A BIJECTION BETWEEN TYPE $D_n^{(1)}$ CRYSTALS AND RIGGED CONFIGURATIONS

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ABSTRACT. Hatayama et al. conjectured fermionic formulas associated with tensor products of $U'_q(\mathfrak{g})$-crystals $B^{r,s}$. The crystals $B^{r,s}$ correspond to the Kirillov–Reshetikhin modules which are certain finite dimensional $U'_q(\mathfrak{g})$-modules. In this paper we present a combinatorial description of the affine crystals $B^{r,1}$ of type $D_n^{(1)}$. A statistic preserving bijection between crystal paths for these crystals and rigged configurations is given, thereby proving the fermionic formula in this case. This bijection reflects two different methods to solve lattice models in statistical mechanics: the corner-transfer-matrix method and the Bethe Ansatz.

1. INTRODUCTION

The quantized universal enveloping algebra $U_q(\mathfrak{g})$ associated with a symmetrizable Kac–Moody Lie algebra $\mathfrak{g}$ was introduced independently by Drinfeld [3] and Jimbo [6] in their study of two dimensional solvable lattice models in statistical mechanics. The parameter q corresponds to the temperature of the underlying model. Kashiwara [8] showed that at zero temperature or $q = 0$ the representations of $U_q(\mathfrak{g})$ have bases, which he coined crystal bases, with a beautiful combinatorial structure and favorable properties such as uniqueness and stability under tensor products.

The irreducible finite-dimensional $U'_q(\mathfrak{g})$-modules were classified by Chari and Pressley [1, 2] in terms of Drinfeld polynomials. The Kirillov–Reshetikhin modules $W^{r,s}$, labeled by a Dynkin node r and a positive integer s, form a special class of these finite-dimensional modules. They naturally correspond to the weight $s\Lambda_r$, where Λ_r is the r-th fundamental weight of $\mathfrak{g}$. Recently, Hatayama et al. [5, 4] conjectured that the Kirillov–Reshetikhin modules $W^{r,s}$ have a crystal basis denoted by $B^{r,s}$. The existence of such crystals allows the definition of one dimensional configuration sums, which play an important role in the study of phase transitions of two dimensional exactly solvable lattice models. For $\mathfrak{g}$ of type $A_n^{(1)}$, the existence of the crystal $B^{r,s}$ was settled in [11], and the one dimensional configuration sums contain the Kostka polynomials, which arise in the theory of symmetric functions, combinatorics, the study of subgroups of finite abelian groups, and Kazhdan–Lusztig theory. In certain limits they are branching functions of integrable highest weight modules.

In [5, 4] fermionic formulas for the one dimensional configuration sums were conjectured. Fermionic formulas originate in the Bethe Ansatz of the underlying exactly solvable lattice model. The term fermionic formula was coined by the Stony Brook group [14, 15], who interpreted fermionic-type formulas for characters and branching functions of conformal field theory models as partition functions of quasiparticle systems with "fractional" statistics obeying Pauli's exclusion principle. For type $A_n^{(1)}$ the fermionic formulas were

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proven in [17] using a generalization of a bijection between crystals and rigged configurations of Kirillov and Reshetikhin [16]. In [19] similar bijections were used to prove the fermionic formula for nonexceptional types for crystals $B^{1,1}$. Rigged configurations are combinatorial objects which label the solutions to the Bethe equations. The bijection between crystals and rigged configurations reflects two different methods to solve lattice models in statistical mechanics: the corner-transfer-matrix method and the Bethe Ansatz.

The theory of virtual crystals [18, 20] provides a realization of crystals of type X as crystals of type Y, based on well-known natural embeddings $X \hookrightarrow Y$ of affine algebras [7]:

$$\begin{array}{lcl} C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, D_{n+1}^{(2)} & \hookrightarrow & A_{2n-1}^{(1)} \\ A_{2n-1}^{(2)}, B_n^{(1)} & \hookrightarrow & D_{n+1}^{(1)} \\ E_6^{(2)}, F_4^{(1)} & \hookrightarrow & E_6^{(1)} \\ D_4^{(3)}, G_2^{(1)} & \hookrightarrow & D_4^{(1)}. \end{array}$$

Note that under these embeddings every affine Kac–Moody algebra is embedded into one of simply-laced type $A_n^{(1)}$, $D_n^{(1)}$ or $E_6^{(1)}$. Hence, by the virtual crystal method the combinatorial structure of any finite-dimensional affine crystal can be expressed in terms of the combinatorial crystal structure of the simply-laced types. Whereas the affine crystals $B^{r,s}$ of type $A_n^{(1)}$ are already well-understood [21], this is not the case for $B^{r,s}$ of types $D_n^{(1)}$ and $E_6^{(1)}$.

In this paper we discuss the affine crystals $B^{r,1}$ for type $D_n^{(1)}$ and the corresponding rigged configurations. The crystal structure of $B^{r,1}$ of type $D_n^{(1)}$ is given by Koga [13]. However, here we need a different combinatorial description of these crystals compatible with rigged configurations. We present a bijection between crystal paths and rigged configurations of type $D_n^{(1)}$, thereby proving the fermionic formulas of [5, 4] for type $D_n^{(1)}$ corresponding to tensor products of crystals of the form $B^{r,1}$.

The paper is structured as follows. In Section 2 the main definitions regarding crystals are reviewed. Section 3 deals with the finite-dimensional crystals of type $D_n^{(1)}$ explicitly. In particular $B^{k,1}$ as a set is given by single column tableaux of height k with the action of the Kashiwara operators as specified in Theorem 3.2. The bijection between crystal paths and rigged configurations is stated in Section 4. The proof of the main Theorem 4.1 is relegated to Appendix A. In Section 5 statistics are defined on both crystal paths and rigged configurations. It is shown in Theorem 5.1 that the bijection preserves the statistics. This yields a proof of the fermionic formulas as presented in Section 5.1. Appendices B and C are devoted to proofs of various lemmas. In Appendix D the combinatorial R-matrix is given explicitly for highest weight elements.

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2. Crystal bases

2.1. **Axiomatic definition of crystals.** Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra, P the weight lattice, I the index set for the vertices of the Dynkin diagram of $\mathfrak{g}$, $\{\alpha_i \in P \mid i \in I\}$ the simple roots, and $\{h_i \in P^* \mid i \in I\}$ the simple coroots. Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}$ [10]. A $U_q(\mathfrak{g})$-crystal is a nonempty set B

equipped with maps $\mathrm{wt} : B \to P$ and $e_i, f_i : B \to B \cup \{\varnothing\}$ for all $i \in I$, satisfying

$$f_i(b) = b' \Leftrightarrow e_i(b') = b \text{ if } b, b' \in B \tag{2.1}$$

$$\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i \text{ if } f_i(b) \in B \tag{2.2}$$

$$\langle h_i\,,\, \mathrm{wt}(b)\rangle = \varphi_i(b) - \epsilon_i(b). \tag{2.3}$$

Here for $b \in B$

$$\epsilon_i(b) = \max\{n \geq 0 \mid e_i^n(b) \neq \varnothing\}$$
$$\varphi_i(b) = \max\{n \geq 0 \mid f_i^n(b) \neq \varnothing\}.$$

(It is assumed that $\varphi_i(b), \epsilon_i(b) < \infty$ for all $i \in I$ and $b \in B$.) A $U_q(\mathfrak{g})$-crystal B can be viewed as a directed edge-colored graph (the crystal graph) whose vertices are the elements of B, with a directed edge from b to b' labeled $i \in I$, if and only if $f_i(b) = b'$.

