On the group of homotopy equivalences of simply connected five manifolds

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The group $\mathcal{E}(X)$ of homotopy equivalences of a space X is the set of homotopy classes of homotopy equivalences $X \to X$. The group structure is induced by map-composition. The group $\mathcal{E}(X)$, i.e. the group of automorphisms of the homotopy type of X, can be regarded as the homotopy symmetry group of the space X. In the literature there has been a lot of interest in the computation of such groups, compare for example the excellent survey article of M.Arkowitz [1]. For a differential manifold M the group $\mathcal{E}(M)$ is comparable with the group $\pi_0 \mathit{Diff}(M)$ of isotopy classes of diffeomorphisms of M. In fact, via the J-homomorphism there is a striking similarity between these groups as is shown in Baues [5] §10.

In this paper we study the group $\mathcal{E}(M)$ where M is a simply connected closed 5-manifold, or more generally a simply connected Poincaré complex of dimension 5. Such manifolds and Poincaré complexes are classified by Barden [2] and Stöcker [16] respectively.

Let $\dot{M} = M - \mathring{e}$ be the complement of a small open cell $\mathring{e}$ in M and let $\mathcal{E}(M|\dot{M})$ be the subgroup of $\mathcal{E}(M)$ which can be represented by an orientation preserving map $M \to M$ which is the identity on $\dot{M}$. In Baues [5] it is shown that $\mathcal{E}(M|\dot{M})$ is a finite abelian group for any simply connected Poincaré complex M and that one has the "fundamental extension" of groups

$$(*) \qquad 0 \longrightarrow \mathcal{E}(M|\dot{M}) \rightarrowtail \mathcal{E}(M) \twoheadrightarrow \mathcal{E}(\dot{M}, \pm f) \longrightarrow 0$$

Here $f : \partial e \to \dot{M}$ is the inclusion of the boundary of e and $\mathcal{E}(\dot{M}, \pm f)$ is the subgroup of elements in $\mathcal{E}(M)$ compatible with f up to sign.

Recently Cochran-Habegger [11] computed the fundamental extension for $dim(M) = 4$, see also Baues [6]. We here compute for $dim(M) = 5$ the abelian group $\mathcal{E}(M|\dot{M})$ as an $\mathcal{E}(\dot{M}, \pm f)$-module and we compute $\mathcal{E}(\dot{M}, \pm f)$ so that $\mathcal{E}(M)$ is determined via $(*)$ up to an extension problem; for special cases we are able to solve the extension problem too. We now describe explicitely some of the results of this paper.

The classification of Barden yields the following simply connected (differential) 5-manifolds M_q and X_q which are indecomposable with respect to connected sums $\sharp$. For a prime power $q = p^i$, $p \geq 2$, the indecomposable manifold M_q is characterized by the second Stiefel-Whitney class and the second homology:

$$\omega_2(M_q) = 0 \quad \text{and} \quad H_2(M_q) = \mathbb{Z}_q \oplus \mathbb{Z}_q \, .$$

Moreover $M_\infty = S^2 \times S^3$ is a product of spheres. The indecomposable manifold X_q is characterized by

$$\omega_2(X_q) \neq 0 \quad \text{and} \quad H_2(X_q) = \begin{cases} \mathbb{Z}_2 & \text{for } q = -1 \\ \mathbb{Z}_q \oplus \mathbb{Z}_q & \text{for } q = 2^i \geq 4 \end{cases}$$

Let X_∞ be the nontrivial 3-sphere bundle over the 2-sphere. The manifold X_{-1} is the Wu manifold and we set $X_2 = X_{-1} \sharp X_{-1}$. Now let $\mathcal{M} = \{M_q \,|\, q = \infty \text{ or } q = p^i \text{ a prime power }\}$ and $\mathcal{X} = \{X_q \,|\, q = \infty \text{ or } q = 2^i \geq 2 \text{ or } q = -1\}$ be the corresponding sets of manifolds. Any simply connected 5-manifold M has the <u>normal form</u>

$$M = M_1 \sharp \ldots \sharp M_r \text{ with } M_1, \ldots, M_{r-1} \in \mathcal{M}$$

and either $M_r \in \mathcal{M}$, $r \geq 0$, for $\omega_2(M) = 0$ or $M_r \in \mathcal{X}$, $r \geq 1$, for $\omega_2(M) \neq 0$; here we set $M = S^5$ for $r = 0$. Compare (10.1) in [16].

<u>Theorem (A):</u> For the manifolds $M = X_q$, M_q the abelian group $\mathcal{E}(M|\dot{M})$ is given by

$$\mathcal{E}(X_q|\dot{X}_q) = \begin{cases} \mathbb{Z}_2 & q = -1, \infty \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & q = 4, 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & q = 2^i \geq 8 \end{cases}$$

$$\mathcal{E}(M_q|\dot{M}_q) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & q = 2, \infty \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & q = 2^i \geq 4 \\ \mathbb{Z}_3 \oplus \mathbb{Z}_3 & q = 3^i \geq 3 \\ 0 & q = p^i,\ p \neq 2, 3 \end{cases}$$

The next result gives an explicit computation of the abelian group $\mathcal{E}(M|\dot{M})$ for any simply connected 5-manifold M. Let $A * B$ be the torsion product of abelian groups A, B.

<u>Theorem (B):</u> For $\omega_2(N) = 0$ and $H_2N = B$ we have an isomorphism of abelian groups

$$\mathcal{E}(N|\dot{N}) \cong B_0 \oplus (B/(B * \mathbb{Z}_2)) \otimes \mathbb{Z}_2 \quad \text{with}$$
$$B_0 = (B * \mathbb{Z}_2) \oplus (B * \mathbb{Z}_3) \oplus (B/\mathrm{Tor}B) \otimes \mathbb{Z}_2 \quad .$$

For $\omega_2(M) \neq 0$ we can choose N as above such that $M = N \sharp X_q$ for appropiate q. Then we get

$$\mathcal{E}(M|\dot{M}) \cong B_0 \oplus B_q \oplus \mathcal{E}(X_q|\dot{X}_q)$$

with B_0 as above and

$$B_q = \begin{cases} (B/(B * \mathbb{Z}_2)) \otimes \mathbb{Z}_2 & \text{for } q = \infty \text{ or } q = 2^i \geq 8\ , \\ (B/(B * \mathbb{Z}_4)) \otimes \mathbb{Z}_4 & \text{for } q = -1 \text{ or } q = 2\ , \\ (B/(B * \mathbb{Z}_8)) \otimes \mathbb{Z}_2 & \text{for } q = 4\ . \end{cases}$$

Let $\mathcal{E}_H(M)$ be the subgroup of elements in $\mathcal{E}(M)$ which induce the identity in Homology. The group $\mathcal{E}_H(M)$ is part of the exact sequence of groups

$$0 \longrightarrow \mathcal{E}_H(M) \longrightarrow \mathcal{E}(M) \xrightarrow{H_*} H_*\mathcal{E}(M) \longrightarrow 0$$

where $H_*\mathcal{E}(M)$ is the group of automorphisms of $H_*(M)$ which are realizable by maps $M \to M$.

<u>Theorem (C):</u> For a simply connected 5-dimensional Poincaré complex M one has an isomorphism of groups

$$H_*\mathcal{E}(M) \cong \mathrm{Aut}(A, \pm b, \omega, e)$$

Here the right hand side denotes the group of automorphisms of $A = H_2M$ compatible with the invariants $\pm b, \omega, e$ of M given by the linking form b of M, the Stiefel-Whitney class $\omega = \omega_2(M)$ and the exotic characteristic class $e = e(M)$. The class $e(M)$ vanishes if and only if M has the homotopy type of a manifold, see §3. We point out that the methods of Stöcker [16] are not suitable for a proof of theorem (C).

We introduce the "torus construction"

$$t : \mathcal{E}_H(M) \longrightarrow \Gamma_6(A)$$

where $\Gamma_6(A)$ for $A = H_2M$ is defined by the free commutative ring with divided powers generated by A. For the homotopy group $\Pi_5\dot{M}$, which we compute explicitely below, let

$$\Sigma\Pi_5\dot{M} \quad = \quad image(\Sigma : \Pi_5\dot{M} \rightarrow \Pi_6\Sigma\dot{M})$$

be the image of the suspension homomorphism. The group $\Sigma\Pi_5\dot{M}$ depends only on the homology group $A = H_2M$.

Theorem (D): For $\omega_2(M) = 0$ one has the following commutative diagram with exact rows.

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Sigma\Pi_5\dot{M} & \longrightarrow & \mathcal{E}(M|\dot{M}) & \longrightarrow & A * \mathbb{Z}_3 & \longrightarrow & 0 & & \\ & & \| & & \cap & & \cap & & & & \\ 0 & \longrightarrow & \Sigma\Pi_5\dot{M} & \longrightarrow & \mathcal{E}_H(M) & \longrightarrow & \Gamma_6 A & \longrightarrow & \Lambda^2(A \otimes \mathbb{Z}_2) & \longrightarrow & 0 \end{array}$$

For example any simply connected 5-dimensional Brieskorn manifold M satisfies $\omega_2(M) = 0$.

Contents:

§1 Homotopy groups of Moore spaces and homology groups of Eilenberg-Mac Lane spaces.

We here describe the homotopy groups $\Pi_n M(A,2)$ of a Moore space $M(A,2)$ and the homology $H_n K(A,2)$ of an Eilenberg-Mac Lane space $K(A,2)$ for low degrees n. For this we have to introduce some purely algebraic functors from abelian groups to abelian groups. These functors are classical, but we describe various properties which are new.

For each abelian group A we have the free ring $\Gamma_*(A)$ with divided powers, this is the commutative graded ring which is generated by the elements $\gamma_t(x)$ for each $x \in A$ and each non-negative integer t, of degree $2t$, subject to the relations

$$(1.1)\qquad \begin{cases} \gamma_0(x) = 1, \quad \gamma_t(rx) = r^t\gamma_t(x) \quad \text{for } r \in \mathbb{Z} \\ \gamma_t(x+y) = \sum\limits_{i+j=t} \gamma_i(x)\gamma_j(y) \\ \gamma_s(x)\gamma_t(x) = \binom{s+t}{s}\gamma_{s+t}(x) \end{cases}$$

Compare §18 [14]. We set $\gamma_1(x) = x$ so that $\Gamma_2 A = A$, moreover $\gamma_2\colon A \to \Gamma_4 A = \Gamma A$ is J.H.C.Whitehead's universal quadratic map. The natural homomorphisms

$$(1.2)\qquad A \otimes A \xrightarrow{[1,1]} \Gamma A \xrightarrow{\sigma} A \otimes Z_2$$

are defined by $[1,1](x \otimes y) = \gamma_2(x+y) - \gamma_2 x - \gamma_2 y = [x,y]$ and by $\sigma\gamma_2(x) = x \otimes 1$ where $1 \in \mathbb{Z}$ is the generator, $x, y \in A$.

The group $\Gamma_6 A$ is generated by the elements $\gamma_2(x) \cdot y$ and $\gamma_3(y)$. Let $\langle\gamma_3 A\rangle \subset \Gamma_6 A$ be the subgroup generated by all elements $\gamma_3(x)$, $x \in A$. We introduce the functor Γ_2^2 by

$$(1.3)\qquad \Gamma_2^2(A) \quad = \quad (\Gamma(A) \otimes \mathbb{Z}_2 \oplus \Gamma(A) \otimes A)/\sim$$

where the relations are given by

$$\begin{cases} \gamma_2(x) \otimes x \quad \sim \quad 0 \\ [x,y] \otimes 1 + (\gamma_2 x) \otimes y + [y,x] \otimes x \quad \sim \quad 0 \end{cases}$$

Next let $L(A,1)_n$ be the group of Lie elements of degree n in the graded tensoralgebra $T(A)$ where A is concentrated in degree 1. The Lie bracket is given by $[x,y] = xy - (-1)^{|x||y|}yx$. For example we have

$$(1.4)\qquad L(A,1)_3 \quad = \quad \mathrm{Im}\{\ [[1,1],1]\colon A^{\otimes 3} \to A^{\otimes 3}\}$$

where $[[1,1],1]$ carries $xyz = x\otimes y\otimes z$ to

$$\begin{aligned}[[x,y],z] &= (xy+yx)z - z(xy+yx)\\ &= xyz + yxz - zxy - zyx \quad \in \quad A^{\otimes 3}\end{aligned}$$

There is a natural isomorphism (see [8])

$$\Gamma_2^2(A) \cong L(A,1)_3 \oplus \Gamma(A)\otimes \mathbb{Z}_2 \tag{1.5}$$

which carries $\gamma_2(x)\otimes y$ to $-[[y,x],x]+[x,y]\otimes 1$ and which carries $\gamma_2(x)\otimes 1$ to $\gamma_2(x)\otimes 1$. Next we have the exact sequence

$$\Gamma(A) \xrightarrow{H} A\otimes A \xrightarrow{q} \Lambda^2(A) \longrightarrow 0 \tag{1.6}$$

where $H(\gamma_2 x) = x\otimes x$ and $w(x\otimes y) = x\wedge y$ for the exterior square $\Lambda^2(A)$. Let $A * \mathbb{Z}_n = \{x\in A,\ nx = 0\}$ be the n-torsion of A. One has the following commutative diagram of natural homomorphisms in which all rows are exact and in which all columns are short exact sequences, see [10].

$$\begin{array}{ccccc}
\langle\gamma_3 A\rangle & \longrightarrow & \Gamma(A)\otimes A & \overset{W}{\twoheadrightarrow} & L(A,1)_3\oplus\Lambda^2(A\otimes\mathbb{Z}_2)\\
\downarrow & & \downarrow & & \downarrow \iota\\
\Gamma_6(A) & \xrightarrow{\chi} & \Gamma(A)\otimes A\oplus\Gamma(A)\otimes\mathbb{Z}_2 & \overset{q}{\twoheadrightarrow} & \Gamma_2^2 A\\
\downarrow \sigma' & & \downarrow & & \downarrow\\
\Lambda^2(A\otimes\mathbb{Z}_2) & \overset{[1,1]}{\rightarrowtail} & \Gamma(A)\otimes\mathbb{Z}_2 & \overset{\sigma}{\twoheadrightarrow} & A\otimes\mathbb{Z}_2
\end{array} \tag{1.7}$$

Here one has an isomorphism $A * \mathbb{Z}_3 \cong \text{kernel}(\chi)$, unnaturally. The map q is the quotient map and χ is defined by

$$\begin{cases}\chi(\gamma_3 x) &= \gamma_2(x)\otimes x\\ \chi(\gamma_2(x)y) &= \gamma_2(x)\otimes y + [y,x]\otimes x + [x,y]\otimes 1\end{cases} \tag{1}$$

We define σ' by $\sigma'\gamma_3(x) = 0$ and $\sigma'(\gamma_2(x)y) = (x\otimes 1)\wedge(y\otimes 1)$ and we set $[1,1](x\otimes 1\wedge y\otimes 1) = [x,y]\otimes 1$. The inclusion i is defined by (1.5) and (1.1). All the other arrows in the diagram are the obvious maps. In particular we obtain W by

$$W(\gamma_2(x)\otimes y) = -[[y,x],x] + (x\otimes 1)\wedge(y\otimes 1) \tag{2}$$

see (1.5). Now exactness, in particular, yields the short exact sequence

(1.8) <u>Lemma:</u> $A * \mathbb{Z}_3 \rightarrowtail \langle\gamma_3 A\rangle \twoheadrightarrow \text{kernel}(W)$.

Let <u>Ab</u> be the category of abelian groups and let F: <u>Ab</u> $\to$ <u>Ab</u> be a functor which needs not to be additive. Then the <u>left derived functors</u>

$$L_n F:\ \underline{\underline{\mathrm{Ab}}} \longrightarrow \underline{\underline{\mathrm{Ab}}} \tag{1.9}$$

are defined, see [12]. For example for $F = \Gamma$ the left derived functor $L_1\Gamma = \Gamma T$: <u>Ab</u> $\to$ <u>Ab</u> is the Γ-torsion considered in III.2.1 of [6].

An Eilenberg-Mac Lane space $K(A,n)$ is a CW-space with homotopy groups $\Pi_n K(A,n) = A$ and $\Pi_j K(A,n) = 0$ for $j \neq n$. A Moore space $M(A,n)$, $n \geq 2$, is a simply connected CW-space with homology groups $H_n M(A,n) = A$ and $\tilde{H}_j M(A,n) = 0$ for $j \neq n$. Using the work of Eilenberg-Mac Lane we have the natural isomorphisms

$$(1.10) \qquad H_i K(A,2) = \begin{cases} A & i=2 \\ 0 & i=3 \\ \Gamma(A) & i=4 \\ \Gamma T(A) & i=5 \\ \Gamma_6(A) & i=6 \end{cases}$$

which we use as identifications. Using (1.10) we can identify the coordinates $(\chi_1, \chi_2) = \chi$ with $\chi_2 = [1,1]\sigma'$ in (1.7) with the following natural operations. Let Sq_2: $H_n(X) \to H_{n-2}(X, \mathbb{Z}_2)$ be the integral Steenrod square and let $k_A \cap$ be the cap product with the canonical element $k_A \in H^2(K(A,2),A)$. Then we have

$$(1.11) \qquad \begin{cases} \chi_1: & \Gamma_6 A = H_6 K(A,2) \xrightarrow{k_A \cap} H_4(K(A,2),A) = \Gamma(A) \otimes A \\ \chi_2: & \Gamma_6 A = H_6 K(A,2) \xrightarrow{Sq_2} H_4(K(A,2),\mathbb{Z}_2) = \Gamma(A) \otimes \mathbb{Z}_2 \end{cases}$$

The homotopy groups $\Pi_n M(A,2)$ of the Moore space $M(A,2)$ are more complicated than the homology groups of an Eilenberg-Mac Lane space $K(A,2)$. We have the following connection between these groups. The first k-invariant of $M(A,2)$ is a natural map k: $M(A,2) \to K(A,2)$ in the homotopy category inducing the isomorphism

$$(1.12) \qquad k_*: \Pi_2 M(A,2) \cong \Pi_2 K(A,2) = A$$

which we use as an identification. For a simply connected CW-complex X we have Whitehead's certain exact sequence

$$(1.13) \qquad \ldots \longrightarrow H_{n+1}X \xrightarrow{b} \Gamma_n X \xrightarrow{i} \Pi_n X \xrightarrow{h} H_n X \xrightarrow{b} \Gamma_{n-1} X \longrightarrow \ldots$$

where h is the Hurewicz map and $\Gamma_n X = Im\{\Pi_n X^{n-1} \to \Pi_n X^n\}$. Since this sequence is natural in X we obtain the natural homomorphism ($n \geq 3$)

$$(1.14) \qquad Q = b^{-1}\Gamma_n(k)i^{-1}: \quad \Pi_n M(A,2) \longrightarrow H_{n+1}K(A,2)$$

induced by k above. Now one has natural isomorphisms

$$(1.15) \qquad \Gamma(A) \overset{\eta^*}{\cong} \Pi_3 M(A,2) \overset{Q}{\cong} H_4 K(A,2)$$

where η^* carries $\gamma_2(X)$ to the composition $x\eta_2$ where η_2 is the Hopf map, $x \in A = \Pi_2 M(A,2)$, compare [18] and [14]. Moreover we have the natural short exact sequence

$$(1.16) \qquad 0 \longrightarrow \Gamma_2^2(A) \xrightarrow{i} \Pi_4 M(A,2) \xrightarrow{Q} \Gamma T(A) \longrightarrow 0 \quad ,$$

compare (III.2.4) in [6]. Here Q is given by (1.14) and (1.10). Moreover the injective homomorphism i in (1.16) carries $\gamma_2(x) \otimes y \in \Gamma(A) \otimes A$ to the Whitehead product $[x\eta_2, y]$ and carries $\gamma_2(x) \otimes 1$ to the composition $x\eta_2\eta_3$, $\eta_3 = \Sigma\eta_2$. For $n = 5$ the homomorphism Q in (1.14) is not surjective, but we have by [10]

$$(1.17) \qquad Q\Pi_5 M(A,2) \cong A * \mathbb{Z}_3 \subset \Gamma_6 A$$

Using the spectral sequence of Dreckmann [13] for $\Pi_* M(A,2)$ one also obtains the exact sequence

$$L_2\Gamma_2^2(A) \xrightarrow{d_2} \Gamma_2^3(A) \longrightarrow \Pi_5 M(A,2) \longrightarrow L_1\Gamma_2^2(A) \longrightarrow 0$$

$$\text{(1.18)} \qquad \text{with} \qquad \Gamma_2^3(A) = L(A,1)_4 \oplus \Gamma_2^2(A) \otimes \mathbb{Z}_2$$

which is useful for the computation of $\Pi_5 M(A,2)$. The sequences (1.16) and (1.18) can also be derived from the EHP sequence for homotopy groups of mapping cones in [7], see [9]. The homomorphism d_2 in (1.18) is a differential in the spectral sequence of Dreckmann which in general is not trivial. The image D of the differential d_2 is the subgroup of $\Gamma(A) \otimes \mathbb{Z}_2 \subset \Gamma_2^2(A)$ given by the image of the natural map $q\Gamma(\lambda)$,

$$\text{(1.19)} \qquad D = Im\{\Gamma(A * \mathbb{Z}_2) \xrightarrow{\Gamma(\lambda)} \Gamma(A \otimes \mathbb{Z}_2) \xrightarrow{q} \Gamma(A) \otimes \mathbb{Z}_2\} \quad .$$

Here q is the quotient map and λ: $A * \mathbb{Z}_2 \subset A \twoheadrightarrow A \otimes \mathbb{Z}_2$ is given by $\lambda(x) = x \otimes 1$, $1 \in \mathbb{Z}_2$.

