Plancherel formula for Berezin deformation of L^2 on Riemannian symmetric space

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0.Introduction

0.1. Kernel representations. Let G be a *classical* real group and let K be its maximal compact subgroup. Consider the Riemannian noncompact symmetric space G/K. There exists a hermitian symmetric space

$$\widetilde{G}/\widetilde{K} \supset G/K$$

such that

$$\dim_{\mathbb{R}} G/K = \frac{1}{2} \dim_{\mathbb{R}} \widetilde{G}/\widetilde{K}$$

and G/K is a totally real submanifold in $\widetilde{G}/\widetilde{K}$ (the list of embeddings $G/K \to \widetilde{G}/\widetilde{K}$ see in Section 6). We say that the symmetric space $\widetilde{G}/\widetilde{K} \supset G/K$ is the *hermitization* of the symmetric space G/K.

We define a *kernel representation* ρ of the group G as a restriction of an unitary highest weight representation $\widetilde{\rho}$ of $\widetilde{G}$ to the subgroup G. By well-known Harish-Chandra construction, highest weight representations of $\widetilde{G}$ are natural representations in spaces of (scalar-valued or vector-valued) holomorphic functions on $\widetilde{G}/\widetilde{K}$. We say that ρ is a *scalar valued kernel representation* if $\widetilde{\rho}$ is realized in scalar-valued holomorphic functions on $\widetilde{G}/\widetilde{K}$.

The kernel representations are deformations of $L^2(G/K)$ in some precise sense explained in Subsection 1.13.

The purpose of this paper[2] is to obtain the Plancherel formula for scalar-valued kernel representations (see formula (2.6)–(2.15)).

There were different reasons for interest, which was attracted by kernel representations in last 5 years (see [2], [7], [8], [33]–[36], [38], [39]–[40], [52], [56]), and we will formulate reasons which are the most closed to the author. In first place, there are many explicit analytical formulas related to kernel representations (I hope that this paper also confirms this statement). Secondly, spectra of kernel representations are very rich[3]. Thirdly, the kernel representations have

[1]supported by grants RFBR-98-01-00303 and RFBR 96-01-96249

[2]This work is continuation of works [35], [36] but logically it is independent on these papers. Our main result was announced in [36].

[3]The most interesting spectral problems of noncommutative harmonic analysis which were intensively investigated in last 20 years are

some interaction with function theory[4], see [38], [34], [36]. Forth, the kernel representations also are closely related to Olshanskii constructions of representations of infinite dimensional groups $\mathrm{U}(p,\infty)$, $\mathrm{O}(p,\infty)$, $\mathrm{Sp}(p,\infty)$ (see [38]).

0.2. Bibliographical comments. Let G/K be itself an hermitian symmetric space (i.e. $G = \mathrm{U}(p,q)$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}^*(2n)$, $\mathrm{SO}(n,2)$). Then its hermitization $\widetilde{G}/\widetilde{K}$ is $G/K \times G/K$. A kernel representation of G in this case is a tensor product of a highest weight representation $\widetilde{\rho}_\alpha$ of G and a lowest weight representation $\widetilde{\rho}^*_\beta$ of G. In short paper [4] published in 1978 Berezin announced nice Plancherel formula for the *sufficiently large parameter* $\alpha = \beta$ *of highest weight* (see below Subsections 1.10-1.11). In this case[5] the kernel representation is equivalent to the representation of G in $L^2(G/K)$. Berezin died soon after this and the proof never was published[6]. Berezin work didn't attract serious interest in this time (see only papers [15],[46] on related subjects).

Secondly, the kernel representations appeared in G.I.Olshanskii and my work which was partially announced in [30], [43] and partially published in [38]. This work cocerned in vector valued kernel representations of the groups $G = \mathrm{O}(p,q)$, $\mathrm{U}(p,q)$, $\mathrm{Sp}(p,q)$ for *small values* of the highest weight. The main topic of our work was investigation of discrete part of spectra of the kernel representations and construction of "exotic" unitary representations of G by simple functional theoretical tools.

In the middle of 90-s interest to kernel representations increases (we list some publications: [52], [2], [33], [39], [40], [7], [8],[56]). In 1994 Upmeier and Unterberger [52] published proof of Berezin formula (see also [2])[7]. Van Dijk and Hille [7] obtained the complete Plancherel formula for rank 1 groups. Olaffson and Orsted [39] proved that for a large highest weight a scalar valued kernel representation of G is equivalent to the representation of G in $L^2(G/K)$.

In paper [35] there was defined B-function for arbitrary classical noncompact Riemannian symmetric space. For the symmetric cones $\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n)$, $\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$, $\mathrm{GL}(n,\mathbb{H})/\mathrm{Sp}(n)$ these B-integrals coincides with Gindikin B-function constructed in [11](1964) (see also exposition in [9]). The construction

— L^2 on pseudoriemannian symmetric spaces

— Howe dual pairs (and the problem of decomposition of L^2 on Stiefel manifolds which are in some sense equivalent to Howe dual pairs)

Each representation, which occurs in spectra of Howe dual pairs, occurs in spectra of some kernel representation. Converse statement is false. The `a priori` explanation of this phenomenon is contained in [33].

I think that spectra of kernel representations and spectra of L^2 on pseudoriemannian symmetric spaces essentially differs. `A priori` embedding of spectra of $L^2(\mathrm{U}(p,q,\mathbb{K})/\mathrm{U}(r,\mathbb{K}) \times \mathrm{U}(p-r,q,\mathbb{K}))$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ to spectra of kernel representations is discussed in [38], [33].

[4]For instance, in the work [38] we use functional theoretical arguments for construction of singular unitary representations of groups $\mathrm{U}(p,q,\mathbb{K})$

[5]Tensor products for $\mathrm{SL}_2(\mathbb{R}) = \mathrm{U}(1,1)$ were earlier investigated by Pukanszky [45] and by Vershik, Gelfand and Graev [53], see also [27]

[6]20 years later I heard some reminiscences about this proof but I can not reconstruct proof itself. It essentially differs from Unterberger–Upmeier [52] proof and my proof [35].

[7]Their result also covers groups E_6, E_7.

of [35] for special cases of parameters gives some integrals of Siegel [48], Hua Loo Keng [21], Unterberger–Upmeier [52], Arazy–Zhang [2]. The Plancherel formula for scalar-valued kernel representations of all classical groups for large values of parameter is easily reduced to these B-integrals. In this case a kernel representation is equivalent to the representation of G in $L^2(G/K)$ and the spectrum of the kernel representation is supported by the principal nondegenerate unitary series.

The case of small values of highest weight was discussed in paper [36]. In this case the spectrum of kernel representation is quite intricate and work [36] contains natural decomposition of a kernel representation on subrepresentations having relatively simple spectra.

The purpose of the present parer is to obtain the complete Plancherel formula for the kernel representations in the scalar-valued case.

0.3. Contents. Main part (Sections 1–5) of the paper deals with the series $G = \mathrm{O}(p, q)$[8].

Section 1 of the paper contains preliminaries. We discuss the definition of the kernel representations and simple a priori properties of the Plancherel formula. We also formulate some necessary properties of spherical functions.

Basic results are formulated in Section 2. For large values of α (where α is the parameter of a highest weight) the Plancherel measure ν_α has the form

$$E(\alpha) \prod_{k=1}^{p} |\Gamma(\frac{1}{2}(\alpha + (p+q)/2 + s_k)|^2 \mathfrak{R}(s) ds$$

where $\mathfrak{R}(s)$ is the Gindikin-Karpelevich density (see(1.42)–(1.43)),

$$s_1, \ \ldots, \ s_p \in i\mathbb{R} \tag{0.1}$$

(this notation means that $\operatorname{Re} s_j = 0$) and $E(\alpha)$ is a meromorphic factor.

Assume $q-p$ be sufficiently large. Let us move the parameter α from $+\infty$ to 0. After passing across the point $\alpha = \frac{1}{2}(p+q) - 1$ there appears an additional piece of the support of the Plancherel measure. This piece is defined by the conditions

$$s_1 = \alpha - \tfrac{1}{2}(p+q) + 1; \quad s_2, \ldots, s_p \in i\mathbb{R} \tag{0.2}$$

After passing across the point $\alpha = \frac{1}{2}(p+q) - 2$ the third piece of the support of the Plancherel measure appears:

$$s_1 = \alpha - \tfrac{1}{2}(p+q) + 1, \, s_2 = \alpha - \tfrac{1}{2}(p+q) + 2; \quad s_3, \ldots, s_p \in i\mathbb{R} \tag{0.3}$$

[8]This case is the most complicated and all difficulties existing for other series exist also for $\mathrm{O}(p, q)$. For all other series our proof is more simple.

After passing across the point $\alpha=\frac{1}{2}(p+q)-3$ we obtain two additional components of the support

$$s_1=\alpha-\tfrac{1}{2}(p+q)+1,\ s_2=\alpha-\tfrac{1}{2}(p+q)+2,\ s_3=\alpha-\tfrac{1}{2}(p+q)+3;\quad s_4,\dots,s_p\in i\mathbb{R}$$

and

$$s_1=\alpha-\tfrac{1}{2}(p+q)+3;\quad s_2,\dots,s_p\in i\mathbb{R} \tag{0.4}$$

etc. After passing across the point $\alpha=(q-p)/2$ we obtain the first one-point piece

$$s_1=\alpha-\tfrac{1}{2}(p+q)+1;\quad s_2=\alpha-\tfrac{1}{2}(p+q)+2,\dots,s_p=\alpha-\tfrac{1}{2}(p+q)+p$$

This means that our representation has a subrepresentation entering discretely.

At the point $\alpha=p-1$ the component (0.1) of the support disappears. At the point $\alpha=p-2$ components (0.2), (0.4) also disappear, etc.

Theorems 2.2–2.4 contain the complete description of this process and give the Plancherel density on each component of the support. Interpretation of these pieces is given in [36], in the present paper this is not discussed.

The nature of spectra of kernel representations is explained in [36][9].

For an integer negative α our construction gives the Plancherel formula for some finite dimensional representation of $\mathrm{O}(p,q)$ (see Subsection 2.6 of the paper).

Section 3 is based on [35] and contains evaluation of the B-integral (see formula (3.2)-(3.4)). For instance, in the case $p=q$ our B-integral is given by

$$\int_{R+R^t>0}\prod_{j=1}^{p}\frac{\det[(R+R^t)/2]_j^{\lambda_j-\lambda_{j+1}}}{\det[1+R]_j^{\sigma_j-\sigma_{j+1}}}\cdot\det(R+R^t)^{-p}dR= \tag{0.5}$$
$$=\mathrm{const}\cdot\frac{\Gamma(\lambda_k-(p+k)/2+1)\Gamma(\sigma_k-\lambda_k-(p-k)/2)}{\Gamma(\sigma_k-p+k)}$$

where the integration is given over the space of dissipative $p\times p$ real matrices R and the symbol $[A]_j$ denotes the left upper $j\times j$ block of a matrix A.

B-Integral allows to obtain the Plancherel formula for $\alpha>\frac{1}{2}(p+q)-1$. In Section 4 we construct the analytic continuation of the Plancherel formula to arbitrary α.

In Section 5 we prove positive definiteness of spherical functions which appears in the right side of the Plancherel formula.

[9] The discrete part of spectra of kernel representations consists of singular unitary representations having quite interesting properties. For instance, these infinite dimensional (non highest weight) representations have Gelfand–Tsetlin bases, see [43], [28]; the problems of decomposition of restrictions and tensor products for these representations also seems rich, see [30], [38]). In [43], [38] it was shown, that these representations admit inductive limits as $q\to\infty$. Certain representations of this type appear in spectra discussed in [50], [25]

Section 6 contains a discussion of other series of classical groups. The B-integrals for other series or real classical groups are evaluated in [35], and a generalization of the consideration of Subsections 1, 4, 5 to other series is quite trivial. Hence, we give only short remarks and also give the Plancherel formula in the form which slightly differs from Theorem 2.2. Author intentionally considers the series $\mathrm{O}(p, q)$ (and not so-called 'general case') to do the exposition more or less self-closed. I try to avoid formal logical dependence on recent papers and also minimize using machinery of representation theory of semisimple groups as far as it is possible[10]; I also try to avoid notations demanding long explanations.

Acknowledgements. I am very grateful to G.I.Olshanskii, V.F.Molchanov, and B.Orsted for numerous discussions of the subject. I thanks H.Schlichtkrull, G. van Dijk, and A.Dvorsky for discussions, comments and references.

1. Preliminaries.

A. Positive definite kernels.

The subject of the paper is the analysis in a family of hilbert spaces defined by positive definite kernels. The notion of positive definite kernel and associated machinery are quite old (see [51], [6], [26]) but not widely known. In this Section we briefly discuss elementary properties of the positive definite kernels and associated hilbert spaces.

1.1. Positive definite kernels. Let H be a hilbert space with a scalar product $< \cdot, \cdot >$, let X be a subset in H. Consider the function $L(x, y)$ on $X \times X$ defined by

$$L(x, y) = < x, y >$$

Obviously for all $x_1, \dots, x_n \in X$ we have

$$\det \begin{pmatrix} L(x_1, x_1) & \cdots & L(x_1, x_n) \\ \cdots & \cdots & \cdots \\ L(x_n, x_1) & \cdots & L(x_n, x_n) \end{pmatrix} \geqslant 0 \qquad (1.1)$$

Let X be an abstract set. A function $L(x, y)$ on $X \times X$ is called a *positive definite kernel* if it satisfies the conditions

1. $L(x, y) = \overline{L(y, x)}$

2. For any $x_1, x_2, \dots, x_n \in X$ inequality (1.1) holds

Let $L(x, y)$ be a positive definite kernel on X. Then where exists a hilbert space $H = H[K]$ and a system of vectors $v_x \in H$ enumerated by points $x \in X$ such that

[10] we needs in some basic properties of spherical functions, all necessary information is contained in Helgason book [17], chapter 4

1. $< v_x, v_y >_H = L(x,y)$
2. the linear span of the vectors v_x is dense in H.
The family v_x is called a *supercomplete basis*[11].

This construction is natural in the following sense. Let H' be another hilbert space and let v'_x be another system of vectors satisfying the same conditions. Then there exists the unique unitary operator $U : H \to H'$ such that $Uv_x = v'_x$ for all $x \in X$.

If X is a separable metric space and the kernel $L(x,y)$ is continuous, then the hilbert space $H[L]$ is separable.

In Subsections 1.2-1.3 and 1.4 we discuss two ways of "materialization" of the space $H[L]$.

1.2. Scalar product in the space of complex-valued measures. Assume X be a separable complete metric space. Let μ be a complex-valued measure (charge) on X with a compact support. Consider a vector

$$v(\mu) = \int_X v_x d\mu(x) \in H[L]$$

Thus, we obtain a way to represent elements of $H[L]$. Obviously

$$< v(\mu), v(\nu) >_{H[L]} = \int_{X\times X} L(x,y)\, d\mu(x)\, \overline{d\nu(y)}$$

Let us say the same construction more formally. Consider the linear space $\mathcal{M}(X)$ of all compactly supported complex-valued measures on X. Consider the scalar product in $\mathcal{M}(X)$ defined by the formula

$$< \mu, \nu > = \int_{X\times X} L(x,y)\, d\mu(x)\, \overline{d\nu(y)}$$

We obtain a structure of a prehilbert space in $\mathcal{M}(X)$ and the space $H[L]$ is the hilbert space associated with the prehilbert space $\mathcal{M}(X)$ (elements of the supercomplete basis corresponds to measures supported at points).

1.3. Scalar products in spaces of distributions. Assume X be a smooth manifold and the kernel $L(x,y)$ be smooth. Denote by $\mathcal{D}$ the space of compactly supported distributions on X. Consider the scalar product in $\mathcal{D}$ given by the formula

$$< \chi, \varphi > = \{K(x,y), \chi(x) \otimes \overline{\varphi(y)}\} \tag{1.2}$$

where brackets $\{\cdot,\cdot\}$ denote the pairing of smooth functions and distributions. Consider the hilbert space H associated with the prehilbert space $\mathcal{D}$. Denote δ-distribution supported at a point x by δ_x. Obviously

$$< \delta_x, \delta_y > = L(x,y)$$

[11] Other terms for v_x are *overfilled basis* or *system of coherent states.*

Hence, we can identify H with $H[L]$ and the vectors δ_x with elements of the supercomplete basis v_x.

REMARK 1. The space of distributions equipped with scalar product (1.2) is not complete. This means that some vectors of $H[L]$ can not be represented by distributions.

REMARK 2. Scalar product (1.2) in $\mathcal{D}$ can be degenerated. This means that a vector $h \in H$ can be represented by a distribution in various ways.

1.4. The embedding of $H[L]$ to the space of functions on X. For arbitrary $h \in H[L]$ we consider the function

$$f_h(x) := < h, v_x >_{H[L]}$$

on the space X. Obviously, the map $h \mapsto f_h$ is an embedding of $H[L]$ to the space of functions on X. We denote the image of the embedding by $H^\circ[L]$. By construction, the space $H^\circ[L]$ has structure of a hilbert space.

LEMMA 1.1. *Assume X be a separable metric space and the kernel $L(x, y)$ be continuous. Let h_j converges to h. Then f_{h_j} converges to f_h uniformly on compacts.*

PROOF. Let $Y \subset X$ be a compact set. Let $x \in Y$. Then

$$\begin{aligned} &|f_{h_j}(x) - f_h(x)| = | < (h_j - h, v_x >_{H[L]} | \leqslant \\ &\leqslant \|h_j - h\| \cdot \|v_x\| = \|h_j - h\| \cdot \sqrt{L(x,x)} \leqslant \|h_j - h\| \cdot \sqrt{\max_{x \in Y} L(x,x)} \qquad \boxtimes \end{aligned}$$

Obviously, the function $\varphi_a(x) \in H^\circ[L]$ associated with the vector $v_a \in H[L]$ is given by the formula

$$\varphi_a(x) = L(x, a) \tag{1.3}$$

LEMMA 1.2. *Let X be a locally compact metric space and the kernel $L(x, y)$ be continuous. Then functions $f_h \in H^\circ[L]$ are continuous.*

PROOF. The linear span of the functions φ_a is dense in $H^\circ[L]$. Then we apply Lemma 1.1. $\boxtimes$

LEMMA 1.3. (REPRODUCING PROPERTY) *For any $f \in H^\circ[L]$, $x \in X$ the following identity holds*

$$f(x) = < f, \varphi_x >_{H^\circ[L]} \tag{1.4}$$

PROOF. Let $f = f_h$. Then

$$< f_h, \varphi_x >_{H^\circ[L]} = < h, v_x >_{H[L]} = f_h(x) \qquad \boxtimes$$

REMARK. Equation (1.4) gives a nonexplicit description of the scalar product in $H^\circ[L]$ and this description is sufficient for many purposes. Another way of description is the following identity. Let $e_n(x) \in H^\circ[L]$ be an orthonormal basis. Then

$$L(x,y) = \sum \overline{e_n(x)} e_n(y)$$

(proof: consider $< \varphi_x, e_n >$)

LEMMA 1.4. *Let Ω be an open domain in $\mathbb{C}^n$ and let $L(z,u)$ be a positive definite kernel on Ω. Let $L(z,u)$ be holomorphic in the variable u and anti-holomorphic in the variable z. Then all elements of the space $H^\circ[L]$ are holomorphic functions on Ω.*

PROOF. It is a corollary of Lemma 1.1. ⊠

1.5. Operations with positive definite kernels.

LEMMA 1.5. a) *Let $L_1(x,y)$, $L_2(x,y)$ be positive definite kernels on X. Then $L_1(x,y) + L_2(x,y)$ is a positive definite kernel*

b) *Let $L_1(x,y)$, $L_2(x,y)$ be positive definite kernels on X. Then the kernel $L_1(x,y)L_2(x,y)$ is positive definite*

c) *Let $L_j(x,y)$ be positive definite kernels and $L_j(x,y)$ converges to $L(x,y)$ point-wise. Then $L(x,y)$ is a positive definite kernel.*

d) *Let $K_m(x,y)$ be a family of positive definite kernels enumerated by points of some measure space M with positive measure μ. Assume that the integral*

$$K^*(x,y) = \int_M K_m(x,y)d\mu(m)$$

converges for all $x,y \in X$. Then $K^*(x,y)$ is positive definite.

e) *Let $L(x,y)$ be a positive definite kernel and let $\lambda(x)$ be a function on X. Then the kernel $M(x,y) = \lambda(x)\overline{\lambda(y)}L(x,y)$ is positive definite.*

PROOF. a) Let v_x (resp. w_x) be the supercomplete basis in $H[L_1]$ (respectively $H[L_2]$). We consider the system of vectors $v_x \oplus w_x \in H[L_1] \oplus H[L_2]$. Then

$$L_1(x,y) + L_2(x,y) = < v_x \oplus w_x, v_y \oplus w_y >$$

b) Proof is similar, $H[L_1L_2] \subset H[L_1] \otimes H[L_2]$

d) This is consequence of a) and c).

e) Indeed, $H[M] = H[L]$ and the supercomplete basis in $H[M]$ consists of vectors $\gamma(x)v_x$ where v_x is the supercomplete basis in $H[L]$. ⊠

1.6. Positive definite kernels on homogeneous spaces. Let Γ be a group acting on X and let a positive definite kernel $L(x,y)$ be Γ-invariant

$$L(gx, gy) = L(x,y) \qquad \text{for all} \qquad g \in \Gamma, x,y \in X$$

Obviously, for each $g \in \Gamma$ where exists the unique unitary operator $U(g) : H[L] \to H[L]$ such that

$$Uv_x = v_{gx} \qquad \text{for all} \qquad x \in X$$

Then

$$U(g_1g_2) = U(g_1)U(g_2)$$

Hence, $U(g)$ is an unitary representation of Γ.

