SEMIINFINITE SYMMETRIC POWERS

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(0.1) In this note we develop a theory of measures, differential forms and Fourier transforms on some infinite-dimensional real vector spaces, by generalizing the following two constructions:

(a) The construction of the semiinfinite wedge power of a space V equipped with a class of commensurable subspaces [ADK] [KP] [S]. Recall that it is obtained as a certain double direct limit of the exterior algebras of finite-dimensional subquotients of V.

(b) The construction of the space of measures on a nonarchimedean local field K with maximal ideal $\mathbf{m}$ as a double inverse limit of the spaces of measures (which in this case are the same as functions) on finite subquotients $\mathbf{m}^i/\mathbf{m}^j$ of K.

The importance of generalizing the construction (b) was emphasized by A.N. Parshin [P2-3]. Among other things, he pointed out that all the 4 types of possible combinations of inductive and projective limits of the spaces of measures on $\mathbf{m}^i/\mathbf{m}^j$ have very transparent analytic meaning (Schwartz-Bruhat functions vs. distributions, compact support vs. arbitrary support) and called for a generalization of this approach to higher local fields. This note can be seen as step in that direction, treating 2-dimensional local fields such as $\mathbb{R}((t)), \mathbb{C}((t))$. More general 2-dimensional local fields and adeles (see [P1] for background) will be considered in a subsequent paper.

(0.2) Let us describe our constructions more precisely. Our setting is exactly the same as the one needed in the theory of the "Japanese group $GL(\infty)$" and the semiinfinite Grassmannians which encompasses (a). Though this is not universally known, the relevant class of structured infinite-dimensional spaces was introduced in the 1942 book of S. Lefschetz [L] under the name of locally linearly compact spaces. We adopt this framework which predates by 40 years the relatively recent interest in "semiinfinite" structures.

For a locally linearly compact $\mathbb{R}$-vector space V we introduce the space $M(V)$ of "smooth measures" on V as a double projective limit of the spaces of smooth measures on the finite-dimensional subquotients U_1/U_2 where U_i run over open, linearly compact subspaces of V. So the procedure is quite similar to (b). Here, however, one encounters the difficulty similar to that familiar in (a), namely that smooth measures can be restricted onto a subspace only after tensoring with some 1-dimensional space of Haar measures, and this leads to the fact that $M(V)$ will be only a projective representation of the group $GL(V)$ of continuous automorphisms of V. This can be pinned down more precisely by saying that we have a version $M_h(V)$ of $M(V)$ for each "Haar theory" h on V, an object of a certain gerbe, see (2.3).

The category $\mathcal{L}$ of locally linearly compact spaces possesses a perfect duality $V \mapsto V^\vee$ (the Lefschetz-Chevalley generalization of Pontryagin duality) and we construct a Fourier transform from $M_h(V)$ to $M_{h^\vee}(V^\vee)$ where $h^\vee$ is a Haar theory on $V^\vee$ naturally associated to h.

Further, we consider the de Rham complexes of forms on the subquotients U_1/U_2 as before, and by taking their double projective limit we construct the semiinfinite de Rham complex $\Omega^\bullet(V)$. Its "anomaly" (i.e., the central extension of $GL(V)$ arising from the projective action) is much simpler than that of $M(V)$: it reduces to a $\mathbb{Z}/2$-central extension coming from the orientation, and in the case of $\mathbb{C}$-spaces vanishes altogether. The complex $\Omega^\bullet(V)$ is similar to Ω^{ch}, the chiral de Rham complex of Malikov, Schechtman and Vaintrob [MSV]. A treatment of Ω^{ch} itself from a similar standpoint will be given in the forthcoming paper [KV].

(0.3) The construction of (0.1)(a) can be included in our framework as a particular case, if we extend it to encompass super-vector spaces, see [M]. In particular, for a purely odd finite-dimensional super-vector space W the space of measures is the exterior algebra of W, and the integration is understood in the sense of Berezin. If we apply our approach to finite-dimensional subqotients of a purely odd locally linearly compact supervector space V, we get precisely the semiinfinite wedge space. Therefore it is natural to call our space $M(V)$ (for any V) the *semiinfinite symmetric power of* V. Unless we are in the purely odd case, it is of a pronounced analytic flavor. If V is equipped with a positive definite quadratic form q, one can specify a more algebraic subspace $AG(V, q) \subset M(V)$ invariant under the orthogonal group $O(q)$ by considering measures which are almost Gaussian, i.e., are product of a polynomial and a Gaussian measure.

Another natural candidate for a "semiinfinite symmetric power" of V would be an irreducible module over the Heisenberg algebra of $V \oplus V^\vee$. For example any open linearly compact subspace $U \subset V$ gives a "vacuum module". But unless V is purely odd (so that we have a Clifford algebra) these vacuum modules are not isomorphic to each other, so there is no preferred one. We show that our $M(V)$ (or, rather, the dual space $D(V)$ formed by distributions) contains naturally all such vacuum modules.

(0.4) It is perhaps worth emphasizing the difference of our approach with the more traditional one of probability theory and cylindric measures, cf. e.g., [K]. Namely, the latter amounts to viewing an infinite-dimensional (say linear) space V as a pro-object, a certain completion of a projective limit of finite-dimensional spaces W_i. The approach of cylindric measures is (in modern language) an elaboration of the naive idea of formally putting

$$\mathrm{Meas}\,(\varprojlim W_i) = \varprojlim \mathrm{Meas}\,(W_i),$$

where Meas stands for the space of measures. What we do is similar, except that we consider our spaces not as pro-objects but rather as ind-pro-objects which is the natural structure present on local fields.

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§1. Locally linearly compact spaces

(1.1) Definitions. We recall some classical concepts, due to Lefschetz and Chevalley [L]. Let k be a field, considered with discrete topology. A topological k-vector space is called *linearly topological* if it has a basis of neighborhoods of 0 formed by linear subspaces. All linearly topological vector spaces will be assumed Hausdorff. A linearly topological vector space V is called *linearly compact*, if any family $A_i \subset V$, $i \in I$ of closed affine subspaces such that $\bigcap_{i\in J} A_i \neq \emptyset$ for any finite $J \subset I$, has $\bigcap_{i\in I} A_i \neq \emptyset$. More generally, one says that V is *locally linearly compact*, if it has a basis of neighborhoods of 0 formed by linearly compact open (automatically closed) subspaces. We will drop "linearly topological" when speaking about (locally) linearly compact spaces.

(1.1.1) Examples. (a) Any finite-dimensional k-vector space with discrete topology is linearly compact. The space $k[[t]]$ of formal Taylor series with the t-adic topology is linearly compact. In fact, any product of linearly compact spaces is linearly compact.

(b) The space $k((t))$ of formal Laurent series is locally linearly compact.

We introduce the following categories:

$\mathrm{Vect}_0(k)$: finite-dimensional k-vector spaces.

$\mathrm{Vect}(k)$: all k-vector spaces (considered with discrete topology)

$\widehat{\mathrm{Vect}}(k)$: linearly compact k-spaces.

$\mathcal{L}(k)$: locally linearly compact spaces.

Clearly, $\mathrm{Vect}(k)$, $\widehat{\mathrm{Vect}}(k)$ are full subcategories of $\mathcal{L}(k)$ with intersection $\mathrm{Vect}_0(k)$.

