CM points on products of Drinfeld modular curves

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Abstract

We prove an analogue of the André-Oort conjecture for subvarieties of products of Drinfeld modular curves over general base rings A of odd characteristic. More precisely, we classify those subvarieties containing a Zariski-dense set of CM points.

1 Introduction

1.1 Basic notations

The following notations will be used throughout this paper. Let $\mathbb{F}_q$ denote the finite field with q elements, where q is a power of the *odd* prime p. Let k be a global function field with field of constants $\mathbb{F}_q$. Fix a place ∞ of k, and denote by k_∞ the completion of k at ∞, and by $\mathbb{C}_\infty = \hat{\bar{k}}_\infty$ the completion of an algebraic closure of k. Let $A = \{x \in k \mid x \text{ is regular outside } \infty\}$, it is a Dedekind domain of finite class number. We denote by $|\cdot|$ the absolute value corresponding to ∞, and note that for $a \in A$, we have $|a| = q^{\deg(a)} = |A/\langle a\rangle|$.

Unless stated otherwise, a *Drinfeld module* always means a rank 2 Drinfeld A-module defined over $\mathbb{C}_\infty$ of generic characteristic. See [8] and [10] for an overview of Drinfeld modules.

We denote by $\hat{A} = \varprojlim A/\mathfrak{n}$ the profinite completion of A, and by $\mathbb{A}_f = \hat{A} \otimes_A k$ the ring of finite adèles of k. Let $\mathcal{K}$ be a subgroup of finite index (i.e. an open subgroup) of $\mathrm{GL}_2(\hat{A})$, then we denote by $\mathcal{M}_A^2(\mathcal{K})$ the coarse moduli scheme parametrizing rank 2 Drinfeld A-modules with level $\mathcal{K}$-structure, it is an affine curve over $\mathrm{Spec}(A)$, which is not in general irreducible. A *Drinfeld modular curve* over $\mathbb{C}_\infty$ is an irreducible component of some $\mathcal{M}_A^2(\mathcal{K})_{\mathbb{C}_\infty} = \mathcal{M}_A^2(\mathcal{K}) \times_A \mathbb{C}_\infty$. If $\mathcal{K} = \mathrm{GL}_2(\hat{A})$ we will just write $\mathcal{M}_A^2 = \mathcal{M}_A^2(\mathrm{GL}_2(\hat{A}))$, which is the coarse moduli space of Drinfeld modules without level structure.

Let $\Omega := \mathbb{P}^1(\mathbb{C}_\infty) - \mathbb{P}^1(k_\infty)$ denote the *Drinfeld upper half-plane*, which has a rigid analytic structure. The group $\mathrm{GL}_2(k_\infty)$ acts on Ω via fractional linear transformations. Let X be a Drinfeld modular curve, then it is known that, as rigid analytic varieties, we have

$$X(\mathbb{C}_\infty)^{\mathrm{an}} \cong \Gamma\backslash\Omega \tag{1.1}$$

for some arithmetic subgroup $\Gamma \subset \mathrm{GL}_2(k)$. We denote by $[\omega] \in X(\mathbb{C}_\infty)$ the point corresponding to $\omega \in \Omega$.

1.2 Main results

Let $X = X_1 \times \cdots \times X_n$ be a product of Drinfeld modular curves. A point $x \in X(\mathbb{C}_\infty)$ is called a *CM point* if it corresponds to an n-tuple of Drinfeld modules with complex multiplication. These in turn correspond to points $\omega \in \Omega$ with $[k(\omega) : k] = 2$ via (1.1).

A subcurve $Y \subset X$ is called a *special subcurve* if there exists a partition $\{1, \ldots, n\} = I_0 \coprod I_1$, $I_1 \neq \emptyset$, and elements $g_i \in \mathrm{GL}_2(k)$, for $i \in I_1$, such that $Y = \{x\} \times Y'$, where $x \in \prod_{i \in I_0} X_i(\mathbb{C}_\infty)$ is a CM point and $Y' \subset \prod_{i \in I_1} X_i$ is a curve such that $Y'(\mathbb{C}_\infty)$ is the image of the map

$$\Omega \longrightarrow \prod_{i \in I_1} X_i(\mathbb{C}_\infty); \quad \omega \longmapsto \big([g_i(\omega)]\big)_{i \in I_1}.$$

The curve Y is called a *pure special subcurve* if $I_1 = \emptyset$, i.e. if the projections $p_i : Y \to X_i$ are surjective for all $i = 1, \ldots, n$.

A subvariety $Y \subset X$ is called a *special subvariety* if there exists a partition $\{1, \ldots, n\} = \coprod_{j=0}^{g} I_j$ such that $Y = \{x\} \times \prod_{j=1}^{g} Y_j$, where $x \in \prod_{i \in I_0} X_i(\mathbb{C}_\infty)$ is a CM point, and each $Y_j \subset \prod_{i \in I_j} X_i$ is a pure special subcurve, for $j = 1, \ldots, g$. We see that CM points are just special subvarieties of dimension zero.

The aim of this paper is to prove the following result.

Theorem 1.2 *Let $X = X_1 \times \cdots \times X_n$ be a product of Drinfeld modular curves. Then an irreducible algebraic subvariety $Y \subset X$ contains a Zariski-dense subset of CM points if and only if Y is a special subvariety.*

It is easy to see that a special subvariety contains a Zariski-dense set of CM points, the hard part is to prove the converse. In the special case where Y is a curve, we actually prove an effective result, namely that Y is special if and only if Y contains a CM point of sufficiently large *CM height*, see Theorem 3.21.

Theorem 1.2 is an analogue of the André-Oort Conjecture for products of classical modular curves, see [1] and [4]. In fact, our proof is closely modelled on Edixhoven's approach [4].

Theorem 1.2 was proved in [3] for the special case $A = \mathbb{F}_q[T]$. In this paper we show how to adapt the arguments of [3] to the case of general A (but still of odd characteristic). This result, which was stated as Conjecture 7.1 in [2], is also the main step towards extending the non-triviality result for higher Heegner points on elliptic curves over rational function fields [2, Thm. 1] to arbitrary global fields, at least of odd characteristic, see [2, Thm. 2]. The remaining step, concerning the non-triviality of certain linear combinations of automorphic forms, will be treated in a separate paper.

2 Some preliminaries

We begin by gathering some basic results which will be needed in the proof of Theorem 1.2.

2.1 Complex multiplication and CM heights

Let ϕ be a Drinfeld module with complex multiplication by the ring R, and let K be the quotient field of R. Then K/k is a quadratic *imaginary* extension, which means that only one prime of K lies above ∞, which we again call ∞. Denote by $\mathcal{O}_K$ the ring of integers of K, i.e. the integral closure of A in K. Then $A \subset R \subset \mathcal{O}_K$, and R is a projective A-module of rank 2, hence by the invariant factor theorem, $R = A + \mathfrak{f}\mathcal{O}_K$, for some ideal $\mathfrak{f} \subset A$, which we will call the conductor of R. Note that $\mathfrak{f}\mathcal{O}_K$ is the largest $\mathcal{O}_K$-ideal which is also an R-ideal, which is the definition of conductor usually found in the literature.

The ring R is an order in K, and is not in general integrally closed. However, one may still view ϕ as a Drinfeld R-module of rank 1, after Hayes [9], and in fact we have $\mathcal{M}_R^1 = \mathrm{Spec}(\mathcal{O}_{H_R})$, where $\mathcal{O}_{H_R}$ is the ring of integers of the class field H_R corresponding to the class group $K^\times \backslash \mathbb{A}_{f,K}^\times / \hat{R}^\times \cong \mathrm{Pic}(R)$. The field H_R is also known as the *ring class field* of R, and H_R/K is unramified outside $\mathfrak{f}\mathcal{O}_K$. Note that we only deal with the *finite* adèles $\mathbb{A}_{f,K}$, so that H_R/K splits completely at ∞. The action of $\mathbb{A}_{f,K}^\times$ on $\mathrm{Spec}(\mathcal{O}_{H_R})$ coincides with the action of $\mathrm{Gal}(H_R/K)$ on $\mathcal{M}_R^1$ given by class field theory. Hence ϕ is defined over H_R and isogenies act like Galois. In particular, we have

Proposition 2.1 *Let ϕ be a Drinfeld module with complex multiplication by R. Let $\mathfrak{n} \subset A$ be a non-zero ideal such that every prime factor of $\mathfrak{n}$ splits in R, and let $\sigma_\mathfrak{n} = (\mathfrak{n}, H_R/k) \in \mathrm{Gal}(H_R/k)$ be the corresponding Frobenius element. Then ϕ and $\phi^{\sigma_\mathfrak{n}}$ are linked by a cyclic isogeny of degree $\mathfrak{n}$.*

Proof. As every prime factor of $\mathfrak{n}$ splits in R, we may choose an ideal $\mathfrak{N}$ of R such that $R/\mathfrak{N} \cong A/\mathfrak{n}$. Now an isogeny of ϕ as a rank 2 Drinfeld A-module with kernel $A/\mathfrak{n}$ is also an isogeny of ϕ as a rank 1 Drinfeld R-module with kernel $R/\mathfrak{N}$. The result now follows from the above discussion. Note that $\sigma_\mathfrak{n} = (\mathfrak{N}, H_R/K)$, which is well-defined. □

Denote by $|\mathfrak{f}| = |A/\mathfrak{f}|$, then we define the *CM height* of ϕ by

$$H_{\mathrm{CM}}(\phi) := q^g |\mathfrak{f}|, \tag{2.2}$$

where g is the genus of K. Note that this definition differs from [3, Def. 3.7] by a power of 1/2. If $x = (x_1, \ldots, x_n) \in \mathcal{M}_A^2(\mathcal{K}_1)(\mathbb{C}_\infty) \times \cdots \times \mathcal{M}_A^2(\mathcal{K}_n)(\mathbb{C}_\infty)$, then we define

$$H_{\mathrm{CM}}(x) := \max\big(H_{\mathrm{CM}}(x_1), \ldots, H_{\mathrm{CM}}(x_n)\big).$$

The following result shows that H_{CM} is a counting function on the set of CM points in $\mathcal{M}_A^2(\mathbb{C}_\infty)$, which justifies the terminology.

Proposition 2.3 *For every $B > 0$, the set*

$$\{\phi \in \mathcal{M}_A^2(\mathbb{C}_\infty) \mid \phi \text{ is CM and } H_{\mathrm{CM}}(\phi) < B\}$$

is finite.

Proof. For a given $g \geq 0$ there are only finitely many global function fields K with genus g and field of constants contained in $\mathbb{F}_{q^2}$. For each such field K, there are only finitely many orders of the form $R = A + \mathfrak{f}\mathcal{O}_K$ with bounded $\mathfrak{f}$. And for each such R, there are only $|\mathrm{Pic}(R)|$ Drinfeld modules ϕ with $\mathrm{End}(\phi) \cong R$. $\square$

We will need the following class-number estimate

Proposition 2.4 *Let ϕ be a Drinfeld module with complex multiplication by R. Then for every $\varepsilon > 0$ there exists a computable constant $C_\varepsilon > 0$ such that*

$$|\mathrm{Pic}(R)| > C_\varepsilon H_{\mathrm{CM}}(\phi)^{1-\varepsilon}.$$

Proof. Let K be the quotient field of R, and g its genus. Firstly, we have [3, Prop. 3.1]

$$|\mathrm{Pic}(\mathcal{O}_K)| = h(K) \geq \frac{(q-1)(q^{2g} - 2gq^g + 1)}{2g(q^{g+1} - 1)}.$$

Secondly, the exact sequence [13, §I.12]

$$1 \to \mathcal{O}_K^\times / R^\times \longrightarrow (\mathcal{O}_K/\mathfrak{f}\mathcal{O}_K)^\times/(R/\mathfrak{f}R)^\times \longrightarrow \mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(\mathcal{O}_K) \to 1$$

leads, as in the classical case, to the pleasing expression

$$|\mathrm{Pic}(R)| = \frac{|\mathrm{Pic}(\mathcal{O}_K)|}{[\mathcal{O}_K^\times : R^\times]} \cdot |\mathfrak{f}| \prod_{\mathfrak{p}|\mathfrak{f}} \left(1 - \chi(\mathfrak{p})|\mathfrak{p}|^{-1}\right), \tag{2.5}$$

where

$$\chi(\mathfrak{p}) = \begin{cases} 1 & \text{if } \mathfrak{p} \text{ splits in } K/k \\ -1 & \text{if } \mathfrak{p} \text{ is inert in } K/k \\ 0 & \text{if } \mathfrak{p} \text{ is ramified in } K/k. \end{cases}$$

Here one uses the fact that $R/\mathfrak{f}R \cong A/\mathfrak{f}$. The estimate now follows easily. $\square$

2.2 Drinfeld modular curves

Let $\mathcal{K} \subset \mathrm{GL}_2(\hat{A})$ be a subgroup of finite index, and recall that $\mathcal{M}_A^2(\mathcal{K})$ denotes the coarse moduli scheme parametrizing Drinfeld modules with level-$\mathcal{K}$ structure. Identifying Drinfeld modules over $\mathbb{C}_\infty$ with their associated rank 2 lattices in $\mathbb{C}_\infty$, and parametrizing the space of such lattices adèlically, one arrives at the following analytic parametrization (see e.g. [7]).

$$\begin{aligned}\mathcal{M}_A^2(\mathcal{K})(\mathbb{C}_\infty)^{\mathrm{an}} &\cong \mathrm{GL}_2(k)\backslash\Omega \times \mathrm{GL}_2(\mathbb{A}_f)/\mathcal{K} \\ &\cong \coprod_{s\in S} \Gamma_s\backslash\Omega, \end{aligned} \tag{2.6}$$

where $S \subset \mathrm{GL}_2(\mathbb{A}_f)$ denotes a (finite) set of representatives for $\mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A}_f)/\mathcal{K}$, and $\Gamma_s = s\mathcal{K}s^{-1} \cap \mathrm{GL}_2(k)$.