Let X be a Γ-homogeneous space, $X = \Gamma/K$, let x_0 be a H-fixed point.

Let $L(x, y)$ be a Γ-invariant function. Then L is completely defined by the function

$$l(y) := L(x_0, y)$$

Indeed, let $u, z \in X$. Then $u = gx_0$ for some element g in Γ and

$$L(u, z) = L(gx_0, z) = L(x_0, g^{-1}z) = l(g^{-1}z)$$

Moreover, for any $\gamma \in K$ we have

$$l(y) = L(x_0, y) = L(\gamma x_0, \gamma y) = L(x_0, \gamma y) = l(\gamma y)$$

We see that the function l_0 is a K-invariant function on Γ/K.

We also can consider a K-invariant function $l(y)$ as function on double cosets $K \setminus \Gamma/K$.

We see that *there is the canonical correspondence between 3 following sets:*

– Γ*-invariant functions on* $\Gamma/K \times \Gamma/K$

– K*-invariant functions on* Γ/K

– *functions on* $K \setminus \Gamma/K$.

We say that a K-invariant function on Γ/K or a function on $K \setminus \Gamma/K$ is *positive definite* if the associated kernel on $\Gamma/K \times \Gamma/K$ is positive definite.

1.7. On K-invariant vectors in representations of Γ. Let Γ, K, x_0 be the same as above. Let ρ be an unitary representation of Γ in a hilbert space H. Assume that there exists a K-invariant vector $v \in H$ and assume v be a *cyclic vector*[12].

Consider the map $\Gamma/K \to H$ given by the formula

$$gx_0 \mapsto \rho(g)v$$

(the image of the map is the Γ-orbit of the vector v). Then the function

$$L(g_1x_0, g_2x_0) :=< \rho(g_1)v, \rho(g_2)v >_H$$

is a Γ-invariant positive definite kernel on Γ/K.

Hence, we can identify the hilbert space H with the space $H[L]$; the supercomplete basis in H consists of vectors $\rho(g)v$.

The function $l(y)$ in our case is the matrix element $< \rho(g)v, v >$ and our construction (Segal–Gelfand–Naimark construction) reconstructs the representation ρ by its matrix element.

[12]This means that the linear span of vectors $\rho(g)v$ is dense in H.

B. Kernel representations.

Assume $p \leqslant q$

1.8. Pseudoorthogonal group $\mathrm{O}(p,q)$**.** Consider the linear space $\mathbb{C}^p \oplus \mathbb{C}^q$ equipped with the indefinite hermitian form

$$J((x,y),(u,v)) = \sum_{j=1}^{p} x_j \overline{u}_j - \sum_{j=1}^{q} y_k \overline{v}_k; \qquad (x,y),(u,v) \in \mathbb{C}^p \oplus \mathbb{C}^q \tag{1.5}$$

The *pseudounitary group* $\mathrm{U}(p,q)$ is the group of all linear operators $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $\mathbb{C}^p \oplus \mathbb{C}^q$ preserving the form $J(\cdot,\cdot)$. In other words, a matrix $g \in \mathrm{U}(p,q)$ satisfies the condition

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.6}$$

The *pseudoorthogonal group* $\mathrm{O}(p,q)$ is the subgroup of $\mathrm{U}(p,q)$ consisting of real matrices. Below in Sections 1-5 by the symbol G we denote the group

$$G = \mathrm{O}(p,q)$$

By K we denote the subgroup $\mathrm{O}(p) \times \mathrm{O}(q) \subset G$ consisting of matrices having the form $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$. It is a maximal compact subgroup in G.

1.9. Matrix balls. By $\mathrm{B}_{p,q}(\mathbb{C})$ we denote space of all complex $p \times q$ matrices z having norm < 1 (where *a norm* is the norm of the operator $v \mapsto vz$ from the euclidean space $\mathbb{C}^p$ to the euclidean space $\mathbb{C}^q$; remind that $\|z\|^2$ is the maximal eigenvalue of $z^* z$).

By $\mathrm{B}_{p,q}(\mathbb{R})$ we denote the space of real $p \times q$ matrices with norm < 1.

The group $\mathrm{U}(p,q)$ acts on the matrix ball $\mathrm{B}_{p,q}(\mathbb{C})$ by *fractional linear* transformations

$$z \mapsto z^{[g]} := (\alpha + z\gamma)^{-1}(\beta + z\delta) \tag{1.7}$$

This action is transitive and the stabilizer of the point $z = 0$ is the subgroup $\mathrm{U}(p) \times \mathrm{U}(q)$. Hence, $\mathrm{B}_{p,q}(\mathbb{C})$ is the symmetric space

$$\mathrm{B}_{p,q}(\mathbb{C}) = \mathrm{U}(p,q)/\mathrm{U}(p) \times \mathrm{U}(q)$$

In the same way, $\mathrm{B}_{p,q}(\mathbb{R})$ is the symmetric space

$$\mathrm{B}_{p,q}(\mathbb{R}) = G/K = \mathrm{O}(p,q)/\mathrm{O}(p) \times \mathrm{O}(q)$$

Arbitrary symmetric space admits unique up to factor invariant measure. For the space $\mathrm{B}_{p,q}(\mathbb{R})$ the $\mathrm{O}(p,q)$-invariant measure is given by the formula

$$d\lambda(z) = \det(1 - z^* z)^{-(p+q)/2} d\mu(z) \tag{1.8}$$

where $d\mu(z)$ is the Lebesgue measure on $\mathrm{B}_{p,q}(\mathbb{R})$.

1.10. Berezin kernels. THEOREM 1.6[13] . *The kernel*

$$L_\alpha(z,u) = \det(1 - z^*u)^{-\alpha}$$

on the matrix ball $\mathrm{B}_{p,q}(\mathbb{C})$ *is positive definite if and only if*

$$\alpha = 0, 1, 2, \dots, p-1 \quad \text{or} \quad \alpha > p-1 \tag{1.9}$$

Thus, for α satisfying the Berezin condition (1.9), we obtain the hilbert spaces $H_\alpha := H[L_\alpha]$ and $H_\alpha^\circ := H^\circ[L_\alpha]$. The function $L_\alpha(z,u)$ is anti-holomorphic in z and hence by Lemma 1.4 the space H_α° consists of holomorphic functions on the matrix ball $\mathrm{B}_{p,q}(\mathbb{C})$.

REMARK. For $\alpha > p+q-1$ the scalar product in H_α° can be represented in the form

$$< f, g >_\alpha = C(\alpha) \int_{\mathrm{B}_{p,q}} f(z)\overline{g(z)} \det(1 - z^*z)^{\alpha-p-q} d\mu(z)$$

where $d\mu(z)$ is the Lebesgue measure on $\mathrm{B}_{p,q}(\mathbb{C})$ and $C(\alpha)$ is the meromorphic factor defined by the condition $< 1, 1 >_\alpha = 1$. In particular H_{p+q}° is the Bergman space. For $\alpha = q$ we obtain the Hardy space H^2. The scalar product in this case is given by the formula

$$< f(z), g(z) >_q = \int_{z^*z=1} f(z)\overline{g(z)} d\nu(z)$$

where $d\nu(z)$ is the unique $\mathrm{U}(p) \times \mathrm{U}(q)$-invariant measure on the set[14] $zz^* = 1$. For other values of parameters there exist integral formulas including partial derivatives but they are not simple (see [1]).

REMARK. For $\alpha > p-1$ the space H_α contains all polynomials on $\mathrm{B}_{p,q}$. For $\alpha = 0, 1, \dots, p-1$ all functions $f \in H_\alpha$ satisfy some system of partial differential equations. For $\alpha = 0$ our space contains only constants.

PROPOSITION 1.7. ([3], [47]) a)*For any* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{U}(p,q)$ *the operator*

$$\widetilde{T}_\alpha(g)(z)f(z) = f((a+zc)^{-1}(b+zd)) \det(a+zc)^{-\alpha} \tag{1.10}$$

is unitary in H_α°

REMARK. If α is not integer, then

$$\begin{aligned} \det(a+zc)^{-\alpha} &= \det a^{-\alpha} \det(1 + zca^{-1})^{-\alpha} = \\ &= |\det a|^{-\alpha} \cdot e^{-\alpha(i \arg \det a + 2\pi k i)} \det(1 + zca^{-1})^{-\alpha} \end{aligned} \tag{1.11}$$

[13]see Berezin [3] (1975), see also Gindikin [11], Rossi, Vergne [47], Wallach [55], see also a recent exposition in [9].

[14]Schtiefel manifold

is a multi-valued function. It is easy to show that $\|ca^{-1}\| < 1$. Hence, $(1 + zca^{-1})^{-\alpha}$ is a well-defined single-valued function on the matrix ball $\mathrm{B}_{p,q}(\mathbb{C})$. Hence, expression (1.11) has countable family of holomorphic branches on $\mathrm{B}_{p,q}(\mathbb{C})$ and formula (1.10) defines a countable family of well-defined operators which differs by constant factors $e^{2\pi k\alpha i}$.

PROOF. Consider the supercomplete basis $\varphi_x(z) = \det(1 - x^*z)^{-\alpha}$ in H_α°. A calculation show that

$$\widetilde{T}_\alpha(g^{-1})\varphi_x(z) = \det(a + zc)^\alpha \varphi_{x^{[g]}}(z)$$

The simple identity

$$\det(1 - x^{[g]}(y^{[g]})^*) = \det(1 - xy^*)\det(a + xc)^{-1}\det(a + yc)^{-1}$$

implies

$$< \varphi_x, \varphi_y >_{H_\alpha^\circ} = < \widetilde{T}_\alpha(g^{-1})\varphi_x, \widetilde{T}_\alpha(g^{-1})\varphi_y >_{H_\alpha^\circ} \qquad \boxtimes$$

Obviously

$$\widetilde{T}_\alpha(g_1)\widetilde{T}_\alpha(g_2) = e^{2\pi m\alpha i}\widetilde{T}_\alpha(g_1 g_2), \qquad \text{where} \quad m \in \mathbb{Z}$$

If α is integer, then $\widetilde{T}_\alpha$ is a linear representation of $\mathrm{U}(p,q)$. If α is not integer, then $\widetilde{T}_\alpha$ is a projective representation of $\mathrm{U}(p,q)$ or a linear representation of the universal covering group $\mathrm{U}(p,q)^\sim$ of the group $\mathrm{U}(p,q)$.

1.11. Kernel representations of $\mathrm{O}(p,q)$. *Kernel representation* T_α of the group $G = \mathrm{O}(p,q)$ is the restriction of the representation $\widetilde{T}_\alpha$ to the subgroup $\mathrm{O}(p,q)$. We also say that the function $f(z) = 1$ is *the marked vector* in H_α°. We denote this vector by Ξ.

REMARK. A kernel representation is a linear representation. Indeed, we can wright $|\det a|^{-\alpha}(1 + zca^{-1})^{-\alpha}$ instead of (1.11).

LEMMA 1.8. *The vector Ξ is $\mathrm{O}(p,q)$-cyclic.*

PROOF. Let $Q \subset H_\alpha$ be a subspace containing the G-orbit of Ξ. This orbit consists of functions (1.11) and hence the functions $\det(1 + zca^{-1})^{-\alpha}$ are contained in Q. But the point ca^{-1} is the image of 0 under the fractional linear transformation (1.7). Since the action of $\mathrm{O}(p,q)$ is transitive on $\mathrm{B}_{p,q}(\mathbb{C})$, the subspace Q contains all functions $v_u = \det(1 + zu^*)^{-\alpha}$ where $u \in \mathrm{B}_{p,q}(\mathbb{R})$. Furthermore, since the family v_u depends on u holomorphically, $v_u \in Q$ for all $u \in \mathrm{B}_{p,q}(\mathbb{C})$. But v_u is the supercomplete basis in H_α°. Hence, $Q = H_\alpha^\circ$. $\boxtimes$

LEMMA 1.9. *Any $\mathrm{O}(p,q)$-invariant subspace in H_α° contains an $\mathrm{O}(p,q)$-invariant vector.*

PROOF. Assume $H_\alpha^\circ = R \oplus Q$ where R, Q are invariant subspaces. Assume that R hasn't an $\mathrm{O}(p) \times \mathrm{O}(q)$-invariant vector. Then the projection of Ξ to R is zero, and hence $\Xi \in Q$. But Ξ is cyclic. Thus, $Q = H_\alpha^\circ$. $\boxtimes$

1.12. Another description of the kernel-representations. Let α satisfies Berezin conditions (1.9). By Lemma 1.5.e) the kernel

$$M_\alpha(z,u) = \frac{\det(1-zz^*)^{\alpha/2}\det(1-uu^*)^{\alpha/2}}{\det(1-zu^*)^\alpha} \tag{1.12}$$

on $\mathrm{B}_{p,q}(\mathbb{R})$ is positive definite. A simple calculation show that the kernel M_α is $\mathrm{O}(p,q)$-invariant. Hence, we obtain an unitary representation of the group $\mathrm{O}(p,q)$ in the hilbert space $H[M_\alpha] \simeq H^\circ[M_\alpha]$ (see Subsection 1.6). The group $\mathrm{O}(p,q)$ acts in $H^\circ[M_\alpha]$ by substitutions

$$f(z) \mapsto f((a+zc)^{-1}(b+zd)) \tag{1.13}$$

The *marked vector* Ξ in this model is the element of the supercomplete basis corresponding to the point $0 \in \mathrm{B}_{p,q}(\mathbb{R})$.

Let us define the canonical unitary $\mathrm{O}(p,q)$-intertwining operator

$$A : H^\circ[L_\alpha] \to H^\circ[M_\alpha]$$

Let $f \in H^\circ[L_\alpha]$, let $z \in \mathrm{B}_{p,q}(\mathbb{R})$. Then

$$Af(z) = f(z)\det(1-zz^*)^{\alpha/2}$$

This map transforms elements of the supercomplete basis in $H^\circ[L_\alpha]$ to elements of the supercomplete basis in $H^\circ[M_\alpha]$.

1.13. Limit as $\alpha \to \infty$. Let λ be the $\mathrm{O}(p,q)$-invariant measure on $\mathrm{B}_{p,q}(\mathbb{R})$ (see (1.8)). Denote by C_0 the space of continuous functions on $\mathrm{B}_{p,q}(\mathbb{R})$ with a compact support. If $\varphi \in C_0$, then $\varphi(z)\lambda(z)$ is a complex valued measure on $\mathrm{B}_{p,q}$. Hence (see Subsection 1.2), we obtain the scalar product in the space C_0 given by

$$<\varphi,\psi>= A_\alpha \int_{\mathrm{B}_{p,q}(\mathbb{R})\times \mathrm{B}_{p,q}(\mathbb{R})} M_\alpha(z,u)\varphi(z)\overline{\varphi(u)}\,d\lambda(z)d\lambda(u) \tag{1.14}$$

Let us define the normalization constant A_α by the condition

$$A_\alpha = \Bigl(\int_{\mathrm{B}_{p,q}(\mathbb{R})} (1-zz^*)^\alpha dz\Bigr)^{-1}$$

(it is a Hua Loo Keng integral, see (3.5)). Obviously $M_\alpha(z,z) = 1$ and $M_\alpha(z,u) < 1$ if $z \neq u$. It is easy to see that the sequence $A_\alpha M_\alpha(z,u)$ approximates the distribution $\delta(z-u)$. Thus, the limit of scalar products (1.14) as $\alpha \to \infty$ is

$$<\varphi,\psi>= \int_{\mathrm{B}_{p,q}(\mathbb{R})} \varphi(z)\overline{\psi(z)}\,d\lambda(z)$$

In this sense *the limit of kernel representations as $\alpha \to \infty$ is the space $L^2(G/K)$.*

We emphasis that the action of $\mathrm{O}(p,q)$ in $L^2(\mathrm{O}(p,q)/\mathrm{O}(p)\times\mathrm{O}(q))$ and in all spaces $H^\circ[M_\alpha]$ is given by the same formula (1.13) and only scalar product in the space of functions varies. We will see that the spectrum of the representation T_α and the structure of Plancherel formula essentially depends on α.

1.14. Preliminary remarks on the Plancherel formula. Our purpose is to obtain a decomposition of the kernel representation T_α on irreducible representations.

An irreducible representation of $G=\mathrm{O}(p,q)$ is called *spherical* if it contains a K-fixed vector. This vector is called *spherical vector*. Remind that the space of K-fixed vectors for G has dimension 0 or 1 (Gelfand theorem, see for instance [17], Theorem 4.3.1 and Lemma 4.3.6). Denote the set of all unitary spherical representations of $\mathrm{O}(p,q)$ by $\widehat{G}_{sph}$

REMARK. The explicit description of this set is not known. Parametrization of all (generally speaking nonunitary) spherical representations of $\mathrm{O}(p,q)$ is simple and it is given below in Subsection 1.17).

By H_ρ we denote the space of a spherical representation ρ, by $\xi(\rho)$ we denote the spherical vector in H_ρ whose length is 1.

LEMMA 1.10. *Decomposition of the kernel-representation T_α has the form*

$$T_\alpha(g)=\int_{\rho\in\widehat{G}_{sph}}\rho(g)d\nu_\alpha(g) \tag{1.15}$$

where ν_α is a Borel measure on $\widehat{G}_{sph}$.

REMARK. For the definition of direct integrals of representations and the abstract Plancherel formula see, for instance, [23], 8.4.

PROOF. By Lemma 1.9, the decomposition contains only spherical representations. Hence, by the abstract Plancherel theorem the representation $T_\alpha(g)$ has the form

$$T_\alpha(g)=\bigoplus_{j=1}^{\varkappa}R_j$$

where

$$R_j=\int_{\rho\in\widehat{G}_{sph}}\rho(g)d\nu_\alpha^j(g)$$

and the measure ν_α^{j+1} is absolutely continuous with respect to ν_α^j for all j. The number $\varkappa$ can be 1,2, … , ∞. We must prove that $\varkappa=1$.

All K-fixed vectors in R_j are functions having the form $\varphi_j(\rho)\xi(\rho)$ where $\varphi_j(\rho)$ is a ν_α^1-measurable function on $\widehat{G}_{sph}$.

Consider the projection $\Xi^{(1,2)}$ of the marked vector Ξ to $R_1\oplus R_2$. Since the vector Ξ is cyclic in whole space, its projection must be cyclic in $R_1\oplus R_2$. The vector $\Xi^{(1,2)}$ has the form

$$(\varphi_1(\rho)\xi(\rho),\varphi_2(\rho)\xi(\rho))\in R_1\oplus R_2$$

Obviously the cyclic span of $\Xi^{(1,2)}$ in $R_1 \oplus R_2$ contains only vectors

$$(q_1(\rho)\xi(\rho), q_2(\rho)\xi(\rho))$$

satisfying the condition

$$\varphi_2(\rho)q_1(\rho) = \varphi_1(\rho)q_2(\rho)$$

If $\nu_\alpha^2 \neq 0$ we obtain a contradiction, since the cyclic span of $\Xi^{(1,2)}$ is a proper subspace in $R_1 \oplus R_2$. ⊠

1.15. Normalization of the Plancherel measure. The measure ν_α in (1.15) is defined up to equivalence of measures[15].

The image of the marked vector Ξ in the direct integral (1.15) is some function $\varphi(\rho)\xi(\rho)$ where $\xi(\rho)$ is an unit K-fixed vector in H_ρ. It is convenient to assume

$$\varphi(\rho) = 1 \tag{1.16}$$

This assumption uniquely defines the measure ν_α.