(1.1.2) Remark. From the modern point of view, all the above categories can be obtained from $\mathrm{Vect}_0(k)$ by the purely algebraic constructions of passing to the categories of ind- and pro-objects, see, e.g., [AM]. More precisely, in the notations of *loc. cit.* $\mathrm{Vect}(k) = \mathrm{Ind}(\mathrm{Vect}_0(k))$ and $\widehat{\mathrm{Vect}}(k) = \mathrm{Pro}(\mathrm{Vect}_0(k))$. An embedding in $\widehat{\mathrm{Vect}}(k)$ is open iff it is induced (in the sense of forming a Cartesian square) from an embedding in $\mathrm{Vect}_0(k)$. Finally, $\mathcal{L}(k)$ is identified with the full subcategory in $\mathrm{Ind}(\mathrm{Pro}(\mathrm{Vect}_0(k)))$ formed by inductive systems over $\mathrm{Pro}(\mathrm{Vect}_0(k))$ consisting of open (in the above sense) embeddings. General definitions of locally compact ind-pro-objects (with $\mathrm{Vect}_0(k)$ replaced by more general categories) were proposed by Beilinson [Be] and earlier by Kato [Ka]. We tried to keep this paper as short and elementary as possible and thus avoided any systematic use of ind/pro-objects.

(1.2) Dim and Det. We start by reformulating, in the form needed for us, the basic constructions of [ADK]. Let V be a locally linearly compact k-vector space. We denote by $G(V)$ and call the *semiinfinite Grassmannian of* V, the set of open linearly compact subspaces in V. If $U_1, U_2 \in G(V)$ and $U_1 \subset U_2$, then (U_2/U_1) is finite-dimensional. Moreover, for arbitrary $U_1, U_2 \in G(V)$ both $U_1 \cap U_2$ and $U_1 + U_2$ are in $G(V)$.

By a *dimension theory* on V we mean a map $d : G(V) \to \mathbb{Z}$ such that, whenever $U_1, U_2 \in G(V)$ and $U_1 \subset U_2$, we have $d(U_2) = d(U_1) + \dim(U_2/U_1)$. The set of dimension theories will be denoted $\mathrm{Dim}(V)$. The group $\mathbb{Z}$ acts on $\mathrm{Dim}(V)$ by adding constant functions. It is clear from the above that this makes $\mathrm{Dim}(V)$ into a $\mathbb{Z}$-torsor.

For $W \in \mathrm{Vect}_0(k)$ let $\det(W)$ be the top exterior power of W. Recall that for any short exact sequence

$$0 \to W' \to W \to W'' \to 0$$

in $\mathrm{Vect}_0(k)$ we have a natural identification

$$m_{W'WW''} : \det(W') \otimes \det(W'') \to \det(W), \tag{1.2.1}$$

and these identifications are associative for any filtration $W_1 \subset W_2 \subset W$ of length 2.

(1.2.2) Definition. *Let V be a locally linearly compact k-vector space. A determinantal theory on V is a rule Δ which associates to each $U \in G(V)$ a 1-dimensional k-vector space $\Delta(U)$, to each embedded pair $U_1 \subset U_2$, $U_i \in G(V)$, an isomorphism*

$$\Delta_{U_1U_2} : \Delta(U_1) \otimes \det(U_2/U_1) \to \Delta(U_2)$$

so that for any nested triple $U_1 \subset U_2 \subset U_3$ the obvious diagram

$$\begin{array}{ccc}
\Delta(U_1) \otimes \det(U_2/U_1) \otimes \det(U_3/U_3) & \longrightarrow & \Delta(U_1) \otimes \det(U_3/U_1) \\
\downarrow & & \downarrow \\
\Delta(U_2) \otimes \det(U_3/U_2) & \longrightarrow & \Delta(U_3)
\end{array}$$

is commutative.

We denote by $\mathrm{Det}(V)$ the category (groupoid) formed by all determinantal theories on V and their isomorphisms (in the obvious sense). If $\phi : \Delta \to \Delta'$ is an isomorphism of determinantal theories and $\lambda \in k^*$, then $\lambda\phi$ is also an isomorphism. One easily sees that:

(1.2.3) Proposition. *The above action of k^* on the morphisms makes* $\mathrm{Det}(V)$ *into a k^*-gerbe [Bre] [Bry], i.e., each* $\mathrm{Hom}_{\mathrm{Det}(V)}(\Delta, \Delta')$ *becomes a k^*-torsor and the composition is bilinear.*

(1.2.4) Example. To compare our approach with that of [ADK], note that any $U \in G(V)$ defines a unique dimension theory d_U such that $d_U(U) = 0$. The difference (in the $\mathbb{Z}$-torsor $\mathrm{Dim}(V)$) of two such elements $d_U, d_{U'}$ is the integer denoted in [ADK] by $[U' : U]$ (the relative dimension of U and U').

Similarly, a choice of U gives rise to a canonical determinantal theory Δ_U such that $\Delta_U(U) = k$ and the values of Δ_U on other elements of $G(V)$ is recovered uniquely (that is, up to unique isomorphism) from the axioms of Definition 1.2.2. The Hom-torsor (in the k^*-gerbe $\mathrm{Det}(V)$) between two such theories $\Delta_U, \Delta_{U'}$ is the k^*-torsor corresponding to the 1-dimensional vector space denoted in [ADK] by $[U'|U]$ (the relative determinant of U' and U).

(1.3) The group $GL(\infty)$. Let V be a locally linearly compact k-vector space. We write $GL(V)$ for $\mathrm{Aut}_{\mathcal{L}(k)}(V)$, i.e., for the group of continuous automorphisms of V. When $V = k((t))$, this is the so-called "Japanese group $GL(\infty)$".

The group $GL(V)$ acts on the k^*-gerbe $\mathrm{Det}(V)$ and any object $\Delta \in \mathrm{Det}(V)$ gives, in a standard way, a central extension

$$1 \to k^* \to \widetilde{GL}_\Delta(V) \to GL(V) \to 1. \tag{1.3.1}$$

More precisely, see, e.g., [Bry], an element of $\widetilde{GL}_\Delta(V)$ is a pair (g, ϕ) where $g \in GL(V)$ and $\phi : \Delta \to g(\Delta)$ is an isomorphism. The composition is $(g_1, \phi_1)(g_2, \phi_2) = (g_1, g_2, g_2(\phi_1) \circ \phi_2)$. Let us recall its standard properties.

(1.3.2) Proposition. *(a) The extension (1.3.1) splits iff Δ can be made into a $GL(V)$-equivariant object of* $\mathrm{Det}(V)$.

(b) If we consider the action of $\widetilde{GL}_\Delta(V)$ on $\mathrm{Det}(V)$ *via the projection onto $GL(V)$, then Δ canonically has the structure of a $\widetilde{GL}_\Delta(V)$-equivariant object.*

(c) If Δ, Δ' are two determinantal theories, then $\mathrm{Hom}_{\mathrm{Det}(V)}(\Delta, \Delta')$ *is identified with* $\mathrm{Hom}(\widetilde{GL}_\Delta(V), \widetilde{GL}_{\Delta'}(V)0$ *in the category of central extensions. In parituclar, all the $\widetilde{GL}_\Delta(V)$ are isomorhic to each other.*

(1.3.3) Examples. (a) A choice of a reference subspace $U \in G(V)$ produces an object $\Delta_U \in \mathrm{Det}(V)$ and hence a particular central extension.

(b) If $V = k((t))$, $U = k[[t]]$, then $\widetilde{GL}_{\Delta_U}(V)$ is the standard central extension of the Japanese group [ADK].

(c) If V is discrete, than $0 \in G(V)$ so Δ_0 is an equivariant object of $\mathrm{Det}(V)$ and the extension splits. Similarly, if V is linearly compact, then $V \in G(V)$ and the extension splits.