2.2. **Tensor products of crystals.** Let $B_1, B_2, \dots, B_L$ be $U_q(\mathfrak{g})$-crystals. The Cartesian product $B_L \times \cdots \times B_2 \times B_1$ has the structure of a $U_q(\mathfrak{g})$-crystal using the so-called signature rule. The resulting crystal is denoted $B = B_L \otimes \cdots \otimes B_2 \otimes B_1$ and its elements $(b_L, \dots, b_1)$ are written $b_L \otimes \cdots \otimes b_1$ where $b_j \in B_j$. The reader is warned that our convention is opposite to that of Kashiwara [10]. Fix $i \in I$ and $b = b_L \otimes \cdots \otimes b_1 \in B$. The i-signature of b is the word consisting of the symbols $+$ and $-$ given by

$$\underbrace{- \cdots -}_{\varphi_i(b_L)\text{ times}} \quad \underbrace{+ \cdots +}_{\epsilon_i(b_L)\text{ times}} \quad \cdots \quad \underbrace{- \cdots -}_{\varphi_i(b_1)\text{ times}} \quad \underbrace{+ \cdots +}_{\epsilon_i(b_1)\text{ times}}.$$

The reduced i-signature of b is the subword of the i-signature of b, given by the repeated removal of adjacent symbols $+-$ (in that order); it has the form

$$\underbrace{- \cdots -}_{\varphi\text{ times}} \quad \underbrace{+ \cdots +}_{\epsilon\text{ times}}.$$

If $\varphi = 0$ then $f_i(b) = \varnothing$; otherwise

$$f_i(b_L \otimes \cdots \otimes b_1) = b_L \otimes \cdots \otimes b_{j+1} \otimes f_i(b_j) \otimes \cdots \otimes b_1$$

where the rightmost symbol $-$ in the reduced i-signature of b comes from b_j. Similarly, if $\epsilon = 0$ then $e_i(b) = \varnothing$; otherwise

$$e_i(b_L \otimes \cdots \otimes b_1) = b_L \otimes \cdots \otimes b_{j+1} \otimes e_i(b_j) \otimes \cdots \otimes b_1$$

where the leftmost symbol $+$ in the reduced i-signature of b comes from b_j. It is not hard to verify that this well-defines the structure of a $U_q(\mathfrak{g})$-crystal with $\varphi_i(b) = \varphi$ and $\epsilon_i(b) = \epsilon$ in the above notation, with weight function

$$\mathrm{wt}(b_L \otimes \cdots \otimes b_1) = \sum_{j=1}^{L} \mathrm{wt}(b_j). \tag{2.4}$$

This tensor construction is easily seen to be associative. The case of two tensor factors is given explicitly by

$$f_i(b_2 \otimes b_1) = \begin{cases} f_i(b_2) \otimes b_1 & \text{if } \epsilon_i(b_2) \geq \varphi_i(b_1) \\ b_2 \otimes f_i(b_1) & \text{if } \epsilon_i(b_2) < \varphi_i(b_1) \end{cases} \tag{2.5}$$

and

$$e_i(b_2 \otimes b_1) = \begin{cases} e_i(b_2) \otimes b_1 & \text{if } \epsilon_i(b_2) > \varphi_i(b_1) \\ b_2 \otimes e_i(b_1) & \text{if } \epsilon_i(b_2) \leq \varphi_i(b_1). \end{cases} \tag{2.6}$$

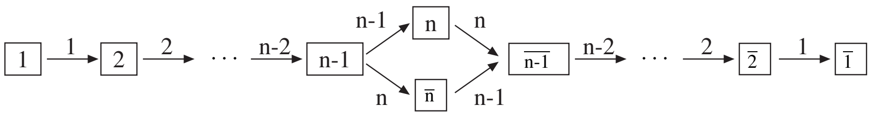


FIGURE 1. Crystal $B(\omega_1)$ of the vector representation

2.3. **Combinatorial R-matrix.** Let B_1 and B_2 be two $U_q(\mathfrak{g})$-crystals such that $B_1 \otimes B_2$ and $B_2 \otimes B_1$ are connected and such that there is a weight $\lambda \in P$ with unique elements $u \in B_1 \otimes B_2$ and $\tilde{u} \in B_2 \otimes B_1$ of weight $\mathrm{wt}(u) = \mathrm{wt}(\tilde{u}) = \lambda$. Then there is a unique crystal isomorphism

$$R : B_1 \otimes B_2 \to B_2 \otimes B_1,$$

called the combinatorial R-matrix. Explicitly, $R(u) = \tilde{u}$ and the action of R on any other element in $B_1 \otimes B_2$ is determined by using

$$R(e_i(b)) = e_i(R(b)) \qquad \text{and} \qquad R(f_i(b)) = f_i(R(b)). \tag{2.7}$$

3. THE CRYSTAL $B^{k,1}$ OF TYPE $D_n^{(1)}$

From now on we restrict our attention to crystals of type D_n and $D_n^{(1)}$.

3.1. **Dynkin data.** For type D_n, the simple roots are

$$\begin{aligned} \alpha_i &= \varepsilon_i - \varepsilon_{i+1} && \text{for } 1 \leq i < n \\ \alpha_n &= \varepsilon_{n-1} + \varepsilon_n \end{aligned}$$

and the fundamental weights are

$$\begin{aligned} \Lambda_i &= \varepsilon_1 + \cdots + \varepsilon_i && \text{for } 1 \leq i \leq n-2 \\ \Lambda_{n-1} &= (\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n)/2 \\ \Lambda_n &= (\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon_n)/2 \end{aligned}$$

where $\varepsilon_i \in Z^n$ is the i-th unit standard vector. We also define

$$\begin{aligned} \omega_i &= \Lambda_i = \varepsilon_1 + \cdots + \varepsilon_i && \text{for } 1 \leq i \leq n-2 \\ \omega_{n-1} &= \Lambda_{n-1} + \Lambda_n = \varepsilon_1 + \cdots + \varepsilon_{n-1} \\ \omega_n &= 2\Lambda_n = \varepsilon_1 + \cdots + \varepsilon_n \\ \overline{\omega}_n &= 2\Lambda_{n-1} = \varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n. \end{aligned}$$

3.2. **The crystals of antisymmetric tensor representations.** The $U_q(D_n)$-crystal graphs of the antisymmetric tensor representations [12] are denoted by $B(\omega_\ell)$ (resp. $B(\overline{\omega}_n)$). The crystal graph $B(\omega_1)$ of the vector representation is given in Figure 1. The crystal $B(\omega_\ell)$ (resp. $B(\overline{\omega}_n)$) is the connected component of $B(\omega_1)^{\otimes \ell}$ (resp. $B(\omega_1)^{\otimes n}$) containing the element $\ell \otimes \ell - 1 \otimes \cdots \otimes 1$ (resp. $\overline{n} \otimes n-1 \otimes \cdots \otimes 1$).