§2 The fundamental extension and the torus construction

Let M be a simply connected Poincaré complex of dimension m. Then $M = \dot{M} \cup_f e^m$ is the mapping cone of a map $f\colon S^{m-1} \to \dot{M}$ which for $m = 5$ is of the form

$$(2.1) \qquad f\colon \; S^4 \longrightarrow \dot{M} = M(A,2) \vee M(B',3)$$

where $A = H_2 M$ and $B' = H_3 M \cong A/\mathrm{Tor}A$. As in [5] §1 we have the fundamental extension of groups

$$(2.2) \qquad 0 \longrightarrow \mathcal{E}(M|\dot{M}) \longrightarrow \mathcal{E}(M) \xrightarrow{r} \mathcal{E}(\dot{M}, \pm f) \longrightarrow 0$$

Here $\mathcal{E}(M|\dot{M})$ is the subgroup of $\mathcal{E}(M)$ consisting of all elements which can be represented by orientation preserving maps $M \to M$ which restrict to the identity of $\dot{M}$, or equivalently which are maps under $\dot{M}$. Moreover let $\mathcal{E}(\dot{M}, \pm f)$ be the group of all pairs $x = (x, \varepsilon) \in \mathcal{E}(\dot{M}) \times \{+1, -1\}$ for which $x_*\colon \Pi_{m-1}\dot{M} \to \Pi_{m-1}\dot{M}$ satisfies $x_* f = \varepsilon f$. Here $f \in \Pi_{m-1}\dot{M}$ is the homotopy class of the attaching map. We also write $\varepsilon = deg\, x$, clearly $deg\, x$ is determined by $x \in \mathcal{E}(\dot{M})$ if $2f \neq 0$. The group $\mathcal{E}(M|\dot{M})$ is finite abelian and endowed with a surjective homomorphism of groups

$$(1) \qquad 1^+\colon \Pi_m(\dot{M}) \twoheadrightarrow \mathcal{E}(M|\dot{M})$$

defined by $1^+(\alpha) = 1_M + i_*\alpha$, see [5]. The map $i\colon \dot{M} \hookrightarrow M$ is the inclusion and $+$ is given by the coaction $M \to M \vee S^5$. Moreover the structure of $\mathcal{E}(M|\dot{M})$ as a left $\mathcal{E}(\dot{M}, \pm f)$-module in the extension can be described by the following formula where $a = 1^+(\alpha) \in \mathcal{E}(M|\dot{M})$ and $x \in \mathcal{E}(\dot{M}, \pm f)$,

$$(2) \qquad x \cdot a = 1^+\big(deg(x) \cdot x_*(\alpha)\big)$$

The fundamental extension leads to three problems for the computation of $\mathcal{E}(M)$: First one has to compute the group $\mathcal{E}(\dot{M}, \pm f)$, then one has to compute $\Pi_m \dot{M}$ and the kernel of 1^+, and finally one has to solve the extension problem for (2.2).

Now let $\mathcal{E}_H(M)$ be the subgroup of elements in $\mathcal{E}(M)$ which induce the identity on homology groups H_*M and let $H_*\mathcal{E}(M)$ be the group of those automorphisms of the homology H_*M which are realizable by maps $M \to M$. Then we get the following commutative diagram in which the rows are short exact sequences of groups

$$(2.3) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}(M|\dot{M}) & \longrightarrow & \mathcal{E}(M) & \xrightarrow{r} & \mathcal{E}(\dot{M}, \pm f) & \longrightarrow & 0 \\ & & \cap & & \| & & \downarrow h & & \\ 0 & \longrightarrow & \mathcal{E}_H(M) & \longrightarrow & \mathcal{E}(M) & \xrightarrow{H_*} & H_*\mathcal{E}(M) & \longrightarrow & 0 \end{array}$$

Here H_* is given by the homology functor and the surjective homomorphism h is induced by H_*. By (2.3) we obtain the extension of groups

$$(2.4) \qquad 0 \longrightarrow \mathcal{E}(M|\dot{M}) \longrightarrow \mathcal{E}_H(M) \xrightarrow{r} \mathrm{kernel}(h) \longrightarrow 0$$

which is a subextension of (2.2). We shall see that kernel(h) is actually an abelian group in case $dim(M) = 5$. We can study the group $\mathcal{E}_H(M)$ by the <u>torus construction</u> which is a homomorphism

$$(2.5) \qquad t\colon \; \mathcal{E}_H(M) \longrightarrow H_{m+1}K(A,2) \quad \text{with } A = H_2(M).$$

For this let $u: M \to M$ be a map which represents an element in $\mathcal{E}_H(M)$ and let $k: M \to K(A,2)$ be the first k-invariant of M. Then the diagram

$$(1) \qquad \begin{array}{ccc} M & & \\ \downarrow u & \searrow k & K(A,2) \\ M & \nearrow k & \end{array}$$

homotopy commutes since $\{u\} \in \mathcal{E}_H(M)$. Let $G_u: k \simeq ku$ be a homotopy. Then we have the following push out diagram of spaces, $I = [1,0] \subset \mathbb{R}$,

$$(2) \qquad \begin{array}{ccccc} I \times M & \xrightarrow{I_u} & T_u M & \xrightarrow{g_u} & K(A,2) \\ \uparrow (i_0, i_1) & \text{push} & \uparrow & \nearrow k & \\ M \cup M & \xrightarrow{(1,u)} & M & & \end{array}$$

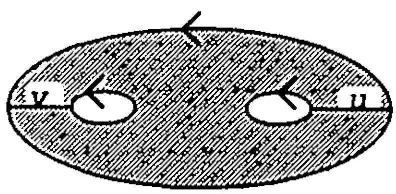
Here the push out $T_u M$ is the mapping torus of u and the map g_u is induced by (k, G_u). The mapping torus $T_u M$ is again a Poincaré complex with fundamental class $I \times [M] \in H_{m+1}(T_u M)$ given by the fundamental class $[M]$ of M. Now we define the function t in (2.5) by

$$(3) \qquad t(u) = (g_u)_*(I \times [M])$$

where $(g_u)_*: H_{m+1}(T_u M) \to H_{m+1} K(A,2)$ is induced by the map g_u in (2).

(2.6) **Lemma:** The torus construction t is a well defined homomorphism between groups.

Proof: We have $g_u \in H^2(T_u M, A) = \mathrm{Hom}(H_2 T_u M, A)$ since $H_1 T_u M = \mathbb{Z}$. Here $H_2 M \to H_2 T_u M$ is an isomorphism so that g_u is well defined up to homotopy. Next we see that t is a homomorphism by the following sketches, $u, v \in \mathcal{E}_H M$.



which describes a bordism between $T_u M + T_v M$ and $T_{vu} M$. ///

(2.7) **Theorem:** Let M be a simply connected Poincaré complex with $A = H_2 M$ and let $g: M(A,2) \to M$ be a map which induces the identity on $H_2 M$. Then the composition

$$\Pi_m M(A,2) \xrightarrow{j_*} \Pi_m(\dot{M}) \xrightarrow{1^+} \mathcal{E}(M|\dot{M}) \xrightarrow{t} H_{m+1} K(A,2)$$

coincides with the homomorphism Q in (1.14).

Proof: Let $u = 1_M + i_* \alpha: M \to M$ be a map under $\dot{M}$ which represents $1^+\alpha$. We consider the following commutative diagram with $ri = id$.

$$\begin{array}{ccccccc}
 & & M \cup_{\dot M} M & \xrightarrow{(1,u)} & M & & \\
 & & \uparrow r & \text{push} & \uparrow \bar r & \searrow k & \\
S^m & \xrightarrow{\omega} & M \cup S^1 \times \dot M \cup M & \xrightarrow{q} & (T_u M)^m & \xrightarrow{g_*} & K(A,2)^m \\
 & & \uparrow i & \text{push} & \uparrow & \nearrow & \downarrow \\
 & & M \cup_{\dot M} M & \xrightarrow{(1,u)} & M & \xrightarrow{k} & K(A,2)
\end{array}$$

Here r is given by the projection $S^1 \times \dot M \to \dot M$ and ω is given by the attaching map ω_f of the top cell in $I \times M$, see II.8.11 in [3]. Since u is a map under $\dot M$ we see that the m-skeleton of $T_u M$ is a push out as indicated in the diagram. Now g_u can be choosen such that $g_u q \,|\, S^1 \times \dot M$ is the projection. Hence $g_u = k\bar r$. Moreover it is easy to see that

$$(1,u)r\omega \quad = \quad i_*\alpha$$

so that $\{k\, i_*\alpha\} \in \Gamma_m K(A,2) \cong H_{m+1}K(A,2)$ represents $t(u)$. This completes the proof of (2.7). In fact we proved the

Addendum(2.8) : The following diagram commutes

$$\begin{array}{ccccc}
\Pi_m \dot M & \twoheadrightarrow \Gamma_m M & \xrightarrow{k_*} & \Gamma_m K(A,2) \\
\downarrow 1^+ & & & \uparrow b\ \cong \\
\mathcal{E}(M|\dot M) & & \xrightarrow{t} & H_{m+1}K(A,2)
\end{array}$$

Here t is the restriction of the torus construction and 1^+ is the homomorphism in (2.2)(1).

§3 On the groups $H_*\mathcal{E}(M)$ and $\mathcal{E}(\overset{\bullet}{M}, \pm f)$.

The homotopy type of a simply connected 5-dimensional Poincaré complex M is determined by its Stöcker invariant (A, b, ω, e). We here describe the groups $H_*\mathcal{E}(M)$ and $\mathcal{E}(\overset{\bullet}{M}, \pm f)$ of section (§2) in terms of this invariant.

(3.1) **Definition:** A Stöcker invariant is a tuple (A, b, ω, e) consisting of a finitely generated abelian group A, a nonsingular skew-symmetric bilinear form b: $\mathrm{Tor}A \times \mathrm{Tor}A \to \mathbb{Q}/\mathbb{Z}$, a homomorphism ω: $A \to \mathbb{Z}_2$ and an element $e \in A \otimes \mathbb{Z}_2$. These data satisfy $\omega(x) = b(x, x)$, $x \in \mathrm{Tor}A$, with $\mathbb{Z}_2 \subset \mathbb{Q}/\mathbb{Z}$ generated by $\frac{1}{2}$ and $(\omega \otimes id)(e) = 0$.

It is the main result of Stöcker in [16] that there is a 1-1 correspondence between homotopy types of simply connected 5-dimensional Poincaré complexes M and isomorphism classes of Stöcker invariants. The correspondence carries M to (A, b, ω, e) with $A = H_2M$, here b is the linking form of M, and $\omega = \omega_2(M)$ is the Stiefel-Whitney class and $e = e(M)$ the exotic characteristic class which vanishes if and only if M has the homotopy type of a manifold.

(3.2) **Definition:** Let $\mathrm{Aut}(A)$ be the group of automorphisms of the abelian group A and $\{+1, -1\} = \mathrm{Aut}(\mathbb{Z})$. Then $\mathrm{Aut}(A, \pm b, \omega, e)$ is the subgroup of $\mathrm{Aut}(A) \times \{+1, -1\}$ consisting of all pairs (u, ε) for which $b(ux, uy) = \varepsilon b(x, y)$ for $x, y \in \mathrm{Tor}(A)$ and $\omega(uz) = \omega(z)$ for $z \in A$ and $(u \otimes 1)e = e$. We denote this group by $\mathrm{Aut}(A, \pm b, \omega)$ if $e = 0$, then $\mathrm{Aut}(A, b, \omega)$ is the subgroup of all (u, ε) with $\varepsilon = 1$.

One has a homomorphism of groups

$$H_2\colon \quad \mathcal{E}(M) \longrightarrow \mathrm{Aut}(A, \pm b, \omega, e)$$

which carries an element $\overline{u} \in \mathcal{E}(M)$ to the induced automorphism $H_2(\overline{u})$ of $A = H_2M$.

(3.3) **Theorem:** For a simply connected 5-dimensional Poincaré-complex M one has an isomorphism of groups

$$H_2\colon \quad H_*\mathcal{E}(M) \cong \mathrm{Aut}(A, \pm b, \omega, e) \quad .$$

Compare the definition of $H_*\mathcal{E}(M)$ in (2.3). For the proof of (3.3) we show that each element (u, ε) in $\mathrm{Aut}(A, \pm b, \omega, e)$ is realizable by a homotopy equivalence $\overline{u} \in \mathcal{E}(M)$ with $H_2\overline{u} = u$ and $\overline{u}_*[M] = \varepsilon[M]$.

The explicit construction of $\overline{u}$ is fairly intricate and relies on a complete knowledge of certain generators and composition laws computed in [8], see (§6) below.

(3.4) **Remark:** Let M be a simply connected, closed, smooth 5-manifold and let $\Pi_0\mathrm{Diff}_+(M)$ be the group of isotopy classes of orientation preserving diffeomorphisms of M. Then it is a result of Barden [2] that

$$H_2\colon \quad \Pi_0\mathrm{Diff}_+(M) \longrightarrow \mathrm{Aut}(A, b, \omega)$$

is surjective. Therefore theorem (3.3) can be deduced from Barden's result if $e = 0$. Since we also deal with the more general case of Poincaré-complexes we give an independent new proof of (3.3) also in the case of manifolds.

Next we consider the group $\mathcal{E}(\overset{\bullet}{M}, \pm f)$ in (2.3). For this we need the canonical homomorphisms

$$(3.5) \qquad \begin{cases} W' & : \Gamma(A) \otimes A \longrightarrow L(A,1)_3 \\ W'' + \Gamma(A) \otimes \omega & : \Gamma(A) \otimes A \longrightarrow \Gamma(A) \otimes \mathbb{Z}_2 \end{cases}$$

with $W'(\gamma_2(x) \otimes y) = -[[y,x],x]$ and $W''(\gamma_2(x) \otimes y) = [x,y] \otimes 1$. The homomorphisms W' and W'' correspond to the coordinates of W in (1.7). Moreover $\Gamma(A) \otimes \omega$ is given by the identity of $\Gamma(A)$ and by the Stiefel-Whitney class $\omega = \omega_2 \colon A \to \mathbb{Z}_2$.

(3.6) **Theorem:** For a simply connected 5-dimensional Poincaré complex M one has a short exact sequence (see(2.3))

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(\omega) & \longrightarrow & \mathcal{E}(\dot{M}, \pm f) & \longrightarrow & \mathrm{Aut}(A, \pm b, \omega, e) & \longrightarrow & 0 \\ & \text{with} & K(\omega) & = & \mathrm{kernel}(W') \cap \mathrm{kernel}(W'' + \Gamma(A) \otimes \omega) & \cong & \mathrm{kernel}\, h & & \end{array}$$

Here $K(\omega) \subset \Gamma(A) \otimes A$ is an $\mathrm{Aut}(A, \pm b, \omega, e)$-module by $(u, \varepsilon) \cdot z = \varepsilon(\Gamma(u) \otimes u)(z)$ for $z \in K(\omega)$. For $\omega = 0$ we have $K(0) = \mathrm{kernel}(W') \cap \mathrm{kernel}(W'') = \mathrm{kernel}(W) \cong (\gamma_3 A)/A * \mathbb{Z}_3$, see (1.8).

(3.7) **Proof:** Let

$$(1) \qquad C \overset{d}{\rightarrowtail} C \oplus D \twoheadrightarrow A$$

be a short free resolution of A where $D \cong A/\mathrm{Tor}A$ and where $C \rightarrowtail C \twoheadrightarrow \mathrm{Tor}A$ is a resolution. Then the cellular chain complex of $\dot{M} = M(A,2) \vee M(D',3)$ is

$$C_3\dot{M} = C \oplus D' \xrightarrow{(d,0)} C \oplus D = C_2\dot{M}$$

with $C_i\dot{M} = 0$ otherwise. Moreover we have the cofiber sequence

$$(2) \qquad M(C \oplus D, 2) \longrightarrow \dot{M} \xrightarrow{q} M(C \oplus D', 3)$$

where the left hand side is the 2-skeleton of $\dot{M}$ and where q is the quotient map. Now we have the short exakt sequence of groups

$$(3) \qquad \begin{cases} 0 \longrightarrow H^3(\dot{M}, \Gamma A) \xrightarrow{1^+} \mathcal{E}(\dot{M}) \xrightarrow{H_*} \mathrm{Aut}(A) \times \mathrm{Aut}(D') \longrightarrow 0 \\ \text{with} \qquad H^3(\dot{M}, \Gamma A) = \mathrm{Ext}(A, \Gamma A) \oplus \mathrm{Hom}(D', \Gamma A) \end{cases}$$

Here H_* is given by the homology functors H_2 and H_3. An element $\alpha \in \mathrm{Ext}(A, \Gamma A)$ is represented by a homomorphism $a \in \mathrm{Hom}(C, \Gamma A)$. Then a and $b \in \mathrm{Hom}(D', \Gamma A)$ yield a homomorphism $c = (a,b) \colon C \oplus D' \to \Gamma A$ and hence by (1.15) a map

$$(4) \qquad c = (a,b) \colon \quad M(C \oplus D', 3) \longrightarrow M(A,2) \subset \dot{M}$$

The homomorphism 1^+ in (2.2)(1) carries (α, b) to the sum $1 + cq$. Here 1 is the identity of the space $\dot{M}$ which is a suspension and q is the quotient map in (2). For $u \in \mathrm{Aut}(A)$ and $v \in \mathrm{Aut}(D')$ the module structure in the extension (3) is given by

$$(5) \qquad (u,v) \cdot (\alpha, b) = (\Gamma(u)_*(u^{-1})^*(\alpha), \Gamma(u) b v^{-1})$$

We point out that the extension (3) in general is not split.