(1.4) Further properties of Dim and Det. The category $\mathcal{L}(k)$ possesses the following extra structures:

(1.4.1) Duality, which is an antiequivalence $V \mapsto V^\vee = \mathrm{Hom}(V, k)$ (the space of continuous functionals). The space $V^\vee$ is again locally linearly compact and $G(V)$ and $G(V^\vee)$ are in order-reversing bijection $U \mapsto U^\perp$ (the orthogonal complement).

(1.4.2) The structure of an exact category in the sense of Quillen [Q], i.e., a class of admissible short exact sequences

$$0 \to V' \xrightarrow{\alpha} V \xrightarrow{\beta} V'' \to 0. \tag{1.4.3}$$

More precisely, one calls a sequence (1.4.3) admissible, if it is exact as a sequence of algebraic vector spaces, if V' is closed in V and the topology on V'' is the quotient one.

These structures are compatible in the obvious sense: the dual of an admissible short exact sequence is again admissible. As in [Q], we will also speak about admissible filtrations etc.

On the other hand, for an abelian group A the category of A-torsors is a symmetric monoidal category with duality, see [Bre]. The monoidal operation (tensor product of torsors over A) will be denoted $\otimes$ for $A = k^*$ and $\odot$ for $A = \mathbb{Z}$. The dual of a torsor T will be denoted $T^\vee = \mathrm{Hom}(T, A)$. For $t \in T$ let $t^\vee \in T^\vee$ be the unique morphism taking t to the unit element of A.

(1.4.4) Proposition. *(a) For $V \in \mathcal{L}(k)$ we have a canonical identification of $\mathbb{Z}$-torsors $\mathrm{Dim}(V^\vee) = \mathrm{Dim}(V)^\vee$.*

(b) For each admissible short exact sequence (1.4.3) we have a natural identification of $\mathbb{Z}$-torsors

$$\mathrm{Dim}(V') \odot \mathrm{Dim}(V'') \to \mathrm{Dim}(V)$$

and these identifications are associative in any admissible filtration of length 2.

Proof: (a) For a dimension theory d on V we have a dimension theory $d^\vee$ on $V^\vee$ given by $d^\vee(U) = -d(U^\perp)$, $U \in G(V^\vee)$.

(b) Given dimension theories d' on V' and d'' on V'', we have a dimension theory d on V given by

$$d(U) = d'(\alpha^{-1}(U)) + d''(\beta(U)).$$

We leave the details to the reader.

For any k^*-gerbes $\mathfrak{G}', \mathfrak{G}''$ we denote $\mathfrak{G}' \boxtimes \mathfrak{G}''$ the gerbe whose class of objects is $\mathrm{Ob}(\mathfrak{G}') \times \mathrm{Ob}(\mathfrak{G}'')$ and

$$\mathrm{Hom}_{\mathfrak{G}' \boxtimes \mathfrak{G}''}((x', x''), (y', y'')) = \mathrm{Hom}_{\mathfrak{G}'}(x', y') \otimes \mathrm{Hom}_{\mathfrak{G}''}(x'', y'').$$

We also denote by $\mathfrak{G}^{op}$ the gerbe opposite to $\mathfrak{G}$. We think of it as having for objects formal symbols $x^\vee, x \in \mathrm{Ob}(\mathfrak{G})$ with $\mathrm{Hom}_{\mathfrak{G}^{op}}(x^\vee, y^\vee) = \mathrm{Hom}_{\mathfrak{G}}(y, x)$.

(1.4.5) Proposition. *(a) For $V \in \mathcal{L}(k)$ we have a canonical equivalence of k^*-gerbes*

$$\mathrm{Det}(V^\vee) \sim \mathrm{Det}(V)^{op}.$$

(b) for an admissible short exact sequence (1.4.3) we have a natural equivalence of k^*-gerbes

$$\delta_{V'VV''} : \mathrm{Det}(V') \boxtimes \mathrm{Det}(V'') \to \mathrm{Det}(V)$$

and natural transformation of "associativity" for these equivalences for any admissible filtration of $V_1 \subset V_2 \subset V$ of length 2:

$$\begin{array}{ccc} \mathrm{Det}(V_1) \boxtimes \mathrm{Det}(V_2/V_1) \boxtimes \mathrm{Det}(V/V_2) & \longrightarrow & \mathrm{Det}(V_1) \otimes \mathrm{Det}(V/V_1) \\ \downarrow & \swarrow \varepsilon_{V_1 \subset V_2 \subset V} & \downarrow \\ \mathrm{Det}(V_2) \boxtimes \mathrm{Det}(V/V_2) & \longrightarrow & \mathrm{Det}(V) \end{array}$$

The transformations $\varepsilon_{V_1 \subset V_2 \subset V}$ fit into a commutative cube for any admissible length 3 filtration.

Proof: (a) For a determinantal theory Δ on V we have a determinantal theory $\Delta^\vee$ on $V^\vee$ given by $\Delta^\vee(U) = \Delta(U^\perp)^*$, $U \in G(V^*)$.

(b) (Sketch) Given determinantal theories Δ' on V' and Δ'' on V'', we have a determinantal theory $\Delta = \delta_{V'VV''}(\Delta', \Delta'')$ on V defined by

$$\Delta(U) = \Delta'(\alpha^{-1}(U)) \otimes \Delta''(\beta(U)).$$

The (somewhat lengthy) checking of details as well as the construction of the $\varepsilon_{V_1 \subset V_2 \subset V}$ and verification of their properties, are left to the reader.

§2. Measures on locally linearly compact spaces

(2.0) Orientation issues. From now on we take $k = \mathbb{R}$ and write Vect_0 for $\mathrm{Vect}_0(\mathbb{R})$, as well as $\mathcal{L}$ for $\mathcal{L}(\mathbb{R})$ etc. Recall that 1-dimensional $\mathbb{R}$-vector spaces are essentially the same as $\mathbb{R}^*$-torsors. If L is such a space, we denote by $|L|$ the 1-dimensional $\mathbb{R}$-vector space whose corresponding torsor is induced from that of L by the homomorphism

$$\mathbb{R}^* \to \mathbb{R}^*, \quad x \mapsto |x|.$$

Alternatively, $|L|$ can be identified with the space of functions $s : L - \{0\} \to \mathbb{R}$ such that $s(\lambda x) = |\lambda|^{-1} s(x)$ for any $\lambda \in \mathbb{R}^*$.

Let $W \in \mathrm{Vect}_0$. Note that the space $|\det(W)^*|$ is canonically identified with the space of Haar measures on W. Further, let $\mathrm{OR}(W)$ be the $\{\pm 1\}$-torsor of orientations of W. Its two elements can be viewed as the two connected components of the space $\Lambda^{\dim(W)}(W) - \{0\}$. Alternatively, they can be viewed as the connected components of the space of all bases of W. Any $\{\pm 1\}$-torsor O gives rise to a 1-dimensional $\mathbb{R}$-vector space $O_{\mathbb{R}}$ via the canonical embedding $\{\pm 1\} \subset \mathbb{R}^*$. Explicitly, $O_{\mathbb{R}}$ can be viewed as consisting of odd (i.e., $\{\pm 1\}$-equivariant) functions $s : O \to \mathbb{R}$.

For any $W \in \mathrm{Vect}_0$ we have a canonical identification

$$|\det(W)^*| \simeq \det(W)^* \otimes \mathrm{OR}(W)_{\mathbb{R}}$$

which expresses the fact that a volume form in the presense of an orientation gives a measure.

(2.1) The finite-dimensional case. Let $W \in \mathrm{Vect}_0$ be a finite-dimensional $\mathbb{R}$-vector space. We introduce the following function spaces:

$S(W)$: the Schwartz space of smooth rapidly decreasing functions $W \to \mathbb{R}$, see [H].

$D(W)$: the topological dual of $S(W)$, i.e., the space of Schwartz distributions.

