On the Scattering Length Spectrum for Real Analytic Obstacles

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1 Introduction

Let K be an *obstacle* in $\mathbb{R}^n$ $(n \geq 3)$, i.e. a compact subset of $\mathbb{R}^n$ with C^∞ boundary ∂K such that $\Omega_K = \overline{\mathbb{R}^n \setminus K}$ is connected. A *scattering ray* γ in Ω_K is an unbounded in both directions generalized geodesic in the sense of Melrose and Sjöstrand [MS1], [MS2] (cf. also Sect. 24.3 in [H]). We denote by T_γ the *sojourn (travelling) time* of γ (cf. Sect. 2). If $\omega \in \mathbf{S}^{n-1}$ is the incoming direction of γ and $\theta \in \mathbf{S}^{n-1}$ its outgoing direction, γ will be called an (ω, θ)*-ray*. The *scattering length spectrum* (SLS) of K is by definition the family of sets of real numbers $\mathcal{SL}_K = \{\mathcal{SL}_K(\omega,\theta)\}_{(\omega,\theta)}$ where (ω,θ) runs over $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ and $\mathcal{SL}_K(\omega,\theta)$ is the set of sojourn times T_γ of all (ω,θ)-rays γ in Ω_K. Thus, $\mathcal{SL}_K$ is a map which assigns to each pair of directions (ω,θ) a set $\mathcal{SL}_K(\omega,\theta)$ of non-negative real numbers. It is known (cf. [St2]) that for $n \geq 3$, n odd, we have $\mathcal{SL}_K(\omega,\theta) = \text{sing supp } s_K(t,\theta,\omega)$ for almost all (ω,θ). Here s_K is the scattering kernel related to the scattering operator for the wave equation in $\mathbb{R} \times \Omega_K$ with Dirichlet boundary condition on $\mathbb{R} \times \partial\Omega_K$ (cf. [LP1], [M]).

It is a natural and important problem in inverse scattering by obstacles to get information about the obstacle K from its SLS. It follows from results of A. Majda [Ma] (see also Majda and Ralston [MaR]) and P. Lax and R. Phillips [LP2] that the *convex hull* $\hat{K}$ of K can be recovered from $\mathcal{SL}_K$. Consequently, in the class of convex obstacles (as well as in the class of connected obstacles with analytic boundaries), K is completely determined by its SLS. However, as an example of M. Livshits shows (cf. Ch. 5 in [M]), in general $\mathcal{SL}_K$ does not determine K uniquely.

Following [St3], we will say that two obstacles K and L have *almost the same SLS* if there exists a subset $\mathcal{R}$ of full Lebesgue measure in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ such that $\mathcal{SL}_K(\omega,\theta) = \mathcal{SL}_L(\omega,\theta)$ for all $(\omega,\theta) \in \mathcal{R}$. Let $\text{Trap}(\Omega_K)$ be the set of *trapped points* of the *generalized geodesic flow* $\mathcal{F}_t^{(K)}$ in $T^*(\Omega_K)$ (cf. Sect. 2) and let U be an open ball in $\mathbb{R}^n$ containing both K and L. It was shown in [St3] (cf. Theorem 2.1 below) that if K and L have almost the same SLS and satisfy some non-degeneracy conditions, then the flows $\mathcal{F}_t^{(K)}$ and $\mathcal{F}_t^{(L)}$ are conjugate on their phase spaces minus the set of trapping points $\text{Trap}(\Omega_K) = \text{Trap}(\Omega_L)$ by a time preserving conjugacy which is smooth and symplectic on an open dense subset. Using this conjugacy, some relationships between K and L (under various conditions) were derived in [St3].

In this paper we consider obstacles with real analytic boundary ∂K. This assumption implies that the normal curvature of ∂K does not vanish of infinite order and in particular (cf. [MS1]) the GHF $\mathcal{F}_t^{(K)}$ is well defined as a global flow. We will also assume that K satisfies the following non-degeneracy conditions: $\gamma_K(\sigma) = \{\mathcal{F}_t^{(K)}(\sigma) : t \in \mathbb{R}\}$ is a non-degenerate simply reflecting ray for almost all $\sigma \in S^*(\mathbb{R}^n \setminus U)$ such that $\gamma_K(\sigma)$ has at least one transversal reflection point, and for almost all $(x,\xi) \in S^*(\partial K)$ the set $\{t : \mathcal{F}_t^{(K)}(x,\xi) \in S^*(\partial K)\}$ is connected. (The latter means that the ray $\gamma_K(\sigma)$ either has one single point of tangency to ∂K or it has one single gliding segment on ∂K and no other tangencies.) We will denote by $\mathcal{K}^{\text{reg}}$ the class of ostacles K satisfying these non-degeneracy conditions. It is not difficult to see that $\mathcal{K}^{\text{reg}}$ is of second Baire category in the space of all obstacles with the C^∞ Whitney topology[1].

We will denote by $\partial K^{(\mathrm{ob})}$ the *observable part* of the boundary ∂K, namely $\partial K^{(\mathrm{ob})}$ is the union of all connected components of ∂K that have a common point with at least one scattering ray in Ω_K. The obstacle K will be called *observable*, if $\partial K = \partial K^{(\mathrm{ob})}$. It should be stressed that these definitions are only apropriate for the real analytic case.

The main result in this paper is the following.

Theorem 1.1. *Let $K, L \in \mathcal{K}^{reg}$ be obstacles in $\mathbb{R}^n$ with real analytic boundaries and let K and L have almost the same SLS. If K is such that $S^*(C) \setminus Trap(\Omega_K)$ is connected, then $\partial K^{(\mathrm{ob})} = \partial L^{(\mathrm{ob})}$. If in addition both K and L are observable, then $K = L$.*

In general it is not clear how restrictive the condition about the connectivity of $S^*(C) \setminus \mathrm{Trap}(\Omega_K)$ is. Proposition 5.5 below shows that this condition is satisfied when K is a finite disjoint union of strictly convex domains with C^∞ boundaries. Combining it with Theorem 1.1, one gets the following.

Corollary 1.2. *Assume that K is a finite disjoint union of strictly convex domains with analytic boundaries and let $L \in \mathcal{K}^{reg}$ be such that K and L have almost the same SLS. Then $K = L$.*

It should be remarked that even the case considered in the above corollary is non-trivial. When the number of convex components of K is large and they are "densily packed" (in a sense), then the flow $\mathcal{F}_t^{(K)}$ has a lot of tangencies that make its study rather difficult even though the boundary ∂K is locally very simple. It is still unknown whether the statement of Corollary 1.2 remains true without assuming real analyticity of the boundary.

Most of the paper actually deals with obstacles with C^∞ boundaries. In Sect. 3 we study the sets of points $\sigma \in S^*(\mathbb{R}^n \setminus U)$ that generate trajectories having tangencies to ∂K. It turns out that those σ that generate trajectories having tangencies of order higher than 1 form a relatively small subset $\mathcal{G}_K$. Roughly speaking, $\mathcal{G}_K$ has codimension 2 in $S^*(\mathbb{R}^n \setminus U)$, so it does not divide topologically the space $S^*(\mathbb{R}^n \setminus U)$. Using this fact, in Sect. 4 we prove Theorem 1.1.

In Sect. 5 we begin with an observation which applies locally near any difractive tangent point $(x_0, \xi_0) \in S^*(\partial K)$ and any K with C^∞ boundary. Namely, we show that there exists a local strictly convex smooth hypersurface Y near x_0 such that the unit normal field to Y consists of vectors that are tangent to ∂K on an open neighbourhood of x_0 in ∂K. We use this fact in the case when K is a finite disjoint union of strictly convex domains with C^∞ boundaries to show that the set $\mathcal{T}_0$ of ponts $\sigma \in S^*(\Omega_K)$ generating trajectories with at least two different tangent points is relatively small. This implies (cf. Proposition 5.5) that $S^*(C) \setminus \mathrm{Trap}(\Omega_K)$ is connected. In fact, more precise information about the set $\mathrm{Trap}(\Omega_K)$ is obtained. For example, if X is a local smooth convex surface in the interior of Ω_K with distinguished unit normal field ν_X and $\widetilde{X} = \{(x, \nu_X(x)) : x \in X\}$ is such that $\widetilde{X} \cap \mathcal{T}_0$ has codimension at least 2 in $\widetilde{X}$ (which is probably always the case but we do not prove it here), then $\dim(\widetilde{X} \cap \mathrm{Trap}(\Omega_K)) \leq n-3$, and in fact $\dim(\widetilde{X} \cap \mathrm{Trap}(\Omega_K) \setminus \mathcal{T}_0) = 0$. Here dim is the *topological dimension.* It should be mentioned that at least in the last relation dim cannot be replaced by the *Hausdorff dimension* $\dim_H$; one can easily show that in many cases $\dim_H(\widetilde{X} \cap \mathrm{Trap}(\Omega_K) \setminus \mathcal{T}_0) > 0$ and it is not easy to estimate $\dim_H$ from above for the sets that we deal with.

[1]It is very likely that the non-degeneracy of $\gamma_K(\sigma)$ for almost all σ such that $\gamma_K(\sigma)$ has at least one transversal reflection point follows automatically from the real analitycity of ∂K.

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2 Preliminaries

Let K be an obstacle in $\mathbb{R}^n$ with C^∞ boundary ∂K. The following conditions guarantees (cf. [MS2]) that $\mathcal{F}_t^{(K)}$ is well-defined as a global flow: for each $(x,\xi) \in T^*(\partial K) \setminus \{0\}$ if the normal curvature of ∂K at x vanishes of infinite order in direction ξ, then all points (y,η) sufficiently close to (x,ξ) are diffractive points (roughly speaking, this means that ∂K is convex at y in the direction of η). Denote by $\mathcal{K}$ the *class of obstacles* that have this property. Notice that if ∂K is real analytic, then $K \in \mathcal{K}$.

We refer the reader to [MS1], [MS2] (or Sect. 24.3 in [H]) for the definition of the generalized Hamiltonian (bicharacteristic) flow on a symplectic manifold with boundary. In the case of scattering by an obstacle K, this flow is generated by the principal symbol of the wave operator in $\mathbb{R} \times \Omega_K$. It defines in a natural way a flow $\mathcal{F}_t^{(K)} : T_b^*(\Omega_K) \setminus \{0\} \longrightarrow T_b^*(\Omega_K) \setminus \{0\}$, called the *generalized geodesic flow.* Here $T_b^*(\Omega_K) = T^*(\Omega_K)/\sim$ is the quotient space with respect to the following equivalence relation on $T^*(\Omega_K)$: $(x,\xi) \sim (y,\eta)$ iff $x = y$ and either $\xi = \eta$ or ξ and η are symmetric with respect to the tangent plane to ∂K at x. For $K \in \mathcal{K}$ the flow $\mathcal{F}_t^{(K)}$ on $T_b^*(\Omega_K)\setminus\{0\}$ is continuous ([MS2]). However, in this paper we prefer to work with $\mathcal{F}_t^{(K)}$ on $T^*(\Omega_K) \setminus \{0\}$. The image $S_b^*(\Omega_K)$ of the *unit cosphere bundle* $S^*(\Omega_K)$ under the natural projection is invariant with respect to $\mathcal{F}_t^{(K)}$.

