Quantum symmetric spaces

J. Donin
S. Shnider

Dept. of Math.
Bar-Ilan University
52900 Ramat-Gan

Israel

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

Germany

MPI / 94-73

Quantum symmetric spaces

J.Donin and S.Shnider

1 Introduction

Let G be a semisimple Lie group, $\mathfrak{g}$ its Lie algebra, and $r \in \wedge^{\otimes 2}\mathfrak{g}$ the Drinfeld–Jimbo R-matrix (see Section 2). Suppose H is a closed subgroup of G and $M = G/H$. Then the action of G on M defines a mapping $\rho : \mathfrak{g} \to \mathrm{Vect}(M)$. So, the element $(\rho \otimes \rho)(r)$ induces a bivector field on M which determines a bracket $\{\cdot,\cdot\}$ on the algebra $C^\infty(M)$ of smooth functions on M. In some cases this will be a Poisson bracket which we will call an R-matrix Poisson bracket. The natural question arises whether this bracket can be quantized.

The first case when $\{\cdot,\cdot\}$ is a Poisson bracket is when the Lie algebra of H contains a maximal nilpotent subalgebra. In [DGM] it is proven that in this case there exists a quantization of $\{\cdot,\cdot\}$, i.e. there is an associative multiplication μ_h in $C^\infty(M)$ of the form $\mu_h = m + h\{\cdot,\cdot\} + o(h)$ where m is the usual multiplication in $C^\infty(M)$. Moreover, this multiplication will be invariant under action of the Drinfeld–Jimbo quantum group $U_h\mathfrak{g}$. This means that μ_h satisfies the condition

$$x\mu_h(a,b) = \mu_h\tilde{\Delta}_h(x)(a \otimes b),$$

where $a, b \in C^\infty(M)$, $x \in U_h\mathfrak{g}$, and $\tilde{\Delta}_h$ is the comultiplication in $U_h\mathfrak{g}$. In [DG1] it is shown that in such a way one can obtain the $U_h\mathfrak{g}$-invariant quantization of the algebra of holomorphic sections of line bundles over the flag manifold of G.

In the present paper we consider the case when M is a symmetric space. It turns out that in this case $\{\cdot,\cdot\}$ also will be a Poisson bracket and there is a $U_h\mathfrak{g}$-invariant quantization of this bracket.

Moreover, if M is equipped with a G-invariant Poisson bracket $\{\cdot,\cdot\}_{inv}$, then there exists a simultaneous $U_h\mathfrak{g}$-invariant quantization $\mu_{\nu,h}$ of these brackets in the form

$$\mu_{\nu,h} = m + \nu\{\cdot,\cdot\}_{inv} + h\{\cdot,\cdot\} + o(\nu, h).$$

This is the case, for example, when M is a Hermitian symmetric space. Then $\{\cdot,\cdot\}_{inv}$ coincides with the Kirillov bracket. The usual deformation quantization of the Kirillov bracket, μ_ν, is invariant under G and $U\mathfrak{g}$. Thus, one may consider the multiplication $\mu_{\nu,h}$ as such a quantization of the Kirillov bracket which is invariant under the action of the quantum group $U_h\mathfrak{g}$.

Note that the Kirillov bracket is also generated by r in the following way. Let $\{\cdot,\cdot\}'$ be a bracket on $C^\infty(G)$ that generates by the left-invariant extension of r as a bivector field on G. Using the projection $G \to G/H = M$ we can consider $C^\infty(M)$ as a subalgebra of $C^\infty(G)$. One can to check that $C^\infty(M)$ is invariant under $\{\cdot,\cdot\}'$ if H is a Levi subgroup. For such H the difference $\{\cdot,\cdot\} - \{\cdot,\cdot\}'$ gives a Poisson bracket on M, the so-called Sklyanin–Drinfeld Poisson bracket. The quantization of this Poisson bracket is given in [DG2]. In case M is a symmetric space the bracket $\{\cdot,\cdot\}'$ will be a Poisson one itself and coincides with the Kirillov bracket $\{\cdot,\cdot\}_{inv}$ (see [DG2]). In [GP] there is given a classification of all orbits in the coadjoint representation of G on which r induses the Poisson bracket.

The authors wish to thank D.Gurevich for stimulating discussions about subject of the paper.

2 R-matrix Poisson brackets on symmetric spaces

Let $\mathfrak{g}$ be a simple Lie algebra over the field of complex numbers $\mathbb{C}$. Fixed a Cartan decomposition of $\mathfrak{g}$ and the corresponding root system Ω, we consider the Drinfeld–Jimbo R-matrix

$$r = \sum_{\alpha\in\Omega^+} X_\alpha \wedge X_{-\alpha} \in \wedge^2\mathfrak{g},$$

where X_α are the elements from the Cartan–Chevalley basis of $\mathfrak{g}$ corresponding to Ω, and Ω^+ denotes the set of positive roots. For shortness we will write this R-matrix as $r = r_1 \otimes r_2$.

That r satisfies the so-called modified classical Yang-Baxter equation which means that the Schouten bracket of r with itself is equal to an invariant element $\varphi \in \wedge^3\mathfrak{g}$:

$$[r,r]_{Sch} = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = \varphi. \tag{1}$$

Here we use the usual notation: $r^{12} = r_1 \otimes r_2 \otimes 1$, $r^{13} = r_1 \otimes 1 \otimes r_2$, and so on. Note that any invariant element in $\wedge^3\mathfrak{g}$ is dual up to a multiplicity to the three-form $(x, [y, z])$ on $\mathfrak{g}$, where $(\cdot, \cdot)$ denotes the Killing form. Therefore, φ will be also invariant under all automorphisms of the Lie algebra $\mathfrak{g}$.

The R-matrix r obviously satisfies the conditions: a) it is invariant under the Cartan subalgebra $\mathfrak{c}$, and b) $\theta r = -r$ where θ is the Cartan involution of $\mathfrak{g}$, so that $\theta X_\alpha = -X_{-\alpha}$. These conditions determine r uniquely up to a multiple (see [SS]).