$M(W) = S(W) \otimes |\det(W)^*|$: the space of smooth rapidly decreasing measures on W.

If $\beta : W \to W''$ is a surjection in Vect_0, we have the direct image map (integration along the fibers):

$$\beta_* : M(W) \to M(W''). \tag{2.1.1}$$

If $\alpha : W' \to W$ is an injection in Vect_0, then the restriction of functions induces a map

$$\alpha^* : M(W) \to M(W') \otimes |\det(\operatorname{Coker}(\alpha))^*|. \tag{2.1.2}$$

The following is then straightforward.

(2.1.3) Proposition. *(a) For two composable surjections β_1, β_2 we have $(\beta_1\beta_2)_* = \beta_{1*}\beta_{2*}$.*
(b) For two composable injections α_1, α_2 we have $(\alpha_1\alpha_2)_* = \alpha_{1*}\alpha_{2*}$, the equality understood with respect to the canonical indetification

$$\det(\mathrm{Coker}(\alpha_1\alpha_2)) \simeq \det(\mathrm{Coker}(\alpha_1)) \otimes \det(\mathrm{Coker}(\alpha_1))$$

(c) Let

$$\begin{array}{cccc} & W & \xrightarrow{\alpha_2} & W_1 & \\ \beta_2 & \downarrow & & \downarrow & \beta_1 \\ & W_2 & \xrightarrow{\alpha_1} & W_{12} & \end{array}$$

be a Cartesian square in Vect_0 with β_i being surjections and α_i injections. (Such a square is authomatically cocartesian.) Then, with respect to the identification $\mathrm{Coker}(\alpha_1) \simeq \mathrm{Coker}(\alpha_2)$, we have the equality

$$\alpha_1^*\beta_{1*} = \beta_{2*}\alpha_1^* : M(W_1) \otimes |\det(\mathrm{Coker}(\alpha_2))^*| \to M(W_2).$$

In a similar way, an injection $\alpha : W' \to W$ defines the direct image map on distributions

$$\alpha_* : D(W') \to D(W), \tag{2.1.4}$$

while a surjection $\beta : W \to W''$ gives rise to the inverse image map

$$\beta^* : D(W') \otimes |\det(\mathrm{K}er(\beta))^*| \to D(W). \tag{2.1.5}$$

These maps, being dual to (2.1.1-2), satisfy the properties similar to Propoisiton 2.1.3.

(2.2) Linearly compact case. Let now U be a linearly compact space. We define the space of measures on U as

$$M(U) = \lim_{\longleftarrow U_1 \subset U} M(U/U_1), \tag{2.2.1}$$

where U_1 runs over open subspaces of U (so that $U/U_1 \in \mathrm{Vect}_0$) and the limit is taken with respect to the maps (2.1.1) associated to the surjections $U/U_1 \to U/U_2$, $U_2 \subset U_1$. In a dual fashion, we define the space of distributions on U to be

$$D(U) = \lim_{\longrightarrow U_1 \subset U} D(U/U_1) \otimes |\det(U/U_1)|, \tag{2.2.2}$$

where limit is now taken with respect to the maps obtained by tensoring (2.1.5).

Let now $\alpha : U' \to U$ be an open embedding of linearly compact spaces. For an open $U_1 \subset U$ let $U_1' = \alpha^{-1}(U)$. Then we have an embedding $\alpha_1 : U'/U_1' \to U/U_1$ of finite-dimensional spaces. If, further, $U_1 \subset U_2 \subset U$ are open and $U_2' = \alpha^{-1}(U_2)$, then the square

$$\begin{array}{cccc} & U'/U_1' & \xrightarrow{\alpha_1} & U/U_1 \\ \beta' & \downarrow & & \downarrow\beta \\ & U'/U_2' & \xrightarrow{\alpha_2} & U/U_2 \end{array}$$

is Cartesian, so satisfies the assumptions of Proposition 2.1.3(c). We get, therefore:

(2.2.3) Proposition. *An open embedding $\alpha : U' \to U$ gives rise to the inverse image map on measures*

$$\alpha^* : M(U) \otimes |\det(\mathrm{Coker}(\alpha))^*| \to M(U')$$

and the direct image map on distributions

$$\alpha_* : D(U') \otimes |\det(\mathrm{Coker}(\alpha))^*| \to D(U),$$

and these maps are compatible with compositions.

(2.3) Locally linearly compact case. Let now V be a locally linearly compact $\mathbb{R}$-vector space and $G(V)$ the set of its open linearly compact subspaces. By a *Haar theory* on V we will understand a rule h which to each $U \in G(V)$ associates a 1-dimensional $\mathbb{R}$-vector space $h(U)$ and to each pair $U_1 \subset U_2$ an isomorphism $h(U_1) \otimes |\det(U_2/U_1)^*| \to h(U_2)$ satisfying a condition for nested triples similar to one given in Definition 1.2.2. It is clear that Haar theories form an $\mathbb{R}^*$-gerb, in particular, any such theory h gives rise to a central extension $\widetilde{GL}_h(V)$ of $GL(V)$ by $\mathbb{R}^*$. It is also clear that each determinantal theory Δ on V gives a Haar theory $|\Delta^*|$, and $\widetilde{GL}_{|\Delta|^*}(V)$ is just the extension induced from $\widetilde{GL}_\Delta(V)$ via the automorphism $x \mapsto |x|^{-1}$ of $\mathbb{R}^*$.

Fix a Haar theory h on V. We define the spaces of measures and distributions on V associated to h to be

$$M_h(V) = \lim_{\longleftarrow U \subset V} M(U) \otimes h(U)^* = \lim_{\longleftarrow U \subset V} \lim_{\longleftarrow U_1 \subset U} M(U/U_1) \otimes h(U)^*, \tag{2.3.1}$$

$$D_\Delta(V) = \lim_{\longrightarrow U \subset V} D(U) \otimes h(U) = \lim_{\longrightarrow U \subset V} \lim_{\longrightarrow U_1 \subset U} D(U/U_1) \otimes h(U). \tag{2.3.2}$$

(2.3.3) Example. If V is discrete and $\Delta = \Delta_0$ is the determinantal theory described in (1.3.3)(c), then $M_{|\Delta_0|^*}(V)$ is the space of Schwartz functions on V, i.e., the inverse limit of the Schwartz spaces on finite-dimensional subspaces of V. Similarly, if V is linearly compact and $\Delta = \Delta_V$, then $M_{|\Delta_V|^*}(V) = M(V)$ is the inverse limit of the spaces of Schwartz measures on finite-dimensional quotients of V. Thus in the general case, $M_h(V)$ is a certain mixture of the spaces of functions and measures.

By construction, $M(V)$, $D(V)$ are representations of $\widetilde{GL}_h(V)$, formally dual to each other. More precisely, we have a nondegenerate equivariant pairing

$$m, \phi \mapsto (m, \phi), \quad M(V) \otimes D(V) \to \mathbb{R}, \tag{2.3.4.}$$

The evaluation of any particular (m, ϕ) reduces to the pairing of a function and a distribution on some finite-dimensional subquotient of V.

(2.4) Heisenberg action on $M_h(V)$. Let $H(V)$ be the Heisenberg algebra generated by symbols $L_v, v \in V$, $L_f, f \in V^*$, linearly depending on v and f and subject to the relations

$$[L_f, L_v] = f(v).$$

It can be viewed as the algebra of polynomial differential operators on V.