A point $\sigma = (x,\omega) \in S^*(\Omega_K)$ is called *non-trapped* if both curves $\{\mathrm{pr}_1(\mathcal{F}_t^{(K)}(\sigma)) : t \leq 0\}$ and $\{\mathrm{pr}_1(\mathcal{F}_t^{(K)}(\sigma)) : t \geq 0\}$ in Ω_K are unbounded. Otherwise σ is called a *trapped point.* Here we use the notation $\mathrm{pr}_1(y,\eta) = y$ and $\mathrm{pr}_2(y,\eta) = \eta$. Denote by $\mathrm{Trap}(\Omega_K)$ the *set of all trapped points.* As the example of M. Livshits (cf. [M]) shows, in general $\mathrm{Trap}(\Omega_K)$ may have positive Lebesgue measure and non-empty interior in $S^*(\Omega_K)$. Let $\sigma = (x,\xi) \in S^*(\Omega_K)$ be such that $\gamma_K(\sigma) = \{\mathrm{pr}_1(\mathcal{F}_t^{(K)}(\sigma)) : t \in \mathbb{R}\}$ is a *simply reflecting ray,* i.e. it has no tangencies to ∂K. We shall say that $\gamma_K(\sigma)$ is *non-degenerate* if for every $t \neq 0$ with $\mathrm{pr}_1(\mathcal{F}_t^{(K)}(\sigma)) \notin K$ the map $\mathbb{R}^n \ni y \mapsto \mathrm{pr}_2(\mathcal{F}_t^{(K)}(y,\xi)) \in \mathbf{S}^{n-1}$ is a submersion at $y = x$, that is its differential at $y = x$ has rank $n-1$.

Denote by $\mathcal{K}_0$ the class of all obstacles $K \in \mathcal{K}$ so that ∂K does not contain non-trivial open flat subsets and such that $\gamma_K(\sigma)$ is a non-degenerate simply reflecting ray for almost all $\sigma \in S^*(\mathbb{R}^n\setminus U)$ such that $\gamma_K(\sigma)$ has at least one transversal reflection point. One can show that $\mathcal{K}_0$ is of second Baire category in $\mathcal{K}$ with respect to the C^∞ Whitney topology in $\mathcal{K}$. That is, for every $K \in \mathcal{K}$, applying suitable arbitrarily small C^∞ deformations to ∂K, one gets obstacles from the class $\mathcal{K}_0$ and in fact "most" deformations have this property.

Set $\dot{T}^*(\Omega) = T^*(\Omega) \setminus \{0\}$ and denote by $\dot{T}_b^*(\Omega)$ the projection of $\dot{T}^*(\Omega)$ in $T^*(\Omega)/\sim$.

The folowing theorem gives a necessary and sufficient condition for two obstacles in the class $\mathcal{K}_0$ to have almost the same SLS.

Theorem 2.1.([St3]) *If the obstacles $K, L \in \mathcal{K}_0$ have almost the same SLS, then there exists a homeomorphism*

$$\Phi : \dot{T}_b^*(\Omega_K) \setminus \mathit{Trap}(\Omega_K) \longrightarrow \dot{T}_b^*(\Omega_L) \setminus \mathit{Trap}(\Omega_L)$$

which defines a symplectic map on an open dense subset of $\dot{T}_b^*(\Omega_K) \setminus \mathit{Trap}(\Omega_K)$, maps $S_b^*(\Omega_K) \setminus \mathit{Trap}(\Omega_K)$ onto $S_b^*(\Omega_L) \setminus \mathit{Trap}(\Omega_L)$, and such that $\mathcal{F}_t^{(L)} \circ \Phi = \Phi \circ \mathcal{F}_t^{(K)}$ for all $t \in \mathbb{R}$ and $\Phi = id$

on $\dot{T}_b^*(\mathbb{R}^n \setminus U) \setminus Trap(\Omega_K) = \dot{T}_b^*(\mathbb{R}^n \setminus U) \setminus Trap(\Omega_L)$. Conversely, if $K, L \in \mathcal{K}_0$ are two obstacles for which there exists a homeomorphism $\Phi : S_b^*(\Omega_K) \setminus Trap(\Omega_K) \longrightarrow S_b^*(\Omega_L) \setminus Trap(\Omega_L)$ such that $\mathcal{F}_t^{(L)} \circ \Phi = \Phi \circ \mathcal{F}_t^{(K)}$ for all $t \in \mathbb{R}$ and $\Phi = id$ on $S^*(\mathbb{R}^n \setminus U) \setminus Trap(\Omega_K)$, then K and L have the same SLS.

It is clear from Livshits' example that the above conclusion is not true without any assumption about the sets of trapped points. In general it seems unlikely that $\mathcal{SL}_K$ provides substantial information about the size of the set $\mathrm{Trap}(\Omega_K)$.

Denote by $\mathcal{K}^{\mathrm{reg}}$ the set of those $K \in \mathcal{K}_0$ such that for almost all $(x, \xi) \in S^*(\partial K)$ the set $\{t : \mathcal{F}_t^{(K)}(x, \xi) \in S^*(\partial K)\}$ is connected[2]. One can show that $\mathcal{K}^{\mathrm{reg}}$ is of second Baire category in $\mathcal{K}$, i.e. the above property holds for generic obstacles.

Let U be a large ball containing the obstacles K and L. Given $\xi \in \mathbf{S}^{n-1}$ denote by Z_ξ *the hyperplane in $\mathbb{R}^n$ orthogonal to ξ and tangent to U* such that U is contained in the open half-space R_ξ determined by Z_ξ and having ξ as an inner normal. Given an (ω, θ)-ray γ in Ω, the *sojourn (travelling) time* T_γ of γ is defined by $T_\gamma = T'_\gamma - 2a$, where T'_γ is the length of that part of γ which is contained in $R_\omega \cap R_{-\theta}$ and a is the radius of the ball U. It is known that this definition does not depend on the choice of the ball U.

3 Tangential singularities

Throughout this section K denotes an obstacle in $\mathbb{R}^n$ such that the normal curvature of K does not vanish of infinite order. For brevity we will use the notation $\mathcal{F}_t = \mathcal{F}_t^{(K)}$. Let ϕ be a defining function for ∂K in a small neighbourhood V_0 of ∂K. That is, $\varphi : V_0 \longrightarrow \mathbb{R}$ is smooth, $d\varphi \neq 0$ on ∂K, and $\varphi^{-1}(0) = \partial K$. Define the Hamiltonian function $p : T^*(\mathbb{R}^n) \longrightarrow \mathbb{R}$ by $p(x, \xi) = \frac{1}{2}(|\xi|^2 - 1)$. The corresponding Hamiltonian vector field is

$$H_p = (\xi_1, \ldots, \xi_n; 0, \ldots, 0) = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i} \, .$$

As before, U will denote a *fixed open ball* in $\mathbb{R}^n$ containing K. Set $C = \partial U$.

In this section we first study the sets

$$\mathcal{T}_k = \{\sigma \in S^*(V_0) : H_p^j \varphi(\sigma) = 0 \text{ for } \ j = 0, 1, \ldots, k \text{ and } \ H_p^{k+1}\varphi(\sigma) \neq 0 \ \},$$

where k is a positive integer. In general these sets are not manifolds, however locally each of them is contained in a submanifold of codimension 2 in $S^*(V_0)$. What is more important, it turns out that for $k \geq 2$ the set $\mathcal{T}_k$ is locally contained in a submanifold of codimension 3 (see Proposition 3.1 below). An important consequence of this is that the set of those $\sigma \in S^*(\mathbb{R}^n \setminus U)$ that generate trajectories containing gliding segments on ∂K can be covered by a countable family of submanifolds of codimension 2 in $S^*(\mathbb{R}^n \setminus U)$, so topologically it does not divide $S^*(\mathbb{R}^n \setminus U)$.

Proposition 3.1. *For each $k \geq 1$ and each $\sigma \in \mathcal{T}_k$ there exists an open neighbourhood $V(\sigma)$ of σ in $T^*(V_0)$ and a smooth submanifold $\Gamma(\sigma)$ of $V(\sigma)$ such that $\mathcal{T}_k \cap V(\sigma) \subset \Gamma(\sigma) \subset S^*(V_0)$ and the codimension of $\Gamma(\sigma)$ in $S^*(V_0)$ is 3 for $k = 1$ and 4 for $k \geq 2$. Consequently, as a submanifold of $S^*(V_0)$ the codimension of $\Gamma(\sigma)$ is 2 for $k = 1$ and 3 for $k \geq 2$.*

[2] This means that the ray $\gamma_K(\sigma)$ either has one single point of tangency to ∂K or it has one single gliding segment on ∂K and no other tangencies.

Proof. Denote by $V'(\sigma)$ the set of those $\rho \in T^*(V_0)$ such that $\xi \neq 0$ and $H_p^{k+1}\varphi(\rho) \neq 0$, and let $g : V'(\sigma) \longrightarrow \mathbb{R}$ be defined by $g(\rho) = H_p\varphi(\rho)$. In coordinates we have

$$g(x,\xi) = \sum_{i=1}^{n} \xi_i \frac{\partial \varphi}{\partial x_i}(x) .$$

First, consider the case $k = 1$.We claim that

$$\Gamma = \Gamma(\sigma) = \{\rho \in V'(\sigma) : p(\rho) = \varphi(\rho) = g(\rho) = 0\}$$

is a submanifold of $V(\sigma) = V'(\sigma)$. For this of course one has to show that $dp(\rho)$, $d\varphi(\rho)$ and $dg(\rho)$ are linearly independent on Γ. Let $\rho = (x,\xi) \in \Gamma$ and assume that

$$u\; dp(\rho) + a\; d\varphi(\rho) + b\; dg(\rho) = 0 \tag{1}$$

for some $u, a, b \in \mathbb{R}$. Here $d = d_{(x,\xi)}$. Considering derivatives with respect to x_j, (1) implies

$$a\; \frac{\partial \varphi}{\partial x_i}(x) + b\; \sum_{i=1}^{n} \xi_i \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x) = 0.$$

Multiplying the latter by ξ_j and summing up gives

$$a\; H_p\varphi(\rho) + b\; H_p^2\varphi(\rho) = 0.$$

Since $H_p\varphi(\rho) = 0$ and $H_p^2\varphi(\rho) \neq 0$, the above implies $b = 0$. Now considering derivatives with respect to ξ_j in (1), we get $u = 0$ and therefore $a = 0$, too. Hence Γ is a submanifold of $V(\sigma)$ of codimension 3. Clearly, $\mathcal{T}_1 \cap V(\sigma) \subset \Gamma$. Moreover $S^*(V_0) = p^{-1}(0) \cap T^*(V_0)$ shows that $\Gamma \subset S^*(V_0)$.