In case $\mathfrak{g}$ is a semisimple Lie algebra with a Cartan decomposition, let $r \in \wedge^2\mathfrak{g}$ satisfy the equation (1) for some invariant $\varphi \in \wedge^3\mathfrak{g}$ and the previous conditions a) and b). Then r will be a linear combination of the Drinfeld–Jimbo R-matrices on the simple components of $\mathfrak{g}$. We will also call such r the Drinfeld–Jimbo R-matrix.

Let $\mathfrak{g}_\mathbb{R}$ be a real form of a semisimple Lie algebra $\mathfrak{g}$, and G a connected Lie group with $\mathfrak{g}_\mathbb{R}$ as its Lie algebra. Suppose σ is an involutive automorphism of G, and H is a subgroup of G such that $G_0^\sigma \subset H \subset G^\sigma$, where G^σ is the set of fixed points of σ and G_0^σ is the identity component of G^σ. The automorphism σ induces an automorphism of the both Lie algebras $\mathfrak{g}_\mathbb{R}$ and $\mathfrak{g}$ which we will also denote by σ. Thus, the space of left cosets $M = G/H$ turns into a symmetric space (see [He]). We denote by o the image of unity by the natural projection $G \to M$. The mapping $\tau : M \to M, gH \mapsto \sigma(g)H$, is well defined and has o as an isolated fixed point, therefore, the differential $\dot{\tau} : T_o \to T_o$ of τ at the point o multiplies the vectors of the tangent space T_o by (-1).

The action of G on M defines the mapping of $\mathfrak{g}_\mathbb{R}$ into the Lie algebra of real vector fields on M, $\rho : \mathfrak{g}_\mathbb{R} \to \mathrm{Vect}_\mathbb{R}(M)$, that extends up to the mapping $\rho : \mathfrak{g} \to \mathrm{Vect}(M)$ of the complexification of $\mathfrak{g}_\mathbb{R}$ into the Lie algebra of complex vector fields on M.

The mapping ρ induces on M a skew-symmetric bivector field in the following way. The element $\rho(r_1) \otimes \rho(r_2) \in \wedge^2\mathrm{Vect}(M)$ generates a bracket on the algebra $C^\infty(M)$ of smooth complex-valued functions on M, $\{f, g\} =$

$\rho(r_1)f \cdot \rho(r_2)g$, where $f, g \in C^\infty(M)$ and $\rho(r_1)f$ is the derivative of f along the vector field $\rho(r_1)$. It is obvious that this bracket is skew-symmetric and satisfies the Leibniz rule. Therefore it is defined by a bivector field which we denote by $\rho(r)$.

From now on we will suppose that the invariant element $\varphi \in \wedge^3\mathfrak{g}$ is invariant under σ as well. In case $\mathfrak{g}$ is a simple Lie algebra this will be satisfied automatically.

Proposition 2.1 *The bracket $\{\cdot,\cdot\}$ is a Poisson bracket on M.*

Proof Indeed, the bracket is obviously skew-symmetric and satisfying the Leibniz rule. Further, $\rho(\varphi)$ is a G-invariant three-vector field on M, therefore it is defined by its value at the point o, $\rho(\varphi)_o$. Since φ is σ-invariant, $\rho(\varphi)$ has to be τ-invariant, which implies that $\dot{\tau}\rho(\varphi)_o = \rho(\varphi)_o$. But the operator $\dot{\tau}$ acts on T_o by multiplying by (-1), so that $\dot{\tau}\rho(\varphi)_o = -\rho(\varphi)_o$. Therefore, $\rho(\varphi) = 0$. It means that the Schouten bracket $[\rho(r), \rho(r)]$ is equal to zero, and this is equivalent for the bracket $\{\cdot,\cdot\}$ to satisfy the Jacobi identity. ■

We will call the bracket $\{\cdot,\cdot\}$ an R-matrix Poisson bracket. Note that this bracket is not $\mathfrak{g}$-invariant and is degenerate in some points of M.

Suppose now that there is on M a $\mathfrak{g}$-invariant Poisson bracket $\{\cdot,\cdot\}_{inv}$. The case will be, for instance, if M is a Hermitian symmetric space. Then the imaginary part of the Hermitian form gives a symplectic form ω on M, and the dual to ω bivector field determines on M a $\mathfrak{g}$-invariant Poisson bracket $\{\cdot,\cdot\}_{inv}$.

Proposition 2.2 *The R-matrix and any invariant Poisson brackets are compatible, i.e. for any $a, b \in \mathbb{C}$ the bracket $a\{\cdot,\cdot\} + b\{\cdot,\cdot\}_{inv}$ is a Poisson one.*

Proof The straightforward computation (see [DGM]). ■

3 Three monoidal categories

We recall that a monoidal category is a triple $(\mathcal{C}, \otimes, \phi)$ where $\mathcal{C}$ is a category equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a tensor product functor, and a functorial isomorphism $\phi : (X \otimes Y) \otimes Z) \to X \otimes (Y \otimes Z)$ called associativity constraint, which satisfies the pentagonal identity, i.e. the diagram

$$\begin{array}{ccccc} ((X \otimes Y) \otimes Z) \otimes U & \xrightarrow{\phi} & (X \otimes Y) \otimes (Z \otimes U) & \xrightarrow{\phi} & X \otimes (Y \otimes (Z \otimes U)) \\ \phi \otimes id \downarrow & & & & \uparrow id \otimes \phi \\ (X \otimes (Y \otimes Z)) \otimes U & & \xrightarrow{\phi} & & X \otimes ((Y \otimes Z) \otimes U) \end{array} \quad (1)$$

is commutative.

If $(\tilde{\mathcal{C}}, \tilde{\otimes}, \tilde{\phi})$ is another monoidal category, then a morphism from $\mathcal{C}$ to $\tilde{\mathcal{C}}$ is given by a pair (α, β) where $\alpha : \mathcal{C} \to \tilde{\mathcal{C}}$ is a functor and $\beta : \alpha(X \otimes Y) \to \alpha(X) \tilde{\otimes} \alpha(Y)$ is a functorial isomorphism such that the diagram

$$\begin{array}{ccccc} \alpha((X \otimes Y) \otimes Z) & \xrightarrow{\beta} & \alpha(X \otimes Y) \tilde{\otimes} \alpha(Z) & \xrightarrow{\beta \tilde{\otimes} id} & (\alpha(X) \tilde{\otimes} \alpha(Y)) \tilde{\otimes} \alpha(Z) \\ \alpha(\phi) \downarrow & & & & \downarrow \tilde{\phi} \\ \alpha(X \otimes (Y \otimes Z)) & \xrightarrow{\beta} & \alpha(X) \tilde{\otimes} \alpha(Y \otimes Z) & \xrightarrow{id \tilde{\otimes} \beta} & \alpha(X) \tilde{\otimes} (\alpha(Y) \tilde{\otimes} \alpha(Z)) \end{array} \quad (2)$$

is commutative.