Since M is a Poincaré complex with fundamental class $[M]$ we have the duality isomorphism $\cap[M]$:

$$(6) \qquad \mathrm{Ext}(A, \Gamma A) \oplus \mathrm{Hom}(D', \Gamma A) = H^3(M, \Gamma A) \cong H_2(M, \Gamma A) = \Gamma(A) \otimes A$$

We denote the inverse of this isomorphism by $D_M = (\cap[M])^{-1}$. Let

$$(7) \qquad \begin{cases} q': & \Gamma A \otimes A = \Pi_3 M(A,2) \otimes \Pi_3 M(A,2) \xrightarrow{[\,,\,]} \Pi_4 M(A,2) \subset \Pi_4 \dot{M} \\ q'': & \Gamma A \otimes \mathbb{Z}_2 = \Pi_3 M(A,2) \otimes \mathbb{Z}_2 \xrightarrow{\eta_3^*} \Pi_4 M(A,2) \subset \Pi_4 \dot{M} \end{cases}$$

be given by the Whitehead product and the Hopf map η_3 respectively. Then $(q', q'') = iq$ where i is the inclusion in (1.16) and where q is the quotient map in (1.7). Recall that $f = f_M \in \Pi_4 \dot{M}$ denotes the attaching map for M, see (2.1). Then we have for $\{c\} \in H^3(\dot{M}, \Gamma A)$ and $1+cq = 1^+\{c\} \in \mathcal{E}(M)$ the formula

$$(8) \qquad (1+cq)_* f_M = f_M - q'(\{c\} \cap [M]) + q''\langle \omega, \{c\} \cap [M] \rangle$$

Here $\omega \in H^2(M; \mathbb{Z}_2)$ is the Stiefel Whitney class and $\{c\} \cap [M] \in H_2(M, \Gamma A)$ by (6). The formula is a consequence of (7.11)(a) and (6.4) in [16]. By definition of $\mathcal{E}(\dot{M}, \pm f)$ and h in (2.3) we have

$$(9) \qquad 1+cq \in \text{kernel}(h) \iff (1+cq)_* f_M = f_M$$

$$(10) \qquad \iff 0 = q'(\{c\} \cap [M]) + q''\langle \omega, \{c\} \cap [M] \rangle$$

$$(11) \qquad \iff z = \{c\} \cap [M] \in \Gamma A \otimes A \text{ satisfies } z \in K(\omega)$$

Here (10) is a consequence of (8) and (11) follows since i in (1.16) is injective. Thus we get the following commutative diagram with short exact rows

$$(12) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & K(\omega) & \longrightarrow & \mathcal{E}(\dot{M}, \pm f) & \xrightarrow{h} & H_*\mathcal{E}(M) & \longrightarrow & 0 \\ & & \downarrow D_M & & \downarrow i & & \downarrow j & & \\ 0 & \longrightarrow & H^3(M, \Gamma A) & \xrightarrow{1^+} & \mathcal{E}(\dot{M}) \times \{+1,-1\} & \longrightarrow & \text{Aut}(A) \times \text{Aut}(D') \times \{+1,-1\} & \longrightarrow & 0 \end{array}$$

Here i and j are the inclusions and D_M is given by (6). The bottom row in (12) is the product $H_* \times \{+1,-1\}$ with H_* in (3). Now we use theorem (3.3) for $H_*\mathcal{E}(M)$ and we get by (12) the exact sequence in (3.6).

We now consider the structure of $K(\omega)$ in (12) as an $H_*\mathcal{E}(M)$-module. Let $\overline{u} \in \mathcal{E}(M)$ be a realization of $(u, \varepsilon) \in \text{Aut}(A, \pm b, \omega, c)$ and $z \in K(\omega)$. Then we get by (5)

$$(13) \qquad \begin{aligned} (u,\varepsilon) \cdot z &= D_M^{-1}\Big((u,v) \cdot D_M(z)\Big) \\ &= D_M^{-1}\Big((u,v) \cdot \{c\}\Big) \\ &= D_M^{-1}\Big(\Gamma(u)_*(\overline{u}^{-1})^*\{c\}\Big) \\ &= \Gamma(u)_*((\overline{u}^{-1})^*\{c\} \cap [M]) \\ &= \Gamma(u)\overline{u}_*(\{c\} \cap (\overline{u}^{-1})_*[M]) \\ &= \varepsilon\Gamma(u)_*\overline{u}_*(\{c\} \cap [M]) = \varepsilon(\Gamma(u) \otimes u)(z) \end{aligned}$$

For (13) see page 254 in [15]. This completes the proof of (3.6). ///

Next we consider the torus construction t for a 1-connected 5-dimensional Poincaré complex M. We obtain the following diagram of homomorphisms between groups where the left hand side is given by the extension in theorem (3.6)

$$
(3.8)\qquad
\begin{array}{ccccc}
\mathcal{E}(M|\dot{M}) & \xrightarrow{t'} & \text{kernel}(\chi) & & \\
\downarrow & & \downarrow & & \\
\mathcal{E}_H(M) & \xrightarrow{t} & \Gamma_6(A) & = & H_6K(A,2) \\
\downarrow & & \downarrow q & & \\
K(\omega) & \overset{t''}{\rightarrowtail} & \Gamma_6(A)/\text{kernel}(\chi) & & \\
\cap & & \cap\chi & & \\
\Gamma(A)\otimes A & \xrightarrow{(i,\Gamma(A)\otimes\omega_2)} & \Gamma(A)\otimes A\oplus\Gamma(A)\otimes\mathbb{Z}_2 & &
\end{array}
$$

Here t' is the restriction of t which we obtain by (1.17) and (2.7). Hence also the quotient map t'' is defined by t. The injection χ in the diagram is induced by χ in the exact sequence (1.7).

(3.9) **Lemma:** Diagram (3.8) commutes, in particular $\chi t''$ coincides on $K(\omega)$ with the map $(i, \Gamma(A)\otimes\omega)$ where i is the inclusion of $\Gamma(A)\otimes A$. This implies that t'' is injective.

We also have the following commutative diagram, in which the columns are exact.

$$
(3.10)\qquad
\begin{array}{ccc}
K(\omega) & \overset{t''}{\rightarrowtail} & \Gamma_6(A)/\text{kernel}(\chi) \\
\cap & & \cap\chi \\
\Gamma(A)\otimes A & \overset{(i,\Gamma(A)\otimes\omega)}{\rightarrowtail} & \Gamma(A)\otimes A\oplus\Gamma(A)\otimes\mathbb{Z}_2 \\
\downarrow{\scriptstyle (W',W''+\Gamma(A)\otimes\omega)} & & \downarrow{\scriptstyle W_0} \\
L(A,1)_3\oplus\Gamma(A)\otimes\mathbb{Z}_2 & = & L(A,1)_3\oplus\Gamma(A)\otimes\mathbb{Z}_2
\end{array}
$$

Here we set $W_0(x\oplus y) = W'(x)\oplus(W''(x)+y)$ for $x\in\Gamma(A)\otimes A$ and $y\in\Gamma(A)\otimes\mathbb{Z}_2$. The exactness of the left hand side is given by definition of $K(\omega)$ in (3.6). Exactness of the right hand side follows from (1.7) and (1.5). One readily checks that the bottom sequence of the diagram commutes. We derive from (3.9) and (3.8) the exact sequence of groups

$$
(3.11)\qquad 0 \longrightarrow \text{kernel}\,(t') \longrightarrow \mathcal{E}_H(M) \xrightarrow{t} \Gamma_6(A) \longrightarrow \text{Coker}\,(t'') \longrightarrow 0
$$

For $\omega = 0$ we compute Coker(t'') as follows

(3.12) **Lemma:** For $\omega = 0$ one has an issomorphism

$$
\text{Coker}(t'') \cong \Lambda^2(A\otimes\mathbb{Z}_2)
$$

and the map $\Gamma_6(A) \twoheadrightarrow \text{Coker}(t'') \cong \Lambda^2(A\otimes\mathbb{Z}_2)$ given by (3.11) coincides with σ' in (1.7).

Proof of (3.12) : For $\omega = 0$ the map (W', W'') on the left hand side corresponds via (1.5) to iW in (1.7). Hence $K(0) = \langle\gamma_3 A\rangle/Ker(\chi)$. Moreover the map W_0 corresponds via (1.5) to q in (1.7). Now (1.7) shows that the cokernel of

$$
\langle\gamma_3 A\rangle \longrightarrow \chi\Gamma_6(A)
$$

is $\Lambda^2(A\otimes\mathbb{Z}_2)$. ///

(3.13) **Theorem:** Let M be a simply connected 5 dimensional Poincaré complex and assume the 3-torsion of H_2M is non trivial. Then the fundamental extension for $\mathcal{E}(M)$ is non split.

Proof: For $A * \mathbb{Z}_3 \neq 0$ we show that the extension

$$0 \longrightarrow \mathcal{E}(M|\dot{M}) \longrightarrow \mathcal{E}_H(M) \longrightarrow K(\omega) \longrightarrow 0$$

is not split. In fact if Z_n is a direct summand of A generated by x with $n = 3^i \geq 3$, then $\gamma_3 x$ generates a direct summand Z_{3n} in $\Gamma_6(A)$. Moreover $\mathbb{Z}_n * \mathbb{Z}_3 \subset \mathbb{Z}_n$ is then a direct summand of kernel(χ) generated by $\frac{n}{3}1_{\mathbb{Z}_n}$ and the inclusion i: kernel(χ) $\rightarrow \Gamma_6(A)$ takes $\frac{n}{3}1_{\mathbb{Z}_n}$ to $2n\gamma_3 x$. This can be seen via the computation of the left derived functor $L_1L(A,1)_3$, see [10]. We observe that $\gamma_2(x) \otimes x \in K(\omega)$. Therefore the proposition follows from the commutativity of the diagram (3.8). ///

(3.14) **Proof of (3.9):** Let χ_1, χ_2 be given by the coordinates of χ in (1.7)(1). Then (1.7) and (1.11) show that χ_2 corresponds to Sq_2, that is

(1) $$\chi_2\colon\ \Gamma_6(A) = H_6K(A,2) \xrightarrow{Sq_2} H_4(K(A,2),\mathbb{Z}_2) = \Gamma(A) \otimes \mathbb{Z}_2$$

Now let $\{c\} \in H^3(\dot{M}, \Gamma A)$ and consider v: $\dot{M} \rightarrow \dot{M}$ with $v = 1 + cq$ as in the proof of (3.6) and let u: $M \rightarrow M$ be an extension of v. The extension u exists by (3.6) if and only if

$$a = \{c\} \cap [M] \in K(\omega) \subset \Gamma(A) \otimes A$$

We have to show

(2) $$\chi_2 t''(a) = (\Gamma(A) \otimes \omega)(a)$$

We now have the commutative diagram

(3) $$\begin{array}{ccccc} H_5M & \overset{\psi}{\cong} & H_6T_uM & \xrightarrow{g_{u*}} & H_6K(A,2) \\ \downarrow{\scriptstyle Sq_2} & & \downarrow{\scriptstyle Sq_2} & & \downarrow{\scriptstyle Sq_2} \\ H_3(M,\mathbb{Z}_2) & \cong & H_4(T_uM,\mathbb{Z}_2) & \xrightarrow{g_{u*}} & H_4(K(A,2),\mathbb{Z}_3) \\ \| & & \| & \nearrow {\scriptstyle g_{v*}} & \\ H_3(\dot{M},\mathbb{Z}_2) & \overset{\psi}{\cong} & H_4(T_v\dot{M},\mathbb{Z}_2) & & \end{array}$$

Here $T_v\dot{M}$ and g_v are defined in the same way as in (2.5)(2). Clearly g_v is a restriction of g_u in (2.5)(2). The isomorphisms ψ in the diagram are induced by the quotient maps $T_uM \rightarrow \Sigma M$ and $T_v\dot{M} \rightarrow \Sigma\dot{M}$ respectively and by the suspension isomorphism of homology. We now consider $\dot{M}$ as a mapping cone as in (3.6)(2) so that $1 + cq = v$ is given by the coaction on $\dot{M}$ as well. We can apply similar arguments as in the proof of (2.7) for the computation of g_{v*}, this yields the formula, $x \in H_3(\dot{M}, \mathbb{Z}_2)$,

(4) $$g_{v*}\psi(x) = \langle\{c\}, x\rangle \in \Gamma(A) \otimes \mathbb{Z}_2$$

Therefore the commutativity of the diagram shows

$$
\begin{array}{lrcl}
 & \chi_2 t''(a) & = & Sq_2 g_{u*}\psi[M] \\
 & & = & g_{u*}\psi(Sq_2[M]) \\
 & & = & \langle\{c\}, Sq_2[M]\rangle \\
 & & = & \langle Sq^2\{c\}, [M]\rangle \\
(5) & & = & \langle\omega \cup \{c\}, [M]\rangle \\
 & & = & \langle\omega, \{c\} \cap [M]\rangle \\
 & & = & \langle\omega, a\rangle \quad = \quad (\Gamma(A) \otimes \omega)(a)
\end{array}
$$

In (5) we use the Wu formula $\omega \cup u = Sq^2 u$, see 5.3 in [16]. Hence the proof of (3) is complete. We now deal with the first coordinate χ_1 which can be identified with $k_A \cap$ as in (1.11). Hence we have the commutative diagram, $k' = g_u^* k_A$,

$$
\begin{array}{ccccc}
 & & H_6 T_u M & \xrightarrow{g_{u*}} & H_6 K(A,2) \\
 & & \downarrow{\scriptstyle k'\cap} & & \downarrow{\scriptstyle k_A\cap} \\
 & & H_4(T_u M, A) & \xrightarrow{g_{u*}} & H_4(K(A,2), A) \\
 & & \| & & \| \\
H_3(\dot{M}, A) & \overset{\psi}{\cong} & H_4(T_v \dot{M}, A) & \xrightarrow{g_{v*}} & \Gamma(A) \otimes A
\end{array}
$$

As in (4) we see that $g_{v*}\psi$ carries $x \in H_3(\dot{M}, A)$ to the element

$$(6) \qquad g_{v*}\psi(x) \quad = \quad \langle\{c\}, x\rangle \quad \in \quad \Gamma A \otimes A$$

We now observe that for $k \in H^2(M, A)$ and for $i\colon M \subset T_u(M)$ we have $i^* k' = k \in H^2(M, A)$ and

$$(7) \qquad \psi(k \cap [M]) \quad = \quad k' \cap (I \times [M])$$

If u is the identity of M this follows from the multiplicativity formula for the cap product since in this case $I \times [M] = [S^1] \times [M]$, the formula holds also in general since we can assume that u induces the identity on cellular chains. Now we get for a in (2) the formulas

$$
\begin{array}{rcl}
\chi_1 t''(a) & = & \chi_1 (g_u)_* I \times [M] \\
 & = & k_A \cap (g_u)_* I \times [M] \\
 & = & g_{u*} k' \cap I \times [M] \\
 & = & g_{u*} \psi k \cap I \times [M] \\
 & = & g_{v*} \psi k \cap I \times [M] \\
 & = & \langle\{c\}, k \cap [M]\rangle \\
 & = & \langle\{c\} \cup k, [M]\rangle \\
 & = & T\langle k \cup \{c\}, [M]\rangle, \quad T\colon \Gamma A \otimes A \cong A \otimes \Gamma A \\
 & = & T\langle k, \{c\} \cap [M]\rangle \\
 & = & T\langle k, a\rangle \quad = \quad a
\end{array}
$$

This completes the proof of (3.9). ///

§4 On the group $\mathcal{E}(M|\dot{M})$.

In this section we describe results on the group $\mathcal{E}(M|\dot{M})$ where M is an $(n-1)$-connected m-dimensional Poincaré complex, $n \geq 2$, $m < 3n$. In this case $\dot{M} = \Sigma A$ is a suspension so that $M = \Sigma A \cup_f e^m$. We say that M is Σ-reducible if $\Sigma f \simeq 0$.

We use the following notation. Let $[U, X]$ be the set of homotopy classes of basepoint preserving maps $U \to X$. Then $[\Sigma U, X]$ is a group, this is the homotopy group $\Pi_m X$ if $U = S^{m-1}$ is a sphere. For $u \in [\Sigma U, X]$, $v \in [\Sigma V, X]$ we have the Whitehead product $[u, v] \in [\Sigma U \wedge V, X]$ where $U \wedge V$ is the smash product. We also need the James-Hopf-invariants $\gamma_n \beta \in [\Sigma U, \Sigma B^{\wedge n}]$ for $\beta \in [\Sigma U, \Sigma B]$. Here $B^{\wedge n}$ is the n-fold smash product $B \wedge \ldots \wedge B$ and the James-Hopf-invariant $\gamma_n \beta$ is defined with respect to the lexicographical ordering from the left, see [4]. Moreover we use for the one point union $U \vee V$ the partial suspension, $m \geq 2$,

$$E : \Pi_{m-1}(U \vee V)_2 \longrightarrow \Pi_m(\Sigma U \vee V)_2 \quad .$$

Here $\Pi_k(U \vee V)_2$ denotes the kernel of r_*: $\Pi_k(U \vee V) \to \Pi_k(V)$ where $r = (0, 1)$: $U \vee V \to V$ is the retraction. Using the cone CU of U and the pinch map π_0: $CU \to CU/U = \Sigma U$ we obtain E by the composition

$$\begin{array}{ccccc} \Pi_{m-1}(U \vee V)_2 & \cong & \Pi_m(CU \vee V, U \vee V) & & \\ & & \big\downarrow (\pi_0 \vee 1)_* & & \\ & & \Pi_m(\Sigma U \vee V, V) & \cong & \Pi_m(\Sigma U \vee V)_2 \end{array}$$

compare (II.11.8)[3]. Let i_1 resp. i_2 be the inclusion of U, resp. V, into $U \vee V$. We define the difference operator

$$\begin{array}{rcll} \nabla\colon \Pi_{m-1}(\Sigma A) & \longrightarrow & \Pi_{m-1}(\Sigma A \vee \Sigma A)_2 & \text{by} \\ \nabla(f) & = & -f^*(i_2) + f^*(i_2 + i_1) & . \end{array}$$

The next theorem is based on results in the book algebraic homotopy [3], see [5].