The determinant map induces an isomorphism

$$\det : \mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A}_f)/\mathcal{K} \xrightarrow{\sim} k^\times\backslash\mathbb{A}_f^\times/\det(\mathcal{K}) \quad (\cong \mathrm{Pic}(A) \text{ if } \det(\mathcal{K}) = \hat{A}^\times).$$

We next describe a scheme-theoretic version of the determinant map. Let $f \in A$, and denote by $\mathcal{K}(f) \subset \mathrm{GL}_2(\hat{A})$ the kernel of reduction mod f. The Weil pairing for Drinfeld modules [12, Chap. 5] gives us a map of $A[1/f]$-schemes

$$\mathcal{M}_A^2(\mathcal{K}(f)) \xrightarrow{w_f} \mathcal{M}_A^1(\det(\mathcal{K}(f))) = \mathrm{Spec}(\mathcal{O}_{H_f}), \tag{2.7}$$

where $\mathcal{O}_{H_f}$ denotes the ring of integers of the class field H_f of k corresponding to the class group $k^\times\backslash\mathbb{A}_f^\times/\det(\mathcal{K}(f)) = k^\times\backslash\mathbb{A}_f^\times/\{x \in \hat{A}^\times \mid x \equiv 1 \bmod f\}$. Under this map, the $\mathbb{A}_f^\times\mathrm{GL}_2(\hat{A})$-action on $\mathcal{M}_A^2(\mathcal{K}(f))$ corresponds to its determinant $\mathbb{A}_f^\times$-action on $\mathrm{Spec}(\mathcal{O}_{H_f})$, which in turn corresponds to the $\mathrm{Gal}(k^{\mathrm{ab}}/k)$-action on both sides given by the reciprocity map. It follows that $\mathcal{M}_A^2(\mathcal{K}(f)) \times k$ is defined over k, and all its $\bar{k}$-irreducible components are defined over H_f.

Suppose now that $\mathcal{K}$ contains $\mathcal{K}(f)$ for some $f \in A$, so $\mathcal{K}$ is a *congruence subgroup*. Then we may divide out by the action of $\mathcal{K}$ and $\det(\mathcal{K})$ in the left and right hand sides of (2.7), respectively. As $\mathcal{M}_A^2(\mathcal{K}) = \mathcal{M}_A^2(\mathcal{K}(f))/\mathcal{K}$, we thus get a map of $A[1/f]$-schemes

$$\mathcal{M}_A^2(\mathcal{K}) \xrightarrow{w_\mathcal{K}} \mathcal{M}_A^1(\det(\mathcal{K})) = \mathrm{Spec}(\mathcal{O}_{H_\mathcal{K}}), \tag{2.8}$$

where this time $H_\mathcal{K}$ is the class field corresponding to $k^\times\backslash\mathbb{A}_f^\times/\det(\mathcal{K})$. As before, it follows that $\mathcal{M}_A^2(\mathcal{K}) \times k$ is defined over k, and its $\bar{k}$-irreducible components are defined over $H_\mathcal{K}$.

We point out that, analytically, the maps w_f and $w_\mathcal{K}$ above send the rank 2 lattice L to the rank 1 lattice $\wedge^2 L$.

In the case $\mathcal{K} = \mathrm{GL}_2(\hat{A})$, recall that we use the notation $\mathcal{M}_A^2 := \mathcal{M}_A^2(\mathrm{GL}_2(\hat{A}))$. Let H denote the Hilbert class field of k, corresponding to the class group $k^\times\backslash\mathbb{A}_f^\times/\hat{A}^\times \cong \mathrm{Pic}(A)$. It is the maximal unramified abelian extension of k in which ∞ splits completely. Then we see that the irreducible components of $\mathcal{M}_{A,\mathbb{C}_\infty}^2$ are defined over H. We also see that its set of irreducible components corresponds to $\mathrm{Pic}(A)$ in such a way that the component corresponding to $[\mathfrak{a}] \in \mathrm{Pic}(A)$ parametrizes lattices isomorphic to $A \oplus \mathfrak{a}$ as projective A-modules.

We may choose our representatives S such that $1 \in S$, in which case $\Gamma_1 = \mathrm{GL}_2(A)$, and one of the irreducible components of $\mathcal{M}^2_{A,\mathbb{C}_\infty}$ thus corresponds to $\mathrm{GL}_2(A)\backslash\Omega$ via (2.6). We call it the *identity component*, and denote it by $\mathbb{M}$. It is the simplest modular curve. If $A = \mathbb{F}_q[T]$ then in fact $\mathbb{M} \cong \mathbb{A}^1$ is the affine line. In general, we have seen that $\mathbb{M}$ is a smooth irreducible affine curve defined over H.

2.3 The curves $Y(\mathfrak{n})$, $Y_0(\mathfrak{n})$ and $Y_2(\mathfrak{n})$

We denote by Z the center of the algebraic group GL_2. Let $\mathfrak{n} \subset A$ be a non-zero ideal, and consider the following open subgroups of $\mathrm{GL}_2(\hat{A})$.

$$\begin{aligned}
\mathcal{K}(\mathfrak{n}) &= \ker\big(\mathrm{GL}_2(\hat{A}) \to \mathrm{GL}_2(\hat{A}/\mathfrak{n})\big),\\
\mathcal{K}_0(\mathfrak{n}) &= \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\hat{A}) \mid c \in \mathfrak{n}\},\\
\mathcal{K}_2(\mathfrak{n}) &= \{\gamma \in \mathrm{GL}_2(\hat{A}) \mid (\gamma \bmod \mathfrak{n}) \in Z(\hat{A}/\mathfrak{n})\}.
\end{aligned}$$

These lead to coarse moduli schemes $\mathcal{M}^2_A(\mathcal{K}(\mathfrak{n}))$ (also denoted $\mathcal{M}^r_A(\mathfrak{n})$ in the literature), $\mathcal{M}^2_A(\mathcal{K}_0(\mathfrak{n}))$ and $\mathcal{M}^2_A(\mathcal{K}_2(\mathfrak{n}))$. Similarly to the previous section, we define the Drinfeld modular curves $Y(\mathfrak{n})$, $Y_0(\mathfrak{n})$ and $Y_2(\mathfrak{n})$ as the identity components of the respective moduli schemes tensored with $\mathbb{C}_\infty$.

We have isomorphisms of rigid analytic varieties

$$Y(\mathfrak{n})(\mathbb{C}_\infty)^{\mathrm{an}} \cong \Gamma(\mathfrak{n})\backslash\Omega, \quad Y_0(\mathfrak{n})(\mathbb{C}_\infty)^{\mathrm{an}} \cong \Gamma_0(\mathfrak{n})\backslash\Omega, \quad Y_2(\mathfrak{n})(\mathbb{C}_\infty)^{\mathrm{an}} \cong \Gamma_2(\mathfrak{n})\backslash\Omega,$$

for the arithmetic groups

$$\begin{aligned}
\Gamma(\mathfrak{n}) &= \{\gamma \in \mathrm{GL}_2(A) \mid \gamma \equiv 1 \bmod \mathfrak{n}\},\\
\Gamma_0(\mathfrak{n}) &= \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(A) \mid c \in \mathfrak{n}\},\\
\Gamma_2(\mathfrak{n}) &= \{\gamma \in \mathrm{GL}_2(A) \mid (\gamma \bmod \mathfrak{n}) \in Z(A/\mathfrak{n})\}.
\end{aligned}$$

As in the classical case, $Y_0(\mathfrak{n})$ parametrizes pairs of Drinfeld modules linked by cyclic $\mathfrak{n}$-isogenies.

As $\mathcal{K}(\mathfrak{n})$, $\mathcal{K}_0(\mathfrak{n})$ and $\mathcal{K}_2(\mathfrak{n})$ are all congruence subgroups of $\mathrm{GL}_2(\hat{A})$ (pick any $0 \neq f \in \mathfrak{n}$), it follows from the Weil-pairing (2.8) that $Y_0(\mathfrak{n})$ is defined over H, whereas the curves $Y(\mathfrak{n})$ and $Y_2(\mathfrak{n})$ are defined over the class fields corresponding to

$$k^\times\backslash\mathbb{A}_f^\times/\{x \in \hat{A}^\times \mid x \equiv 1 \bmod \mathfrak{n}\}, \quad \text{and}$$

$$k^\times\backslash\mathbb{A}_f^\times/\{x \in \hat{A}^\times \mid x \text{ is a square mod } \mathfrak{n}\},$$

respectively.

For a ring $R \supset \mathbb{F}_q$, and algebraic group G, we define

$$G^1(R) := \{g \in G(R) \mid \det(g) \in \mathbb{F}_q^\times\}.$$

The *degree* of an ideal $\mathfrak{n} \subset A$ is $\deg(\mathfrak{n}) := \log_q |\mathfrak{n}| = \log_q |A/\mathfrak{n}|$.

Proposition 2.9 *We have covers* $Y(\mathfrak{n}) \to Y_2(\mathfrak{n}) \to Y_0(\mathfrak{n}) \to \mathbb{M}$. *Moreover,* $Y(\mathfrak{n})$ *and* $Y_2(\mathfrak{n})$ *are Galois covers of* $\mathbb{M}$ *with Galois groups*

$$\begin{aligned} \mathrm{Gal}(Y(\mathfrak{n})/\mathbb{M}) &\cong \mathrm{GL}_2^1(A/\mathfrak{n})/Z(\mathbb{F}_q) \\ \mathrm{Gal}(Y_2(\mathfrak{n})/\mathbb{M}) &\cong \mathrm{GL}_2^1(A/\mathfrak{n})/Z^1(A/\mathfrak{n}) \cong \mathrm{PGL}_2^1(A/\mathfrak{n}). \end{aligned}$$

Note that $\mathrm{PGL}_2^1(A/\mathfrak{n}) \cong \mathrm{PSL}_2(A/\mathfrak{n})$ *if every prime factor of* $\mathfrak{n}$ *has even degree.*

Proof. Clearly, $\Gamma(\mathfrak{n}) \subset \Gamma_2(\mathfrak{n}) \subset \Gamma_0(\mathfrak{n})$, whence the coverings. Furthermore, $\Gamma(\mathfrak{n})$ and $\Gamma_2(\mathfrak{n})$ are normal subgroups of $\mathrm{GL}_2(A)$, so the respective coverings are Galois with Galois groups

$$\mathrm{Gal}(Y(\mathfrak{n})/\mathbb{M}) \cong \mathrm{GL}_2(A)/\Gamma(\mathfrak{n}) \cong \mathrm{GL}_2^1(A/\mathfrak{n})/Z(\mathbb{F}_q), \quad \text{and}$$

$$\mathrm{Gal}(Y_2(\mathfrak{n})/\mathbb{M}) \cong \mathrm{GL}_2(A)/\Gamma_2(\mathfrak{n}) \cong \mathrm{GL}_2^1(A/\mathfrak{n})/Z^1(A/\mathfrak{n}) \cong \mathrm{PGL}_2^1(A/\mathfrak{n}).$$

Lastly, if every prime factor of $\mathfrak{n}$ has even degree, then every element of $\mathbb{F}_q$ is a square in $A/\mathfrak{n}$ and $\mathrm{PGL}_2^1(A/\mathfrak{n}) = \mathrm{PSL}_2(A/\mathfrak{n})$. $\square$

2.4 Special subcurves of $\big(\mathcal{M}^2_{A,\mathbb{C}_\infty}\big)^n$

In the Introduction we defined the notion of (pure) special subcurves of a product of n Drinfeld modular curves, in particular in $\mathbb{M}^n$.