Next, consider the case $k \geq 2$. Define another function f on $V(\sigma)$ by

$$f(\rho) = H_p^k\varphi(\rho) = \sum_{i_1,i_2,\ldots,i_k=1}^{n} \xi_{i_1}\xi_{i_2}\ldots\xi_{i_k} \frac{\partial^k \varphi}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_k}}(x) .$$

Set

$$\Gamma = \{\rho \in V'(\sigma) : p(\rho) = \varphi(\rho) = g(\rho) = f(\rho) = 0\}.$$

Clearly $\mathcal{T}_k \cap V'(\sigma) \subset \Gamma \subset S^*(V_0)$. We will show that dp, $d\varphi$, dg and df are linearly independent at any point $\rho \in \mathcal{T}_k \cap V'(\sigma)$ (in particular at σ). Let $\rho = (x,\xi) \in \mathcal{T}_k \cap V'(\sigma)$ and let

$$u\; dp(\rho) + a\; d\varphi(\rho) + b\; dg(\rho) + c\; df(\rho) = 0 \tag{2}$$

for some $u, a, b, c \in \mathbb{R}$. Considering derivatives with respect to x_m, (2) implies

$$a\; \frac{\partial \varphi}{\partial x_m}(x) + b\; \sum_{i=1}^{n} \xi_i \frac{\partial^2 \varphi}{\partial x_i \partial x_m}(x) + c \sum_{i_1,i_2,\ldots,i_k=1}^{n} \xi_{i_1}\xi_{i_2}\ldots\xi_{i_k} \frac{\partial^{k+1} \varphi}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_k}\partial x_m}(x) = 0$$

for all $m = 1, \ldots, n$. Multiplying the latter by ξ_m and summing up, we get

$$a\; H_p\varphi(\rho) + b\; H_p^2\varphi(\rho) + c\; H_p^{k+1}\varphi(\rho) = 0.$$

Since $\rho \in \mathcal{T}_k$ and $k \geq 2$, we have $H_p\varphi(\rho) = H_p^2\varphi(\rho) = 0$ and $H_p^{k+1}\varphi(\rho) \neq 0$. Hence $c = 0$. Next, considering the terms in (2) corresponding to derivatives with respect to ξ_i, we get

$$u\; \xi_i + b\; \frac{\partial \varphi}{\partial x_i}(x) = 0$$

for all i. Multiplying this by ξ_i and summing up gives $0 = u\,|\xi|^2 + b\, H_p\varphi(\rho) = u\,|\xi|^2$, so we must have $u = 0$ (since $\xi \neq 0$ on $V'(\sigma)$). Returning to the previous equality and using the fact that $d\varphi(\rho) = 0$, one gets $b = 0$. Then (2) yields $a = 0$ as well. This shows in particular that dp, $d\varphi$, dg and df are linearly independent at σ. By continuity, there exists an open neighbourhood $V(\sigma)$ of σ in $V'(\sigma)$ such that dp, $d\varphi$, dg and df are linearly independent on $V(\sigma)$. Then $\Gamma(\sigma) = \Gamma' \cap V(\sigma)$ is a submanifold of codimension 4 in $V(\sigma)$ and clearly $\mathcal{T}_k \cap V(\sigma) \subset \Gamma(\sigma)$.■

Next, we are going to show that for $k \geq 2$, the set $\mathcal{T}_k$ can be covered by a countable family of codimension 2 submanifolds of $S^*(C)$.

Fix for a moment $k \geq 2$ and $m \geq 0$. Denote by $\mathcal{T}_k^{(m)}$ the *set of those* $\sigma \in \mathcal{T}_k$ such that there exists $t > 0$ with $\mathcal{F}_t(\sigma) \in S^*(C)$ and the trajectory $\{\mathcal{F}_s(\sigma) : 0 < s \leq t\}$ has no common points with $\cup_{r=2}^{\infty}\mathcal{T}_r$ and has exactly m transversal reflection points at ∂K (and possibly some tangent points that belong to $\mathcal{T}_1$). For $\sigma \in \mathcal{T}_k^{(m)}$ denote by $s(\sigma)$ the *minimal number* $t > 0$ with $\mathcal{F}_t^{(K)}(\sigma) \in S^*(C)$.

Lemma 3.2. *For every $\sigma_0 \in \mathcal{T}_k^{(m)}$ there exists an open neighbourhood $U_k^{(m)}(\sigma_0)$ such that the set*

$$N_k^{(m)}(\sigma_0) = \{\mathcal{F}_{s(\sigma)}(\sigma) : \sigma \in \mathcal{T}_k^{(m)} \cap U_k^{(m)}(\sigma)\}$$

is contained in a smooth codimension 2 submanifold of $S^*(C)$.

Proof. We will use the above proposition and an argument from [St2].

Fix an arbitrary $\sigma_0 \in \mathcal{T}_k^{(m)}$ and let $s_1 < s_2 < \ldots < s_m$ be the times of the transversal reflections of $\{\mathcal{F}_s(\sigma_0) : 0 < s \leq s(\sigma_0)\}$. Clearly $0 < s_1$ and $s_m < s(\sigma)$. For each $j = 1, \ldots, m$ fix two numbers a_j and b_j close to s_j and such that

$$b_0 = 0 < a_1 < s_1 < b_1 < a_2 < s_2 < b_2 < \ldots < a_m < s_m < b_m < a_{m+1} = s(\sigma_0).$$

For each $j = 1, \ldots, m$ choose arbitrary smooth cros-section (i.e. submanifolds of $S^*(\Omega)$ of codimension 1) $\mathcal{A}_j$ and $\mathcal{B}_j$ to the trajectory $\{\mathcal{F}_s(\sigma_0) : 0 < s \leq s(\sigma_0)\}$ such that $\mathcal{F}_{a_j}(\sigma_0) \in \mathcal{A}_j$ and $\mathcal{F}_{b_j}(\sigma_0) \in \mathcal{B}_j$. We assume that $\mathcal{A}_j$ and $\mathcal{B}_j$ are so small and so close to the reflection point $\mathcal{F}_{s_j}(\sigma_0)$ that for any $\rho \in \mathcal{A}_j$ the trajectory of ρ under $\mathcal{F}_t$ makes exactly one (transversal) reflection at ∂K before intersecting transversally $\mathcal{B}_j$.

Let $V(\sigma_0)$ be an open neighbourhood of σ_0 in $S^*(V_0)$ with the properties described in Proposition 3.1. For ρ in a small neighbourhood U of σ_0 in $V(\sigma_0)$ we denote by $\psi_t(\rho)$ the unique curve in $S^*(\mathbb{R}^n$ for which there exists a sequence of numbers

$$b_0(\rho) = 0 < a_1(\rho) < s_1 < b_1(\rho) < a_2(\rho) < \ldots < a_m(\rho) < s_m < b_m(\rho) < a_{m+1}(\rho)$$

with the following properties:

(i) $\psi_{a_j(\rho)}(\rho) \in \mathcal{A}_j$ and $\psi_{b_j(\rho)}(\rho) \in \mathcal{B}_j$ for all $j = 1, \ldots, m$, $\psi_{a_{m+1}(\rho)}(\rho) \in S^*(C)$, and $\psi_s(\rho) \in S^*(U)$ for $s \in (0, a_{m+1}(\rho))$;

(ii) for each $j = 0, 1, \ldots, m$, the curve $\{\psi_t(\rho) : b_j(\rho) \leq t \leq a_{j+1}(\rho)\}$ is a trajectory of the vector field H_p in $\mathbb{R}^n$;

(iii) for each $j = 1, \ldots, m$, the curve $\{\psi_t(\rho) : a_j(\rho) \leq t \leq b_j(\rho)\}$ is a trajectory of the GHF $\mathcal{F}_t$ in Ω.

It is clear that if the neighbourhood U of σ_0 in $V(\sigma_0)$ is sufficiently small, then the curve $\psi_t(\rho)$ is well-defined for all $\rho \in U$.

Set $\Lambda(\sigma_0) = \{\rho \in U : H_p^k\varphi(\rho) = 0\ \}$. We assume that on $V(\sigma_0)$ we have $H_p^{k+1}\varphi \neq 0$ (which follows from the construction of $V(\sigma_0)$ in the proof of Proposition 3.1). Then $\Lambda(\sigma_0)$ is a codimension 1 submanifold of U transversal to the vector field H_p. Consequently, the map $\Lambda(\sigma_0) \ni \rho \mapsto \psi_{a_1(\rho)}(\rho) \in \mathcal{A}_1$ is smooth and a local bijection, so it defines a local diffeomorphism. Dealing in the same way with the shift along the curve $\psi_t(\rho)$ between successive cros-sections, one derives that the map $\Psi_k^{(m)}(\rho) = \psi_{a_{m+1}(\rho)}(\rho)$ from U to $S^*(C)$ is a local diffeomorphism.

Set $U_k^{(m)}(\sigma_0) = U$. It follows from Proposition 3.1 that $\Gamma(\sigma_0) \cap U_k^{(m)}$ is a codimension 2 submanifold of $\Lambda(\sigma_0)$. Hence $\Psi_k^{(m)}(\Gamma(\sigma_0) \cap U_k^{(m)}(\sigma_0))$ is a a codimension 2 submanifold of $S^*(C)$. It remains to show that $N_k^{(m)}(\sigma_0) = \Psi_k^{(m)}(\Gamma(\sigma_0) \cap U_k^{(m)}(\sigma_0))$. To check this, observe that for any $\rho \in U_k^{(m)}(\sigma_0) \cap \mathcal{T}_k^{(m)}$ we have $\psi_s(\rho) = \mathcal{F}_s^{(K)}(\rho)$ for all $s \in [0, a_{m+1}(\rho)]$. In particular $s(\sigma) = a_{m+1}(\rho)$ and therefore $N_k^{(m)}(\sigma_0) = \Psi_k^{(m)}(\Gamma(\sigma_0) \cap U_k^{(m)}(\sigma_0))$. This proves the lemma. ■

Denote by $\mathcal{G}_K$ the *set of those* $\sigma \in S^*(C)$ such that $\mathcal{F}_t(\sigma) \in \mathcal{T}_k$ for some $t \in \mathbb{R}$ and some $k \geq 2$. Clearly $\mathcal{G}_K$ contains any $\sigma \in S^*(C)$ that generates a trajectory containing a gliding segment on ∂K.

As an immediate consequence of the above lemma we get the following.