(4.1) Theorem: Let M be a 1-connected Poincaré complex with $\dot{M} = \Sigma A$ and let f: $S^{m-1} \to \Sigma A$ be the attaching map. Then 1^+ in (2.2)(1) induces an isomorphism

$$\begin{array}{rcll} \mathcal{E}(M|\dot{M}) & \cong & \Pi_m(\Sigma A)/\mathcal{J} & \text{where} \\ \mathcal{J} = \mathcal{J}_M & = & Im\,\nabla(1, f) + Im\, f_* & . \end{array}$$

Here f_*: $\Pi_m(S^{m-1}) \to \Pi_m(\Sigma A)$ is induced by f and $\nabla(1, f)$ is the homomorphism

$$\nabla(1, f)\colon [\Sigma^2 A, \Sigma A] \longrightarrow \Pi_m(\Sigma A)$$

which is defined by the formulas

$$\begin{array}{rcl} \nabla(1, f)(\xi) & = & (E\nabla f)^*(\xi, 1) \\ & = & \xi \circ (\Sigma f) \; + \; [\xi, 1](\Sigma\gamma_2 f) \; + \; [[\xi, 1], 1](\Sigma\gamma_3 f) \; + \; \ldots \end{array}$$

Here $1 = 1_{\Sigma A}$ is the identity of ΣA and the sum is taken over all summands $\omega_n \circ (\Sigma\gamma_n f)$, $n \geq 1$, with $\omega_1 = \xi$ and $\omega_n = [\omega_{n-1}, 1]$ for $n \geq 2$.

Clearly $\omega_n \circ (\Sigma\gamma_n f)$ is trivial if n is sufficiently large since ΣA is 1-connected.

For the delicate dimension $m = 3n - 1$ we need the following condition $(*)$.

(4.2) **Definition:** We say that $f\colon S^{m-1} \to \Sigma A$ satisfies condition (*) if the equation

$$(*) \qquad Ker[1,1]_* + Ker\,\Sigma + Im\,\overline{u}_{\Sigma A} \quad = \quad \Pi_m \Sigma A \wedge A$$

holds. Here we use the homomorphisms $[1,1]_*\colon \Pi_m \Sigma A\wedge A \to \Pi_m \Sigma A$, $\Sigma\colon \Pi_m \Sigma A \wedge A \to \Pi_{m+1}\Sigma^2 A\wedge A$, $\overline{u}_{\Sigma A}\colon [\Sigma^2 A, \Sigma A] \to \Pi_m \Sigma A \wedge A$, $\overline{u}_{\Sigma A}(\xi) = (\xi \wedge 1_A)\circ(\Sigma\gamma_2 f)$.

(4.3) **Theorem:** Let $M = \Sigma A \cup_f e^m$ be an $(n-1)$-connected Poincaré-complex of dimension $m = 2n+k < 3n$, $n \geq 2$. If M is Σ-reducible and if for $m = 3n-1$ condition (*) is satisfied for f then one has an isomorphism

$$\mathcal{E}(M|\dot{M}) \quad \cong \quad \Pi_m(\Sigma A)/W\ .$$

Here W is the subgroup generated by all compositions $S^m \xrightarrow{\alpha} \Sigma A^{\wedge t} \xrightarrow{w^t} \Sigma A$, $2 \leq t \leq 4$, where w^t is any t-fold Whitehead-product of the identity $1_{\Sigma A}$.

This result is proved in (3.7) [5]. Next we consider the connected sum of Poincaré-complexes. Let $M_0 = \Sigma A \cup_f e^m$ and $M_1 = \Sigma B \cup_g e^m$ be both $(n-1)$-connected Poincaré-complexes of dimension $m < 3n$. Then the connected sum is given by

$$M \quad = \quad M_0 \# M_1 \quad \simeq \quad (\Sigma A \vee \Sigma B)\cup_{i_A f + i_B g} e^m\ .$$

Here i_A (resp. i_B) is the inclusion of ΣA (resp. ΣB) into $\Sigma A \vee \Sigma B$. With these notations we get

(4.4) **Theorem:** Assume M_0 is Σ-reducible and $m < 3n-2$, then one has an isomorphism

$$\mathcal{E}(M|\dot{M}) \quad \cong \quad \mathcal{E}(M_0|\dot{M}_0)/V \oplus \mathcal{E}(M_1|\dot{M}_1)$$

Here V is the image of the homomorphism

$$(\Sigma^2 g)^*\colon \Sigma[\Sigma^2 B, \Sigma A] \quad \longrightarrow \quad \Sigma(\Pi_m \Sigma A) \quad = \quad \mathcal{E}(M_0|\dot{M}_0)$$

which carries an element $\Sigma\xi$ to the composition $(\Sigma\xi)\circ(\Sigma^2 g)$.

This theorem is a consequence of the following more intricate result in which we also deal with the delicate dimensions $m = 3n-2$ and $m = 3n-1$.

(4.5) **Theorem:** Let $m < 3n$ and let M_0 be Σ-reducible. Moreover assume that the attaching map f of M_0 satisfies condition (*) in (4.2). Then one has an isomorphism

$$\mathcal{E}(M|\dot{M}) \quad \cong \quad (\mathcal{E}(M_0|\dot{M}_0) \oplus W(A,B))/V \oplus \mathcal{E}(M_1|\dot{M}_1)\ .$$

Here $W(A,B)$ is the direct sum of homotopy groups

$$W(A,B) \quad = \quad \Sigma(\Pi_m \Sigma A\wedge B) \oplus \Pi_m \Sigma A \wedge B \wedge B \oplus \Pi_m \Sigma A \wedge B \wedge B \wedge B$$

and V is the subgroup generated by the following elements where we identify

$$(\Pi_m \Sigma A)/W \quad = \quad \mathcal{E}(M_0|\dot{M}_0)$$

as in (4.3):

(a) $\Sigma T_{21}(\xi_1^0 \wedge A)\circ \Sigma^2\gamma_2 f + T_{312}(\gamma_2 \xi_1^0 \wedge A)\circ \Sigma\gamma_2 f$,

(b) $\xi_{011}^1 \circ \Sigma g + (\xi_{011}^1 \wedge B)\circ \Sigma\gamma_2 g$,

(c) $\Sigma\xi_{01}^1 \circ \Sigma^2 g + (\xi_{01}^1 \wedge B)\circ\Sigma\gamma_2 g + (\xi_{01}^1 \wedge B \wedge B)\circ \Sigma\gamma_3 g$,

(d) $\{\xi_0^1 \circ \Sigma g\} + \Sigma(\xi_0^1 \wedge B)\circ \Sigma^2 \gamma_2 g + (\xi_0^1 \wedge B \wedge B)\circ\Sigma\gamma_3 g$.

Here the curly bracket denotes the coset modulo W and $\xi_1^0 \in [\Sigma^2 A, \Sigma B]$, $\xi_0^1 \in [\Sigma^2 B, \Sigma A]$, $\xi_{01}^1 \in [\Sigma^2 B, \Sigma A \wedge B]$, $\xi_{011}^1 \in [\Sigma^2 B, \Sigma A \wedge B \wedge B]$.

We shall use this result for the computation of $\mathcal{E}(M|\dot{M})$ in case M is a simply-connected 5-dimensional manifold. The proof of (4.4) and (4.5) is based on similar idears as the proof of (4.3) in (3.7) [5]. Moreover we shall use notation and facts from the proof of (3.7) [5] in the following proofs.

Proof of (4.4): Since $m < 3n-2$ we see that all triple products in (4.5) vanish and $W(A, B) = \Sigma\Pi_m \Sigma A \wedge B$. Moreover V in (4.5) is generated by the following elements.

$$\begin{array}{ll} (a) & \Sigma T_{21}(\xi_1^0 \wedge A) \circ \Sigma^2 \gamma_2 f \\ (c) & \Sigma \xi_{01}^1 \circ \Sigma^2 g \\ (d) & \{\xi_0^1 \circ \Sigma g\} + \Sigma(\xi_0^1 \wedge B) \circ \Sigma^2 \gamma_2 g \end{array}$$

As in diagram (3.7)(1) [5] we can identify u_X and $\overline{u}_{\Sigma X}$ so that by (a) each element of $W(A, B)$ is in V. This implies by (d) that

$$\left(\mathcal{E}(M_0|\dot{M}_0) \oplus W(A, B)\right)/V = \mathcal{E}(M_0|\dot{M}_0)/(\Sigma^2 g)^* \Sigma[\Sigma^2 B, \Sigma A]$$

///

Proof of (4.5): We use the computation of $\mathcal{E}(M|\dot{M})$ in (2.2) [5]. For this we can decompose $\Pi_m(\Sigma A \vee \Sigma B)$ by the Hilton-Milnor-formula. This leads to the following direct summands, which are embedded by iterated Whitehead-products as indicated in the notation.

$$\begin{array}{rcll} \Pi_m(\Sigma A \vee \Sigma B) & = & (i_A)_* \Pi_m \Sigma A \oplus (i_B)_* \Pi_m \Sigma B & (1) \\ & \oplus & [i_A, i_B]_* \Pi_m \Sigma A \wedge B & (2) \\ & \oplus & [[i_A, i_B], i_B]_* \Pi_m \Sigma A \wedge B \wedge B & (3) \\ & \oplus & [[i_A, i_B], i_A]_* \Pi_m \Sigma A \wedge B \wedge A & (4) \\ & \oplus & [[[i_A, i_B], i_B], i_B]_* \Pi_m \Sigma A \wedge B \wedge B \wedge B & (5) \\ & \oplus & [[[i_A, i_B], i_B], i_A]_* \Pi_m \Sigma A \wedge B \wedge B \wedge A & (6) \\ & \oplus & [[[i_A, i_B], i_A], i_A]_* \Pi_m \Sigma A \wedge B \wedge A \wedge A & (7) \end{array}$$

The retractions r_A and r_B of $\Sigma^2(A \vee B)$ onto $\Sigma^2 A$ and $\Sigma^2 B$ respectively induce the isomorphism $(r_A, r_B)^*$

(8) $$[\Sigma^2(A \vee B), \Sigma(A \vee B)] = [\Sigma^2 A, \Sigma(A \vee B)] \oplus [\Sigma^2 B, \Sigma(A \vee B)] \quad .$$

The right hand side has the following direct sum decomposition, where we set $\varepsilon = 0$ in case $X = A$ and $\varepsilon = 1$ in case $X = B$.

$$\begin{array}{rclll} [\Sigma^2 X, \Sigma(A \vee B)] & \cong & (i_A)_* [\Sigma^2 X, \Sigma A] & \ni \xi_0^\varepsilon & (9) \\ & \oplus & (i_B)_* [\Sigma^2 X, \Sigma B] & \ni \xi_1^\varepsilon & (10) \\ & \oplus & [i_A, i_B]_* [\Sigma^2 X, \Sigma A \wedge B] & \ni \xi_{01}^\varepsilon & (11) \\ & \oplus & [[i_A, i_B], i_B]_* [\Sigma^2 X, \Sigma A \wedge B \wedge B] & \ni \xi_{011}^\varepsilon & (12) \\ & \oplus & [[i_A, i_B], i_A]_* [\Sigma^2 X, \Sigma A \wedge B \wedge A] & \ni \xi_{010}^\varepsilon & (13) \end{array}$$

Thus we have 10 direct summands of (8). These yield the following 10 equations which describe the subgroup $Im\nabla(i_Af + i_Bg)$ of $\mathcal{J}$. In the notation on the right hand side we refer to the direct summands in (1)...(7) and here we omit the embeddings given by iterated Whitehead-products.

$$
\begin{array}{llll}
\nabla(i_A\xi_0^1 r_B) & = & i_A\xi_0^1\Sigma g & \in (1) \quad (14)\\
 & + & (\xi_0^1\wedge B)\Sigma\gamma_2 g & \in (2)\\
 & + & T_{132}(\gamma_2\xi_0^1\wedge B)\Sigma\gamma_2 g & \in (4)\\
 & + & (\xi_0^1\wedge B\wedge B)\Sigma\gamma_3 g & \in (3)\\
\nabla(i_B\xi_1^1 r_B) & = & i_B\xi_1^1\Sigma g & \in (1) \quad (15)\\
 & + & i_B[1,1](\xi_1^1\wedge B)\Sigma\gamma_2 g & \in (1)\\
 & + & i_B[[1,1],1]T_{132}(\gamma_2\xi_1^1\wedge B)\Sigma\gamma_2 g & \in (1)\\
 & + & i_B[[1,1],1](\xi_1^1\wedge B\wedge B)\Sigma\gamma_3 g & \in (1)\\
\nabla([i_A,i_B]\xi_{01}^1 r_B) & = & \xi_{01}^1\Sigma g & \in (2) \quad (16)\\
 & + & (\xi_{01}^1\wedge B)\Sigma\gamma_2 g & \in (3)\\
 & + & (\xi_{01}^1\wedge B\wedge B)\Sigma\gamma_3 g & \in (5)\\
\nabla([[i_A,i_B],i_B]\xi_{011}^1 r_B) & = & \xi_{011}^1\Sigma g & \in (3) \quad (17)\\
 & + & (\xi_{011}^1\wedge B)\Sigma\gamma_2 g & \in (5)\\
\nabla([[i_A,i_B],i_A]\xi_{010}^1 r_B) & = & \xi_{010}^1\Sigma g & \in (4) \quad (18)\\
 & + & (-[1_{\Sigma A\wedge B},1_{\Sigma A\wedge B}]T_{3412}(\xi_{010}^1\wedge B)\Sigma\gamma_2 g) & \in (2)\\
 & + & T_{1243}(\xi_{010}^1\wedge B)\Sigma\gamma_2 g & \in (6)\\
\nabla(i_A\xi_0^0 r_A) & = & i_A[1,1](\xi_0^0\wedge A)\Sigma\gamma_2 f & \in (1) \quad (19)\\
 & + & i_A[[1,1],1]T_{132}(\gamma_2\xi_0^0\wedge A)\Sigma\gamma_2 f & \in (1)\\
 & + & i_A[[1,1],1](\xi_0^0\wedge A\wedge A)\Sigma\gamma_3 f & \in (1)\\
\nabla(i_B\xi_1^0 r_A) & = & (-T_{21}(\xi_1^0\wedge A)\Sigma\gamma_2 f) & \in (2) \quad (20)\\
 & + & (-T_{312}(\gamma_2\xi_1^0\wedge A)\Sigma\gamma_2 f) & \in (3)\\
 & + & (-T_{213}(\xi_1^0\wedge A\wedge A)\Sigma\gamma_3 f) & \in (4)\\
\nabla([i_A,i_B]\xi_{01}^0 r_A) & = & (\xi_{01}^0\wedge A)\Sigma\gamma_2 f & \in (4) \quad (21)\\
 & + & (\xi_{01}^0\wedge A\wedge A)\Sigma\gamma_3 f & \in (7)\\
\nabla([[i_A,i_B],i_B]\xi_{011}^0 r_A) & = & (\xi_{011}^0\wedge A)\Sigma\gamma_2 f & \in (6) \quad (22)\\
\nabla([[i_A,i_B],i_A]\xi_{010}^0 r_A) & = & (\xi_{010}^0\wedge A)\Sigma\gamma_2 f & \in (7) \quad (23)
\end{array}
$$

The equations (14)...(23) are obtained by computing the left hand side via the formula for $\nabla(\xi)$ in (4.1) where we set $\nabla = \nabla(1, i_Af + i_Bg)$. For this we use the distributivity laws for Whitehead-products as in (3.7) [5]. Moreover in (18) we use the Jacobi identity and in (20) we use the antisymmetry of Whitehead-products.

The subgroup $Im\,(i_Af + i_Bg)_*$ of $\mathcal{J}$ is generated by

$$(i_Af + i_Bg)_*\eta_{m-1} \quad = \quad i_Af\eta_{m-1} + i_Bg\eta_{m-1} \quad \in \quad (1) \qquad (24)$$

Thus the subgroup $\mathcal{J}$ is generated by all elements (14)...(24).

The proposition in (4.5) is equivalent to the equation

$$\begin{array}{rcll} \mathcal{J} & = & i_A\mathcal{J}_{M_0}+i_B\mathcal{J}_{M_1} & \subset\ (1) \\ & + & V' & \subset\ (1)\oplus(2)\oplus(3)\oplus(5) \\ & + & V'' & \subset\ (2) \\ & + & (4)+(6)+(7) & \end{array} \tag{25}$$

Here the right hand side denotes a sum of subgroups of $\Pi_m(\Sigma A\vee\Sigma B)$ in (1). We set

$$V'' \quad = \quad \text{kernel}\,\{\Sigma\colon \Pi_m(\Sigma A\wedge B)\to\Pi_{m+1}(\Sigma^2 A\wedge B)\}$$

and the group V' is generated by all elements $(a),(b),(c),(d)$ in (4.5) where we omit the curly brackets for the left hand term of (d).

We first consider the inclusion $\subset$ for the equation in (25). For this we observe that (14) corresponds to (d) so that $(14)\in V'+i_AW+V''+(4)$ where $W=\mathcal{J}_{M_0}$, see (3.7)[5] with condition $(*)$ for f. Moreover $(15)\in i_B\mathcal{J}_{M_1}$ by definition of $\mathcal{J}_{M_1}$. Now (16) corresponds to (c) so that $(16)\in V'+V''$. The element (17) corresponds to (b) so that $(17)\in V'$. Clearly $(18)\in(4)+V''+(6)$. Next $(19)\in i_A\mathcal{J}_{M_0}$ by definition of $\mathcal{J}_{M_0}$ since $\Sigma f\simeq 0$. The element (20) corresponds to (a) in V' so that $(20)\in V'+V''+(4)$. Clearly $(20),(21),(23)\in(4)\oplus(6)\oplus(7)$. Finally $(24)\in i_A\mathcal{J}_{M_0}+i_B\mathcal{J}_{M_1}$ by definition of $\mathcal{J}_{M_0}$ and $\mathcal{J}_{M_1}$.

The proof of the inclusion $\supset$ for (25) is more complicated. Using duality as in (3.7)[5] we see that $(6)\subset\mathcal{J}$ and $(7)\subset\mathcal{J}$ by (22) and (23) respectively. Now (21) shows $(4)\subset\mathcal{J}$ again by a duality argument. Since the elements (18)(2) generate V'' we see by (18) that also $V''\subset\mathcal{J}$. Since $W=Im\,\nabla(1,f)$ by (3.7)[5] and condition $(*)$ for f we see by (19) that $i_A\mathcal{J}_{M_0}=i_AW\subset\mathcal{J}$. Next we see by (24) and (14) that $i_B\mathcal{J}_{M_1}\subset\mathcal{J}$. Here we observe that the first summand of (24) is an element in $i_A\mathcal{J}_{M_0}$ which we have seen to be a subgroup of $\mathcal{J}$. By $(14),(16),(17)$ and (20) we see $V'\subset\mathcal{J}$. For this we use the inclusions $(4)\subset\mathcal{J}$ and $V''\subset\mathcal{J}$ which we already have seen to be true. ///

§5 The normal form of a simply connected 5-dimensional Poincaré complex.