The morphism (α, β) of monoidal categories allow us to transfer additional structures given on objects of $\mathcal{C}$ to objects from $\tilde{\mathcal{C}}$. For example, let $X \in Ob(\mathcal{C})$. A morphism will be called $\mathcal{C}$**−associative** if we have the following equality of morphisms of $(X \otimes X) \otimes X \to X$

$$\mu(\mu \otimes id) = \mu(id \otimes \mu)\phi.$$

Then, for $\alpha(X) \in Ob(\tilde{\mathcal{C}})$ the naturally defined morphism $\alpha(\mu)\beta^{-1} : \alpha(X) \otimes \alpha(X) \to \alpha(X)$ will be associative in the category $\tilde{\mathcal{C}}$.

Let A be a commutative algebra with unit, B a unitary A-algebra. The category of representations of B in A-modules, i.e. the category of B-modules, will be a monoidal category if the algebra B is equipped with additional structures [Dr]. Suppose $\Delta : B \to B \otimes_A B$ is an algebra morphism, a comultiplication, and $\Phi \in B^{\otimes 3}$ is an invertible element. Suppose Δ and Φ satisfy the conditions

$$(id \otimes \Delta)(\Delta(b)) \cdot \Phi = \Phi \cdot (\Delta \otimes id)(\Delta(b)), \quad b \in B, \quad (3)$$

$$(id^{\otimes 2} \otimes \Delta)(\Phi) \cdot (\Delta \otimes id^{\otimes 2})(\Phi) = (1 \otimes \Phi) \cdot (id \otimes \Delta \otimes id)(\Phi) \cdot (\Phi \otimes 1). \quad (4)$$

We define a tensor product functor which we will denote $\otimes_{\mathcal{C}}$ for $\mathcal{C}$ the category of B modules or simply $\otimes$ when there can be no confusion in the following way: given B-modules M, N $M \otimes_{\mathcal{C}} N = M \otimes_A N$ as an A-module with the action of B defined as $b(m \otimes n) = b_1 m \otimes b_2 n$ where $b_1 \otimes b_2 = \Delta(b)$. The element Φ gives an associativity constraint $\Phi : (M \otimes N) \otimes P \to M \otimes (N \otimes P), (m \otimes n) \otimes p \mapsto \Phi_1 m \otimes (\Phi_2 n \otimes \Phi_3 p)$, where $\Phi_1 \otimes \Phi_2 \otimes \Phi_3 = \Phi$. By virtue of (3) Φ induces an isomorphism of B-modules, and by virtue of (4) the pentagonal identity (1) holds. We call the triple (B, Δ, Φ) a Drinfeld algebra. Thus, the category $\mathcal{C}$ of B-modules for B a Drinfeld algebra becomes a monoidal category. When it becomes necessary to be more explicit we shall denote $\mathcal{C}(B, \Delta, \Phi)$.

Let (B, Δ, Φ) be a Drinfeld algebra and $F \in B^{\otimes 2}$ an invertible element. Put

$$\tilde{\Delta}(b) = F\Delta(b)F^{-1}, \quad b \in B, \tag{5}$$

and

$$\tilde{\Phi} = (1 \otimes F) \cdot (id \otimes \Delta)(F) \cdot \Phi \cdot (\Delta \otimes id)(F^{-1}) \cdot (F \otimes 1)^{-1}. \tag{6}$$

Then $\tilde{\Delta}$ and $\tilde{\Phi}$ are satisfying (3) and (4), therefore the triple $(B, \tilde{\Delta}, \tilde{\Phi})$ also becomes a Drinfeld algebra which generates the corresponding monoidal category $\tilde{\mathcal{C}}(B, \tilde{\Delta}, \tilde{\Phi})$. Note that the categories $\mathcal{C}$ and $\tilde{\mathcal{C}}$ consist of the same objects as B-modules, and the tensor products of two objects coincide as A-modules. The categories $\mathcal{C}$ and $\tilde{\mathcal{C}}$ will be equivalent. The equivalence $\mathcal{C} \to \tilde{\mathcal{C}}$ is given by the pair $(\alpha, \beta) = (Id, F)$, where $Id : \mathcal{C} \to \tilde{\mathcal{C}}$ is the identity functor of the categories (considered without the monoidal structures, but only as categories of B-modules), and $F : M \otimes_{\mathcal{C}} N \to M \otimes_{\tilde{\mathcal{C}}} N$ is defined by $m \otimes n \mapsto F_1 m \otimes F_2 n$ where $F_1 \otimes F_2 = F$. By virtue of (5) F gives an isomorphism of B-modules, and (6) implies the commutativity of diagram (2).

Assume M is a B-module with a multiplication $\mu : M \otimes_A M \to M$ which is a homomorphism of A-modules. We say that μ is invariant with respect to B and Δ if it is a morphism in the monoidal category $\mathcal{C}(B, \Delta, \Phi)$. This means that

$$b\mu(x \otimes y) = \mu\Delta(b)(x \otimes y) \quad \text{for } b \in B, \ x, y \in M. \tag{7}$$

When μ is $\mathcal{C}$-associative, $\mathcal{C} = \mathcal{C}(B, \Delta, \Phi)$, then we shall also say that μ is a Φ-associative multiplication, i.e. we have the equality

$$\mu(\mu \otimes id)(x \otimes y \otimes z) = \mu(id \otimes \mu)\Phi(x \otimes y \otimes z) \quad \text{for } x, y, z \in M. \tag{8}$$

Since the pair (Id, F) realizes an equivalence of the categories, the multiplication $\tilde{\mu} = \mu F^{-1} : M \otimes_A M \to M$ will be $\tilde{\Phi}$-associative and invariant in the category $\tilde{\mathcal{C}}$.

