $E + \widetilde{T}_i + \Theta_j$ has a positive eigenvalue and hence $\kappa(\widetilde{S}, E + \widetilde{T}_i + \Theta_j) = 2$.*

In particular, $\pi_1(S^o)$ is finite (cf. Lemma 1.12).

(4) *There is a $\mathbf{P}^1$-fibration $\varphi : \widetilde{S} \to \mathbf{P}^1$ such that $C + D$ and all singular fibers of φ are given in one of the Figures 1, 2, 3 or 4 (cf. end of the paper).*

Hence $\pi_1(S^o)$ is finite. (cf. Lemma 1.13).

(5) *One of two cases in Lemma 1.11 occurs. Hence $\pi_1(S^o)$ is finite. (cf. lemma 1.14).*

Theorem 1.1 is a consequence of the following Lemmas 1.2, 1.3, 1.8, Theorems 1.9 and 1.10, and Lemmas 1.11, 1.12, 1.13, 1.14.

Lemma 1.2. *Suppose that Δ_i is a linear chain with D_i as a tip for $i = 1$, or 2. Then $\pi_1(S^o)$ is finite.*

Proof. Suppose Δ_1 is a linear chain with D_1 as a tip. As $rank S = 1, C + \Delta_1 + \Delta_2$ supports a divisor with strictly positive self-intersection. By Lemma 1.10 of part **I**, we have a surjection $\pi_1(U - \Delta_1 - \Delta_2) \to \pi_1(\widetilde{S} - D)$, where U is a small tubular neighborhood of $C \cup \Delta_1 \cup \Delta_2$. We can write $U = U_1 \cup U_2$, where U_i is a small neighborhood of $C \cup \Delta_i$. It is easy to see that $U_i - \Delta_1 - \Delta_2$ contains a small neighborhood N_i of Δ_i as a strong deformation retract for $i = 1, 2$. By assumption, $\pi_1(N_i - \Delta_i)$ is finite for $i = 1, 2$ and by Mumford's presentation (cf. [3]), $\pi_1(N_1 - \Delta_1)$ is a cyclic group generated by "the" loop γ_1 in $C - \Delta_1 - \Delta_2$ around the point $C \cap \Delta_1$. Now an easy application of Van-Kampen's theorem for the covering $U_1 - \Delta_1 - \Delta_2$ and $U_2 - \Delta_1 - \Delta_2$ of $U - \Delta_1 - \Delta_2$ shows that $\pi_1(U - \Delta_1 - \Delta_2)$ is finite and hence so is $\pi_1(\widetilde{S} - D)$.

Lemma 1.3. *Suppose that Δ_1 contains $G_i (i = 1, \cdots, s; s \geq 3)$ such that $G_i^2 = -2, G_1 = D_1, G_j.G_{j+1} = G_{s-2}.G_s = 1 (j = 1, \cdots, s-2)$ (This is the case if Δ_1 consists of only (-2)-curves but D_1 is not a tip of Δ_1). Then Theorem 1.1, (2) or (3) is true with $E = C$.*

Proof. Let $S_0 = 2(C + G_1 + \cdots + G_{s-2}) + G_{s-1} + G_s$ and let $\varphi : \widetilde{S} \to \mathbf{P}^1$ be the $\mathbf{P}^1$-fibration with S_0 as a singular fiber. If $\Delta_1 = \sum_i G_i$, then Theorem 1.1,(2) is true with $E = C, \sum_i A_i = \sum G_i, \sum_i B_i = B_1 = D_2$. Otherwise, Theorem 1.1, (3) is true with $E = C, \Theta_i = \Delta_i, \widetilde{T}_2 = 0$; indeed, $C + \Delta_1$ has a positive eigenvalue.

From now on till the end of the section, we shall assume the following hypothesis :

($*$) *neither the case of Lemma 1.2 nor the case of Lemma 1.3 occurs.*

Let $C + T_i$ $(i = 1, 2)$ be the maximal twig of $C + \Delta_i$. Hence $T_i = 0$ if D_i is not a tip of Δ_i and T_i is the maximal twig of Δ_i containing D_i otherwise. By the maximality of $C + T_i$ and by the hypothesis$(*)$, there are irreducible components H_i, H_{i1}, H_{i2} in $\Delta_i - T_i$ such that

$$T_i.(\Delta_i - T_i) = T_i.H_i = 1, H_i.H_{i1} = H_i.H_{i2} = 1.$$

Let $\sigma : \widetilde{S} \to \widetilde{T}$ be the smooth blowing-down of curves in $C + T_1 + T_2$ such that

(1) *$\sigma(C + \Delta_1 + \Delta_2)$ consists of exactly one (-1)-curve $\widetilde{C}$, with $\widetilde{C} \leq \sigma(C + T_1 + T_2)$, and several $(-n_i)$-curves with $n_i \geq 2$, and*

(2) *the condition(1) will not be satisfied if σ is replaced by the composite of σ and the blowing-down of $\widetilde{C}$.*

Thus, $\sigma = id$ if and only if D_1 is not a tip of Δ_1. If $\sigma \neq id$, then C is contracted by σ and $\sigma'(\widetilde{C}) \leq D$.

Let $\widetilde{D} = D$ (resp. $\widetilde{\Delta}_i := \sigma(\Delta_i)$) if $\sigma = id$, and $\widetilde{D} = \sigma(D) - \widetilde{C}$ (resp. $\widetilde{\Delta}_i := \sigma(\Delta_i) - \widetilde{C}$) otherwise. Let $\widetilde{H}_i = \sigma(H_i), \widetilde{H}_{ij} = \sigma(H_{ij})$, etc. By the definition of σ there is an irreducible component J_i in $T_i + H_i$ such that

$$\widetilde{C}.\widetilde{D} = \widetilde{C}.(\tilde{J}_1 + \tilde{J}_2) = 2, \quad \widetilde{C}.\tilde{J}_i = 1.$$

$\widetilde{T}$ is contractible to quotient singularities with, say $g : \widetilde{T} \to T$ the contraction morphism, and T is again a log del Pezzo surface of rank one with g as a minimal desingularization (cf. [4, Lemma 4.3]). So, Lemma 1.1 of part **I** is true for T. In particular, we have

$$g^*K_T \equiv K_{\widetilde{T}} + \widetilde{D}^*, \ -E.(K_{\widetilde{T}} + \widetilde{D}^*) > 0$$

for every (-1)-curve E on $\widetilde{T}$. Here M^* is an effective $\mathbf{Q}$-divisor with support contained in M.

Suppose that there are two smooth blowing-downs $\sigma_1 : \widetilde{S} \to \widetilde{S}_1$, $\sigma_2 : \widetilde{S}_1 \to \widetilde{T}$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Let E be the unique (-1)-curve in $\sigma_1(C + \Delta_1 + \Delta_2)$. Let $M := D$ if $\sigma_1 = id$ and $M := \sigma_1(D) - E$ otherwise. The same result [4, Lemma 4.3] implies the following :

Lemma 1.4. *M is contractible to quotient singularities with, say $f_1 : \widetilde{S}_1 \to S_1$ the contraction morphism, and S_1 is again a log del Pezzo surface of rank one with f_1 as a minimal desingularization. In particular, we have*

$$f_1^* K_{S_1} \equiv K_{\widetilde{S}_1} + M^*, -E.(K_{\widetilde{S}_1} + M^*) > 0,$$

where M^* is an effective $\mathbf{Q}$-divisor with support contained in M.

Suppose that for $a = 1$ or 2, we have $J_a = H_a$ and $\widetilde{H}_a^2 = -2$. Let $\widetilde{G} \sim K_{\widetilde{T}} + 2(\widetilde{C} + \widetilde{H}_a) + \widetilde{H}_{a1} + \widetilde{H}_{a2} + \widetilde{J}_b$ where $\{a, b\} = \{1, 2\}$ as sets. Note that $H^2(\widetilde{T}, \widetilde{G}) \cong H^0(\widetilde{T}, -(2(\widetilde{C} + \widetilde{H}_a) + \widetilde{H}_{a1} + \widetilde{H}_{a2} + \widetilde{J}_b)) = 0$. Note also that $\widetilde{G}.B = 0$ for $B = \widetilde{C}, \widetilde{H}_a, \widetilde{H}_{a1}, \widetilde{H}_{a2}, \widetilde{J}_b$. Hence $\widetilde{G}^2 = \widetilde{G}.K_{\widetilde{T}}$. Now the Riemann-Roch theorem implies that

$$h^0(\widetilde{T}, \widetilde{G}) \geq \frac{1}{2}\widetilde{G}.(\widetilde{G} - K_{\widetilde{T}}) + 1 = 1.$$

We may assume that $\widetilde{G} \geq 0$.

Lemma 1.5. *Assume the above conditions and notations. We have :*

(1) $\widetilde{G}$ is a nonzero effective divisor.

(2) $\widetilde{G} \cap (\widetilde{C} + \widetilde{H}_a + \widetilde{H}_{a1} + \widetilde{H}_{a2} + \widetilde{J}_b) = \phi$. In particular, $\widetilde{G}_1.\widetilde{G} = \widetilde{G}_1.K_{\widetilde{T}}$ for every $\widetilde{G}_1 \leq \widetilde{G}$.

(3) We can decompose $\widetilde{G}$ into $\widetilde{G} = \widetilde{\Sigma} + \widetilde{\Delta}$ such that Supp $\widetilde{\Delta}$ is contained in Supp $\widetilde{D}$ and $\widetilde{\Sigma} = \sum_{i=1}^r \widetilde{\Sigma}_i$ ($r \geq 1$) where $\widetilde{\Sigma}_i$ is a (-1)-curve.

(4) Write $\sigma^*\widetilde{G} \sim \sigma^*(K_{\widetilde{T}} + 2(\widetilde{C} + \widetilde{H}_a) + \widetilde{H}_{a1} + \widetilde{H}_{a2} + \widetilde{J}_b) = K_{\widetilde{S}} + sC +$ (an effective divisor with support in D). Then $r \leq s - 1$.

(5) Let $B \leq \widetilde{D} - (\widetilde{H}_a + \widetilde{H}_{a1} + \widetilde{H}_{a2} + \widetilde{J}_b)$. Then $B.\widetilde{G} > 0$ if and only if $B^2 \leq -3$ or $B.(\widetilde{H}_{a1} + \widetilde{H}_{a2} + \widetilde{J}_b) > 0$.

(6) If $\widetilde{\Sigma}$ is a reduced divisor, then $\widetilde{G} = \widetilde{\Sigma}$ and $\widetilde{\Sigma}$ is a disjoint union of $\widetilde{\Sigma}_i$'s.

Proof. From the defnition of $\widetilde{G}$, one can calculate that :

Claim(1). $\widetilde{G}.B = 0$ if B is one of $\widetilde{C}, \widetilde{H}_a, \widetilde{H}_{a1}, \widetilde{H}_{a2}$ and $\widetilde{J}_b$. Moreover, $\widetilde{G}.B \geq 0$ for every irreducible component B of $\widetilde{D}$.

Claim(2). $|K_{\widetilde{T}} + \widetilde{C} + \widetilde{D}| = \phi$.

This follows from that $|K_{\widetilde{S}} + C + D| = \phi$ and the definition of σ.

(1) By the hypothesis(*) after Lemma 1.3, $\widetilde{J}_b$ meets an irreducible component B of $\widetilde{\Delta}_b$. So, $\widetilde{G}.B = (K_{\widetilde{T}} + \widetilde{J}_b).B \geq 1$. Hence $\widetilde{G} > 0$.

(2) Suppose $\widetilde{G} \cap \widetilde{C} \neq \phi$. Then $\widetilde{C} \leq \widetilde{G}$ by Claim(1). Now, $\widetilde{H}_a \leq \widetilde{G} - \widetilde{C}$ because $\widetilde{H}_a.(\widetilde{G} - \widetilde{C}) = -\widetilde{H}_a.\widetilde{C} = -1 < 0$. This leads to $0 \leq \widetilde{G} - \widetilde{C} - \widetilde{H}_a \in |K_{\widetilde{T}} + \widetilde{C} + \widetilde{H}_a + \widetilde{H}_{a1} + \widetilde{H}_{a2} + \widetilde{J}_b| \subseteq |K_{\widetilde{T}} + \widetilde{C} + \widetilde{D}|$, a contradiction to Claim(2). So, $\widetilde{G} \cap \widetilde{C} = \phi$. One applies this argument and can prove (2).

(3) Decompose $\widetilde{G}$ into $\widetilde{G} = \widetilde{\Sigma} + \widetilde{\Delta}$ where Supp $\widetilde{\Delta} \subseteq$ Supp $\widetilde{D}$ and $\widetilde{\Sigma}$ contains no irreducible components of $\widetilde{D}$. First, by Claim(1), we have $\widetilde{G}.\widetilde{\Delta}_i \geq 0$ for every $\widetilde{\Delta}_i \leq \widetilde{\Delta}$. Hence $0 \leq \widetilde{G}.\widetilde{\Delta} = \widetilde{\Sigma}.\widetilde{\Delta} + \widetilde{\Delta}^2 < \widetilde{\Sigma}.\widetilde{\Delta}$ because Supp $\widetilde{\Delta} \subseteq$ Supp $\widetilde{D}$ and $\widetilde{D}$ is negative definite. This proves that $\widetilde{\Sigma} \neq 0$.