Remark. Assumption (1.16) is not restrictive. Indeed, let us assume that the image of Ξ in (1.15) is a function $\gamma(\rho)\xi(\rho)$. Then the Plancherel measure is completely defined by this assumption and it equals to $\frac{1}{\sqrt{|\gamma|}}\nu_\alpha$.

After normalization (1.16) we obtain the following equality of matrix elements

$$< T_\alpha(g)\Xi, \Xi >_{H_\alpha} = < \left[\int_{\widehat{G}_{sph}} \rho(g) d\nu_\alpha(\rho)\right] \cdot 1, 1 > \tag{1.17}$$

or

$$< T_\alpha(g)\Xi, \Xi >_{H_\alpha} = \int_{\widehat{G}_{sph}} < \rho(g)\xi(\rho), \xi(\rho) >_{H_\rho} d\nu_\alpha(\rho) \tag{1.18}$$

Conversely, assume that we know a measure ν_α on $\widehat{G}_{sph}$ satisfying condition (1.18). Then it satisfies condition (1.17). Hence, the representations T_α and $\int_{\widehat{G}_{sph}} \rho(g) d\nu_\alpha$ have the same matrix elements, and therefore they are canonically equivalent (see Subsection 1.7).

The marked vector Ξ is K-invariant, therefore (see Subsection 1.6) we can consider the matrix element

$$\mathcal{B}_\alpha(g) := < T_\alpha(g)\Xi, \Xi >$$

as a function on G/K or a function on $K \setminus G/K$. Vectors Ξ and $T_\alpha(g)\Xi$ are elements of the supercomplete basis in $H[M_\alpha]$, therefore the function $\mathcal{B}_\alpha$ can be easily evaluated.

[15] measures μ, ν are equivalent if there exists a function χ such that $\chi \neq 0$ almost everywhere (in sense of ν) and $\mu = \chi\nu$.

In the matrix ball model of G/K the function $\mathcal{B}_\alpha$ is given by the formula

$$\mathcal{B}_\alpha(z) = \det(1 - zz^*)^{\alpha/2}; \qquad z \in \mathrm{B}_{p,q} \tag{1.19}$$

Let us obtain the formula for $\mathcal{B}_\alpha$ as function on $K \setminus G/K$. Denote by a_t the element of $\mathrm{O}(p,q)$ given by the matrix

$$a_t = \begin{pmatrix} \cosh t_1 & & & \sinh t_1 & & & 0 & \cdots & 0 \\ & \ddots & & & \ddots & & & \ddots & \\ & & \cosh t_p & & & \sinh t_p & 0 & \cdots & 0 \\ \sinh t_1 & & & \cosh t_1 & & & 0 & \cdots & 0 \\ & \ddots & & & \ddots & & & \ddots & \\ & & \sinh t_p & & & \cosh t_p & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & \\ & \ddots & & & \ddots & & & \ddots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & \cdots & 1 \end{pmatrix} \tag{1.20}$$

It is easy to show that arbitrary element g of $G = \mathrm{O}(p,q)$ can be represented in the form

$$g = k_1 a_t k_2; \qquad \text{where} \qquad k_1, k_2 \in K$$

The collection of parameters $t = (t_1, \dots, t_p)$ is uniquely defined up to permutations of t_j and reflections

$$(t_1, \dots, t_p) \mapsto (\sigma_1 t_1, \dots, \sigma_p t_p) \tag{1.21}$$

where $\sigma_j = \pm 1$.

We denote by $\mathcal{A}$ the subgroup in $\mathrm{O}(p,q)$ consisting of all elements a_t. We denote by D_p the group of transformations of $\mathbb{R}^p$ generated by permutations of coordinates and reflections (1.21).

We identify the set $K \setminus G/K$ with the set of D_p-orbits on $\mathcal{A}$.

In coordinates $(t_1, \dots, t_p)$ the matrix element $\mathcal{B}_\alpha$ is given by the formula

$$\mathcal{B}_\alpha(t_1, \dots, t_p) = \prod_{k=1}^{p} \cosh^{-\alpha} t_k \tag{1.22}$$

Hence, we must obtain the expansion (1.18) of the function $\mathcal{B}_\alpha$ given by formula (1.19) or (1.22) in positive definite spherical functions.

Our purpose in Section C is to give an expression for spherical functions.

C. Spherical representations and spherical transform

1.16. Parabolic subgroup. Consider the space $\mathbb{R}^p \oplus \mathbb{R}^q$ equipped with the indefinite symmetric form J defined by formula (1.5). A subspace $V \subset \mathbb{R}^p \oplus \mathbb{R}^q$ is called *isotropic* if the form J is zero on V.

An *isotropic flag* $\mathcal{V}$ in $\mathbb{R}^p \oplus \mathbb{R}^q$ is a family of isotropic subspaces

$$\mathcal{V}: \quad V_1 \subset V_2 \subset \dots \subset V_p; \qquad \text{where} \quad \dim V_j = j$$

The *flag manifold* $\mathcal{F}$ is the space of all isotropic flags in $\mathbb{R}^p \oplus \mathbb{R}^q$.

The space $\mathcal{F}$ is an $\mathrm{O}(p,q)$-homogeneous space. A *minimal parabolic subgroup* is the stabilizer of a point in $\mathcal{F}$. Let us give more explicit description of the minimal parabolic subgroup.

For this let us consider the basis $v_1, \dots, v_p, w_1, \dots, w_{q-p}, v'_1, \dots, v'_p$ in $\mathbb{R}^p \oplus \mathbb{R}^q$ defined by

$$v_j = \frac{1}{\sqrt{2}}(e_j + e_{q+j}); \qquad v'_j = \frac{1}{\sqrt{2}}(e_j - e_{q+j}); \quad w_k = e_{p+k} \tag{1.23}$$

Then

$$J(e_k, e'_k) = 1; \qquad J(f_k, f_k) = 1$$

and the scalar products of all other pairs of basic vectors are zero.

Denote by L_k the subspace in $\mathbb{R}^p \oplus \mathbb{R}^q$ generated by the basic vectors $v_1, \dots, v_k$. We denote by $P \subset \mathrm{O}(p,q)$ the stabilizer in $\mathrm{O}(p,q)$ of the isotropic flag

$$\mathcal{L}: \; L_1 \subset \dots \subset L_p \tag{1.24}$$

The subgroup P is a minimal parabolic subgroup in $\mathrm{O}(p,q)$ and

$$\mathcal{F} \simeq \mathrm{O}(p,q)/P$$

Elements of the parabolic subgroup P in the basis (1.23) have the form

$$\begin{pmatrix} A & * & * \\ 0 & C & * \\ 0 & 0 & A^{t-1} \end{pmatrix} \tag{1.25}$$

where A is an upper triangular matrix and $C \in \mathrm{O}(q-p)$.

Elements of the subgroup $\mathcal{A}$ (see Subsection 1.15) in new basis are diagonal matrices with eigenvalues

$$e^{t_1}, \dots, e^{t_p}, 1, \dots, 1, e^{-t_1}, \dots, e^{-t_p}$$

Remark. Let us change the order of basic elements (1.23) to $v_1, \dots, v_p$ $,w_1, \dots, w_{q-p}$, $v'_p, \dots, v'_1$. Then elements of the parabolic subgroup P will be upper triangular matrices.

1.17. Spherical representations. Denote by Gr_k the space of all k-dimensional isotropic subspaces in $\mathbb{R}^p \oplus \mathbb{R}^q$. Consider the tautological embedding of the flag space $\mathcal{F}$ to the product of the Grassmannians $\times_{k=1}^{p}\mathrm{Gr}_k$ (to each point $\mathcal{V}: V_1 \subset \cdots \subset V_p$ we assign the point $(V_1, \ldots, V_p) \in \times_{k=1}^{p}\mathrm{Gr}_k$).

Consider the natural action of $\mathrm{O}(p,q)$ on Gr_k. For $g \in G$ we denote by $j_k(g,V)$ the Jacobian of the transformation g at the point $V \in \mathrm{Gr}_k$. By $J(g,\mathcal{V})$ we denote the Jacobian of the transformation g on the flag space $\mathcal{F}$.

Fix $s_1, \ldots, s_p \in \mathbb{C}$. Assume $s_0 = s_{p+1} = 0$. We define the representation $\widetilde{\pi}_s$ of the group $\mathrm{O}(p,q)$ in the space of functions on $\mathcal{F}$ by the formula

$$\widetilde{\pi}_s(g)f(V_1,\ldots,V_p) = f(gV_1,\ldots,gV_p)J(g,\mathcal{V})^{1/2}\prod_{k=1}^{p} j_k(g,V_k)^{(s_{j-1}-2s_j+s_{j+1})/2}$$

REMARK. The representation $\widetilde{\pi}_s$ is a Harish-Chandra module[16] and hence a topology in the space of functions on $\mathcal{F}$ is not essential. For instance, we can consider the space $L^2(\mathcal{F})$, the space of smooth functions $C^\infty(\mathcal{F})$, the space of distributions $\mathcal{D}(\mathcal{F})$, the space of hyperfunctions etc.

REMARK. Consider the δ-function $\delta_{\mathcal{L}}$ supported at the point $\mathcal{L} \in \mathcal{F}$ (see (1.24)). It is easy to observe that the function $\delta_{\mathcal{L}}$ is an eigenfunction of P and for a matrix $g \in P$ given by (1.25) we have

$$\widetilde{\pi}_s(g)\delta_{\mathcal{L}}(\mathcal{V}) = \exp\Big\{\sum_{j=1}^{p} t_j(s_j - (q+p)/2 + j)\Big\}\delta_{\mathcal{L}}(\mathcal{V}) \qquad (1.26)$$

where e^{t_j} are the absolute values of the eigenvalues of the matrix A (see (1.25)).

We want to define a canonical irreducible subquotient π_s in $\widetilde{\pi}_s$.

REMARK. For generic $s \in \mathbb{C}^n$ the representation $\widetilde{\pi}_s$ is irreducible and hence $\pi_s \simeq \widetilde{\pi}_s$.

Consider the function $f_0(\mathcal{V}) = 1$ on the space $\mathcal{F}$, it is the unique K-invariant function on $\mathcal{F}$ (since $\mathcal{F}$ is K-homogeneous). Denote by S the cyclic span of f_0. Denote by R the sum of all proper $\mathrm{O}(p,q)$-submodules in S.

LEMMA 1.11. $R \neq S$.

PROOF. Indeed there is the unique K-fixed vector in S and this vector is cyclic. Hence, it can't be element of a proper submodule. Hence, a proper submodule in S hasn't K-fixed vector. Hence, R also hasn't K-fixed vectors and hence $f_0 \notin R$. ⊠

We define the $\mathrm{O}(p,q)$-module π_s by

$$\pi_s = S/R$$

THEOREM 1.12.[17] *The representations π_s are precisely all spherical representations of* $\mathrm{O}(p,q)$. *Moreover*

$$\pi_s \simeq \pi_{s'} \quad \text{iff there exists} \quad \gamma \in D_p \quad \text{such that} \quad \gamma s = s'$$

[16]This means (for instance, see [24]) that the spectrum of the maximal compact subgroup K in the space of functions on $\mathcal{F}$ has finite multiplicities.

[17]for instance, see [17], Theorem 4.4.3

Hence, we can consider our Plancherel measure ν_α as a measure on $\mathbb{C}^p/D_p$. *It will be more convenient for us to consider the Plancherel measure as a D_p-invariant measure on $\mathbb{C}^p$ or any measure on on $\mathbb{C}^p$ whose D_p-average is ν_α.*

1.18. Unitary spherical representations. LEMMA 1.13. *Assume the representation π_s be unitary. Then for any j*

$$\operatorname{Re} s_j = 0 \qquad \text{or} \qquad \operatorname{Im} s_j = 0 \tag{1.27}$$

PROOF. The representation dual to π_s is π_{-s}. The complex conjugate representation to π_s is $\pi_{\overline{s}}$. If π_s is unitary, then the dual representation is equivalent to the complex conjugate representation. Hence, $-s = \gamma\overline{s}$ for some $\gamma \in D_p$. ⊠

If $s_1, \dots, s_p$ are pure imaginary, then the representation π_s is unitary in $L^2(\mathcal{F})$. These representations are called *representations of the principal nondegenerate series.*

For some other values of s representations π_s also are unitary, but scalar product in these cases is more complicated.

THEOREM 1.14. (*see* [17], 4.8.1) *Denote by ρ the vector*

$$\big((q+p)/2-1, (q+p)/2-2, \dots, (q-p)/2)\big) \in \mathbb{R}^p$$

Denote by Q the convex polyhedron in $\mathbb{R}^p$ with vertices $\gamma\rho$ where $\gamma \in D_p$. Then for each unitary representation π_s

$$(\operatorname{Re} s_1, \dots, \operatorname{Re} s_p) \in Q \tag{1.28}$$

Moreover, spherical function of a spherical representation π_s is bounded if and only if condition (1.28) holds.

Our next purpose is to give the integral formula for the spherical functions in an explicit form. For this we must give another realization of G/K.

1.19. Matrix wedges. First, consider the case $p = q$. Consider the matrix ball $\mathrm{B}_{q,q}(\mathbb{R})$. Consider the *Cayley transform*

$$\operatorname{Cay} : z \mapsto \frac{1-z}{1+z} \tag{1.29}$$

Then the map Cay transfers the matrix ball $\mathrm{B}_{q,q}(\mathbb{R})$ to the wedge W_q consisting of matrices R satisfying the condition[18]

$$R + R^t > 0$$

(where the notation $Q > 0$ means that a matrix Q is positive definite). It is convenient to wright R in the form

$$R = T + S \qquad \text{where} \qquad T = T^t > 0; \quad S = -S^t$$

[18]A matrix satisfying the condition $R + R^t > 0$ is called *dissipative.*

The group $\mathrm{O}(q,q)$ acts on $\mathrm{B}_{q,q}(R)$ and hence it acts on $\mathrm{W}_{q,q}$. For description of the last action we consider the basis (1.23) in $\mathbb{R}^q \oplus \mathbb{R}^q$. In our case $p=q$, and hence the basic elements w_j are lacking. Hence, $\mathrm{O}(q,q)$ becomes the group of real $(q+q)\times(q+q)$-matrices having the form $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ and satisfying the condition

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}^t=\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$$

The group $\mathrm{O}(q,q)$ acts on W_q by fractional linear transformations

$$R\mapsto R^{[g]}:=(a+Rc)^{-1}(b+Rd)$$

In this model, the parabolic subgroup $P\subset \mathrm{O}(q,q)$ becomes the group of real matrices having the form

$$\begin{pmatrix} a & b \\ 0 & a^{t-1}\end{pmatrix}; \qquad \text{where} \qquad d \text{ is upper triangular} \tag{1.30}$$

Hence, the parabolic subgroup acts on W_q by affine transformations

$$R\mapsto a^{-1}Ra^t+a^{-1}b$$

We emphasis that $a^{-1}b$ is a skew-symmetric matrix.

We can easily wright the eigenfunctions of the group P on the wedge W_q

$$\Psi_{s_1,\dots,s_q}(R)=\prod_{j=1}^q \det[T]_j^{(-\theta_j+s_j-s_{j+1})/2} \quad \text{where} \quad \theta_1=\dots=\theta_{p-1}=1,\ \theta_p=0 \tag{1.31}$$

Here the symbol $[T]_j$ denotes the left upper $j\times j$ block of the matrix T.

Consider $g\in P$ given by formula (1.30). Let $e^{t_1},\dots,e^{t_p}$ be the absolute values of the diagonal elements of the block a. Then

$$\Psi_{s_1,\dots,s_q}(R^{[g]})=\exp\{\sum_{j=1}^q (s_j-q+j)t_j\}\Psi_{s_1,\dots,s_q}(R) \tag{1.32}$$

REMARK. Compare (1.32) and (1.26).

1.20. Sections of wedges. Consider arbitrary group $\mathrm{O}(p,q)$. Let us represent a point $z\in \mathrm{B}_{p,q}(\mathbb{R})$ as block $p\times(p+(q-p))$-matrix $z=(z_1\ z_2)$. Consider block $(p+(q-p))\times(p+(q-p))$ matrix

$$\widetilde{z}=\begin{pmatrix} 0 & 0 \\ z_1 & z_2\end{pmatrix}\in \mathrm{B}_{q,q}(\mathbb{R})$$

Thus, we realized the matrix ball $\mathrm{B}_{p,q}(\mathbb{R})$ as a submanifold of $\mathrm{B}_{q,q}(\mathbb{R})$. The image $\mathrm{SW}_{p,q}$ of $\mathrm{B}_{p,q}(\mathbb{R})$ under the Cayley transform (1.29) is the set of $(p+(q-p))\times(p+(q-p))$-matrices $R\in \mathrm{W}_q$ having block structure

$$R=\begin{pmatrix}1&0\\Q&H\end{pmatrix} \tag{1.33}$$

The condition $R+R^t>0$ for matrix (1.33) is equivalent to the condition

$$\frac{1}{2}(H+H^t)-LL^t>0 \tag{1.34}$$

(spaces $\mathrm{SW}_{p,q}$ are real sections of so-called Siegel domains of the second type, see [44])

We will wright matrices $R\in \mathrm{SW}_{p,q}$ in the form

$$R=\begin{pmatrix}1&0\\2L&M+N\end{pmatrix};\qquad M=M^t, N=-N^t$$

The condition (1.34) can be represented in the form

$$M-LL^t>0 \qquad \text{or} \qquad \begin{pmatrix}1&L^t\\L&M\end{pmatrix}>0 \tag{1.35}$$

The eigenfunctions of the parabolic subgroup $P\subset \mathrm{O}(p,q)$ in this model are given by the formula

$$\Psi_{s_1,\dots,s_p}(R)=\prod_{j=1}^{p}\det\Bigl[\begin{pmatrix}1&L^t\\L&M\end{pmatrix}\Bigr]_{q-p+j}^{(-\theta_j+s_j-s_{j+1})/2}=\prod_{j=1}^{p}\det[M-LL^t]_j^{(-\theta_j+s_j-s_{j+1})/2} \tag{1.36}$$

where $\theta_1=\dots=\theta_{p-1}=1,\ \theta_p=\frac{1}{2}(q-p)$ (see simple calculations in [35])

The Berezin kernel L_α (see Subsection 1.10) in the models W_q, $\mathrm{SW}_{p,q}$ is given by the formula

$$L_\alpha(R_1,R_2)=\Bigl[\frac{\det(R_1+R_2^t)}{\det(1+R_1)\det(1+R_2)}\Bigr]^{-\alpha}$$

This gives the following expression for the function $\mathcal{B}_\alpha(R)$ in our coordinates

$$\mathcal{B}_\alpha(R)=\frac{\det 2^\alpha\begin{pmatrix}1&L^t\\L&M\end{pmatrix}^{\alpha/2}}{\det(1+M+N)^\alpha} \tag{1.37}$$

The $\mathrm{O}(p,q)$-invariant measure on $\mathrm{SW}_{p,q}$ is

$$\det\begin{pmatrix}1&L^t\\L&M\end{pmatrix}^{-(p+q)/2}dL\,dM\,dN$$

where dL, dM, dN are Lebesgue measures on the spaces of matrices.

1.21. Canonical embedding of the spherical G-module π_s to the space $C^\infty(G/K)$. First, we define the canonical intertwining operator J_s from $\widetilde{\pi}_s$ to $C^\infty(G/K)$. This operator is uniquely defined by the following property

$$J_s : \quad \delta_{\mathfrak{L}} \mapsto \Psi_s$$

where P-eigenfunctions $\delta_{\mathfrak{L}}$, Ψ_s were defined in Subsections 1.17, 1.19–1.20. By the intertwining property we obtain

$$J_s \delta_{g\mathfrak{L}}(R) = \Psi_s(R^{[g]})$$

and this defines the operator J_s on all δ-functions. Then we extend J_s by linearity and continuity to the whole space of distributions on $\mathcal{F}$.

LEMMA 1.15. *The operator J_s induces an embedding of the subquotient π_s to $C^\infty(G/K)$.*

Le us denote by $d\mathbf{k}$ the Haar measure on $K = \mathrm{O}(p) \times \mathrm{O}(q)$. We assume that the measure of the whole group is 1.

PROOF. Let R, S be the same as in Subsection 1.17. Let $Q \subset C^\infty(G/K)$ be a G-invariant closed subspace. Then for any function $f \in R$, its *average*

$$f^K(R) = \int_{\mathbf{k}\in K} f([R]^{\mathbf{k}})d\mathbf{k}$$

is contained in Q. Hence, Q contains a K-invariant function.