Proposition 3.3. *There exists a countable family* $\{N_i\}$ *of codimension* 2 *submanifolds of* $S^*(C)$ *such that* $\cup_i N_i$ *contains all points* $\sigma \in S^*(C)$ *such that* $\mathcal{F}_t^{(K)}(\sigma) \in \mathcal{T}_k$ *for some* $k \geq 2$.

Proof. Cover the set $\mathcal{T}_k^{(m)}$ with a countable family $U_k^{(m)}(\sigma_j^{(k,m)})$ of open subsets of $S^*(V_0)$ with the properties listed in Lemma 3.2. Then

$$\mathcal{G}_K \subset \cup_{j=1}^{\infty} \cup_{m=0}^{\infty} \cup_{k=1}^{\infty} N_k^{(m)}(\sigma_j^{(k,m)}) ,$$

so the statement follows from Lemma 3.2.■

Finally, it remains to deal with the case $k = 1$.

Lemma 3.4. *For every* $\sigma_0 \in \mathcal{T}_1$ *there exists an open neighbourhood* $U(\sigma)$ *such that the set*

$$N(\sigma_0) = \{\mathcal{F}_{s(\sigma)}(\sigma) : \sigma \in \mathcal{T}_1 \cap U(\sigma_0)\}$$

is contained in a smooth codimension 1 *submanifold of* $S^*(C)$.

Proof. This is essentially a repetition of the proof of Lemma 3.2 with minor modifications and less technicalities, since there are no futher tangencies that have to be avoided. We omit the details. ■

An important consequence of the last lemma is the following.

Proposition 3.5. *There exists a countable family* $\{M_i\}$ *of codimension* 1 *submanifolds of* $S^*(C) \setminus (Trap(\Omega_K) \cup \mathcal{G}_K)$ *such that every* $\sigma \in S^*(C) \setminus (Trap(\Omega_K) \cup \mathcal{G}_K \cup \cup_i M_i)$ *generates a simply reflecting trajectory in* Ω. *Moreover the family* $\{M_i\}$ *is locally finite in* $S^*(C) \setminus (Trap(\Omega_K) \cup \mathcal{G}_K)$, *that is any compact subset of* $S^*(C) \setminus (Trap(\Omega_K) \cup \mathcal{G}_K)$ *has common points with only finitely many of the submanifolds* M_i.

Proof. For every $\sigma \in \mathcal{T}_1 \setminus \mathrm{Trap}(\Omega_K)$ choose an open neighbourhood $U(\sigma)$ as in Lemma 3.4. Shrinking $U(\sigma)$ if necessary, we may assume that $H_p^2\varphi \neq 0$ on the closure of $U(\sigma)$ and therefore it does not have common points with $\mathcal{T}_k$ for any $k \geq 2$. Also, we choose $U(\sigma)$ such that $U(\sigma) \cap \mathrm{Trap}(\Omega_K) = \emptyset$. Consequently, $\mathrm{dist}(N(\sigma), \mathrm{Trap}(\Omega_K) \cup \mathcal{G}_K) > 0$, and so there exists a smooth codimension 1 submanifold $M(\sigma)$ of $S^*(C)$ such that $N(\sigma) \subset M(\sigma)$ and the closure of $M(\sigma)$ has no common points with $\mathrm{Trap}(\Omega_K) \cup \mathcal{G}_K$.

Choose a countable set of elements $\sigma_i \in \mathcal{T}_1 \setminus \mathrm{Trap}(\Omega_K)$ such that $\mathcal{T}_1 \setminus \mathrm{Trap}(\Omega_K) \subset \cup_i U(\sigma_i)$ and $\{U(\sigma_i) \cap \mathcal{T}_1\}$ is a locally finite family in $\mathcal{T}_1 \setminus \mathrm{Trap}(\Omega_K)$. Denote $M_i = M(\sigma_i)$. It is now clear that any $\sigma \in S^*(C) \setminus (\mathrm{Trap}(\Omega_K) \cup \mathcal{G}_K \cup \cup_i M_i)$ generates a simply reflecting trajectory in Ω. It remains to show that $\{M_i\}$ is locally finite in $S^*(C) \setminus (\mathrm{Trap}(\Omega_K) \cup \mathcal{G}_K)$. Let L be a compact subset of $S^*(C) \setminus (\mathrm{Trap}(\Omega_K) \cup \mathcal{G}_K)$. Then the sojourn (travelling) times of trajectories generated by elements of L are uniformly bounded. Assume that there exists $\rho_m \in M_{i_m} \cap L$ for infinitely many i_m. Choosing a subsequence, we may assume that $\rho_m \to \rho \in L$ as $m \to \infty$. Since $\rho_m \in M_{i_m}$, there exists $t_m \in \mathbb{R}$ with $\mathcal{F}_{t_m}(\rho_m) \in U(\sigma_{i_m}) \cap \mathcal{T}_1$. Since the sequence $\{t_m\}$ is bounded, we may assume that $t_m \to t \in \mathbb{R}$ as $t \to \infty$. Then $\mathcal{F}_{t_m}(\rho_m) \to \mathcal{F}_t(\rho)$, so for $\sigma = \mathcal{F}_t(\rho)$ we must have $H_p\varphi(\sigma) = 0$. On the other hand, $\rho \in L$ shows that $\rho \notin \mathrm{Trap}(\Omega_K) \cup \mathcal{G}_K$, so the trajectory generated by ρ is not trapped and does not have common points with $\mathcal{T}_k$ for any $k \geq 2$. Hence $\sigma \in \mathcal{T}_1 \setminus \mathrm{Trap}(\Omega_K)$, and therefore there exists a compact neighbourhood W of σ in $S^*(\mathbb{R}^n)$ such that $W \cap \mathcal{T}_1 \cap \mathrm{Trap}(\Omega_K) = \emptyset$. Then $\mathcal{F}_{t_m}(\rho_m) \to \sigma$ implies that W has common points with $U(\sigma_{i_m}) \cap \mathcal{T}_1$ for all sufficiently large m, contradiction with the local finiteness of the family $\{U(\sigma_i) \cap \mathcal{T}_1\}$ in $\mathcal{T}_1 \setminus \mathrm{Trap}(\Omega_K)$. Hence L can have common points with only finitely many M_i's. ∎

4 Proof of Theorem 1.1

Assume that $K \in \mathcal{K}^{\mathrm{reg}}$ and $L \in \mathcal{K}^{\mathrm{reg}}$ are two obstacles in $\mathbb{R}^n$ with real analytic boundaries that have almost the same SLS and such that every connected component of K (resp. L) has at least one common point with a scattering ray in Ω_K (resp. Ω_L). As before, U will denote a *fixed open ball* containing K and L and C will be the *boundary sphere* of U.

For K we assume in addition that the set $\mathrm{Trap}(\Omega_K)$ is not topologically dividing $S^*(C)$, that is, $S^*(C) \setminus \mathrm{Trap}(\Omega_K)$ is connected. Then L has the same property, since K and L having almost the same SLS and Theorem 2.1 imply $\mathrm{Trap}(\Omega_L) = \mathrm{Trap}(\Omega_K)$.

Under these assumptions we will show that $L = K$.

Clearly some connected components of K coincide with connected components of L. Indeed, it follows from result of Majda [Ma] and Lax and Phillips [LP2] (and also from the more general Theorem 2.1 above) that the extremal points of K and L coincide. The real analyticity of ∂K and ∂L then implies that any connected component of K containing extremal points of K coincides with a corresponding connected component of L.

Fix a countable family $\{N_i\}$ of codimension 2 submanifolds of $S^*(C)$ with the properties described in Proposition 3.3 and a countable family $\{M_i\}$ of codimension 1 submanifolds of $S^*(C) \setminus (\mathrm{Trap}(\Omega_K) \cup \mathcal{G}_K)$ with the properties described in Proposition 3.5.

Denote by X the *union of all connected components* of $\partial K^{(\mathrm{ob})}$ contained in ∂L. Then $Y = \partial K^{(\mathrm{ob})} \setminus X$ is a finite union of connected components of ∂K, so Y (if not empty) is a compact real analytic submanifold of $\mathbb{R}^n$ and $X \cap Y = \emptyset$. Our aim is to show that $Y = \emptyset$.

Assume that $Y \neq \emptyset$. It then follows from the assumptions about K that there exists $\sigma' \in S^*(C)$ such that the trajectory

$$\gamma_K(\sigma') = \{\mathrm{pr}_1(\mathcal{F}_t^{(K)}(\sigma')) : t \in \mathbb{R}\}$$

has at least one common point with Y. Since both $\mathcal{G}_K \cup \cup_i M_i$ and $\mathrm{Trap}(\Omega_K) \cap S^*(C)$ have Lebesgue measure zero in $S^*(C)$ (cf. [St2]), slightly changing σ', we may assume that $\gamma_K(\sigma')$ is a simply reflecting ray having a common point with Y (which is therefore a transversal reflection point). Clearly, there exists points $\sigma \in S^*(C)$ such that $\gamma_K(\sigma)$ has no common points with ∂K. Fix an arbitrary $\sigma_0 \in S^*(C)$ with this property.

Since $\sigma_0, \sigma' \in S^*(C) \setminus \mathrm{Trap}(\Omega_K)$ and this set is open and connected (by assumption), there exists a smooth curve $\sigma(s)$, $0 \leq s \leq 1$, in $S^*(C) \setminus \mathrm{Trap}(\Omega_K)$ such that $\sigma(0) = \sigma_0$, $\sigma(1) = \sigma'$. Since $\mathrm{Trap}(\Omega_K) \cap S^*(C)$ is a compact set, there exists a whole compact neighbourhood W of $\gamma = \{\sigma(s) : s \in [0,1]\}$ in $S^*(C)$ such that $W \cap \mathrm{Trap}(\Omega_K) = \emptyset$. Slightly perturbing the curve γ in the interior of W, we may assume that γ is transversal to any of the submanifolds M_i and N_i (cf. [Hi]). Now the fact that the codimension of each M_i in $S^*(C)$ is 2 implies that $\gamma \cap M_i = \emptyset$ for each i. Thus, $\gamma \cap \mathcal{G}_K = \emptyset$. On the other hand, the set $\mathcal{G}_K \cap W$ is closed in W (since W is away from $\mathrm{Trap}(\Omega_K)$), and so $\mathcal{G}_K \cap W$ is a compact subset of W having no common points with γ. Then shrinking W, we may assume that $W \cap \mathcal{G}_K = \emptyset$. With this, we have $W \cap (\mathcal{G}_K \cup \mathrm{Trap}(\Omega_K)) = \emptyset$. Now the fact that the family $\{M_i\}$ is locally finite in $S^*(C) \setminus (\mathrm{Trap}(\Omega_K) \cup \mathcal{G}_K)$ implies that $\gamma \cap M_i \neq \emptyset$ only for finitely many i. Consequently, there exist only finitely many $s_1, \ldots, s_m \in [0,1]$ such that $\gamma_K(\sigma(s_j))$ contains a tangent point to ∂K (and that must be a point from $\mathcal{T}_1$). In general it may happen that some of the trajectories $\gamma_K(\sigma(s_j))$ have more than one tangency to ∂K. However, using the condition $K \in \mathcal{K}^{\mathrm{reg}}$, we can slightly perturb the curve γ near each s_j, so that $\gamma_K(\sigma(s))$ has a single point of tangency to ∂K for any $s \in [0,1]$ with $\sigma(s) \in \cup_i M_i$.