Let $\widetilde{\Sigma}_i$ be an irreducible component of $\widetilde{\Sigma}$. Note that $\widetilde{\Sigma}_i.K_{\widetilde{T}} \leq \widetilde{\Sigma}_i.(K_{\widetilde{T}} + \widetilde{D}^*) < 0$ (cf. Lemma 1.1). So, if $\widetilde{\Sigma}_i^2 < 0$, then $\widetilde{\Sigma}_i$ is a (-1)-curve. Suppose that $\widetilde{\Sigma}_i^2 \geq 0$. Then, by (1), $\widetilde{\Sigma}_i^2 \leq \widetilde{\Sigma}_i.\widetilde{G} = \widetilde{\Sigma}_i.K_{\widetilde{T}} < 0$. We reach a contradiction. This proves (3).

(4) By Claim(2), $\sigma^*\widetilde{\Sigma}_i$ is again a (-1)-curve and $\sigma^*(\widetilde{\Delta}) \subseteq D$. Write $f(C) \equiv c(-K_S)$, $f(\sigma^*\widetilde{\Sigma}_i) \equiv e_i(-K_S)$, where $c > 0, e_i > 0$. Then $(sc - 1)(-K_S) \equiv f(\sigma^*\widetilde{G}) \equiv \sum_{i=1}^r e_i(-K_S)$. Since $K_S^2 > 0$, we have

$$sc - 1 = \sum_i e_i \geq rc$$

by the minimality of $-C.(K_{\widetilde{S}} + D^*) = c(K_{\widetilde{S}} + D^*)^2 = c(K_S)^2$. Hence $(s-r)c \geq 1 > 0$. (4) then follows.

(5) follows from the following calculation : $B.\widetilde{G} = B.(K_{\widetilde{T}} + \widetilde{H}_{a1} + \widetilde{H}_{a2} + \widetilde{J}_b)$.

(6) By the condition, $\widetilde{\Sigma}_i \neq \widetilde{\Sigma}_j$ if $i \neq j$. So,

$$-1 = \widetilde{\Sigma}_i^2 = \widetilde{\Sigma}_i.\widetilde{G} - \widetilde{\Sigma}_i.(\widetilde{\Delta} + \sum_{j \neq i} \widetilde{\Sigma}_j) \leq \widetilde{\Sigma}_i.\widetilde{G} = \widetilde{\Sigma}.K_{\widetilde{T}} = -1.$$

Thus, $\widetilde{\Sigma}_i.(\widetilde{\Delta} + \sum_{j \neq i} \widetilde{\Sigma}_j) = 0$ for every i. So, $\widetilde{\Sigma}$ is a disjoint union of $\widetilde{\Sigma}_i$'s and $\widetilde{\Sigma} \cap \widetilde{\Delta} = \phi$. In particular, $\widetilde{G}.\widetilde{\Delta} = \widetilde{\Delta}^2$. By Claim(1), we have $\widetilde{G}.\widetilde{\Delta} \geq 0$. So, $\widetilde{\Delta}^2 \geq 0$. Since $\widetilde{\Delta}$ is contained in $\widetilde{D}$ and $\widetilde{D}$ is negative definite, we have $\widetilde{\Delta} = 0$. This proves (6).

Lemma 1.5 is proved.

Corollary 1.6. *Assume that σ is the contraction of curves in $C + T_1$. Assume further that $J_1 = H_1$ and $\widetilde{H}_1^2 = -2$ (hence $J_2 = D_2$ and the hypothesis* *in Lemma 1.5 is satisfied with $a = 1$). Then $K_{\widetilde{T}} + 2(\widetilde{C} + \widetilde{H}_1) + \widetilde{H}_{11} + \widetilde{H}_{12} + \widetilde{J}_2 \sim \widetilde{G} = \widetilde{\Sigma} = \widetilde{\Sigma}_1$, i.e., $\widetilde{G}$ is reduced and a (-1)-curve.*

Proof. We apply Lemma 1.5 to $\widetilde{G} \sim K_{\widetilde{T}} + 2(\widetilde{C} + \widetilde{H}_1) + \widetilde{H}_{11} + \widetilde{H}_{12} + \widetilde{J}_2$. By the hypothesis, $\sigma^*\widetilde{G} \sim K_{\widetilde{S}} + 2C+$ (an effective divisor with support in D). Then Corollary 1.5 follows from Lemma 1.5.

Lemma 1.7. *Assume that σ is the contraction of curves in $C + T_2$. Assume further that $J_2 = H_2$ and $\widetilde{H}_2^2 = -2$ (hence $J_2 = D_2$ and the hypothesis in Lemma 1.5 is satisfied with $a = 2$). Then $\sigma^*\widetilde{G} \sim \sigma^*(K_{\widetilde{T}} + 2(\widetilde{C} + \widetilde{H}_2) + \widetilde{H}_{21} + \widetilde{H}_{22} + \widetilde{J}_1) = K_{\widetilde{S}} + sC+$ (an effective divisor with support in D) with $s = -D_2^2$.*

Proof. The result follows from the hypothesis on σ.

Lemma 1.8. *Suppose the case (II-4) in Remark 3.11 occurs. Then one of the following two cases occurs :*

(1) Theorem 1.1,(2) or (3) is true with $E = C$.

(2) $(\widetilde{J}_a^2, \widetilde{J}_b^2) = (-2,-2), (-2,-3)$ or $(-2,-4)$ where $\{a, b\} = \{1, 2\}$ as sets. If $\widetilde{J}_k^2 = -2$ (this is the case if $k = a$), then $J_k = H_k$ and $H_{kj}^2 \leq -3$ for $j = 1$ or 2.

Proof. By [4, Lemma 4.4], $\widetilde{J}_a^2 = -2$ for $a = 1$ or 2. Let $\{a, b\} = \{1, 2\}$ as sets.

Case(1) $\widetilde{J}_b^2 = -2$. If $\widetilde{J}_s$ is a tip of $\widetilde{\Delta}_s$, say $s = b$, i.e., $J_b \neq H_b$, then Theorem 1.1,(3) occurs with $E = C$. Indeed, $\widetilde{C} + \widetilde{J}_b + \widetilde{\Delta}_a$ has a positive eigenvalue and so does $C + T_b + \Delta_a$. Thus, may assume $J_a = H_a, J_b = H_b$.

Suppose $H_{s1}^2 = H_{s2}^2 = -2$ for $s = a$ or b, say $s = a$. Let $S_0 := 2(\widetilde{C} + \widetilde{H}_a) + \widetilde{H}_{a1} + \widetilde{H}_{a2}$ and let $\psi : \widetilde{S} \to \mathbf{P}^1$ be the $\mathbf{P}^1$-fibration with S_0 as a singular fiber. If $\widetilde{\Delta}_a = \widetilde{H}_a + \widetilde{H}_{a1} + \widetilde{H}_{a2}$, then Theorem 1.1,(2) is true with $E = C, \varphi = \psi \cdot \sigma, \sum_i B_i = B_1 = H_b$. If $\widetilde{\Delta}_a > \widetilde{H}_a + \widetilde{H}_{a1} + \widetilde{H}_{a2}$, Theorem 1.1,(3) is true with $E = C$. Indeed, $\widetilde{C} + \widetilde{\Delta}_a$ then has a positive eigenvalue and so does $C + T_b + \Delta_a$. Thus, may assume that $H_{aj}^2 \leq -3$ for $j = 1$ or 2. The same argument works for $s = b$. So, Lemma 1.8 is true in this case.

Case(2) $\widetilde{J}_b^2 \leq -3$. Then by the definition of σ (cf. the second condition), $J_a = H_a$, i.e., $\widetilde{J}_a$ is not a tip of $\widetilde{\Delta}_a$. If $H_{a1}^2 = H_{a2}^2 = -2$, then by the arguments in the above paragraph, Theorem 1.1,(2) or (3) is true with $E = C$. So, may assume that $H_{aj}^2 \leq -3$ for $j = 1$ or 2, say $j = 1$.

To finish the proof, it remains to prove that $d := -\widetilde{J}_b^2 \leq 4$. Since it is impossible that $\widetilde{\Delta}_b$ is a linear chain with $\widetilde{J}_b$ as a tip (cf. the hypothesis(*) after Lemma 1.3), we have $\widetilde{D}^* \geq (d-2)/(d-1)\widetilde{J}_b + 3/7\widetilde{H}_{a1} + 2/7\widetilde{H}_a + 1/7\widetilde{H}_{a1}$. So, $0 < -\widetilde{C}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - \widetilde{C}.((d-2)/(d-1)\widetilde{J}_b + 2/7\widetilde{H}_a) = 1/(d-1) - 2/7$. Hence $d \leq 4$. This proves Lemma 1.8.

Theorem 1.9. *Suppose the case(2) in Lemma 1.8 occurs. Then it is impossible that* $\widetilde{J}_1^2 = \widetilde{J}_2^2 = -2$.

Proof. We consider the case where $\widetilde{J}_1^2 = \widetilde{J}_2^2 = -2$. By the hypothesis, we have $J_i = H_i, \widetilde{H}_i^2 = -2$ for $i = 1, 2$ and may assume that $H_{11}^2 \leq -3, H_{21}^2 \leq -3$.

Case(1) σ is the contraction of curves contained in $C + T_1$.

Then the conditions of Corollary 7.6 are satisfied. Hence $K_{\widetilde{T}} + 2(\widetilde{C} + \widetilde{H}_1) + \widetilde{H}_{11} + \widetilde{H}_{12} + \widetilde{H}_2 \sim \widetilde{G} = \widetilde{\Sigma}$ where $\widetilde{\Sigma}$ is a (-1)-curve. Note that $\widetilde{\Sigma}.\widetilde{H}_{21} = \widetilde{G}.\widetilde{H}_{21} = (K_{\widetilde{T}} + \widetilde{H}_2).\widetilde{H}_{21} \geq 1 + 1$ (cf. Lemma 1.5, (2)). Let $\Sigma := \sigma^*(\widetilde{\Sigma})$. Then Σ is again a (-1)-curve (cf. Lemma 1.5,(2)) with $\Sigma.H_{21} \geq 2$. On the other hand, $D^* \geq 1/2D_2 + 1/2H_{21}$ because $D_2^2 \leq -3, H_{21}^2 \leq -3$. This leads to $0 < -\Sigma.(K_{\widetilde{S}} + D^*) \leq 1 - \Sigma.1/2H_{21} \leq 0$, a contradicion. So, the case(1) is impossible.

Case(2) σ contracts at least one irreducible component of the maximal twig T_2 of Δ_2.

By noting that $D_1^2 = -2, D_2^2 \leq -3$, there are two smooth blowing-downs $\sigma_1 : \widetilde{S} \to \widetilde{S}_1, \sigma_2 : \widetilde{S}_1 \to \widetilde{T}$ such that $\sigma = \sigma_2 \cdot \sigma_1$ and that :

(1) $\sigma_1(T_1 + C + T_2) = T_1' + E + T_2'$ where E is a (-1)-curve and $T_i' \leq \sigma_1(T_i)$,

(2) $T_1' + \sigma_1(H_1) = \sum_{i=1}^{s} L_i, E.L_1 = L_i.L_{i+1} = 1 (i = 1, \cdots, s-1; s \geq 2), L_s = \sigma_1(H_1), L_1^2 = -2, L_2^2 = -(t+1), L_j^2 = -2(j > 2, j \neq s)$, and

(3) $T_2' + \sigma_1(H_2) = \sum_{i=1}^{t} M_i, E.M_1 = M_i.M_{i+1} = 1 (i = 1, \cdots, t-1; t \geq 2), M_t = \sigma_1(H_2), M_1^2 = -3, M_t^2 = -s, M_j^2 = -2(j \geq 2, j \neq t)$.

Now applying Lemma 1.4, we get $-E.(K_{\widetilde{S}_1} + M^*) > 0$. Since $\sigma_1(\Delta_1 + \Delta_2) - E$ can be contractible to quotient singularities (cf. Lemma 1.4), we have $(s, t) = (2, 2), (2, 3), (3, 2)$.

Case(2-1) $(s, t) = (2, 2)$. Then $M^* \geq 2/5L_1 + 4/5\sigma_1(H_1) + 3/5\sigma_1(H_{11}) + 2/5\sigma_1(H_{12}) + 3/5M_1 + 4/5\sigma_1(H_2) + 3/5\sigma_1(H_{21}) + 2/5\sigma_1(H_{22})$. This leads to

$0 < -E.(K_{\widetilde{S}_1} + M^*) \leq 1 - E.(2/5L_1 + 3/5M_1) = 0$, a contradiction. So, the case(2-1) does not occur.

Case(2-2) $(s,t) = (2,3)$. Then $M^* \geq 7/16L_1 + 14/16\sigma_1(H_1) + 10/16\sigma_1(H_{11}) + 7/16\sigma_1(H_{12}) + 10/17M_1 + 13/17M_2 + 16/17\sigma_1(H_2) + 11/17\sigma_1(H_{21}) + 8/17\sigma_1(H_{22})$. This leads to $0 < -E.(K_{\widetilde{S}_1} + M^*) \leq 1 - E.(7/16L_1 + 10/17M_1) = 1 - 7/16 - 10/17 < 0$, a contradiction. So, the case(2-2) does not occur.

Case(2-3) $(s,t) = (3,2)$. Then $B^* \geq 9/23L_1 + 18/23L_2 + 22/23\sigma_1(H_1) + 15/23\sigma_1(H_{11}) + 11/23\sigma_1(H_{12}) + 7/11M_1 + 10/11\sigma_1(H_2) + 7/11\sigma_1(H_{21}) + 5/11H_{22}$. This leads to $0 < -E.(K_{\widetilde{S}_1} + B^*) \leq 1 - E.(9/23L_1 + 7/11M_1) = 1 - 9/23 - 7/11 < 0$, a contradiction. So, the case(2-3) does not occur.