Explicitly, the elements of $B(\omega_\ell)$ for $1 \leq \ell < n$ are single columns of height ℓ with entries from the alphabet $\{1 < 2 < \cdots < n-1 < \frac{n}{n} < \overline{n-1} < \cdots < \overline{1}\}$. In Young tableau notation a column is represented by

$$\begin{array}{|c|}\hline m_1 \\ \hline m_2 \\ \hline \vdots \\ \hline m_\ell \\ \hline \end{array} \qquad \text{which we abbreviate by } m_\ell \dots m_1.$$

If $m_\ell \dots m_1$ is in $B(\omega_\ell)$ it has to satisfy

$$\begin{aligned} &1.\ (m_j, m_{j+1}) = (n, \overline{n}), (\overline{n}, n) \text{ or } m_j < m_{j+1} \text{ for all } j; \\ &2.\ \text{If } m_a = p \text{ and } m_b = \overline{p}, \text{ then } \operatorname{dist}(\overline{p}, p) \le p. \end{aligned} \tag{3.1}$$

Here $\operatorname{dist}(\overline{p}, p) = \ell + 1 + a - b$ if $m_a = p$ and $m_b = \overline{p}$.

The elements of $B(\omega_n)$ (resp. $B(\overline{\omega}_n)$) are single columns of height n. An element $m_n \dots m_1 \in B(\omega_n)$ (resp. $B(\overline{\omega}_n)$) satisfies (3.1) and in addition

$$\begin{aligned} &3.\ \text{If } m_j = n, \text{ then } n - j \text{ is even (resp. odd);} \\ &\quad \text{if } m_j = \overline{n}, \text{ then } n - j \text{ is odd (resp. even).} \end{aligned} \tag{3.2}$$

The action of e_i, f_i for $1 \le i \le n$ is given by embedding $B(\omega_\ell) \hookrightarrow B(\omega_1)^{\otimes \ell}$ (resp. $B(\overline{\omega}_n) \hookrightarrow B(\omega_1)^{\otimes n}$) in the natural way by mapping $m_\ell \cdots m_1$ to $m_\ell \otimes \cdots \otimes m_1$ and using the tensor product rules.

3.3. **The crystals of spinor representations.** There are two crystals $B(\Lambda_n)$ and $B(\Lambda_{n-1})$ associated with the spinor representations of D_n [12]. The elements of $B(\Lambda_n)$ (resp. $B(\Lambda_{n-1})$) are given by single columns $m_n \dots m_1$ such that

$$\begin{aligned} &1.\ m_j < m_{j+1}; \\ &2.\ m \text{ and } \overline{m} \text{ cannot occur simultaneously;} \\ &3.\ \text{If } m_j = n, \text{ then } n - j \text{ is even (resp. odd);} \\ &\quad \text{if } m_j = \overline{n}, \text{ then } n - j \text{ is odd (resp. even).} \end{aligned} \tag{3.3}$$

If $b \in B(\Lambda_n)$ or $B(\Lambda_{n-1})$ and b contains i and $\overline{i+1}$ for $1 \le i < n$, then $f_i b$ is obtained by replacing i by $i+1$ and $\overline{i+1}$ by $\overline{i}$. Else $f_i b = 0$. If $b \in B(\Lambda_n)$ or $B(\Lambda_{n-1})$ and b contains $n-1$ and n, then $f_n b$ is obtained by replacing $n-1$ by $\overline{n}$ and n by $\overline{n-1}$. Otherwise $f_n b = 0$.

Note that for $k = n-1, n$ there is the following embedding of crystals

$$B(\Lambda_k) \hookrightarrow B(2\Lambda_k)$$

where e_i and f_i act as e_i^2 and f_i^2, respectively.

3.4. **Affine crystals.** As a classical crystal the affine crystal $B^{k,1}$ is isomorphic to

$$B^{k,1} \cong \begin{cases} B(\Lambda_k) \oplus B(\Lambda_{k-2}) \oplus \cdots \oplus B(0) & \text{if } k \text{ is even, } 1 \le k \le n-2 \\ B(\Lambda_k) \oplus B(\Lambda_{k-2}) \oplus \cdots \oplus B(\Lambda_1) & \text{if } k \text{ is odd, } 1 \le k \le n-2 \\ B(\Lambda_k) & \text{if } k = n-1, n. \end{cases} \tag{3.4}$$

The rule for f_0 on $B^{k,1}$ for $k = n, n-1$ is as follows. If $b = \overline{12} m_{n-2} \dots m_1$ then $f_0 b = m_{n-2} \dots m_1 21$. Otherwise $f_0 b = 0$. The operator f_0 on $B^{k,1}$ when $1 \le k < n-1$ was given by Koga [13] in terms of a tensor product of two spinor representations. For our purposes, we will need a combinatorial description of f_0 on elements $b \in B^{k,1}$ represented by a column of height k. This description of b will also be used in the description of the bijection from crystals to rigged configurations.

For $1 \le k \le n-2$, we want to represent $B^{k,1}$ as the set of all column tableaux of height k satisfying (3.1) point 1. By (3.4) $B^{k,1}$ is the union of the sets corresponding to $B(\Lambda_\ell)$ with $k - \ell \equiv 0 \pmod 2$. We will define maps from $B(\Lambda_\ell)$ to column tableaux of height k. As explained in Section 3.2, $b \in B(\Lambda_\ell)$ can be represented by a column of height ℓ. If $b \in B(\Lambda_\ell) \subset B^{k,1}$ then fill the column of height ℓ of b successively by a pair $(\overline{i}_j, i_j)$ for $1 \le j \le (k-\ell)/2$ in the following way to obtain a column of height k. Set $i_0 = 0$. Let $i_{j-1} < i_j$ be minimal such that

(1) neither i_j nor $\overline{i_j}$ is in the column;
(2) adding i_j and $\overline{i_j}$ to the column we have $\mathrm{dist}(\overline{i_j}, i_j) \ge i_j + j$;
(3) adding i_j and $\overline{i_j}$ to the column, all other pairs $(\overline{a}, a)$ in the new column with $a > i_j$ satisfy $\mathrm{dist}(\overline{a}, a) \le a + j$.