For a subset $I \subset \{1, \dots, m\}$ we denote by $p_I : \mathbb{M}^n \to \mathbb{M}^I$ the projection onto the coordinates listed in I. We will also write $p_{\{i\}} = p_i$ and $p_{\{i,j\}} = p_{i,j}$.

The curve $Y_0(\mathfrak{n})$ parametrizes Drinfeld modules linked by a cyclic isogeny of degree $\mathfrak{n}$. If (ϕ, ϕ') is such a pair, with $\phi \in \mathbb{M}(\mathbb{C}_\infty)$, then $\phi' \in \mathbb{M}(\mathbb{C}_\infty)$ if and only if $\mathfrak{n}$ is a principal ideal. Thus, for $\mathfrak{n} = \langle N \rangle$ principal, we obtain a map $Y_0(\mathfrak{n}) \to \mathbb{M}^2$ whose image is denoted by $Y_0'(N)$. Analytically, this map is given by

$$\begin{aligned} \Gamma_0(\mathfrak{n})\backslash\Omega &\longrightarrow \big(\mathrm{GL}_2(A)\backslash\Omega\big) \times \big(\mathrm{GL}_2(A)\backslash\Omega\big) \\ [\omega] &\longmapsto \big([\omega], [N\omega]\big). \end{aligned} \tag{2.10}$$

In particular, we see that $Y_0'(N)$ is an irreducible pure special subcurve of $\mathbb{M}^2$. Moreover, every irreducible pure special subcurve of $\mathbb{M}^2$ is of the form $Y_0'(N)$ for some $N \in A$. Indeed, for any $g \in \mathrm{GL}_2(k)$, we denote by $N = \deg(g)$ the determinant of $a \cdot g$, with $a \in A$ chosen in such a way that the entries of $a \cdot g$ are in A and have no common factor. This N is defined up to the square of a unit, and is called the *degree* of g. Now, let $g_1, g_2 \in \mathrm{GL}_2(k)$ and $N := \deg(g_2 g_1^{-1})$. Then the special curve corresponding to the map $\omega \mapsto \big([g_1(\omega)], [g_2(\omega)]\big)$ is just $Y_0'(N) \subset \mathbb{M}^2$.

Notice that an irreducible curve $Y \subset \mathbb{M}^n$ is a pure special subcurve if and only if $p_{i,j}(Y) = Y_0'(N_{ij})$ for some $N_{ij} \in A$ for all $i \neq j$.

Next, it will be convenient to study the space $\mathcal{SC}_n$ of all special subcurves in $\big(\mathcal{M}^2_{A,\mathbb{C}_\infty}\big)^n$. Analytically, this will turn out to be the following double coset space

$$\mathcal{SC}_n(\mathbb{C}_\infty)^{\mathrm{an}} := \mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A}_f)^n \times \Omega/(Z(\mathbb{A}_f)\mathrm{GL}_2(\hat{A}))^n, \tag{2.11}$$

where $(Z(\mathbb{A}_f)\mathrm{GL}_2(\hat{A}))^n$ acts from the right on $\mathrm{GL}_2(\mathbb{A}_f)^n$ in the usual way, and trivially on Ω, while $\mathrm{GL}_2(k)$ acts from the left on Ω in the usual way, and diagonally on $\mathrm{GL}_2(\mathbb{A}_f)^n$.

Choose a set of representatives $T \subset \mathrm{GL}_2(\mathbb{A}_f)^n$ for the double quotient

$$\mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A}_f)^n/(Z(\mathbb{A}_f)\mathrm{GL}_2(\hat{A}))^n. \tag{2.12}$$

Note that T is infinite, and we again choose $1 \in T$. For each $t = (t_1, \dots, t_n) \in T$ let $\Gamma_t = \cap_{i=1}^n t_i \mathrm{GL}_2(\hat{A}) t_i^{-1} \cap \mathrm{GL}_2(k)$. Then we have a canonical bijection

$$\begin{array}{rcl} \mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A}_f)^n \times \Omega/(Z(\mathbb{A}_f)\mathrm{GL}_2(\hat{A}))^n & \stackrel{\sim}{\longrightarrow} & \displaystyle\coprod_{t\in T} \Gamma_t\backslash\Omega \\ {[h,\omega]} & \longmapsto & [g^{-1}(\omega)]_t, \end{array}$$

where $t \in T$ is such that $[t] = [h]$ in (2.12), and $g \in \mathrm{GL}_2(k)$ is such that $h = gta$, for some $a \in (Z(\mathbb{A}_f)\mathrm{GL}_2(\hat{A}))^n$. We see that $\mathcal{SC}_n$ is the disjoint union of an infinite family of Drinfeld modular curves, parametrized by T.

Next, we describe a map from $\mathcal{SC}_n(\mathbb{C}_\infty)$ into $\big(\mathcal{M}_A^2(\mathbb{C}_\infty)\big)^n$.

$$\begin{array}{rcl} \mathcal{SC}_n(\mathbb{C}_\infty) & \stackrel{\theta}{\longrightarrow} & \big(\mathcal{M}_A^2(\mathbb{C}_\infty)\big)^n \\ \mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A}_f)^n \times \Omega/(Z(\mathbb{A}_f)\mathrm{GL}_2(\hat{A}))^n & \longrightarrow & \mathrm{GL}_2(k)^n\backslash\mathrm{GL}_2(\mathbb{A}_f)^n \times \Omega^n/(Z(\mathbb{A}_f)\mathrm{GL}_2(\hat{A}))^n \\ & & = \big(\mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A}_f) \times \Omega/\mathrm{GL}_2(\hat{A})\big)^n \\ {[(h_1,\dots,h_n),\omega]} & \longmapsto & \big([h_1,\omega],\dots,[h_n,\omega]\big); \end{array}$$

$$\begin{array}{rcl} \displaystyle\coprod_{t\in T} \Gamma_t\backslash\Omega & \longrightarrow & \displaystyle\coprod_{s\in S^n} \big(\Gamma_{s_i}\backslash\Omega\big)_{i=1}^n \\ {[\omega]_t} & \longmapsto & \big([g_i^{-1}(\omega)]_{s_i}\big)_{i=1}^n, \end{array} \tag{2.13}$$

where $s_i \in S$ such that $[s_i] = [t_i]$ in $\mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A}_f)/\mathrm{GL}_2(\hat{A})$, and $g_i \in \mathrm{GL}_2(k)$ such that $g_i s_i a_i = t_i$ for some $a_i \in \mathrm{GL}_2(\hat{A})$, for $i = 1, \dots, n$.

We note that $\mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A}_f)/(Z(\mathbb{A}_f)\mathrm{GL}_2(\hat{A})) = \mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A}_f)/\mathrm{GL}_2(\hat{A})$, as for any $x \in \mathbb{A}_f$ we have $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right) \equiv \prod_{\mathfrak{p}} \left(\begin{smallmatrix} \pi_{\mathfrak{p}}^{n_{\mathfrak{p}}} & 0 \\ 0 & \pi_{\mathfrak{p}}^{n_{\mathfrak{p}}} \end{smallmatrix}\right) \bmod \mathrm{GL}_2(\hat{A})$, where $\pi_{\mathfrak{p}} \in k$ denotes a chosen uniformizer at $\mathfrak{p}$, and $n_{\mathfrak{p}} = \min(0, v_{\mathfrak{p}}(x))$ is zero for almost all $\mathfrak{p}$. This latter scalar is in $\mathrm{GL}_2(k)$, and, as scalars commute in $\mathrm{GL}_2(\mathbb{A}_f)$, is killed by the left $\mathrm{GL}_2(k)$ action. The same principle does not hold when we have $n > 1$ copies of $\mathrm{GL}_2(\mathbb{A}_f)$ and $\mathrm{GL}_2(\hat{A})$ but only one copy of $\mathrm{GL}_2(k)$, which is why the $Z(\mathbb{A}_f)$ appears explicitly in the definition of $\mathcal{SC}_n(\mathbb{C}_\infty)^{\mathrm{an}}$.

Let

$$T^0 := \ker\big(T \to \mathrm{GL}_2(k)^n\backslash\mathrm{GL}_2(\mathbb{A}_f)^n/\mathrm{GL}_2(\hat{A})^n\big).$$

Then for every $t \in T^0$, the Drinfeld modular curve $\Gamma_t\backslash\Omega$ is mapped into $\mathbb{M}^n(\mathbb{C}_\infty)$ by θ, where its image is a pure special curve, as defined in the Introduction.

Conversely, all pure special curves in $\mathbb{M}^n$ arise in this way:

Proposition 2.14 *We have bijections*

$$\begin{array}{ccccc} T^0 & \longleftrightarrow & (Z(k)\mathrm{GL}_2(A))^n\backslash \mathrm{GL}_2(k)^n/\mathrm{GL}_2(k) & \longleftrightarrow & \{\textit{Isomorphism classes of pure} \\ & & & & \textit{special curves in } \mathbb{M}^n\} \\ t & \longmapsto & [g]=[(g_1,\ldots,g_n)] & \longmapsto & \textit{Pure special curve defined by} \\ & & & & \omega\mapsto[(g_1(\omega),\ldots,g_n(\omega))], \end{array}$$

with $g\in\mathrm{GL}_2(k)^n$ *such that* $g=at^{-1}$ *for some* $a\in(Z(\mathbb{A}_f)\mathrm{GL}_2(\hat{A}))^n$.

Proof. In the first bijection, let $t\in T^0$, then by definition there exist $g\in\mathrm{GL}_2(k)^n$ and $a^{-1}\in(Z(\mathbb{A}_f)\mathrm{GL}_2(\hat{A}))^n$ such that $gta^{-1}=1$. Then $[g]=[at^{-1}]$ is well-defined, since if also $g'=a't^{-1}$, then $g'=a'a^{-1}g$ with $a'a^{-1}\in(Z(\mathbb{A}_f)\mathrm{GL}_2(\hat{A}))^n\cap\mathrm{GL}_2(k)^n=(Z(k)\mathrm{GL}_2(A))^n$. Moreover, given $g\in\mathrm{GL}_2(k)^n$, there exist $h\in\mathrm{GL}_2(k)$, $t\in T^0$ and $a\in(Z(\mathbb{A}_f)\mathrm{GL}_2(\hat{A}))^n$ such that $g^{-1}=hta$, hence the map is surjective. If also $g^{-1}=h't'a'$, then it is clear that $t=t'$, by definition of T.

The second bijection follows as $(g_1,\ldots,g_n)$ and $(g_1',\ldots,g_n')$ in $\mathrm{GL}_2(k)^n$ define the same pure special curve in $\mathbb{M}^n$ if and only if $g_i'=\gamma_i g_i\sigma$ for all $i=1,\ldots,n$, for some $(\gamma_1,\ldots,\gamma_n)\in(Z(k)\mathrm{GL}_2(A))^n$ and $\sigma\in\mathrm{GL}_2(k)$. $\square$

This proposition justifies our definition of $\mathcal{SC}_n$ as the space of all pure special subcurves of $(\mathcal{M}^2_{A,\mathbb{C}_\infty})^n$. We also denote by $\mathcal{SC}_n^0\subset\mathcal{SC}_n$ the subfamily of those Drinfeld modular curves corresponding to T^0, i.e. $\mathcal{SC}_n^0(\mathbb{C}_\infty)^{\mathrm{an}}\cong\coprod_{t\in T^0}\Gamma_t\backslash\Omega$, which is the space of all pure special subcurves of $\mathbb{M}^n$.

Lastly, we point out that special subvarieties are simply products of CM points and pure special subcurves of subproducts.

2.5 Pure special curves and trees

Let $\mathfrak{p}\subset A$ be a prime. Recall (e.g. [14]) that the *Brûhat-Tits tree* $\mathcal{T}_\mathfrak{p}$ of $\mathrm{GL}_2(k_\mathfrak{p})$ is the $(|\mathfrak{p}|+1)$-regular tree whose vertices represent homothety classes of $A_\mathfrak{p}$-lattices in $k_\mathfrak{p}^2$, and two vertices representing classes Λ and Λ' are joined by an (unoriented) edge if there exist representative lattices $L\in\Lambda$ and $L'\in\Lambda'$ such that $L'\subset L$ and $L/L'\cong A/\mathfrak{p}$. The group $\mathrm{GL}_2(k_\mathfrak{p})$ acts transitively on $\mathcal{T}_\mathfrak{p}$ and the stabilizer of the "origin" $v_{o,\mathfrak{p}}$, which is the vertex corresponding to the lattice $A_\mathfrak{p}^2$, is $Z(k_\mathfrak{p})\mathrm{GL}_2(A_\mathfrak{p})$. Thus we get a bijection $\mathcal{T}_\mathfrak{p}\leftrightarrow\mathrm{GL}_2(k_\mathfrak{p})/Z(k_\mathfrak{p})\mathrm{GL}_2(A_\mathfrak{p})$.