This proves Theorem 1.9.

Theorem 1.10. *Suppose that the case in Corollary 1.6 occurs. Suppose further that the case(2) in Lemma 1.8 occurs with* $(\widetilde{J}_a^2, \widetilde{J}_b^2) = (-2,-3)$ *or* $(-2,-4)$ *(hence* $a = 1, b = 2, J_1 = H_1, J_2 = D_2$*). Then either Theorem 1.1 (3) is true with* $E = C$*, or Theorem 1.1,(4) is true.*

Proof. By the hypothesis, may assume that $\widetilde{H}_{11}^2 \leq -3$. By Corollary 1.6, $K_{\widetilde{T}} + 2(\widetilde{C} + \widetilde{H}_1) + \widetilde{H}_{11} + \widetilde{H}_{12} + \widetilde{J}_2 \sim \widetilde{G} = \widetilde{\Sigma}$ where $\widetilde{\Sigma}$ is a (-1)-curve.

Claim(1). (1) $\widetilde{D}^* \geq 3/7\widetilde{H}_{11} + 2/7\widetilde{H}_1 + 1/7\widetilde{H}_{12} + (a-2)/(a-1)\widetilde{J}_2$. Here $a := -\widetilde{J}_2^2 \geq 3$ and hence $(a-2)/(a-1) \geq 1/2$.

(2) $\widetilde{\Delta}_1$ is a linear chain.

(3) Either $\widetilde{\Delta}_2$ is a linear chain or $\widetilde{\Delta}_2$ is a fork with $\widetilde{J}_2$ as a tip.

(4) $\widetilde{\Delta}_1 - \widetilde{H}_{11}$ consists of (-2)-curves.

(5) $\widetilde{\Delta}_2 - \widetilde{J}_2$ consists of (-2)-curves.

Since $\widetilde{H}_{11}^2 \leq -3$ and since it is imposible that $\widetilde{\Delta}_2$ is a linear chain with $\widetilde{J}_2$ as a tip (cf. the hypothesis($*$) after Lemma 1.3), (1) follows.

If $\widetilde{\Delta}_1$ is not a linear chain, then also $\widetilde{D}^* \geq 1/2\widetilde{H}_1 + 1/2\widetilde{H}_{11}$. This leads to $0 < -\widetilde{C}.(K_{\widetilde{S}} + \widetilde{D}^*) \leq 1 - \widetilde{C}.(1/2\widetilde{H}_1 + 1/2\widetilde{J}_2) = 0$, a contradiction. So, (2) of Claim(1) is true.

Suppose (3) of Claim(1) is false, then $\widetilde{\Delta}_2$ contains $L_i (i = 1, \cdots, s; s \geq 4)$ such that $L_2 = \widetilde{J}_2, L_i.L_{i+1} = L_{s-2}.L_s = 1 (i = 1, \cdots, s-2)$. So, we have $\widetilde{D}^* \geq 1/3L_1 + 2/3\sum_{i=2}^{s-2} L_i + 1/3L_{s-1} + 1/3L_s$. On the other hand, for $i = 1,3$ (and also for $i = 4$ if $s = 4$), we have $L_i.\widetilde{\Sigma} = L_i.(K_{\widetilde{T}} + \widetilde{D}_2) \geq 1$ (cf. Lemma 1.5). This leads to $0 < -\widetilde{\Sigma}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - \widetilde{\Sigma}.(1/3L_1 + 2/3\sum_{i=2}^{s-2} L_i + 1/3L_{s-1} + 1/3L_s) \leq 0$. We reach a contradiction. Thus, (3) of Claim(1) is true.

Suppose $\tilde{\Delta}_1 - \widetilde{H}_{11}$ contains a $(-n)$-curve B with $n \geq 3$. If B and $\widetilde{H}_{12}$ are in the same connected component of $\tilde{\Delta}_1 - \widetilde{H}_1$, then $\widetilde{D}^* \geq 1/2\widetilde{H}_1 + 1/2\tilde{J}_2$ and hence $0 < -\tilde{C}.(K_{\tilde{T}} + \widetilde{D}^*) \leq 1 - \tilde{C}.(1/2\widetilde{H}_1 + 1/2\tilde{J}_2) = 0$, a contradiction. If B and $\widetilde{H}_{11}$ are in the same connected component of $\tilde{\Delta}_1 - \widetilde{H}_1$, we let $L_1 + \cdots + L_s$ be a linear chain in $\tilde{\Delta}_1$ such that $L_1 = \widetilde{H}_{11}, L_s = B, L_i.L_{i+1} = 1 (i = 1, \cdots, s-1)$. Then one has $D^* > 1/2\sum_i L_i$. Moreover, $L_i.\tilde{\Sigma} = L_i.(K_{\tilde{T}} + \widetilde{H}_{11}) \geq 1$ for $i = 2, s$ and $L_2.\tilde{\Sigma} \geq 2$ if $s = 2$. This leads to $0 < -\tilde{\Sigma}.(K_{\tilde{T}} + \widetilde{D}^*) \leq 1 - \tilde{\Sigma}.1/2\sum_i L_i \leq 0$, a contradiction. Therefore, (4) of Claim(1) is true.

Suppose that $\tilde{\Delta}_2 - \tilde{J}_2$ contains a $(-n)$-curve B with $n \geq 3$. Let $L_1 + \cdots + L_s$ be a linear chain contained in $\tilde{\Delta}_2$ such that $L_1 = \tilde{J}_2, L_s = B, L_i.L_{i+1} = 1 (i = 1, \cdots, s-1)$. Then we have $\widetilde{D}^* \geq 1/2\sum_i L_i$. Note that for $i = 2, s$, we have $L_i.\tilde{\Sigma} = L_i.(K_{\tilde{T}} + \tilde{J}_2) \geq 1$. Moreover, $L_2.\tilde{\Sigma} \geq 2$ if $s = 2$. This leads to $0 < -\tilde{\Sigma}.(K_{\tilde{T}} + \widetilde{D}^*) \leq 1 - \tilde{\Sigma}.(1/2\sum_i L_i) \leq 0$. We reach a contradiction. Therefore, (5) of Claim(1) is true.

This proves Claim(1).

Claim(2). Suppose that $\tilde{J}_2^2 = -4$. Then Theorem 7.1,(3) is true with $E = C$.

We consider the case $\tilde{J}_2^2 = -4$. Then $\widetilde{D}^* \geq 2/3\tilde{J}_2$ by Claim(1). If $\widetilde{H}_{11}$ is not a tip of $\tilde{\Delta}_1$ (resp. $\widetilde{H}_{12}$ is not a tip, or $H_{11}^2 < -3$) then $D^* \geq 6/11\widetilde{H}_{11} + 4/11\widetilde{H}_1 + 2/11\widetilde{H}_{12}$ (resp. $D^* \geq 4/9\widetilde{H}_{11} + 3/9\widetilde{H}_1 + 2/9\widetilde{H}_{12}$, or $D^* \geq 3/5\widetilde{H}_{11} + 2/5\widetilde{H}_1 + 1/5\widetilde{H}_{12}$). Either of the three cases implies that $0 < -\tilde{C}.(K_{\tilde{T}} + \widetilde{D}^*) \leq 1 - (1/3\widetilde{H}_1 + 2/3\tilde{J}_2) = 0$, a contradiction.

Thus, $\tilde{\Delta}_1 = \widetilde{H}_1 + \widetilde{H}_{11} + \widetilde{H}_{12}$ and $\widetilde{H}_1^2 = -2, H_{11}^2 = -3, H_{12}^2 = -2$ (cf. Claim(1)). If $\tilde{J}_2$ is a tip of $\tilde{\Delta}_2$, i.e., if $J_2 \neq H_2$, then Theorem 7.1,(3) is true with $E = C$. Because $\tilde{C} + \tilde{J}_2 + \tilde{\Delta}_1$ and hence $C + T_2 + \Delta_1$ have a positive eigenvalue.

We may assume that $J_2 = H_2$. Then $\widetilde{D}^* \geq 2/3\widetilde{H}_2 + 1/3\widetilde{H}_{21} + 1/3\widetilde{H}_{22}$ (cf. Claim(1)). We shall show that this would lead to a contradiction. By Claim(1), $\tilde{\Delta}_2$ is now a linear chain. If H_{2j} is not tip of $\tilde{\Delta}_2$ for $j = 1$ and 2, then $D^* \geq 2/4\widetilde{H}_{21} + 3/4\widetilde{H}_2 + 2/4\widetilde{H}_{22}$. This leads to $0 < -\tilde{C}.(K_{\tilde{T}} + \widetilde{D}^*) \leq 1 - \tilde{C}.(2/7\widetilde{H}_1 + 3/4\widetilde{H}_2) = 1 - 2/7 - 3/4 < 0$, a contradiction.

So, may assume that H_{21} is a tip of $\tilde{\Delta}_2$. If $\tilde{\Delta}_2$ has more than four irreducible components, then $D^* \geq 4/11\widetilde{H}_{21} + 8/11\widetilde{H}_2 + 6/11\widetilde{H}_{22}$. This leads to $0 < -\tilde{C}.(K_{\tilde{T}} + \widetilde{D}^*) \leq 1 - \tilde{C}.(2/7\widetilde{H}_1 + 8/11\widetilde{H}_2) = 1 - 2/7 - 8/11 < 0$, a contradiction. Therefore, $H := \tilde{\Delta}_2 - (\widetilde{H}_{21} + \widetilde{H}_2 + \widetilde{H}_{22})$ is zero or a (-2)-curve adjacent to $\widetilde{H}_{22}$ (cf. Claim(1)).

Note that $\tilde{\Sigma}.\widetilde{H}_{2j} = (K_{\tilde{T}} + \widetilde{H}_2) = 1$ for $j = 1$ and 2 (cf. Lemma 7.5). If $B.\tilde{\Sigma} > 0$ for some irreducible component B of $\widetilde{D} - (\widetilde{H}_{21} + \widetilde{H}_{22})$, then B is not contained in $\tilde{\Delta}_1$ or $\tilde{\Delta}_2$, $B^2 \leq -3$ and $B.\tilde{\Sigma} = B.K_{\tilde{T}}$ (cf. Lemma 1.5,(5)). Hence $\widetilde{D}^* \geq 1/3B$. This leads to $0 < -\tilde{\Sigma}.(K_{\tilde{T}} + D^*) \leq 1 - \tilde{\Sigma}.(1/3B + 1/3\widetilde{H}_{21} + 1/3\widetilde{H}_{22}) = 0$, a contradiction. So, $\tilde{\Sigma}$ meets only $\widetilde{H}_{21}$ and $\widetilde{H}_{22}$ in $\widetilde{D}$.

Let $S_0' := 2\tilde{\Sigma} + \widetilde{H}_{21} + \widetilde{H}_{22}$ and let $\psi : \tilde{T} \to \mathbf{P}^1$ be the $\mathbf{P}^1$-fibration with S_0' as a singular fiber. Let S_1' be the singualr fiber containing $\tilde{C} + \tilde{\Delta}_1$. Then there is a (-1)-curve E such that $E.\widetilde{H}_{11} = 1$ and $S_1' = 2(\tilde{C} + \widetilde{H}_1) + \widetilde{H}_{11} + \widetilde{H}_{12} + E$. Since $\rho(\tilde{T}) = 1$ and since every irreducible component of $\widetilde{D} - (H + \widetilde{H}_2)$ is contained in singular fibers of ψ, every singular fiber S_2' other than S_1' consists of one (-1)-curve and several irreducible components of $\widetilde{D}$ (cf. Lemma 1.1,(4) of part **I**). Here $H := \tilde{\Delta}_2 - (\widetilde{H}_{21} + \widetilde{H}_2 + \widetilde{H}_{22})$. Moreover, $H \neq 0$. So, H is a (-2)-curve adjacent to $\widetilde{H}_{22}$. Since H is a cross-section $H.E = 1$ and S_0', S_1' are the only singular fibers of ψ for otherwise H would meet a (-1)-curve F in some singular fiber S_2' and F has multiplicity at least two.

Let $\tau : \tilde{T} \to \Sigma_2$ be the smooth blowing-down of curves in singular fibers of ψ such that $\tau(H)^2 = -2$. On the one hand, $\widetilde{H}_2$ is a 2-section with $\widetilde{H}_2 \cap H = \phi$ and hence $\tau(\widetilde{H}_2)^2 = 8$. On the other hand, a calculation shows that $\tau(\widetilde{H}_2)^2 = \widetilde{H}_2^2 + 1 + 7 = 4$. We reach a contradiction.

This proves Claim(2).

In view of Claim(2), may assume that $\tilde{J}_2^2 = -3$. If $\tilde{J}_2$ is a tip of $\tilde{\Delta}_2$, i.e., if $J_2 \neq H_2$, then Theorem 1.1,(3) is true with $E = C$. Indeed, then $\tilde{C} + \tilde{J}_2 + \tilde{\Delta}_1$ and hence $C + T_2 + \Delta_1$ have a positive eigenvalue.