Now we return to the situation of Section 2. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with a fixed Cartan decomposition and an involution σ. Let $U\mathfrak{g}$ be the universal enveloping algebra with the usual comultiplication $\Delta : U\mathfrak{g} \to U\mathfrak{g}^{\otimes 2}$ generated as a morphism of algebras by the equations $\Delta(x) = 1 \otimes x + x \otimes 1$ for $x \in \mathfrak{g}$ and extended multiplicatively.

We will deal with the category $\mathrm{Rep}(U\mathfrak{g})$. Objects of this category are representations of $U\mathfrak{g}[[h]]$ in $\mathbb{C}[[h]]$-modules of the form $E[[h]]$ for some vector space E. We denote here by $E[[h]]$ the set of formal power series in an indeterminate h with coefficients in E. By tensor product of two $\mathbb{C}[[h]]$-modules we mean the completed tensor product in h-adic topology, i.e. for two vector spaces E_1 and E_2 we have $E_1[[h]] \otimes E_2[[h]] = (E_1 \otimes_{\mathbb{C}} E_2)[[h]]$. As usual, morphisms in this category are morphisms of $\mathbb{C}[[h]]$-modules that commute with the action of $U\mathfrak{g}[[h]]$. A representation of $U\mathfrak{g}[[h]]$ on $E[[h]]$ can be given by a power series $R_h = R_0 + hR_1 + \cdots + h^n R_n + \cdots \in End(E)[[h]]$ where R_0 is a $\mathbb{C}$ representation of $U\mathfrak{g}$ in E and $R_i \in \mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{g}), End(E))$. Hence, R_h may be considered as a deformation of R_0. By misuse of language, we will say that R_h is a representation of $U\mathfrak{g}$ in the space E.

Since the comultiplication Δ on $U\mathfrak{g}$ gives rise to a comultiplication on $U\mathfrak{g}[[h]]$ and is coassociative, the triple $(U\mathfrak{g}[[h]], \Delta, 1 \otimes 1 \otimes 1 = \mathbf{1})$ becomes a Drinfeld algebra and the category $\mathrm{Rep}(U\mathfrak{g})$ turns into a monoidal category $\mathrm{Rep}(U\mathfrak{g}, \Delta, \mathbf{1})$ with the identity associativity constraint. This is the classical way to introduce a monoidal structure in the category $\mathrm{Rep}(U\mathfrak{g})$. Another possibility arises from the theory of quantum groups due to Drinfeld. In the following proposition we suppose that the element $\varphi = [r, r]_{Sch}$ is invariant under the involution σ.

Proposition 3.1 *1. There is an invariant element* $\Phi_h \in U\mathfrak{g}[[h]]^{\otimes 3}$ *of the form* $\Phi_h = 1 \otimes 1 \otimes 1 + h^2\varphi + \cdots$ *satisfying the following properties:*

a) it depends on h^2, *i.e.* $\Phi_h = \Phi_{-h}$;

b) it satisfies the equations (3) and (4) with the usual Δ;

c) $\Phi_h^{-1} = \Phi_h^{321}$, *where* $\Phi^{321} = \Phi_3 \otimes \Phi_2 \otimes \Phi_1$ *for* $\Phi = \Phi_1 \otimes \Phi_2 \otimes \Phi_3$;

d) Φ_h *is invariant under the Cartan involution* θ *and* σ;

e) $\Phi_h \Phi_h^s = 1$, *where* s *is an antiinvolution of* $U\mathfrak{g}$ *such that* $s(x) = -x$ *for* $x \in \mathfrak{g}$, *an antiautomorphism of* $\mathfrak{g}$, *and* $\Phi_h^s = (s \otimes s \otimes s)(\Phi_h)$.

2. There is an element $F_h \in U\mathfrak{g}[[h]]^{\otimes 2}$ of the form $F_h = 1 \otimes 1 + hr + \cdots$ satisfying the following properties:

a) it satisfies the equation (6) with the usual Δ and with $\tilde{\Phi} = 1 \otimes 1 \otimes 1$;

b) it is invariant under the Cartan subalgebra $\mathfrak{c}$;

c) $F_{-h} = F_h^\theta = F_h^{21}$;

d) $F_h(F_h^s)^{21} = 1$

Proof Existence and properties a)–c) for Φ are proven by Drinfeld [Dr]. From his proof which is purely cohomological it is seen that Φ can be chosen invariant under all those automorphisms under which the element φ is invariant. This proves 1 d). 1 e) can also be deduced from the cohomologcal construction.

Existence and the property a) for F are also proven by Drinfeld [Dr]. In his proof he used the explicit existence of the Drinfeld–Jimbo quantum group $U_h\mathfrak{g}$. A purely cohomological construction of F, not assuming the existence of the Drinfeld–Jimbo quantum group, and establishing the properties listed in 2 b)–2 d) is given in [DS]. ∎

So, we obtain two nontrivial Drinfeld algebras: $(U\mathfrak{g}, \Delta, \Phi)$ with the usual comultiplication and Φ from Proposition 3.1, and $(U\mathfrak{g}, \tilde{\Delta}, id)$ where $\tilde{\Delta}(x) = F_h\Delta(x)F_h^{-1}$ for $x \in U\mathfrak{g}$. The corresponding monoidal categories $\mathrm{Rep}(U\mathfrak{g}, \Delta, \Phi)$ and $\mathrm{Rep}(U\mathfrak{g}, \tilde{\Delta}, \mathbf{1})$ are isomorphic, the isomorphism is given by the pair (Id, F_h). Note that the bialgebra $(U\mathfrak{g}[[h]], \tilde{\Delta})$ is coassociative one and is isomorphic to Drinfeld-Jimbo quantum group $U_h\mathfrak{g}$. So that the category $\mathrm{Rep}(U\mathfrak{g}, \tilde{\Delta}, \mathbf{1})$ with the trivial associativity constraint is called the category of representation of quantum group. Underline ones again that if "to forget" of monoidal structures all three categories are isomorphic to the category $\mathrm{Rep}(U\mathfrak{g})$.