By this reason, J_s maps the submodule R to 0 (since R hasn't K-invariants). Assume that J_s is zero on S. Then J_s is an operator from $\widetilde{\pi}_s/S$ to $C^\infty(G/K)$. But the module $\widetilde{\pi}_s/S$ hasn't K-invariant vectors. Hence, J_s is identical zero and this contradicts to its definition. ⊠

Obviously the K-fixed function $f_0 = 1$ on $\mathcal{F}$ can be represented in the form

$$f_0(\mathcal{V}) = \int_{\mathbf{k}\in K} \delta_{\mathfrak{L}}(\mathbf{k}\mathcal{V})d\mathbf{k}$$

Hence, its image under J_s is the K-average of Ψ_s. This gives the integral formula for spherical function given in the next Subsection.

1.22. Integral formula for spherical functions. Spherical functions are K-averages of P-eigenfunctions on G/K

$$\Phi_{s_1,\dots,s_p}(R) = \int_{\mathbf{k}\in \mathrm{O}(p)\times\mathrm{O}(q)} \Psi_{s_1,\dots,s_p}(R^{[\mathbf{k}]})d\pi(\mathbf{k}) \tag{1.38}$$

LEMMA 1.16.

$$|\Phi_{s_1,\dots,s_p}(t)| \leqslant \Phi_{\mathrm{Re}\, s_1,\dots,\mathrm{Re}\, s_p}(t) \tag{1.39}$$

PROOF is obvious. ⊠

1.23. Spherical transform. Let $f(z)$ be a K-invariant function on G/K. Then the *spherical transform* of f is defined by the formula

$$\widehat{f}(s) = \int_{G/K} \Phi_s(z) f(z) d\lambda(z) \tag{1.40}$$

where λ is the G-invariant measure on G/K.

If $f \in L^2 \cap L^1(G/K)$, then the *Gindikin–Karpelevich inversion formula* (see [13], [14], [17], [9]) is valid

$$f(z) = C \cdot \int_{i\mathbb{R}^p} \widehat{f}(s) \Phi_s(z) \mathfrak{R}(s) ds \tag{1.41}$$

where C is a known constant (see [17], formula (4.6.40)) and $\mathfrak{R}(s)$ is the Gindikin–Karpelevich density. For $G = \mathrm{O}(p,q)$ it is given by the formula

$$\cdot\, \mathfrak{R}(s) = \prod_{k=1}^{p} \frac{\Gamma((q-p)/2 + s_k)\Gamma((q-p)/2 - s_k)}{\Gamma(s_k)\Gamma(-s_k)} \times \tag{1.42}$$

$$\times \prod_{1 \leqslant k < l \leqslant p} \frac{\Gamma(\frac{1}{2}(1 + s_l + s_k))\Gamma(\frac{1}{2}(1 + s_l - s_k))\Gamma(\frac{1}{2}(1 - s_l + s_k))\Gamma(\frac{1}{2}(1 - s_l - s_k))}{\Gamma(\frac{1}{2}(s_l + s_k))\Gamma(\frac{1}{2}(s_l - s_k))\Gamma(\frac{1}{2}(-s_l + s_k))\Gamma(\frac{1}{2}(-s_l - s_k))} \tag{1.43}$$

REMARK. This expression is an elementary function. For instance, using the complement formula for Γ, we reduce factor (1.43) to the form

$$\prod_{1 \leqslant k < l \leqslant p} (s_k^2 - s_l^2) \tan \pi(s_k + s_l) \tan \pi(s_k - s_l) \tag{1.44}$$

If $q - p$ is even, then (1.42) equals to

$$\prod_{k=1}^{p} \Big\{ \prod_{\tau=0}^{(q-p)/2-1} (\tau^2 - s_k^2) \Big\} \tag{1.45}$$

If $(q - p)$ is odd, then (1.42) equals to

$$\prod_{k=1}^{p} \Big\{ s_k \tan \pi s_k \prod_{\tau=0}^{(q-p-3)/2} ((\tau + 1/2)^2 - s_j^2) \Big\} \tag{1.46}$$

For pure imaginary s we can replace (1.42)-(1.43) by

$$\prod_{k=1}^{p} \Big| \frac{\Gamma((q-p)/2 + s_k)}{\Gamma(is_k)} \Big|^2 \prod_{1 \leqslant k < l \leqslant p} \Big| \frac{\Gamma(\frac{1}{2}(1 + s_l + s_k))\Gamma(\frac{1}{2}(1 + s_l - s_k))}{\Gamma(\frac{1}{2}(s_l + s_k))\Gamma(\frac{1}{2}(s_l - s_k))} \Big|^2$$

since s_k is imaginary. Nevertheless long expression (1.42)–(1.43) is more convenient for our calculations.

1.24. Another formula for spherical transform. By integral formula for spherical functions (1.38) we can wright spherical transform (1.40) in the following form

$$\widehat{f}(s) = \int_{G/K} f(z)\Psi_s(z)d\lambda(z) \tag{1.47}$$

1.25. Further structure of the paper. We want to obtain an expansion of the function $\mathcal{B}_\alpha(z)$ in spherical functions. If $\mathcal{B}_\alpha \in L^1 \cap L^2(\mathrm{B}_{p,q}(\mathbb{R}))$ (or $\alpha > p+q-1$), then it is sufficient to evaluate the spherical transform of the function $\mathcal{B}_\alpha(z)$, and the Gindikin-Karpelevich inversion formula gives required expansion.

In Section 3 we evaluate the spherical transform of $\mathcal{B}_\alpha(z)$ using formula (1.47). The final result is given in Theorem 2.1. Then in Section 4 we construct the analytic continuation of our formula to arbitrary α. As result, we obtain an expansion of $\mathcal{B}_\alpha$ in spherical functions. In Section 5 we prove positive definiteness of these spherical functions.

D. Deformation of L^2 on Riemannian compact symmetric space and kernel representations of $\mathrm{O}(p+q)$.

This subject is a supplement to the main topic of the paper.

1.26. The symmetric spaces $\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)$ and $\mathrm{O}(p+q)/\mathrm{O}(p)\times\mathrm{O}(q)$. Consider the group $\mathrm{U}(p+q)$ consisting of all complex block $(p+q)\times(p+q)$-matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying the condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Consider the subgroup $\mathrm{O}(p+q) \subset \mathrm{U}(p+q)$ consisting of real matrices.

Consider the Grassmannians $\mathrm{Gr}_{p,q}(\mathbb{C})$ and $\mathrm{Gr}_{p,q}(\mathbb{R})$ consisting of p-dimensional subspaces in $\mathbb{C}^{p+q}$ and $\mathbb{R}^{p+q}$ respectively. Obviously, these Grassmannians are symmetric spaces

$$\mathrm{Gr}_{p,q}(\mathbb{C}) = \mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)$$
$$\mathrm{Gr}_{p,q}(\mathbb{R}) = \mathrm{O}(p+q)/\mathrm{O}(p)\times\mathrm{O}(q)$$

Denote by $\mathrm{Mat}_{p,q}(\mathbb{C})$ (resp. $\mathrm{Mat}_{p,q}(\mathbb{R})$) the space of all $p\times q$-matrices over $\mathbb{C}$ (resp. over $\mathbb{R}$). For any $z \in \mathrm{Mat}_{p,q}$ we define its graph $\mathrm{GRAPH}_z \subset \mathrm{Gr}_{p,q}$.

Obviously the map $z \mapsto \mathrm{GRAPH}_z$ is an embedding of $\mathrm{Mat}_{p,q} \to \mathrm{Gr}_{p,q}$ and the image of the embedding is dense in $\mathrm{Gr}_{p,q}$.

In the coordinate $z \in \mathrm{Mat}_{p,q}$, the action of the group $\mathrm{U}(p+q)$ on Grassmannian is given by the formula

$$z \mapsto z^{[g]} = (a + zc)^{-1}(b + zd) \tag{1.48}$$

coinciding with formula (1.7).

1.27. Representations $\widetilde{T}_{-n}$. Fix $n = 0, 1, 2, \dots$. Denote by $\varphi_a(z)$ the polynomial on $\mathrm{Mat}_{p,q}$ given by the formula

$$\varphi_a(z) = \det(1 + a^* z)^n; \qquad \text{where} \quad a \in \mathrm{Mat}_{p,q}$$

Denote by H_{-n} the linear span of all polynomials $\varphi_a(z)$. Obviously the space H_{-n} is finite dimensional (since degree of the polynomial $\varphi_a(z)$ is pn).

Consider the action of the group $\mathrm{U}(p,q)$ in the space H_{-n} given by the formula

$$\widetilde{T}_{-n} \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = f\big((a + zc)^{-1}(b + zd)\big) \det(a + zc)^n \tag{1.49}$$

coinciding with formula (1.10). It is easy to check, that the transformations $\widetilde{T}_{-n}(g)$ preserve the space H_{-n}.

Consider the scalar product in H_{-n} given by

$$< f_1(z), f_2(z) >_{-n} = C_n \int_{\mathrm{Mat}_{p,q}} f_1(z) \overline{f_2(z)} \det(1 + z^* z)^{-n-p-q} dz$$

where the normalization constant C_n is defined by the condition $< 1, 1 >_{-n} = 1$.

It is easy to check that the operators $\widetilde{T}_{-n}(g)$ are unitary with respect to this scalar product and

$$< \varphi_a, \varphi_b >_{-n} = \det(1 + a^* b)^n \tag{1.50}$$

Hence, the (finite dimensional) hilbert space H_{-n} is the hilbert space H° defined by the positive definite kernel

$$L_{-n}(a, b) = \det(1 + a^* b)^n; \qquad a, b \in \mathrm{Mat}_{p,q} \tag{1.51}$$

1.28. Kernel representations of $\mathrm{O}(p+q)$. The *kernel representation* T_{-n} of the group $\mathrm{O}(p+q)$ is the restriction of the representation $\widetilde{T}_{-n}$ to the subgroup $\mathrm{O}(p+q)$.

The also define the *marked vector* Ξ

$$\Xi : \qquad f(z) = 1$$

1.29. Limit as $n \to \infty$. Let us consider the kernel

$$M_{-n}(z,u) = \frac{\det(1+z^t u)^n}{\det(1+z^t z)^{n/2}\det(1+u^t u)^{n/2}}$$

on $\mathrm{Mat}_{p,q}(\mathbb{R})$. By Lemma 1.5.e) the kernel M_{-n} is positive definite. A simple calculation shows, that the kernel is $\mathrm{O}(p+q)$-invariant. Consider the hilbert space $H^\circ[M_{-n}]$. The operator

$$Af(z) = \det(1+z^* z)^{n/2} f(z)$$

defines the canonical unitary $\mathrm{O}(p+q)$-intertwining operator $H^\circ[L_\alpha] \to H^\circ[M_\alpha]$.

The arguments given in Subsection 1.13 show that a natural limit of the spaces H_{-n} as $n \to \infty$ is

$$L^2\big(\mathrm{O}(p+q)/\mathrm{O}(p)\times\mathrm{O}(q)\big)$$

1.30. Preliminary remarks on the Plancherel formula. By Subsection 1.7 the matrix element

$$\mathcal{B}_{-n}(g) = < T_{-n}(g)\Xi, \Xi >_{H_{-n}}$$

is a function on $\mathrm{O}(p+q)/\mathrm{O}(p)\times\mathrm{O}(q) \simeq \mathrm{Gr}_{p,q}(\mathbb{R})$. In the coordinate $z \in \mathrm{Mat}_{p,q}(\mathbb{R})$ it is given by

$$\mathcal{B}_{-n}(z) = \det(1+zz^t)^{-n/2}$$

Denote by $\widehat{\mathrm{O}(p+q)}_{sph}$ the set of all irreducible representations of $\mathrm{O}(p+q)$ having an $\mathrm{O}(p)\times\mathrm{O}(q)$-invariant vector (*spherical vector*); description of this (countable) set is given by Helgason theorem, [17], Theorem 5.4.1. By H_ρ we denote the space of a spherical representation $\rho \in \widehat{\mathrm{O}(p+q)}_{sph}$. Denote by ξ_ρ the spherical vector in H_ρ having unit length.

Arguments given in Subsections 1.13-1.14 show, that the decomposition of T_{-n} in irreducible representations has the form

$$T_{-n}(g) = \bigoplus_{\rho\in\Delta_n} \rho(g)$$

where Δ_n is a finite subset in $\widehat{\mathrm{O}(p+q)}_{sph}$.

The scalar product in $\bigoplus_{\rho\in\Delta_n} H_\rho$ has the form

$$< \bigoplus_{\rho\in\Delta_n} v_\rho, \bigoplus_{\rho\in\Delta_n} w_\rho > = \sum_{\rho\in\Delta_n} \nu_\rho^n < v_\rho, w_\rho >_{H_\rho} \tag{1.52}$$

where $v_\rho, w_\rho \in H_\rho$ and ν_ρ^n are positive constants. The formula (1.52) is called the *Plancherel formula.*

We normalize the constants ν_ρ^n by the assumption

$$\text{the image of} \quad \Xi \quad \text{in} \quad \bigoplus_{\rho\in\Delta_n} H_\rho \quad \text{is} \quad \bigoplus_{\rho\in\Delta_n} \xi_\rho$$

The constants ν_ρ^n are evaluated in Section 2 as a corollary of the Plancherel formula for kernel-representations of $\mathrm{O}(p,q)$.

E. An interpolation between $\mathbf{L^2}(\mathrm{O}(\mathbf{p},\mathbf{q}))/\mathrm{O}(\mathbf{p})\times\mathrm{O}(\mathbf{q}))$ and $\mathbf{L^2}(\mathrm{O}(\mathbf{p}+\mathbf{q}))/\mathrm{O}(\mathbf{p})\times\mathrm{O}(\mathbf{q}))$?

The purpose of the Section is a formulation of a strange problem.

1.31. General representations $\widetilde{T}_\alpha$. Denote by $Hol(\mathrm{B}_{p,q})$ the space of holomorphic functions in $\mathrm{B}_{p,q}(\mathbb{C})$ equipped with topology of the uniform convergence on compacts.

Consider arbitrary $\alpha\in\mathbb{C}$ and consider the action of $\mathrm{U}(p,q)$ in $Hol(\mathrm{B}_{p,q})$ given by the formula

$$\widehat{T}_\alpha \begin{pmatrix} a & b\\ c & d\end{pmatrix} f(z) = f\big((a+zc)^{-1}(b+zd)\big)\det(a+zc)^{-\alpha} \tag{1.53}$$

Denote by $\mathcal{H}_\alpha^\circ$ the cyclic span of the function $f(z)=1$. Denote by $\widetilde{T}_\alpha$ the restriction of $\widehat{T}_\alpha$ to $\mathcal{H}_\alpha^\circ$. It is easy to observe that $\widetilde{T}_\alpha$ is an irreducible Harish-Chandra module.

Let us denote by H_α^{fin} the space of polynomials contained in $\mathcal{H}_\alpha^\circ$. Consider the action of the Lie algebra $u(p,q)$ in H_α^{fin}. The space H_α^{fin} is an irreducible $u(p,q)$-module with a highest weight. For $\alpha\in\mathbb{R}$ there exists the unique $u(p,q)$-invariant hermitian form in H_α^{fin} (it is called *Shapovalov form*[19]). In general this form is indefinite.

If $\alpha\in\mathbb{R}$ satisfies Berezin conditions (1.9), then the Shapovalov form is positive definite. It coincides with the Berezin scalar product, and representations $\widetilde{T}_\alpha$ coincides with representations $\widetilde{T}_\alpha$ constructed in Subsection 1.10. If α is a negative integer, then $\widetilde{T}_\alpha$ is finite dimensional. By the unitary Weyl trick, there is no difference between finite dimensional representations of $\mathrm{U}(p,q)$, holomorphic finite dimensional representations of $\mathrm{GL}(p+q,\mathbb{C})$ and finite dimensional representations of $\mathrm{U}(p+q)$. The representations $\widetilde{T}_\alpha$ for negative integer α differs from the representations $\widetilde{T}_{-n}$ from Subsection 1.27 by a nonessential change of notations.

[19] Its definition for highest weight modules of various group is uniform, see for instance [32].

1.32. Nonunitary kernel representations of $\mathrm{O}(p,q)$**?** Consider the restriction T_α of $\widetilde{T}_\alpha$ to the subgroup $\mathrm{O}(p,q)$. It is a well-defined representation of the group $\mathrm{O}(p,q)$ in the space $\mathcal{H}_\alpha^\circ$.

We have seen that

$$\lim_{\alpha\to+\infty} T_\alpha \simeq L^2\big(\mathrm{O}(p,q))/\mathrm{O}(p)\times \mathrm{O}(q)\big)$$
$$\lim_{n\to-\infty} T_n \simeq L^2\big(\mathrm{O}(p+q))/\mathrm{O}(p)\times \mathrm{O}(q)\big)$$

It seems that the Plancherel formula (2.5)-(2.15) gives the decomposition of the kernel representation T_α for any complex α. Unfortunately it is a result of 'mathematical physics level' . This is the solution of a problem which hasn't a satisfactory formulation (since the definition of the abstract Plancherel formula doesn't exist for nonunitary representations).

REMARK. For the case $p=1$ the space $\mathcal{H}_\alpha^\circ$ equipped with the Shapovalov form is a Pontryagin space[20] and in this case our Plancherel formula is really the Plancherel formula for arbitrary real α.

The questions of this type are discussed for a long time and they arise to deep Molchanov work [27](1980) containing the Plancherel decomposition of tensor products of unitary representations of $\mathrm{SL}(2,\mathbb{R})$ (see also [8] and references in this paper). It was clear, that Molchanov formula give formal interpolation between tensor products of unitary representations of $\mathrm{SL}(2,\mathbb{R})$ and tensor products of finite dimensional representations. Unfortunately before our time a quite satisfactory group-theoretical interpretation of this interpolation doesn't exist.

2. Formulation of results.

2.1. Large α**.** THEOREM 2.1.*Let* $\alpha > (p+q)/2-1$. *Then the spectrum of the kernel-representation* T_α *is supported by nondegenerate principal series and*

[20]This means that negative inertia index of the Shapovalov form is finite.

the Plancherel decomposition is given by the formula

$$\prod_{k=1}^{p}\cosh^{-\alpha} t_j = C\cdot 2^{\alpha}\frac{1}{\prod_{j=1}^{p}\Gamma(\alpha-j+1)}\times \tag{2.1}$$

$$\int_{i\mathbb{R}^p}\prod_{k=1}^{p}\left\{\Gamma(\frac{1}{2}(\alpha-(p+q)/2+1+s_k))\Gamma(\frac{1}{2}(\alpha-(p+q)/2+1-s_k))\right\}\times \tag{2.2}$$

$$\times\prod_{k=1}^{p}\frac{\Gamma((q-p)/2+s_k)\Gamma((q-p)/2-s_k)}{\Gamma(s_k)\Gamma(-s_k)}\times \tag{2.3}$$

$$\prod_{1\leqslant k<l\leqslant p}\frac{\Gamma(\frac{1}{2}(1+s_l+s_k))\Gamma(\frac{1}{2}(1+s_l-s_k))\Gamma(\frac{1}{2}(1-s_l+s_k))\Gamma(\frac{1}{2}(1-s_l-s_k))}{\Gamma(\frac{1}{2}(s_l+s_k))\Gamma(\frac{1}{2}(s_l-s_k))\Gamma(\frac{1}{2}(-s_l+s_k))\Gamma(\frac{1}{2}(-s_l-s_k))}\times \tag{2.4}$$

$$\times\,\Phi_{s_1,\dots,s_p}(t_1,\dots,t_p)\,ds_1 ds_2\dots ds_p \tag{2.5}$$

where C is a constant.

Remark. Factor (2.3)–(2.4) is the Gindikin–Karpelevich density, it is an elementary function, see (1.44)–(1.46).

2.2. Analytic formula for arbitrary α**.** Fix $m=0,1,\dots,p$. Consider nonnegative integers

$$u_1\leqslant u_2\leqslant\dots\leqslant u_m$$

satisfying the condition

$$\alpha+2u_m+m<\frac{1}{2}(p+q)$$

(if $m=0$, then a collection $\{u\}$ is empty).