In this way, we proved the existence of a smooth curve $\sigma(s)$ $(s \in [0,1])$ in $S^*(C) \setminus (\mathcal{G}_K \cup \mathrm{Trap}(\Omega_K))$ such that $\sigma(0) = \sigma_0$, $\sigma(1) = s'$, there exist only finitely many $s \in [0,1]$ with $\sigma(s) \in \cup_i M_i$ and for any such s the trajectory $\gamma_K(\sigma(s))$ has at most one tangent point to ∂K.

Since $\gamma \cap \mathrm{Trap}(\Omega_K) = \emptyset$ and Y is compact, there exists

$$a = \min\{s \in [0,1] : \gamma_K(\sigma(s)) \cap Y \neq \emptyset\}.$$

Clearly $a > 0$ and for $\rho = \sigma(a)$ we have $\gamma_K(\rho) \cap Y \neq \emptyset$. The choice of a shows that every common point of $\gamma_K(\rho)$ with Y must be a tangent point to Y. So, $a \in M_i$ for some i. Consequently, $\gamma_K(\rho)$ has only one tangent point y to Y, and this is its only common point with Y. Now we no longer nead the curve γ. There exists a point $\rho' \in W$ close to ρ such that $\gamma_K(\rho')$ is a simply reflecting ray in Ω_K and has exactly one common point y' with Y, y' close to y.

We claim that y' and all points of Y close to y' belong to ∂L. Indeed, let $y_1, \ldots, y_{j-1}, y_j = y', \ldots, y_m$ be the successive reflection points of $\gamma(\rho')$. By the construction of ρ', we have $y_i \in X$ for all $i \neq j$. The definition of X then implies $y_i \in \partial K \cap \partial L$ and the tangent planes to ∂K and ∂L at y_i coincide for all $i \neq j$. From this and Theorem 2.1 above one concludes that $\gamma_K(\rho') = \gamma_L(\rho')$, so we must have $y' \in \partial L$. In the same way, considering points $\rho'' \in W$ sufficiently close to ρ', one shows that all $y'' \in Y$ sufficiently close to y' belong to ∂L. Now the real analyticity of ∂K and ∂L implies that y' belongs to a connected component of ∂K that is contained in ∂L, contradiction with the definition of the sets X and Y. Hence we must have $Y = \emptyset$, i.e. $\partial K^{(\mathrm{ob})} \subset \partial L$. By symmetry, $\partial L^{(\mathrm{ob})} \subset \partial K$, so $\partial K^{(\mathrm{ob})} = \partial L^{(\mathrm{ob})}$. ■

5 The case of several strictly convex bodies

In this section we assume that K is an obstacle in $\mathbb{R}^n$ $(n \geq 3)$ of the form

$$K = K_1 \cup K_2 \cup \ldots \cup K_s \,, \tag{3}$$

where K_i are compact strictly convex disjoint domains in $\mathbb{R}^n$ with C^∞ boundaries ∂K_i. Set

$$d = \min_{i \neq j} \mathrm{dist}(K_i, K_j),$$

and $\mathcal{F} = \mathcal{F}^{(K)}$.

Lemma 5.1. *Let X be a C^∞ smooth submanifold of codimension 1 in $\mathbb{R}^n$, and let $x_0 \in X$ and $\xi_0 \in T_xX$, $\|\xi_0\| = 1$, be such that the normal curvature of X at x_0 in the direction ξ_0 is non-zero. Then for every $\epsilon > 0$ there exist an open neighbourhood V of x_0 in X, a smooth map $V \ni x \mapsto \xi(x) \in T_xX$ and a smooth positive function $t(x) \in [\delta, \epsilon]$ on V for some $\delta > 0$ such that $Y = \{y(x) = x + t(x)\xi(x) : x \in V\}$ is a smooth strictly convex surface with unit normal field $\mu(y(x)) = \xi(x)$, $x \in V$. That is, the normal field of Y consists of vectors tangent to X at the corresponding points of V.*

Proof. Considering X with the Riemann metric induced by $\mathbb{R}^n$, there exists a local smooth codimension 1 submanifold X' of X containing x and perpendicular to ξ_0 at x_0 and such that the second fundamental form of X' in X with respect to the normal ξ_0 is negative definite at x_0. (For example, we can take another strictly convex $(n-1)$-dimensional submanifold Z of $\mathbb{R}^n$ containing x_0 and having "outward" unit normal ξ_0 at x_0, and set $X' = X \cap Z$.) Parametrize X' by $h(u')$, $u' = (u_2, \dots, u_{n-1})$, $h(0) = x_0$, and let $\xi(u')$ be a continuous unit normal field to X' with $\xi(u') \in T_{h(u')}X$ for all u' and $\xi(0) = \xi_0$. For any u' let $c(t; u')$ be the geodesic on X parametrized by arc length t such that $c(0; u') = h(u')$ and $\dot{c}(t; u') = \xi(u')$. Define $r(u_1, u_2, \dots, u_{n-1}) = c(u_1; u')$. It then follows that for $|u_1|$ and $\|u'\|$ small enough, $r(u)$ is a smooth parametrization of an open neighbourhood V of x_0 in X such that $x_0 = r(0)$, $\xi_0 = \frac{\partial r}{\partial u_1}(0)$, and

$$\left\| \frac{\partial r}{\partial u_1}(u) \right\| = 1 \quad , \quad \left\langle \frac{\partial r}{\partial u_1}(u), \frac{\partial r}{\partial u_i}(u) \right\rangle = 0 \quad , \; i > 1 \; , \; u \in U \; . \tag{4}$$

(cf. e.g. Theorems 3 and 4 in Sect. 8.1 of [BC] for the second relation). Shrinking the neighbourhood V of x_0 if necessary, we may assume that u runs over some open ball U in $\mathbb{R}^{n-1}$. Notice that the second fundamental form of X' in X at x_0 has the form

$$II'(v') = \sum_{i,j=2}^{n-1} v_i v_j \left\langle \xi_0, \frac{\partial^2 r}{\partial u_i \partial u_j}(0) \right\rangle \; ,$$

where $v' = (v_2, \dots, v_{n-1}) \in \mathbb{R}^{n-2}$. So the choice of X' implies $II'(v') < 0$ whenever $v' \neq 0$.

Fix a small $\epsilon > 0$ and set $y(u) = r(u) + (\epsilon - u_1)\frac{\partial r}{\partial u_1}(u)$, $u \in U$. Shrinking the ball U if necessary, we may assume that $|u_1| < \epsilon/2$ for all $u \in U$. It will become clear later how small ϵ should be.

We claim that $Y = \{y(u) : u \in U\}$ is a smooth submanifold of $\mathbb{R}^n$ (provided $\epsilon > 0$ and U are small enough), $y(u)$ is a smooth parametrization of Y and $\mu(u) = \frac{\partial r}{\partial u_1}(u)$ is a normal vector to Y at $y(u)$.

First, notice that differentiating (3) with respect to u_j implies

$$\left\langle \frac{\partial r}{\partial u_1}(u), \frac{\partial^2 r}{\partial u_1 \partial u_j}(u) \right\rangle = 0 \quad , \quad 1 \leq j \leq n-1 \; , \tag{5}$$

$$\left\langle \frac{\partial^2 r}{\partial u_1 \partial u_j}(u), \frac{\partial r}{\partial u_i}(u) \right\rangle + \left\langle \frac{\partial r}{\partial u_1}(u), \frac{\partial^2 r}{\partial u_i \partial u_j}(u) \right\rangle = 0 \quad , \; 2 \leq i \leq n-1 \; , \; 1 \leq j \leq n-1 \; . \tag{6}$$

From these two equalities, it follows in particular that $\frac{\partial^2 r}{\partial u_1^2}(u) \perp \frac{\partial r}{\partial u_i}(u)$ for any $i = 1, \dots, n-1$, and therefore $\frac{\partial^2 r}{\partial u_1^2}(u)$ is a normal vector to X at $r(u)$. On the other hand, the assumption that the normal curvature of X at $x_0 = r(0)$ in the direction of $\xi_0 = \frac{\partial r}{\partial u_1}(0)$ is non-zero implies that $\frac{\partial^2 r}{\partial u_1^2}(0) \neq 0$. In particular, the vectors $\frac{\partial^2 r}{\partial u_1^2}(0), \frac{\partial r}{\partial u_2}(0), \dots, \frac{\partial r}{\partial u_{n-1}}(0)$ in $\mathbb{R}^n$ are linearly independent. Without loss of generality we may assume that the matrix formed by the first $n-1$ coordinates of these vectors has a non-zero determinant.

Since

$$\frac{\partial y}{\partial u_1}(u) = (\epsilon - u_1)\frac{\partial^2 r}{\partial u_1^2}(u) \quad , \quad \frac{\partial y}{\partial u_i}(u) = \frac{\partial r}{\partial u_i}(u) + (\epsilon - u_1)\frac{\partial^2 r}{\partial u_1 \partial u_i}(u) \quad , \; i > 1, \tag{7}$$

(4) and (5) imply that $\mu(u) = \frac{\partial r}{\partial u_1}(u)$ is a normal vector to Y at $y(u)$, provided Y is a smooth submanifold with parametrization $y(u)$.

To prove that Y is locally a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, it is enough to show that if $\epsilon > 0$ is small enough, then the vectors $\frac{\partial y}{\partial u_i}(0)$ $(1 \leq i \leq n-1)$ are linearly independent. Assume the contrary, i.e. for any $\epsilon \in (0, a]$ for some $a > 0$, the corresponding vectors $\frac{\partial y}{\partial u_i}(0)$ are linearly dependent. Let $y(u) = (y_1(u), \dots, y_n(u))$ and $r(u) = (r_1(u), \dots, r_n(u))$. Define $z(u) = (y_1(u), \dots, y_{n-1}(u))$ and $h(u) = (r_1(u), \dots, r_{n-1}(u))$. Then the vectors $\frac{\partial z}{\partial u_i}(0)$ $(1 \leq i \leq n-1)$ are also linearly dependent, so we must have

$$0 = \det \frac{\partial z}{\partial u}(0) = \det \begin{pmatrix} \frac{\partial z}{\partial u_1}(0) \\ \frac{\partial z}{\partial u_2}(0) \\ \cdots \\ \frac{\partial z}{\partial u_{n-1}}(0) \end{pmatrix} = \det \begin{pmatrix} \epsilon \frac{\partial^2 h}{\partial u_1^2}(0) \\ \frac{\partial h}{\partial u_2}(0) + \epsilon \frac{\partial^2 h}{\partial u_1 \partial u_2}(0) \\ \cdots\cdots \\ \frac{\partial h}{\partial u_{n-1}}(0) + \epsilon \frac{\partial^2 h}{\partial u_1 \partial u_{n-1}}(0) \end{pmatrix} ,$$

where the rows in the matrices above are vectors in $\mathbb{R}^{n-1}$. Dividing by ϵ the first row of the determinant in the right-hand side and then letting $\epsilon \to 0$, we obtain that the matrix formed by the first $n-1$ coordinates of the vectors $\frac{\partial^2 r}{\partial u_1^2}(0), \frac{\partial r}{\partial u_2}(0), \dots, \frac{\partial r}{\partial u_{n-1}}(0)$ has a zero determinant – contradiction with our assumption above.