Hence we have bijections

$$T\longleftrightarrow\mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A}_f)^n/(Z(\mathbb{A}_f)\mathrm{GL}_2(\hat{A}))^n\longleftrightarrow\mathrm{GL}_2(k)\backslash{\prod_{\mathfrak{p}}}'\mathcal{T}_\mathfrak{p}^n. \tag{2.15}$$

Here the restricted product $\prod_\mathfrak{p}'$ denotes the subset of those families $(v_{1,\mathfrak{p}},\ldots,v_{n,\mathfrak{p}})_\mathfrak{p}$ of n-tuples of vertices $v_{i,\mathfrak{p}}\in\mathcal{T}_\mathfrak{p}$ such that $v_{1,\mathfrak{p}}=\cdots=v_{n,\mathfrak{p}}=v_{o,\mathfrak{p}}$ for almost all primes $\mathfrak{p}$.

We also have the following non-canonical bijection

$$\mathrm{GL}_2(k)\backslash{\prod_{\mathfrak{p}}}'\mathcal{T}_{\mathfrak{p}}^n \longleftrightarrow S \times {\prod_{\mathfrak{p}}}'\mathrm{GL}_2(k)\backslash\mathcal{T}_{\mathfrak{p}}^n \quad \Big(\leftrightarrow \mathrm{Pic}(A) \times {\prod_{\mathfrak{p}}}'\mathrm{GL}_2(k_{\mathfrak{p}})\backslash\mathcal{T}_{\mathfrak{p}}^n \Big) \tag{2.16}$$

which arises from

$$\begin{gathered}\mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A}_f)^n/\big(Z(\mathbb{A}_f)\mathrm{GL}_2(\hat{A})\big)^n \xrightarrow{\sim} \\ \mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A}_f)/Z(\mathbb{A}_f)\mathrm{GL}_2(\hat{A}) \quad \times \quad \mathrm{GL}_2(\mathbb{A}_f)\backslash\mathrm{GL}_2(\mathbb{A}_f)^n/\big(Z(\mathbb{A}_f)\mathrm{GL}_2(\hat{A})\big)^n; \\ [g_1, \ldots, g_n] \longmapsto \big([g_1], [1, g_1^{-1}g_2, \ldots, g_1^{-1}g_n]\big),\end{gathered}$$

which is readily verified. This bijection may also be of help to the reader puzzling over [3, §1.3].

The moduli interpretation of (2.15) and (2.16) is the following.

We may view $\mathcal{T}_{\mathfrak{p}}$ as the tree of $\mathfrak{p}$-isogenies of Drinfeld modules as in the elliptic curve case, but with the added subtlety that different vertices will map to different irreducible components of $\big(\mathcal{M}^2_{A,\mathbb{C}_\infty}\big)^n$ when $\mathfrak{p} \subset A$ is not principal. Now let $t \in T$ correspond to the pure special subcurve $C_t \overset{\theta}{\hookrightarrow} \big(\mathcal{M}^2_{A,\mathbb{C}_\infty}\big)^n$. Let $(s, \underline{v}) \in S \times \prod_{\mathfrak{p}}'\mathrm{GL}_2(k)\backslash\mathcal{T}_{\mathfrak{p}}^n$ correspond to t via (2.16). We may choose a representative family $(v_{1,\mathfrak{p}}, \ldots, v_{n,\mathfrak{p}})_{\mathfrak{p}}$ of the class $\underline{v}$ such that $v_{1,\mathfrak{p}} = v_{o,\mathfrak{p}}$ for all $\mathfrak{p}$. Then a typical point $x = (x_1, \ldots, x_n) \in C_t(\mathbb{C}_\infty)$ satisfies: x_1 lies on the s-component of $\mathcal{M}^2_{A,\mathbb{C}_\infty}$, and the isogenies $x_1 \to x_2 \to \cdots \to x_n$ correspond to $(v_{2,\mathfrak{p}}, \ldots, v_{n,\mathfrak{p}})_{\mathfrak{p}}$, respectively.

In the case $n = 3$ we have a particularly pleasing combinatorial description of $\mathrm{GL}_2(k)\backslash\mathcal{T}_{\mathfrak{p}}^3$. Any triple of vertices $(v_1, v_2, v_3) \in \mathcal{T}_{\mathfrak{p}}^3$ has a well-defined *center* v_c, defined by the property that the three paths (possibly of length zero) from v_c to the v_i are disjoint. Denote by $(n_1, n_2, n_3) \in \mathbb{N}_0^3$ the lengths of these paths. Furthermore, $\mathrm{GL}_2(k_{\mathfrak{p}})$ acts 3-transitively on the set of ends of $\mathcal{T}_{\mathfrak{p}}$, and it follows that a triple of vertices (v_1, v_2, v_3) with corresponding center v_c and triple of path-lengths (n_1, n_2, n_3) is mapped by $\mathrm{GL}_2(k)$ to another triple (v_1', v_2', v_3') with center v_c' and path-lengths (n_1', n_2', n_3') if and only if $(n_1, n_2, n_3) = (n_1', n_2', n_3')$. Thus we have bijections

$$\mathrm{GL}_2(k)\backslash\mathcal{T}_{\mathfrak{p}}^3 \longleftrightarrow \mathbb{N}_0^3; \quad {\prod_{\mathfrak{p}}}'\mathrm{GL}_2(k)\backslash\mathcal{T}_{\mathfrak{p}}^3 \longleftrightarrow \mathcal{I}_A^3; \quad T \longleftrightarrow \mathrm{Pic}(A) \times \mathcal{I}_A^3, \tag{2.17}$$

where $\mathcal{I}_A$ denotes the semigroup of non-zero A-ideals. The second bijection is given by

$$(v_{1,\mathfrak{p}}, v_{2,\mathfrak{p}}, v_{3,\mathfrak{p}})_{\mathfrak{p}} \mapsto (n_{1,\mathfrak{p}}, n_{2,\mathfrak{p}}, n_{3,\mathfrak{p}})_{\mathfrak{p}} \mapsto (\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3) = \prod_{\mathfrak{p}} \big(\mathfrak{p}^{n_{1,\mathfrak{p}}}, \mathfrak{p}^{n_{2,\mathfrak{p}}}, \mathfrak{p}^{n_{3,\mathfrak{p}}}\big).$$

Let Y be the special curve corresponding to the data $\big([\mathfrak{a}], (\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3)\big) \in \mathrm{Pic}(A) \times \mathcal{I}_A^3$. Then Y maps into $\mathbb{M}^3$ if and only if $[\mathfrak{a}] = 1$ and $\mathfrak{n}_i = \langle N_i \rangle$ is principal for $i = 1, 2, 3$. In this case we have $p_{i,j}(Y) = Y_0'(N_iN_j) \subset \mathbb{M}^2$. In general, if a pure modular curve $Y \subset \big(\mathcal{M}^2_{A,\mathbb{C}_\infty}\big)^3$ corresponds to a triple $(\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3) \in \mathcal{I}_A^3$, then $p_{i,j}(Y)$ is isomorphic to a suitable irreducible component of $\mathcal{M}^2_A(\mathcal{K}_0(\mathfrak{n}_i\mathfrak{n}_j))_{\mathbb{C}_\infty}$.

The following result will be useful later on.

Proposition 2.18 *Let $Y \subset \mathbb{M}^3$ be a pure special curve corresponding to a triple $(\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3)$, and fix $(x_1, x_2) \in p_{1,2}(Y(\mathbb{C}_\infty))$. Then*

$$|p_{1,2}^{-1}(x_1, x_2) \cap Y(\mathbb{C}_\infty)| \geq \prod_{\mathfrak{p}|\mathfrak{n}_3} \frac{(|\mathfrak{p}|-1)(|\mathfrak{p}|+1)^{n_{3,\mathfrak{p}}-1}}{2n_{3,\mathfrak{p}}+1},$$

where $\mathfrak{n}_i = \prod_{\mathfrak{p}} \mathfrak{p}^{n_{i,\mathfrak{p}}}$.

Proof. Let $(v_{1,\mathfrak{p}}, v_{2,\mathfrak{p}}, v_{3,\mathfrak{p}})_{\mathfrak{p}} \in \prod'_{\mathfrak{p}} \mathcal{T}_{\mathfrak{p}}^3$ be a representative family of tuples corresponding to the special curve Y, and denote by $(v_{c,\mathfrak{p}})_{\mathfrak{p}}$ its family of centers. Fixing x_1 and x_2 amounts to fixing $v_{1,\mathfrak{p}}$, $v_{2,\mathfrak{p}}$ for all $\mathfrak{p}$. We now count the possible vertex families $(v_{3,\mathfrak{p}})_{\mathfrak{p}}$ for which $(v_{1,\mathfrak{p}}, v_{2,\mathfrak{p}}, v_{3,\mathfrak{p}})_{\mathfrak{p}}$ represents Y. As $(\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3)$ is prescribed, so is $v_{c,\mathfrak{p}}$ for every $\mathfrak{p}$, and from (2.17) follows that we must count the paths of length $n_{3,\mathfrak{p}}$ from $v_{c,\mathfrak{p}}$ which are disjoint from the two paths leading to $v_{1,\mathfrak{p}}$ and $v_{2,\mathfrak{p}}$, which gives at least $(|\mathfrak{p}|-1)(|\mathfrak{p}|+1)^{n_{3,\mathfrak{p}}-1}$ for each $\mathfrak{p}$. Thus the number of valid cyclic $\mathfrak{n}_3$-isogenies from x_c (corresponding to $(v_{c,\mathfrak{p}})_{\mathfrak{p}}$) to x_3 is given by $\prod_{\mathfrak{p}|\mathfrak{n}_3}(|\mathfrak{p}|-1)(|\mathfrak{p}|+1)^{n_{3,\mathfrak{p}}-1}$.

But if x_c has complex multiplication by an order in K, then distinct isogenies may well lead to the same x_3, they correspond to non-trivial endomorphisms $f \in \mathrm{End}(x_c)$ of norm $N_{K/k}(f) = \mathfrak{n}_3^2$. There are at most $\prod_{\mathfrak{p}}(2n_{3,\mathfrak{p}}+1)$ such endormorphisms, which completes the proof. $\square$

2.6 Main results and reduction to $\mathbb{M}^n$

Let $\mathcal{K}_i \subset \mathrm{GL}_2(\hat{A})$ be subgroups of finite index, for $i = 1, \ldots, n$. With a bit more effort, we could give a general treatment of pure special subcurves of $\prod_{i=1}^n \mathcal{M}_A^2(\mathcal{K}_i)_{\mathbb{C}_\infty}$ as in the previous two sections. But the following definition is much easier:

Consider the morphism

$$\pi : \prod_{i=1}^n \mathcal{M}_A^2(\mathcal{K}_i)_{\mathbb{C}_\infty} \longrightarrow \left(\mathcal{M}_{A,\mathbb{C}_\infty}^2\right)^n$$

induced by the inclusions $\mathcal{K}_i \hookrightarrow \mathrm{GL}_2(\hat{A})$. Then an irreducible subvariety $Y \subset \prod_{i=1}^n \mathcal{M}_A^2(\mathcal{K}_i)_{\mathbb{C}_\infty}$ is a *special subvariety* if and only if the image $\pi(Y)$ is a special subvariety of $\left(\mathcal{M}_{A,\mathbb{C}_\infty}^2\right)^n$. Similarly for (pure) special subcurves.

We restate our main result as follows

Theorem 1.2′ *Let $\mathcal{K}_i \subset \mathrm{GL}_2(\hat{A})$ be subgroups of finite index for $i = 1, \ldots, n$. Let $Y \subset \prod_{i=1}^n \mathcal{M}_A^2(\mathcal{K}_i)_{\mathbb{C}_\infty}$ be an irreducible subvariety. Then $Y(\mathbb{C}_\infty)$ contains a Zariski-dense subset of CM points if and only if Y is a special subvariety.*

As every irreducible component of $\mathcal{M}_A^2(\mathcal{K}_i)_{\mathbb{C}_\infty}$ is a Drinfeld modular curve corresponding to $\Gamma_s \backslash \Omega$, for the arithmetic subgroup $\Gamma_s = s\mathcal{K}_i s^{-1} \cap \mathrm{GL}_2(k)$ with some $s \in \mathrm{GL}_2(\mathbb{A}_f)$, we see that the above Theorem is equivalent to Theorem 1.2.

Next we want to show that it suffices to prove Theorem 1.2 for $X = \mathbb{M}^n$.