Theorem 2.2. *Let $p\neq q$ and α be arbitrary, or $p=q$ and $\alpha\in\mathbb{R}\setminus\{1,2,\dots,p-1\}$. Then*

$$\prod_{k=1}^{p}\cosh^{-\alpha} t_j = C\cdot\sum_{\substack{m;\\ u_1\leqslant\dots\leqslant u_m<\frac{1}{4}(p+q)-\frac{m}{2}-\frac{1}{2}\alpha}} E_m(\alpha,u)\times$$

$$\times\int_{i\mathbb{R}^{p-m}} Y_m(\alpha,u;s)\mathfrak{R}_m(s)\Phi_{\alpha-(p+q)/2+1+2u_1,\dots,\alpha-(p+q)/2+m+2u_m,s_{m+1},\dots,s_m}(t_1,\dots,t_p)\times$$

$$\times\, ds_{m+1}\dots ds_p \tag{2.6}$$

where C is the same as above,

$$E_m(\alpha,u)=(2\pi)^m\frac{p!}{m!}2^\alpha\prod_{j=1}^{p}\frac{1}{\Gamma(\alpha-j+1)}\times \tag{2.7}$$

$$\times\prod_{\tau=1}^{m}\frac{(-\alpha+\frac12(p+q)-2u_\tau-\tau)\Gamma(\alpha-p+\tau+2u_\tau)\Gamma(-\alpha+q-\tau-2u_\tau)}{(u_\tau-u_{\tau-1})!\Gamma(-\alpha+\frac12(p+q)-\tau+1-u_\tau-u_{\tau-1})}\times \tag{2.8}$$

$$\times\prod_{1\leqslant\sigma<\tau\leqslant m}\Big\{(-\alpha+\tfrac12(p+q)-\tfrac12(\tau+\sigma)-u_\sigma-u_\tau)(\tfrac12(\tau-\sigma)+u_\tau-u_\sigma)\times \tag{2.9}$$

$$\times\frac{\Gamma(\frac12(\tau-\sigma+1)+u_\tau-u_\sigma)\Gamma(-\alpha+\frac12(p+q)-\frac12(\tau+\sigma)-u_\tau-u_\sigma+\frac12)}{\Gamma(\frac12(\tau-\sigma)+u_\tau-u_{\sigma-1})\Gamma(-\alpha+\frac12(p+q)-\frac12(\sigma+\tau)-u_\sigma-u_\tau)}\Big\} \tag{2.10}$$

$$Y_m(\alpha,u;s)=$$

$$=\prod_{k=m+1}^{p}\Big\{\Gamma(\tfrac12(\alpha-\tfrac12(p+q)+m+1+s_k))\Gamma(\tfrac12(\alpha-\tfrac12(p+q)+m+1-s_k))\Big\}\times \tag{2.11}$$

$$\times\prod_{\tau\leqslant m;\ k>m}\Bigg\{(\tfrac12(-\alpha+\tfrac12(p+q)-\tau-2w_\tau+s_k))(\tfrac12(-\alpha+\tfrac12(p+q)-\tau-2w_\tau-s_k))\times \tag{2.12}$$

$$\times\frac{\Gamma(\frac12(-\alpha+\frac12(p+q)-(\tau-1)-2u_\tau+s_k)\Gamma(\frac12(-\alpha+\frac12(p+q)-(\tau-1)-2u_\tau-s_k)}{\Gamma(\frac12(-\alpha+\frac12(p+q)-\tau+2+2w_{\tau-1}+s_k)\Gamma(\frac12(-\alpha+\frac12(p+q)-\tau+2+2w_{\tau-1}-s_k)}\Bigg\} \tag{2.13}$$

and

$$\mathfrak{R}_m(s)=\prod_{k=m+1}^{p}\frac{\Gamma((q-p)/2+s_k)\Gamma((q-p)/2-s_k)}{\Gamma(s_k)\Gamma(-s_k)}\times \tag{2.14}$$

$$\times\prod_{m+1\leqslant k<l\leqslant p}\frac{\Gamma(\frac12(1+s_l+s_k))\Gamma(\frac12(1+s_l-s_k))\Gamma(\frac12(1-s_l+s_k))\Gamma(\frac12(1-s_l-s_k))}{\Gamma(\frac12(s_l+s_k))\Gamma(\frac12(s_l-s_k))\Gamma(\frac12(-s_l+s_k))\Gamma(\frac12(-s_l-s_k))} \tag{2.15}$$

REMARKS. a) The factor $\mathfrak{R}_m(s)$ is an elementary function.
b) More convenient notations are used in Section 4 (see 4.13).
c) The formula, which is not so explicit, but short is given in Section 6.

REMARK. The summand corresponding $m=0$ coincides with integral (2.1)–(2.5). For summands corresponding $m=p$, the integration is given by one point set and hence these summands are spherical functions $\Phi_{\dots}$ with some coefficients.

2.3. The case $\alpha = p-1, p-2, \dots, 1$. In this case some summands disappear.

PROPOSITION 2.3. *Let $\alpha = p-h$ where $h \leqslant p$. Then the factor $E_m(\alpha, u)$ is nonzero if and only if*

$$m \geqslant h; \qquad u_1 = u_2 = \dots = u_h = 0$$

PROOF. Vanishing of $E_m(\alpha, u)$ is completely defined by a behavior of the factor

$$\frac{\prod_{\tau=1}^{m} \Gamma(\alpha - p + \tau + 2u_\tau)}{\prod_{j=1}^{p} \Gamma(\alpha - j + 1)} \tag{2.16}$$

The denominator has a pole of order h at $\alpha = p-h$. If the fraction is nonvanishing, then the numerator has a pole of order h at the same point. *boxtimes*

2.4. The case $\alpha = -1, -2, -3, \dots$. Assume $E_m(\alpha, u) \neq 0$. The denominator of (2.16) has a pole of order p in α. Hence, the numerator also has a pole of order p. Hence,

$$m = p$$

This means that all integrals in Plancherel formula (2.6) vanish and we have only finite sum of spherical functions with some coefficients. The coefficient $E_m(\alpha, u)$ is nonzero iff

$$m + 2u_m \leqslant -\alpha$$

2.5. The Plancherel formula for the kernel representations T_α of $\mathrm{O}(p,q)$.

THEOREM 2.4. *Let α satisfies Berezin conditions* (1.9). *Then*

a)*if $E_m(\alpha, u) \neq 0$ (see Subsection* 2.3*), then all spherical functions*

$$\Phi_{\alpha-(p+q)/2+1+2u_1, \dots, \alpha-(p+q)/2+m+u_m, s_{m+1}, \dots, s_m}$$

are positive definite.

b)*formula* (2.6)-(2.15) *is really the Plancherel formula*

2.6. The Plancherel formula for kernel-representations of $\mathrm{O}(p+q)$. For a negative integer $\alpha = -n$ (see Subsection 2.4 above) formula (2.6) gives the expansion of $\det(1 - z^* z)^n$ in $\mathrm{O}(p) \times \mathrm{O}(q)$-spherical functions of $\mathrm{O}(p,q)$ and this is equivalent to the Plancherel formula for the kernel representations of $\mathrm{O}(p+q)$.

2.7. The case of indefinite Shapovalov form. For noninteger $\alpha < p-1$ we obtain the problem discussed in Section 1.E.

3. B-function of the space $\mathrm{O}(p,q)/\mathrm{O}(p)\times\mathrm{O}(q)$

In this section we construct a matrix imitation of the B-integral

$$\mathrm{B}(x,y)=\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt$$

for the symmetric spaces $\mathrm{O}(p,q)/\mathrm{O}(p)\times\mathrm{O}(q)$. For symmetric spaces $\mathrm{GL}(n,\mathbb{K})/\mathrm{U}(n,\mathbb{K})$ the B-integrals were defined by Gindikin [11] (see also exposition in [9]), for other symmetric spaces B-integrals were derived in [35].

3.1. B-integral. Let

$$\lambda_1,\dots,\lambda_p,\sigma_1,\dots,\sigma_p\in\mathbb{C}$$

We also assume

$$\lambda_{p+1}=\sigma_{p+1}=0$$

Let $\mathrm{SW}_{p,q}$ be the section of wedge defined in Subsection 1.20.

THEOREM 3.1. *Let λ_k, σ_k satisfy the inequalities*

$$\frac12(q+k)/2+1<\lambda_k<\sigma_k-\frac12(p-k) \tag{3.1}$$

Then

$$\int\limits_{\mathrm{SW}_{p,q}(\mathbb{R})}\prod_{j=1}^{p}\frac{\det[M-LL^t]_j^{\lambda_j-\lambda_{j+1}}}{\det[1+M+N]_j^{\sigma_j-\sigma_{j+1}}}\cdot\det(M-LL^*)^{-(p+q)/2}dM\,dN\,dL= \tag{3.2}$$

$$=\int\limits_{\substack{M-LL^t>0\\ N=-N^t}}\prod_{j=1}^{n}\frac{\det\begin{bmatrix}1&L^t\\L&M\end{bmatrix}_{q-p+j}^{\lambda_j-\lambda_{j+1}}}{\det[1+M+N]_j^{\sigma_j-\sigma_{j+1}}}\det\begin{pmatrix}1&L^t\\L&M\end{pmatrix}^{-(p+q)/2}dL\,dM\,dN= \tag{3.3}$$

$$==\prod_{k=1}^{p}\pi^{k-(q-p)/2-1}\frac{\Gamma(\lambda_k-(q+k)/2+1)\Gamma(\sigma_k-\lambda_k-(p-k)/2)}{\Gamma(\sigma_k-p+k)} \tag{3.4}$$

The proof of the Theorem is given in Subsections 3.2-3.6.

REMARK. For $p=q$ we have $L=0$ and integral (3.2)–(3.3) has more simple form, see (0.5). In this case the calculation given below also is simpler. The main simplification is the expression for matrix (3.13): the first block row and the first block column are lacked.

REMARK. We have $M=M^t>0$, $N=-N^t$. Hence,

$$\det(1+M+N)>0$$

Indeed, for any $v\in\mathbb{C}^p$ we have $\operatorname{Re} v(M+N)v^* = vMv^*>0$. Hence, the eigenvalues λ_j of $M+N$ satisfy the condition $\operatorname{Re}\lambda_j>0$. Hence, the eigenvalues of $1+M+N$ are nonzero.

REMARK. Hua Loo Keng in [21] evaluated the integrals[21]

$$\int_{\mathrm{B}_{p,q}(\mathbb{R})} \det(1-zz^*)^\tau\, dz \qquad (3.5)$$

Cayley transform reduces the *Hua integral* to the following partial case of our integral

$$\mathrm{const}\cdot \int_{SW_q(\mathbb{R})} \frac{\det\begin{pmatrix}1 & L^t\\ L & M\end{pmatrix}^\tau}{\det(1+M+N)^{2\tau}}\, dL\, dM\, dN$$

Our calculation in this case is not homotopic to Hua calculations.

3.2. Replacement of notations. Firstly, we call to mind the standard formula (see [10]) for determinant of block $(m+n)\times(m+n)$-matrix

$$\det\begin{pmatrix}A & B\\ C & D\end{pmatrix} = \det A\cdot\det(D-BA^{-1}C) \qquad (3.6)$$

Let us represent M,N as block $((p-1)+1)\times((p-1)+1)$ matrices, and L as a block $((p-1)+1)\times(q-p)$ matrix:

$$M=\begin{pmatrix}P & q^t\\ q & r\end{pmatrix};\qquad N=\begin{pmatrix}A & -b^t\\ b & 0\end{pmatrix};\qquad L=\begin{pmatrix}H\\ l\end{pmatrix}$$

Then for $j\leqslant p-1$

$$\begin{bmatrix}1 & L^t\\ L & M\end{bmatrix}_{q-p+j} \text{ coincides with } \begin{bmatrix}1 & H^t\\ H & P\end{bmatrix}_{q-p+j}$$

$$[1+M+N]_j \qquad \text{coincides with} \qquad [1+P+A]_j$$

and by (3.6)

$$\det\begin{pmatrix}1 & L^t\\ L & M\end{pmatrix} = \det\begin{pmatrix}1 & H^t & l^t\\ H & P & q^t\\ l & q & r\end{pmatrix} =$$

$$= \det\begin{pmatrix}1 & H^t\\ H & P\end{pmatrix}\cdot\left[r-\begin{pmatrix} l & q\end{pmatrix}\begin{pmatrix}1 & H^t\\ H & P\end{pmatrix}^{-1}\begin{pmatrix}l^t\\ q^t\end{pmatrix}\right]$$

$$\det(1+M+N)=\det(1+P+A)\cdot\big(1+r-(q+b)(1+P+A)^{-1}(q^t-b^t)\big)$$

[21] Hua integrals also can be reduced to the Selberg B-integrals by integration over $K\setminus G/K$.

By the Sylvester criterion the condition $\begin{pmatrix}1 & L^t\\ L & M\end{pmatrix} > 0$ (see (1.35)) in new notations has the form

$$\begin{pmatrix}1 & H^t\\ H & P\end{pmatrix} > 0; \qquad r - \begin{pmatrix} l & q\end{pmatrix}\begin{pmatrix}1 & H^t\\ H & P\end{pmatrix}^{-1}\begin{pmatrix} l^t\\ q^t\end{pmatrix} > 0 \tag{3.7}$$

By the remark given in Subsection 3.1

$$\det(1+P+A) > 0$$

3.3. Substitution. Let us replace the variable r to the variable

$$u = r - \begin{pmatrix} l & q\end{pmatrix}\begin{pmatrix}1 & H^t\\ H & P\end{pmatrix}^{-1}\begin{pmatrix} l^t\\ q^t\end{pmatrix}$$

(all other variables are the same). By (3.7) we have $u > 0$. The Jacobian of the substitution is 1. Our integral converts to the form

$$\int dP\, dA\, dH \Bigg(\Xi(A,P,H) \times \tag{3.8}$$

$$\times \int\limits_{u>0,\, q,b\in\mathbb{R}^{p-1}, l\in\mathbb{R}^{q-p}} u^{\lambda_p-(p+q)/2}\Bigg\{1+u+\begin{pmatrix} l & q\end{pmatrix}\begin{pmatrix}1 & H^t\\ H & P\end{pmatrix}^{-1}\begin{pmatrix} l^t\\ q^t\end{pmatrix} + \tag{3.9}$$

$$+\begin{pmatrix} q & b\end{pmatrix}\begin{pmatrix} -(1+P+A)^{-1} & -(1+P+A)^{-1}\\ (1+P+A)^{-1} & (1+P+A)^{-1}\end{pmatrix}\begin{pmatrix} q^t\\ b^t\end{pmatrix}\Bigg\}^{-\sigma_p} du\, dl\, dq\, db\Bigg) \tag{3.10}$$

where

$$\Xi(A,P,H) = \prod_{j=1}^{p-2}\frac{\det\begin{bmatrix}1 & H^t\\ H & P\end{bmatrix}_{q-p+j}^{\lambda_j-\lambda_{j+1}}}{\det[1+P+A]_j^{\sigma_j-\sigma_{j+1}}}\cdot\frac{\det\begin{pmatrix}1 & H^t\\ H & P\end{pmatrix}^{\lambda_{n-1}-(p+q)/2}}{\det(1+P+A)^{\sigma_{n-1}}} \tag{3.11}$$

is an expression independent on u, b, l, q.

Firstly, we want to evaluate interior integral (3.9)–(3.10)

3.4. Transformation of the integrand. Denote by S the expression

$$S = 1+P+A$$

Let us represent the expression in the curly brackets in (3.10) in the form

$$\left\{1+u+\begin{pmatrix} l & q & b\end{pmatrix} X \begin{pmatrix} l^t\\ q^t\\ b^t\end{pmatrix}\right\} \tag{3.12}$$

where

$$X = \begin{pmatrix} \begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{-1} + \begin{pmatrix} 0 & 0 \\ 0 & -S^{-1} \end{pmatrix} & \begin{pmatrix} 0 \\ -S^{-1} \end{pmatrix} \\ \begin{pmatrix} 0 & S^{-1} \end{pmatrix} & S^{-1} \end{pmatrix} \tag{3.13}$$

(we wright a block matrix whose elements are block matrices itself). The last summand in the curly brackets is a quadratic form in the variables b, q, l. But the matrix X is not symmetric and it is more natural to re-wright expression (3.12) in the form

$$\left\{ 1 + u + \begin{pmatrix} l & q & b \end{pmatrix} \frac{1}{2}(X + X^t) \begin{pmatrix} l^t \\ q^t \\ b^t \end{pmatrix} \right\} \tag{3.14}$$

3.5. Separation of variables.
LEMMA 3.2.

$$\det\left(\frac{1}{2}(X + X^t)\right) = \det \begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{-1} \cdot \det(1 + P + A)^{-2}$$

PROOF. $\det\left(\frac{1}{2}(X + X^t)\right) =$

$$= \det \begin{pmatrix} \begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{-1} + \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2}S^{t-1} - \frac{1}{2}S^{-1} \end{pmatrix} & \begin{pmatrix} 0 \\ -\frac{1}{2}S^{-1} + \frac{1}{2}S^{t-1} \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2}S^{-1} - \frac{1}{2}S^{t-1} \end{pmatrix} & \frac{1}{2}S^{-1} + \frac{1}{2}S^{t-1} \end{pmatrix}$$

Adding the third row to the second row and the third column to the second column, we obtain

$$\det \begin{pmatrix} \begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{-1} & \begin{pmatrix} 0 \\ (1 + P - A)^{-1} \end{pmatrix} \\ \begin{pmatrix} 0 & (1 + P + A)^{-1} \end{pmatrix} & \frac{1}{2}(1 + P + A)^{-1} + \frac{1}{2}(1 + P - A)^{-1} \end{pmatrix}$$

Formula (3.6) reduces the determinant to the form

$$\det \begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{-1} \cdot \det\Big(\frac{1}{2}(1 + P + A)^{-1} + \frac{1}{2}(1 + P - A)^{-1} -$$
$$- \begin{pmatrix} 0 & (1 + P + A)^{-1} \end{pmatrix} \begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix} \begin{pmatrix} 0 \\ (1 + P - A)^{-1} \end{pmatrix}\Big) =$$
$$= \det \begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{-1} \det(1 + P + A)^{-1} \det(1 + P - A)^{-1} \times$$
$$\times \det\left(\tfrac{1}{2}(1 + P - A) + \tfrac{1}{2}(1 + P + A) - P\right)$$

The last factor is 1. We also observe

$$(1+P+A)^t = 1+P-A$$

and hence their determinants coincides. ⊠

LEMMA 3.3. $X + X^t > 0$.

PROOF. In the identity

$$\det(1+M+N) = \det(1{+}P{+}A) \cdot \left[1 + r - (q+b)(1{+}P{+}A)^{-1}(q^t - b^t)\right]$$

we have $\det(1+M+N) > 0$, $\det(1+P+A) > 0$. Hence, the factor in the square brackets is positive. Hence, expression (3.12) is positive for all $u > 0$, and all q, b, l. Quantity (3.12) coincides with quantity (3.14). Hence, the matrix $X + X^t$ is nonnegative defined. By Lemma 3.2 its determinant is nonzero and we obtain the required statement. ⊠

Consider the linear substitution

$$\begin{pmatrix} l & q & b \end{pmatrix} \sqrt{\tfrac{1}{2}(X+X^t)} = h \in \mathbb{R}^{q-p} \oplus \mathbb{R}^{p-1} \oplus \mathbb{R}^{p-1}$$

to interior integral (3.9)–(3.10). Its Jacobian is

$$\det \begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{1/2} \cdot \det(1+P+A)$$

and hence the interior integral coverts to the form

$$\det \begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{1/2} \cdot \det(1+P+A) \times \tag{3.15}$$

$$\times \int\limits_{u>0,\, h\in\mathbb{R}^{q+p-2}} u^{\lambda_p - (p+q)/2} \left\{1 + u + |h|^2\right\}^{-\sigma_p} du\, dh \tag{3.16}$$

The first factor (3.15) adds to the product $\Xi(A, P, H)$ (see (3.11)) and we reduce our B-integral (3.3) to the product of the integrals

$$\int_{P-HtH>0,\, A=-A^t} \Xi(A,P,H) \det \begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{1/2} \cdot \det(1+P+A) dA\, dP\, dH \times$$
$$\times \int\limits_{u>0,\, h\in\mathbb{R}^{q+p-2}} u^{\lambda_p - (p+q)/2} \left\{1 + u + |h|^2\right\}^{-\sigma_p} du\, dh$$

Let us denote B-integral (3.3) by

$$I_{p,q}(\alpha_1, \ldots, \alpha_p; \sigma_1, \ldots, \sigma_p) \tag{3.17}$$

and let us denote factor (3.16) by $J_{p,q}(\alpha_p;\sigma_p)$. We obtain the following recurrent identity

$$I_{p,q}(\alpha_1,\dots,\alpha_p;\sigma_1,\dots,\sigma_p) =$$
$$= I_{p-1,q-1}(\alpha_1,\dots,\alpha_{p-2},\alpha_{p-1}-\tfrac{1}{2};\sigma_1,\dots,\sigma_{p-2},\sigma_{p-1}-1)J_{p,q}(\alpha_p;\sigma_p)$$

3.6. Evaluation of $J_{p,q}(\alpha_p;\sigma_p)$. This problem is trivial. Firstly, we consider spherical coordinates in $\mathbb{R}^{p+q-2}$ in the variable h. Then $J_{p,q}(\alpha_p;\sigma_p)$ converts to the form

$$\frac{2\pi^{(p+q)/2-1}}{\Gamma((p+q)/2-1)}\int_{u>0}\int_{r>0} u^{\lambda_p-(p+q)/2} r^{p+q-3}\{1+u+r^2\}^{-\sigma_p}\,dr\,du$$

The substitution $v=r^2$ reduces our integral to a special case of the Dirichlet B-integral

$$\int_{u>0,v>0}\frac{u^{a-1}v^{b-1}}{(1+u+v)^{a+b+c}}du\,dv=\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}$$

This completes the proof of Theorem 3.1.