(2.4.1) Proposition. *The space $M_h(V)$ has a natural structure of a right $H(V)$-module, and $D_h(v)$ of a left $H(V)$-module.*

Proof: Let W be a finite-dimensional $\mathbb{R}$-vector space and $H(W)$ be its Heisenberg algebra, defined as before. Then, $H(W)$ is the algebra of polynomial linear differential operators on W. As such, it acts on the left in $S(W)$ (functions) and on the right in $M(W)$ and $D(W)$ (measures and distributions). This would be the case for any smooth manifold.

Let now $v \in V$ and $U \subset V$ be any linearly compact open subspace containing v. We define the operator L_v on the space $M(U) = \varprojlim_{U' \subset U} M(U/U')$ to be induced by the morphism of projective systems which on each $M(U/U')$ is given by the right action of the constant vector field $v \bmod U'$. Then, we extend the action to $M(U) \otimes h(U)^*$ by viewing $h(U)^*$ as a "constant" vector space of multiplicities. If $v \in U \subset U_1$, then the resulting operators on $M(U) \otimes h(U)^*$ and $M(U_1) \otimes h(U_1)^*$ are compatible under the restriction map. Therefore we get an operator, still denoted L_v, on $M_h(V) = \varprojlim_U M(U) \otimes h(U)^*$. The operators L_f are defined in a similar way. Q.E.D.

For any linearly compact open subspace $U \subset V$ we denote by N_U the vacuum representation of $H(V)$ corresponding to U. By definition, it is generated by one vector $|U\rangle$ ("vacuum") subject to the relations:

$$L_v|U\rangle = 0, \quad v \in U, \qquad L_f|U\rangle = 0, \quad f \in U^\perp. \tag{2.4.2}$$

We can view it as the space of distributions on V which have support in U and which are smooth along U. For different U these modules are not isomorphic to each other. In particular, the group $\widetilde{GL}(V)$ does not act, even projectively, in any of the N_U (even though its Lie algebra does, via is embedding into a completion of $H(V)$ by vertex operators [KR]).

(2.4.3) Proposition. *N_U is naturally embeddeded into the space $D_{|\Delta_U|^*}(V)$ as an $H(V)$-submodule. Here Δ_U is the determinantal theory associated to U, see (1.2.4).*

Proof: We exhibit an element δ_U ("delta function along U") in $D_{|\Delta_U|^*}(V)$ satisfying the relations (2.4.2). Then the embedding would be uniquely determined by sending

$|U\rangle$ to δ_U. We represent $D_{|\Delta_U|^*}(V)$ as $\varinjlim_{U_1 \supset U} \varinjlim_{U_2 \subset U_1} D(U_1/U_2) \otimes |\Delta_U(U_1)^*|$. Consider the term of this double inductive system corresponding to $U_1 = U_2 = U$. This term is canonically identified with $\mathbb{R}$ and we define δ_U as the image of $1 \in \mathbb{R}$ in the inductive limit. The relations (2.4.2) are verified straightforwardly. Q.E.D.

If h is some other Haar theory, then we have not a canonical embedding of N_U into $D_h(V)$ but a canonical 1-dimensional space formed by such embeddings.

(2.5) Bilinear and quadratic forms. Let $V \in \mathcal{L}$ be a locally linearly compact $\mathbb{R}$-space. We want to compare two possible approaches to defining quadratic forms on V. First of all, by a bilinear form on V we will mean a morphism $b : V \to V^\vee$, where $V^\vee \in \mathcal{L}$ is the topological dual of V. As usual, for such a b we have the transposed form $b^t : V \to V^*$ and b is called symmetric if $b^t = b$. We denote by $B(V)$ the space of all symmetric bilinear forms on V. An element $b \in B(V)$ can be regarded as a continuous function $b : V \times V \to \mathbb{R}$ and we denote by $q_b : V \to \mathbb{R}$ the associated quadratic form $q_b(x) = b(x,x)$. We call b nondegenerate if it is an isomorphism $V \to V^\vee$. In this case we have a bilinear form b^{-1} on $V^\vee$, which is symmetric if b is. We call b positive definite, if $q_b(x) > 0$ for any $x \neq 0$. We denote by $B_{nd}(V)$ the set of all nondegenerate symmetric bilinear forms on V and by $B_+(V) \subset B_{nd}(V)$ the subset of forms which are both nondegenerate and positive definite.

Consider now the case of finite-dimensional vector spaces $W \in \mathrm{Vect}_0$. In this case the above concepts have their usual meaning. Let $\alpha : W' \to W$ be an injection in Vect_0 and $b \in B(W)$. Then we have the restriction $\alpha^* b \in B(W')$. Note that for a nondegenerate b the form $\alpha^* b$ may be degenerate, but if b is positive definite, then so is $\alpha^* b$ (and, in particular, $\alpha^* b$ is nondegenerate). Further, let $\beta : W \to W''$ be a surjection in Vect_0 and b be a *positive definite* symmetric bilinear form on W. Then we define the direct image $\beta_*(b) \in B(W'')$ as follows. Consider the injection $\beta^t : (W'')^* \to W^*$. dual to β and define

$$\beta_*(b) = \left((\beta^t)^*(b^{-1})\right)^{-1}. \tag{2.5.1}$$

Here we used the nondegeneracy of the positive definite forms b and $(\beta^t)^*(b^{-1})$. The following is then elementary.

(2.5.2) Proposition. *Let q_b be the quadratic form corresponding to b and similarly for $\beta_*(b)$. Then for $w'' \in W''$ we have*

$$q_{\beta_* b}(w'') = \max_{\beta(w) = w''} q_b(w'').$$

(2.5.3) Corollary. *(a) The operations α^* and β_* on positive definitite symmetric bilinear forms (on finite-dimensional spaces) is compatible with the composition of injections (resp. surjections).*

(b) Consider a Cartesian square in $\mathrm{Vect}_0(\mathbb{R})$ *as in (2.1.3)(c). Then, for any* $q_1 \in Q(W_1)$ *we have* $\alpha_1^*\beta_{1*}(q') = \beta_{2*}\alpha_1^*(q')$.

For $W \in \mathrm{Vect}_0$ we denote by $Q(W)$ the cone of positive definite quadratic forms on W. It is canonically identified with $B_+(W)$. So we can think of operations α_*, β_* as defined on elements of $Q(W)$.

Let now $V \in \mathcal{L}$ be a locally linearly compact $\mathbb{R}$-space. We define

$$Q(V) = \lim_{\longleftarrow U \subset V} \lim_{\longleftarrow U' \subset U} Q(U/U'), \tag{2.5.4}$$

where the limits are taken with respect to the operations α^* and β_* on quadratic forms. Elements of $Q(V)$ will be called (positive-definite) quadratic forms on V.

(2.5.5) Proposition. *The set* $Q(V)$ *is naturally identified with* $B_+(V)$.

Proof: Let b be a symmetric bilinear form on V. The topology in V is that of inductive limit of its linearly compact open subspaces $U \subset V$. Further, the basis of linearly compact neighborhoods of 0 in $V^\vee$ is formed by the orthogonals $U^\perp$ where U is a linearly compact neighborhood of 0 in V. Accordingly,

$$\mathrm{Hom}_{\mathcal{L}}(V, V^\vee) = \lim_{\longleftarrow U_1 \subset V} \lim_{\longrightarrow U_2 \subset V} \mathrm{Hom}(U_1, U_2^\perp).$$

Note that in the above we can take $U_2 \subset U_1$, and in this case a morphism from U_1 to $U_2^\perp$ gives (after restriction to U_1) a bilinear form b_{U_1/U_2} on the finite-dimensional space U_1/U_2. Assuming that b is nondegenerate and positive definite, we find that these forms on finite-dimensional subquotients are positive definite and compatible with respect to the restrictions and projections. In other words, the system of their associated quadratic forms is an element of the double inverse limit (2.5.4). The converse is similar.