For our explicit computations below we need the normal form of a 1-connected 5-dimensional Poincaré complex as described by Stöcker in §10 of [16]. We use the following notation. For a natural number n let $P_n = S^1 \cup_n e^2$ be the pseudo-projective plane. Then $P_n^{d+1} = \Sigma^{d-1} P_n = M(\mathbb{Z}_n, d)$ is a Moore space of the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. For $n = \infty$ we set $\mathbb{Z}_\infty = \mathbb{Z}$ and $P_\infty = S^1$ so that $\Sigma^{d-1} P_\infty = M(\mathbb{Z}, d) = S^d$ is a sphere. We write $C = (\mathbb{Z}_n)x = \mathbb{Z}_n x$ if C is a cyclic group of order n with generator $x \in C$. If the abelian group A is a direct sum of cyclic groups $(n_1, \ldots, n_r \leq \infty)$

(5.1) $$\begin{cases} A = (\mathbb{Z}_{n_1})x_1 \oplus \ldots \oplus (\mathbb{Z}_{n_r})x_r & \text{, then} \\ P_A = P_{n_1} \vee \ldots \vee P_{n_r} \end{cases}$$

is the corresponding one point union of pseudo projective planes. Then the $(d-1)$-fold suspension

(1) $$M(A, d) = \Sigma^{d-1} P_A$$

is the "normal form" of the Moore space $M(A, d)$. The generator x_i of A yields the corresponding inclusion of Moore spaces

(2) $$x_i\colon \ M(\mathbb{Z}_{n_i}, d) = \Sigma^{d-1} P_{n_i} \subset \Sigma^{d-1} P_A = M(A, d)$$

We denote the inclusion of a bottom sphere by i and a pinch map by q. In particular we have

(3) $$\begin{cases} i = i_n\colon & S^d \subset P_n^{d+1} \\ q = q_n\colon & P_n^{d+1} \longrightarrow S^{d+1} \end{cases}$$

and for a smash product $P_n \wedge P_m$ we use

(4) $$i = i_{nm} = \Sigma(i_n \wedge i_m)\colon \ S^3 \subset \Sigma P_n \wedge P_m$$

As introduced by Toda [17] we have the Hopf maps $\eta_n = \Sigma^{n-2}\eta_2 \in \Pi_{n+1} S^n$, $n \geq 2$, and the iterated Hopf maps η_n^k with $\eta_n^1 = \eta_n$ and $\eta_n^k = \eta_n^{k-1}\eta_{n+k-1}$. Moreover $\iota_n \in \Pi_n S^n$ is given by the identity of S^n. We have $\Pi_3 \Sigma P_n = \mathbb{Z}_{(n^2, 2n)} i\eta_2$ where (n, m) is the greatest common divisor. Moreover we describe in (§6) elements ξ_n and ξ_{nm} such that for finite n, m:

(5.2)
$$\Pi_4(\Sigma P_n) = \begin{cases} (\mathbb{Z}_4)\xi_2 \quad (2\xi_2 = i\eta_2^2) & \text{for } n = 2 \\ (\mathbb{Z}_2)\xi_n \oplus (\mathbb{Z}_2)\, i\eta_2^2 & \text{for } n = 2^i \geq 4 \\ 0 & \text{for } n \text{ odd} \end{cases}$$

$$\Pi_4(\Sigma P_n \wedge P_m) = \begin{cases} (\mathbb{Z}_4)\xi_{22} \quad (2\xi_{22} = i\eta_3) & \text{for } n = m = 2 \\ (\mathbb{Z}_n * \mathbb{Z}_m)\xi_{nm} \oplus (\mathbb{Z}_2)\, i\eta_3 & \text{for } n = 2^i,\ m = 2^j,\ i + j > 2 \\ (\mathbb{Z}_n * \mathbb{Z}_m)\xi_{nm} & \text{for } n \text{ or } m \text{ odd} \end{cases}$$

Using the elements ξ_n, ξ_{nm} we describe in the following list all basic 1-connected 5-dimensional Poincaré complexes M. They are all indecomposable exept X_2 with $X_2 = X_{-1} \# X_{-1}$. The generators y, y' of $H_2 M \oplus H_3 M$ denote also the corresponding inclusions of Moore spaces into $\dot{M}$, see (5.1)(2). Let $f_M = f$ be the attaching map of $M = \dot{M} \cup_f e^5$.

(5.3)

M	$\overset{\bullet}{M}$	f_M	H_2M	H_3M
M_∞	$S^2 \vee S^3$	$[y, y']$	$\mathbb{Z}y$	$\mathbb{Z}y'$
X_∞	$S^2 \vee S^3$	$[y, y'] + y'\eta_3$	$\mathbb{Z}y$	$\mathbb{Z}y'$
M'_∞	$S^2 \vee S^3$	$[y, y'] + y\eta_2^2$	$\mathbb{Z}y$	$\mathbb{Z}y'$
X_{-1}	ΣP_2	$y\xi_2$	$\mathbb{Z}_2 y,\ y = y'$	0
$q = p^i,\ M_q$	$\Sigma P_q \vee \Sigma P_q$	$[y, y']\xi_{qq}$	$\mathbb{Z}_q y \oplus \mathbb{Z}_q y'$	0
$q = 2^i,\ X_q$	$\Sigma P_q \vee \Sigma P_q$	$[y, y']\xi_{qq} + y\xi_q$	$\mathbb{Z}_q y \oplus \mathbb{Z}_q y'$	0
$q = 2^i,\ X'_q$	$\Sigma P_q \vee \Sigma P_q$	$[y, y']\xi_{qq} + y(\xi_q + i\eta_2^2)$	$\mathbb{Z}_q y \oplus \mathbb{Z}_q y'$	0
$q = 2^i,\ M'_q$	$\Sigma P_q \vee \Sigma P_q$	$[y, y']\xi_{qq} + yi\eta_2^2$	$\mathbb{Z}_q y \oplus \mathbb{Z}_q y'$	0

Here p is a prime. By the classification theorem of Stöcker [16] the following is a complete list of homotopy types of simply connected 5-dimensional Poncaré complexes M. Let $q = 2^i$ and $t = 2^j$ be powers of 2 or ∞.

(5.4)

(I)	P	
(II)	$X_{-1}\#P$	
(III)	$X_q\#P$	for $2 \le q \le \infty$
(IV)	$M'_q\#P$	for $2 \le q \le \infty$
(V)	$X_{-1}\#M'_q\#P$	for $2 \le q \le \infty$
(VI)	$X_q\#M'_t\#P$	for $2 \le q, t \le \infty$
(VII)	$X'_q\#P$	for $2 \le q < \infty$

Here $P = M_\infty\#\ldots\#M_\infty\#M_{p_1}\#\ldots\#M_{p_r}$ (s times M_∞) with prime powers $p_1, \ldots, p_r$ and $r, s \ge 0$ (if $r = s = 0$, then $P = S^5$). We call (5.4) the <u>normal form</u> of M. According to (5.4) the manifold M is of the form $M = M_{(0)}\#M_{(1)}\#\ldots\#M_{(k)}$ where we have $M_{(0)} = X_{-1}$ in case (II), (V) and $M_{(0)} = S^5$ otherwise. Moreover $M_{(1)}, \ldots, M_{(k)}$ are basic Poincaré complexes as described in (5.3). The homology $H_2M \oplus H_3M$ has the basis $x_0, x_1, \ldots, x_{2k}$ in case (II), (V) and $x_1, \ldots, x_{2k}$ otherwise. Here $y_0 = y'_0 = x_0$ is a basis of $H_2M_{(0)}$ and $\{y_i = x_{2i-1},\ y'_i = x_{2i}\}$ is a basis of $H_2M_{(i)} \oplus H_3M_{(i)}$ for $i \ge 1$. As above the basis elements y_i, y'_i as well denote inclusions of Moore spaces. The attaching map

(5.5) $$f = f_M\colon\ S^4 \longrightarrow M(H_2M, 2) \vee M(H_3M, 3) = \overset{\bullet}{M}$$

for $M = \overset{\bullet}{M} \cup_f e^5$ is by (5.3) of the form

$$f_M = \varepsilon_M + \sum_{i=1}^{k}[y_i, y'_i]\xi_{(y_i)}$$

Here we set $\xi_{(y)} = \iota_4$ if the order of y is infinite and we set $\xi_{(y)} = \xi_{qq}$ if the order of y is $q < \infty$. Moreover we have by (5.3) and (5.4) the following values of ε_M

(I)	0			
(II)	$y_0\xi_2$			
(III)	$y_1\xi_y$	for $q < \infty$	$y_1'\eta_3$	for $q = \infty$
(IV)	$y_1 i\eta_2^2$			
(V)	$y_0\xi_2 + y_1 i\eta_2^2$			
(VI)	$y_1\xi_q + y_2 i\eta_2^2$	for $q < \infty$	$y_1'\eta_3 + y_2 i\eta_2^2$	for $q = \infty$
(VII)	$y_1(\xi_q + i\eta_2^2)$			

The Stöcker invariant (A, b, ω, e) of M in (5.4) is given as follows. Clearly $A = H_2M$ is the abelian group with generators $x \in \{x_0, \dots, x_{2k}\}$ with $|x| = 2$ as above. Moreover the only non trivial values of b on generators are $b(y_i, y_i') = 1/q$ where $q < \infty$ is the order of y_i. The only non trivial value of ω: $A \to \mathbb{Z}_2$ on generators is $\omega(y_0) = 1$ for (II), (V) and $\omega(y_1) = 1$ for (III), (VI) and (VII) and moreover for (III), (VI) $\omega(y_1') = 1$ if $q = \infty$. Finally we get $e \in A \otimes \mathbb{Z}_2$ by $e = y_1 \otimes 1$ for (IV), (V), (VII) and $e = y_2 \otimes 1$ for (VI), moreover $e = 0$ otherwise. Thus e corresponds exactly to the summand η_2^2 in $(I), \dots, (VII)$. Hence we have $e(M) = 0$ and $\omega_2(M) = 0$ if and only if $M = P$. This shows

(5.6) **Lemma:** A 1-connected 5-dimensional Poincaré complex M is Σ-reducible if and only if $e(M) = 0$ and $\omega_2(M) = 0$.

§6 Realizability of homology isomorphisms.

Let again M be a simply connected 5-dimensional Poincaré complex and $\dot{M}$ its 3 skeleton. Moreover let $A = H_2M$ and $D' = A/\mathrm{Tor}A$. We will now construct the isomorphism H_2: $H_*\mathcal{E}(M) \cong \mathrm{Aut}(A, \pm b, \omega, e)$ stated in theorem (3.3). For this we consider the following commutative diagram of groups.

(6.1)

$$\begin{array}{ccccccc} & & \mathcal{E}(\dot{M}, \pm f) & \overset{\pi'}{\twoheadrightarrow} & H_*\mathcal{E}(M) & \underset{\cong}{\overset{H_2}{\longrightarrow}} & \mathrm{Aut}(A, \pm b, \omega, e) \\ & & \downarrow i & & \downarrow j & \swarrow \psi & \\ H^3(\dot{M}, \Gamma A) & \overset{1^+}{\rightarrowtail} & \mathcal{E}(\dot{M}) \times \mathrm{Aut}\mathbb{Z} & \overset{\pi}{\twoheadrightarrow} & \mathrm{Aut}(A) \times \mathrm{Aut}(D') \times \mathrm{Aut}\mathbb{Z} & & \end{array}$$

We obtain this diagram from (3.7)(12) if we can show the commutativity of the triangle. Every element in $H_*\mathcal{E}(M)$ is given by a triple of isomorphisms (u_2, u_3, u_5), $u_i \in \mathrm{Aut}(H_iM)$, for which we have $u_3 = (u_2|D')^*$ is the adjoint automorphism of u_2 restricted to D'. We may define $\psi(u, \varepsilon) = (u, (u|D')^*, \varepsilon)$ and get $\psi \circ H_2 = j$. This also shows the injectivity of H_2 since j is injective. We remark that the map π' is surjective and the bottom row is exact.

To prove surjectivity of H_2 we will choose a splitting σ for π in (6.1) and obatin elements $\sigma\psi(u, \varepsilon)$. Now we are free to choose an element $\delta = \delta(u, \varepsilon) \in H^3(\dot{M}, \Gamma A)$ to construct $(\overline{u}, \varepsilon) = ((1^+)(\delta))_* \sigma\psi(u, \varepsilon)$ for which we also have $\pi(\overline{u}, \varepsilon) = \psi(u, \varepsilon)$. If we can do this choice in a way that $\overline{u}_* f_M = \varepsilon f_M$ holds for the attaching map f_M of M, then $(\overline{u}, \varepsilon)$ is in the image of i and hence we obtain an element mapping to (u, ε) by $H_2\pi'$.

In order to make these choices and computations explicit we will use the basis of A and the normal form of M defined in (5.4) and (5.5).

(6.2) <u>The choice of generators in the fourth homotopy group of Moore spaces.</u>

We fix the generators in the fourth homotopy group. For this we need the canonical splitting function

(6.3) $$B_2\colon \quad \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_r) \longrightarrow [\Sigma P_n, \Sigma P_r]$$

defined in [8] (4.4) which maps each homomorphism φ to the suspension of a principal map which induces φ in homology. This way we obtain for the canonical generators $\chi_r^n \in \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_r)$ the suspensions $\mathcal{X}_r^n = B_2(\chi_r^n) \in [\Sigma P_n, \Sigma P_r]$.

For ΣP_n we have the inclusions of the bottom sphere and the pinch map, such that $S^2 \overset{i_n}{\hookrightarrow} \Sigma P_n \overset{q}{\longrightarrow} S^3$ is a cofiber sequence.

(6.4) **<u>Definition:</u>** Let $\eta_2^2 \in \Pi_4 S^2$ and $\eta_3 \in \Pi_4 S^3$ be the (iterated resp. suspended) Hopf maps. We set

(1)	ε_n	$=$	$i_n\eta_2^2$	$\in \Pi_4\Sigma P_n$	
(2)	ε_{nm}	$=$	$(i_n \# i_m)\eta_3$	$\in \Pi_4\Sigma P_n \wedge P_m$	Moreover we choose a generator
(3)	ξ_2	$\in$	$\Pi_4\Sigma P_2$	$\cong \mathbb{Z}_4$	and define
	ξ_n	$=$	$(\mathcal{X}_n^2)_*\xi_2$		By use of the James-Hopf invariant let
(4)	ξ_{22}	$=$	$\gamma_2(\xi_2)$		

and we require for a generator $\xi_{nm} \in \Pi_4 \Sigma P_n \wedge P_m$

$$(5) \qquad \begin{aligned} (\mathcal{X}_2^n \# \mathcal{X}_2^m)_* \xi_{nm} &= 0 \quad \text{for } (n,m) \neq (2,2) \\ h(\xi_{nm}) &= \overline{\xi}_{nm} \end{aligned}$$

Here $h\colon \Pi_4 \Sigma P_n \wedge P_m \to H_4 \Sigma P_n \wedge P_m$ is the Hurewiçz map and $\overline{\xi}_{nm} \in \mathbb{Z}_n * \mathbb{Z}_m$ is the canonical generator.

(6.5) **Definition:** Let $A = \mathbb{Z}_n(x)$ and $B = \mathbb{Z}_m(y)$ be cyclic groups with fixed generators x and y respectively and let $\varphi \in \mathrm{Hom}(A, B)$. We choose a number $N(\varphi)$ which satisfies $\varphi(x) = N(\varphi)y$ and let $\overline{N}(\varphi)$ be the unique number with $0 \leq \overline{N}(\varphi) < m$ and $\varphi(x) = \overline{N}(\varphi)y$. Moreover we define $M(\varphi)$ by $nN(\varphi) = mM(\varphi)$. For example $\mathbb{Z}_n$, $\mathbb{Z}_n \otimes \mathbb{Z}_m$, $\mathbb{Z}_n * \mathbb{Z}_m = \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_m)$ are cyclic groups with canonically fixed generators.

(6.6) With these coefficients we can write the following rules for $\varphi_r^n \in \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_r)$, $\psi_s^m \in \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_s)$.

$$\begin{array}{lrcl}
(1) & \gamma_2(\varepsilon_n) & = & \varepsilon_{nn} \\
 & \gamma_2(\xi_n) & = & \frac{n}{2}\xi_{nn} \\
(2) & T_{21}\varepsilon_{nm} & = & \varepsilon_{mn} \\
 & T_{21}\xi_{nm} & = & \xi_{mn} \\
(3) & [1,1]\varepsilon_{nn} & = & 0 \\
 & [1,1]\xi_{nn} & = & 0 \\
(4) & B_2(\varphi_m^n)\varepsilon_n & = & N(\varphi_m^n)\varepsilon_m \\
 & B_2(\varphi_m^n)\xi_n & = & M(\varphi_m^n)M(\varphi_m^n)\xi_m \\
(5) & (\alpha+\beta)\xi_n & = & \alpha\xi_n + \beta\xi_n - [\alpha,\beta]\frac{n}{2}\xi_{nn} + [[\alpha,\beta],\beta]\frac{n^2}{4}i_{nnn}, \ \alpha,\beta \in [\Sigma P_n, U] \\
(6) & (B_2(\varphi_r^n)\# B_2(\psi_s^m))\varepsilon_{nm} & = & N(\varphi_r^n)N(\psi_s^m)\varepsilon_{rs} \\
 & (B_2(\varphi_r^n)\# B_2(\psi_s^m))\xi_{nm} & = & \overline{N}(\varphi_r^n * \psi_s^m)\xi_{rs} \\
(7) & [\alpha, \beta_1+\beta_2]\xi_{nm} & = & [\alpha,\beta_1]\xi_{nm} + [\alpha,\beta_2]\xi_{nm} + \iota_{nm}, \ \alpha \in [\Sigma P_n, U], \beta_1, \beta_2 \in [\Sigma P_m, U]
\end{array}$$

$$\iota_{nm} = \begin{cases} \frac{n}{2}[[\alpha,\beta_1],\beta_2]i_{nmm} & n = m = 2^i \\ 0 & \text{else} \end{cases}$$

$$[\alpha_1+\alpha_2,\beta]\xi_{nm} = [\alpha_1,\beta]\xi_{nm} + [\alpha_2,\beta]\xi_{nm} + \iota'_{nm}, \ \alpha_1,\alpha_2 \in [\Sigma P_n, U], \beta \in [\Sigma P_m, U]$$

$$\iota'_{nm} = \begin{cases} \frac{n}{2}[[\alpha_1,\beta],\alpha_2]i_{nmn} & n = m = 2^i \\ 0 & \text{else} \end{cases}$$

The map T_{21} denotes the suspended interchange map $T_{21} = \Sigma T'_{21}\colon \Sigma A \wedge B \to \Sigma B \wedge A$. Moreover for maps $f\colon \Sigma A \to \Sigma A'$, $g\colon \Sigma B \to \Sigma B'$ between suspensions we define their smash product $f\# g$ to be the composition

$$\Sigma A \wedge B \xrightarrow{f \wedge B} \Sigma A' \wedge B \xrightarrow{T_{21}} \Sigma B \wedge A \xrightarrow{g \wedge A'} \Sigma B' \wedge A' \xrightarrow{T_{21}} \Sigma A' \wedge B'$$

Let $A = \bigoplus_i \mathbb{Z}_{a_i}(x_i)$ and $B = \bigoplus_j \mathbb{Z}_{b_j}(y_j)$ be finitely generated abelian groups with basis $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_r\}$ and let $\varphi = (\varphi_j^i)_{i,j} \in \mathrm{Hom}(A, B)$ be a homomorphism. Then we may use the mapping B_2 in (6.3) to construct a map $s\varphi$ between Moore spaces in "normal form",(cf. (5.1)(1)).

$$(6.7) \qquad \begin{aligned} s\varphi\colon \quad M(A,2) \ = \ \textstyle\bigvee_i \Sigma P_{a_i} \ &\longrightarrow \ M(B,2) \ = \ \textstyle\bigvee_j \Sigma P_{b_j} \\ \text{where } (s\varphi)(x_i) \ &= \ \textstyle\sum_j^{<} y_j B_2(\varphi_j^i) \end{aligned}$$

is the ordered sum, this is, the order of the summands is taken according to the increasing j. Here x_i is the inclusion $x_i\colon \Sigma P_{a_i} \hookrightarrow M(A,2)$ and $y_j\colon \Sigma P_{b_j} \hookrightarrow M(B,2)$. This way s is a splitting for H_2 in $[M(A,2), M(B,2)] \underset{s}{\overset{H_2}{\rightleftarrows}} \mathrm{Hom}(A,B)$, the function s however is no homomorphism.