Thus, may assume that $J_2 = H_2$. Then $\tilde{\Delta}_2$ is a linear chain (cf. Claim(1)). We have also

$$\widetilde{D}^* \geq 3/7\widetilde{H}_{11} + 2/7\widetilde{H}_1 + 1/7\widetilde{H}_{12} + 1/4\widetilde{H}_{21} + 2/4\widetilde{H}_2 + 1/4\widetilde{H}_{22}.$$

Note that $H.\tilde{\Sigma} = H.(K_{\widetilde{H}} + \widetilde{H}_{11} + \widetilde{H}_{12} + \widetilde{H}_2) = 1$ (cf. Lemma 1.5) if H is an irreducible component of $\widetilde{D} - \widetilde{H}_1$ adjacent to one of $\widetilde{H}_{11}, \widetilde{H}_{12}$ and $\widetilde{H}_2$. In particular, $\tilde{\Sigma}.\widetilde{H}_{21} = \tilde{\Sigma}.\widetilde{H}_{22} = 1$.

Claim(3). $\widetilde{D} - (\widetilde{H}_{11} + \widetilde{H}_2)$ consists of (-2)-curves.

Suppose to the contrary that Claim(3) is false. Then $\widetilde{D} - (\tilde{\Delta}_1 + \tilde{\Delta}_2)$ contains a $(-n)$-curve B with $n \geq 3$ (cf. Claim(1)). By Lemma 1.5, we have

$B.\tilde{\Sigma} = B.K_{\tilde{T}} = n-2$. Note that $\widetilde{D^*} \geq (n-2)/nB$ and $0 < -\tilde{\Sigma}.(K_{\tilde{T}} + \widetilde{D^*}) \leq 1 - \tilde{\Sigma}.(n-2)/nB = 1 - (n-2)^2/n$. So, $n = 3$ and $B.\tilde{\Sigma} = 1$.

If $\widetilde{D} - \widetilde{H}_1$ has an irreducible component H adjacent to $\widetilde{H}_{11}$, then $\widetilde{D^*} \geq 3/11H + 6/11\widetilde{H}_{11} + 4/11\widetilde{H}_1 + 2/11\widetilde{H}_{12}$. This leads to $0 < -\tilde{\Sigma}.(K_{\tilde{S}} + \widetilde{D^*}) \leq 1 - \tilde{\Sigma}.(1/3B + 3/11H + 1/4\widetilde{H}_{21} + 1/4\widetilde{H}_{22}) = 1 - 1/3 - 3/11 - 1/4 - 1/4 < 0$. We reach a contradiction. So, $\widetilde{H}_{11}$ is a tip of $\tilde{\Delta}_1$.

If $\widetilde{D} - \widetilde{H}_1$ has an irreducible component H adjacent to $\widetilde{H}_{12}$ but H is not a tip of $\tilde{\Delta}_1$, then $\widetilde{D^*} \geq 2/11H + 3/11\widetilde{H}_{12} + 4/11\widetilde{H}_1 + 5/11\widetilde{H}_{11}$. This leads to $0 < -\tilde{\Sigma}.(K_{\tilde{T}} + \widetilde{D^*}) \leq 1 - \tilde{\Sigma}.(1/3B + 2/11H + 1/4\widetilde{H}_{21} + 1/4\widetilde{H}_{22}) = 1 - 1/3 - 2/11 - 1/4 - 1/4 < 0$. We reach a contradiction again. Thus, $H := \tilde{\Delta}_1 - (\widetilde{H}_{11} + \widetilde{H}_1 + \widetilde{H}_{12})$ is zero or a (-2)-curve adjacent to $\widetilde{H}_{12}$ (cf. Claim(1)).

Let $S_0' := 2\tilde{\Sigma} + \widetilde{H}_{21} + \widetilde{H}_{22}$ and let $\psi : \tilde{T} \to \mathbf{P}^1$ be the $\mathbf{P}^1$- fibration with S_0' as a singular fiber. Let S_1' be the singular fiber containing $\widetilde{C} + \widetilde{H}_1 + \widetilde{H}_{11} + \widetilde{H}_{12}$.

Suppose $H_{11}^2 = -3$. Then there is a (-1)-curve E such that $E.\widetilde{H}_{11} = 1$ and $S_1' = 2(\widetilde{C} + \widetilde{H}_1) + \widetilde{H}_{12} + \widetilde{H}_{11} + E$. Since B is a 2-section, we have $B.E = 2$. This leads to $0 < -E.(K_{\tilde{D}} + D^*) \leq 1 - E.(1/3B + 3/7\widetilde{H}_{11}) = 1 - (1/3)\cdot 2 - 3/7 < 0$, a contradiction. So, $H_{11}^2 \leq -4$.

Suppose $\sigma \neq id$. Let $\sigma_2 : \tilde{S}_1 \to \tilde{T}$ be the blowing-up of the point $P_2 := \widetilde{C} \cap \widetilde{H}_2$ and set $E := \sigma_2^{-1}(P_2)$. Then by the hypothesis, there is a smooth blowing-down $\sigma_1 : \tilde{S} \to \tilde{S}_1$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Now we apply Lemma 1.4. In particular, we have $-E.(K_{\tilde{S}_1} + M^*) > 0$. On the other hand, $M^* \geq 2/3\sigma_2'\widetilde{H}_{11} + 2/3\sigma_2'\widetilde{H}_1 + 1/3\sigma_2'\widetilde{H}_{12} + 1/3\sigma_2'\widetilde{C} + 1/3\sigma_2'\widetilde{H}_{21} + 2/3\sigma_2'\widetilde{H}_2 + 1/3\sigma_2'\widetilde{H}_{22}$. This leads to $0 < -E.(K_{\tilde{S}_1} + M^*) \leq 1 - E.(1/3\sigma_2'\widetilde{C} + 2/3\sigma_2'\widetilde{H}_2) = 0$, a contradiction. So, $\sigma = id$. Hence $\tilde{T} = \tilde{S}, H_i = D_i (i = 1, 2)$.

Let $S_0 := 3C + 2D_1 + H_{12} + D_2$ and let $\varphi : \tilde{S} \to \mathbf{P}^1$ be the $\mathbf{P}^1$- fibration with S_0 as a singular fiber. Then $\tilde{\Sigma}$ and the (-3)-curve B are contained in the same singular fiber of φ, say S_1. By the minimality of $-C.(K_{\tilde{S}} + D^*)$ and by noting that C has multiplicity 3 in S_0 and the summation of the multiplicities of (-1)-curves in S_1 is at least 3

(cf. [4, Lemma 1.6]), every (-1)-curve F in S_1, especially Σ, satisfies $-F.(K_{\tilde{S}} + D^*) = -C.(K_{\tilde{S}} + D^*)$. So, every singular fiber of the previous fibration ψ defined above has one of two types in Lemma 6.12, part **I**. However, S_1' above contains a curve H_{11} with $H_{11}^2 < -3$. We reach a contradiction.

This proves Claim(3).

Let

$$S_0 := 3\tilde{C} + 2\widetilde{H}_1 + \widetilde{H}_{12} + \widetilde{H}_2$$

and let $\varphi : \widetilde{T} \to \mathbf{P}^1$ be the $\mathbf{P}^1$-fibration with S_0 as a singular fiber. $\widetilde{H}_{21}, \widetilde{H}_{22}$ (resp. $\widetilde{H}_{11}$) is a cross-section (resp. 2-section). Denote by S_1 the singular fiber containing $\tilde{\Sigma}$. Let

$$S_i \ (i = 0, 1, \cdots, r)$$

be all singular fibers of φ. By Claim(3), every singular fiber S_i $(i \geq 1)$ consists of only (-1) or (-2)-curves. So, S_i has one of two types in Lemma 6.12, part **I**.

Claim(4). Suppose that S_k has the second type in Lemma 6.12 of part **I** for some $k \geq 1$. Then Case(4-1) of Theorem 1.1 occurs.

Suppose S_1 has the second type in Lemma 6.12, part **I**. Then $\tilde{\Sigma}$ is the unique (-1)-curve in S_1. Then the 2-section $\widetilde{H}_{11}$ meets two multiplicity-one or one multiplicity-two irreducible component(s) other than $\tilde{\Sigma}$ in S_1. This leads to that $\tilde{\Delta}_1$ is a fork (cf. Lemma 1.1,(4), part **I**), a contradiction to Claim(1). So, S_1 consists of two (-1)-curves $\tilde{\Sigma}, E$ and several (-2)-curves.

Suppose that S_k has the second type in Lemma 6.12, part **I** for some $k \geq 2$, say $k = 2$. Let F be the unique (-1)-curve in S_2. Since $\widetilde{H}_{2j}.S_2 = 1$ $(j = 1, 2)$, there are two (-2)-curves $G_j (j = 1, 2)$ such that $F.G_j = 1, \widetilde{H}_{2j}.G_j = \widetilde{H}_{11}.F = 1$ and

$$S_2 = 2F + G_1 + G_2.$$

Now we have (cf. Claim(1)) :

$$\tilde{\Delta}_2 = G_1 + \widetilde{H}_{21} + \widetilde{H}_2 + \widetilde{H}_{22} + G_2.$$

We have also $\widetilde{D}^* \geq 1/5 G_1 + 2/5 \widetilde{H}_{21} + 3/5 \widetilde{H}_2 + 2/5 \widetilde{H}_{22} + 1/5 G_2.$

If H is an irreducible component of $\tilde{\Delta}_1 - \widetilde{H}_1$ adjacent to $\widetilde{H}_{12}$, then H is a cross-section and $H.G_j = 1$ for $j = 1$ or 2. This leads to $\tilde{\Delta}_1 = \tilde{\Delta}_2$, a contradiction. So, $\widetilde{H}_{12}$ is a tip of $\tilde{\Delta}_1$.

If H is an irreducible component of $\tilde{\Delta}_1 - \widetilde{H}_1$ adjacent to $\widetilde{H}_{11}$, then $\widetilde{D}^* \geq 3/11 H + 6/11 \widetilde{H}_{11} + 4/11 \widetilde{H}_1 + 2/11 \widetilde{H}_{12}$. This leads to $0 < -\tilde{\Sigma}.(K_{\widetilde{T}} + D^*) \leq 1 - \tilde{\Sigma}.(3/11 H + 2/5 \widetilde{H}_{21} + 2/5 \widetilde{H}_{22}) = 1 - 3/11 - 2/5 - 2/5 < 0$, a contradiction. So, $\widetilde{H}_{11}$ is tip of $\widetilde{H}_1$.

Therefore,

$$\tilde{\Delta}_1 = \widetilde{H}_1 + \widetilde{H}_{11} + \widetilde{H}_{12}.$$

In particular, $\tilde{\Sigma}$ meets only $\widetilde{H}_{2j} (j = 1, 2)$ in $\widetilde{D}$ (cf. Lemma 1.5 and Claim(3)). So,

$$S_1 = \tilde{\Sigma} + E$$

with $\tilde{\Sigma}.E = 1$ and $\widetilde{H}_{11}.E = 2$.

If $\widetilde{H}_{11}^2 \leq -4$, then $D^* > 1/2\widetilde{H}_{11}$ and $0 < -E.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - E.1/2\widetilde{H}_{11} = 0$, a contradiction. So, $\widetilde{H}_{11}^2 = -3$.

For every $i \geq 3$, since $\widetilde{H}_{21}$ meets a (-1)-curve of multiplicity one in S_i, S_i has the first type in Lemma 6.12, pat **I**. Since $\widetilde{D} - (\widetilde{H}_{21} + \widetilde{H}_{22} + \widetilde{H}_{11})$ are contained in singular fibers of φ and since $\rho(T) = 1$, $r = 3$ and

$$S_i \ (i = 0, 1, 2, 3)$$

are all singular fibers of φ (cf. Lemma 1.5(1) in [4]). Let $E_j (j = 1, 2)$ be the two (-1)-curves in S_3.

Let $\tau : \widetilde{T} \to \Sigma_2$ be the smooth blowing-down of curves in singular fibers such that $\tau(\widetilde{H}_{21})^2 = -2$. Then $\tau(\widetilde{H}_{22}) \sim \tau(\widetilde{H}_{21}) + 2\tau(S_0)$ and $\tau(\widetilde{H}_{11}) \sim 2\tau(\widetilde{H}_{21}) + 4\tau(S_0)$. In particular, $\tau(\widetilde{H}_{22})^2 = 2$ and $\tau(\widetilde{H}_{11})^2 = 8$. So, may assume that $\widetilde{H}_{2j}.E_j = \widetilde{H}_{11}.E_j = 1 \ (j = 1, 2)$. Moreover,

$$S_3 = E_1 + G_3 + G_4 + E_2$$

where $G_3 + G_4$ is a connected component of $\widetilde{D}$ with two (-2)-curves (cf. Lemma 1.1,(4), part **I**) and with $E_j.G_{j+2} = 1$.

Now $\widetilde{H}_{11}^2 = -3$, and

$$\tilde{\Delta}_1, \, \tilde{\Delta}_2, \, G_3 + G_4$$

are all connected components of $\widetilde{D}$ (cf. Lemma 1.1, (4), part **I**). To show that Case(4-1) of Theorem 1.1 occurs, it suffices to show that $\sigma = id$. Let $\sigma_2 : \widetilde{S}_1 \to \widetilde{T}$ be the blowing-up of the point $P_2 := \widetilde{C} \cap \widetilde{H}_2$ and let $L := \sigma_2^{-1}(P_2)$. Suppose to the contrary that $\sigma \neq id$. Then by the hypothesis, there is a smooth blowing-down $\sigma_1 : \widetilde{S} \to \widetilde{S}_1$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Now applying Lemma 1.4, we get $-L.(K_{\widetilde{S}_1} + M^*) > 0$. On the other hand, $M^* = 1/2\sigma_2'\widetilde{H}_{11} + 1/2\sigma_2'\widetilde{H}_1 + 1/4\sigma_2'\widetilde{H}_{12} + 1/4\sigma_2'\widetilde{C} + 1/4\sigma_2'G_1 + 2/4\sigma_2'\widetilde{H}_{21} + 3/4\sigma_2'\widetilde{H}_2 + 2/4\sigma_2'\widetilde{H}_{22} + 1/4\sigma_2'G_2$. This leads to $-L.(K_{\widetilde{S}_1} + M^*) = 1 - L.(1/4\sigma_2'\widetilde{C} + 3/4\sigma_2'\widetilde{H}_2) = 0$. We reach a contradiction. So, $\sigma = id$ and Case(4-1) of Theorem 1.1 occurs.