Remark. Define for the category $\mathrm{Rep}(U\mathfrak{g}, \Delta, \Phi)$ a category Rep' with the reversed tensor product, $V \otimes' W = W \otimes V$, and the associstivity constraint $\Phi'((V \otimes' W) \otimes' U) = \Phi^{-1}(U \otimes (W \otimes V))$. Denote by $S : V \otimes W \to W \otimes V$ the usual permutation, $v \otimes w \mapsto w \otimes v$, which we will perceive as a mapping $V \otimes W \to V \otimes' W$. Then the condition 1 c) for Φ implies that the pair (Id, S) defines an equivalence of the categories $\mathrm{Rep}(U\mathfrak{g}, \Delta, \Phi)$ and Rep'.

The antiinvolution s defines an antipode on the bialgebra $U\mathfrak{g}$. The existance of the antipode together with the property 1 e) for Φ_h makes $\mathrm{Rep}(U\mathfrak{g}, \Delta, \Phi)$

into a rigid monoidal category, and the equivalence between $\mathrm{Rep}(U\mathfrak{g}, \Delta, \Phi)$ and Rep′ will be an equivalence between rigid monoidal categories.

The property 2 c) for F_h provides the equivalence of the categories $\mathrm{Rep}(U\mathfrak{g}, \Delta, \Phi)$ and $\mathrm{Rep}(U\mathfrak{g}, \tilde{\Delta}, \mathbf{1})$ as rigid monoidal categories (see [DS] for more details).

4 Quantization

Let A be the sheaf of smooth functions on a smooth manifold M. Let $\mathrm{Diff}(M)$ be the sheaf of linear differential operators on M. A $\mathbb{C}$-linear mapping $\lambda : \otimes^n_{\mathbb{C}} A \to A$ is called n-differential if there exists an element $\hat{\lambda} \in \otimes^n_A \mathrm{Diff}(M)$ such that $\lambda(a_1 \otimes \ldots \otimes a_n) = \hat{\lambda}_1 a_1 \cdot \hat{\lambda}_2 a_2 \cdots \hat{\lambda}_n a_n$, where $\hat{\lambda} = \hat{\lambda}_1 \otimes \cdots \otimes \hat{\lambda}_n$ (summation understood). It is easy to see that the element $\hat{\lambda}$ is uniquely determined by the form λ. We say that λ is "null on constants", if $\lambda(a_1 \otimes \ldots \otimes a_n) = 0$ in case one of a_i is a constant. Such λ is presented by $\hat{\lambda} \in \otimes^n_A \mathrm{Diff}(M)_0$ where $\mathrm{Diff}(M)_0$ denotes differential operators which are zero on constants. From now on we only consider n-differential forms that are zero on the constants. These forms are elements of the space of differencial n-cochains of the Hochschild complex of the algebra A. Denote by $H^n(A)$ the Hochschild cohomology defined by the complex of such spaces.

It is known that the space $H^n(A)$ is isomorphic to the space of the antisymmetric n-vector fields on M. Suppose that a group G acts on M and there exists a G-invariant connection on M. In this case Lichnerowicz proved ([Li]) for $n \leq 3$ that $H^n_G(A)$ is isomorphic to the space of the G-invariant antisymmetric n-vector fields on M. Here $H_G(A)$ is the cohomology of the subcomplex of G-invariant cochains.

We will consider forms $\lambda_h : A[[h]]^{\otimes 2} \to A[[h]]$ given by power series from $\mathrm{Diff}(M)^{\otimes 2}[[h]]$ of the form $D_1 \otimes D_2 = 1 \otimes 1 + \sum h^i \lambda_{1i} \otimes \lambda_{2i}$. It means that $\lambda_h(a, b) = D_1 a \cdot D_2 b$ for $a, b \in A$. So, $\lambda_0(a, b) = ab$. We will also write $\lambda_h : A^{\otimes 2} \to A$. The form $\mu_h : A^{\otimes 2}$ is called equivalent to λ_h if there exists a form $\xi_h : A \to A$, $\xi_h = 1 + \sum h^i \xi_i$ such that $\mu_h(a \otimes b) = \xi_h^{-1} \lambda_h(\xi_h a \otimes \xi_h b)$.

Let M be a symmetric space, as in Section 2. Consider the space $A = C^\infty(M)$ as an object of the category $\mathrm{Rep}(U\mathfrak{g}, \Delta, \Phi_h)$ where Φ_h is from Proposition 3.1.

Proposition 4.1 *There is a multiplication μ_h on A with the properies:*

a) μ_h is Φ_h-associative, i.e.

$$\mu_h(\mu_h \otimes id)(a \otimes b \otimes c) = \mu_h(id \otimes \mu_h)\Phi_h(a \otimes b \otimes c), \quad a, b, c \in A;$$

b) μ_h has the form

$$\mu_h(a \otimes b) = ab + \sum_{i \geq 4} h^i \mu_i(a \otimes b),$$

where μ_i are two-differential cochains, null on constants. Moreover, μ_h depends only on h^2, i.e. $\mu_h = \mu_{-h}$;

c) μ_h is invariant under $\mathfrak{g}$ and τ;

d) μ_h is commutative, i.e.

$$\mu_h(a \otimes b) = \mu_h(b \otimes a).$$

The multiplication with such properties is unique up to equivalence.

Proof We use arguments from [Li], proceeding by induction. We may put $\mu_1 = \mu_2 = 0$, because the usual multiplication $m(a \otimes b) = ab$ satisfies a) modulo h^4. This follows because Φ_h is a series in h^2 and the h^2-term $\varphi = 0$ on M. Suppose we have constructed μ_i for even $i < n$, such that $\mu_h^n = \sum_{\text{even} i<n} \mu_i h^i$ satisfies a)–d) modulo h^n. Then,

$$\mu_h^n(\mu_h^n \otimes id) = \mu_h^n(id \otimes \mu_h^n)\Phi_h + h^n \eta \mod h^{n+2}, \tag{1}$$

where η is an invariant three-form.