(2.6) Gaussian measures. Let $W \in \mathrm{Vect}_0$. Any $q \in Q(W)$. $q \in Q(W)$ gives rise to a Haar measure $\mathrm{dVol}_q \in |\det(W^*)|$ (the measure of a q-orthocube is 1). We get therefore the *Gaussian measure*

$$\gamma_q = \frac{1}{(2\pi)^{\dim(W)/2}} e^{-q(x)/2} \mathrm{dVol}_q \in M(W). \tag{2.6.1}$$

Recall the standard properties of Gaussian integrals.

(2.6.2) Proposition. *(a) If* $\beta : W \to W''$ *is a surjection and* $q \in Q(W)$, *then* $\beta_*(\gamma_q)$ *(the direct image of measures as in (2.1.1)) is equal to* $\gamma_{\beta_*(q)}$.

(b) If $\alpha : W' \to W$ *is an injection,* $q \in Q(W)$ *and* $\mathrm{dVol}_{\alpha,q} \in |\det(\mathrm{Coker}(\alpha))^*|$ *is the Haar measure induced by* q *on the orthogonal complement of* $\mathrm{Im}(\alpha)$, *then*

$$\alpha^*(\gamma_q) = \gamma_{\alpha^* q} \otimes \mathrm{dVol}_{\alpha,q}.$$

Let now $V \in \mathcal{L}$ and $q \in Q(V)$. Note that q trivializes the space $|\det(U_2/U_1)^*|$ for any open, linearly comapct $U_2 \subset U_2 \subset V$. This is because U_2/U_1 is identified with the q-orthogonal complement of U_1 in U_2 and the latter comes equipped with the Haar measure dVol_q. If $U_1 \subset U_2 \subset U_3 \subset V$, then these trivializations are compatible with the exact sequence

$$0 \to U_2/U_1 \to U_3/U_1 \to U_3/U_2 \to 0.$$

It follows that for any two open, linearly compact $U_1, U_2 \subset V$ the Haar theories $|\Delta_{U_1}|^*, |\Delta_{U_2}|^*$ are canonically identified. Further, Proposition 2.6.2 implies the following fact.

(2.6.3) Proposition. *Fix $U \in G(V)$. Then any $q \in Q(V)$ gives rise to an element $\gamma_q \in M_{|\Delta_U|^*}(V)$ (the Gaussian measure) invariant under the orthogonal group $O(V,q)$.*

Let $W \in \mathrm{Vect}_0$ and $q \in Q(W)$. A measure $\nu \in M(W)$ will be called *almost Gaussian* (with respect to q) if it has the form $\nu = f \cdot \gamma_q$ where f is a real polynomial function on W. We denote $AG(W,q) \subset M(W)$ the space of almost Gaussian measures. It is classical ("Wick's theorem") that the class of almost Gaussian measures is preserved under the operations of direct and inverse image, i.e., we have maps

$$\beta_* : AG(W,q) \to AG(W'', \beta_* q), \quad \beta : W \twoheadrightarrow W'', \tag{2.6.4}$$

$$\alpha^* : AG(W,q) \to AG(W', \alpha^* q) \otimes |\det(\mathrm{Coker}(\alpha))^*|, \quad \alpha : W' \hookrightarrow W. \tag{2.6.5}$$

(2.6.6) Definition. *Let $V \in \mathcal{L}$ and $q \in Q(V)$. We define $AG(V,q)$, the space of almost Gaussian measures on V (with respect to q) to be*

$$AG(V,q) = \lim_{\longleftarrow U \subset V} \lim_{\longleftarrow U' \subset U} AG(U/U', q_{U/U'}),$$

where $q_{U/U'}$ is the quadratic form induced by q on U/U', see (2.5.4) and the limits are taken with respect to the maps (2.6.4-5).

By construction, $AG(V,q)$ is a representation of the group $O(q)$. It can be regarded as a kind of "algebraic semiinfinite symmetric power" of V.

(2.7) The Fourier transform. Let $W \in \mathrm{Vect}_0$. We denote by $S_{\mathbb{C}}(W)$, $M_{\mathbb{C}}(W)$ the complexifications of the spaces of Schwartz functions and measures on W. Then, we have the Fourier transform which is an isomorphism

$$\mathcal{F}_W : M_{\mathbb{C}}(W) \to S_{\mathbb{C}}(W^*), \quad \mathcal{F}(\mu)(y) = \int_{x \in W} e^{i(x,y)} d\mu. \tag{2.7.1}$$

Let now $V \in \mathcal{L}$ and Δ be a determinantal theory on V. Let also $h^\vee$ be the dual determinantal theory on $V^\vee$, defined as in (1.4.5). Then, the maps (2.7.1) for finite-dimensional subquotients of V are assembled together into an isomorphism

$$\mathcal{F} = \mathcal{F}_V : M_h(V) \otimes \mathbb{C} \to M_{h^\vee}(V^\vee) \otimes \mathbb{C} \tag{2.7.2}$$

which we will also call the Fourier transform. The following is an immediate consequence of the standard properties of the Fourier transform.

(2.7.3) Proposition. *(a) The composition $\mathcal{F}_{V^\vee} \circ \mathcal{F}_V$ is equal to the automorphism of $M_h(V)$ induced by the action of the element $(-1) \in GL(V)$.*

(b) If $q \in Q(V)$ and $q^{-1} \in Q(V^\vee)$ corresponds to the inverse bilinear form, then $\mathcal{F}_V(\gamma_q) = \gamma_{q^{-1}}$.

(c) In the situation of (b), the Fourier transform takes the space $AG(V, q) \otimes \mathbb{C}$ into $AG(V^\vee, q^{-1}) \otimes \mathbb{C}$.

§3. Forms on locally linearly compact spaces.

(3.1) The finite-dimensional case. Let $W \in \mathrm{Vect}_0$ be a finite-dimensional $\mathbb{R}$-vector space. Let $\Omega^\bullet(W)$ be the de Rham complex of differential forms on W whose components (with respect to some linear coordinate system) are Schwartz functions. We denote

$$\widetilde{\Omega}^\bullet(W) = \Omega^\bullet(W) \otimes_{\mathbb{R}} \mathrm{OR}(W)_{\mathbb{R}}. \tag{3.1.1}$$

Then for each surjection $\beta : W \to W''$ in Vect_0 we have the direct image map (integration of forms along the fibers in the presense of orientation)

$$\beta_* : \widetilde{\Omega}^\bullet(W) \to \widetilde{\Omega}^\bullet(W'')[d], \quad d = \dim(\mathrm{Ker}(\beta)). \tag{3.1.2}$$

Here $[d]$ means the shift of grading by d. Similarly, for an injection $\alpha : W' \to W$ we have the restriction map

$$\alpha^* : \widetilde{\Omega}^\bullet(W) \to \widetilde{\Omega}^\bullet(W) \otimes \mathrm{OR}(\mathrm{Coker}(\alpha))_{\mathbb{R}}. \tag{3.1.3}$$

(3.1.4) Remark. If W is a finite-dimensional $\mathbb{C}$-vector space (considered as an $\mathbb{R}$-vector space) then $\mathrm{OR}(W)$ is canonically trivialized. Accordingly, for surjections or injections in $\mathrm{Vect}_0(\mathbb{C})$ we have functorialities β_*, α^* sithout any twist (but with a shift of grading for β_*).

(3.1.5) Proposition. *(a) Each β_* and α^* is a morphism of complexes.*

(b) The direct and inverse images of forms commute with compositions of surjections (resp. injections).

(c) An analog of Proposition 2.1.3(c) holds.