This proves Claim(4).

In view of Claim(4), may assume that each singular fiber $S_i \ (i = 1, \cdots, r)$ has the first type in Lemma 6.12, part **I**. Then the number of singular fibers containing two (-1)-curves is one less than the number of sectional-components of $\widetilde{D}$ because $\rho(T) = 1$. So, $r = 2$ and S_0, S_1, S_2 are all singular fibers if $\widetilde{H}_{12}$ is a tip of $\widetilde{\Delta}_1$, or $r = 3$ and S_0, S_1, S_2, S_3 are all singular fibers otherwise. Let

$$\mu : \widetilde{T} \to \Sigma_2$$

be the smooth blowing-down of curves in singular fibers of φ such that $\mu(\widetilde{H}_{21})^2 = -2$. Write $\mu(\widetilde{H}_{ij}) = \overline{H}_{ij}, \mu(S_i) = \overline{S}_i$, etc. Then $\overline{H}_{22} \sim \overline{H}_{21} + 2\overline{S}_0$ and $\overline{H}_{11} \sim 2\overline{H}_{21} + 4\overline{S}_0$. In particular, $\overline{H}_{22}^2 = 2, \overline{H}_{11}^2 = 8, \overline{H}_{22}.\overline{H}_{11} = 4$.

Claim(5). Suppose that $\widetilde{H}_{11}$ is not a tip. Then Case(4-2) of Theorem 1.1 occurs.

One can see that $\widetilde{H_{11}}$ is a $(-3)-$curve, as in the proof of Claim (4) above. Note that $r \geq 2$ and we can write

$$S_1 = \widetilde{\Sigma} + \sum_{i=1}^{s} G_i + E$$

such that $E^2 = -1, G_i^2 = -2, \widetilde{H}_{11}.G_1 = \widetilde{\Sigma}.G_1 = G_j.G_{j+1} = G_s.E = 1$ $(j = 1, \cdots, s-1)$ (cf. Lemma 1.5),

$$S_2 = E_1 + \sum_{i=s+1}^{s+t} G_i + E_2$$

such that $E_i^2 = -1, G_j^2 = -2, E_1.G_{s+1} = G_j.G_{j+1} = G_{s+t}.E_2 = 1 (j \leq s+t-1)$. Note that $\widetilde{H}_{11}.E = 1$ for $\widetilde{H}_{11}.S_1 = 2$.

Note that $\widetilde{D}^* \geq 2/11\widetilde{H}_{12} + 4/11\widetilde{H}_1 + 6/11\widetilde{H}_{11} + 3/11G_1$. If $F.\widetilde{H}_{11} \geq 2$ for some (-1)-curve F, then $0 < -F.(K_{\widetilde{T}} + D^*) \leq 1 - F.6/11\widetilde{H}_{11} \leq 1 - (6/11)\cdot 2 < 0$, a contradiction. So, $F.\widetilde{H}_{11} \leq 1$ for every (-1)-curve F and the equality holds if F is in S_i $(i \geq 2)$ because $\widetilde{H}_{11}.S_i = 2$ (cf. Claim(1),(2)).

Case(5.1) $\widetilde{H}_{12}$ is a tip of $\widetilde{\Delta}_1$ while $\widetilde{H}_{2j}$ is not a tip of $\widetilde{\Delta}_2$ for $j = 1$ or 2, say $j = 1$. Then $r = 2$. May assume $\widetilde{H}_{21}.G_{s+1} = 1$. Since $\overline{H}_{22}^2 = 2$, one gets $\widetilde{H}_{22}.E_2 = 1$ and $t = 4$. This leads to $\widetilde{D}^* \geq 1/10G_{s+4} + 2/10G_{s+3} + 3/10G_{s+2} + 4/10G_{s+1} + 5/10\widetilde{H}_{21} + 6/10\widetilde{H}_2 + 3/10\widetilde{H}_{22}$ and $0 < -\widetilde{\Sigma}.(K_{\widetilde{T}} + D^*) \leq 1 - \widetilde{\Sigma}.(5/10\widetilde{H}_{21} + 3/10\widetilde{H}_{22} + 3/11G_1) = 1 - 5/10 - 3/10 - 3/11 < 0$, a contradiction. So, Case(5.1) is impossible.

Case(5.2). $\widetilde{H}_{12}$ is a tip of $\widetilde{\Delta}_1$ and both $\widetilde{H}_{21}$ and $\widetilde{H}_{22}$ are tips of $\widetilde{\Delta}_2$. Then $r = 2$, i.e.,

$$S_i \ (i = 0, 1, 2)$$

are all singular fibers of φ, and

$$\widetilde{\Delta}_1 = \widetilde{H}_{12} + \widetilde{H}_1 + \widetilde{H}_{11} + \sum_{i=1}^{s} G_i,\ \widetilde{\Delta}_2 = \widetilde{H}_{21} + \widetilde{H}_2 + \widetilde{H}_{22},$$

because $\widetilde{\Delta}_i$'s are linear chains. Moreover,

$$\widetilde{\Delta}_1,\ \widetilde{\Delta}_2,\ \sum_{i=s+1}^{s+t} G_i$$

are all connected components of $\widetilde{D}$ (cf. Lemma 1.1, (4), part I). We shall show that Case(4-2) of Theorem 1.1 occurs. May assume that $\widetilde{H}_{21}.E_1 = 1$. By the same reasoning as in the previous case, we have $\widetilde{H}_{22}.E_2 = 1$ and $t = 3$. Then $8 = \overline{H}_{11}^2 = \widetilde{H}_{11}^2 + 2 + (s+4) + 4$. Hence $s = -\widetilde{H}_{11}^2 - 2$. If $s \geq 2$, then $\widetilde{H}_{11}^2 \leq -4$ and $\widetilde{D}^* \geq 1/4\widetilde{H}_{12} + 2/4\widetilde{H}_1 + 3/4\widetilde{H}_{11} + 2/4G_1 + 1/4G_2$. This leads to $0 < -\widetilde{C}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - \widetilde{C}.(1/2\widetilde{H}_1 + 1/2\widetilde{H}_2) = 0$, a contradiction. So, $s = 1, \widetilde{H}_{11}^2 = -3$.

Now $s = 1, t = 3, \widetilde{H}_{11}^2 = -3$. To show that Case(4-2) of Theorem 1.1 occurs, it is sufficient to show that $\sigma = id$. Let $\sigma_2 : \widetilde{S}_1 \to \widetilde{T}$ be the blowing-up of the point $P_2 := \widetilde{C} \cap \widetilde{H}_2$ and let $F := \sigma_2^{-1}(P_2)$. Suppose to the contrary that $\sigma \neq id$. Then by the hypothesis, there is a smooth blowing-down $\sigma_1 : \widetilde{S} \to \widetilde{S}_1$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Applying Lemma 1.4, we get $0 - F.(K_{\widetilde{S}_1} + M^*) > 0$. On the other hand, $M^* = 1/3\sigma_2'G_1 + 2/3\sigma_2'\widetilde{H}_{11} + 2/3\sigma_2'\widetilde{H}_1 + 1/3\sigma_2'\widetilde{H}_{12} + 1/3\sigma_2'\widetilde{C} + 1/3\sigma_2'\widetilde{H}_{21} + 2/3\sigma_2'\widetilde{H}_2 + 1/3\sigma_2'\widetilde{H}_{22}$. Hence $0 < -F.(K_{\widetilde{S}_1} + M^*) = 1 - F.(1/3\sigma_2'\widetilde{C} + 2/3\sigma_2'\widetilde{H}_2) = 0$. We reach a contradiction. Therefore, $\sigma = id$ and Case(4-2) of Theorem 1.1 occurs.

Case(5.3). $\widetilde{H}_{12}$ is not a tip of $\widetilde{\Delta}_1$. Let H be the irreducible component of $\widetilde{D} - \widetilde{H}_1$ adjacent to $\widetilde{H}_{12}$. Then $\widetilde{D}^* \geq 1/7H + 2/7\widetilde{H}_{12} + 3/7\widetilde{H}_1 + 4/7\widetilde{H}_{11} + 2/7G_1$. Note that H is a cross-section and $H.\widetilde{\Sigma} = H.(K_{\widetilde{T}} + \widetilde{H}_{12}) = 1$ (cf. Lemma 1.5).

If $\widetilde{H}_{2j}$ is not a tip of $\widetilde{\Delta}_2$ for $j = 1$ or 2, say $j = 1$, then also $\widetilde{D}^* \geq 4/11\widetilde{H}_{21} + 6/11\widetilde{H}_2 + 3/11\widetilde{H}_{22}$, this leads to $0 < -\widetilde{\Sigma}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - \widetilde{\Sigma}.(4/11\widetilde{H}_{21} + 3/11\widetilde{H}_{22} + 1/7H + 2/7G_1) = 1 - 4/11 - 3/11 - 1/7 - 2/7 < 0$, a contradiction. So, $\widetilde{H}_{2j}$'s are tips of $\widetilde{\Delta}_2$ and hence $\widetilde{\Delta}_2 = \widetilde{H}_{21} + \widetilde{H}_2 + \widetilde{H}_{22}$.

If G_1 or H is not a tip of $\widetilde{\Delta}_1$ (resp. if $\widetilde{H}_{11}^2 \leq -4$), then $\widetilde{D}^* \geq 3/19H + 6/19\widetilde{H}_{12} + 9/19\widetilde{H}_1 + 12/19\widetilde{H}_{11} + 8/19G_1$ or $\widetilde{D}^* \geq 4/17H + 6/17\widetilde{H}_{12} + 8/17\widetilde{H}_1 +$

$10/17\widetilde{H}_{11} + 5/17G_1$ (resp. $\widetilde{D}^* \geq 2/11H + 4/11\widetilde{H}_{12} + 6/11\widetilde{H}_1 + 8/11\widetilde{H}_{11} + 4/11G_1$) and hence $-\tilde{\Sigma}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - \tilde{\Sigma}.(3/19H + 8/19G_1 + 1/4\widetilde{H}_{21} + 1/4\widetilde{H}_{22}) = 1 - 3/19 - 8/19 - 1/4 - 1/4 < 0$, or $\leq 1 - \tilde{\Sigma}.(4/17H + 5/17G_1 + 1/4\widetilde{H}_{21} + 1/4\widetilde{H}_{22}) = 1 - 4/17 - 5/17 - 1/4 - 1/4 < 0$ (resp. $\leq 1 - \tilde{\Sigma}.(2/11H + 4/11G_1 + 1/4\widetilde{H}_{21} + 1/4\widetilde{H}_{22}) = 1 - 2/11 - 4/11 - 1/4 - 1/4 < 0$. We reach a contradiction in either of the cases. So, $s = 1, \tilde{\Delta}_1 = H + \widetilde{H}_{12} + \widetilde{H}_1 + \widetilde{H}_{11} + G_1, \widetilde{H}_{11}^2 = -3$.

Note that $r = 3$. Let E_1, E_2 (resp. E_3, E_4) be the (-1)-curves in S_2 (resp. S_3). Let $t_i + 2$ be the number of irreducible components of S_i. May assume that $\widetilde{H}_{21}.E_j = 1$ for $j = 1$ and 3. Note that $8 = \overline{H}_{11}^2 = \widetilde{H}_{11}^2 + 2 + (1 + 4) + (t_1 + 1) + (t_2 + 1)$. So, $t_1 + t_2 = 2$. $\overline{H}_{22}^2 = 2$ implies that $\widetilde{H}_{22}.E_j = 1$ for $j = 2, 4$. But then it is impossible that $\overline{H}^2 = \overline{H}.\overline{H}_{22} = 2$. So, Case(5-3) is impossible.

This proves Claim(5).

In view of Claim(5), may assume that

$$\widetilde{H}_{11} \textit{ is a tip of } \tilde{\Delta}_1.$$

Thus,

$$S_1 = \tilde{\Sigma} + E$$

where E is a (-1)-curve such that $E.\tilde{\Sigma} = 1$ and $E.\widetilde{H}_{11} = S_1.\widetilde{H}_{11} = 2$ (cf. Lemma 1.5,(5)). If $\widetilde{H}_{11}^2 \leq -4$, then $\widetilde{D}^* \geq 1/2\widetilde{H}_{11}$ and $0 < -E.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - E.1/2\widetilde{H}_{11} = 0$, a contradiction. So,

$$\widetilde{H}_{11}^2 = -3.$$

Claim(6). Suppose that $\widetilde{H}_{12}$ is a tip. Then Case(4-3) of Theorem 1.1 occurs.

In this case, we have $r = 2$, i.e.,

$$S_i \ (i = 0, 1, 2)$$

are all singular fibers of φ and

$$\tilde{\Delta}_1 = \widetilde{H}_1 + \widetilde{H}_{11} + \widetilde{H}_{12}.$$

Hence $\tilde{\Sigma}$ meets only $\widetilde{H}_{2j}$ $(j = 1, 2)$ in $\widetilde{D}$(cf. Lemma 1.5,(5) and Claim(3)).