3.7. Spherical transform of $\mathcal{B}_\alpha$.

COROLLARY 3.4. *Let $\alpha>p+q-1$. Then spherical transform of $\mathcal{B}_\alpha$ is*

$$\frac{2^\alpha}{\prod_{1\leqslant j\leqslant p}\Gamma(\alpha-j+1)}\prod_{k=1}^{p}\Gamma(\tfrac{1}{2}(\alpha-\tfrac{1}{2}(p+q)+1+s_k))\Gamma(\tfrac{1}{2}(\alpha-\tfrac{1}{2}(p+q)+1-s_k)) \tag{3.18}$$

PROOF. The function $\mathcal{B}_\alpha$ is given by the formula (1.37). By Subsection 1.24 we must evaluate the integral

$$\int_{G/K}\mathcal{B}_\alpha(z)\Psi_s(z)\,d\lambda(z)$$

But the integral is a special case of our B-integral.

3.8. Proof of Theorem 2.1. By the Gindikin-Karpelevich inversion formula and Corollary 3.4 we obtain the statement of the theorem for $\alpha>(p+q)-1$.

For $\alpha>(p+q)/2-1$ the statement of the Theorem follows from trivial Lemma 4.1 proved below.

4. Formal analytic continuation.

We proved the Plancherel formula (2.1)-(2.5) for large values of the parameter α. Its left part $\prod\cosh^{-\alpha}(t_j)$ depends analytically on $\alpha\in\mathbb{C}$. The integrand in the right part has singularities on the lines

$$\operatorname{Re}\alpha=\tfrac{1}{2}(p+q)-1-2\varkappa;\qquad\text{where}\quad\varkappa=0,1,2,\dots \tag{4.1}$$

Thus, the right part of formula (2.1)–(2.5) may be nonalytic for these values of α.

Our next purpose is to construct the analytic continuation of the right part to arbitrary complex α.

4.1. Analyticity. Let us denote the right part of the formula (2.1)–(2.5) by

$$\mathfrak{F}(\alpha) := \mathfrak{F}(\alpha; t) = E(\alpha) \int_{i\mathbb{R}^p} Y(\alpha; s)\mathfrak{R}(s)\Phi_s(t)\, ds \tag{4.2}$$

where the meromorphic factor $E(\alpha)$ is given by formula (2.1), the factor $Y(\alpha; s)$ is defined by (2.2) and $\mathfrak{R}(s)$ is Gindikin–Karpelevich density (2.3)–(2.4). *In this Section we fix the variable t and we omit the argument t from the notation $\mathfrak{F}(\alpha; t)$.*

Consider domains $\Pi_0, \Pi_1, \dots$ in $\mathbb{C}$ defined by

$$\Pi_0 : \qquad \operatorname{Re}\alpha > \tfrac{1}{2}(p+q) - 1$$
$$\Pi_k : \qquad \tfrac{1}{2}(p+q) - 1 - 2k < \operatorname{Re}\alpha < \tfrac{1}{2}(p+q) - 1 - 2(k-1) \quad \text{where } k > 0$$

LEMMA 4.1. *The function $\mathfrak{F}(\alpha)$ is an analytical function on $\Pi_\varkappa$ for all $\varkappa = 0, 1, 2 \dots$.*

PROOF. a) *Convergence of integral* (4.2). First, the Gindikin-Karpelevich factor $\mathfrak{R}(s)$ has a polynomial growth in s, see formulas (1.44)–(1.46).

By the formula (see [19],1.18.6)

$$|\Gamma(a+iy)| = (2\pi)^{1/2}|y|^{a-1/2}\exp\{\tfrac{1}{2}\pi|y|\}(1+o(1)); \qquad |y| \to \infty \tag{4.3}$$

the factor $Y(\alpha; s)$ exponentially decreases.

A spherical function $\Phi_s(t)$ is a spherical function of an unitary representation and hence we have $|\Phi_s(t)| \leqslant 1$[22].

Hence, the integrand exponentially decreases and the integral absolutely converges.

b)*Existence of* $\frac{\partial}{\partial\alpha}\mathfrak{F}(\alpha)$. It is sufficient to prove uniform convergence of the integral

$$\int_{i\mathbb{R}^p} \frac{\partial}{\partial\alpha} Y(\alpha; s)\mathfrak{R}(s)\Phi_s(t)ds \tag{4.4}$$

in small neighborhood of a fixed point $\widetilde{\alpha}$. For this we needs in uniformity by a of $o(1)$ in (4.3). In fact, the asymptotics is really uniform but formally we have no possibility to refer to [19]. Formula (4.3) is derived from the Binet formula (see [19],(1.9.4))

$$\ln\Gamma(z) \;=\; (z - \frac{1}{2})\ln z \,-\, z \,+\, \frac{1}{2}\ln(2\pi) \,+\, \int_0^\infty \left[\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right] t^{-1}e^{-tz}dt$$

[22]For following inductive steps this arguments must be replaced by inequality (1.39)

This formula easily implies an uniform estimate of the form

$$|\Gamma(a+iy)| \leqslant \mathrm{const}\cdot \exp(-(\tfrac{1}{2}\pi-\varepsilon)|y|); \qquad |a-\widetilde{a}|<\delta$$

The Cauchy integral for derivative

$$-2\pi i f'(z) = \int_L \frac{f(z)}{(z-u)^2}dz$$

implies the same estimate for derivative of Γ-function.

We observe that integrand (4.4) is dominated by some function having the form

$$P(s)\exp(-b\sum|s_j|)$$

where $P(s)$ is a polynomial and $b>0$.

Thus, the function $\mathfrak{F}(\alpha)$ has a derivative in the complex variable α and this completes the proof. ⊠

LEMMA 4.2 *Let $p\neq q$. Then the function $\mathfrak{F}(\alpha)$ is continuous on the line $\alpha\in\mathbb{R}$.*

PROOF. Let $h=\frac{1}{2}(p+q)-1-2\varkappa$ be one of our singular points. The singularity of the integrand near this point has the form

$$\mathrm{const}\cdot\prod_{k=1}^{p}\frac{s_k^2}{(\alpha-h)^2-s_k^2}\prod_{1\leqslant k<l\leqslant p}(s_k^2-s_l^2)^2(1+o(1))$$

Remind that s_k are pure imaginary. Hence, the integrand is bounded in a neighborhood of the point $\alpha=h, s=0$. As we have seen in the previous proof, the integrand has an integrable majorant in a domain $|s_k|>A$, $|\alpha-h|<\varepsilon$ $(\alpha\in\mathbb{R})$. By Lebesgue theorem about dominant convergence, expression (4.2) is continuous at the point $\alpha=h$. ⊠

REMARK. The function $\mathfrak{F}(\alpha)$ is continuous at the real points $\alpha=\frac{1}{2}(p+q)-1-2\varkappa$ but it is not smooth at these points.

We denote the restriction of the function $\mathfrak{F}(\alpha)$ to the domain $\Pi_\varkappa$ by

$$\mathfrak{F}_\varkappa(\alpha)$$

4.2. Analytic continuation of $\mathfrak{F}_\varkappa(\alpha)$ through a point of a line $\mathrm{Re}\,\alpha=\frac{1}{2}(p+q)-1-2\varkappa$. The following lemma is main in this Section. Its proof is given in Subsections 4.2-4.5.

LEMMA 4.3. *Let α_0 satisfies the condition*

$$\mathrm{Re}\,\alpha_0=\tfrac{1}{2}(p+q)-1-2\varkappa; \qquad \mathrm{Im}\,\alpha_0\neq 0$$

Then

a) *the function* $\mathfrak{F}_\varkappa(\alpha)$ *admits the analytic continuation to some small neighborhood*

$$\mathcal{O}_\delta : \qquad |\alpha - \alpha_0| < \delta \qquad \text{where} \quad \delta < \min\{1/1000, |\operatorname{Im}\alpha|/1000\} \tag{4.5}$$

of the point α_0

b) *for any*

$$\alpha \in \mathcal{O} \cap \Pi_{\varkappa+1}$$

we have

$$\mathfrak{F}_{\varkappa+1}(\alpha) - \mathfrak{F}_\varkappa(\alpha) = \frac{\pi p}{\varkappa!} \prod_{j=1}^{p} \frac{1}{\Gamma(\alpha - j + 1)} \times \tag{4.6}$$

$$\times \frac{(-\alpha + \frac{1}{2}(p+q) - 2\varkappa - 1)\Gamma(\alpha - p + 1 + 2\varkappa)\Gamma(-\alpha + q - 1 - 2\varkappa)}{\Gamma(-\alpha + \frac{1}{2}(p+q) - \varkappa)} \times \tag{4.7}$$

$$\times \int_{i\mathbb{R}^{p-1}} \prod_{2\leqslant k\leqslant p, \pm} \Gamma(\frac{1}{2}(\alpha - \frac{1}{2}(p+q) + 2 + 2\varkappa \pm s_k)) \times \tag{4.8}$$

$$\times \prod_{2\leqslant k\leqslant p,\pm} \frac{(\frac{1}{2}(-\alpha + \frac{1}{2}(p+q) - 1 \pm s_k))\Gamma(\frac{1}{2}(-\alpha + \frac{1}{2}(p+q) - 2\varkappa \pm s_k))}{\Gamma(\frac{1}{2}(-\alpha + \frac{1}{2}(p+q) + 1 \pm s_k))} \times \tag{4.9}$$

$$\times \prod_{2\leqslant k\leqslant p,\pm} \frac{\Gamma(\frac{1}{2}(q-p) \pm s_k)}{\Gamma(\pm s_k)} \times \tag{4.10}$$

$$\times \prod_{2\leqslant k<l\leqslant p,\pm} \frac{\Gamma(\frac{1}{2}(1 + s_k \pm s_l))\Gamma(\frac{1}{2}(1 - s_k \pm s_l)}{\Gamma(\frac{1}{2}(s_k \pm s_l)\Gamma(\frac{1}{2}(-s_k \pm s_l)} \times \tag{4.11}$$

$$\times \Phi_{\alpha-(p+q)/2+1+2\varkappa, s_2, \ldots, s_p}(t) ds_2 \ldots ds_p \tag{4.12}$$

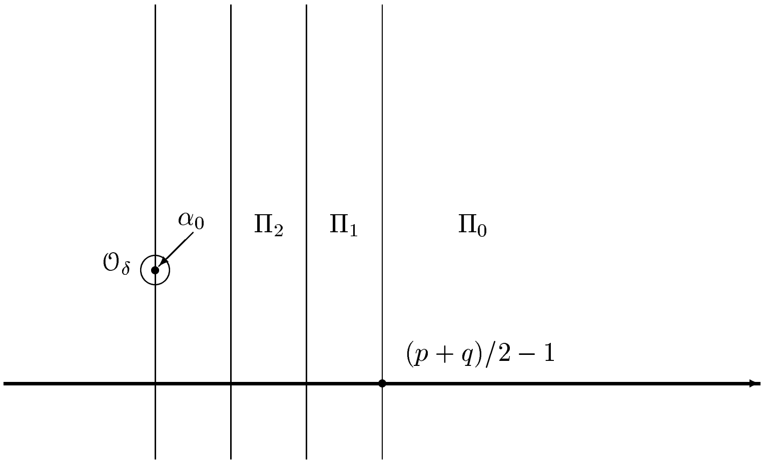
In the last formula we use the following notation

$$\prod_{\pm} \Gamma(a \pm s) := \Gamma(a+s)\Gamma(a-s) \tag{4.13}$$

4.3. Existence of the analytic continuations. Let us represent expression (4.2) (or (2.1)–(2.5)) for $\mathfrak{F}_\varkappa(\alpha)$ in the form

$$E(\alpha) \int \prod_{k,\pm} \Gamma(\tfrac{1}{2}(\alpha - \tfrac{1}{2}(p+q) + 1 \pm s_k)) \cdot \mathfrak{R}(s) \Phi_s(t) ds$$

Let $\varepsilon_1 > \varepsilon_2 \cdots > \cdots > \varepsilon_p \geqslant 0$ be very small (for instance $\varepsilon_1 < \delta/10$ where δ was defined in (4.5)). Consider the function



Picture 1. The complex plane α. The lines $\alpha = (p+q)/2 - 1 - 2\varkappa$ and the domains Π_k

$$\mathfrak{F}_\varkappa(\alpha;\varepsilon) \;=\; E(\alpha)\int_{i\mathbb{R}^p}\prod_{k,\pm}\Gamma(\frac{1}{2}(\alpha+\varepsilon_k-\tfrac{1}{2}(p+q)+1\pm s_k))\cdot\mathfrak{R}(s)\Phi_s(t)ds \tag{4.14}$$

in the domain

$$\Pi^\varepsilon_\varkappa: \qquad -2\varkappa < \mathrm{Re}(\alpha-\frac{1}{2}(p+q)+1) < -2(\varkappa-1)-\varepsilon_1$$

LEMMA 4.4. a) *The function $\mathfrak{F}_\varkappa(\alpha;\varepsilon)$ admits the holomorphic continuation to the domain $\mathcal{O}_\delta$ (see (4.5)).*

b) *The functions $|\mathfrak{F}_\varkappa(\alpha;\varepsilon)|$ in $\mathcal{O}_\delta$ are bounded by a constant independent on ε.*

PROOF. a) The factor $\mathfrak{R}(s)$ is holomorphic in the domain $|\operatorname{Re}(s_j)| < 1/4$ and its poles are very far from the contour $\mathcal{L}$ which is described below.

Consider

$$\alpha \in \mathcal{O}_\delta \cap \Pi^\varepsilon_\varkappa \tag{4.15}$$

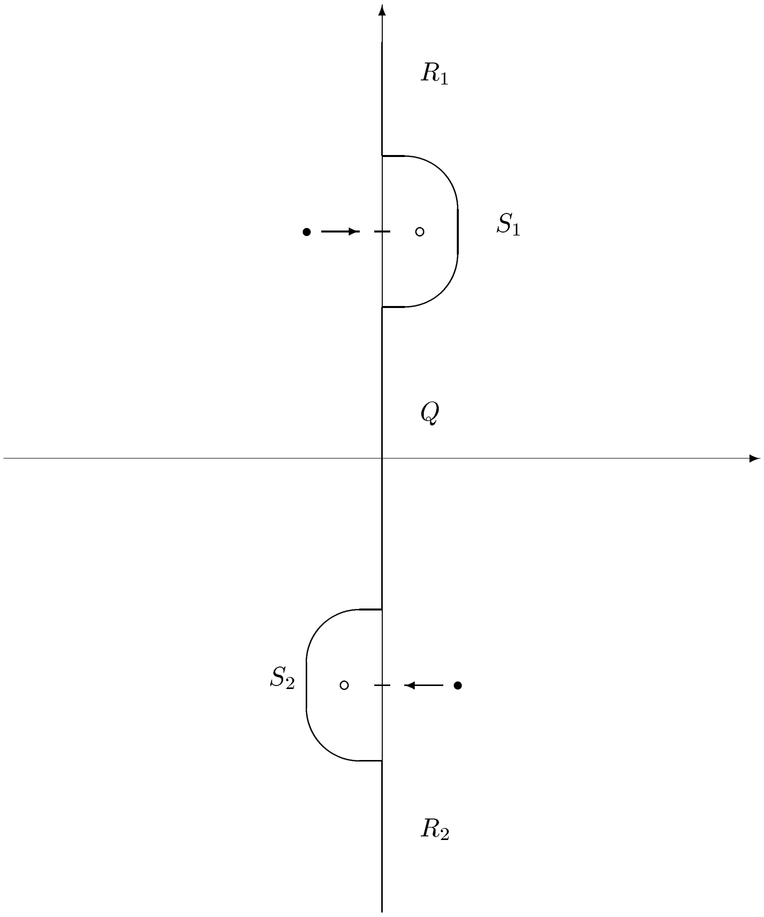
Then integrand (4.14) has poles on hyperplanes

$$s_k = \pm(\tfrac{1}{2}(p+q)-1-\alpha-2u-\varepsilon_k); \qquad \text{where} \quad u = 0,1,\dots$$

If $u=\varkappa$, then the poles are lying near points $\pm\operatorname{Im}\alpha_0$, on Picture 2 the poles are marked as black circles. The arrows show the direction of their motion if $\operatorname{Re}\alpha$ decreases. The white circles show rough position of the poles than $\alpha\in\mathcal{O}_\delta\cap\Pi^\varepsilon_{\varkappa+1}$

Consider the contour L_k on the complex plane $s_k \in \mathbb{C}$ given by Picture 2. Let $\mathcal{L}\subset\mathbb{C}^p$ be the product of the contours L_k. Obviously, for $\alpha\in\mathcal{O}_\delta$, we can replace the integration over $i\mathbb{R}^p$ in the formula (4.14) by the integration over $\mathcal{L}$. But the integral

$$\int_{\mathcal{L}} = \int_{\mathcal{L}}\prod_{k,\pm}\Gamma(\frac{1}{2}(\alpha+\varepsilon_k-\tfrac{1}{2}(p+q)+1\pm s_k))\cdot\mathfrak{R}(s)\Phi_s(t)ds$$



Picture 2. The contour $L_k = R_1 \cup S_1 \cup Q \cup S_2 \cup R_2$ on the complex plane s_k. The centers of the semicircles S_1, S_2 are $\pm \operatorname{Im} \alpha_0$, the radius of the semicircles is 10δ.

obviously is holomorphic with respect to α in the domain $\mathcal{O}_\delta$ (indeed, the surface $\mathcal{L}$ doesn't intersect the singularities and the integrand exponentially decreases as $|s| \to \infty$).

b) Consider the parameter $\theta_k := \operatorname{Im} s_k$ on the contour L_k.

LEMMA 4.5. *There exist constants $A = A(t)$, N such that*

$$|\prod_{k,\pm} \Gamma(\tfrac{1}{2}(\alpha+\varepsilon_k - \tfrac{1}{2}(p+q)+1 \pm s_k)) \cdot \mathfrak{R}(s)\Phi_s(t)| \leqslant A \prod_{j=1}^{p} (1+|\theta_j|)^N \exp(-\pi|\theta_j|)$$

for all $(s_1, \dots, s_p) \in \mathcal{L}$.

PROOF OF LEMMA 4.5. Let us estimate all factors in the left part of the inequality.

a) *The Gindikin–Karpelevich factor* $\mathfrak{R}(s)$. By the formulas (1.45)–(1.46) for any imaginary s_k we have

$$\Big|\frac{\Gamma(\frac{1}{2}(q-p)+s_k)\Gamma(\frac{1}{2}(q-p)-s_k)}{\Gamma(s_k)\Gamma(-s_k)}\Big| \leqslant \mathrm{const} \cdot (1+|\theta_k|)^{2(q-p)}$$

The same expression is bounded on the semi-circles S_1, S_2.

We also must estimate the factor (1.44). Firstly,

$$|\prod_{k=1}^{p} (s_k^2 - s_l^2)| \leqslant \mathrm{const} \prod_k (1+|\theta_k|)^{2(p-1)}$$

Secondly, let us estimate the factors

$$\tan(\pi(s_k \pm s_l))$$

of (1.44). If s_k, s_l are imaginary, then $|\tan(\pi(s_k \pm s_l))| < 1$. If $s_k, s_l \in S_1, S_2$, then this expression is bounded (since S_1, S_2 are compact sets). Let s_k be imaginary and $s_l \in S_1, S_2$. Then we obtain a value having the form $|\tan(x+iy)|$ where $x, y \in \mathbb{R}$, $|x| < 10\pi\delta$. Then

$$|\tan(x+iy)| = \Big|\frac{\tan x + \tan iy}{1+\tan x \tan iy}\Big| \leqslant |\tan x + \tan iy| \leqslant |\tan x| + 1 \leqslant \tan(10\pi\delta)+1$$

b) *The* Γ*-factor* $Y(\alpha; s)$. By formula (4.3) for imaginary s_k we have

$$|\Gamma(\tfrac{1}{2}(\alpha+\varepsilon_k - \tfrac{1}{2}(p+q)+1+s_k))\Gamma(\tfrac{1}{2}(\alpha+\varepsilon_k-\tfrac{1}{2}(p+q)+1-s_k)| \leqslant \\ \leqslant \mathrm{const} \cdot (1+|\theta_k|)^{\operatorname{Re}\alpha+\varepsilon_k-\frac{1}{2}(p+q)+1} \exp(-\pi|\theta_k|)$$

For $s_k \in S_1, S_2$ the same expression is bounded (but very large).

c)*Spherical functions* $\Phi_s(t)$. By estimation (1.39) we have

$$|\Phi_s(t)| \leqslant \Phi_{\operatorname{Re} s}(t) \leqslant \max \Phi_{r_1,\dots,r_p}(t)$$

where maximum is given over all real vectors $(r_1, \dots, r_p)$ satisfying the condition $|r_j| \leqslant 10\delta$. Hence, for a fixed t the spherical function in integrand is dominated by a constant.