(3.2) The locally linearly compact case. Let now $V \in \mathcal{L}$ and $G(V)$, as in (1.2), denote the set of open linearly compact subspaces in V. We call an *orientation theory* on V a rule O which to any $U \in G(V)$ associates a $\{\pm 1\}$-torsor $O(U)$ and to any pair $U_1 \subset U_2$ an isomorphism $U(U_1) \otimes \mathrm{OR}(U_2/U_1) \to O(U_2)$ so that for any nested triple $U_1 \subset U_2 \subset U_3$ the diagram analogous to (1.2.2) commutes.

Every determinantal theory Δ on V defines an orientation theory $\mathrm{OR}(\Delta)$ via the change of structure groups given by $\mathrm{sgn} : \mathbb{R}^* \to \{\pm 1\}$.

(3.2.1) Example. Let $V \in \mathcal{L}(\mathbb{C})$ be a locally linearly compact $\mathbb{C}$-space and $V_{\mathbb{R}} \in \mathcal{L}$ be V considered as an $\mathbb{R}$-space. Then any $U \in G(V_{\mathbb{R}})$ contains a $\mathbb{C}$-subspace $U' \in G(V)$ so that $\dim_{\mathbb{R}}(U/U') < \infty$. We set $C(U) = \mathrm{OR}(U/U')$. A different'choice of U' leads to a canonically isomorphic $\{\pm 1\}$-torsor because a finite-dimensional $\mathbb{C}$-space has a canonical orientation. It is clear that $C : U \mapsto C(U)$ is an orientation theory on $V_{\mathbb{R}}$.

(3.2.2) Definition. *Let $V \in \mathcal{L}$ and O be an orientation theory on V. The (semi-infinite) de Rham complex of V associated to O is*

$$\Omega^\bullet_O(V) = \varprojlim_{U\subset V} \varprojlim_{U_1\subset U} \widetilde{\Omega}^\bullet(U/U_1) \otimes \mathrm{OR}(U/U_1)_{\mathbb{R}}[\dim(U/U_1)].$$

It is graded by the $\mathbb{Z}$-torsor $\mathrm{Dim}(V)$.

If $V \in \mathcal{L}(\mathbb{C})$ and $O = C$ is the canonical orientation theory on $V_{\mathbb{R}}$ then we can write the above more simply and denote it by

$$\Omega^\bullet(V) = \Omega^\bullet_C(V_{\mathbb{R}}) = \varprojlim_{U\subset V} \varprojlim_{U_1\subset U} \Omega^\bullet((U/U_1)_{\mathbb{R}})[2\dim_{\mathbb{C}}(U/U_1)], \tag{3.2.3}$$

where U, U_1 now run over open, linearly compact $\mathbb{C}$-subspaces of V. This complex can be seen as an analog of the chiral de Rham complex of Malikov, Schechtman and Vaintrob [MSV].

§4. Measures on locally linearly compact superspaces and complexes

(4.1) Superspaces and Berezin integration. Let k be a field. By $\mathrm{SVect}_0(k)$ we denote the category of finite-dimensional k-supervector spaces [M]. Thus an object of $\mathrm{SVect}_0(k)$ is a finite-dimensional k-vector space W together with a $\mathbb{Z}/2$-grading $W = W^{\overline{0}} \oplus W^{\overline{1}}$; here we denote elements of $\mathbb{Z}/2$ by $\overline{0}, \overline{1}$. By Π we denote the functor of shift of $\mathbb{Z}/2$-grading; $(\Pi W)^{\overline{i}} = W^{\overline{i+1}}$.

Similarly, we denote by $S\mathcal{L}(k)$ the category of locally linearly compact k-supervector spaces $V = V^{\overline{0}} \oplus V^{\overline{1}}$. It can be identified with the category of locally compact ind-pro-objects (in the sense of [Be] [K]) of $\mathrm{SVect}_0(k)$.

From now on we take $k = \mathbb{R}$ and write $\mathrm{SVect}_0, S\mathcal{L}$ etc. Let $W \in \mathrm{SVect}_0$. According to the general principles of super-analysis [M] we define $S(W)$, the Schwartz space of W as

$$S(W) = S(W^{\overline{0}}) \otimes_{\mathbb{R}} \Lambda(W^{\overline{1}*}), \tag{4.1.1}$$

where $S(W^{\overline{0}})$ is the usual Schwartz space and $\Lambda(W^{\overline{1}*})$ is the exterior algebra. similarly, the space of distributions on W is defined as

$$D(W) = D(W^{\overline{0}}) \otimes \Lambda(W^{\overline{1}}). \tag{4.1.2}$$

The 1-dimensional $\mathbb{R}$-space of Haar measures on W is defined to be

$$\mu(W) = |\det(W^{\overline{0}})^*| \otimes \det(W^{\overline{1}}). \tag{4.1.3}$$

This is justified by the existence of the integration map

$$\int_W : S(W) \otimes \mu(W) \to \mathbb{R}, \tag{4.1.4}$$

which on $S(W^{\overline{0}}) \otimes |\det(W^{\overline{0}})^*|$ is the usual integration over $W^{\overline{0}}$ and on $\Lambda(W^{\overline{1}*}) \otimes \det(W^{\overline{1}})$ is the "Berezin integration" [M] which is, algebraically, just the projection

$$\Lambda(W^{\overline{1}*}) \otimes \det(W^{\overline{1}}) \to \Lambda^{\max}(W^{1*} \otimes \Lambda^{\max}(W^{\overline{1}}) \overset{\sim}{\to} \mathbb{R}. \tag{4.1.5}$$

We denote $M(W) = S(W) \otimes \mu(W)$ can call it, as in (2.1) the space of (smooth, rapidly decreasing) measures on W. Generalizing (4.1.4), for any surjection $\beta : W \to W''$ in SVect_0 we have the direct image map

$$\beta_* : M(W) \to M(W''), \tag{4.1.6}$$

and for any injection $\alpha : W' \to W$ the inverse image map

$$\alpha^* : M(W) \to M(W') \otimes \mu(\mathrm{Coker}(\alpha)), \tag{4.1.7}$$

which satisfy the properties similar to those listed in Proposition 2.1.3.

(4.2) Measure on locally linearly compact superspaces. Let now $V = V^{\overline{0}} \oplus V^{\overline{1}} \in S\mathcal{L}$. We denote by $G(V)$ the set of open linearly compact sub-superspaces $U \subset V$. As every such U is a direct sum $U = U^{\overline{0}} \oplus U^{\overline{1}}$, we have that $G(V) = G(V^{\overline{0}}) \times G(V^{\overline{1}})$. The concept of a Haar theory from (2.3) generalizes trivially to the case of a superspace such as V. Namely, a Haar theory on V is a rule h which associates to any $U \in G(V)$ a 1-dimensional $\mathbb{R}$-vector space $h(U)$ and to any pair $U_1 \subset U_2$ an isomorphism $h(U_1) \otimes \mu(U_2/U_1) \to h(U_2)$ such that the properties of Definition 1.2.2 hold.

Given a Haar theory h on V, we define the spaces of measures and distributions on V, similarly to (2.3), as

$$M_h(V) = \lim_{\longleftarrow U \subset V} M(U) \otimes h(U)^* = \lim_{\longleftarrow U \subset V} \lim_{\longleftarrow U_1 \subset U} M(U/U_1) \otimes h(U)^*, \tag{4.2.1}$$

$$D_\Delta(V) = \lim_{\longrightarrow U \subset V} D(U) \otimes h(U) = \lim_{\longrightarrow U \subset V} \lim_{\longrightarrow U_1 \subset U} D(U/U_1) \otimes h(U), \tag{4.2.2}$$

where the limits are taken with respect to the maps (4.1.6-7) and similar maps on distributions.