It is not hard to see that f_i and the filling map commute, that is, the classical crystal graph with edges $f_1, \dots, f_n$ does not change. Denote the filling map to height k by F_k or simply F.

Let D_k or D, the dropping map, be the inverse of F_k. Explicitly, given a one-column tableau b of height k, let $i_0 = 0$ and successively find $i_j > i_{j-1}$ minimal such that the pair $(\overline{i_j}, i_j)$ is in b and $\mathrm{dist}(\overline{i_j}, i_j) \ge i_j + j$. Drop all such pairs $(\overline{i_j}, i_j)$ from b.

For the definition of f_0 we need slight variants of F_k and D_k, denoted by $\tilde{F}_k$ and $\tilde{D}_k$, respectively, which act on columns that do not contain $1, 2, \overline{2}, \overline{1}$. On these columns $\tilde{F}_k$ and $\tilde{D}_k$ are defined by replacing $i \mapsto i-2$ and $\overline{i} \mapsto \overline{i-2}$, then applying F_k and D_k, and finally replacing $i \mapsto i+2$ and $\overline{i} \mapsto \overline{i+2}$.

Example 3.1. Let $b = \overline{35}532 \in B(\Lambda_5) \subset B^{9,1}$. Then $F_9(b) = \overline{3567}76532$. Similarly if $b = \overline{2345}65421 \in B^{9,1}$ then $D_9(b) = \overline{3}61$. Finally, $\tilde{F}_5(\overline{4}64) = \overline{45}654$.

Theorem 3.2. *Let $b \in B^{k,1}$. Then*

$$f_0 b = \begin{cases} F_k(\tilde{D}_{k-2}(x)) & \text{if } b = \overline{12}x \\ \tilde{F}_{k-1}(x)2 & \text{if } b = \overline{12}x2 \\ \tilde{F}_{k-1}(x)1 & \text{if } b = \overline{12}x1 \\ \tilde{F}_{k-2}(x)21 & \text{if } b = \overline{12}x21 \\ F_k(\tilde{D}_{k-1}(x)2) & \text{if } b = \overline{1}x \\ F_k(\tilde{D}_{k-1}(x)1) & \text{if } b = \overline{2}x \\ x21 & \text{if } b = \overline{1}x1 \text{ and } \tilde{D}_{k-2}(x) = x \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

$$e_0 b = \begin{cases} F_k(\tilde{D}_{k-2}(x)) & \text{if } b = x21 \\ \overline{2}\tilde{F}_{k-1}(x) & \text{if } b = \overline{2}x21 \\ \overline{1}\tilde{F}_{k-1}(x) & \text{if } b = \overline{1}x21 \\ \overline{12}\tilde{F}_{k-2}(x) & \text{if } b = \overline{12}x21 \\ F_k(\overline{2}\tilde{D}_{k-1}(x)) & \text{if } b = x1 \\ F_k(\overline{1}\tilde{D}_{k-1}(x)) & \text{if } b = x2 \\ \overline{12}x & \text{if } b = \overline{1}x1 \text{ and } \tilde{D}_{k-2}(x) = x \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

where x does not contain $1, 2, \overline{2}, \overline{1}$.

Proof. In [13], Koga gives the crystal $B^{k,1}$ in terms of a tensor product of two spinor crystals. Let us denote the map from our description in terms of columns of height k to Koga's picture by ψ. Since F_k commutes with all e_i, f_i for $1 \le i \le n$, $B^{k,1} \cong \psi(B^{k,1})$ as classical crystals. Hence ψ is specified by its action on highest weight vectors which is given by

$$\psi(\overline{\ell+1}\ \cdots \overline{p}\, p \cdots \ell+1\ \ell \cdots 21) = \overline{\ell+1} \cdots \overline{n}\ \ell \cdots 21\ \otimes\ n \cdots 21$$

where $p = (k+\ell)/2$.

Note that f_0 commutes with f_i for $3 \le i \le n$. Hence, to show that the above action of f_0 on $B^{k,1}$ is equivalent to the action given by Koga, it suffices to show that $\psi(f_0 b) = f_0 \psi(b)$ where b is highest weight with respect to e_i for $3 \le i \le n$, that is $e_i b = 0$ for these i. Call such b J'-highest weight where $J' = \{3, 4, \ldots, n\}$. This can be done explicitly for each case. $\square$

Alternatively, the action of f_0 and e_0 is given by $f_0 = \sigma \circ f_1 \circ \sigma$ and $e_0 = \sigma \circ e_1 \circ \sigma$ where σ is an involution on $B^{k,1}$ based on the type $D_n^{(1)}$ Dynkin diagram automorphism that interchanges nodes 0 and 1. If $b \in B^{k,1}$ contains either 1 or $\overline{1}$, $\sigma(b)$ is obtained from b by interchanging 1 and $\overline{1}$. If $b = \overline{1}x1$, then $\sigma(b) = \hat{F}_k(D(x))$. If b contains neither 1 nor $\overline{1}$ then $\sigma(b) = F_k(\hat{D}(b))$. Here $\hat{D}$ and $\hat{F}$ act on columns not containing 1 and $\overline{1}$ and are defined in the same way as $\tilde{D}$ and $\tilde{F}$ with the maps $i \mapsto i \pm 2, \overline{i} \mapsto \overline{i \pm 2}$ replaced by $i \mapsto i \pm 1, \overline{i} \mapsto \overline{i \pm 1}$.

For later use, it will be convenient to define the sets $\hat{B}^{k,1}$ for $1 \le k \le n$ and $\hat{\hat{B}}^{n,1}$. For $1 \le k \le n-2$, $\hat{B}^{k,1} = B^{k,1}$. $\hat{B}^{n-1,1}$ is the set of all single columns of height $n-1$ satisfying condition (3.1) point 1. $\hat{B}^{n,1}$ (resp. $\hat{\hat{B}}^{n,1}$) is the set of all single columns of height n satisfying (3.1) point 1 and (3.2). An affine crystal structure can be defined on $\hat{B}^{n-1,1}$, $\hat{B}^{n,1}$ and $\hat{\hat{B}}^{n,1}$ by extending the rules of Theorem 3.2.