Proposition 2.19 *Let $\Gamma_i' \subset \Gamma_i \subset \mathrm{GL}_2(k)$ be arithmetic subgroups corresponding to Drinfeld modular curves X_i' and X_i, respectively, for $i = 1, \ldots, n$. Let $X = X_1 \times \cdots \times X_n$ and $X' = X_1' \times \cdots \times X_n'$. Then the canonical map $f : X' \to X$ preserves special subvarieties. In other words, a subvariety $Y \subset X'$ is special if and only if $f(Y)$ is special.*

Proof. This is immediate, as special subvarieties are defined purely in terms of isogeny relations between coordinates, and the property that any constant projections $Y \to X_i'$ have CM points as images. These properties are independent of any level structures. □

Now let $\Gamma_i \subset \mathrm{GL}_2(k)$ be an arithmetic subgroup such that $X_i(\mathbb{C}_\infty)^{\mathrm{an}} \cong \Gamma_i \backslash \Omega$, for each $i = 1, \ldots, n$. Then each $\Gamma_i' = \mathrm{GL}_2(A) \cap \Gamma_i$ is again an arithmetic subgroup, and we have maps

$$\Gamma_i' \backslash \Omega \longrightarrow \Gamma_i \backslash \Omega, \quad \Gamma_i' \backslash \Omega \longrightarrow \mathrm{GL}_2(A) \backslash \Omega$$

corresponding to morphisms of Drinfeld modular curves

$$f_i : X_i' \longrightarrow X_i, \quad g_i : X_i' \longrightarrow \mathbb{M},$$

for each $i = 1, \ldots, n$.

Applying Proposition 2.19 to the maps $\prod_i f_i$ and $\prod_i g_i$ shows that Theorem 1.2 holds for $X_1 \times \cdots \times X_n$ if and only if it holds for $\mathbb{M}^n$.

With this reduction step out of the way, the rest of our proof of Theorem 1.2 will follow [3] very closely, with $\mathbb{M}^n$ playing the role of $\mathbb{A}^n$. The main geometric aspect of $\mathbb{A}^n$ used in [3] is its structure as the product of affine lines, and it turns out that the product structure of $\mathbb{M}^n$ will suffice, with some caveats. Firstly, $\mathbb{M}$ is not defined over k but over H, the Hilbert class field of k. (Classically, the elliptic modular curve $Y(1)$ is defined over the Hilbert class field of $\mathbb{Q}$, which is just $\mathbb{Q}$ itself. In our case, $k \neq H$ in general, which illustrates the "metamathematical" phenomenon of splitting of the two different roles of $\mathbb{Q}$ into two different fields (k and H) in characteristic p. See [8, Preface] for a discussion).

Secondly, in general the isomorphism

$$\mathrm{GL}_2(A) \backslash \Omega \xrightarrow{\sim} \mathbb{M}(\mathbb{C}_\infty)^{\mathrm{an}}$$

is no longer induced by a well-behaved "j-invariant" $j : \Omega \to \mathbb{C}_\infty$, as is the case for $A = \mathbb{F}_q[T]$. Consequently, we cannot take advantage of the analytic properties of the j-function, and will say nothing about the Weil heights of CM points.

3 Hecke correspondences and CM points

3.1 The general formalism

We briefly describe Hecke correspondences on $\mathcal{M}_A^2(\mathcal{K})$ in general. Let $g \in \mathrm{GL}_2(\mathbb{A}_f)$, and set $\mathcal{K}_g = \mathcal{K} \cap g^{-1}\mathcal{K}g$. Then g acts from the left on $\mathcal{M}_A^2(\mathcal{K}_g)$ (by letting g^{-1} act from the *right* on Drinfeld modules with full level structures, see [7, II.3]). Combined with the standard projection $\pi : \mathcal{M}^2_A(\mathcal{K}_g) \to \mathcal{M}^2_A(\mathcal{K})$ this gives rise to the *Hecke correspondence* on $\mathcal{M}^2_A(\mathcal{K})$:

$$T_g : \mathcal{M}^2_A(\mathcal{K}) \xleftarrow{\pi} \mathcal{M}^2_A(\mathcal{K}_g) \xrightarrow{\pi\circ g} \mathcal{M}^2_A(\mathcal{K}). \tag{3.1}$$

This is an algebraic correspondence of finite degree $[\mathcal{K} : \mathcal{K}_g]$.

Analytically, this correspondence is described as follows. The group $\mathrm{GL}_2(\mathbb{A}_f)$ acts from the left on $\mathrm{GL}_2(\mathbb{A}_f) \times \Omega$ by $g \cdot (h, \omega) = (hg^{-1}, \omega)$. This gives rise to the correspondence

$$\begin{array}{ccc} \mathrm{GL}_2(\mathbb{A}_f) \times \Omega & \xrightarrow{\quad g \quad} & \mathrm{GL}_2(\mathbb{A}_f) \times \Omega \\ \downarrow & & \downarrow \\ \mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A}_f) \times \Omega/\mathcal{K} & & \mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A}_f) \times \Omega/\mathcal{K}, \end{array}$$

which is easily seen to factor through $\mathrm{GL}_2(k)\times\mathcal{K}_g$, thus describing the finite correspondence

$$\begin{tikzcd} & \mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A}_f) \times \Omega/\mathcal{K}_g \arrow[dl, "\pi"'] \arrow[dr, "\pi\circ g"] & \\ \mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A}_f) \times \Omega/\mathcal{K} & T_g \arrow[l, dotted] \arrow[r, dotted] & \mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A}_f) \times \Omega/\mathcal{K}. \end{tikzcd} \tag{3.2}$$

Explicitly, this correspondence acts as follows. Choose a finite set $\{g_i \in \mathrm{GL}_2(\mathbb{A}_f) \mid i \in I_g\}$ of representatives for the cosets $\mathcal{K}\backslash\mathcal{K}g\mathcal{K}$. Let $[h, \omega] \in \mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A}_f) \times \Omega/\mathcal{K} \cong \mathcal{M}^2_A(\mathcal{K})(\mathbb{C}_\infty)$. Then T_g maps the point $[h, \omega]$ to the set $\{[hg_i^{-1}, \omega] \mid i \in I_g\}$.

At first sight, the g-action does not seem to effect the Ω part of the above spaces, but we will show next how T_g acts on $\mathcal{M}^2_A(\mathcal{K})(\mathbb{C}_\infty)^{\mathrm{an}} \cong \coprod_{s\in S} \Gamma_s\backslash\Omega$. For simplicity, we assume that $\det(\mathcal{K}_g) = \det(\mathcal{K})$. That way, $\mathcal{M}^2_A(\mathcal{K}_g)_{\mathbb{C}_\infty}$ and $\mathcal{M}^2_A(\mathcal{K})_{\mathbb{C}_\infty}$ have the same number of irreducible components, and we may choose the same set $1 \in S \subset \mathrm{GL}_2(\mathbb{A}_f)$ of representatives for $\mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A}_f)/\mathcal{K}$ and for $\mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A}_f)/\mathcal{K}_g$. We may also assume that $h \in S$.

For each $i \in I_g$, there exist $f_{h,g,i} \in \mathrm{GL}_2(k)$ and $k_{h,g,i} \in \mathcal{K}$ such that $f_{h,g,i}hg_i^{-1}k_{h,g,i} = s_{h,g} \in S$. Now $[hg_i^{-1}, \omega] = [s_{h,g}, f_{h,g,i}(\omega)]$, and this element maps to $[f_{h,g,i}(\omega)] \in \Gamma_{s_{h,g}}\backslash\Omega$. Moreover, we can check that $s_{h,g}$ here only depends on h and on g, not on i, so the correspondence T_g maps the h-component $\Gamma_h\backslash\Omega$ to the $s_{h,g}$-component $\Gamma_{s_{h,g}}\backslash\Omega$. Within each component, T_g maps the point $[\omega]$ to the set $\{[f_{h,g,i}(\omega)] \mid i \in I_g\}$, and in fact the set $\{f_{h,g,i} \mid i \in I_g\}$ is a set of representatives for $\Gamma_{s_{h,g}}\backslash\Gamma_{s_{h,g}}f_{h,g}\Gamma_h$, where $f_{h,g} \in \mathrm{GL}_2(k)$ is such that $f_{h,g}hg^{-1}k_{h,g} = s_{h,g} \in S$ for some $k_{h,g} \in \mathcal{K}$. Hence, for each $h \in S$, the correspondence T_g on $\mathcal{M}^2_A(\mathcal{K})$ induces the following correspondence from the h-component to the $s_{h,g}$-component of $\mathcal{M}^2_A(\mathcal{K})(\mathbb{C})^{\mathrm{an}}$:s

$$\begin{array}{ccc} \Omega & \xrightarrow{f_{h,g}} & \Omega \\ \downarrow & & \downarrow \\ \Gamma_h\backslash\Omega & & \Gamma_{s_{h,g}}\backslash\Omega \end{array}$$

which factors through

$$\Gamma_h \cap f_{h,g}^{-1} \Gamma_{s_{h,g}} f_{h,g} = h\mathcal{K}_g h^{-1} \cap \mathrm{GL}_2(k) = \Gamma_h \cap \Gamma_{hg},$$

thus giving the correspondence

$$\begin{tikzcd}[column sep=small] & (\Gamma_h \cap \Gamma_{hg})\backslash\Omega \arrow[dl, "\pi"'] \arrow[dr, "\pi\circ f_{h,g}"] & \\ \Gamma_h\backslash\Omega & & \Gamma_{s_{h,g}}\backslash\Omega \end{tikzcd} \tag{3.3}$$

which has finite degree $[\Gamma_h : \Gamma_h \cap \Gamma_{hg}] = [\mathcal{K} : \mathcal{K}_g]$. Thus the correspondence (3.3) gives a component by component description of the correspondence T_g on $\mathcal{M}_A^2(\mathcal{K})_{\mathbb{C}_\infty}$ of (3.1).

Remark 3.4 Let $s_1, s_2 \in S$, and set $g = s_2^{-1}s_1$. Then $f_{s_1,g} = k_{s_1,g} = 1$, and so T_g maps the s_1-component of $\mathcal{M}_A^2(\mathcal{K})_{\mathbb{C}_\infty}$ to the s_2-component, while leaving the ω unchanged in that component.

We also notice that $s_{h,g} = h$ if and only if $g \in \mathcal{K}h^{-1}\mathrm{GL}_2(k)h$.

3.2 Hecke correspondences on $\mathbb{M}^n$

In the remainder of this section, we closely follow [3, §2]. From now on, we will assume that $\mathcal{K} = \mathrm{GL}_2(\hat{A})$ and $g = \mathrm{diag}(N, 1, \ldots, 1)$ for some $N \in A$ with $|N| > 1$. In this case, we see that $\mathcal{K}_g = \mathcal{K}_0(\langle N\rangle) =: \mathcal{K}_0(N)$, $\Gamma_1 = \mathrm{GL}_2(A)$, $\Gamma_1 \cap \Gamma_g = \Gamma_0(N)$, and furthermore, $T_N := T_g$ acts trivially on the set of irreducible components on $\mathcal{M}^2_{A,\mathbb{C}_\infty}$, in particular, it induces a correspondence, again denoted T_N, on $\mathbb{M}$. The correspondence is also described by the inclusion $Y_0'(N) \subset \mathbb{M}^2$ from (2.10), and it maps the point $x \in \mathbb{M}(\mathbb{C}_\infty)$ to the set of those points $y \in \mathbb{M}(\mathbb{C}_\infty)$ corresponding to Drinfeld modules linked to x via cyclic $\langle N\rangle$-isogenies.

We define $T_{\mathbb{M}^n,N}$ to be the correspondence on $\mathbb{M}^n$ which is the product of the correspondences T_N on each factor $\mathbb{M}$. When there is no risk of confusion, we also denote it by T_N. We say $T_{\mathbb{M}^n,N}$ *stabilizes* an algebraic subvariety $Y \subset \mathbb{M}^n$ if $Y \subset T_{\mathbb{M}^n,N}(Y)$. In this case, we define the *restriction of* T_N *to* Y, denoted $T_{Y,N}$, to be the union of the irreducible components of $T_{\mathbb{M}^n,N} \cap (Y \times Y)$ of maximal dimension. It is a correspondence on Y which is still surjective, in the sense that the two projections $T_{Y,N} \to Y$ are surjective. Whenever we mention $T_{Y,N}$, it is implicit that T_N stabilizes Y.