Thus, for $\epsilon > 0$ sufficiently small the vectors $\frac{\partial y}{\partial u_i}(0)$ $(1 \leq i \leq n-1)$ are linearly independent. Shrinking U if necessary, we may assume that $\frac{\partial y}{\partial u_i}(u)$ $(1 \leq i \leq n-1)$ are linearly independent for any $u \in U$. Then Y is a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ and $y(u)$ is a smooth parametrization of Y. As we observed above, $\mu(u) = \frac{\partial r}{\partial u_1}(u)$ is then a unit normal to Y at $y(u)$. Moreover, by the definition of $y(u)$, the segment $[r(u), y(u)]$ is tangent to X at $r(u)$.

It remains to show that the normal curvature of Y with respect to the normal field $\mu(u)$ is negative. For this we need the second derivatives of $y(u)$ which we get from (7):

$$\frac{\partial^2 y}{\partial u_1^2}(0) = -\frac{\partial^2 r}{\partial u_1^2}(0) + \epsilon \frac{\partial^3 r}{\partial u_1^3}(0) ,$$

$$\frac{\partial^2 y}{\partial u_1 \partial u_i}(0) = \epsilon \frac{\partial^3 r}{\partial u_1^2 \partial u_i}(0) \quad , \; 2 \leq i \leq n-1 ,$$

$$\frac{\partial^2 y}{\partial u_i \partial u_j}(0) = \frac{\partial^2 r}{\partial u_i \partial u_j}(0) + \epsilon \frac{\partial^3 r}{\partial u_1 \partial u_i \partial u_j}(0) \quad , \; 2 \leq i, j \leq n-1 .$$

Hence the coefficients $c_{ij} = \left\langle \mu(0), \frac{\partial^2 y}{\partial u_i \partial u_j}(0) \right\rangle$ of the second fundamental form of Y at $y(0)$ have the form:

$$c_{1j} = \epsilon \left\langle \frac{\partial r}{\partial u_1}(0), \frac{\partial^3 r}{\partial u_1 \partial u_1^2 \partial u_j}(0) \right\rangle \quad , \quad 2 \leq j \leq n-1 ,$$

$$c_{ij} = \left\langle \frac{\partial r}{\partial u_1}(0), \frac{\partial^2 r}{\partial u_i \partial u_j}(0) \right\rangle + \epsilon \left\langle \frac{\partial r}{\partial u_1}(0), \frac{\partial^3 r}{\partial u_1 \partial u_i \partial u_j}(0) \right\rangle \quad , \quad 2 \leq i, j \leq n-1 \; .$$

On the other hand, differentiating (5) with respect to u_i, one gets

$$\left\langle \frac{\partial r}{\partial u_1}(0), \frac{\partial^3 r}{\partial u_1 \partial u_i \partial u_j}(0) \right\rangle = - \left\langle \frac{\partial^2 r}{\partial u_1 \partial u_i}(0), \frac{\partial^2 r}{\partial u_1 \partial u_j}(0) \right\rangle \; .$$

Thus, the second fundamental form $II(v)$ ($v = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n-1}$) of Y at $y(0)$ has the form

$$\begin{aligned} II(v) &= \sum_{i,j=1}^{n-1} v_i v_j c_{ij} = \sum_{i,j=1}^{n-1} v_i v_j \left\langle \frac{\partial r}{\partial u_1}(0), \frac{\partial^3 r}{\partial u_1 \partial u_i \partial u_j}(0) \right\rangle + \sum_{i,j=2}^{n-1} v_i v_j \left\langle \frac{\partial r}{\partial u_1}(0), \frac{\partial^2 r}{\partial u_i \partial u_j}(0) \right\rangle \\ &= - \sum_{i,j=1}^{n-1} v_i v_j \left\langle \frac{\partial^2 r}{\partial u_1 \partial u_i}(0), \frac{\partial^2 r}{\partial u_1 \partial u_j}(0) \right\rangle + II'(v') = -\|w\|^2 + II'(v'), \end{aligned}$$

where $v' = (v_2, \ldots, v_{n-1})$ and $w = \sum_{i=1}^{n-1} v_i \frac{\partial^2 r}{\partial u_1 \partial u_i}(0)$. Since by the choice of the submanifold X' we have $II'(v') < 0$ for $v' \neq 0$, it now follows that $II(v) < 0$ whenever $v \neq 0$. Thus, Y is strictly convex at $y(0)$ with respect to the normal $\mu(0) = \frac{\partial r}{\partial u_1}(0)$, and therefore the same conclusion holds for any $y(u)$ close enough to $y(0)$. So, shrinking U (and therefore V) we get that the whole submanifold Y is strictly convex in $\mathbb{R}^n$. ■

Let $\sigma_0 = (x_0, \xi_0) \in S^*(\partial K)$ be such that $\rho_0 = \mathcal{F}_{t_0}(\sigma_0) \in S^*(\partial K)$ for some $t_0 > 0$ and t_0 is minimal with this property. Let $\varphi(x)$ and $p(\xi)$ be as in the beginning of Sect. 3. On some small neighbourhoods U_0 and W_0 of σ_0 and ρ_0 in $T^*(\mathbb{R}^n)$ define

$$M = \{\sigma \in U_0 : p(\sigma) = H_p\varphi(\sigma) = 0\} \quad , \quad N = \{\sigma \in W_0 : p(\sigma) = H_p\varphi(\sigma) = 0\},$$

$M_0 = M \cap \varphi^{-1}(0)$, $N_0 = N \cap \varphi^{-1}(0)$. It follows from Sect. 3 that M and M_0 (resp. N and N_0) are smooth submanifolds of U_0 (resp. W_0) of codimensions 2 and 3. The strict convexity of ∂K implies that the Poisson bracket $\{p, H_p\varphi\} = H_p^2\varphi \neq 0$ on each of the sets M and N. Therefore M and N are codimension 2 symplectic submanifolds of U_0 and W_0, respectively. Moreover, $H_\varphi p = 0$ and $H_\varphi(H_p\varphi)(x,\xi) = \sum_{i=1}^n \left(\frac{\partial \varphi}{\partial x_i}\right)^2 > 0$ imply that M_0 and N_0 are codimension 1 submanifolds of M and N, respectively.

Let $t_1 < \ldots < t_m$ be the times of reflections of the trajectory $\gamma = \{\mathcal{F}_t(\sigma_0) : 0 < t < t_0\}$. Take a small open neighbourhood U_0 of σ_0 in $T^*(\mathbb{R}^n)$ and for each $\sigma \in U_0$ define the curve $\psi_t(\sigma)$, $0 \leq t \leq t_0 + d/2$, as the billiard trajectory of σ in $\mathbb{R}^n$ which makes (transversal) reflections at ∂K near the points $\mathcal{F}_{t_i}(\sigma_0)$ ($i = 1, \ldots, m$) but disregards ∂K near σ_0 and ρ_0. Let $\Phi : M' \longrightarrow N'$ be the map defined by shift along the curves ψ_t, that is for any $\sigma \in M'$, $\Phi(\sigma) = \psi_t(\sigma)$, where t is such that $|t - t_0| < d/2$ and $\psi_t(\sigma) \in N'$. If U_0 and V_0 are small enough, then Φ is well-defined and smooth and $\Phi(M) \subset N$. Moreover, the restriction of Φ ot M determines a symplectic map $\Phi : M \longrightarrow N$.

Lemma 5.2. *Under the assumptions and with the notation above, if the open neighbourhoods U_0 and W_0 of σ_0 and ρ_0 in $T^*(\mathbb{R}^n)$ are sufficiently small, then $M_0 \cap \Phi^{-1}(N_0)$ is a submanifold of M_0 of positive codimension.*

Proof. We will need the submanifolds

$$M' = \{\sigma \in U_0 : H_p\varphi(\sigma) = 0\} \quad , \quad N' = \{\sigma \in W_0 : H_p\varphi(\sigma) = 0\},$$

of codimension 1 in U_0 and W_0, respectively.

It follows from Lemma 5.1 that there exists an open neighbourhood V of x_0 in ∂K, a smooth map $V \ni x \mapsto \xi(x) \in T_x\partial K$ and a smooth positive function $t(x) \in [\delta, \epsilon]$ on V for some $\epsilon > \delta > 0$ such that $Y = \{y(x) = x + t(x)\xi(x) : x \in V\}$ is a smooth strictly convex surface with unit normal field $\mu(y(x)) = \xi(x)$, $x \in V$. Taking V sufficiently small, we may assume that Y is entirely in the exterior of K. Then $\widetilde{Y} \subset p^{-1}(0)$ and $\widetilde{Y}$ is an $(n-1)$-dimensional submanifold of $T^*(\mathbb{R}^n)$. It is contained in the codimension 1 submanifold $L' = T^*_Y(\mathbb{R}^n)$ of $T^*(\mathbb{R}^n)$. Clearly, $(y_0, \eta_0) = (y(x_0), \mu(x_0)) = \mathcal{F}_{s_0}(\sigma_0)$ for $s_0 = t(x_0)$.

Define the local maps $\Xi : M' \longrightarrow L'$ and $\Psi : L' \longrightarrow N'$ as the shifts along the curves ψ_t; then $\Phi = \Psi \circ \Xi$. Notice that

$$\Xi^{-1}(\widetilde{Y}) = \{(x, \xi(x)) : x \in V\} \subset M_0 \subset M$$

is an $(n-1)$-dimensional (Lagrangian) submanifold of M.