Theorem 3.3. *As affine crystals*

$$\begin{aligned}
\hat{B}^{n-1,1} &\cong B^{n,1} \otimes B^{n-1,1}\\
\hat{B}^{n,1} &\cong B^{n,1} \otimes B^{n,1}\\
\hat{\hat{B}}^{n,1} &\cong B^{n-1,1} \otimes B^{n-1,1}.
\end{aligned}$$

Proof. The proof is similar to the proof of Theorem 3.2. $\square$

There are affine crystal embeddings

$$\mathrm{emb}_B : B^{k,1} \hookrightarrow B^{k,1} \otimes B^{k,1} \tag{3.7}$$

mapping $u(B^{k,1}) \mapsto u(B^{k,1}) \otimes u(B^{k,1})$ for $1 \le k \le n$ and sending e_i and f_i to e_i^2 and f_i^2, respectively, for $0 \le i \le n$. Here $u(B^{k,1}) = k(k-1)\cdots 1$ for $1 \le k \le n-2$, $u(B^{n-1,1}) = \overline{n}(n-1)\cdots 21$ and $u(B^{n,1}) = n(n-1)\cdots 21$. Note that the elements in the image of this embedding are aligned (see [18, Section 6.4]), meaning in this case that $\varphi_i(b)$ and $\epsilon_i(b)$ are even for all $0 \le i \le n$. Denote the image of $B^{k,1}$ under this embedding by $E^{k,1} = \mathrm{emb}_B(B^{k,1})$. For $k = n-1, n$ we view $E^{n-1,1} \subset \hat{\hat{B}}^{n,1}$ and $E^{n,1} \subset \hat{B}^{n,1}$.

As classical crystals

$$\begin{aligned}
\hat{B}^{n-1,1} &\cong B(\omega_{n-1}) \oplus B(\omega_{n-3}) \oplus B(\omega_{n-5}) \oplus \cdots\\
\hat{B}^{n,1} &\cong B(\omega_n) \oplus B(\omega_{n-2}) \oplus B(\omega_{n-4}) \oplus \cdots\\
\hat{\hat{B}}^{n,1} &\cong B(\overline{\omega}_n) \oplus B(\omega_{n-2}) \oplus B(\omega_{n-4}) \oplus \cdots .
\end{aligned}$$

3.5. **Paths.** Let $B = B^{k_L,1} \otimes B^{k_{L-1},1} \otimes \cdots \otimes B^{k_1,1}$ be a type $D_n^{(1)}$ crystal with $1 \le k_i \le n$. For a dominant integral weight λ define the set of paths as follows

$$\mathcal{P}(\lambda, B) = \{ b \in B \mid \mathrm{wt}(b) = \lambda,\, e_i(b) = \varnothing \text{ for all } i \in J \}$$

where $J = I \backslash \{0\} = \{1, 2, \ldots, n\}$.

3.6. **Dual crystals.** Let ω_0 be the longest element in the Weyl group of D_n. The action of ω_0 on the weight lattice P of D_n is given by

$$\begin{aligned} \omega_0(\Lambda_i) &= -\Lambda_{\sigma(i)} \\ \omega_0(\alpha_i) &= -\alpha_{\sigma(i)} \end{aligned}$$

where $\sigma : J \to J$ is the identity if n is even and interchanges $n-1$ and n and fixes all other Dynkin nodes if n is odd. The automorphism σ extends to the Dynkin diagram of $D_n^{(1)}$ by defining $\sigma(0) = 0$.

There is a unique involution $* : B \to B$, called dual map, satisfying

$$\begin{aligned} \mathrm{wt}(b^*) &= \omega_0 \mathrm{wt}(b) \\ e_i(b)^* &= f_{\sigma(i)}(b^*) \\ f_i(b)^* &= e_{\sigma(i)}(b^*). \end{aligned}$$

Let $u(B)$ be the highest weight element in the highest component of B (for $B = B^{k,1}$ with $1 \le k \le n-2$ the highest component is $B(\Lambda_k)$). Uniqueness of $*$ follows from the connectedness of B and the fact that $u(B)$ is the unique vector of its weight. We have

$$(B_1 \otimes B_2)^* \cong B_2 \otimes B_1$$

with $(b_1 \otimes b_2)^* \mapsto b_2^* \otimes b_1^*$.

Explicitly, on $B(\Lambda_1)$ the involution $*$ is given by

$$i \longleftrightarrow \overline{i}$$

except for $i = n$ with n odd in which case $n \leftrightarrow n$ and $\overline{n} \leftrightarrow \overline{n}$. A column $m_k \cdots m_1 \in B^{k,1}$ is mapped to $m_1^* \cdots m_k^*$.

4. Bijection between paths and rigged configurations

Let $B = B^{k_L,1} \otimes B^{k_{L-1},1} \otimes \cdots \otimes B^{k_1,1}$ be a type $D_n^{(1)}$ crystal and λ a dominant integral weight. In this section we define the set of rigged configurations $\mathrm{RC}(\lambda, B)$ and a bijection $\Phi : \mathrm{RC}(\lambda, B) \to \mathcal{P}(\lambda, B)$.

4.1. **Rigged configurations.** Denote by L^a the number of factors $B^{a,1}$ in B, so that $B = \bigotimes_{a=1}^n (B^{a,1})^{\otimes L^a}$. For later use, let us also define $\hat{L}^{n-1}$ (resp. $\hat{L}^n$, $\hat{\hat{L}}^n$) to be the number of tensor factors $\hat{B}^{n-1,1}$ (resp. $E^{n,1}$, $E^{n-1,1}$) in B. Let

$$\boldsymbol{L} = \sum_{a \in J} L^a \Lambda_a + \sum_{a \in \{n-1,n\}} \hat{L}^a \omega_a + \hat{\hat{L}}^n \overline{\omega}_n. \tag{4.1}$$

Then the sequence of partitions $\nu^\bullet = (\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(n)})$ is a (λ, B)-configuration if

$$\sum_{\substack{a \in J \\ i \ge 0}} i m_i^{(a)} \alpha_a = \boldsymbol{L} - \lambda, \tag{4.2}$$

where $m_i^{(a)}$ is the number of parts of length i in partition $\nu^{(a)}$. A (λ, B)-configuration is admissible if $p_i^{(a)} \geq 0$ for all $a \in J$ and $i \geq 0$, where $p_i^{(a)}$ is the vacancy number

$$p_i^{(a)} = \big(\alpha_a \mid \boldsymbol{L} - \sum_{b \in J} \alpha_b \sum_{j \geq 1} \min(i,j) m_j^{(b)}\big), \tag{4.3}$$

where $(\cdot \mid \cdot)$ is the normalized invariant form on P such that $(\alpha_i \mid \alpha_j)$ is the Cartan matrix. Denote the set of admissible (λ, B)-configurations by $\mathrm{C}(\lambda, B)$. A rigged configuration $(\nu^\bullet, J^\bullet)$ is a sequence of partitions $\nu^\bullet = \{\nu^{(a)}\}_{a \in J} \in \mathrm{C}(\lambda, B)$ together with a double sequence of partitions $J^\bullet = \{J^{(a,i)}\}_{\substack{a \in J \\ i \geq 0}}$ such that the partition $J^{(a,i)}$ is contained in a box of size $m_i^{(a)} \times p_i^{(a)}$. The set of rigged configurations is denoted by $\mathrm{RC}(\lambda, B)$.

4.2. **The bijection Φ.** Let B' be a tensor product of type $D_n^{(1)}$ crystals of the form $B^{a,1}$ or $\hat{B}^{n-1,1}$.