This completes the proof of the Lemma 4.5. $\boxtimes$

Now we can complete the proof of Lemma 4.4.b). By Lemma 4.5 we have

$$|\mathfrak{F}_\varkappa(\alpha;\varepsilon)| \leqslant A(t) \int_{\mathbb{R}^p} \prod_{j=1}^{p} \left[(1+|\theta_j|)^N \exp(-\pi|\theta_j|)\right] \prod_{j=1}^{p} \chi(\theta_j) d\theta_1 \dots d\theta_p$$

there the function $\chi(\theta)$ is given by the formula

$$\chi(\theta) := \frac{ds}{d\theta} = \begin{cases} 1 & \text{if} \quad \big|\mathrm{Im}\,\alpha_0 - |\theta|\big| \geqslant 10\delta \\ (100\delta^2 - (\mathrm{Im}\,\alpha_0 - \theta)^2)^{-1/2} & \text{if} \quad \big|\mathrm{Im}\,\alpha_0 - |\theta|\big| \leqslant 10\delta \end{cases}$$

Hence

$$|\mathfrak{F}_\varkappa(\alpha;\varepsilon)| \leqslant A(t)^p \left(\int_{-\infty}^{\infty} (1+|\theta|)^N \exp(-\pi|\theta|)\chi(\theta) d\theta\right)^p$$

This completes the proof of uniform boundedness of the functions $\mathfrak{F}_\varkappa(\alpha;\varepsilon)$ for a fixed t (Lemma 4.4b). $\boxtimes$

Now we are ready to prove existence of the analytic continuation of the function $\mathfrak{F}_\varkappa$.

Proof of Lemma 4.3.b. Let us denote by ε/n the vector $(\varepsilon_1/n, \dots, \varepsilon_p/n)$. Consider the sequence of functions

$$g_n(\alpha) = \mathfrak{F}_\varkappa(\alpha;\varepsilon/n)$$

in the circle $\mathcal{O}_\delta$. Since the functions $g_n(\alpha)$ are uniformly bounded, by Montel theorem there exists a subsequence g_{n_j} which is uniformly convergent on each smaller circle. Let $g(\alpha)$ be its limit. By Weierstrass theorem $g(\alpha)$ is holomorphic in $\mathcal{O}_\delta$. It remains to notice that

$$\lim_{n\to\infty} \mathfrak{F}_\varkappa(\alpha;\varepsilon/n) = \mathfrak{F}_\varkappa(\alpha)$$

for $\alpha \in \Pi_\varkappa \cap \mathcal{O}_\delta$. Hence, $g(\alpha)$ is the analytic continuation of $\mathfrak{F}_\varkappa(\alpha)$ to the circle $\mathcal{O}_\delta$.

4.4. Forcing of poles. First, we want to obtain an explicit formula for the analytic continuation of $\mathfrak{F}_\varkappa(\alpha;\varepsilon)$ to the domain $\Pi^\varepsilon_{\varkappa+1}$.

Let the contours L_k be the same as above. Let $i\mathbb{R}_k$ be the imaginary axis on the complex plane s_k. Consider the surface

$$\mathcal{L}_k = i\mathbb{R}_1 \times \dots \times i\mathbb{R}_{k-1} \times L_k \times \dots \times L_p \subset \mathbb{C}^p$$

We have $\mathcal{L}_1 = \mathcal{L}$, $\mathcal{L}_p = i\mathbb{R}^p$.

Consider $\alpha \in \mathcal{O}_\delta \cap \Pi_{\varkappa+1}$. Then

$$\mathfrak{F}_{\varkappa+1}(\alpha;\varepsilon) = E(\alpha) \int\limits_{i\mathbb{R}^p} \prod_{k,\pm} \Gamma(\tfrac{1}{2}(\alpha+\varepsilon_k - \tfrac{1}{2}(p+q)+1 \pm s_k)) \cdot \mathfrak{R}(s)\Phi_s(t)ds \tag{4.16}$$

$$\mathfrak{F}_{\varkappa}(\alpha;\varepsilon) = E(\alpha) \int\limits_{\mathfrak{L}} \prod_{k,\pm} \Gamma(\tfrac{1}{2}(\alpha+\varepsilon_k - \tfrac{1}{2}(p+q)+1 \pm s_k)) \cdot \mathfrak{R}(s)\Phi_s(t)ds \tag{4.17}$$

Hence,

$$\mathfrak{F}_{\varkappa+1}(\alpha;\varepsilon) - \mathfrak{F}_{\varkappa}(\alpha;\varepsilon) = \int\limits_{i\mathbb{R}^p} - \int\limits_{\mathfrak{L}} = \sum_{\sigma} \Big[\int\limits_{\mathfrak{L}_{\sigma+1}} - \int\limits_{\mathfrak{L}_\sigma} \Big] \tag{4.18}$$

Looking to Picture 2 we observe

$$\int_{\mathfrak{L}_{\sigma+1}} - \int_{\mathfrak{L}_\sigma} = \tag{4.19}$$

$$= 2\pi i \int\limits_{\substack{s_1 \in i\mathbb{R}_1, \dots, s_{\sigma-1} \in i\mathbb{R}_{\sigma-1} \\ s_{\sigma+1} \in L_{\sigma+1}, \dots, \sigma_p \in L_p}} \Big[\operatorname*{Res}_{s_\sigma = \alpha+\varepsilon_\sigma - \frac{1}{2}(p+q)+1+2\varkappa} - \tag{4.20}$$

$$- \operatorname*{Res}_{s_\sigma = -\alpha-\varepsilon_\sigma + \frac{1}{2}(p+q)-1-2\varkappa} \Big] ds_1 \dots ds_{\sigma-1} ds_{\sigma+1} \dots ds_p \tag{4.21}$$

The integrand in (4.16)–(4.17) is an even function in s_σ and hence two residues in (4.20)–(4.21) differ only by sign. The order of poles

$$s_\sigma = \pm(\alpha+\varepsilon_\sigma - \tfrac{1}{2}(p+q)+1+2\varkappa)$$

of the integrand is 1 and hence the residues can be evaluated by a simple substitution

$$\operatorname*{Res}_{s_\sigma = \alpha+\varepsilon_\sigma - \frac{1}{2}(p+q)+1+2\varkappa} = H_\sigma(\alpha,\varepsilon,s) :=$$

$$= E(\alpha) \Big[\frac{\Gamma(\frac{1}{2}(\alpha+\varepsilon_\sigma - \frac{1}{2}(p+q)+1-s_\sigma))}{s_\sigma - \alpha - \varepsilon_\sigma + \frac{1}{2}(p+q) - 1 - 2\varkappa} \Gamma(\tfrac{1}{2}(\alpha+\varepsilon_\sigma - \tfrac{1}{2}(p+q)+1+s_\sigma)) \times$$

$$\times \prod_{\pm;k\neq\sigma} \Gamma(\tfrac{1}{2}(\alpha+\varepsilon_k - \tfrac{1}{2}(p+q)+1 \pm s_k)) \cdot \mathfrak{R}(s)\Phi_s(t) \Big] \Big|_{s_\sigma = \alpha+\varepsilon_\sigma - \frac{1}{2}(p+q)+1+2\varkappa} \tag{4.22}$$

In this way, we reduce sum (4.18) to

$$2 \sum_{\sigma=1}^{p} \int\limits_{i\mathbb{R}_1 \times \dots \times i\mathbb{R}_{\sigma-1} \times L_{\sigma+1} \times \dots \times L_p} H_\sigma(\alpha,\varepsilon,s) ds_1 \dots ds_{\sigma-1} ds_{\sigma+1} \dots ds_p$$

We obtain an expression for $\mathfrak{F}_{\varkappa+1}(\alpha;\varepsilon)-\mathfrak{F}_{\varkappa}(\alpha;\varepsilon)$. Unfortunately the domains of integration is yet complicated. By this reason we apply transformation (4.18) to each summand in the last expression. We obtain $p(p-1)/2$ additional summands which are integrals over $(p-2)$-dimensional surfaces. Each integral can be easily evaluated by residues. After this we apply our arguments again, again, again.

It is possible to wright the final expression (it is slightly long). Fortunately, this is not necessary. The only goal of our interest is

$$\lim_{\varepsilon\to 0}(\mathfrak{F}_{\varkappa+1}(\alpha;\varepsilon)-\mathfrak{F}_{\varkappa}(\alpha;\varepsilon)) \tag{4.23}$$

For instance, consider the summand obtained by the substitution

$$s_\sigma=\alpha+\varepsilon_\sigma-\frac{1}{2}(p+q)+1+2\varkappa$$
$$s_\upsilon=\alpha+\varepsilon_\upsilon-\frac{1}{2}(p+q)+1+2\varkappa$$

Then the integrand contains the factors

$$\frac{1}{\Gamma(\pm\frac{1}{2}(s_\sigma-s_\upsilon))}\Bigg|_{\substack{s_\sigma=\alpha+\varepsilon_\sigma-\frac{1}{2}(p+q)+1+2\varkappa\\ s_\upsilon=\alpha+\varepsilon_\upsilon-\frac{1}{2}(p+q)+1+2\varkappa}}=\frac{1}{\Gamma(\pm\frac{1}{2}(\varepsilon_\sigma-\varepsilon_\upsilon))}$$

This factors tend to 0 if $\varepsilon\to 0$. Of course, it is necessary to check lacking of poles of the numerator in dangerous for us domain.

Therefore a nonzero contribution to the limit (4.23) can be given only by the terms

$$2\sum_{\sigma=1}^{p}\int_{i\mathbb{R}^p} H_\sigma(\alpha,\varepsilon,s)ds_1\dots ds_{\sigma-1}ds_{\sigma+1}\dots ds_p \tag{4.24}$$

Our expression is symmetric under permutations of s_j and hence all summands of (4.28) give the same contribution to the limit. Thus, deleting ε in (4.24) we obtain the formula

$$\mathfrak{F}_{\varkappa+1}(\alpha)-\mathfrak{F}_{\varkappa}(\alpha)=$$
$$\frac{2\pi i\cdot 2p\cdot 2^\alpha}{\prod\Gamma(\alpha-j+1)}\int_{\mathbb{R}^{p-1}}\Bigg[\frac{\Gamma(\frac{1}{2}(\alpha-\frac{1}{2}(p+q)+1-s_1))}{s_1-\alpha+\frac{1}{2}(p+q)-1-2\varkappa}\Gamma(\tfrac{1}{2}(\alpha-\tfrac{1}{2}(p+q)+1+s_1))\times$$
$$\times\prod_{2\leqslant k\leqslant p,\pm}\Gamma(\tfrac{1}{2}(\alpha-\tfrac{1}{2}(p+q)+1\pm s_k))\cdot\prod_{2\leqslant k\leqslant p,\pm}\frac{\Gamma(\frac{1}{2}(q-p)\pm s_k)}{\Gamma(\pm s_k)}\times$$
$$\times\prod_{2\leqslant k<l\leqslant p,\pm}\frac{\Gamma(\frac{1}{2}(1+s_k\pm s_l))\Gamma(\frac{1}{2}(1-s_k\pm s_l))}{\Gamma(\frac{1}{2}(s_k\pm s_l))\Gamma(\frac{1}{2}(-s_k\pm s_l))}\Phi_s(t)\Bigg]\Bigg|_{s_1=\alpha-\frac{1}{2}(p+q)+1+2\varkappa}ds_2\dots ds_p$$

where $\alpha \in \Pi_{\varkappa+1} \cap \mathcal{O}_\delta$.

4.5. Calculations. Lemma 4.3.b is an obvious corollary of the last formula. Nevertheless we present some elements of the calculation, since it is essential for understanding Subsection 4.7.

$$1) \qquad \left[\frac{\Gamma(\frac{1}{2}(\alpha+\frac{1}{2}(p+q)-1\pm s_k))}{\Gamma(\frac{1}{2}(s_1\pm s_k))\Gamma(\frac{1}{2}(-s_1\pm s_k))}\right]\Big|_{s_1=\alpha-\frac{1}{2}(p+q)+2+2\varkappa} = \\ = \frac{\frac{1}{2}(-\alpha+\frac{1}{2}(p+q)-1\pm s_k)-\varkappa}{\Gamma(\frac{1}{2}(-\alpha+\frac{1}{2}(p+q)+1\pm s_k))}$$

We observe that the factor $Y(\alpha;s)$ (see (2.2)) is canceled. This factor was the origin of singularities in our integral (2.1)–(2.5).

$$2) \qquad \Gamma(\tfrac{1}{2}(1+s_1\pm s_k))\Big|_{s_1=\alpha-\frac{1}{2}(p+q)+1+2\varkappa} = \Gamma(\tfrac{1}{2}(\alpha-\tfrac{1}{2}(p+q)+1+2\varkappa\pm s_k))$$

We observe appearance of factor (4.8) which is very similar to the factor $Y(\alpha;s)$. Later it will be an origin of new singularities.

$$3) \qquad \left[\frac{\Gamma(\frac{1}{2}(\alpha-\frac{1}{2}(p+q)+1+s_1))}{\Gamma(s_1)\Gamma(-s_1)}\right]\Big|_{s_1=\alpha-\frac{1}{2}(p+q)+1+2\varkappa} = \\ = \frac{(-\alpha+\frac{1}{2}(p+q)-2\varkappa-1)}{\Gamma(-\alpha+\frac{1}{2}(p+q)-\varkappa)}$$

This gives formula (4.6)-(4.12), and completes the proof of Lemma 4.3.

4.6. Analytic continuation through the line $\operatorname{Re}\alpha < \frac{1}{2}(p+q)-1-2\varkappa$**.** Lemma 4.3 gives the analytic continuation of $\mathfrak{F}_\varkappa$ to $\mathcal{O}_\delta \cap \Pi_{\varkappa+1}$. Evidentely the expression for the analytic continuation is analytic in the strip

$$-2\varkappa-1 < \alpha-\tfrac{1}{2}(p+q)+1 < -2(\varkappa-1) \tag{4.25}$$

and hence we obtain the analytic continuation of $\mathfrak{F}_\varkappa$ to the whole strip (4.25).

4.7. Proof of Theorem 2.2. The Plancherel formula (2.1)–(2.5) is correct if $\operatorname{Re}\alpha > \frac{1}{2}(p+q)-1$. We want to construct the analytic continuation of its right part to the domain $\operatorname{Re}\alpha < \frac{1}{2}(p+q)-1$. Let us move α to the left side.

Firstly we pass across the line $\alpha = \frac{1}{2}(p+q)-1$. Then we obtain the additional summand $\mathfrak{F}^0(\alpha) := \mathfrak{F}_1(\alpha)-\mathfrak{F}_0(\alpha)$ given by formula (4.6)-(4.12) for $\varkappa=0$. This is the summand of the Plancherel formula corresponding $m=1$, u_1=0.

Let us compare the formula (4.6)–(4.12) for $\mathfrak{F}^0(\alpha)$ and (2.1)–(2.5). First, we have in (4.6)–(4.12) additional factor (4.9). This factor has singularities but all these singularities are lying in the domain $\alpha > \frac{1}{2}(p+q)-1$. The factors (2.2) and (4.8) are very similar (α is changed to $\alpha + 1$). The factors (2.14)–(2.15) and (4.10)–(4.11) also are very similar. In fact (4.10)–(4.11) is the Gindikin–Karpelevich density for $\mathrm{O}(p-1, q-1)$.

Hence, we can construct the analytical continuation of $\mathfrak{F}^0(\alpha)$ in the same way as above. The first singularity of $\mathfrak{F}^0(\alpha)$ on our way is the line $\operatorname{Re}\alpha = \frac{1}{2}(p+q)-2$. After passing across the line we obtain one more summand $\mathfrak{F}^{00}(\alpha)$ corresponding $m = 2$, $u_1 = u_2 = 0$.

The line $\operatorname{Re}\alpha = \frac{1}{2}(p+q) - 3$ contains singularities of the integral $\mathfrak{F}(\alpha)$ and also singularities of $\mathfrak{F}^{00}(\alpha)$. Hence, we obtain two additional summands $\mathfrak{F}^1(\alpha)$ and $\mathfrak{F}^{000}(\alpha)$ corresponding $m = 1$, $u_1 = 1$ and $m = 3$, $u_1 = u_2 = u_3 = 0$ etc. etc. etc.

Formally, we must give complete description of the inductive step but it literally repeats the arguments of Subsections 4.1–4.6.

5. Positive definiteness of spherical functions.

We obtained the expansion of $\mathcal{B}_\alpha(s)$ in spherical functions having the form

$$\mathcal{B}_\alpha(z) = \int_{\mathbb{C}^p} \Phi_s(z)\, d\mu_{our}(s) \tag{5.1}$$

where the positive D_p-invariant measure $\mu_{our}(z)$ is described in Theorem 2.2. Our purpose is to prove positive definiteness of spherical functions $\Phi_s(z)$ which are contained in the support of the measure μ_{our}.

By the abstract Plancherel theorem, there exists the unique expansion

$$\mathcal{B}_\alpha(z) = \int_{\mathbb{C}^p} \Phi_s(z)\, d\mu_{truth}(s) \tag{5.2}$$

where μ_{truth} is a positive D_p-invariant measure on $\mathbb{C}^n$ supported by the space $\widehat{G}_{sph}$ of positive definite spherical functions.

Substitute $z = 0$ to (5.2). Then $\mathcal{B}_\alpha(0) = 1$, $\Phi_s(0) = 1$ and hence

$$\int_{\mathbb{C}^p} d\mu_{truth} = 1 \tag{5.3}$$

We denote by $\operatorname{supp}\mu_{truth}$ and $\operatorname{supp}\mu_{our}$ the supports of the measures μ_{truth} and μ_{our}.

5.1. Preliminary remarks on the supports of the measures. Consider the bounded polyhedron $Q \subset \mathbb{R}^p$ described in Theorem 1.14. Consider the tube $\widetilde{Q} \subset \mathbb{C}^p$ defined by the condition

$$s \in \widetilde{Q} \qquad \text{iff} \qquad \operatorname{Re} s \in Q$$

. Then

$$\operatorname{supp}\mu_{truth}\subset\widetilde{Q};\qquad \operatorname{supp}\mu_{our}\subset\widetilde{Q} \tag{5.4}$$

(the first is corollary of Theorem 1.14, the second is corollary of Theorem 2.2).

Denote by $\mathbb{R}\cup i\mathbb{R}$ the union of the real and imaginary axises in $\mathbb{C}$. Then

$$\operatorname{supp}\mu_{our}\subset(\mathbb{R}\cup i\mathbb{R})\times\cdots\times(\mathbb{R}\cup i\mathbb{R});\qquad \operatorname{supp}\mu_{truth}\subset(\mathbb{R}\cup i\mathbb{R})\times\cdots\times(\mathbb{R}\cup i\mathbb{R}) \tag{5.5}$$

(the first is the corollary of Theorem 2.2 and the second is the Corollary of Lemma 1.13).

5.2. Heat kernel. Let $\Delta_1,\dots,\Delta_p$ be Laplace operators (see [17], Section 2.5) on the symmetric space G/K. The operator Δ_j is some G-invariant partial differential operator of order $2j$ on $G/K=\mathrm{B}_{p,q}$ with rational coefficients. The operator Δ_1 is the usual Laplace-Beltrami operator on G/K (see [17], Section 2.2.4).

Spherical functions are joint eigenfunctions of the operators Δ_j (see [17], Section 4.2). We have equalities

$$\Delta_j\Phi_s(z)=a_j(s)\Phi_s(z)$$

where a_j are some polynomials invariant with respect to the group D_p (see Subsection 1.15) consisting of permutations and changing of signs. If $\Phi_s\neq\Phi_{s'}$, then $a_j(s)\neq a_j(s')$ for some j.