The following direct computation using the pentagon identity for Φ_h shows that η is a Hochschild cocycle. By definition

$$d\eta = m(id \otimes \eta) - \eta(m \otimes id^{\otimes 2} + \eta(id \otimes m \otimes id) - \eta(id^{\otimes 2} \otimes m) + m(\eta \otimes id).$$

Using (1) and calculating modulo h^{n+2} we can replace m with μ_h^n. Furthermore, the G-invariance of μ_h^n implies that

$$\begin{aligned}
\Phi(\mu_h^n \otimes id^{\otimes 2}) &= (\mu_h^n \otimes id^{\otimes 2})(\Delta \otimes id^{\otimes 2})\Phi, \\
\Phi(id \otimes \mu_h^n \otimes id) &= (id \otimes \mu_h^n \otimes id)(id \otimes id \otimes \Delta)\Phi, \\
\Phi(id^{\otimes 2} \otimes \mu_h^n) &= (id^{\otimes 2} \otimes \mu_h^n)(id^{\otimes 2} \otimes \Delta)\Phi.
\end{aligned}$$

Therefore we have the following equations modulo h^{n+2},

$$\mu_h^n(id\otimes\mu_h^n)(id\otimes\mu_h^n\otimes id)-\mu_h^n(id\otimes\mu_h^n)(id^{\otimes 2}\otimes\mu_h^n)(1\otimes\Phi_h)=h^n m(id\otimes\eta)$$
$$\mu_h^n(\mu_h^n\otimes id)(\mu_n^h\otimes id^{\otimes 2})-\mu_h^n(id\otimes\mu_h^n)(\mu_h^n\otimes id^{\otimes 2})(\Delta\otimes id^{\otimes 2})(\Phi_h)=h^n\eta(m\otimes id^{\otimes 2})$$
$$\mu_h^n(\mu_h^n\otimes id)(id\otimes\mu_h^n\otimes id)-\mu_h^n(id\otimes\mu_h^n)(id\otimes\mu_h^n\otimes id)(id\otimes\Delta\otimes id)(\Phi_h)$$
$$=h^n\eta(id\otimes m\otimes id)$$
$$\mu_h^n(\mu_h^n\otimes id)(id^{\otimes 2}\otimes\mu_h^n)-\mu_h^n(id\otimes\mu_h^n)(id^{\otimes 2}\otimes\mu_h^n)(id^{\otimes 2}\otimes\Delta)(\Phi_h)=h^n\eta(id^{\otimes 2}\otimes m)$$
$$\mu_h^n(\mu_h^n\otimes id)(\mu_h^n\otimes id^{\otimes 2})-\mu_h^n(\mu_h^n\otimes id)(id\otimes\mu_h^n\otimes id)(\Phi_h\otimes 1)=h^n m(\eta\otimes id).$$

Since the equations are congruences modulo h^{n+2} and $h^n\Phi=1 \mod h^{n+2}$ the equations remain valid if we multiply on the left by any expression in Φ and leave the right side unchanged. Multiply the left side of the first equation by $((id\otimes\Delta\otimes id)\Phi)(\Phi\otimes 1)$, the left side of the third equation by $\Phi\otimes 1$, the left side of the fourth equation by $(\Delta\otimes id\otimes id)\Phi$, leave the remaining equations unchanged, then add the five equations with alternating signs. Using the pentagon identity in Φ and the identity $(\mu_h^n\otimes id)(id\otimes id\otimes\mu_h^n)=\mu_h^n\otimes\mu_h^n=(id\otimes\mu_h^n)(\mu_h^n\otimes id\otimes id)$, we conclude that $d\eta=0$.

Since $\mathfrak{g}$ is semisimple the cochains invariant under $\mathfrak{g}$ and τ form a subcomplex which is a direct summand. The arguments from the proof of Proposition 2.1 show that there are no three-vector fields on M invariant under $\mathfrak{g}$ and $\dot{\tau}$. Hence the cohomology of this subcomplex is equal to zero, i.e. η is a coboundary. Further, there is a $\mathfrak{g}$ and τ invariant connection on M (see [He] 4.A.1). The property $\Phi_h^{-1}=\Phi_h^{321}$ and commutativity of μ_i imply that $\eta(a\otimes b\otimes c)=\eta(c\otimes b\otimes a)$. It follows from this that there is an invariant commutative two-form μ_n such that $d\mu_n=\eta$, which gives that $\mu_h^n+h^n\mu_n$ satisfies to a)–d) modulo h^{n+2}. Therefore, proceeding step-by-step we can build the multiplication μ_h.

Uniqueness of such a multiplication can be proven by the similar cohomological arguments. ∎

Now we suppose that on the algebra A there is a $\mathfrak{g}$ and τ invariant multiplication $\mu_\nu:(A\otimes A)[[\nu]]\to A[[\nu]]$ which is associative in the usual sense and such that $\mu_0=m$ where m is the usual multiplication on A. Denote by A_ν the corresponding algebra. Let $H^n(A_\nu)$ be the Hochschild cohomology of this algebra. Since $H^3_{G,\tau}(A_0)=0$ it is easy to see that $H^3_{G,\tau}(A_\nu)=0$ as well. It allows us, using the arguments from the proof of Proposition 4.1, to state the following

Proposition 4.2 *There is a multiplication $\mu_{\nu,h}$ on A depending on two formal variables with the properies:*

a) $\mu_{\nu,h}$ is Φ_h-associative, i.e.

$$\mu_{\nu,h}(\mu_{\nu,h}\otimes id)(a\otimes b\otimes c)=\mu_{\nu,h}(id\otimes\mu_{\nu,h})\Phi_h(a\otimes b\otimes c),\quad a,b,c\in A;$$

b) $\mu_{\nu,h}$ has the form

$$\mu_{\nu,h}(a\otimes b)=\mu_\nu(a\otimes b)+\sum_{i\geq 4}h^i\mu_{\nu,i}(a\otimes b),$$

where $\mu_{\nu,i}:(A\otimes A)[[\nu]]\to A[[\nu]]$ are two-differential forms being null on constants. Moreover, $\mu_{\nu,h}$ depends only of h^2, i.e. $\mu_{\nu,h}=\mu_{\nu,-h}$;

c) $\mu_{\nu,h}$ is invariant under $\mathfrak{g}$ and τ;

d) $\mu_{0,h}$ coincides with μ_h from Proposition 4.1.