Theorem 4.1. *There exists a unique family of bijections $\Phi_B : \mathrm{RC}(\lambda, B) \to \mathcal{P}(\lambda, B)$ such that:*

(1) *Suppose $B = B^{1,1} \otimes B'$. Then the diagram*

$$\begin{array}{ccc} \mathrm{RC}(\lambda, B) & \xrightarrow{\Phi_B} & \mathcal{P}(\lambda, B) \\ \delta \big\downarrow & & \big\downarrow \mathrm{lh} \\ \bigcup_{\lambda^-} \mathrm{RC}(\lambda^-, B') & \xrightarrow{\Phi_{B'}} & \bigcup_{\lambda^-} \mathcal{P}(\lambda^-, B') \end{array}$$

commutes.

(2) *Suppose $B = \hat{B}^{k,1} \otimes B'$ with $2 \leq k \leq n-1$ and let $\widetilde{B} = B^{1,1} \otimes \hat{B}^{k-1,1} \otimes B'$. Then the diagram*

$$\begin{array}{ccc} \mathrm{RC}(\lambda, B) & \xrightarrow{\Phi_B} & \mathcal{P}(\lambda, B) \\ \mathrm{tj} \big\downarrow & & \big\downarrow \mathrm{ts} \\ \mathrm{RC}(\lambda, \widetilde{B}) & \xrightarrow{\Phi_{\widetilde{B}}} & \mathcal{P}(\lambda, \widetilde{B}) \end{array}$$

commutes.

(3) *Suppose $B = B^{k,1} \otimes B'$ with $k = n-1, n$. Then the diagram*

$$\begin{array}{ccc} \mathrm{RC}(\lambda, B) & \xrightarrow{\Phi_B} & \mathcal{P}(\lambda, B) \\ \delta_s \big\downarrow & & \big\downarrow \mathrm{lh}_s \\ \bigcup_{\lambda^{-s}} \mathrm{RC}(\lambda^{-s}, B') & \xrightarrow{\Phi_{B'}} & \bigcup_{\lambda^{-s}} \mathcal{P}(\lambda^{-s}, B') \end{array}$$

commutes.

A proof of this theorem is given in Appendix A. In the following we define the various yet undefined maps.

4.3. **Various maps.** The map $\mathrm{lh} : \mathcal{P}(\lambda, B^{1,1} \otimes B') \to \bigcup_{\lambda^-} \mathcal{P}(\lambda^-, B')$ in point 1 of Theorem 4.1 is given by $b \otimes b' \mapsto b'$. The union is over all λ^- such that $\lambda - \lambda^-$ is the weight of an element in $B^{1,1}$.

Similarly, the map $\mathrm{lh}_s : \mathcal{P}(\lambda, B) \to \bigcup_{\lambda^{-s}} \mathcal{P}(\lambda^{-s}, B')$ of point 3 is given by $b \otimes b' \mapsto b'$. The union is over all λ^{-s} such that $\lambda - \lambda^{-s}$ is the weight of an element in $B^{k,1}$ for $k = n-1$ or n.

The map top split $\mathrm{ts} : \mathcal{P}(\lambda, B) \to \mathcal{P}(\lambda, \widetilde{B})$ in point 2 of the theorem is obtained by sending $m_k m_{k-1} \cdots m_1 \otimes b' \in \mathcal{P}(\lambda, \hat{B}^{k,1} \otimes B')$ to $m_k \otimes m_{k-1} \cdots m_1 \otimes b' \in \mathcal{P}(\lambda, B^{1,1} \otimes \hat{B}^{k-1,1} \otimes B')$ for $1 \le k \le n-1$. Similarly we may also define ts to map $m_n m_{n-1} \cdots m_1 \otimes b' \in \mathcal{P}(\lambda, E^{n-1,1} \otimes B')$ or $\mathcal{P}(\lambda, E^{n,1} \otimes B')$ to $m_n \otimes m_{n-1} \cdots m_1 \otimes b' \in \mathcal{P}(\lambda, B^{1,1} \otimes \hat{B}^{n-1,1} \otimes B')$.

Define $\mathrm{tj} : \mathrm{RC}(\lambda, B) \to \mathrm{RC}(\lambda, \widetilde{B})$ in the following way. Let $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B)$. If $B = \hat{B}^{k,1} \otimes B'$ for $1 \le k \le n-1$, $\mathrm{tj}(\nu^\bullet, J^\bullet)$ is obtained from $(\nu^\bullet, J^\bullet)$ by adding a singular string of length 1 to each of the first $k-1$ rigged partitions of $\nu^\bullet$. For $B = E^{n,1} \otimes B'$, add a singular string of length 1 to $(\nu^\bullet, J^\bullet)^{(a)}$ for $1 \le a \le n-1$. For $B = E^{n-1,1} \otimes B'$, add a singular string of length 1 to $(\nu^\bullet, J^\bullet)^{(n)}$ and $(\nu^\bullet, J^\bullet)^{(a)}$ for $1 \le a \le n-2$.

Lemma 4.2. tj *is a well-defined injection that preserves the vacancy numbers.*

Proof. To show that tj is well-defined, we need to check that $(\tilde{\nu}^\bullet, \tilde{J}^\bullet) = \mathrm{tj}(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, \widetilde{B})$ for $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B)$. Let us first consider the case $B = \hat{B}^{k,1} \otimes B'$. In this case $\tilde{m}_i^{(a)} = m_i^{(a)} + \chi(1 \le a < k)\delta_{i,1}$ and $\tilde{\boldsymbol{L}} = \boldsymbol{L} + \omega_1 + \omega_{k-1} - \omega_k$. Hence the change of both the left and right hand side of (4.2) is $\sum_{1 \le a \le k-1} \alpha_a = \omega_1 + \omega_{k-1} - \omega_k = \epsilon_1 - \epsilon_k$. Therefore (4.2) holds for $(\tilde{\nu}^\bullet, \tilde{J}^\bullet)$.

Using (4.3) the vacancy numbers transform as

$$\tilde{p}_i^{(a)} = p_i^{(a)} + \big(\alpha_a \mid \omega_1 + \omega_{k-1} - \omega_k - \sum_{1 \le b < k} \alpha_b\big) = p_i^{(a)},$$

which shows that they are preserved under tj.

The cases $B = E^{n,1} \otimes B'$ and $B = E^{n-1,1} \otimes B'$ are analogous. □

The maps δ and δ_s are defined in the next two subsections.

4.4. **Algorithm for δ.** Let $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B^{1,1} \otimes B')$. The map $\delta : \mathrm{RC}(\lambda, B^{1,1} \otimes B') \to \bigcup_{\lambda^-} \mathrm{RC}(\lambda^-, B')$ is defined by the following algorithm [19]. The partition $J^{(a,i)}$ is called singular if it has a part of size $p_i^{(a)}$.