Let $x \in \mathbb{M}(\mathbb{C}_\infty)$, and suppose that $x \in T_N(x)$. Then x admits a cyclic endomorphism, hence is a CM point. For a given $N \in A$, $|N| > 1$, there are only finitely many points stabilized by T_N, which correspond to the points of the diagonal in $\mathbb{M}^2$ which intersect $Y_0'(N)$. On the other hand, given a CM point $x \in \mathbb{M}(\mathbb{C}_\infty)$ with $R = \mathrm{End}(x)$, there are infinitely many $N \in A$ such that $x \in T_N(x)$, namely all those N composed of primes $\mathfrak{p} \subset A$ for which $\mathfrak{p}R$ is a product of two distinct principal prime ideals of R. Equivalently, these primes split completely in the class field corresponding to $\mathrm{Pic}(R)$, hence the set of such primes has density $1/2|\mathrm{Pic}(R)|$, by Čebotarev. Notice also that such primes $\mathfrak{p}$ are necessarily principal in A.

3.3 Some intersection theory

$\mathbb{M}$ is affine, so we may fix an embedding $\mathbb{M} \subset \mathbb{A}^m_{\mathbb{C}_\infty}$. Then we obtain embeddings $\mathbb{M}^n \subset \mathbb{A}^{mn}_{\mathbb{C}_\infty}$ and $\overline{\mathbb{M}^n} \subset \mathbb{P}^{mn}_{\mathbb{C}_\infty}$. (Here $\bar{\cdot}$ denotes the Zariski-closure).

For any irreducible $Y \subset \mathbb{M}^n$, we denote by $\deg(Y)$ the degree of $\overline{Y} \subset \mathbb{P}^{mn}$ in the usual sense. If $Y = \cup_i Y_i$ is a union of irreducible components, then $\deg(Y) := \sum_i \deg(Y_i)$, (regardless of their dimensions).

We denote by $\psi(N) = [\mathrm{GL}_2(A) : \Gamma_0(N)] = \prod_{\mathfrak{p}|N}(1+|\mathfrak{p}|^{-1})$ the degree of the correspondence T_N on $\mathbb{M}$. In particular, $\psi(N) \leq \deg(Y_0'(N)) \leq 2\psi(N)$.

We collect the following facts:

Proposition 3.5 *Let $Y \subset \mathbb{M}^n$ be an algebraic subvariety.*

1. *Y has at most $\deg(Y)$ irreducible components.*
2. *If $Y' \subset \mathbb{M}^n$ is another subvariety, then $\deg(Y \cap Y') \leq \deg(Y)\deg(Y')$.*
3. *$\deg(T_{\mathbb{M}^n,N}(Y)) \leq 2^n\psi(N)^n \deg(Y)$.*
4. *There are only finitely many pure special subcurves of $\mathbb{M}^n$ of degree less than a given bound.*

Proof. (1) is trivial, (2) and (3) follow from [6, 8.4.6], and (4) follows from the fact that a curve $Y \subset \mathbb{M}^n$ is pure special if and only if $p_{i,j}(Y) = Y_0'(N_{ij})$ for some $N_{ij} \in A$ for all $i \neq j$, and $\deg(Y_0'(N_{ij})) \geq \psi(N_{ij})$. $\square$

3.4 Preimages in Ω^n

Denote by π the rigid analytic map $\pi : \Omega^n \to \mathbb{M}^n(\mathbb{C}_\infty)^{\mathrm{an}}$, which is the quotient for the $\mathrm{GL}_2(A)^n$-action. For each irreducible component Y_i of Y, we choose an irreducible rigid analytic variety $Z_i \subset \Omega^n$ with $\pi(Z_i) = Y_i(\mathbb{C}_\infty)^{\mathrm{an}}$. The group $\mathrm{GL}_2(k_\infty)^n$ acts on Ω^n, and the $\mathrm{GL}_2(A)^n$-orbit of Z_i is $\pi^{-1}(Y_i)$. We next describe the action of T_N in Ω^n.

We may choose a set $T \subset \mathrm{GL}_2(k)$ of representatives for $\mathrm{GL}_2(A)\backslash\mathrm{GL}_2(A)\left(\begin{smallmatrix} N & 0 \\ 0 & 1\end{smallmatrix}\right)\mathrm{GL}_2(A)$ such that the following holds: For every pair of principal ideals $\mathfrak{a}, \mathfrak{d} \subset A$ with $\mathfrak{a}\mathfrak{d} = \langle N\rangle$, and set of representatives $\{b_1, \ldots, b_r\} \subset A$ of $A/\mathfrak{d}$, there exist $a, d \in A$ with $\mathfrak{a} = \langle a\rangle$ and $\mathfrak{d} = \langle d\rangle$ such that

$$\begin{pmatrix} a & b_j \\ 0 & d \end{pmatrix} \in T \quad \text{for } j = 1, \ldots, r. \tag{3.6}$$

One easily verifies that these elements indeed represent distinct coset classes. We write $T^n = \{t_j \in \mathrm{GL}_2(k)^n \mid j \in \mathcal{I}^n\}$, where $\mathcal{I} = \{1, \ldots, \psi(N)\}$.

For each Z_i we define $\mathcal{J}_{Z_i} \subset \mathcal{I}^n$ as the set of those indices j for which $t_j(Z_i) \subset \pi^{-1}(Y_i)$. Now let $y \in Y_i(\mathbb{C}_\infty)$ and choose some $z \in Z_i$ with $\pi(z) = y$. Then

$$T_{\mathbb{M}^n,N}(y) = \{\pi(t_j(z)) \mid j \in \mathcal{I}^n\}, \tag{3.7}$$
$$T_{Y,N}(y) = \{\pi(t_j(z)) \mid j \in \mathcal{J}_{Z_i}\}. \tag{3.8}$$

In particular, we see that each $\mathcal{J}_{Z_i}$ is non-empty, as $T_{Y,N}$ is a surjective correspondence. In fact, under suitable conditions, the sets $\mathcal{J}_{Z_i}$ are fairly large:

Proposition 3.9 *Let $Y \subset \mathbb{M}^n$ be a subvariety all of whose irreducible components have the same dimension, and suppose that $Y \subset T_{\mathbb{M}^n,N}(Y)$ for some $N \in A$ such that $\langle N \rangle$ is a product of distinct primes $\mathfrak{p} \subset A$ of even degree satisfying $|\mathfrak{p}| \geq \max(13, \deg(Y))$. Let Y_i be an irreducible component of Y for which the projection $p_I : Y_i \to \mathbb{M}^I$ is dominant, for some $I \subset \{1, \ldots, n\}$, and choose a preimage $Z_i \subset \Omega^n$ of $Y_i(\mathbb{C}_\infty)^{\mathrm{an}}$. Then the projection*

$$p_I : \mathcal{J}_{Z_i} \longrightarrow \mathcal{I}^I$$

is surjective.

Proof. This is [3, Thm. 4], the proof is exactly the same. We briefly explain the hypotheses on N: we use the fact that $\mathbb{M}$ has a $\mathrm{PSL}_2(A/NA) \cong \prod_{\mathfrak{p}} \mathrm{PSL}_2(A/\mathfrak{p})$-covering $Y_2(N)$, of which $Y_0(N)$ is a subcover (Proposition 2.9), and that this group has no proper subgroups of index $\deg(Y)$ or less when $|\mathfrak{p}| \geq \max(13, \deg(Y))$. $\square$

3.5 An interlude in group theory

We remark that the $\mathrm{GL}_2(k_\infty)$ action on Ω induces a $\mathrm{PGL}_2(k_\infty)$ action, as the centre acts trivially. Until now we have found it more convenient to work with GL_2, but in order to continue we will need a number of group-theoretic results, and here it will be simpler to work with PGL_2, as had been done throughout [3].

For the convenience of the reader, we recall here some basic properties of the groups $\mathrm{PGL}_2(k_\infty)$, which we will need. These results were used implicitly in [3], and thus may also help the reader with that paper.

Lemma 3.10 $k_\infty^\times / k_\infty^{\times 2} \cong (\mathbb{Z}/2\mathbb{Z})^2$.

Proof. This is trivial, as $k_\infty \cong \mathbb{F}_q((\varpi))$ for a uniformizer ϖ. $\square$

Proposition 3.11 *Every non-trivial normal subgroup of* $\mathrm{PGL}_2(k_\infty)$ *contains* $\mathrm{PSL}_2(k_\infty)$. *In particular,* $\mathrm{PGL}_2(k_\infty)$ *has no non-trivial discrete normal subgroups.*

Proof. Let $H \triangleleft \mathrm{PGL}_2(k_\infty)$ be a normal subgroup. Then, as $\mathrm{PSL}_2(k_\infty)$ is simple, either $\mathrm{PSL}_2(k_\infty) \subset H$, or $H \cap \mathrm{PGL}_2(k_\infty) = \{1\}$. In the latter case we get an embedding $H \hookrightarrow \mathrm{PGL}_2(k_\infty)/\mathrm{PSL}_2(k_\infty) \cong k_\infty^\times / k_\infty^{\times 2} \cong (\mathbb{Z}/2\mathbb{Z})^2$. It remains to show that $\mathrm{PGL}_2(k_\infty)$ has no normal subgroups isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$. This may be verified with explicit calculations, by conjugating an element of order 2 with $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. $\square$

Proposition 3.12 *Let H be a subgroup of finite index in $G = \mathrm{PGL}_2(k_\infty)$. Then H is normal and contains $\mathrm{PSL}_2(k_\infty)$. In particular, if H is simple then $H = \mathrm{PSL}_2(k_\infty)$.*

Proof. G acts on the cosets G/H, giving a representation $\rho : G \to \mathrm{Aut}(G/H) \cong S_n$ where $n = [G : H]$. Then $K = \ker(\rho)$ is a normal subgroup of G contained in H. As K is infinite, K must intersect non-trivially with $\mathrm{PSL}_2(k_\infty)$, hence contains $\mathrm{PSL}_2(k_\infty)$, as the latter is simple. We will not need the fact that H is normal, but it is known that all the subgroups between $\mathrm{PSL}_2(k_\infty)$ and $\mathrm{PGL}_2(k_\infty)$ are normal in G. $\square$

Corollary 3.13 *Let $\mathrm{PSL}_2(k_\infty) \subset H \subset \mathrm{PGL}_2(k_\infty)$ and suppose that $f : H \hookrightarrow \mathrm{PGL}_2(k_\infty)$ is a monomorphism with image of finite index. Then $f|_{\mathrm{PSL}_2(k_\infty)}$ in an automorphism of $\mathrm{PSL}_2(k_\infty)$.*

Proof. The image $f(\mathrm{PSL}_2(k_\infty))$ has finite index in $\mathrm{PGL}_2(k_\infty)$ and is simple, hence $f(\mathrm{PSL}_2(k_\infty)) = \mathrm{PSL}_2(k_\infty)$. $\square$

The above results may easily be generalized to PGL_2 over other (infinite) fields. Mostly we have just used the fact that $k_\infty^\times / k_\infty^{\times 2}$ is finite.

Proposition 3.14 (Goursat's Lemma) *Let G_1 and G_2 be groups, and $H \subset G_1 \times G_2$ a subgroup such that the two projections $pr_i : H \to G_i$ are surjective. Then $K_i = \ker(pr_i)$ can be considered a normal subgroup of G_j, for $i \neq j$. Then H is the inverse image of the graph of an isomorphism $\rho : G_1/K_2 \xrightarrow{\sim} G_2/K_1$.*

Proof. This is straight forward. The map

$$\begin{aligned} \rho : G_1/K_2 &\longrightarrow G_2/K_1 \\ g_1 &\longmapsto g_2 \text{ with } (g_1, g_2) \in H \end{aligned}$$

is easily checked to be a well-defined isomorphism. Now H is the inverse image of the graph of ρ. $\square$

3.6 Curves stabilized by Hecke correspondences

Our next goal is to characterize special subvarieties of $\mathbb{M}^n$ by their property of being stabilized by suitable Hecke correspondences. The hard part is to prove this for curves.

Theorem 3.15 *Let $Y \subset \mathbb{M}^2$ be an irreducible algebraic curve such that both projections $Y \to \mathbb{M}$ are dominant, and suppose $Y \subset T_{\mathbb{M}^2,N}(Y)$ for some $N \in A$ such that $\langle N \rangle$ is a product of distinct primes $\mathfrak{p} \subset A$ of even degree satisfying $|\mathfrak{p}| \geq \max(13, \deg(Y))$. Then $Y = Y_0'(N')$ for some $N' \in A$.*

Note that we cannot deduce N' from N. Indeed $Y_0'(N')$ is stabilized by $T_{\mathbb{M}^2,N}$ for all $N \in A$ coprime to N'.

Proof. Suppose that the hypotheses of Theorem 3.15 are satisfied. The group $G := \mathrm{PGL}_2(k_\infty)^2$ acts on Ω^2, and we define the following subgroups: $S := \mathrm{PSL}_2(k_\infty)^2$, $\Gamma := \mathrm{PGL}_2(A)^2$ and $\Sigma := \mathrm{PSL}_2(A)^2$. We fix an irreducible rigid analytic curve $Z \subset \Omega^2$ with $\pi(Z) = Y(\mathbb{C}_\infty)^{\mathrm{an}}$. Let $G_Z \subset G$ be the stabilizer of Z, which is a closed analytic subgroup of G. We also define the subgroups $S_Z = G_Z \cap S$, $\Gamma_Z = G_Z \cap \Gamma$ and $\Sigma_Z = G_Z \cap \Sigma$. We intend to prove Theorem 3.15 by investigating the structure of G_Z and S_Z.