Write

$$S_2 = E_1 + \sum_{i=1}^{t} G_i + E_2$$

such that $E_1.G_1 = G_i.G_{i+1} = G_t.E_2 = 1 (i = 1, \cdots, t-1)$. May assume that $\widetilde{H}_{2j}$ does not meet $\sum_i G_i$ for $j = 1$ or 2, say $j = 1$. May assume also that $\widetilde{H}_{21}.E_1 = 1$. $\overline{H}_{22}^2 = 2$ implies that either $t = 3$ and $\widetilde{H}_{22}.E_2 = 1$, or $t = 4$ and $\widetilde{H}_{22}.G_4 = 1$. Since $\overline{H}_{11}^2 = 8$, we must have $t = 4$ and $\widetilde{H}_{11}.E_j = 1$ for $j = 1$ and 2. Now $\widetilde{H}_{11}^2 = -3$,

$$\widetilde{\Delta}_2 = \widetilde{H}_{21} + \widetilde{H}_2 + \widetilde{H}_{22} + G_4 + G_3 + G_2 + G_1,$$

and

$$\widetilde{\Delta}_1, \ \widetilde{\Delta}_2$$

are all connected components of $\widetilde{D}$ (cf. Lemma 1.1,(4), part **I**).

To prove that Case(4-3) of Theorem 1.1 occurs, it is sufficent to show that $\sigma = id$. Let $\sigma_2 : \widetilde{S}_1 \to \widetilde{T}$ be the blowing-up of the point $P_2 := \widetilde{C} \cap \widetilde{H}_2$ and set $F := \sigma_2^{-1}(P_2)$. Suppose to the contrary that $\sigma \neq id$. Then by the hypothesis, there is a smooth blowing-down $\sigma_1 : \widetilde{S} \to \widetilde{S}_1$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Applying Lemma 1.4, we get $-F.(K_{\widetilde{S}_1} + M^*) > 0$. On the other hand, $M^* = 1/2\sigma_2'\widetilde{H}_{11} + 1/2\sigma_2'\widetilde{H}_1 + 1/4\sigma_2'\widetilde{H}_{12} + 1/4\sigma_2'\widetilde{C} + 1/8\sigma_2'G_1 + 2/8\sigma_2'G_2 + 3/8\sigma_2'G_3 + 4/8\sigma_2'G_4 + 5/8\sigma_2'\widetilde{H}_{22} + 6/8\sigma_2'\widetilde{H}_2 + 3/8\sigma_2'\widetilde{H}_{21}$. This leads to $-F.(K_{\widetilde{S}_1} + M^*) = 1 - F.(1/4\sigma_2'\widetilde{C} + 3/4\sigma_2'\widetilde{H}_2) = 0$, a contradiction. Therefore $\sigma = id$ and Case(4-3) of Theorem 1.1 occurs.

This proves Claim(6).

Claim(7). Suppose that $\widetilde{H}_{12}$ is not a tip. Then either Theorem 1.1,(3) is true with $E = C$ or Case(4-4) of Theorem 1.1 occurs.

Then $r = 3$, i.e.,

$$S_i \ (i = 0, 1, 2, 3)$$

are all singular fibers of φ. Write

$$S_2 = E_1 + \sum_{i=1}^{t_1} G_i + E_2,$$

$$S_3 = E_3 + \sum_{i=t_1+1}^{t_1+t_2} G_i + E_4$$

such that $E_j^2 = -1, G_i^2 = -2, E_1.G_1 = G_{t_1}.E_2 = E_3.G_{t_1+1} = G_{t_1+t_2}.E_4 = G_i.G_{i+1} = 1$.

Let H be an irreducible component of $\widetilde{\Delta}_1 - \widetilde{H}_1$ adjacent to $\widetilde{H}_{12}$. If H is not a tip of $\widetilde{\Delta}_1$ then $\widetilde{C} + \widetilde{\Delta}_1$ and hence $C + T_2 + \Delta_1$ have a positive eigenvalue. So, Theorem 1.1,(3) is true. Thus, may assume that H is a tip of $\widetilde{\Delta}_1$. Hence $\widetilde{\Sigma}.H = 1$ and

$$\widetilde{\Delta}_1 = \widetilde{H}_1 + \widetilde{H}_{11} + \widetilde{H}_{12} + H.$$

Note that

$$\widetilde{D}^* = 1/9H + 2/9\widetilde{H}_{12} + 3/9\widetilde{H}_1 + 4/9\widetilde{H}_{11} + (other\, terms).$$

Now one may assume that $E_j.H = 1$ for $j = 2, 4$. Let $\varepsilon : \widetilde{T} \to \Sigma_2$ be the smooth blowing-down of curves in the singular fibers of φ such that $\varepsilon(H)^2 = -2$. Then $\varepsilon(H_{2j})^2 = 2$ $(j = 1, 2)$ and $\varepsilon(H_{11})^2 = 8$.

If $\widetilde{H}_{11}.E_i = 2$ for $i = 1$ or 3, say $i = 1$, then $S_2 = E_1 + E_2, S_3 = E_3 + E_4$ and $\widetilde{H}_{11}.E_k = 1$ for $k = 3$ and 4 because $\varepsilon(H_{11})^2 = 8$. But then $\varepsilon(H_{2j})^2 \leq -2 + 3$ $(j = 1, 2)$, a contradiction. If $\widetilde{H}_{11}.E_i = 2$ for $i = 2$ or 4, then $-E_i.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - E_i.(1/9H + 4/9\widetilde{H}_{11}) = 1 - 1/9 - (4/9) \times 2 = 0$, a contradiction. So, $\widetilde{H}_{11}.E_j = 1$ for $j = 1, 2, 3$ and 4. Now $\varepsilon(H_{11})^2 = 8$ implies that $t_1 + t_2 = 3$.

If $\widetilde{H}_{2j}$ is not a tip of $\widetilde{\Delta}_2$ for both $j = 1$ and 2, then one may assume that $(t_1, t_2) = (1, 2)$ and $\widetilde{H}_{21}.G_1 = 1$. Then it is impossible that $\varepsilon(H_{21})^2 = 2$. So, one may assume that $\widetilde{H}_{21}$ is a tip of $\widetilde{\Delta}_2$.

Since $\varepsilon(H_{21})^2 = 2$, one may assume that $(t_1, t_2) = (1, 2)$ and $\widetilde{H}_{21}.E_j = 1$ for $j = 2$ and 3. Now $\varepsilon(H_{22}).\varepsilon(H_{21}) = 2$ implies that $\widetilde{H}_{22}.E_1 = \widetilde{H}_{22}.G_3 = 1$. So,

$$\widetilde{\Delta}_2 = \widetilde{H}_{21} + \widetilde{H}_2 + \widetilde{H}_{22} + G_3 + G_2,$$

and

$$\widetilde{\Delta}_1, \widetilde{\Delta}_2, G_1$$

are all connected components of $\widetilde{D}$ (cf. Lemma 1.1,(4), part **I**).

Now $(t_1, t_2) = (1, 2)$ and $\widetilde{H}_{11}^2 = -3$. To prove that Case(4-4) takes place, we have only to show that $\sigma = id$. Let $\sigma_2 : \widetilde{S}_1 \to \widetilde{T}$ be the blowing-up of the point $\widetilde{C} \cap \widetilde{H}_2$ and set $F := \sigma_2^{-1}(P_2)$. Suppose to the contrary that $\sigma \neq id$. Then by the hypothesis, there is a smooth blowing-down $\sigma_1 : \widetilde{S} \to \widetilde{S}_1$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Applying Lemma 1.4, we get $-F.(K_{\widetilde{S}_1} + M^*) > 0$. On the other hand, $M^* = 2/9\sigma_2' H + 4/9\sigma_2'\widetilde{H}_{12} + 6/9\sigma_2'\widetilde{H}_1 + 5/9\sigma_2'\widetilde{H}_{11} + 3/9\sigma_2'\widetilde{C} + {} + 4/11\sigma_2'\widetilde{H}_{21} + 8/11\sigma_2'\widetilde{H}_2 + 6/11\sigma_2'\widetilde{H}_{22} + 4/11\sigma_2' G_3 + 2/11\sigma_2' G_2$. This leads to $-F.(K_{\widetilde{S}_1} + M^*) = 1 - F.(1/3\sigma_2'\widetilde{C} + 8/11\sigma_2'\widetilde{H}_2) = 1 - 1/3 - 8/11 < 0$, a contradiction. Therefore, $\sigma = id$ and Case(4-4) of Theorem 1.1 occurs.

This proves Claim(7) and also Theorem 1.10.

Lemma 1.11. *Suppose that the case (2) in Lemma 1.8 occurs with $(\widetilde{J}_a^2, \widetilde{J}_b^2) = (-2,-3)$ or $(-2,-4)$ but the case in Corollary 1.6 does not occur. Then either Theorem 1.1,(3) is true with $E = C$, or one of the following two cases occurs :*

Case(1) Δ_1 is a fork with 4 or 5 irreducible components, and T_1, a maximal twig of Δ_1 consists of a single (-2)-curve D_1. H_1 is the central component of Δ_1 and $H_1^2 = -3$. Every irreducible component of $\Delta_1 - H_1$ is a (-2)-curve. Thus, $\Delta_1 = D_1 + H_1 + H_{11} + H_{12} + H$ where $H = 0$ or an irreducible component adjacent to H_{12}.

Δ_2 is a linear chain with three irreducible components and with D_2 as the middle one. Hence $J_1 = H_1, J_2 = H_2 = D_2, \Delta_2 = D_2 + H_{21} + H_{22}, D_2^2 = H_{21}^2 = -3, H_{22}^2 = -2$. Moreover, σ is the blowing-down of C, $\widetilde{C} = \sigma(D_1), \widetilde{J}_1^2 = -3, \widetilde{J}_2^2 = -2$.

Case(2) Δ_1 is a fork with 5 irreducible components, and T_1, a maximal twig of Δ_1 consists of two (-2)-curves, say $T_1 = D_1 + B_1$. H_1 is the central component of Δ_1 and $H_1^2 = -3$. Every irreducible component of $\Delta_1 - H_1$ is a (-2)-curve. Thus, $\Delta_1 = D_1 + B_1 + H_1 + H_{11} + H_{12}$.

Δ_2 is a linear chain with three irreducible components and with D_2 as the middle one. Hence $J_1 = H_1, J_2 = H_2 = D_2, \Delta_2 = D_2 + H_{21} + H_{22}, D_2^2 = -4, H_{21}^2 = -3, H_{22}^2 = -2$. Moreover, σ is the blowing-down of C, D_1, $\widetilde{C} = \sigma(B_1), \widetilde{J}_1^2 = -3, \widetilde{J}_2^2 = -2$.

Proof. By the hypothesis, $J_a = H_a$ and may assume that $\widetilde{H}_{a1}^2 \le -3$.

Claim(1). It is impossible that $\widetilde{J}_b^2 = -4$.

We consider the case $\widetilde{J}_b^2 = -4$. Since the case in Corollary 1.6 does not occur, we have $\sigma \neq 1$. Let $\tau_i : \widetilde{S}_i \to \widetilde{T}$ be the blowing-up of the point $P_i := \widetilde{C} \cap \widetilde{J}_i$. Let $E_i := \tau_i^{-1}(P_i)$. Then for $t = a$ or b, there is a smooth blowing-down $\sigma_t : \widetilde{S} \to \widetilde{S}_t$ such that $\sigma = \tau_t \cdot \sigma_t$. Now we apply Lemma 1.4. In particular, we have $-E_t.(K_{\widetilde{S}_t} + M^*) > 0$.

Case $t = a$. Then $M^* \ge 8/13\tau_a'\widetilde{H}_a + 7/13\tau_a'\widetilde{H}_{a1} + 4/13\tau_a'\widetilde{H}_{a2} + 2/5\tau_a'\widetilde{C} +$

$4/5\tau_a'\widetilde{J}_b$. This leads to $0 < -E_a.(K_{\widetilde{S}_t} + M^*) \leq 1 - E_a.(8/13\tau_a'\widetilde{H}_a + 2/5\tau_a'\widetilde{C}) = 1 - 8/13 - 2/5 < 0$, a contradiction. So, this case is impossible.

Case $t = b$. Then $M^* \geq 1/4\tau_b'\widetilde{C} + 1/2\tau_b'\widetilde{H}_a + 1/2\tau_b'\widetilde{H}_{a1} + 1/4\tau_b'\widetilde{H}_{a2} + 3/4\tau_b'\widetilde{J}_b$. This leads to $0 < -E_b.(K_{\widetilde{S}_t} + M^*) \leq 1 - E_b.(1/4\tau_b'\widetilde{C} + 3/4\tau_b'\widetilde{J}_b) = 0$, a contradiction. So, this case is impossible.

This proves Claim(1).

Therefore, $\widetilde{J}_b^2 = -3$.

Claim(2). (1) $\widetilde{\Delta}_a$ is a linear chain and the connected component of $\widetilde{\Delta}_a - \widetilde{H}_a$ containing $\widetilde{H}_{a2}$ is a (-2)-chain.

Since it is impossible that $\widetilde{\Delta}_b$ is a linear chain with $\widetilde{J}_b$ as a tip (cf. the hypopthesis($*$) after Lemma 1.3), we have $\widetilde{D}^* \geq 1/2\widetilde{J}_2$. If Claim(2) is false, then we have $\widetilde{D}^* \geq 1/2\widetilde{H}_{a1} + 1/2\widetilde{H}_a + 1/2\widetilde{H}_{a2}$. This leads to $0 < -\widetilde{C}.(K_{\widetilde{T}} + \widetilde{D}^*) \leq 1 - \widetilde{C}.(1/2\widetilde{H}_a + 1/2\widetilde{J}_b) = 0$, a contradiction. So, Claim(2) is true.