Set $\ell^{(0)} = 1$ and repeat the following process for $a = 1, 2, \ldots, n-2$ or until stopped. Find the smallest index $i \ge \ell^{(a-1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $b = a$ and stop. Otherwise set $\ell^{(a)} = i$ and continue with $a+1$.

If the process has not stopped at $a = n-2$, find the minimal indices $i, j \ge \ell^{(n-2)}$ such that $J^{(n-1,i)}$ and $J^{(n,j)}$ are singular. If neither i nor j exist, set $b = n-1$ and stop. If i exists, but not j, set $\ell^{(n-1)} = i$, $b = n$ and stop. If j exists, but not i, set $\ell^{(n)} = j$, $b = \overline{n}$ and stop. If both i and j exist, set $\ell^{(n-1)} = i$, $\ell^{(n)} = j$ and continue with $a = n-2$.

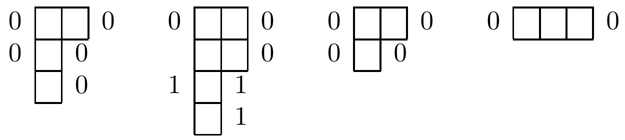
Now continue for $a = n-2, n-3, \ldots, 1$ or until stopped. Find the minimal index $i \ge \overline{\ell}^{(a+1)}$ where $\overline{\ell}^{(n-1)} = \max(\ell^{(n-1)}, \ell^{(n)})$ such that $J^{(a,i)}$ is singular (if $i = \ell^{(a)}$ then there need to be two parts of size $p_i^{(a)}$ in $J^{(a,i)}$). If no such i exists, set $b = \overline{a+1}$ and stop. If the process did not stop, set $b = \overline{1}$. Set all yet undefined $\ell^{(a)}$ and $\overline{\ell}^{(a)}$ to ∞.

The rigged configuration $(\tilde{\nu}^\bullet, \tilde{J}^\bullet) = \delta(\nu^\bullet, J^\bullet)$ is obtained by removing a box from the selected strings and making the new strings singular again. Explicitly

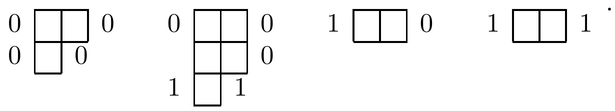
$$m_i^{(a)}(\tilde{\nu}^\bullet) = m_i^{(a)}(\nu^\bullet) + \begin{cases} 1 & \text{if } i = \ell^{(a)} - 1 \\ -1 & \text{if } i = \ell^{(a)} \\ 1 & \text{if } i = \overline{\ell}^{(a)} - 1 \text{ and } 1 \le a \le n-2 \\ -1 & \text{if } i = \overline{\ell}^{(a)} \text{ and } 1 \le a \le n-2 \\ 0 & \text{otherwise.} \end{cases}$$

The partition $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $p_i^{(a)}(\nu^\bullet)$ for $i = \ell^{(a)}$ and $i = \overline{\ell}^{(a)}$, adding a part of size $p_i^{(a)}(\tilde{\nu}^\bullet)$ for $i = \ell^{(a)} - 1$ and $i = \overline{\ell}^{(a)} - 1$, and leaving it unchanged otherwise.

Example 4.3. Let $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B)$ with $\lambda = \Lambda_2$ and $B = (B^{1,1})^{\otimes 2} \otimes (B^{2,1})^{\otimes 3}$ of type $D_4^{(1)}$ given by



Here the vacancy number $p_i^{(a)}$ is written on the left of the parts of length i in $\nu^{(a)}$ and the partition $J^{(a,i)}$ is given by the labels on the right of the parts of length i in $\nu^{(a)}$. In this case $\ell^{(1)} = \ell^{(2)} = \ell^{(3)} = 1$, $\ell^{(4)} = 3$, $\overline{\ell}^{(2)} = \overline{\ell}^{(1)} = \infty$ and $b = \overline{3}$, so that $\delta(\nu^\bullet, J^\bullet)$ is



Altogether

$$\Phi_B(\nu^\bullet, J^\bullet) = \boxed{\overline{3}} \otimes \boxed{\overline{4}} \otimes \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline \overline{1} \\ \hline 1 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}.$$

4.5. **Algorithm for δ_s.** The embedding (3.7) can be extended to paths

$$\mathrm{emb}_{\mathcal{P}} : \mathcal{P}(\lambda, B) \hookrightarrow \mathcal{P}(\hat{\lambda}, \hat{B}) \tag{4.4}$$

where $\hat{\lambda} = 2\lambda$, $B = \bigotimes_{a=1}^n (B^{a,1})^{\otimes L^a}$, and $\hat{B} = \bigotimes_{a=1}^n (B^{a,1})^{\otimes \hat{L}^a}$ where $\hat{L}^a = 2L^a$ for $1 \le a \le n$. A path $b = b_L \otimes \cdots \otimes b_1 \in \mathcal{P}(\lambda, B)$ is mapped to $\mathrm{emb}_{\mathcal{P}}(b) = \mathrm{emb}_B(b_L) \otimes \cdots \otimes \mathrm{emb}_B(b_1)$.

An analogous embedding can be defined on rigged configurations

$$\mathrm{emb}_{\mathrm{RC}} : \mathrm{RC}(\lambda, B) \hookrightarrow \mathrm{RC}(\hat{\lambda}, \hat{B}) \tag{4.5}$$

where $\mathrm{emb}_{\mathrm{RC}}(\nu^\bullet, J^\bullet) = 2(\nu^\bullet, J^\bullet)$, meaning that all parts in $\nu^{(a)}$ and $J^{(a,i)}$ are doubled in length.

Theorem 4.4. *For B a tensor product of crystals of the form $B^{a,1}$ with $1 \le a \le n-2$ the following diagram commutes:*

$$\begin{tikzcd} \mathrm{RC}(\lambda, B) \arrow[r, "\Phi_B"] \arrow[d, "\mathrm{emb}_{\mathrm{RC}}"'] & \mathcal{P}(\lambda, B) \arrow[d, "\mathrm{emb}_B"] \\ \mathrm{RC}(\hat{\lambda}, \hat{B}) \arrow[r, "\Phi_{\hat{B}}"] & \mathcal{P}(\hat{\lambda}, \hat{B}). \end{tikzcd}$$

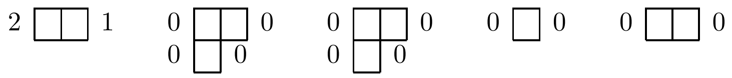
The proof of this theorem is given in Appendix B.