Denote by $pr_i : G_Z \to \mathrm{PGL}_2(k_\infty)$ the two projections for $i = 1, 2$.

Lemma 3.16 *The projection $pr_i : G_Z \to \mathrm{PGL}_2(k_\infty)$ is injective, and $pr_i(\Gamma_Z)$ has finite index in $\mathrm{PGL}_2(A)$, for $i = 1, 2$.*

Proof. Exactly the same as [3, Lemma 2.11]. Here one uses Proposition 3.11. $\square$

Let $i \in \mathcal{J}_Z$, then $t_i(Z) \subset \pi^{-1}(Y)$, so there is some $\gamma_i \in \Gamma$ such that $\gamma_i t_i \in G_Z$. As the projections $pr_1, pr_2 : \mathcal{J}_Z \to \mathcal{I}$ are surjective (Proposition 3.9), we get many non-trivial elements of G_Z this way.

Let $H_1 = pr_1(G_Z)$. Our first goal is to show that $\mathrm{PSL}_2(k_\infty) \subset H_1$. By a slight abuse of notation, we also denote the elements of $pr_1(T) \subset \mathrm{GL}_2(k)$ by t_i, for $i \in \mathcal{I}$. We have seen that, for every $i \in pr_1(\mathcal{J}_Z) = \mathcal{I}$, there exists some $\gamma_i \in \mathrm{PGL}_2(A)$ such that $g_i := \gamma_i t_i \in H_1$. Lemma 3.16 says that $H_1 \cap \mathrm{PGL}_2(A)$ has finite index in $\mathrm{PGL}_2(A)$, and we choose a finite set $R \subset \mathrm{PGL}_2(A)$ of representatives for $\mathrm{PGL}_2(A)/(\mathrm{PGL}_2(A) \cap H_1)$.

We claim that given any string $i_1, \ldots, i_n$ of elements of $\mathcal{I}$, and any $a \in \mathrm{GL}_2(A)$, there exists $\gamma \in R$ such that

$$\gamma t_{i_n} t_{i_{n-1}} \cdots t_{i_1} a \in H_1.$$

Indeed, by induction it suffices to prove the claim for $n = 1$. Let $i_1 \in \mathcal{I}$ and $a \in \mathrm{GL}_2(A)$ be given. $\mathrm{GL}_2(A)$ acts from the right on the set of left cosets $\mathrm{GL}_2(A)\backslash\mathrm{GL}_2(A)\left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)\mathrm{GL}_2(A)$, so we let a act from the right on $\mathrm{GL}_2(A) \cdot t_{i_1}$, obtaining $\mathrm{GL}_2(A) \cdot t_{i_1} a = \mathrm{GL}_2(A) \cdot t_j$ for some $j \in \mathcal{I}$. Thus $t_{i_1} a = \gamma_j' t_j$, and for suitable $\gamma \in R$ and $\gamma' \in H_1 \cap \mathrm{PSL}_2(A)$ we have

$$\gamma t_{i_1} a = \gamma \gamma_j' t_j = \gamma(\gamma_j' \gamma_j^{-1})\gamma_j t_j = \gamma' \gamma_j t_j = \gamma' g_j \in H_1.$$

This proves the claim.

Now, multiplying by a suitable power of $\left(\begin{smallmatrix} N & 0 \\ 0 & N \end{smallmatrix}\right)$, we see from (3.6) that for any $x \in A[1/N]$ and $a \in \mathrm{GL}_2(A)$, there exists some $\gamma_{x,a} \in R$ such that $\gamma_{x,a}\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)a \in H_1$. Denote by $E \subset \mathrm{PSL}_2(A[1/N])$ the subgroup generated by elementary matrices. We have shown that $H_1 \cap E$ has finite index in E. As $A[1/N]$ is dense in k_∞, and $\mathrm{PSL}_2(k_\infty)$ is generated by elementary matrices, it follows that E is dense in $\mathrm{PSL}_2(k_\infty)$.

Next, we see that H_1 is a closed subgroup of $\mathrm{PGL}_2(k_\infty)$, exactly the same way as [3, Lemma 2.12 and the following paragraph], where we need only replace $\mathrm{PSL}_2(A[1/\mathfrak{m}])$ by E. It follows that $H_1 \cap \mathrm{PSL}_2(k_\infty)$ has finite index in $\mathrm{PSL}_2(k_\infty)$, so by Proposition 3.12

$$\mathrm{PSL}_2(k_\infty) \subset H_1, \quad \text{and similarly, } \mathrm{PSL}_2(k_\infty) \subset H_2 = pr_2(G_Z). \tag{3.17}$$

Now, since the projections $pr_1, pr_2 : G_Z \to \mathrm{PGL}_2(k_\infty)$ are injective, Proposition 3.14 implies that

$$G_Z = \{(g, \rho(g)) \mid g \in H_1\},$$

where $\rho : H_1 \xrightarrow{\sim} H_2$ is an isomorphism. From (3.17) and Corollary 3.13 follows that

$$S_Z = G_Z \cap \mathrm{PSL}_2(k_\infty)^2 = \{(g, \rho(g)) \mid g \in \mathrm{PSL}_2(k_\infty)\}, \tag{3.18}$$

where $\rho|_{\mathrm{PSL}_2(k_\infty)}$ is an automorphism of $\mathrm{PSL}_2(k_\infty)$.

It is known that the automorphisms of $\mathrm{PSL}_2(k_\infty)$ are all of the form $g \mapsto hg^\sigma h^{-1}$ for some $h \in \mathrm{PGL}_2(k_\infty)$ and $\sigma \in \mathrm{Aut}(k_\infty)$, see [11]. By the definition of Σ_Z and (3.18), we see that $h \cdot pr_1(\Sigma_Z)^\sigma \cdot h^{-1} \subset \mathrm{PSL}_2(A)$. On the other hand, Lemma 3.16 tells us that $pr_1(\Sigma_Z)$ has finite index in $\mathrm{PSL}_2(A)$. This in turn tells us a lot about h and σ. Fix some $\varpi \in A$ so that ϖ^{-1} is a uniformizer for k_∞, i.e. $k_\infty = \mathbb{F}_q((\varpi^{-1}))$.

Proposition 3.19 *Let G be a subgroup of finite index in $\mathrm{PGL}_2(A)$, and suppose that $hG^\sigma h^{-1} \subset \mathrm{PGL}_2(k)$, for some $h \in \mathrm{PGL}_2(k_\infty)$ and $\sigma \in \mathrm{Aut}(k_\infty)$. Then $h \in \mathrm{PGL}_2(k)$, $\sigma|_{\mathbb{F}_q} \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$, and $\sigma(\varpi^{-1}) = u\varpi^{-1} + v$ for some $u \in \mathbb{F}_q^*$ and $v \in \mathbb{F}_q$.*

Proof. Let $R = \mathbb{F}_q[[\varpi^{-1}]]$ be the unique valuation ring in k_∞. We show that $h \in \mathrm{PGL}_2(k)$, $\sigma(k) \subset k$ and $\sigma(R) \subset R$ exactly as in [3, Prop. 2.13].

It follows that $\sigma(A) \subset A$ and $\sigma(A^\times) \subset A^\times$, hence $\sigma(\mathbb{F}_q) \subset \mathbb{F}_q$. Furthermore, $\sigma(\varpi^{-1}) \in A^{-1}$ must again be a uniformizer for k_∞. But all other uniformizers of k_∞ that lie in A^{-1} are of the form $u\varpi^{-1} + v$ for some $u \in \mathbb{F}_q^*$ and $v \in \mathbb{F}_q$. This completes the proof. $\square$

Now that we have assembled enough ingredients, the proof of Theorem 3.15 follows exactly as in [3, §2.7]. We provide a sketch. We have seen that

$$S_Z = \{(g, hg^\sigma h^{-1}) \mid g \in \mathrm{PSL}_2(k_\infty)\},$$

where $h \in \mathrm{PGL}_2(k)$ and $\sigma \in \mathrm{Aut}(k_\infty)$ is as in Proposition 3.19. In particular, $\sigma(\alpha) = \alpha^{q^t}$, for all $\alpha \in \mathbb{F}_q$ and some $t \geq 0$. Let $f = (\varpi^{-q} - \varpi^{-1})^{q-1}$, and $F := \mathbb{F}_p((f))$, which is a complete subfield of k_∞ on which σ acts trivially. Fix some non-square $\alpha \in \mathbb{F}_q$, and set

$$P = \{z \in \Omega \mid z^2 = \alpha e,\ e \in F\} \subset \Omega.$$

Pick any $z_1 \in P$. Then $S_1 = \mathrm{Stab}_{\mathrm{PSL}_2(F)}(z_1)$ is a one-dimensional Lie group (so far we have made essential use of the assumption that p be odd).

Now let $z_2 \in \Omega$ such that $(z_1, z_2) \in Z$. Then the "S_1-orbit"

$$\{(g(z_1), hg^\sigma h^{-1}(z_2)) \mid g \in S_1\} \subset Z \cap (\{z_1\} \times \Omega)$$

is discrete, but S_1 is not, so there exists $1 \neq g \in S_1$ such that $hg^\sigma h^{-1}$ fixes both z_2 and $h'(z_1)$, where $h' = h \circ \left(\begin{smallmatrix} \alpha^{(p^t-1)/2} & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{PGL}_2(k)$. This means that z_2 and $h'(z_1)$ are conjugate over k_∞, i.e. $z_2 = \pm h'(z_1)$. It follows that either $\pi(z_1, h'(z_1)) \in Y(\mathbb{C}_\infty)$ or $\pi(-z_1, h'(-z_1)) \in Y(\mathbb{C}_\infty)$. Both of these points also lie on $Y_0'(N')$, where $N' = \deg(h')$

is independent of (z_1, z_2). As P is uncountable, whereas the fibres of π are countable, it follows that $Y = Y_0'(N')$. $\square$

To extend Theorem 3.15 to subvarieties $Y \subset \mathbb{M}^n$ of higher dimension we may follow [3, §2.8 and Corollaries 2.8 and 2.9] almost verbatim, no new ingredients are required. We must just keep in mind that $\mathbb{M}$ is defined over H, the Hilbert class field of k. We also use Galois action on CM points as in the following section.

Theorem 3.20 *Let F be a field lying between H and $\mathbb{C}_\infty$. Let $Y \subset \mathbb{M}^n$ be an F-irreducible algebraic subvariety, containing a CM point $x \in Y(\mathbb{C}_\infty)$. Suppose that $Y \subset T_{\mathbb{M}^n,N}(Y)$ for some $N \in A$ such that $\langle N \rangle$ is a product of distinct primes $\mathfrak{p} \subset A$ of even degree satisfying $|\mathfrak{p}| \geq \max(13, \deg(Y))$. Then $Y \subset \mathbb{M}^n$ is a special subvariety.* $\square$

3.7 CM points on curves

We may now start proving our main results, by exploiting the behavior of CM points under Galois action (Proposition 2.1) in conjunction with Theorems 3.15 and 3.20. We first treat the case of curves, where our results are effective.

Theorem 3.21 *Let $X = X_1 \times \cdots \times X_n$ be a product of Drinfeld modular curves. Let F/H be a finite extension, and $d \in \mathbb{N}$. Then there exists an absolutely computable constant $B = B(X, F, d) > 0$ such that the following holds. Let $Y \subset X$ be an irreducible algebraic subcurve of degree d and defined over F. Then Y is a special subcurve if and only if $Y(\mathbb{C}_\infty)$ contains a CM point x satisfying $H_{\mathrm{CM}}(x) > B$.*

This indeed implies Theorem 1.2 for curves, since if $Y \subset X$ contains a Zariski-dense set of CM points, then it contains CM points of arbitrary CM height and Y is defined over some finite extension F/k, as the CM points are all defined over $\bar{k}$.

Proof. We again follow [3, §3.4] very closely, but will provide full details here for the benefit of the reader.

It follows from Proposition 2.19 that we may assume that $X = \mathbb{M}^n$. Furthermore, we may assume that none of the projections $p_i : Y \to \mathbb{M}$ are constant. Using the fact that $Y \subset \mathbb{M}^n$ is a pure special subcurve if and only if $p_{i,j}(Y) \subset \mathbb{M}^2$ is special for all $i < j$, we have reduced the problem to the case $n = 2$.