In particular

$$\Delta_1\Phi_s(z)=(\lambda+s^2)\Phi_s(z)$$

where λ is a constant and

$$s^2:=s_1^2+\cdots+s_p^2$$

By conditions (5.4)–(5.5) *the eigenvalues* $(\lambda+s^2)$ *are real and they are uniformly bounded above on the supports of the measures* μ_{our}, μ_{truth}.

Consider the Cauchy problem for the *heat equation*

$$\left(\frac{\partial}{\partial\tau}-\Delta_1\right)F(z,\tau)=0;\qquad F(z,0)=f(z)$$

on G/K. Let $R_\tau(z,u)$ be the heat kernel. This means that the solution of the Cauchy problem for the heat equation is given by the formula

$$F(z,\tau)=A_\tau f(z):=\int_{G/K}R_\tau(z,u)f(u)\,d\lambda(u)$$

where λ is the G-invariant measure on G/K.

LEMMA 5.1. *For each $\tau > 0$ and N there exists a constant $C(\tau, N)$ independent on z, u such that*

$$R_\tau(z,u) \leqslant C(\tau,N)(1+\operatorname{dist}(z,u))^{-n}$$

where $\operatorname{dist}(\cdot,\cdot)$ *is the distance in G/K associated with Riemannian metric.*

PROOF. Since the heat kernel is G-invariant, we can assume $u = 0$. Then

$$R_\tau(z,0) = \int_{s\in i\mathbb{R}} \exp\{\tau(\lambda+s^2)\}\Phi_s(z)ds$$

By integral formula for spherical functions (1.38)

$$R_\tau(z,0) = \int_{s\in i\mathbb{R}}\int_{\mathbf{k}\in K} \exp\{\tau(\lambda+s^2)\}\Psi_s(z^{[\mathrm{k}]})d\mathrm{k}\, ds$$

Rapid decreasing of the last expression is more or less obvious. ⊠

Similar estimates are valid for partial derivatives of $R_\tau(z,u)$ of any order.

For spherical functions we have the equality

$$A_\tau\Phi_s(z) = \exp\{\tau(\lambda+s^2)\}\Phi_s(z)$$

LEMMA 5.2. *Let μ_{our}, μ_{truth} be the same as above. Then*

$$a) A_\tau\mathcal{B}_\alpha(z) = \int_{G/K} \exp\{\tau(\lambda+s^2)\}\Phi_s(z)\, d\mu_{our}(s) \tag{5.6}$$

$$b) A_\tau\mathcal{B}_\alpha(z) = \int_{G/K} \exp\{\tau(\lambda+s^2)\}\Phi_s(z)\, d\mu_{truth}(s) \tag{5.7}$$

PROOF. We must prove a possibility to change the order of the integration. It is sufficient to show absolute convergence of the integrals

$$\int_{G/K}\int_{\mathbb{C}^p} R_\tau(z,u)\Phi_s(u)\, d\mu_{our}(s)d\lambda(u); \qquad \int_{G/K}\int_{\mathbb{C}^p} R_\tau(z,u)\Phi_s(u)\, d\mu_{truth}(s)d\lambda(u) \tag{5.8}$$

a) For the first integral (5.8) we use estimate $|\Phi_s(u)| \leqslant \Phi_{\operatorname{Re} s}(u)$ (see (1.39)). The measure μ_{our} is supported by a finite family of planes $P^{(j)}$ having the form

$$P_j: \qquad \operatorname{Re} s_1 = \theta_1^{(j)}, \ldots, \operatorname{Re} s_m = \theta_m^{(j)}$$

On each plane P_j the integrand is dominated by some expression

$$\sum R_\tau(z,u)\Phi_{\sigma_j}(u)$$

where σ_j are real vectors. The last expression doesn't depend on s. It remains to notice that the heat kernel rapidly decrease in u for fixed z, spherical functions

$\Phi_{\sigma_j}(u)$ are bounded (see Theorem 1.14), and the density of $\mu_{our}(s)$ exponentially decreases if $|s| \to \infty$.

b) For the second integral we use the inequality $|\Phi_s(z)| \leqslant 1$ and this completes the proof (see (5.3)). ⊠

LEMMA 5.3. *For each polynomial $r(x_1, \dots, x_p)$ we have*

$$r(\Delta_1, \dots, \Delta_p) A_\tau \mathcal{B}_\alpha(z) = \int_{G/K} r(a_1(s), \dots, a_p(s)) \exp\{\tau(\lambda + s^2)\} \Phi_s(z)\, d\mu_{our}(s)$$

$$r(\Delta_1, \dots, \Delta_p) A_\tau \mathcal{B}_\alpha(z) = \int_{G/K} r(a_1(s), \dots, a_p(s)) \exp\{\tau(\lambda + s^2)\} \Phi_s(z)\, d\mu_{truth}(s)$$

PROOF. It is sufficient to prove that all partial derivatives by z of integrals (5.6), (5.7) absolutely converges. It is obvious by the following reasons.

1. The integrand rapidly decreases in the variable s.
2. For a given z partial derivatives of the heat kernel by u rapidly decrease.
3. Spherical functions are bounded. ⊠

5.3. Proof of positive definiteness. Consider $\sigma \in \operatorname{supp} \mu_{our}$. Consider the function

$$\eta(s) = \exp\{\tau(\lambda + s^2)\} \sum_{j=1}^{p} (a_j(s) - a_j(\sigma))^2$$

Let M be the maximum of η on $(\mathbb{R} \cup i\mathbb{R}) \times \dots \times (\mathbb{R} \cup i\mathbb{R})$. Then the function

$$\zeta(s) = M - \eta(s)$$

satisfies conditions

$$\zeta(\sigma) = M; \qquad \zeta(s) < M \quad \text{if} \quad s \neq w\sigma \text{ for all } w \in D_p$$

Consider the sequence of functions

$$\xi_k(s) = C_k (M - \eta(s))^k \exp\{\tau(\lambda + s^2)\}$$

where C_k is determined by the condition $\int \xi_k(s)\, d\nu_{our} = 1$. Obviously the sequence $\xi_k(s)$ converges to distribution $\sum_{w \in D_p} \delta(s - w\sigma)$.

The function $\xi_k(s)$ is a polynomial expression

$$\xi_k(s) = P_k(a_1(s), \dots, a_p(s), \exp\{\tau(\lambda + s^2)\})$$

Consider the operator

$$\Xi_k := P_k(\Delta_1, \dots, \Delta_p, A_\tau)$$

By Lemma 5.3 we have

$$\Xi_k \mathcal{B}_\alpha(z) = \int_{G/K} \xi_k(s) \Phi_s(z) d\nu_{our}(s) \tag{5.9}$$

$$\Xi_k \mathcal{B}_\alpha(z) = \int_{G/K} \xi_k(s) \Phi_s(z) d\nu_{truth}(s) \tag{5.10}$$

We have $\xi_k(s) \geqslant 0$. Hence, by Lemma 1.5d and (5.10) the function $\Xi_k \mathcal{B}_\alpha(s)$ is positive definite. By (5.9) the sequence $\Xi_k \mathcal{B}_\alpha(s)$ converges to $\Phi_\sigma(z)$. Thus, $\Phi_\sigma(z)$ is a point-wise limit of positive definite functions and hence it is positive definite.

6. Other series

6.1. Hermitizations. Below we present the list of hermitizations (see Subsection 0.1)[23].

Each Riemannian noncompact classical symmetric space G/K can be realized as a matrix ball[24]. A *matrix ball* is a space of all matrices of a given size over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ with norm < 1 satisfying (or not satisfying) some symmetry condition. The list of matrix balls is given in the Table 1.

Table 1

G/K	$\mathbb{K}$	size	condition	$\widetilde{G}/\widetilde{K}$
1.$\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n,\mathbb{R})$	$\mathbb{R}$	$n\times n$	$z=z^t$	$\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$
2.$\mathrm{O}(p,q)/\mathrm{O}(p)\times\mathrm{O}(q)$	$\mathbb{R}$	$p\times q$		$\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)$
3.$\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$	$\mathbb{C}$	$n\times n$	$z=z^t$	$\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)\times\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$
4.$\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$	$\mathbb{C}$	$n\times n$	$z=z^*$	$\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$
5.$\mathrm{O}(n,\mathbb{C})/\mathrm{O}(n)$	$\mathbb{R}$	$n\times n$	$z=-z^t$	$\mathrm{SO}^*(2n)/\mathrm{U}(n)$
6.$\mathrm{Sp}(2n,\mathbb{C})/\mathrm{Sp}(n)$	$\mathbb{H}$	$n\times n$	$z=-z^*$	$\mathrm{Sp}(4n,\mathbb{R})/\mathrm{U}(2n)$
7.$\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)$	$\mathbb{H}$	$p\times q$		$[\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)]\times[\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)]$
8.$\mathrm{GL}(n,\mathbb{H})/\mathrm{Sp}(n)$	$\mathbb{H}$	$n\times n$	$z=z^*$	$\mathrm{SO}^*(2n)/\mathrm{U}(n)$
9.$\mathrm{Sp}(p,q)/\mathrm{Sp}(p)\times\mathrm{Sp}(q)$	$\mathbb{C}$	$p\times q$		$\mathrm{U}(2p,2q)/\mathrm{U}(2p)\times\mathrm{U}(2q)$
10.$\mathrm{SO}^*(2n)/\mathrm{U}(n)$	$\mathbb{C}$	$n\times n$	$z=z^t$	$\mathrm{SO}^*(2n)/\mathrm{U}(n)\times\mathrm{SO}^*(2n)/\mathrm{U}(n)$

The last column contains the hermitization $\widetilde{G}/\widetilde{K}$ of G/K. The embedding $G/K \to \widetilde{G}/\widetilde{K}$ in all cases is obvious (we must omit nonholomorphic condition to a matrix z, after this we obtain an hermitian matrix ball).

REMARK. Some spaces of small dimension are present in the left column two times (for instance Lobachevskii plane $\mathrm{O}(2,1)/\mathrm{O}(2)\times\mathrm{O}(1)$). Two associated hermitizations are different.

[23] Classical part of the list is contained in H.Jaffee paper [22]. Olshanskii in [42], [41] observed that all cases (including exceptional cases) can be easily reduced to Nagano work [29]. The list 1-18 is in one-to-one correspondence with the list of compressive semigroups of symmetric spaces and with the list of causal symmetric spaces, see [42], [41]. The list 1-10 is in one-to-one correspondence with the list of real classical categories, see [32], Addendum A.

[24] This observation is present in [30]

Table 2 contains hermitizations related to future tubes

Table 2

	G/K	$\widetilde{G}/\widetilde{K}$
11.	$SO(2,n)/O(n)\times O(2)$	$[SO(2,n)/O(n)\times O(2)]\times[SO(2,n)/O(n)\times O(2)]$
12.	$SO(1,p)\times O(1,q)$	$SO(2,p+q)$

Exceptional hermitizations[25] [26] are given in the Table 3

Table 3

13.	$E\mathrm{III}/SO(10)\times SO(2)$	$[E\mathrm{III}/SO(10)\times SO(2)]\times[E\mathrm{III}/SO(10)\times SO(2)]$
14.	$E\mathrm{VII}/E\mathrm{III}\times SO(2)$	$[E\mathrm{VII}/E\mathrm{III}\times SO(2)]\times[E\mathrm{VII}/E\mathrm{III}\times SO(2)]$
15.	$F\mathrm{II}/\mathrm{Spin}(9)$	$E\mathrm{III}/SO(10)\times SO(2)$
16.	$Sp(2,2)/Sp(2)\times Sp(2)$	$E\mathrm{III}/SO(10)\times SO(2)$
17.	$GL(4,\mathbb{H})/Sp(2)$	$E\mathrm{VII}/E\mathrm{III}\times SO(2)$
18.	$E\mathrm{IV}\times\mathbb{R}/F_4$	$E\mathrm{VII}/E\mathrm{III}\times SO(2)$

6.2. Kernel representations. A *kernel representation*[27] ρ of G is a restriction of a highest weight representation $\widetilde{\rho}$ of $\widetilde{G}$ to G. The constructive description of the scalar-valued kernel representations of $O(p,q)$ given in Subsections 1.11-1.12 is valid for all series 1-10.

Below we discuss only scalar-valued kernel-representations.

6.3. Plancherel formula for large α for the series 1-10 was obtained in [35][28]

Consider for simplicity nonhermitian case or hermitian case[29] $\alpha=\beta$. In all

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[25] EIII and EIV are real form of E_6, EVII is a real forms of E_7, and FII is a real form of F_4, see [54]

[26] Here we observe one of the cases when a phenomenon existing for classical groups doesn't exist for all exceptional groups

[27] This definition was proposed in [33]

[28] The case of hermitian spaces 3,7,10,11 was considered by Berezin [4] for $\alpha=\beta$, see notations in Subsection 0.2 (the proof is published in [52]), For hermitian case $\alpha\neq\beta$ the Plancherel formula independently on [35] was obtained by Zhang [56]. Future tube case is simple exercise. The hermitian cases 13, 14 are covered by [52]; the case 15 is reduced to one of Gindikin (1964) integrals [11]. Probably only for the exceptional cases 16, 17, 18 the formula is not known. Hence, the possibility to obtain the solution in "general case" (i.e. 16–18) is yet preserved.

[29] For hermitian case $\alpha\neq\beta$ we have the formula

$$C(\alpha)C(\beta)\prod_{k=1}^{p}\Gamma(\frac{1}{2}(\alpha-h_1+s_k))\Gamma(\frac{1}{2}(\beta-h_2+s_k))d\mathfrak{Q}(s)$$

where $d\mathfrak{Q}$ is the Shimeno measure [49].

these cases the Plancherel formula has the form

$$\prod_{k=1}^{p} \cosh^{-\alpha} t_k = E(\alpha) \int_{\imath\mathbb{R}^p} \prod_{k\leqslant p,\pm} \Gamma(\frac{1}{2}(\alpha - h + s_k))\mathfrak{R}(s)\Phi_s(t) ds \tag{6.1}$$

where m is the rank of G, $\mathfrak{R}(s)$ is the Gindikin-Karpelevich density, $E(\alpha)$ is a meromorphic factor and h is a constant. In fact, h is the last point of square integrability. This means that $\int_{G/K} |\mathcal{B}_\alpha(z)|^2 dz$ is finite for $\alpha > h$ and infinite for $\alpha = h$.

6.4. The analytic continuation of the Plancherel formula. Our arguments from Sections 4–5 don't depend on series and they are valid for all series 1–10.

In fact considerations of Section 4 prove that the following formal procedure gives correct result. We fix $m = 0, 1, \dots, p := \operatorname{rank} G$ and the collection of numbers $u_1 \leqslant u_2 \leqslant \dots u_m$ such that $\alpha + 2u_m + (m-1)\dim\mathbb{K} < h$

Let

$$\begin{aligned} Q_0^{m;u}(s) &= \prod_{k\leqslant p,\pm} \Gamma(\frac{1}{2}(\alpha - h + s_k)) \cdot \mathfrak{R}(s)\Phi_s(t) \\ Q_k^{m;u}(s) &= \left.\frac{Q_{k-1}^{m;u}(s)}{s_k - \alpha + h - 2u_k - k\dim\mathbb{K}}\right|_{s_k=\alpha-h+2u_k+k\dim\mathbb{K}} \end{aligned} \tag{6.2}$$

Then the analytic continuation of (6.1) has the form

$$\prod_{k=1}^{p} \cosh^{-\alpha} t_k = \sum_{m;u_1,\dots,u_p} E(\alpha) \frac{(2\pi)^m p!}{(p-m)!} \int_{i\mathbb{R}^{p-m}} Q_m^{m;u}(s) ds_{m+1} \dots ds_m \tag{6.3}$$

In the cases $\mathbb{K} = \mathbb{C}, \mathbb{H}$ this formula can be considered as a final result in a closed form.

In the case $\mathbb{K} = \mathbb{R}$ ($G = \mathrm{O}(p,q), \mathrm{Sp}(2n,\mathbb{R}), \mathrm{GL}(n,\mathbb{R})$) substitutions (6.3) are impossible without cancellations and in this case formula (6.3) gives algorithmic procedure of calculation of Plancherel measure.

Remark. For the groups $\mathrm{U}(p,q)$ it is easy to obtain the Plancherel formula (see the explicit final expression in [33]) using Berezin–Karpelevich formula for spherical functions (see [5],[20]) and Molev unitarizability results [28]; partially this idea was also realized in [18].

6.5. Kernel representations of compact groups. Hermitization procedure also is valid for compact Riemannian symmetric spaces. To obtain the list of hermitizations we must replace the groups G, $\widetilde{G}$ in Tables 1-3 to their compact forms. For instance the table 1 transforms to the following table 1'

Table 1'

G/K	$\mathbb{K}$	size	condition	$\widetilde{G}/\widetilde{K}$
$1'.\mathrm{U}(n)/\mathrm{O}(n,\mathbb{R})$	$\mathbb{R}$	$n\times n$	$z=z^t$	$\mathrm{Sp}(n)/\mathrm{U}(n)$
$2'.\mathrm{O}(p+q)/\mathrm{O}(p)\times\mathrm{O}(q)$	$\mathbb{R}$	$p\times q$		$\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)$
$3'.\mathrm{Sp}(n)/\mathrm{U}(n)$	$\mathbb{C}$	$n\times n$	$z=z^t$	$\mathrm{Sp}(n)/\mathrm{U}(n)\times\mathrm{Sp}(n)/\mathrm{U}(n)$
$4'.\mathrm{U}(n)\times\mathrm{U}(n)/\mathrm{U}(n)$	$\mathbb{C}$	$n\times n$	$z=z^*$	$\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$
$5'.\mathrm{O}(n)\times\mathrm{O}(n)/\mathrm{O}(n)$	$\mathbb{R}$	$n\times n$	$z=-z^t$	$\mathrm{O}(2n)/\mathrm{U}(n)$
$6'.\mathrm{Sp}(n)\times\mathrm{Sp}(n)/\mathrm{Sp}(n)$	$\mathbb{H}$	$n\times n$	$z=-z^*$	$\mathrm{Sp}(2n)/\mathrm{U}(2n)$
$7'.\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)$	$\mathbb{H}$	$p\times q$		$[\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)]\times[\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)]$
$8'.\mathrm{U}(2n)/\mathrm{Sp}(n)$	$\mathbb{H}$	$n\times n$	$z=z^*$	$\mathrm{O}(2n)/\mathrm{U}(n)$
$9'.\mathrm{Sp}(p+q)/\mathrm{Sp}(p)\times\mathrm{Sp}(q)$	$\mathbb{C}$	$p\times q$		$\mathrm{U}(2p+2q)/\mathrm{U}(2p)\times\mathrm{U}(2q)$
$10'.\mathrm{O}(2n)/\mathrm{U}(n)$	$\mathbb{C}$	$n\times n$	$z=z^t$	$\mathrm{O}(2n)/\mathrm{U}(n)\times\mathrm{O}(2n)/\mathrm{U}(n)$

The construction of the kernel representation of $\mathrm{O}(p+q)$ given in Subsection 1.28 can be literally translated to all series 1–10. For this purpose we must replace the space $\mathrm{Mat}_{p,q}(\mathbb{R})$ by the space Mat of all matrices over $\mathbb{K}$ (see the second column) having the size given in the third column and satisfying the condition in the forth column. The group G acts on Mat by fractional linear transformations[30].

Integrals evaluated in [35] easily give Plancherel formulas for all kernel-representations in the cases 1'-10'.

The case of hermitian symmetric spaces was earlier considered by Zhang [56].

6.6. Our terminology. a) *The term "hermitization".* The variant G/K *is a real form of* $\widetilde{G}/\widetilde{K}$ (see [22]) contradicts to the generally accepted usage of the term "real form", since $\widetilde{G}/\widetilde{K}$ is not the *complexification* $G_{\mathbb{C}}/K_{\mathbb{C}}$ of G/K.

b) *The term "kernel representation" or "Berezin kernel representation".* There is no common term for this object. One possible term is *"Berezin transform.* It is not suitable in our situation, since there is no "transform" in this paper.

Another term is *"canonical representation"*. The term "canonical representation" can be used in many other senses. For instance in [53] and several successive papers of Vershik, Gelfand and Graev this term was used for multiplicative integral and also for some infinite divisible representations. The complex of

[30]More details on these models of Riemannian compact symmetric spaces are contained in [37]

phenomena related to infinite divisibility has his own interest (see, for instance [32], Chapter 10) but it has small relation with kernel representations.

There is also the term *"Berezin quantization"*. The "quantization" in Berezin sence is an operation on the space of functions. In our picture this operation can be defined only for hermitian case.

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