The multiplication with such properties is unique up to equivalence.

The multiplication μ_ν exists when M is a Hermitian symmetric space. In this case μ_ν can be constructed as the deformation quantization of the Poisson bracket $\{\cdot,\cdot\}_{inv}$ which is the dual to the imaginary part of Hermitian form on M. Such a quantization also can be given using the arguments of Proposition 4.1 and has the form

$$\mu_\nu(a,b)=ab+\frac{1}{2}\nu\{a,b\}_{inv}+o(\nu).$$

Remark. Given μ_ν, the multiplication $\mu_{\nu,h}$ can be constructed directly using μ_h from Proposition 4.1. Namely, let $\hat{\mu}_\nu$ and $\hat{\mu}_h$ are bidifferential operators which present μ_ν and μ_h, respectively. Thanks to the invariance of these operators it is easy to show that the bidifferential operator $\hat{\mu}_{\nu,h}=\hat{\mu}_\nu\cdot\hat{\mu}_h$ can be taken as a presenting one for $\mu_{\nu,h}$ which satisfies Proposition 4.2. Indeed, let Δ denotes a comultiplication which appears on the algebra $\mathrm{Diff}(M)$ due to the G-invariant connection on M. This is an A-linear morphism of algebras, $\Delta:\mathrm{Diff}(M)\to\mathrm{Diff}(M)\otimes_A\mathrm{Diff}(M)$. Then the associativity conditions for μ_ν and μ_h can be expressed through the corresponding bidifferential operators as follows (we again use the notations $\hat{\mu}_\nu=\hat{\mu}_{\nu 1}\otimes\hat{\mu}_{\nu 2}$ and so on):

$$(\Delta\hat{\mu}_{\nu 1}\otimes\hat{\mu}_{\nu 2})(\hat{\mu}_\nu\otimes 1)=(\hat{\mu}_{\nu 1}\otimes\Delta\hat{\mu}_{\nu 2})(1\otimes\hat{\mu}_\nu)\qquad(2)$$

$$(\Delta\hat{\mu}_{h1}\otimes\hat{\mu}_{h2})(\hat{\mu}_h\otimes 1)=(\hat{\mu}_{h1}\otimes\Delta\hat{\mu}_{h2})(1\otimes\hat{\mu}_h)\Phi_h.\qquad(3)$$

Now observe that the operators $\hat{\mu}_\nu \otimes 1$ and $\Delta\hat{\mu}_{h1} \otimes \hat{\mu}_{h2}$ commute. Indeed, one can assume that in the presentation $\hat{\mu}_h = \hat{\mu}_{h1} \otimes \hat{\mu}_{h2}$ (summation understood) operators $\hat{\mu}_{h1}$ are of the form $\rho(x)$ where $x \in U\mathfrak{g}$ and $\rho : U\mathfrak{g} \to \mathrm{Diff}(M)$ induced by the action of G on M. So that the commutativity follows from the G-invariance of $\hat{\mu}_\nu$. Similarly, the operators $1 \otimes \hat{\mu}_\nu$ and $\hat{\mu}_{h1} \otimes \Delta\hat{\mu}_{h2}$ commute. Now, taking the products of left and right sides of (2) and (3), respectively, and using just proved commutativity, we obtain

$$(\Delta\hat{\mu}_{\nu,h1} \otimes \hat{\mu}_{\nu,h2})(\hat{\mu}_{\nu,h} \otimes 1) = (\hat{\mu}_{\nu,h1} \otimes \Delta\hat{\mu}_{\nu,h2})(1 \otimes \hat{\mu}_{\nu,h})\Phi_h,$$

which equivalent to Φ-commutativity of $\mu_{\nu,h}$.

Such a construction of $\mu_{\nu,h}$ gives a more detailed information about this multiplication. For example, if we set $[a,b]_\nu = \mu_\nu(a,b) - \mu_\nu(b,a)$ and $[a,b]_{\nu,h} = \mu_{\nu,h}(a,b) - \mu_{\nu,h}(b,a)$ for $a, b \in A$, then

$$[a,b]_{\nu,h} = [\ ,\ ]_\nu \hat{\mu}_h(a \otimes b). \tag{4}$$

Suppose now that A with the multiplication μ_ν has a trace, i.e. a functional $tr : A \to \mathbb{C}$ which satisfies the property

$$tr(\mu_\nu(a,b)) = tr(\mu_\nu(b,a)) \quad \text{for } a, b \in A.$$

It follows from (4) that the same trace will be also satisfy the property

$$tr(\mu_{\nu,h}(a,b)) = tr(\mu_{\nu,h}(b,a)) \quad \text{for } a, b \in A,$$

i.e. it respects the new multiplication.

Now let us consider $A = C^\infty(M)$ as an object of the category $\mathrm{Rep}(U\mathfrak{g}, \tilde{\Delta}, \mathbf{1})$ of representations of the Drinfeld–Jimbo quantum group $U_h\mathfrak{g}$. As we have seen in Section 3, the multiplications μ_h and $\mu_{\nu,h}$ can be transfered to this category in the following way:

$$\tilde{\mu}_h = \mu_h F_h^{-1}$$

$$\tilde{\mu}_{\nu,h} = \mu_{\nu,h} F_h^{-1}.$$

We may obviously assume that F_h has the form

$$F_h = 1 \otimes 1 - \frac{1}{2}h\{\cdot,\cdot\} + o(h).$$

Then we have the following

Theorem 4.3 *Let M be a symmetric space over a semisimple Lie group. Then the multiplications $\tilde{\mu}_h$ and $\tilde{\mu}_{\nu,h}$ (the second exists when M is a Hermitian symmetric space) satisfy the following properties:*

a) $\tilde{\mu}_h$ and $\tilde{\mu}_{\nu,h}$ are associative;

b) $\tilde{\mu}_h$ and $\tilde{\mu}_{\nu,h}$ have the form

$$\mu_h(a \otimes b) = ab + \frac{1}{2}h\{a,b\} + o(h)$$

$$\mu_{\nu,h}(a \otimes b) = ab + \frac{1}{2}(h\{a,b\} + \nu\{a,b\}_{inv}) + o(h,\nu)$$

c) $\tilde{\mu}_h$ and $\tilde{\mu}_{\nu,h}$ are invariant under $U_h\mathfrak{g}$;

d) $\tilde{\mu}_{0,h}$ coincides with $\tilde{\mu}_h$;

e) Let $\tilde{S} = F_h S F_h^{-1}$ where S denotes the usual transposition, $S(a \otimes b) = b \otimes a$ for $a, b \in A$. Then $\tilde{\mu}_h$ is $\tilde{S}$-commutative:

$$\tilde{\mu}_h(a \otimes b) = \tilde{\mu}_h \tilde{S}(b \otimes a) \quad \textit{for } a, b \in C^\infty(M).$$

The multiplications with such properties are unique up to equivalence.