Observe that for free groups D', E' there is a canonical isomorphism s'

$$[M(D',3), M(E',3)] \quad \underset{s'}{\overset{H_3}{\rightleftarrows}} \quad \mathrm{Hom}(D',E')$$

Now we can define the function σ in (6.1) to be

$$(6.8) \qquad \begin{aligned} \sigma\colon \quad \mathrm{Aut}(A) \times \mathrm{Aut}(D') \times \mathrm{Aut}(\mathbb{Z}) &\longrightarrow \mathcal{E}(\dot{M}) \times \mathrm{Aut}(\mathbb{Z}) \\ \sigma(u,v,\varepsilon) &= ((su) \vee (s'v), \varepsilon) \end{aligned}$$

Remember that we have $\mathcal{E}(\dot{M}) \subset [M(A,2) \vee M(D',3), M(A,2) \vee M(D',3)]$, see (3.7) (1),(3).

Let $u \in \mathrm{Aut}(A)$ be an automorphism of A and $(su) \in \mathcal{E}(\dot{M})$ its realization. For the generators of the fourth homotopy group $\Pi_4 M(A,2)$ arising in the attaching map f_M of M defined in (5.5) we obtain the following laws of composition.

$$\begin{aligned}
&(6.9)(1) & (su)_* y_i \eta_2^2 &= \sum_j N(u_j^i) y_j \eta_2^2 + \sum_{r<s} N(u_r^i) N(u_s^i)[y_r, y_s]\eta_3 \\
&(2) & (su)_* y_i \xi_{n_i} &= \sum_r M(u_r^i) M(u_r^i) y_r \xi_{n_r} \\
& & &\quad + \sum_{r<s} \frac{n_i}{2} (\overline{N}(u_r^i * u_s^i)[y_r, y_s]\xi_{n_r n_s}) \\
& & &\quad + \varrho(n_i = 2) \sum_{s<t\leq k} N(u_s^i) N(u_t^i) N(u_k^i)[[y_s, y_t], y_k]\iota_5 \\
& & (su)_*[y_i, y_{i'}]\xi_{n_i, n_i} &= \sum_{r<s} \Big((\overline{N}(u_r^i * u_s^{i'}) - \overline{N}(u_s^i * u_r^{i'}))[y_r, y_s]\xi_{n_r n_s} \Big) \\
&(3) & &\quad + \varrho(n_i = 2^\lambda) \sum_{r\neq s<t} \frac{n_i}{2} N(u_r^i) N(u_s^{i'}) N(u_t^{i'})[[y_r, y_s], y_t]\iota_5 \\
& & &\quad + \varrho(n_i = 2^\lambda) \sum_{s\neq r<t} \frac{n_i}{2} N(u_r^i) N(u_s^{i'}) N(u_t^i)[[y_r, y_s], y_t]\iota_5
\end{aligned}$$

Here we used the mapping ϱ from the set of logical expressions to $\{0,1\} \subset \mathbb{Z}$, which is defined to be 1 iff the expression is true, to simplify notation.

Next we describe the composition laws necessary to compute $((1^+)(\delta))_*$. From (3.8)(3) we obtain $(1^+)(\delta) = (1 + \delta q)$ where $\delta = \sum_i \delta_i \in [M(C \oplus D', 3), M(A,2)] \cong (C \oplus D') \otimes \Pi_3 M(A,2)$ and $q : \dot{M} \longrightarrow M(C \oplus D', 3)$ is the pinch map (3.8)(2). Since q pinches the 2-skeleton of $\dot{M}$ to a point, every map which can be retracted to the 2-skeleton becomes trivial when composed with q. The analogous result is true if we deal with smash products of $1_{\dot{M}}$ and q. Hence $(1 + \delta q)_* \alpha = \alpha$ for every generator α of $\Pi_4 \dot{M}$ other than in the following list.

$$\begin{aligned}
&(6.10)(1) & (1+\delta q)_* y_i \eta_3 &= y_i \eta_3 + \delta_i \eta_3 \\
&(2) & (1+\delta q)_*[y_i, y_{i'}]\iota_4 &= [y_i, y_{i'}]\iota_4 + [y_i, \delta_{i'}]\iota_4 \\
&(3) & (1+\delta q)_* y_i \xi_{n_i} &= y_i \xi_{n_i} + \delta_i \eta_3 + \frac{n_i}{2}[y_i, \delta_i]\iota_4 \\
&(4) & (1+\delta q)_*[y_i, y_{i'}]\xi_{n_i n_i} &= [y_i, y_{i'}]\xi_{n_i n_i} + [\delta_i, y_{i'}]\iota_4 + [y_i, \delta_{i'}]\iota_4
\end{aligned}$$

For a list of generators see (§5), for these composition laws see [8].

(6.11) **We are now ready to prove theorem (3.3).**

By (6.1) we have to find an element $(\overline{u}, \varepsilon) = (1 + \delta(u, \varepsilon))_* \sigma\psi(u, \varepsilon)$ for which $\overline{u}_* f_M = \varepsilon f_M$ is true. By now we have choosen a splitting σ, see (6.8), we will now choose a suitable $\delta(u, \varepsilon)$ in the following computations.

We only deal with the case that M is a manifold with trivial Stiefel-Whitney class and finite $H_2 M = A$; the other cases can be treated in a similar way giving an explicit choice of $\delta(u, \varepsilon)$. By (5.5) the attaching map f_M is of the form

$$(1) \qquad f_M = \sum_i [y_i, y_{i'}] \xi_{n_i n_i}$$

Using (6.9)(3) we get an expression for $(su)_* f_M$. By assumption $u \in \mathrm{Aut}(A)$ preserves the linking form b up to the sign ε. Since this linking form is induced by f_M we can conclude that

$$\sum_i \sum_{r<s} (\overline{N}(u_r^i * u_s^{i'}) - \overline{N}(u_s^i * u_r^{i'}))[y_r, y_s]\xi_{n_r n_s} = \varepsilon \sum_i \lambda^i(u)[y_i, y_{i'}]\xi_{n_i n_{i'}}$$

with a suitable sign $\lambda^i(u) \in \{+1, -1\} \subset \mathbb{Z}$. This sign may be -1 only if $n_i = n_{i'} = 2$ since this is the only case where the Hurewiçz map h in (6.4)(5) has no splitting. Then the order of $\xi_{n_i n_i}$ is 4 while the order $|x_i| = |x_{i'}| = 2$. We define

$$(2) \qquad \delta^i(u, \varepsilon) = \begin{cases} y_i \eta_2 q_i - [y_i, y_{i'}] q_{i'} & \text{if } \lambda^i(u) = -1 \\ 0 & \text{else} \end{cases}$$

With this $\delta^i(u, \varepsilon)$ we get in case $\lambda^i(u) = -1$ using (6.10)(4)

$$\begin{aligned} (1 + \delta^i(u, \varepsilon))_* [y_i, y_{i'}]\xi_{22} &= [y_i, y_{i'}]\xi_{22} + [y_i \eta_2, y_{i'}] - [y_i, [y_i, y_{i'}]] \\ &= [y_i, y_{i'}]\xi_{22} - [y_i, y_{i'}]\eta_3 - [[y_{i'}, y_i], y_i] + [[y_{i'}, y_i], y_i] \\ &= -[y_i, y_{i'}]\xi_{22} \quad \text{, with } 2\xi_{22} = i_{22}\eta_3, \text{see (5.2)} \end{aligned}$$

Hence we have the suitable sign.

Now regard the summands with generator $[[y_r, y_s], y_t]$ we obtained from (6.9)(3). Since they have the order $gcd(n_r, n_s, n_t)$ they are at most nontrivial with these coefficients if we have $n_r = n_s = n_t = n_i$ and then $\frac{n_i}{2}[[y_r, y_s], y_t]$ has order 2. We define

$$\lambda \begin{pmatrix} i & i' & i'' \\ r & s & t \end{pmatrix}(u) = \frac{n_i}{2} N(u_r^i) N(u_s^{i'}) N(u_t^{i''}) \qquad \text{if } n_i = n_r = n_s = n_t = 2^k$$

and zero otherwise. Let

$$(3) \qquad \begin{aligned} \delta_{r,s,t}^i(u) &= \lambda \begin{pmatrix} i & i' & i' \\ r & s & t \end{pmatrix}(u)[y_r, y_s]q_{t'} & \text{if } s < t,\ r \neq s \\ \overline{\delta}_{r,s,t}^i(u) &= \lambda \begin{pmatrix} i & i' & i \\ r & s & t \end{pmatrix}(u)[y_r, y_s]q_{t'} & \text{if } r < t,\ r \neq s \end{aligned}$$

and zero otherwise and let

$$\delta(u, \varepsilon) = \sum_i \delta^i(u, \varepsilon) + \sum_{r,s<t} \delta_{r,s,t}^i(u) + \sum_{s,r<t} \overline{\delta}_{r,s,t}^i(u)$$

With this $\delta(u,\varepsilon)$ we get

$$(1+\delta(u,\varepsilon))_*\overline{u}_* f_M = \varepsilon f_M \tag{4}$$

Here the δ defined in (3) compensate for the arising generators $[[y_r, y_s], y_t]$ see (6.10)(4), and we can assure the sign ε via the choice of δ in (2). This completes the proof of theorem (3.3) in our case, the other cases can be computed similarly. ///

§7 Computation of $\mathcal{E}(M|\dot{M})$.

In this section we compute explicit results on the groups $\mathcal{E}(M|\dot{M})$ where M is a simply connected 5-dimensional Poincaré complex. The results are achieved using the theorems of (§4) and rely heavily on the complete knowledge of the fifth homotopy group of Moore spaces in degree 2 and on certain homotopy groups of maps between Moore spaces. Moreover we will need the generators and their composition laws in a certain range of dimensions.

Let $P_n = S^1 \cup_n e^2$, $0 < n < \infty$, be the pseudo projective plane, see (§5). With the notation of (§6) we obtain

(7.1) **Theorem:**

$\Pi_5 \Sigma P_n \quad \cong$

$$(1) \quad \begin{cases} (\mathbb{Z}_2)\xi \oplus (\mathbb{Z}_2)\gamma \oplus (\mathbb{Z}_2)\zeta & (\varepsilon = 0) & \text{, for } n = 2 \\ (\mathbb{Z}_2)\xi \oplus (\mathbb{Z}_2)\gamma \oplus (\mathbb{Z}_n)\zeta \oplus (\mathbb{Z}_2)\varepsilon & & \text{, for } n = 2^i \geq 4 \\ (\mathbb{Z}_{3n})\zeta & (\xi = \gamma = \varepsilon = 0) & \text{, for } n = 3^i \geq 3 \\ (\mathbb{Z}_n)\zeta & (\xi = \gamma = \varepsilon = 0) & \text{, for } n = p^i, p \text{ prime } \neq 2,3 \end{cases}$$

Here $\varepsilon = \varepsilon_n \eta_4 = i_n \eta_2^3$, $\xi = \xi_n \eta_4$, $\gamma = [1,1](i_n \# \xi_n)$ and $\zeta = \zeta_n$ is an element choosen with $ggt(3,n)\zeta = [[1,1],1](i_n \# \xi_{nn})$.

Let $n \geq m$:

$\Pi_5 \Sigma P_n \wedge P_m \quad \cong$

$$(2) \quad \begin{cases} (\mathbb{Z}_2)\xi \oplus (\mathbb{Z}_2)\gamma & (\varepsilon = 0) & \text{, for } (n,m) = (2,2) \\ (\mathbb{Z}_2)\xi \oplus (\mathbb{Z}_4)\gamma & (2\gamma = \varepsilon) & \text{, for } (n,m) = (2^i \geq 4, 2) \\ (\mathbb{Z}_2)\xi \oplus (\mathbb{Z}_2)\gamma \oplus (\mathbb{Z}_2)\varepsilon & & \text{, for } (n,m) = (2^i \geq 4, 2^j \geq 4) \\ 0 & (\xi = \gamma = \varepsilon = 0) & \text{, for } n \text{ or } m \text{ odd} \end{cases}$$

Here $\varepsilon = \varepsilon_{nm}\eta_4 = (i_n \# i_m)\eta_3^2$, $\xi = \xi_n \# i_m$, $\gamma = i_n \# \xi_m$.

Let $n \geq m \geq r$ and let $g = gcd(n,m,r)$.

$\Pi_5 \Sigma P_n \wedge P_m \wedge P_r \quad \cong$

$$(3) \quad \begin{cases} (\mathbb{Z}_2)\xi \oplus (\mathbb{Z}_2)\gamma & ,(\varepsilon = 0) & \text{, for } (n,m,r) = (2,2,2) \\ (\mathbb{Z}_2)\xi \oplus (\mathbb{Z}_4)\gamma & ,(2\gamma = \varepsilon) & \text{, for } (n,m,r) = (2^i \geq 4, 2, 2) \\ (\mathbb{Z}_r)\xi \oplus (\mathbb{Z}_r)\gamma \oplus (\mathbb{Z}_2)\varepsilon & & \text{, for } (n,m,r) = (2^i \geq 4, 2^j \geq 4, 2^k \geq 2) \\ (\mathbb{Z}_g)\xi \oplus (\mathbb{Z}_g)\gamma & ,(\varepsilon = 0) & \text{, for } n \text{ or } m \text{ or } r \text{ odd} \end{cases}$$

Here $\varepsilon = (i_n \# i_m \# i_r)\eta_4$, $\xi = \xi_{nm} \# i_r$, $\gamma = i_n \# \xi_{mr}$.

Moreover we have the relations

$$(4) \qquad 0 = \xi_{nm}\eta_4 + \frac{n}{gcd(n,m)}(i_n \# \xi_m) + \frac{m}{gcd(n,m)}(\xi_m \# i_n) + \varrho(m = n = 4)\varepsilon_{nm}\eta_4$$

(5) $$0 = H(r,n,m)(\xi_{nm}\# i_r) - H(n,r,m)(i_n\#\xi_{mr}) + H(m,n,r)T_{132}(\xi_{nr}\# i_m)$$

with $H(n,m,r) = \dfrac{n \cdot gcd(n,m,r)}{gcd(n,m)\cdot gcd(n,r)}$

Proof: These homotopy groups can be computed using the EHP-Sequence in its extended form, see [7]. The generators are constructed using the generators of the fourth homotopy groups fixed in (§6). ///

Let again P_n be a pseudo projective plane, let $2^i, 2^j, 2^t$ be finite numbers. Then we have

(7.2) **Theorem:**

(1) $$[\Sigma^2 P_k, \Sigma P_n] \cong \begin{cases} (\mathbb{Z}_2)\xi \oplus (\mathbb{Z}_2)\gamma & (\varepsilon = 0) & \text{, for } (k,n) = (2,2) \\ (\mathbb{Z}_2)\varepsilon \oplus (\mathbb{Z}_2)\xi \oplus (\mathbb{Z}_g)\gamma & & \text{, for } (k,n) = (2^i \geq 2, 2^j \geq 4) \\ (\mathbb{Z}_4)\xi \oplus (\mathbb{Z}_4)\gamma & (\varepsilon = 2\xi) & \text{, for } (k,n) = (2^i \geq 4, 2) \\ (\mathbb{Z}_2)\varepsilon & (\xi = \gamma = 0) & \text{, for } (k,n) = (2^i \geq 2, \infty) \\ (\mathbb{Z}_g)\gamma & (\xi = \varepsilon = 0) & \text{, else} \end{cases}$$

Here $\mathbb{Z}_g = \mathrm{Hom}(\mathbb{Z}_k, \Gamma\mathbb{Z}_n)$ and the generators are $\varepsilon = \varepsilon_n q = i_n \eta_2^2 q$, $\xi = \xi_n q$ and $\gamma = \gamma_n^k$ defined in [8]. Again q is the pinch map in the cofiber sequence $S^2 \xrightarrow{i_n} \Sigma P_n \xrightarrow{q} S^3$. The group (1) is embeded in the exact sequence

$$\mathrm{Ext}(\mathbb{Z}_k, \Pi_4 \Sigma P_n) \rightarrowtail [\Sigma^2 P_k, \Sigma P_n] \twoheadrightarrow \mathrm{Hom}(\mathbb{Z}_k, \Gamma\mathbb{Z}_n)$$

.