(4.2.3) Example. Let $V = V^{\overline{1}}$ be a purely odd superspace and $\overline{V} = \Pi V$ be the same space but considered as an even one. Then a Haar theory h on V is the same as a determinantal theory Δ on $\overline{V}$. It follows that the space of h-distributions on V has the form

$$D_h(V) = \lim_{\longrightarrow U \subset \overline{V}} \lim_{\longrightarrow U_1 \subset U} \Lambda(U/U_1) \otimes \Delta(U) =: \Lambda_\Delta^{\frac{\infty}{2}+\bullet}(\overline{V}).$$

In other words, this is the semiinfinite exterior power of $\overline{V}$ corresponding to the determinantal theory Δ, see [ADK] [KP] [S]. The space $M_h(V)$ is just the dual of $\Lambda_\Delta^{\frac{\infty}{2}+\bullet}(\overline{V})$ (being a double projective limit). Note that $\Lambda_\Delta^{\frac{\infty}{2}+\bullet}(\overline{V})$ is graded by the $\mathbb{Z}$-torsor $\mathrm{Dim}(\overline{V})$.

Therefore it is natural to think about $D_h(V)$ (even for a purely even V) as a "semiinfinite symmetric power" of V.

(4.3) Measures on complexes. Let $V^\bullet$ be a bounded admissible complex over the exact category $\mathcal{L} = \mathcal{L}(\mathbb{R})$. By assembling together terms of the same parity, we associate to $V^\bullet$ a superspace $V_{\mathrm{sup}} \in S\mathcal{L}$ with

$$V_{\mathrm{sup}}^{\overline{i}} = \bigoplus_{j \equiv i\ (2)} V^j, \quad \overline{I} \in \mathbb{Z}/2. \tag{4.3.1}$$

A Haar theory on $V^\bullet$ is, by definition, a Haar theory on $V_{\rm sup}$. For example, given determinantal theories Δ_i on V^i for each i, we have a Haar theory

$$h = \Big(\bigotimes_{i\equiv 0\,(2)} |\Delta_i|^*\Big) \otimes \Big(\bigotimes_{i\equiv 1\,(2)} \Delta_i\Big) \tag{4.3.2}$$

on V.

Let h be a Haar theory on $V^\bullet$. We define the spaces of measures and distributions on $V^\bullet$ to be those on $V_{\rm sup}$. These spaces are naturally graded by the $\mathbb{Z}/2$-torsor

$$\mathrm{Dim}(V^\bullet)_{\rm sup} = \bigotimes_{i\equiv 1\,(2)} \mathrm{Dim}(V^i)/2\mathbb{Z}. \tag{4.3.3}$$

(4.3.4) Proposition. *The differential d_V of $V^\bullet$ induces a natural differential d in $M_h(V^\bullet)$, of degree $\overline{1}$, satisfying $d^2 = 0$.*

Proof: This can be seen as an instance of the naturality of $M_h(V^\bullet) = M_h(V_{\rm sup})$ with respect to the super-group $\widetilde{GL}_h(V_{\rm sup})$, if we regard d_V as an odd element of the Lie algebra of this group. More precisely, consider the commutative $\mathbb{R}$-superalgebra $\Lambda[\epsilon]$ generated by one odd generator ϵ (with square 0). Then $V_{\rm sup}[\epsilon] = V_{\rm sup} \otimes_{\mathbb{R}} \Lambda[\epsilon]$ is a locally linearly compact $\Lambda[\epsilon]$-supermodule. we denote by $G_{\Lambda[\epsilon]}(V_{\rm sup}[\epsilon])$ the set of linearly compact sub-$\Lambda[\epsilon]$-supermodules $U \subset V_{\rm sup}[\epsilon]$ such that $V_{\rm sup}[\epsilon]/U$ is free over $\Lambda[\epsilon]$. Then, we repeat the definition of Haar theories and the construction of M_h in this setting which is possible because of the functorial nature of the Berezin integration. Our statement is obtained by considering the automorphism $1 + \epsilon d_V$ of $V_{\rm sup}[\epsilon]$ and its action on M_h.

(4.3.5) Example: semiinfinite Koszul complex. Let $V \in \mathcal{L}$ and consider the complex $CV = \{V \stackrel{\rm Id}{\to} V\}$ concentrated in degrees 0 and 1. The gerbe of Haar theories of CV can be written as $|\mathrm{Det}(V)| \boxtimes \mathrm{Det}(V)^{op}$. This is equivalent to the gerbe of orientation theories on V. Let such an orientation theory O be chosen and h be the corresponding Haar theory on CV. The complex $M_h(CV)$ can be called the *semiinfinite Koszul complex* of V. It is similar (but not identical) to the semiinfinite de Rham complex $\Omega^\bullet_O(V)$.

References

[ADK] E. Arbarello, C. De Concini, V.G. Kac, The infinite wedge representation and the reciprocity law for algebraic curves, in:*Proc. Symp. Pure Math.* **49** (1989), part 1, p. 171-190.

[AM] M. Artin, B. Mazur, Etale Homotopy, Lecture Notes in Math. **100**, Springer-Verlag, 1969.

[Be] A. A. Beilinson, On the derived category of perverse sheaves, in: "K-theory, Arithmetic and Geometry" (Y.I. Manin Ed.) (Lect. Notes in Math. **1289**), p. 27-41, Springer-Verlag, 1987.

[Bre] L. Breen, On the Classification of 2-gerbes and 2-stacks, *Astérisque* **225**, Soc. Math. France, 1994.

[Bry] J.-L. Brylinski, Loop Spaces, Characteristic Classes and Geometric Quantization (Progress in Math. **107**), Birkhäuser, Boston, 1993.

[H] L. Hörmander, The Analysis of Linear Partial Differential Operators I. Distribution Theory and Fourier Analysis (Grund. math. Wiss. **256**) Springer-Verlag, 1990.

[Ka] K. Kato, The existence theorem for higher local class field theory, math.AG/0012150

[KP] V.G. Kac, D. Peterson, Spin and wedge representations of infinite-dimensional Lie algebras and groups, *Proc. Nat. Acad. USA*, **78** (1981), 3308-3312.

[KR] V.G. Kac, A.K. Raina, Bombay Lectures on highest-weight representations of infinite-dimensional Lie algebras, World Scientific Publ. Singapore, 1987.

[Ku] H.-H. Kuo, Gaussian Measures in Banach Spaces, Lect. Notes in Math. **463**, Springer-Verlag, 1975.

[KV] M. Kapranov, E. Vasserot, Vertex algebras and the formal loop space, in preparation.

[L] S. Lefschetz, Algebraic Topology, (AMS Colloquium Publications **27**), Amer. Math. Soc. New York, 1942.

[MSV] F. Malikov, V. Schechtman, A. Vaintrob, Chiral de Rham complex, *Comm. Math. Phys.* **204** (1999), 439-473.

[M] Y.I. Manin, Gauge Fields and Complex Geometry, Springer-Verlag, 1985.

[P1] A.N. Parshin, On the arithmetic of 2-dimensional schemes I. Repartitions and residues, *Russian math. Izv.* **40** (1976), 736-773.

[P2] A.N. Parshin, Lectures at Université Paris-Nord, June 1996.

[P3] A.N. Parshin, Higher-dimensional local fields and L-functions, math.AG/0012151.

[Q] D. Quillen, Algebraic K-theory I, in: Lecture Notes in Math. **341**, p. 85-147.

[S] M. Sato, The KP hierarchy and infinite-dimensional Grassmann manifolds, in:*Proc. Symp. Pure Math.* **49** (1989), part 1, p. 51-66.