Remarks. 1. If a trace tr is well defined for the multiplication $\mu_{\nu,h}$ (see the previous Remark), then setting $\tilde{tr}(a,b) = tr(F_h(a \otimes b)F_h^{-1}$ we obtain a trace for $\tilde{\mu}_{\nu,h}$ with the property

$$\tilde{tr}(\tilde{\mu}_{\nu,h}(a,b)) = \tilde{tr}(\tilde{\mu}_{\nu,h}(\tilde{S}(a,b))).$$

2. The action of the real Lie group G and τ on M induces an action on $C^\infty(M)[[h]]$. It follows from Propositions 4.1 and 4.2 that μ_h and $\mu_{\nu,h}$ are invariant under G and τ. This implies that $\tilde{\mu}_h$ and $\tilde{\mu}_{\nu,h}$ will be invariant under a "quantized" action of G and τ. This new action appears by taking of tensor products of $C^\infty(M)$. Namely, let g be either an element of G or $g = \tau$, then for $a, b \in C^\infty(M)$ define $g \circ_h a = g \circ a$, $g \circ_h (a \otimes b) = F_h(g \otimes g)F_h^{-1}(a \otimes b)$, where $\circ$ denotes the usual action. The multiplications $\tilde{\mu}_h$ and $\tilde{\mu}_{\nu,h}$ are invariant under this quantized action, i.e., for example,

$$g \circ_h \tilde{\mu}_{\nu,h}(a,b) = \tilde{\mu}_{\nu,h} g \circ_h (a \otimes b).$$

3. We may consider a complex symmetric space $M = G/H$, where G is a complex semisimple Lie group and H a complex subgroup. As above, one

can construct the multiplications μ_h and $\tilde{\mu}_h$ on the space $C^\infty(M)$ that also will give a multiplication on the space of holomorphic functions on M. The previous remark remains valid for the complex group G.

In particular, the group G itself may be considered as a symmetric space, $G = (G \times G)/D$ where D is the diagonal. The action of $G \times G$ on G is $(g_1, g_2) \circ g = g_1 g g_2^{-1}$, $(g_1, g_2) \in G \times G$, $g \in G$. In this case $\sigma(g_1, g_2) = (g_2, g_1)$, $\tau(g) = g^{-1}$. For $\bar{r}$ to be σ-invariant R-matrix on the Lie algebra $\bar{\mathfrak{g}} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of $G \times G$ it must be of the form $\bar{r} = (r, r) \in \wedge^2 \mathfrak{g}_1 \oplus \wedge^2 \mathfrak{g}_2 \subset \wedge^2(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$ where the Lie subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ correspond to $(G \times 1)$ and $(1 \times G)$. In this example $U\bar{\mathfrak{g}} = (U\mathfrak{g})^{\otimes 2} \supset U\mathfrak{g} \oplus U\mathfrak{g}$ and the elements $\bar{\Phi}_h$ and $\bar{F}_h$ have the forms $\bar{\Phi}_h = (\Phi_h, \Phi_h)$ and $\bar{F}_h = (F_h, F_h)$ corresponding to $\bar{r}$ where Φ_h and F_h are the elements from $U\mathfrak{g}$ appropriated to r. Then, $\rho(\bar{\Phi}_h) = id$, so that for μ_h one can take the usual multiplication m on $C^\infty(G)$, and $\tilde{\mu}_h(a, b) = m(F_h(a \otimes b)F_h^{-1})$. Therefore, $C^\infty(G)$ may be considered as an algebra (and even bialgebra) in the category $\mathrm{Rep}((U\mathfrak{g})^{\otimes 2}, \tilde{\Delta}, \mathbf{1})$ with the multiplication $\tilde{\mu}_h$ (and the comultiplication $a \mapsto F_h\Delta(a)$ where Δ is the usual comultiplication).

References

[SS] S.Shnider and S.Sternberg, "Quantum groups", Int. Press, 1993.

[DGM] J.Donin, D.Gurevich, and Sh.Majid, *R-matrix brackets and their quantisation*, Ann.de l'Institut d'Henri Poincaré, 58 (1993), pp.235-246.

[DG1] J.Donin and D.Gurevich, *Quantization of R-matrix structures*, Selecta Mathematica Sovietica, v.12, n.1(1993), 37–41.

[DG2] J.Donin and D.Gurevich, *Some Poisson structures associated to Drinfeld-Jimbo R-matrix and their quantization*, Israel Math. J., to appear.

[KRR] S.Khoroshkin, A.Radul, and V.Rubtsov, *A family of APoisson structures on compact Hermitian symmetric spaces*, Comm. Math. Phys., v. 152(1993), 299–316.

[GP] D.Gurevich and D.Panyushev, *On Poisson pairs associated to modified R-matrices*, Duke Math. J., v. 73, n. 1 (1994).

[Dr] V.Drinfeld, *Quasi-Hopf algebra*, Leningrad Math.J. 1 (1990), pp.1419-1457.

[He] S.Helgason, "Differential geometry, Lie groups, and symmetric spaces", Academic Press, 1978.

[Li] A.Lichnerowicz, *Déformations d'algèbres associées à une variété symplectique (les $*_\nu$-produits)*, Ann. Inst. Fourier 32, 1(1982), 157-209.

[DS] J.Donin and S.Shnider, *Cohomological construction of quantum groups*, to appear.