Fix an arbitrary $s > 0$ such that $t_m < s + s_0 < t_0$ and consider the set

$$S = \{\mathrm{pr}_1(\psi_s(\rho)) : \rho \in \widetilde{Y}\}.$$

Then

$$\psi_s(\widetilde{Y}) = \{(q, \omega(q)) : q \in S\},$$

where $\omega(q)$ is the unit normal field on Z pointing in the direction of the flow $\mathcal{F}$ on $\psi_s(\widetilde{Y})$. Clearly S is the shift of the strictly convex surface Y along the billiard flow in the exterior of K after fixed time s. It is known in the theory of dispersing billiards (see Sinai [Si1], [Si2]) that the strict convexity of ∂K and that of Y implies that S is a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ which is strictly convex with respect to the normal field determined by the direction of the flow. Assuming that the neighbourhood V of x_0 in ∂K, and therefore Y and S, are sufficiently small, we have $\psi_t(q, \omega(q)) = (q + t\omega(q), \omega(q))$ for $0 \leq t < t_0 + d/2 - (s + s_0)$. It is now easy to see that the set S_0 of those $q \in S$ such that $\psi_t(q, \omega(q)) \in S^*(\partial K)$ for some $t \in (0, t_0 + d/2 - (s + s_0))$ is a smooth submanifold of S of codimension 1 (cf. Lemma 5.3 below for a proof of this fact). Then $\{(q, \omega(q)) : q \in S_0\}$ has the same property in $\psi_s(\widetilde{Y})$, and taking $(\psi_s)^{-1}$, we deduce that the set L_0 of those $\rho \in \widetilde{Y}$ such that $\mathcal{F}_t(\rho) \in S^*(\partial K)$ for some t with $|t - t_0 - s_0| < d/2$ is a smooth codimension 1 submanifold of $\widetilde{Y}$.

Assume that U_0 is so small that $\mathrm{pr}_1(U_0) \subset V$. Notice that $L_0 = \widetilde{Y} \cap \Psi^{-1}(N_0)$. Applying Ξ^{-1} to this relation we get that $\Xi^{-1}(L_0) = \Xi^{-1}(\widetilde{Y}) \cap \Phi^{-1}(N_0)$. Here $L = \Xi^{-1}(\widetilde{Y})$ is a submanifold of M_0 of dimension $n-1$, while $\Xi^{-1}(L_0)$ is a submanifold of M_0 of dimension $n-2$. Since $L \cap \Phi^{-1}(N_0)$ is a proper submanifold of L and $L \subset M_0$, it follows that $M_0 \cap \Phi^{-1}(N_0)$ is a proper submanifold of M_0. ■

Given a smooth $(n-1)$-dimensional submanifold X of $\mathbb{R}^n$ with a specified continuous unit normal field ν_X, we will denote

$$\widetilde{X} = \{(x, \nu_X(x)) : x \in X\} \subset S^*_X(\mathbb{R}^n).$$

The following simple lemma is well-known. We prove it for completeness (it was already used in the proof of Lemma 5.2 above).

Lemma 5.3. *Let Z and X be smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ with specified continuous unit normal fields ν_Z and ν_X such that Z is convex with respect to ν_Z while X is strictly convex with respect to ν_Z. Let $z_0 \in Z$ be such that $x_0 = z_0 + \tau_0\nu_Z(z_0) \in X$ for some $\tau_0 > 0$ and the ray $\theta(z_0) = \{z_0 + t\nu_Z(z_0) : t > 0\}$ is tangent to X at x_0. Then the set Z_0 of* *those $z \in Z$ such that the corresponding ray $\theta(z)$ is tangent to X at some point $x(z) \in \theta(z)$ is an $(n-2)$-dimensional submanifold of Z locally near z_0. Consequently, the set $X_0 = \{x(z) : z \in Z_0\}$ is an $(n-2)$-dimensional submanifold of X locally near x_0.*

Proof. Let $\lambda_0 = (z_0, \nu_Z(z_0)) \in S^*_Z(\mathbb{R}^n)$ and $\sigma_0 = (x_0, \nu_Z(z_0)) \in S^*_Z(\mathbb{R}^n)$. Consider a defining function φ of X near x_0 with $\nabla\varphi = \nu_X$, and let $M = \{\sigma \in U_0 : p(\sigma) = H_p(\sigma) = 0\}$, where U_0 is a small open neighbourhood of σ_0 in $T^*(\mathbb{R}^n)$. Then M is a symplectic submanifold of $T^*(\mathbb{R}^n)$ of codimension 2. Consider the local map $\Lambda : S^*_Z(\mathbb{R}^n) \longrightarrow M$ whoich shifts each $\sigma \in S^*_Z(\mathbb{R}^n)$ along the free flow of H_p in $T^*(\mathbb{R}^n)$ until it hits M. That this map is well-defined, smooth and symplectic near λ_0 follows in a standard way (see e.g. the proof of Lemma 5.2 above). As in Lemma 5.2, set $M_0 = \{\sigma \in M : \varphi(\sigma) = 0\}$; then $\Lambda^{-1}(M_0)$ is a codimension 1 submanifold of $S^*_Z(\mathbb{R}^n)$ (near λ_0). We claim that $\Lambda^{-1}(M_0)$ is transversal in $S^*_Z(\mathbb{R}^n)$ to the $(n-1)$-dimensional (Lagrangian) submanifold $\widetilde{X}$. This is in fact trivial: consider a short strightline segment at y_0 transversal to Y, for example $r(s) = x_0 + s\nu_X(x_0)$, $|s| \leq a$, for some small a. Since Z is convex, the orthogonal projection to Z on a small open neighbourhood of x_0 in $\mathbb{R}^n$ is smooth and has maximal rank. Thus, the image $z(s)$ of $r(s)$ under this projection is a smooth curve on Z with $z(0) = z_0$. Moreover the curve $\sigma(s) = (z(s), \nu_Z(z(s))$ in $\widetilde{Z}$ is not tangent to $\Lambda^{-1}(M_0)$ at z_0. Hence $\widetilde{Z}$ is transversal to $\Lambda^{-1}(M_0)$ at z_0. Since $Z_0 = \mathrm{pr}_1(\widetilde{Z} \cap \Lambda^{-1}(M_0))$, it follows that locally near z_0, Z_0 is a smooth $(n-2)$-dimensional submanifold of Z.

The statement about X_0 follows from the fact that $\Lambda(\widetilde{Z} \cap \Lambda^{-1}(M_0)) = \Lambda(\widetilde{Z}) \cap M_0$ is a smooth $(n-2)$-dimensional submanifold of M_0 and locally near σ_0 the projection $\mathrm{pr}_1 : M_0 \longrightarrow \mathbb{R}^n$ has maximal rank. ■

Given $\sigma \in S^*(\Omega_K)$ we denote $\gamma^+_K(\sigma) = \{\mathrm{pr}_1(\mathcal{F}^{(K)}_t(\sigma)) : t \geq 0\}$.

We are going to use the construction in the proof of the above lemma to show that for obstacles K of the form (3) there are relatively few points $\sigma \in S^*(C)$ that generate trajectories with more than one tangency to ∂K. Here as before C is the boundary sphere of a large open ball U containing K in its interior.

Proposition 5.4. (a) *There exists a countable family $\{R_i\}$ of codimension 2 smooth submanifolds of $S^*(C)$ such that for any $\sigma \in S^*(C) \setminus (\cup_i R_i)$ the trajectory $\gamma_K(\sigma)$ has at most one tangency to ∂K.*

(b) *There exists a countable family $\{R_i\}$ of codimension 2 smooth submanifolds of $S^*_{\partial K}(\Omega_K)$ such that for any $\sigma \in S^*(C) \setminus (\cup_i R_i)$ the trajectory $\gamma_K(\sigma)$ has at most one tangency to ∂K.*

Proof. (a) Denote by Σ the set of those $\sigma \in S^*(C)$ such that $\gamma_K(\sigma)$ has at least two tangencies to ∂K. Given integers $1 \leq p < q$, let $\Sigma(p,q)$ be the set of those $\sigma \in \Sigma$ such that $\gamma_K(\sigma)$ has at least two different tangent points to ∂K, the first of which has the form $\mathcal{F}_{s_1}(\sigma)$ for some $s_1 \in [p\frac{d}{4}, (p+1)\frac{d}{4}]$ and the second one has the form $\mathcal{F}_{s_2}(\sigma)$ for some $s_2 \in [q\frac{d}{4}, (q+1)\frac{d}{4}]$. Clearly for such σ, $s_1 = s_1(\sigma)$ and $s_2(\sigma)$, we have that if $\rho \in \Sigma(p,q)$ is close to σ, then $\mathcal{F}_{s_1(\rho)}(\rho)$ and $\mathcal{F}_{s_1(\sigma)}(\sigma)$ belong to the same convex component of K and the same applies to $\mathcal{F}_{s_2(\rho)}(\rho)$ and $\mathcal{F}_{s_2(\sigma)}(\sigma)$.

It is enough to show that for any p and q, the set $\Sigma(p,q)$ can be covered by a countable union of codimension 2 submanioflds of $S^*(C)$. For this of course it is enough to prove the corresponding local statement: every $\sigma \in \Sigma(p,q)$ has an open neighbourhood U in $S^*(C)$ such that $\Sigma(p,q) \cap U$ is contained in a countable union of codimension 2 submanifolds of U.

Fix integers $q > p \geq 1$ and an arbitrary $\sigma' \in \Sigma(p,q)$. Set $s_1 = s_1(\sigma')$, $s_2 = s_2(\sigma')$, $\sigma_0 = \mathcal{F}_{s_1}(\sigma')$ and $\rho_0 = \mathcal{F}_{s_2}(\sigma')$. Then $\sigma_0, \rho_0 \in S^*(\partial K)$ and $\rho_0 = \mathcal{F}_{t_0}(\sigma_0)$ for $t_0 = s-2-s_1 > 0$, so we can apply Lemma 5.2. The latter gives that there exist open neighbourhoods U_0 and W_0 of σ_0 and

ρ_0 in $T^*(\mathbb{R}^n)$ such that if $M_0 \subset M \subset M'$, $N_0 \subset N \subset N'$ and the map $\Phi : M' \longrightarrow N'$ are defined as before the statement of Lemma 5.2, then $M_0 \cap \Phi^{-1}(N_0)$ is a submanifold of M_0 of positive codimension. Consequently, the codimension of $M_0 \cap \Phi^{-1}(N_0)$ in M is at least 2. Let k be the number of (transversal) reflection points of $\{\mathcal{F}_t(\sigma') : 0 \leq t < s_1\}$. Take a small open neighbourhood U' of σ' in $T^*_C(\mathbb{R}^n)$ and define $\Phi' : U' \longrightarrow M'$ similarly to Φ: given $\sigma \in U'$, we consider the shift of σ' along the flow $\mathcal{F}_t$ until after the kth reflection point, and after that the shift continuous along the free flow of H_p in $\mathbb{R}^n$ until the trajectory hits N'. Clearly, if U' is small enough, then Φ' is well-defined and induces a symplectic map $\Phi' : U = U' \cap S^*(C) \longrightarrow M$. Consequently, $(\Phi')^{-1}(M_0 \cap \Phi^{-1}(N_0))$ is a submanifold of U of codimension at least 2. Since $\Sigma(p,q) \cap U \subset M_0 \cap \Phi^{-1}(N_0)$, the assertion is proved.