Let $x = (x_1, x_2) \in Y(\mathbb{C}_\infty)$ be a CM point. For $i = 1, 2$ we write $R_i = \mathrm{End}(x_i) = A + \mathfrak{f}_i \mathcal{O}_{K_i}$, an order of conductor $\mathfrak{f}_i \subset A$ in the CM field K_i, which has genus g_i. Furthermore, we denote by $K_i(x_i) := H_{R_i}$ the ring class field of R_i, which is a field of definition for x_i, and we have $\mathrm{Gal}(K_i(x_i)/K_i) \cong \mathrm{Pic}(R_i)$, and $K_i(x_i)/K_i$ is unramified outside $\mathfrak{f}_i$. We denote by $K = K_1K_2$ and $K(x_1, x_2) = K_1(x_1)K_2(x_2)$ the composite fields. Denote by F_s the separable closure of k in F (which contains H), and by L the Galois closure of $F_sK(x_1, x_2)$ over k.

Let $\mathfrak{p}$ be a prime of k of even degree which splits completely in F_sK (in particular, $\mathfrak{p}$ is principal, as it splits in H), and suppose $\mathfrak{p} \nmid \mathfrak{f}_1\mathfrak{f}_2$. Let $\mathfrak{P}$ be a prime of L lying above $\mathfrak{p}$, and denote by $\mathfrak{P}_i$ its restriction to $K_i(x_i)$. Denote by $\sigma \in \mathrm{Aut}(FL/FK)$ an extension of the Frobenius element $(\mathfrak{P}, L/k)$, and let $\sigma_i = \sigma|_{K_i(x_i)} = (\mathfrak{P}_i, K_i(x_i)/K_i)$ (remember that $\mathfrak{p}$ splits in K and is unramified in L).

It follows from Proposition 2.1 that x_i and x_i^σ correspond to Drinfeld modules linked by cyclic $\mathfrak{p}$-isogenies, and hence

$$(x_1, x_2) \in Y \cap T_{\mathbb{M}^2,\mathfrak{p}}(Y^\sigma) = Y \cap T_{\mathbb{M}^2,\mathfrak{p}}(Y). \tag{3.22}$$

Moreover, the whole $\mathrm{Gal}(FK(x_1,x_2)/F)$-orbit of (x_1,x_2) lies in this intersection, which thus contains at least $\max\big(|\mathrm{Pic}(R_1)|, |\mathrm{Pic}(R_2)|\big)/[F:k]$ points. On the other hand, from Proposition 3.5, $\deg(Y \cap T_{\mathbb{M}^2,\mathfrak{p}}(Y)) \leq 4(|\mathfrak{p}|+1)^2 \deg(Y)^2$. Therefor, if

$$|\mathrm{Pic}(R_i)|/[F:k] > 4(|\mathfrak{p}|+1)^2 \deg(Y)^2, \quad i = 1 \text{ or } i = 2, \tag{3.23}$$

then the intersection is improper, $Y \subset T_{\mathbb{M}^2,\mathfrak{p}}(Y)$ and hence Y is special (Theorem 3.15), provided also $|\mathfrak{p}| \geq \max(13, \deg(Y))$.

It remains to show that such a suitable prime $\mathfrak{p}$ indeed exists, if $H_{\mathrm{CM}}(x)$ is sufficiently large. Let M be the Galois closure of F_sK over k, and set

$$\pi_M(t) = \#\{\mathfrak{p} \subset A \mid \text{prime, split in } M \text{ and } |\mathfrak{p}| = q^t\}.$$

Let $T \in k$ be a transcendental element such that k is a finite separable geometric extension of $\mathbb{F}_q(T)$, and let $e = [k : \mathbb{F}_q(T)]$. Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_q$ in M, let $n_c = [\mathbb{F} : \mathbb{F}_q]$ be the constant extension degree and $n_g = [M : \mathbb{F}k]$ the geometric extension degree of M/k. The Čebotarev Theorem for function fields [5, Prop. 5.16] says

$$\text{if } n_c | t, \text{ then } \quad |\pi_M(t) - \frac{1}{n_g} q^t/t| < 4\big(e^2 + g_M(e+1)/2 + g_k + 1\big) q^{t/2}, \tag{3.24}$$

where g_M and g_k are the genus of M and k, respectively. We may bound g_M in terms of g_1, g_2 and g_k using the Castelnuovo inequality [15, III.10.3], and eventually obtain

$$\pi_M(t) > C_1 q^t/t - \big(C_2(g_1+g_2) + C_3\big) q^{t/2}, \tag{3.25}$$

where C_1, C_2 and C_3 are absolutely computable positive constants, depending on k and F.

We want $\pi_M(t) > \log_q |\mathfrak{f}_1\mathfrak{f}_2|$, $q^t \geq \max(13, \deg(Y))$ and $2n_c | t$, so that there exists a prime $\mathfrak{p}$ which splits in M (and thus in F_sK), does not divide $\mathfrak{f}_1\mathfrak{f}_2$, and satisfies the hypotheses of Theorem 3.15. We also want (3.23) to hold, for which we employ Proposition 2.4:

$$|\mathrm{Pic}(R_i)| > C_\varepsilon H_{\mathrm{CM}}(x_i)^{1-\varepsilon} = C_\varepsilon (q^{g_i}|\mathfrak{f}_i|)^{1-\varepsilon}, \quad \text{for any } \varepsilon > 0.$$

In summary, we need a simultaneous solution $t \in 2n_c\mathbb{N}$ to the three inequalities

$$q^t \geq \max(13, \deg(Y)),$$

$$C_1 q^t/t - \big(C_2(g_1+g_2) + C_3\big) q^{t/2} > \log_q |\mathfrak{f}_1\mathfrak{f}_2|,$$

and

$$C_\varepsilon(q^{g_i}|\mathfrak{f}_i|)^{1-\varepsilon} > 4[F:k](q^t+1)^2\deg(Y)^2 \quad \text{for some } \varepsilon>0,\ i=1 \text{ or } i=2.$$

Such a solution will always exist if $H_{\mathrm{CM}}(x) = \max(H_{\mathrm{CM}}(x_1), H_{\mathrm{CM}}(x_2))$ is sufficiently large. (Intuitively, q^t must be large compared to $\log_q|\mathfrak{f}_i|$ and g_i, and small compared to $|\mathfrak{f}_i|$ and q^{g_i}). $\square$

3.8 Completing the proof of Theorem 1.2

The proof of Theorem 1.2 now follows exactly as in [3, §3.5]. We sketch the proof here for completeness.

Firstly, as $Y(\mathbb{C}_\infty)$ contains a Zariski-dense subset S of CM points, which are defined over k^{sep}, there exists a finite Galois extension F/k over which Y is defined. We will use induction on $d = \dim(Y)$, the case $d=1$ following from Theorem 3.21, so we assume that $d \geq 2$ and $n \geq 3$. We may furthermore assume that $Y \subset \mathbb{M}^n$ is a hypersurface, as it is an irreducible component of

$$\bigcap_{\substack{I\subset\{1,\ldots,n\}\\ |I|=d+1}} p_I^{-1}p_I(Y).$$

Lastly, we may assume that all the projections $p_i : Y \to \mathbb{M}$ are dominant.

Step 1. For a given constant $B>0$, we may assume that every $x=(x_1,\ldots,x_n)\in S$ satisfies $H_{\mathrm{CM}}(x_i) > B$, $\forall i = 1,\ldots,n$.

Pick one such $x \in S$. **Suppose** that there exist primes $\mathfrak{p}_1,\ldots,\mathfrak{p}_{d-1} \subset A$ of even degree satisfying the following conditions:

1. Each $\mathfrak{p}_j$ splits in F and in $\mathrm{End}(x_i)$, for all $i=1,\ldots,n$.
2. $|\mathfrak{p}_1| \geq \max(13, \deg(Y))$.
3. $|\mathfrak{p}_{j+1}| \geq (\deg(Y))^{2^j}\prod_{m=1}^{j}(2|\mathfrak{p}_m|+2)^{n2^{j-m}}$, for $j=1,\ldots,d-2$.
4. $|\mathrm{Pic}(\mathrm{End}(x_i))| > [F:k]|\mathfrak{p}_{d-1}|^2(2|\mathfrak{p}_{d-1}|+2)^n$, for all $i=1,\ldots,n$.

Then one shows, again using Galois action on x together with Theorem 3.20, that there exists a pure special subvariety $Y_x \subset Y$.

As the points in S are Zariski-dense, it follows that there exists a Zariski-dense family $\mathcal{C}$ of pure special subcurves $C \subset Y$, $C \in \mathcal{C}$.

Step 2. We now show that Y is special. Choose a CM point $x_1 \in \mathbb{M}(\mathbb{C}_\infty)$, and consider the slice

$$Y_1 = Y \cap (\{x_1\}\times\mathbb{M}^{n-1}).$$

Each pure special curve $C \in \mathcal{C}$ intersects Y_1 in at least one CM point, and we denote by Y' the Zariski-closure of these intersection points:

$$Y' = \overline{\cup_{C\in\mathcal{C}}(C \cap Y_1)}$$

We must have $\dim(Y') < \dim(Y)$, so by the induction hypothesis, Y' is special. We write $Y' = Y_1' \cup \cdots \cup Y_r'$ as a union of irreducible components. Replacing $\mathcal{C}$ by a Zariski-dense subfamily and renumbering if necessary, we may assume that Y_1' contains at least $1/r$ of the points of $C \cap Y_1$ for all $C \in \mathcal{C}$.

Now, either $Y_1' \cong \{y\} \times \mathbb{M}^m$ for some CM point $y \in \mathbb{M}^{n-m}(\mathbb{C}_\infty)$, in which case $Y = Y_0'(N') \times \mathbb{M}^{n-2}$ is special, or else at least one pure special curve appears as a factor of Y_1'.

It follows that there exist indices $1 < i < j$ such that $p_{i,j}(Y_1') = Y_0'(M)$ for some *fixed* $M \in A$. Fix $C \in \mathcal{C}$ and let the pure special curve $p_{\{1,i,j\}}(C) \subset \mathbb{M}^3$ correspond to the triple $(N_{C,1}, N_{C,i}, N_{C,j}) \in A^3$ via (2.17). Again restricting $\mathcal{C}$ if necessary, we may assume that $|N_{c,i}| \leq |N_{C,j}|$ for all $C \in \mathcal{C}$.

Now we fix $C \in \mathcal{C}$ and $x_1, x_i \in \mathbb{M}(\mathbb{C}_\infty)$. Then the number of distinct $x_j \in \mathbb{M}(\mathbb{C}_\infty)$ such that $(x_1, x_i, x_j) \in p_{\{1,i,j\}}(C)$ is bounded from below by an increasing function in $|N_{C,j}|$ (Proposition 2.18). But at least $1/r$ of these points x_j must also satisfy $(x_i, x_j) \in Y_0'(M)$, of which there can be at most $\psi(M)$. It follows that $|N_{C,i}|$ and $|N_{C,j}|$ are bounded independently of $C \in \mathcal{C}$. Restricting $\mathcal{C}$ once again, we may assume that $p_{i,j}(C) = Y_0'(N_0)$ for all $C \in \mathcal{C}$.

Now one can show that $Y \cong Y_0'(N') \times \mathbb{M}^{n-2}$, which is special.

Step 3. It remains to show that the primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_{d-1} \subset A$ satisfying (1)-(4) above actually exist, if the constant $B > 0$ is chosen sufficiently large. Let $x = (x_1, \ldots, x_n) \in S$ such that $H_{\mathrm{CM}}(x_i) = q^{g_i}|\mathfrak{f}_i| > B$ for all $i = 1, \ldots, n$.

As before, the problem boils down to finding simultaneous solutions $t_1, \ldots, t_{d-1} \in 2n_c\mathbb{N}$ to the following four inequalities:

$$q^{t_1} \geq \max(13, \deg(Y)),$$

$$C_\varepsilon(q^{g_i}|\mathfrak{f}_i|)^{1-\varepsilon} > [F:k]q^{2t_{d-1}}(2q^{t_{d-1}}+2)^n, \quad \text{for some } \varepsilon > 0 \text{ and some } 1 \leq i \leq n,$$

$$q^{t_{j+1}} \geq (\deg(Y))^{2^j} \prod_{m=1}^{j} (2q^{t_m}+2)^{n2^{j-m}}, \quad \text{for all } 1 \leq j \leq d-1,$$

$$C_1 q^{t_j}/t_j - \big(C_2(g_1 + \cdots + g_n) + C_3\big)q^{t_j/2} > \log_q|\mathfrak{f}_1 \cdots \mathfrak{f}_n|, \quad \text{for all } 1 \leq j \leq d-1.$$

If $B > 0$ is sufficiently large, then such solutions exist. This completes the proof of Theorem 1.2. $\square$

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