(2)
$[\Sigma^2 P_k, \Sigma P_n \wedge P_m] \cong$

$$\begin{cases} (\mathbb{Z}_2)\xi \oplus (\mathbb{Z}_2)\eta & (\varepsilon = 0) & ,(k,n,m) = (2,2,2) \\ (\mathbb{Z}_2)\xi \oplus (\mathbb{Z}_4)\eta & (\varepsilon = 2\eta) & ,(k,n,m) = (2,2^i \geq 4,2),(2,2,2^j \geq 4) \\ (\mathbb{Z}_4)\xi \oplus (\mathbb{Z}_2)\eta & (\varepsilon = 2\xi) & ,(k,n,m) = (2^t \geq 4,2,2) \\ (\mathbb{Z}_2)\varepsilon \oplus (\mathbb{Z}_e)\xi \oplus (\mathbb{Z}_d)\eta & & ,(k,n,m) = (2^t \geq 2, 2^i \geq 2, 2^j \geq 2) \\ & & \text{at most one index equal 2} \\ (\mathbb{Z}_2)\varepsilon \oplus (\mathbb{Z}_d)\eta & (\xi = 0) & ,(k,n,m) = (2^t \geq 2, 2^i \geq 2, \infty) \text{ or } (2^t \geq 2, \infty, 2^j \geq 2) \\ & & \text{at most one index equal 2} \\ (\mathbb{Z}_2)\varepsilon & (\xi = \eta = 0) & ,(k,n,m) = (2^t \geq 2, \infty, \infty) \\ (\mathbb{Z}_4)\eta & (\xi = 0, \varepsilon = 2\eta) & ,(k,n,m) = (2,2,\infty),(2,\infty,2) \\ (\mathbb{Z}_e)\xi \oplus (\mathbb{Z}_d)\eta & (\varepsilon = 0) & \text{, else} \end{cases}$$

Here $\mathbb{Z}_e = \mathrm{Ext}(\mathbb{Z}_k, \mathbb{Z}_n * \mathbb{Z}_m)$, $\mathbb{Z}_d = \mathrm{Hom}(\mathbb{Z}_k, \mathbb{Z}_n \otimes \mathbb{Z}_m)$ and the generators are $\varepsilon = \varepsilon_{nm} q = (i_n \# i_m)\eta_3 q$, $\xi = \xi_{nm} q$ and $\eta = \eta_{nm}^k$. We set

(3) $$\eta_{nm}^k = \begin{cases} i_n \# \mathcal{X}_m^k & \text{if } m = gcd(n,m) \\ -(T_{21})_* \eta_{mn}^k & \text{if } m \neq gcd(n,m) \end{cases}$$

where the $\mathcal{X}_m^k$ are defined in (6.3).

(7.3) **Lemma:** The generator $\gamma_n^k \in [\Sigma^2 P_k, \Sigma P_n]$, $k < \infty$ defined in [8] has the James-Hopf invariant

$$\gamma_2(\gamma_n^k) \quad = \quad \tfrac{2gcd(n,k)}{g}\eta_{nn}^k - \tfrac{k}{g}\xi_{nn}q \qquad , \; g = gcd(n^2, 2n, k).$$

We get for $k < \infty$

$$\begin{array}{lrcl} (1) & \gamma_n^k \circ \Sigma i_k & = & i_n 2\frac{n}{g}\eta_2 \\ (2) & \gamma_n^k \circ \Sigma \xi_k & = & \frac{k}{g}[1,1](\xi_n \# i_n) \\ (3) & (\gamma_n^k \wedge P_k) \circ \Sigma \xi_{kk} & = & \frac{2n}{g}(i_n \# \xi_k) + \lambda_{nk}\varepsilon_{nk}\eta_4 \end{array}$$

$$\text{with } \lambda_{nk} = \begin{cases} 1 & , (n,k) = (4,2),(4,4),(4,8) \\ 0,1 & , (n,k) = (2,4) \text{ unknown} \\ 0 & , \text{else} \end{cases} .$$

In case $k = \infty$ we define $\gamma_n^k = i_n\eta_2$, this is the generator of $[\Sigma^2 P_k, \Sigma P_n] = [S^3, \Sigma P_n]$. Then we have

$$(1') \qquad \gamma_n^k \circ \Sigma i_k = i_n \eta_2$$

Observe that $\frac{2n}{g} = 1$ for n even, finite.

Proof: The composition laws can be proved using [8] as done in [9]. ///

We are now ready to prove theorems on $\mathcal{E}(M|\dot{M})$.

(7.4) **Theorem:** Let M be a Σ-reducible 1-connected 5-dimensional Poincaré complex and let $B = H_2 M$. Then

$$\mathcal{E}(M|\dot{M}) \quad = \quad (B * \mathbb{Z}_2) \oplus (B * \mathbb{Z}_3) \oplus \Big((B/\mathrm{Tor}B) \otimes \mathbb{Z}_2\Big) \oplus \Big((B/B * \mathbb{Z}_2) \otimes \mathbb{Z}_2\Big)$$

As a quotient $\mathcal{E}(M|\dot{M}) = \Pi_5\dot{M}/\mathcal{J}$ is generated by equivalence classes of the elements $y_i\xi_n\eta_4$, $y_i\zeta_n$, $y_i'\eta_3^2$ and $y_i i_n \eta_2^3$. Here we use the generators defined in (7.1)(1) and the inclusions $y_i\colon \Sigma P_n \to \dot{M}$, $y_i'\colon S^3 \to \dot{M}$ of (5.3).

Proof: By (5.6) and (5.5) we have a normal form for $M = \dot{M} \cup_f e^5$ where $\dot{M} = M^3$ is a one point union of suspended pseudo projective spaces and of spheres. Hence we may use the Hilton-Milnor theorem to decompose $\Pi_5\dot{M}$ into a direct sum of groups. It is

$$\Pi_5\dot{M} \quad \cong \quad \bigoplus_{i\in\mathcal{J}} (y_i)_* \Pi_5 \Sigma P_{n_i} \oplus \bigoplus_{i \in \mathcal{J}^0} (y_i')_* \Pi_5 S^3 \oplus \ldots \text{ higher order terms}$$

in the notation of (7.6)(1) below. Supposed we can show condition $(*)$ in (4.2) then we get the result from theorem (4.3). This is since all higher order terms contain Whitehead products. Moreover observe that the generators $[1,1](i_n\#\xi_n)$, $[[1,1],1](i_n\#\xi_{nn})$ as given in (7.1)(1) of a summand $(y_i)_*\Pi_5\Sigma P_{n_i} \subset \Pi_5\dot{M}$ also contain Whitehead products and hence vanish in $\mathcal{E}(M|\dot{M})$. Therefore only the generators of type $\xi_n\eta_4$, $i_n\eta_2^3$, $i\eta_3^2$ and ζ_n with $n = 3^i \geq 3$ remain as representatives of nontrivial equivalence classes. Observe that $B/\mathrm{Tor}B \cong H_3 M$ and that $(B/B * \mathbb{Z}_2) \otimes \mathbb{Z}_2$ is in the cokernel of d_2 in (1.18). We will show condition $(*)$ in the succeding lemma. ///

(7.5) **Lemma:** For the attaching map $f\colon S^4 \to \Sigma A$ of an 1-connected 5-dimensional Poincaré complex $M = \Sigma A \cup_f e^5$ condition $(*)$ of (4.2) holds.

Proof: Let $J = \text{kernel}[1,1]_* + \text{kernel}\Sigma + \text{Im}\,\overline{u}_{\Sigma A}$ with the maps defined in (4.2). We have to show that $\Pi_5 \Sigma A \wedge A \subset J$ holds. By the proof of (3.7)[5] we get for any connected finite CW-complex X a commutative diagram

$$(1)\qquad \begin{array}{ccc} [\Sigma^2 A, \Sigma X] & \xrightarrow{\overline{u}_{\Sigma X}} & \Pi_5 \Sigma X \wedge A \\ \downarrow{\scriptstyle \Sigma^2} & & \downarrow{\scriptstyle \Sigma} \\ [\Sigma^4 A, \Sigma^3 X] & & \Pi_6 \Sigma X \wedge \Sigma A \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ \{\Sigma A, X\} & \underset{\cong}{\xrightarrow{u_X}} & \{S^5, X \wedge \Sigma A\} \end{array}$$

Here $\overline{u}_{\Sigma X}(\xi) = (\xi \wedge 1_A) \circ (\Sigma \gamma_2 f)$ and u_X is its stabilization. Since f is the attaching map of a Poincaré complex u_X is an isomorphism, see (2.4) [5]. Moreover if X is simply connected then $\overline{u}_{\Sigma X}$ is an isomorphism. From the normal form (5.5) we know that we can take ΣA as an one point union of Moore spaces of cyclic groups in dimension 2 and 3. Hence we may decompose $\Pi_5 \Sigma A \wedge A$ using the Hilton-Milnor theorem, and obtain

$$(2)\qquad \begin{aligned} \Pi_5 \Sigma A \wedge A \;\cong\; & \bigoplus_{i,j \in \overline{\mathcal{J}}} (y_i \# y_j)_* \Pi_5 \Sigma P_{n_i} \wedge P_{n_j} \oplus \bigoplus_{\substack{i \in \overline{\mathcal{J}} \\ j \in \mathcal{J}^0}} (y_i \# y_{j'}) \Pi_5 \Sigma P_{n_i} \wedge S^2 \\ \oplus\; & \bigoplus_{\substack{i \in \mathcal{J}^0 \\ j \in \overline{\mathcal{J}}}} (y_{i'} \# y_j)_* \Pi_5 \Sigma S^2 \wedge P_{n_j} \oplus \bigoplus_{i,j \in \mathcal{J}^0} (y_{i'} \# y_{j'}) \Pi_5 S^5 \\ \oplus\; & \text{higher order terms} \end{aligned}$$

in the notation of (7.6)(1) below. Since the higher order terms of the decomposition lie in the kernel of $\Sigma\colon \Pi_5 \Sigma A \wedge A \to \Pi_6 \Sigma A \wedge \Sigma A$ they are contained in J. If we take ΣX to be a Moore space in dimension 3 then X is simply connected and hence $\overline{u}_{\Sigma X}$ is surjective, thus we have $\Pi_5 \Sigma X \wedge A \subset J$. Since any element $\Sigma u \wedge v \in \Pi_5 \Sigma A \wedge A$ is equivalent to $\Sigma v \wedge u$ modulo the kernel of $[1,1]_*\colon \Pi_5 \Sigma A \wedge A \to \Pi_5 \Sigma A$ we may conclude that also $\Pi_5 \Sigma A \wedge X \subset \Pi_5 \Sigma A \wedge A$ is contained in J. The remaining generators of $\Pi_5 \Sigma A \wedge A$ we have not found to be in J yet are of the form $(y_i \# y_j)(i_{n_i} \# i_{n_j})\eta_3^2$, $(y_i \# y_j)(\xi_{n_i} \# i_{n_j})$, $(y_i \# y_j)(i_{n_i} \# \xi_{n_j})$, see (7.1)(2). From these $(y_i \# y_j)(\xi_{n_i} \# i_{n_j})$ is equivalent to $(y_j \# y_i)(i_{n_j} \# \xi_{n_i})$ modulo the kernel of $[1,1]_*$. Hence let G be the subgroup of $\Pi_5 \Sigma A \wedge A$ generated by $(y_i \# y_j)(i_{n_i} \# i_{n_j})\eta_3^2$, $(y_i \# y_j)(\xi_{n_i} \# i_{n_j})$, $i, j \in \overline{\mathcal{J}}$, then $\Pi_5 \Sigma A \wedge A \subset J + G$ by the arguments above.

We may decompose $[\Sigma^2 A, \Sigma A]$ using the Hilton-Milnor theorem. Let H be the subgroup $\bigoplus_{i \in \overline{\mathcal{J}}} [\Sigma^2 A, \Sigma P_{n_i}] \subset [\Sigma^2 A, \Sigma A]$ generated by the elements $y_i i_{n_i} \eta_2^3 q r_j$, $y_i \xi_{n_i} q r_j$, $i \in \overline{\mathcal{J}}$, $j \in \mathcal{J}$, where r_j is the projection to a component of ΣA. Since the elements in H and G are stable under suspension and since these groups are finite $\overline{u}_{\Sigma A} | H \to G$ is an isomorphism by (1). Hence $G \subset \text{Im}\,\overline{u}_{\Sigma A} \subset J$ and we have shown $\Pi_5 \Sigma A \wedge A \subset J$. ///

(7.6) **Theorem:** Let M be a non Σ-reducible 1-connected manifold of dimension 5. Then M has the normal form $M = X_q \# P$ wher P is Σ-reducible and X_q has nontrivial Stiefel-Whitney class. Let $B = H_2 P$, then

$$\begin{aligned} \mathcal{E}(M|\dot{M}) \;\cong\; & (B * \mathbb{Z}_2)(y_i \xi_{n_i} \eta_4) \oplus (B * \mathbb{Z}_3)(y_i \zeta_{n_i}) \oplus \Big((B/\text{Tor}B) \otimes \mathbb{Z}_2\Big)(y_{i'} i \eta_3^2) \\ \oplus\; & B_q \oplus \mathcal{E}(X_q|\dot{X}_q) \end{aligned}$$

$$\text{with } B_q = \begin{cases} (B/B * \mathbb{Z}_2) \otimes \mathbb{Z}_2\ ([y_\omega, y_i]\varepsilon_{2n_i}\eta_4) & \text{for } q = \infty \text{ or } q = 2^i \geq 8 \\ (B/B * \mathbb{Z}_4) \otimes \mathbb{Z}_4\ ([y_\omega, y_i](\xi_2 \# i_{n_i})) & \text{for } q = -1, 2 \\ (B/B * \mathbb{Z}_8) \otimes \mathbb{Z}_2\ ([y_\omega, y_i]\varepsilon_{2n_i}\eta_4) & \text{for } q = 4 \end{cases}$$

The elements we gave in the brackets determine the type of elements representing the equivalence classes which generate $\mathcal{E}(M|\dot{M}) = \Pi_5\dot{M}/J$. The notation is defined in the proof.

<u>**Proof:**</u> From (5.4) we obtain the normal form $M = X_q \# P$. Let $P = \Sigma A \cup_f e^5$ and $X_q = \Sigma B \cup_g e^5$. We define, using the notation of (5.4), the index sets

(1)
$$\begin{aligned} \mathcal{J}^0 &= \{1, \ldots, s\} \times \{0\} \quad \subset \quad \mathcal{J} \\ \mathcal{J}^+ &= \{s+1, \ldots, s+r\} \times \{0\} \quad \subset \quad \mathcal{J} \\ \tilde{\mathcal{J}}^+ &= \{i' \in \mathcal{J} | i \in \mathcal{J}^+\} \\ \overline{\mathcal{J}} &= \mathcal{J}^0 \cup \mathcal{J}^+ \cup \tilde{\mathcal{J}}^+ \end{aligned}$$

which are subsets of $\mathcal{J} = \{0, \ldots, s+r\} \times \{0, 1\}$. For $i = (\iota, \nu) \in \mathcal{J}$ define $i' = (\iota, 1-\nu)$. Moreover we identify $(\iota, 0) \in \mathcal{J} \setminus \{0, \nu\}$ with $2\iota - 1 \in \{1, \ldots, 2k\}$ and $(\iota, 1) \in \mathcal{J}$ with $2\iota \in \{0, \ldots, 2k\}$, $k = s + r$, and $(0, \nu)$ with $0 \in \{0, \ldots, 2k\}$ which are the index sets of a basis of $H_2M \oplus H_3M$, see (5.4). We then obtain with n_i equal p_i of (5.4) for $i \in \mathcal{J}^+ \cup \tilde{\mathcal{J}}^+$ and $n_i = \infty$ for $i \in \mathcal{J}^0$

$$H_2P = \bigoplus_{i \in \overline{\mathcal{J}}} (\mathbb{Z}_{n_i}) y_i, \qquad H_3P = \bigoplus_{i \in \mathcal{J}^0} (\mathbb{Z}_{n_i}) y_{i'}$$

For the attaching map f of P we get

(2)
$$\begin{aligned} f &= \sum_{i \in \mathcal{J}^0} [y_i, y_{i'}]\iota_4 \quad + \quad \sum_{i \in \mathcal{J}^+} [y_i, y_{i'}]\xi_{n_i n_i} \\ \Sigma A &= \bigvee_{i \in \mathcal{J}^0} (S^2 \vee S^3) \quad \vee \quad \bigvee_{i \in \mathcal{J}^+} (\Sigma P_{n_i} \vee \Sigma P_{n_i}) \end{aligned}$$

We will write y_i' for $y_{i'}$ and n for n_i if the index is understood. The second James-Hopf invariant of f may be computed using methods of [4], giving

(3)
$$\begin{aligned} \gamma_2 f &= \sum_{i \in \mathcal{J}^0} ((y_i \# y_i')\iota_4 - (y_i' \# y_i)\iota_4) \\ &+ \sum_{i \in \mathcal{J}^+} ((y_i \# y_i')\xi_{nn} - (y_i' \# y_i)\xi_{nn}) \end{aligned}$$

<u>We will now restrict our computations to the case $X_q = X_{-1}$.</u> The remaining cases can be computed analogous to what follows, but the arising groups are bigger and more complex.

We have $X_{-1} = \Sigma B \cup_g e^5$ with $\Sigma B = \Sigma P_2$, $g = y_\omega \xi_2$ and $P = \Sigma A \cup_f e^5$. Therefore we get

(4)
$$\Sigma g = \Sigma y_\omega \circ \Sigma \xi_2, \qquad \gamma_2 g = (y_\omega \# y_\omega)\xi_{22} \qquad \text{and } \gamma_3 g = 0$$

compare (6.6).