Thus, $\widetilde{H}_{a2}^2 = -2$. If $\widetilde{J}_b$ is a tip of $\widetilde{\Delta}_b$, i.e., if $J_b \neq H_b$, then Theorem 1.1,(3) is true with $E = C$. Indeed, $\widetilde{C} + \widetilde{J}_b + \widetilde{H}_a + \widetilde{H}_{a2}$ is a support of a singular fiber of a $\mathbf{P}^1$-fibration; hence $\widetilde{C} + \widetilde{J}_b + \widetilde{\Delta}_a$ and $C + T_b + \Delta_a$ have a positive eigenvalue.

Therefore, we may assume that $J_b = H_b$. Since the case in Corollary 1.6 does not occur, there are two smooth blowing-downs $\sigma_1 : \widetilde{S} \to \widetilde{S}_1, \sigma_2 : \widetilde{S}_1 \to \widetilde{T}$ such that $\sigma = \sigma_2 \cdot \sigma_1$ and that :

(1) $\sigma_1(T_a + C + T_b) = T_a' + E + T_b'$ where E is a (-1)- curve and $T_i' \leq \sigma_1(T_i)$,

(2) $T_a' + \sigma_1(H_a) = \sum_{i=1}^{s} L_i, E.L_1 = L_i.L_{i+1} = 1 (i = 1, \cdots, s-1; s \geq 1), L_s = \sigma_1(H_a), L_1^2 = -t - 1 \leq -3, L_j^2 = -2 (j > 1)$,

(3) $T_b' + \sigma_1(H_b) = \sum_{i=1}^{t} M_i, E.M_1 = M_i.M_{i+1} = 1 (i = 1, \cdots, t-1; t \geq 2), M_t = \sigma_1(H_b), M_j^2 = -2 (j < t), M_t^2 = -s - 2 \leq -3$, and

(4) σ_1 does not factorize through the blowing-up of the point $P_a := E \cap L_1$.

In particular, we see that $\sigma_1(\Delta_b)$ is a fork and hence $\widetilde{\Delta}_b$ is a linear chain. Now we apply Lemma 1.4. In particular, we have $-E.(K_{\widetilde{S}_1} + M^*) > 0$.

Claim(3). $\sigma_1 = id$. Hence $a = 2, b = 1, C = E, D_1 = M_1, D_2 = L_1, D_2^2 = -t - 1 \leq -3, H_{21}^2 \leq -3$ and $T_1 = \sum_{i=1}^{t-1} M_i$ is a (-2)-twig.

Let $\tau_2 : \widetilde{X} \to \widetilde{S}_1$ be the blowing-up of the point $P_b := E \cap M_1$ and set $F := \tau_2^{-1}(P_b)$. Suppose that Claim(3) is false. Then by the definition of σ_1 (cf. the above condition(3)), there is a smooth blowing-down $\tau_1 : \widetilde{S} \to \widetilde{X}$ such that $\sigma_1 = \tau_2 \cdot \tau_1$. Now we apply Lemma 1.4. In particular, we have $-F.(K_{\widetilde{X}} + N^*) > 0$, where $N = D$ if $\tau_1 = id$ and $N = \tau_1(D) - F$ otherwise.

Since $\tau_1(C + \Delta_1 + \Delta_2) - F$ can be contractible to quotient singularities (cf. Lemma 1.4), we have $s = 1$ or 2, and if $s = 2$ then $t = 2, \widetilde{H}_{a1}^2 = -3$ and $\tau_1(\Delta_a) = \tau_2'(E + \sum_i L_i) + \tau_1(H_{a1} + H_{a2})$.

Suppose $s = 1$. Then $N^* \geq (3t-2)/(6t-2)\tau_2' E + 2(3t-2)/(6t-2)\tau_1(H_a) + (4t-2)/(6t-2)\tau_1(H_{a1}) + (3t-2)/(6t-2)\tau_1(H_{a2}) + \sum_i (t+i)/(2t+1)\tau_2'(M_i) + t/(2t+1)\tau_1(H_{b1}) + t/(2t+1)\tau_1(H_{b2})$. This leads to

$$0 < -F.(K_{\widetilde{X}} + N^*) \leq 1 - F.((3t-2)/(6t-2)\tau_2' E + (t+1)/(2t+1)\tau_2' M_1) =$$

$$1-(3t-2)/(6t-2)-(t+1)/(2t+1) = 1/(6t-2)-1/2(2t+1) = (-2t+4)/2(6t-2)(2t+1) \leq 0,$$

because $t \geq 2$. We reach a contradiction.

Suppose that $s = 2$. Then $N^* \geq 9/23\tau_2' E + 18/23\tau_2'(L_1) + 22/23\tau_1(H_a) + 15/23\tau_1(H_{a1}) + 11/23\tau_1(H_{a2}) + 10/16\tau_2'(M_1) + 14/16\tau_1(H_b) + 7/16\tau_1(H_{b1}) + 7/16\tau_1(H_{b2})$. This leads to

$$0 < -F.(K_{\widetilde{X}} + N^*) \leq 1 - F.(9/23\tau_2' E + 10/16\tau_2' M_1) = 1 - 9/23 - 10/16 < 0.$$

We reach a contradiction.

So, Claim(3) is true.

Claim(4). $s = 1$. Hence Δ_2 is a linear chain, $H_2 = D_2$ and $H_1^2 = -s-2 = -3$.

Suppose $s \geq 3$. Then $s = 3, t = 2, H_{21}^2 = -3, \Delta_2 = D_2(= L_1) + L_2 + H_2(= L_3) + H_{21} + H_{22}$ because Δ_2 is contractible to a quotient singularity. So, we have $D^* \geq 3/7D_1(= M_1) + 6/7H_1(= M_2) + 3/7H_{11} + 3/7H_{12} + 10/17D_2(= L_1) + 13/17L_2 + 16/17H_2(= L_3) + 11/17H_{21} + 8/17H_{22}$. This leads to $0 < -C.(K_{\widetilde{S}} + D^*) \leq 1 - C.(3/7D_1 + 10/17D_2) = 1 - 3/7 - 10/17 < 0$, a contradiction.

Suppose $s = 2$. Then $D^* \geq \sum_i 2i/(2t+1)M_i + t/(2t+1)H_{11} + t/(2t+1)H_{12} + (7t-5)/(7t+1)D_2(= L_1) + 4(2t-1)/(7t+1)H_2(= L_2) + (5t-1)/(7t+1)H_{21} + 2(2t-1)/(7t+1)H_{22}$. This leads to $0 < -C.(K_{\widetilde{S}} + D^*) \leq 1 - C.(2/(2t+1)D_1 + (7t-5)/(7t+1)D_2) = 1 - 2/(2t+1) - (7t-5)/(7t+1) = -2/(2t+1) + 6/(7t+1) = (4-2t)/(2t+1)(7t+1) \leq 0$, because $t \geq 2$. We reach a contradiction.

This proves Claim(4).

Claim(5). $t = 2, 3$. Hence $D_2^2 = -t-1 = -3, -4$.

Note that $D^* \geq i/(t+1)M_i + t/2(t+1)H_{11} + t/2(t+1)H_{12} + (6t-4)/(6t+1)D_2(= L_1) + (4t-1)/(6t+1)H_{21} + (3t-2)/(6t+1)H_{22}$. So, $0 < -C.(K_{\widetilde{S}} + D^*) \leq 1 - C.(1/(t+1)D_1 + (6t-4)/(6t+1)D_2) = 1 - 1/(t+1) -$

$(6t-4)/(6t+1) = -1/(t+1) + 5/(6t+1) = (4-t)/(t+1)(6t+1)$. Hence $t \le 3$. This proves Claim(5).

Claim(6). Case (1) or (2) in Lemma 1.11 occurs.

Consider first the case $D_2^2 = -t-1 = -3$. Then $D^* \ge 1/3D_1(= M_1) + 2/3H_1(= M_2) + 1/3H_{11} + 1/3H_{12} + 7/13H_{21} + 8/13D_2(= H_2) + 4/13H_{22}$. If H_{21} is not a tip (resp. H_{22} is not a tip, or $H_{21}^2 \le -4$), then also $D^* \ge 2/3H_{21} + 2/3D_2 + 1/3H_{22}$ (resp. $D^* \ge 5/9H_{21} + 6/9D_2 + 4/9H_{22}$, or $D^* \ge 2/3H_{21} + 2/3D_2 + 1/3H_{22}$). Either of the three cases leads to $0 < -C.(K_{\widetilde{S}} + D^*) \le 1 - C.(1/3D_1 + 2/3D_2) = 0$, a contradiction. Thus, $\Delta_2 = D_2 + H_{21} + H_{22}$ and $H_{21}^2 = -3$. So, Δ_2 is as described in the case(1) of Lemma 1.11.

Let T_1', T_1'' be twigs of Δ_1 containing H_{11}, H_{12}, respectively. If both T_1' and T_1'' have more than one irreducible components (resp. T_1' or T_1'', say T_1' has more than two irreducible components), then $D^* \ge 3/7D_1 + 6/7H_1 + 4/7H_{11} + 4/7H_{12}$ (resp. $D^* \ge 2/5D_1 + 4/5H_1 + 3/5H_{11} + 2/5H_{12}$). Either of the two cases leads to $0 < -C.(K_{\widetilde{S}} + D^*) \le 1 - C.(2/5D_1 + 8/13D_2) = 1 - 2/5 - 8/13 < 0$, a contradiction.

To show that Δ_1 is as described in the case(1) of Lemma 1.11, it remains to show that $\Delta_1 - H_1$ consists of only (-2)-curves. Indeed, if $H_{1j}^2 \le -3$ for $j = 1$ or 2, say $j = 1$, then $D^* \ge 2/5D_1 + 4/5H_1 + 3/5H_{11} + 2/5H_{12}$. We shall reach a contradiction as in the above paragraph. Note that $H := \Delta_1 - (D_1 + H_1 + H_{11} + H_{12})$ is zero or a single curve. It remains to show that $H^2 = -2$ if exists. Indeed, suppose $H^2 \le -3$ and suppose, without loss of generality, $H \le T_1'$. Then $D^* \ge 3/7D_1 + 6/7H_1 + 4/7H + 5/7H_{11} + 3/7H_{12}$. We shall again reach a contradiction as in the above paragraph.

We have proved that the case(1) in Lemma 1.11 occurs if $D_2^2 = -3$.

Now we consider the case $D_2^2 = -4$. Let $\gamma_1 : \widetilde{S} \to \widetilde{X}$ be the blowing-down of C. Let $\gamma_2 : \widetilde{X} \to \widetilde{T}$ be the smooth blowing-down such that $\sigma = \gamma_2 \cdot \gamma_1$. Now we apply Lemma 1.4. In particular, we have $-F.(K_{\widetilde{X}} + N^*) > 0$ where $F = \gamma_1(D_1)$ is a (-1)-curve and $N = \gamma_1(D) - F$.

Now F meets a (-2)-curve $\gamma(M_2)$ and a (-3)-curve $\gamma(D_2)$. By making use the latter inequality for F and by the arguments for the case $D_2^2 = -3$, we can also prove that $\gamma(\Delta_1 - D_1), \gamma(\Delta_2)$ have the same weighted dual graphs as Δ_1, Δ_2, respectively in Case(1) of Lemma 1.11. To verify that the case(2) in Lemma 1.11 occurs. It remains to show that $H := \Delta_1 - (D_1 + M_2 + H_1 + H_{11} + H_{12}) = 0$. Suppose $H \ne 0$, say H is adjacent to H_{11}. Then $D^* \ge 2/7D_1 + 4/7M_2 + 6/7H_1 + 2/7H + 4/7H_{11} + 3/7H_{12} + 11/19H_{21} + 14/19D_2 + 7/19H_{21}$.

This leads to $0 < -C.(K_{\widetilde{S}} + D^*) \le 1 - C.(2/7D_1 + 14/19D_2) = 1 - 2/7 - 14/19 < 0$, a contradiction.

This proves Claim(6) and hence Lemma 1.11.

Lemma 1.12 *In the Case (3) of Theorem 1.1, $\pi_1(S^0)$ is finite.*

Proof. The argument in this case is similar to the proof of Lemma 6.24 at the end of part **I**. We can assume that $C + \widetilde{T_1} + \Theta_2$ has a positive eigenvalue. Let $\widetilde{T_1} = B_1 + L_2 + \cdots + L_r$ be the twig. If U is a nice tubular neighborhood of $C + \widetilde{T_1} + \Theta_2$, then it is easy to see that $U - D$ has $N - D$ as a strong deformation retract, where N is a tubular neighborhood of $C + \Theta_2$. Now the rest of the argument is exactly as in the proof of Lemma 6.24 in part **I**.

Lemma 1.13. *In the case (4) of Theorem 1.1, $\pi_1(S^0)$ is finite.*

Proof. We will use the description of $C + \Delta_1 + \Delta_2$ which occurs in the proof of Theorem 1.10 (cf. Figures 1, 2, 3, 4).
As before, the intersection form on $C + \Delta_1 + \Delta_2$ has one positive eigenvalue and by Lemma 1.10 of part **I** we have a surjection $\pi_1(U - \Delta_1 - \Delta_2) \to \pi_1(S^0)$, where U is a small neighborhhod of $C \cup \Delta_1 \cup \Delta_2$. We will use the presentation of $\pi_1(U - \Delta_1 - \Delta_2)$ given by Mumford in [3].