(b) The proof is almost the same as that of part (a). Define $\Sigma(p,q)$ as before, however now $p < q$ can be negative or 0 as well. If $\sigma' \in \Sigma(p,q) \cap S^*(\partial K)$, one can simply apply Lemma 5.2. If $\sigma' \in \Sigma(p,q) \cap S^*_{\partial K}(\Omega_K)$, then one can apply the argument in the proof of part (b), replacing $S^*(C)$ by $S^*_{\partial K}(\Omega_K)$. Finally, if $\mathrm{pr}_1(\sigma') \notin \partial K$, one take any transversal cros-section U' of $T^*(\Omega_K)$ at σ' and proceed as in the proof of part (a). We omit the details. ■

As a consequence of the above considerations we get some information about the size of the set $\mathrm{Trap}(\Omega_K)$ in the case when K has the form (3). We will denote by $\mathrm{Trap}^+(K)$ the set of those $\sigma \in S^*(\Omega_K)$ such that the forward trajectory $\gamma^+_K(\sigma)$ is bounded.

Proposition 5.5. *Let $n \geq 3$, let K have the form* (3)*, and let U be an open ball containing K and $C = \partial U$. Then the subspace $S^*(C) \setminus$ Trap(Ω_K) of $S^*(C)$ is connected.*

Proof. Clearly it is enough to prove that if U is a sufficiently large open ball, then $S^*(\mathbb{R}^n \setminus U) \setminus \mathrm{Trap}(\Omega_K)$ is connected. Take an arbitrary $\sigma_0 = (x_0, \xi_0) \in S^*(\mathbb{R}^n \setminus U) \setminus \mathrm{Trap}(\Omega_K)$. It is enough to find a continuous curve $\sigma(t)$ in $S^*(\mathbb{R}^n \setminus U) \setminus \mathrm{Trap}(\Omega_K)$ such that $\sigma(0) = \sigma_0$ and $\sigma(1)$ generates a free trajectory in Ω_K, i.e. a trajectory without any reflections at ∂K.

We will assume that the trajectory $\gamma_K(\sigma_0)$ has common points with $S^*_{\partial K}(\Omega_K)$; otherwise there is nothing to prove. Replacing ξ_0 by $-\xi_0$ if necessary, we will assume that $\gamma^+_K(\sigma_0)$ has common points with $S^*_{\partial K}(\Omega_K)$. Then replacing x_0 by $x_0 - t\xi_0$ for some large $t > 0$, we may assume that there exists a strictly convex smooth hypersurface X in $\mathbb{R}^n \setminus U$ with a continuous unit normal field ν_X such that $x_0 \in X$, $\nu_X(x_0) = \xi_0$, and there exists $x_1 \in X$ such that the ray $\{x_1 + t\nu_X(x_1) : t > 0\}$ has no common points with K. (For X one can take the boundary sphere of a very large ball in $\mathbb{R}^n \setminus U$.) Take a very large closed ball U_1 that contains the orthogonal projection of K onto X and the point x_1 as well. Since the set $\mathrm{Trap}(\Omega_K) \cap S^*(U_1)$ is compact and $\sigma_0 \notin \mathrm{Trap}(\Omega_K)$, there exists and open connected neighbourhood V_0 of σ_0 in $S^*(U_1)$ with $V_0 \cap \mathrm{Trap}(\Omega_K) = \emptyset$.

It follows from Proposition 5.4 (a) that there exists a countable family $\{Q_i\}$ of smooth codimension 2 submanifolds of $S^*(\mathbb{R}^n \setminus U)$ such that for any $\sigma \in S^*(\mathbb{R}^n \setminus U) \setminus (\cup_i Q_i)$, the trajectory $\gamma_K(\sigma)$ has at most one tangency to ∂K. The submanifolds Q_i are obtained from the submanifolds $\mathbb{R}_i$ in Proposition 5.4 (a) by translation along the second (vector) component. So, locally Q_i are invariant under $\mathcal{F}$.

Using Thom's Transversality Theorem (cf. e.g. [H]), we can find an arbitrarily small in the C^∞ Whitney topology deformation Y of X such that $\widetilde{Y}$ is transversal to Q_i in $S^*(V)$ for each i. Since each Q_i is locally invariant under $\mathcal{F}_t$ (for small $|t|$), we may assume that $x_0 \in Y$. Moreover, taking Y sufficiently close to X, we may assume that $Y \cap U_1$ is strictly convex and if η_0 is the normal to Y at x_0, then $(x_0, \eta_0) \in V_0$, so (x_0, η_0) and (x_0, ξ_0) can be joined by a continuous curve in V_0 which therefore has no common points with $\mathrm{Trap}(\Omega_K)$. Finally, we can take Y so close to

X that there exists $y_1 \in Y$ (close to x_1) such that the ray $\{y_1 + t\nu_Y(y_1) : t > 0\}$ has no common points with K.

The above argument shows that without loss of generality we may assume that $\widetilde{X}$ is transversal to each of the submanifolds Q_i. Since $\dim(Q_i) = 2n-3$ and $\dim(\widetilde{X}) = n-1$, it then follows that for each i, $\widetilde{X} \cap Q_i$ is a submanifold of $S^*(V)$ with $\dim(\widetilde{X} \cap Q_i) = (2n-3+n-1)-(2n-1) = n-3$. Using the Sum Theorem for the topological dimension dim (cf. e.g. [HW]), it now follows that for $X' = \widetilde{X} \cap (\cup_i Q_i)$ we have $\dim(X') \leq n-3$.

Next, denote by X_0 the set of those $\sigma \in \widetilde{X} \cap \mathrm{Trap}(\Omega_K)$ such that the trajectory $\gamma_K(\sigma)$ has no tangencies to ∂K. Given integers p, q such that $q \in \{1, \ldots, s\}$ and $p \geq 1$, denote by $X(p,q)$ the set of those $\sigma \in \widetilde{X} \cap \mathrm{Trap}(\Omega_K)$ such that $\gamma_K(\sigma)$ has exactly one tangency to ∂K which is its pth reflection point and it belongs to ∂K_q. We will show that each of the subspaces X_0 and $X(p,q)$ of $\widetilde{X}$ has topological dimension $\dim = 0$.

Let $\widetilde{F} = \prod_{r=1}^{\infty} F$ be endowed with the product topology, where $F = \{1, 2, \ldots, s\}$ with the discrete topology. It is well known (and easy to see) that $\dim(\widetilde{F}) = 0$ and therefore every subspace of $\widetilde{F}$ has topological dimension zero (cf. e.g. [HW]). Consider the map $f : X_0 \longrightarrow \widetilde{F}$, defined by $f(\sigma) = (i_1, i_2, \ldots)$, where the jth reflection point of $\gamma_K(\sigma)$ belongs to ∂K_{i_j} for all $j = 1, 2, \ldots$. Clearly, the map f is continuous and it follows from [St1] that $f^{-1} : f(X_0) \longrightarrow X_0$ is also continuous (it is in fact Lipschitz with respect to an appropriate metric on $\widetilde{F}$). Thus, X_0 is homeomorphic to a subspace of $\widetilde{F}$ and therefore $\dim(X_0) = 0$.

Given p, q, we deal with $X(p,q)$ in a similar way. Define $f : X(p,q) \longrightarrow \widetilde{F}$ as above and notice that in the present case we have $i_q = p$ for any $\sigma \in X(p,q)$, where $f(\sigma) = (i_j)$. This and the definition of $X(p,q)$ imply that f is continuous on $X(p,q)$ and using [St1] again, we get that $X(p,q)$ is homeomorphic to a subspace of $\widetilde{F}$ and so $\dim(X(p,q)) = 0$.

Now the Sum Theorem for dim (cf. [HW]) shows that $\dim(X' \cup X_0 \cup \cup_{p,q} X(p,q)) \leq n-3$. Since $\widetilde{X}$ is an $(n-1)$-dimensional Cantor manifold (cf. Sect. 6.5 in [HW]), it follows that $\widetilde{X} \setminus \mathrm{Trap}(\Omega_K)$ is connected. Since σ_0 and $\sigma_1 = (x_1, \nu_X(x_1))$ both belong to $\widetilde{X} \setminus \mathrm{Trap}(\Omega_K)$, there exists a continuous curve $\sigma(t)$ in $\widetilde{X} \setminus \mathrm{Trap}(\Omega_K)$ with $\sigma(0) = \sigma_0$ and $\sigma(1) = \sigma_1$. This proves the assertion. ■

References

[BC] R. Bishop and R. Crittenden, *Geometry of manifolds*, New York, Academic Press, 1964.

[E] G. Edgar, *Measure, Topology and Fractal Geometry*, Berlin, Springer, 1990

[H] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, vol. III, Berlin, Springer, 1985

[Hi] M. Hirsch, *Differential Topology*, Berlin, Springer, 1976

[HW] W. Hurewicz, H. Wallman, *Dimension Theory*, Princeton University Press, 1948

[LP1] P. Lax, R. Phillips, *Scattering Theory*, Academic Press, New York, 1967

[LP2] P. Lax, R. Phillips, *The scattering of sound waves by an obstacle*, Comm. Pure Appl. Math. **30** (1977), 195-233

[Ma] A. Majda, *A representation formula for the scattering operator and the inverse problem for arbitrary bodies*, Comm. Pure Appl. Math. **30** (1977), 165-194

[MaR] A. Majda, J. Ralston, *An analogue of Weyl's formula for unbounded domains*, Duke Math. J. **45** (1978), 183-196

[M] R. Melrose, *Geometric Scattering Theory*, Cambridge Univ. Press, 1994

[MS1] R. Melrose, J. Sjöstrand *Singularities in boundary value problems*, I. Comm. Pure Appl. Math. **31** (1978), 593-617

[MS2] R. Melrose, J. Sjöstrand, *Singularities in boundary value problems*, II. Comm. Pure Appl. Math. **35** (1982),129-168

[Si1] Ya. Sinai, *Dynamical systems with elastic reflections*, Russian Math. Surveys **25** (1970), 137-190.

[Si2] Ya. Sinai, *Development of Krylov's ideas*, An addendum to the book: N.S.Krylov "Works on the foundations of statistical physics", Princeton Univ. Press, Princeton 1979, pp. 239-281.

[St1] L. Stoyanov, *Exponential instability for a class of dispersing billiards*, Ergodic Theory & Dynamical Systems **19** 1999, 201-226.

[St2] L. Stoyanov, *Generalized Hamiltonian flow and the Poisson relation for the scattering kernel*, Preprint 1996.

[St3] L. Stoyanov, *Rigidity of the scattering length spectrum*, Preprint, 1997.

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