We will use Theorem (4.5) to get our result. Here we are in the right range of dimensions and by Lemma (7.5) condition $(*)$ holds for P. The following tables list the groups arising from (4.5) in our case. So let $G = \Pi_5\Sigma A/W \oplus \Sigma\Pi_5\Sigma A\wedge B \oplus \Pi_5\Sigma A\wedge B\wedge B \oplus \Pi_5\Sigma A\wedge B\wedge B\wedge B$. We have

$$(5)\qquad \begin{aligned}
\Pi_5\Sigma A\wedge B\wedge B\wedge B &\cong \bigoplus_{i\in\overline{\mathcal{J}}}(y_i\#y_w\#y_w\#y_w)_*\Pi_5\Sigma P_n\wedge P_2\wedge P_2\wedge P_2\\
\Pi_5\Sigma A\wedge B\wedge B &\cong \bigoplus_{i\in\overline{\mathcal{J}}}(y_i\#y_w\#y_w)_*\Pi_5\Sigma P_n\wedge P_2\wedge P_2\\
&\oplus \bigoplus_{i\in\mathcal{J}^0}(y_i'\#y_w\#y_w)_*\Pi_5\Sigma^3 P_2\wedge P_2\\
\Sigma\Pi_5\Sigma A\wedge B &\cong \bigoplus_{i\in\overline{\mathcal{J}}}\Sigma(y_i\#y_w)_*\Sigma\Pi_5\Sigma P_n\wedge P_2\\
&\oplus \bigoplus_{i\in\mathcal{J}^0}\Sigma(y_i'\#y_w)_*\Sigma\Pi_5\Sigma^3 P_2\\
\Pi_5\Sigma A/W &\cong \bigoplus_{i\in\overline{\mathcal{J}}}(y_i)_*\Pi_5\Sigma P_n/W \oplus \bigoplus_{i\in\mathcal{J}^0}(y_i')_*\Pi_5 S^3
\end{aligned}$$

Here W denotes the subgroup of elements containing Whitehead products. The groups on the right hand side, as far as they are not listed in (7.1), are as follows $\big((n,2) = gcd(n,2)\big)$.

$$\begin{aligned}
\Pi_5\Sigma P_n\wedge P_2\wedge P_2\wedge P_2 &\cong (\mathbb{Z}_{(n,2)})(i_n\#i_2\#i_2\#i_2)\iota_5\\
\Pi_5\Sigma^3 P_2\wedge P_2 &\cong (\mathbb{Z}_2)\Sigma^3(i_2\#i_2)\iota_5\\
\Pi_5\Sigma^3 P_2 &\cong (\mathbb{Z}_2)\Sigma^2 i_2\eta_4\\
\Pi_5 S^3 &\cong (\mathbb{Z}_2)\eta_3^2
\end{aligned}$$

Moreover we have

$$(6)\qquad \begin{aligned}
[\Sigma^2 B, \Sigma A\wedge B\wedge B] &\cong \bigoplus_{i\in\overline{\mathcal{J}}}(y_i\#y_w\#y_w)_*(r_w)^*[\Sigma^2 P_2, \Sigma P_n\wedge P_2\wedge P_2]\\
[\Sigma^2 B, \Sigma A\wedge B] &\cong \bigoplus_{i\in\overline{\mathcal{J}}}(y_i\#y_w)_*(r_w)^*[\Sigma^2 P_2, \Sigma P_n\wedge P_2]\\
&\oplus \bigoplus_{i\in\bar{\mathcal{J}}^0}(y_i\#y_w)_*(r_w)^*[\Sigma^2 P_2, \Sigma^3 P_2]\\
[\Sigma^2 B, \Sigma A] &\cong \bigoplus_{i\in\overline{\mathcal{J}}}(y_i)_*(r_w)^*[\Sigma^2 P_2, \Sigma P_n]\\
&\oplus \bigoplus_{i\in\bar{\mathcal{J}}^0}(y_i)_*(r_w)^*[\Sigma^2 P_2, S^3]\\
[\Sigma^2 A, \Sigma B] &\cong \bigoplus_{i'\overline{\mathcal{J}}}(y_w)_*(r_{i'})^*[\Sigma^2 P_n, \Sigma P_2]\\
&\oplus \bigoplus_{i'\in\bar{\mathcal{J}}^0}(y_w)_*(r_{i'})^*[S^4, \Sigma P_2]
\end{aligned}$$

Here r_i is the retraction to the component i such that $r_i y_i = id$. The groups on the right hand side are listed in (7.2), resp. are as follows

$$\begin{aligned}
[\Sigma^2 P_2, \Sigma P_n\wedge P_2\wedge P_2] &\cong (\mathbb{Z}_{(n,2)})(i_n\#i_2\#i_2)q\\
[\Sigma^2 P_2, \Sigma^3 P_2] &\cong (\mathbb{Z}_2)(\Sigma^2 i_2)q\\
[\Sigma^2 P_2, S^3] &\cong (\mathbb{Z}_2)\eta_3 q
\end{aligned}$$

The next step is to evaluate the elements given in (4.5)(a)–(d) in our case. For this we will need the composition laws given in (7.3), (6.6) and the following

$$
\begin{aligned}
q \circ \Sigma\xi_n &= \eta_4 \\
(q\wedge 1)\circ \Sigma\xi_{nn} &= \iota_5 \\
(q\wedge 1\wedge 1)\circ \iota_5 &= 0
\end{aligned}
\tag{7}
$$

compare (6.10).

The subgroup $V \subset G$ defined in (4.5) is in our case generated by the elements

(8) $\Sigma(y_i\#y_w)\Sigma(i_n\#i_2)\eta_3^2 + (y_i\#y_w\#y_w)(i_n\#i_2\#i_2)\eta_4$

(9) $\Sigma(y_i\#y_w)\Sigma(i_n\#\xi_2) + (y_i\#y_w\#y_w)(i_n\#\xi_{22})$

(10) (1) $n = 2$ $\quad -(y_i'\#y_w\#y_w)(i_2\#\xi_{22})$

(2) $n = 4$ $\quad \Sigma(y_i'\#y_w)\Sigma(\xi_4\#i_2) + \Sigma(y_i'\#y_w)\lambda_{24}\Sigma(i_4\#i_2)\eta_3^2$
$\quad +(y_i'\#y_w\#y_w)(\xi_{42}\#i_2) - (y_i'\#y_w\#y_w)3(i_4\#\xi_{22})$

(3) $n = 8$ $\quad \Sigma(y_i'\#y_w)\Sigma(\xi_8\#i_2)$
$\quad +(y_i'\#y_w\#y_w)(\xi_{82}\#i_2) - (y_i'\#y_w\#y_w)2(i_8\#\xi_{22})$

(4) $n = 2^i \geq 16$ $\quad \Sigma(y_i'\#y_w)\Sigma(\xi_n\#i_2) + (y_i'\#y_w\#y_w)(\xi_{n2}\#i_2)$

(5) $n = \infty$ $\quad \Sigma(y_i'\#y_w)(i^3\#i_2)\eta_4 + (y_i'\#y_w\#y_w)(i^3\#i_2\#i_2)\iota_5$

(11) $(y_i\#y_w\#y_w)(i_n\#i_2\#i_2)\eta_4 + (y_i\#y_w\#y_w\#y_w)(i_n\#i_2\#i_2\#i_2)\iota_5$

(12) $\Sigma(y_i\#y_w)\Sigma(i_n\#i_2)\eta_3^2 + (y_i\#y_w\#y_w)(i_n\#i_2\#i_2)\eta_4$

(13) $\Sigma(y_i\#y_w)\Sigma\big(\frac{n}{2}(i_n\#\xi_2) + \xi_n\#i_2\big) + (y_i\#y_w\#y_w)(\xi_{n2}\#i_2)$

(14) $\Sigma(y_i\#y_w)\Sigma(i_n\#\xi_2) + (y_i\#y_w\#y_w)(i_n\#\xi_{22})$

(15) $\Sigma(y_k'\#y_w)\Sigma(i^3\#i_2)\eta_4 + (y_k'\#y_w\#y_w)(i^3\#i_2\#i_2)\iota_5$

(16) $\{(y_i)i_n\eta_2^3\} + \Sigma(y_i\#y_w)\Sigma(i_n\#i_2)\eta_3^2$

(17) $\{(y_i)\xi_n\eta_4\} + \Sigma(y_i\#y_w)\Sigma(\xi_n\#i_2)$

(18) $\{(y_k')i^3\eta_3^2\} + \Sigma(y_k'\#y_w)\Sigma(i^3\#i_2)\eta_4$

with $i \in \overline{\mathcal{J}}$ and $k \in \mathcal{J}^0$. These elements are obtained from (4.5) by setting ξ_1^0 equal $y_w i_2\eta_2^2 q r_i'$, $y_w\xi_2 q r_i'$, $y_w\gamma_2^n r_i$ for (8),(9) and (10). Moreover $\xi_{011}^1 = (y_i\#y_w\#y_w)(i_n\#i_2\#i_2)qr_w$ for (11), ξ_{01}^1 equal $(y_i\#y_w)(i_n\#i_2)\eta_3 q r_w$, $(y_i\#y_w)\xi_{n2}qr_w$, $(y_i\#y_w)\eta_{n2}^2 r_w$, $(y_k'\#y_w)(i^3\#i_2)qr_w$ for (12)–(15) and ξ_0^1 equal $y_i i_n\eta_2^2 q r_w$, $y_i\xi_n q r_w$, $y_k'\eta_3 q r_w$ for (16)–(18). The element obtained from (4.5)(d) with $\xi_0^1 = y_i\gamma_n^2 r_w$ is trivial. For the necessary computations we used the relations (7.1)(4),(5), the composition laws in (7.3) and those of (§6).

By use of these elements (8)–(18) the theorem can be shown easily (still in the case $X_q = X_{-1}$). The following computations are carried out in the group

$$
G/V = \Big(\Pi_5\Sigma A/W \oplus \Sigma\Pi_5\Sigma A\wedge B \oplus \Pi_5\Sigma A\wedge B\wedge B \oplus \Pi_5\Sigma A\wedge B\wedge B\wedge B\Big)/V \tag{19}
$$

where we write $\alpha = \beta$ if α and β are equivalent modulo V. Observe that the elements (8) = (12), (9) = (14) and (10)(5) = (15). We have

(20)	$\Sigma(y_i\#y_w)\Sigma(i_n\#i_2)\eta_3^2$	$=$	$2\Sigma(y_i\#y_w)\Sigma(i_n\#\xi_2)$	with (7.1)(2)
(21)	$(y_i\#y_w\#y_w\#y_w)(i_n\#i_2\#i_2\#i_2)\iota_5$	$=$	$(y_i\#y_w\#y_w)(i_n\#i_2\#i_2)\eta_4$	with (11)
		$=$	$\Sigma(y_i\#y_w)\Sigma(i_n\#i_2)\eta_3^2$	with (12)
		$=$	$2\Sigma(y_i\#y_w)\Sigma(i_n\#\xi_2)$	with (20)
(22)	$(y_i\#y_w\#y_w)(i_n\#i_2\#i_2)\eta_4$	$=$	$2\Sigma(y_i\#y_w)\Sigma(i_n\#\xi_2)$	with (12), (20)
(23)	$(y_k'\#y_w\#y_w)(i^3\#i_2\#i_2)\iota_5$	$=$	$\Sigma(y_k'\#y_w)\Sigma(i^3\#i_2)\eta_4$	with (15)
		$=$	$\{y_k' i^3\eta_3^2\}$	with (18)
(24)	$\Sigma(y_k'\#y_w)\Sigma(i^3\#i_2)\eta_4$	$=$	$\{y_k' i^3\eta_3^2\}$	with (18)
(25)	$(y_i\#y_w\#y_w)(i_n\#\xi_{22})$	$=$	$-\Sigma(y_i\#y_w)\Sigma(i_n\#\xi_2)$	with (14)
(26)	$(y_i\#y_w\#y_w)(\xi_{n2}\#i_2)$	$=$	$\Sigma(y_i\#y_w)\Sigma(\frac{n}{2}(i_n\#\xi_2)+\xi_n\#i_2)$	with (13)
		$=$	$\Sigma(y_i\#y_w)\Sigma(\frac{n}{2}(i_n\#\xi_2)+\{y_i\xi_n\eta_4\})$	with (17)
(27)	$\Sigma(y_i\#y_w)\Sigma(\xi_n\#i_2)$	$=$	$\{y_i\xi_n\eta_4\}$	with (17)
(28)	$\{y_i i_n\eta_2^3\}$	$=$	$2\Sigma(y_i\#y_w)\Sigma(i_n\#\xi_2)$	with (16), (20)

From (10) we derive in the different cases

(29) (1)	$n=2$	0	$=$	$-(y_i'\#y_w\#y_w)(i_2\#\xi_{22})$	
			$=$	$\Sigma(y_i'\#y_w)\Sigma(i_2\#\xi_2)$	with (25)
(2)	$n=4$	0	$=$	$\{y_i'\xi_4\eta_4\}+\lambda_{24}\Sigma(y_i'\#y_w)\Sigma(i_4\#i_2)\eta_3^2$	with (27)
			$+$	$\Sigma(y_i'\#y_w)2\Sigma(i_4\#\xi_2)+\{y_i'\xi_4\eta_4\}$	with (26)
			$+$	$\Sigma(y_i'\#y_w)3\Sigma(i_4\#\xi_2)$	with (25)
			$=$	$\lambda_{24}\Sigma(y_i'\#y_w)\Sigma(i_4\#i_2)\eta_3^2+\Sigma(y_i'\#y_w)\Sigma(i_4\#\xi_2)$	
			$=$	$\pm\Sigma(y_i'\#y_w)\Sigma(i_4\#\xi_2)$	with (20)
(3)	$n=8$	0	$=$	$\Sigma(y_i'\#y_w)\Sigma 2(i_8\#\xi_2)$	with (25), (26), (27)
(4)	$n=2^i\geq 16$	trivial by use of (23)			

We now derive our conclusion. Let $J\subset G$ be the subgroup generated by the elements $\{y_i\xi_n\eta_4\}$, $\{y_i\zeta_r\}$, $\{y_k' i^3\eta_3^2\}$, $\Sigma(y_i\#y_w)\Sigma(i_n\#\xi_2)$ where $i\in\mathcal{J}^+\cup\tilde{\mathcal{J}}^+$, $k\in\mathcal{J}^\circ$, $n=2^t\geq 2$, $r=3^k\geq 3$. Hence

$$J\cong(H*\mathbb{Z}_2)\oplus(H*\mathbb{Z}_3)\oplus((H/\mathrm{Tor}H)\otimes\mathbb{Z}_2)\oplus(H\otimes\mathbb{Z}_4)$$

Let $V'\subset J$ be the subgroup generated by the elements

$$\begin{cases}\Sigma(y_i\#y_w)\Sigma(i_n\#\xi_2) & ,\ n=2,4\\ 2\Sigma(y_i\#y_w)\Sigma(i_n\#\xi_2) & ,\ n=8\end{cases}$$

then $J/V'\cong(H*\mathbb{Z}_2)\oplus(H*\mathbb{Z}_3)\oplus((H/\mathrm{Tor}H)\otimes\mathbb{Z}_2)\oplus((H/(H*\mathbb{Z}_4))\otimes\mathbb{Z}_4)$. Now the equations (20) ... (28) define a retraction $\psi: G\longrightarrow J\subset G$, $\varphi|J=id$ by assigning to the generators on the left the generators on the right hand side. The kernel of this map is generated by (8),(9),(11),...,(18) and (7.1)(2). For the image $\psi(V)\subset J$ we have $\psi(V)=V'$, which is clear by inspection of the relations (29) which are given by the image of the subgroup of V generated by (10). Hence passing to the quotients we get a map

$$\psi': G/V\xrightarrow{\cong}J/V'$$

which is an isomorphism. This is the result in case $X_q = X_{-1}$. ///

We complete the computation of $\mathcal{E}(M|\dot{M})$ for 1-connected 5-dimensional Poincaré complexes by citing some theorems. These theorems are proved using (4.1) and (4.5) and the material in this paragraph but they are still more complex. The complete prove can be found in [9].

(7.7) **Theorem:** For the manifolds X_q defined in (5.3) the abelian group $\mathcal{E}(X_q|\dot{X}_q)$ is given by

$$\mathcal{E}(X_q|\dot{X}_q) = \begin{cases} \mathbb{Z}_2 & q = -1, \infty \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & q = 2, 4 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & q = 2^i \geq 8 \end{cases}$$

The generators are equivalence classes of the elements

$$\begin{array}{ll} y'_w i^3 \eta_3^2 & \text{for } q = \infty \\ y_w[1,1](\xi_2 \# i_2) & \text{for } q = -1 \\ y_w[1,1](\xi_q \# i_q),\ y'_w \xi_q \eta_4 & \text{for } q = 2^i \geq 2 \\ [y_w, y'_w](i_q \# i_q)\eta_3^2 & \text{for } q = 2^i \geq 8 \end{array}$$

given in the notation of (7.6).

Theorem (7.7) is a corollary of the more general

(7.8) **Theorem:** Let N be a simply connected 5-dimensional Poincaré complex in the normal form of (5.4) with $P = S^5$. Then we get for N of type I–VII of (5.4)

$$\mathcal{E}(N|\dot{N}) \cong \begin{cases} 0 & (I);\quad (IV)q = \infty;\quad (VII)q = t = \infty \\ \mathbb{Z}_2 & (II);\quad (III)q = \infty \\ 2\mathbb{Z}_2 & (III)q = 2,4;\quad (IV)q = 2;\quad (V)q = \infty;\quad (VI)(t = \infty \wedge q \geq 4); \\ & (VII)q = 2 \vee q \geq 8 \\ 3\mathbb{Z}_2 & (III)q \geq 8;\quad (IV)q \geq 4;\quad (V)q = 2,4;\quad (VI)(t = \infty \wedge q = 2); \\ & (VII)q = 4 \\ 4\mathbb{Z}_2 & (VI)(t \geq q \geq 4) \vee (t = 2 \wedge q = 4) \vee (t = 2,4 \wedge q = 2) \vee (t \geq 2 \wedge q = \infty) \\ 5\mathbb{Z}_2 & (V)q \geq 8;\quad (VI)t < q \geq 8 \\ 6\mathbb{Z}_2 & (VI)(t \geq 8 \wedge q = 2) \end{cases}$$

Here we have $q = 2^i$, $t = 2^j$ finite unless otherwise stated and $\wedge, \vee$ denote logical operators.

(7.9) **Theorem:** Let $M = N \sharp P$ be a simply connected 5-dimensional Poincaré complex in the normal form of (5.4) where P is the Σ-reducible part and where N is as in theorem (7.8). Let $B = H_2 P$, then

$$\mathcal{E}(M|\dot{M}) \cong \tilde{\mathcal{E}}(P|\dot{P}) \oplus B^3(M) \oplus B(M) \oplus \mathcal{E}(N|\dot{N})$$

with

$$\tilde{\mathcal{E}}(P|\dot{P}) \cong (B * \mathbb{Z}_2) \oplus (B * \mathbb{Z}_3) \subset \mathcal{E}(P|\dot{P})$$

$$B^3(M) = \begin{cases} 0 & (IV)q=\infty; \quad (V)q=\infty; \quad (VI)t=\infty \\ (B/\mathrm{Tor}B)\otimes \mathbb{Z}_2 & \text{else} \end{cases}$$

$$B(M) = \begin{cases} 0 & (IV)q=\infty; \quad (VI)(t\geq q\geq 4)\vee(t=\infty\wedge q\geq 4)\vee(t=q=\infty) \\ (B/B*\mathbb{Z}_2)\otimes\mathbb{Z}_2 & (I); \quad (III)q=\infty\vee q\geq 8; \quad (IV)q=2; \quad (VII)q=2,4,\infty \\ (B/B*\mathbb{Z}_4)\otimes\mathbb{Z}_4 & (II); \quad (III)q=2 \\ (B/B*\mathbb{Z}_4)\otimes\mathbb{Z}_2 & (V); \quad (VI)q=2 \\ (B/B*\mathbb{Z}_{\frac{q}{2}})\otimes\mathbb{Z}_2 & (IV)q\geq 4 \\ (B/B*\mathbb{Z}_{2q})\otimes\mathbb{Z}_2 & (III)q=4; \quad (VII)q\geq 8 \\ (B/B*\mathbb{Z}_t)\otimes\mathbb{Z}_2 & (VI)q>t; \quad (VI)(t\geq 2\wedge q=\infty) \end{cases}$$

Here again $q = 2^i$, $t = 2^j$ finite unless otherwise stated. The cases concern the types of M in the normal form of (5.4).

(7.10) **Corollary:** Let M be a simply connected Poincaré complex of dimension 5 with free homology. Then we have

$$\mathcal{E}(M|\dot{M}) \cong \begin{cases} (H_3M\otimes\mathbb{Z}_2)\oplus \mathrm{kernel}(\omega_2(M)\otimes 1) & \text{if } M \text{ is a manifold} \\ 0 & \text{else} \end{cases}$$

Here we use the homomorphism $\omega_2(M)\otimes 1\colon H_2M\otimes\mathbb{Z}_2 \to \mathbb{Z}_2\otimes\mathbb{Z}_2$ with $\omega_2(M) \in H^2(M;\mathbb{Z}_2) \cong \mathrm{Hom}(H_2M,\mathbb{Z}_2)$ beeing the second Stiefel-Whitney class. The complex M is a manifold exactly if its exotic characteristic class $e(M)$ is trivial, see (5.5).

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