Case (4-1) $\pi_1(\partial U)$ is given by generators $e_0, e_1, e_{11}, e_{12}, e_2, e_{21}, e_{22}, g_1, g_2$ corresponding to $C, H_1, H_{11}, H_{12}, H_2, H_{21}, H_{22}, G_1, G_2$ respectively and the following relations :

$$e_{11}^{-3}e_1 = e_{12}^{-2}e_1 = e_{11}e_{12}e_1^{-2}e_0 = e_1e_o^{-1}e_2 = \cdots = 1$$

Hence $e_1 = e_{11}^3 = e_{12}^2$ and $e_{11}e_{12}^{-3}e_0 = 1$.
Now $\pi_1(U - D)$ is obtained by putting $e_0 = 1$ in the relations above. Hence in $\pi_1(U - D)$

$$e_{11} = e_{12}^3, e_2 = e_1^{-1} = e_{12}^{-2}$$

etc. From the remaining relations, we can express g_1, e_{21}, and e_{22} in terms of g_2 and after putting $e_0 = 1, e_2 = g_2^3$ and $g_2^{15} = 1 = e_{12}^7$.

Here, 7 and 15 are the absolute values of the determinants of the intersection forms of Δ_1 and Δ_2 respectively. Hence e_{12} can be expressed in terms

of e_1 and hence $\pi_1(U - D)$ is a finite cyclic group generated by g_2. Hence $\pi_1(S^0)$ is finite cyclic in this case.
Case (4-2) From the proof of Theorem 1.10, Claim (5), Case (5.2) (cf. Figure 2) we know that $\sigma = identity$. We argue exactly as above. The determinant of $\Delta_1 = \pm 11$ and $\pi_1(U - D)$ is generated by e_{21} (corresponding to H_{21}). Again $\pi_1(U - D)$ is finite cyclic.
Case (4-3) By the proof of Theorem 1.10, Claim (6) we have $\sigma = identity$ and the determinant of $\Delta_1 = \pm 7$ (cf. Figure 3). In this case $\pi_1(U - D)$ is a finite group generated by g_1 (corresponding to G_1).
In the above cases, the crucial fact used was the linearity of Δ_1, Δ_2.
Case (4-4) By the proof of Claim 7 in Theorem 1.10, $\sigma = identity$. Now the determinants of Δ_1, Δ_2 are $\pm 9, \pm 14$ respectively (both non-primes).
In this case we use the (-1)-curve E in the singular fiber S_1 (cf. Figure 4). Now $E + \Delta_1$ supports a divisor with a positive self-intersection. E intersects only the curve H_{11} from Δ_1 ($E.H_{11} = 2$) which is a tip of the linear chain Δ_1. Now the proof used for the case $|K + C + D| \neq \phi$ in part **I**, using Lemma 1.14 in part **I** proves that $\pi_1(S^0)$ is finite.

Lemma 1.14 *In the two cases of Lemma 1.11, $\pi_1(S^0)$ is finite.*

Proof. In Case (1) of Lemma 1.11, the determinant of

$\Delta_2 = \pm 13$ and Δ_2 is linear (whether or not $H = \phi$ or $\neq \phi$). In Case (2) of Lemma 1.11, the determinant of $\Delta_2 = \pm 19$ and Δ_2 is linear (cf. Figures 5, 6).
If U is a tubular neighborhood of $C \cup \Delta_1 \cup \Delta_2$, then using Mumford's presentation we see that $\pi_1(U - D)$ is a homomorphic image of $\pi_1(U_1 - \Delta_1)$, where U_1 is a small tubular neighborhood of Δ_1. Since Δ_1 defines a quotient singular point, we deduce the finiteness of $\pi_1(S^0)$.
This completes the proof of Theorem 1.1 and also of the Main Theorem.

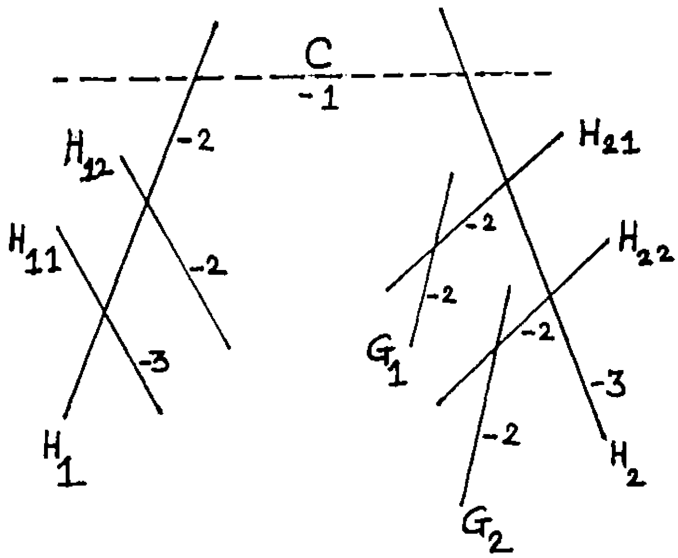

Figure 1

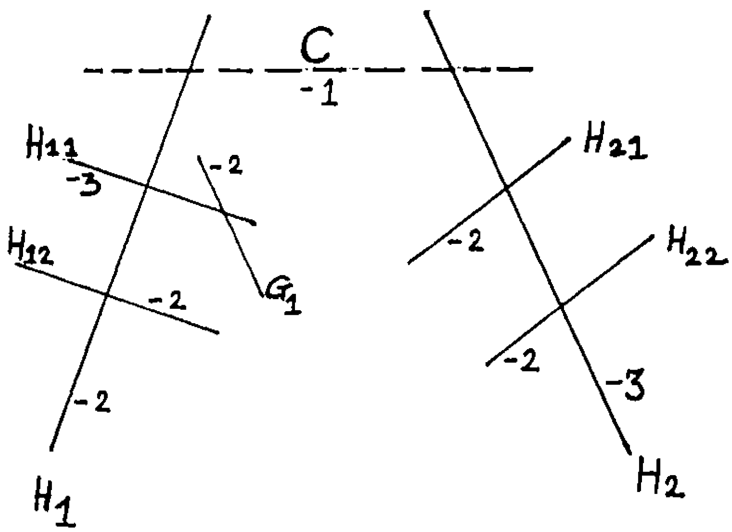

Figure 2

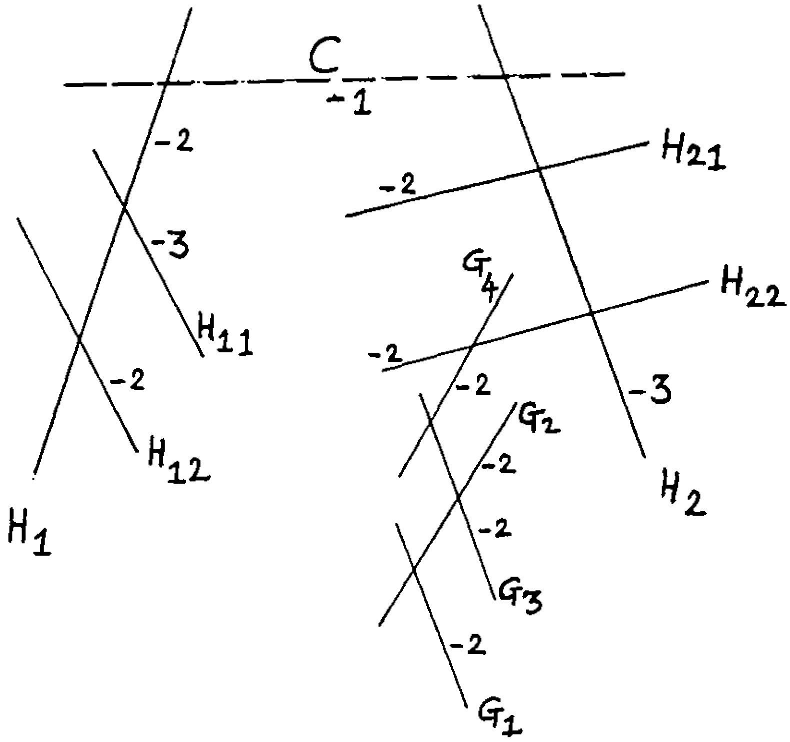

Figure 3

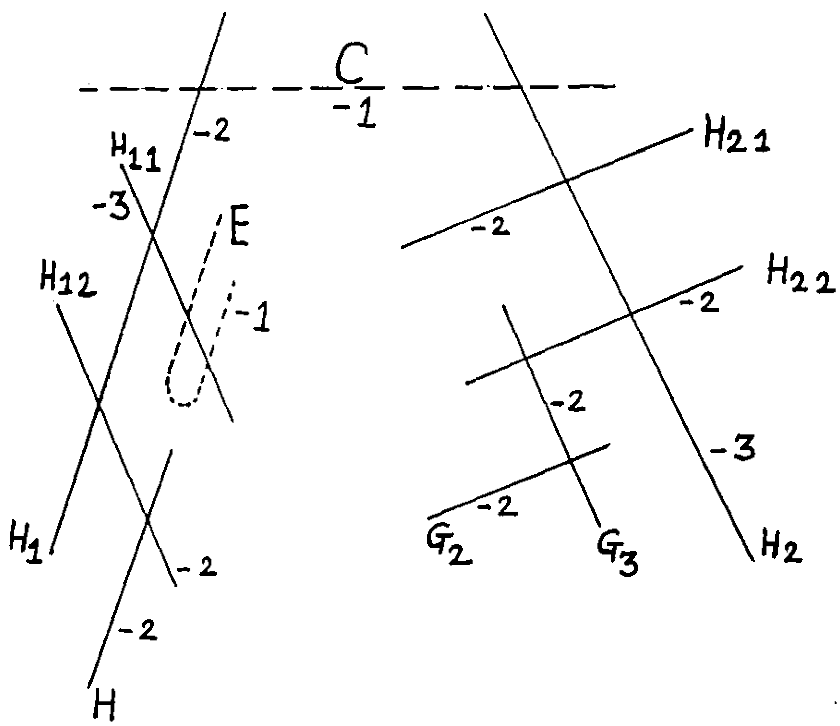

Figure 4

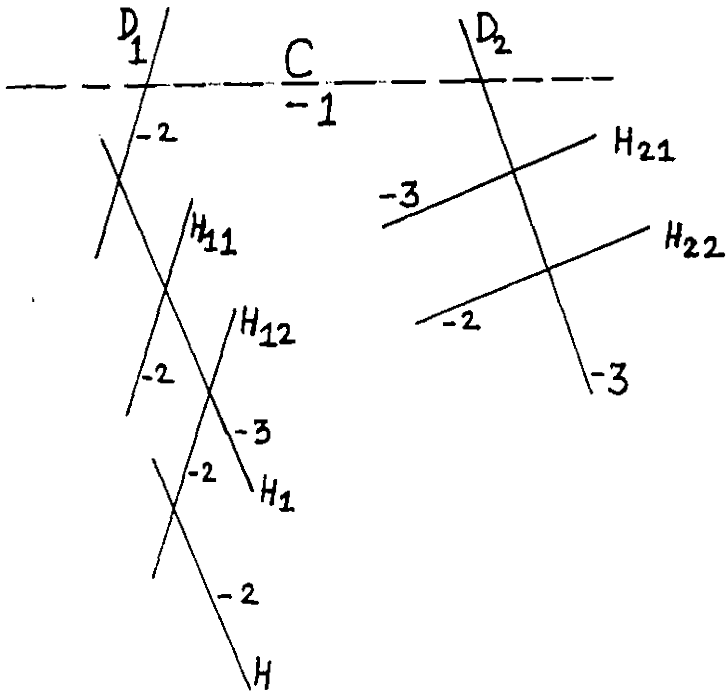

Figure 5

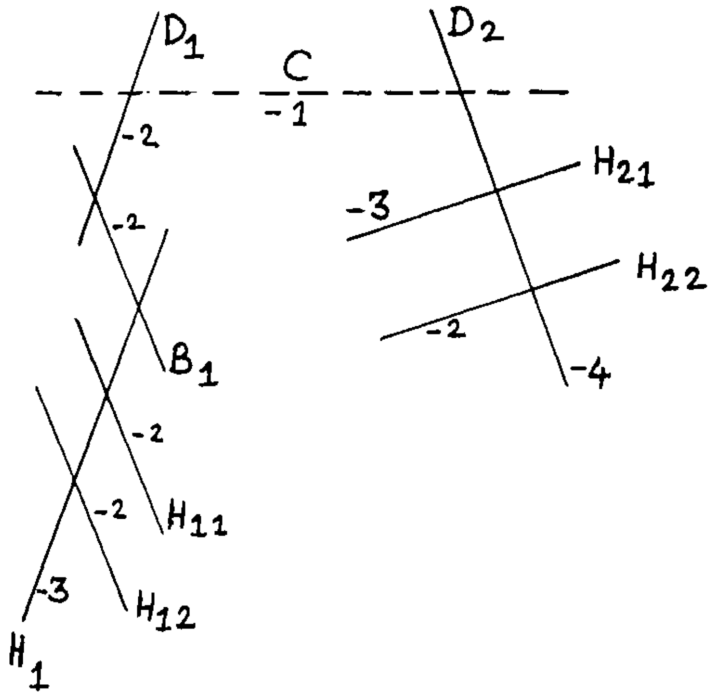

Figure 6

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