The definition of δ_s is given in such a way that Theorem 4.4 also holds for the spinor case. Let us define

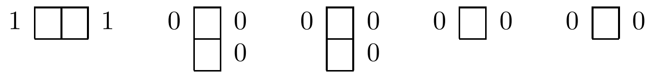
$$\delta_s : \mathrm{RC}(\lambda, B) \to \bigcup_{\lambda^{-s}} \mathrm{RC}(\lambda^{-s}, B')$$

as follows. For $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B)$ apply $\mathrm{emb}_{\mathrm{RC}}$ and then a sequence of points 1 and 2 of Theorem 4.1 to remove the last tensor factor. The claim is that this rigged configuration is in the image of $\mathrm{emb}_{\mathrm{RC}}$ so that one can apply $\mathrm{emb}_{\mathrm{RC}}^{-1}$. The result is $\delta_s(\nu^\bullet, J^\bullet)$. The proof of the well-definedness of δ_s is part of the proof of Theorem 4.1 as given in Appendix A.

Example 4.5. Let $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B)$ with $\lambda = 2\Lambda_1 + \Lambda_4$ and $B = B^{5,1} \otimes B^{2,1} \otimes (B^{1,1})^{\otimes 3}$ of type $D_5^{(1)}$ given by



Then $\delta_s(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda^{-s}, B')$ with $\lambda^{-s} = \Lambda_1 + \Lambda_2$ and $B' = B^{2,1} \otimes (B^{1,1})^{\otimes 3}$ is



The details are given in Table 1. The first entry is $\mathrm{emb}_{\mathrm{RC}}(\nu^\bullet, J^\bullet)$. The next entries are obtained by acting with $\delta \circ \mathrm{tj}$. Acting with $\mathrm{emb}_{\mathrm{RC}}^{-1}$ on the last rigged configuration yields $\delta_s(\nu^\bullet, J^\bullet)$. In the last column of the table we recorded b of the algorithm δ. Hence the step in $B^{5,1}$ corresponding to this rigged configuration is $\overline{2}\overline{5}431$.

5. Statistics on paths and rigged configurations

A statistic can be defined on both paths and rigged configurations. In this section we define the intrinsic energy function on paths and cocharge statistics on rigged configurations and show that the bijection Φ preserves the statistics (see Theorem 5.1). This gives rise to the fermionic formula presented in Section 5.1.

In Section 2.3 we defined the combinatorial R-matrix $R : B_2 \otimes B_1 \to B_1 \otimes B_2$. In addition, there exists a function $H : B_2 \otimes B_1 \to \mathbb{Z}$ called the local energy function, that is unique up to a global additive constant. It is constant on J components and satisfies for all $b_2 \in B_2$ and $b_1 \in B_1$ with $R(b_2 \otimes b_1) = b_1' \otimes b_2'$

$$H(e_0(b_2 \otimes b_1)) = H(b_2 \otimes b_1) + \begin{cases} -1 & \text{if } \epsilon_0(b_2) > \varphi_0(b_1) \text{ and } \epsilon_0(b_1') > \varphi_0(b_2') \\ 1 & \text{if } \epsilon_0(b_2) \le \varphi_0(b_1) \text{ and } \epsilon_0(b_1') \le \varphi_0(b_2') \\ 0 & \text{otherwise.} \end{cases}$$

We shall normalize the local energy function by the condition $H(u(B_2) \otimes u(B_1)) = 0$.

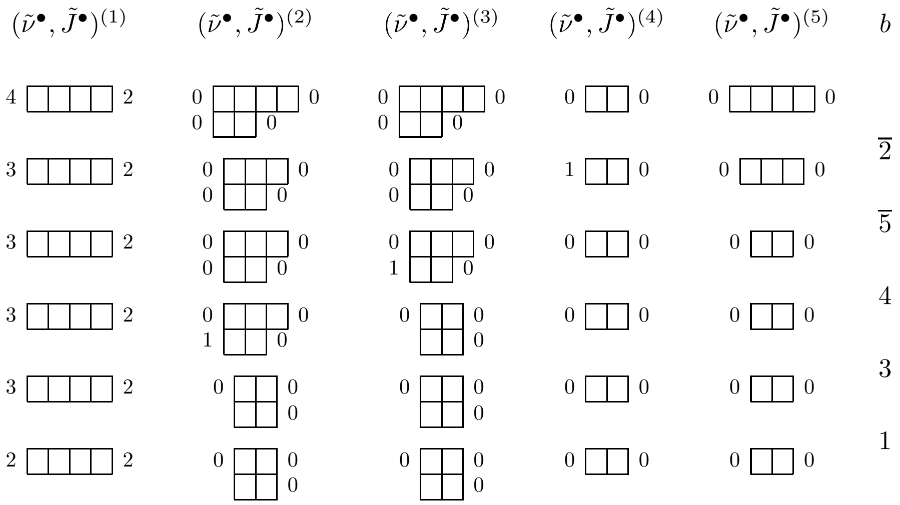


TABLE 1. An example for δ_s

For a crystal $B^{k,1}$ of type $D_n^{(1)}$, the intrinsic energy $D_{B^{k,1}} : B^{k,1} \to \mathbb{Z}$ is defined as follows. Let $b \in B^{k,1}$ which is in the classical component $B(\Lambda_{k-2j})$ (see (3.4)). Then $D_{B^{k,1}}(b) = j$.

On the tensor product $B = B_L \otimes \cdots \otimes B_1$ of simple crystals there is an intrinsic energy function defined $D_B : B \to \mathbb{Z}$ (see for example [20, Section 2.5])

$$D_B = \sum_{1\le i<j\le L} H_i R_{i+1} R_{i+2} \cdots R_{j-1} + \sum_{j=1}^{L} D_{B_j} \pi_1 R_1 R_2 \cdots R_{j-1}. \tag{5.1}$$

Here R_i and H_i denote the combinatorial R-matrix and local energy function acting on the i-th and $(i+1)$-th tensor factors, respectively. π_1 is the projection onto the rightmost tensor factor.

Similarly, there is a statistic on the set of rigged configurations given by $cc(\nu^\bullet, J^\bullet) = cc(\nu^\bullet) + \sum_{a,i} |J^{(a,i)}|$ where $|J^{(a,i)}|$ is the size of the partition $J^{(a,i)}$ and

$$cc(\nu^\bullet) = \frac{1}{2} \sum_{a,b\in J} \sum_{j,k\ge 1} (\alpha_a \mid \alpha_b) \min(j,k) m_j^{(a)} m_k^{(b)}. \tag{5.2}$$

Let $\tilde{\Phi} = \Phi \circ \theta$ where $\theta : \mathrm{RC}(\lambda, B) \to \mathrm{RC}(\lambda, B)$ with $\theta(\nu^\bullet, J^\bullet) = (\nu^\bullet, \tilde{J}^\bullet)$ is the function which complements the riggings, meaning that $\tilde{J}^\bullet$ is obtained from $J^\bullet$ by complementing all partitions $J^{(a,i)}$ in the $m_i^{(a)} \times p_i^{(a)}$ rectangle.

Theorem 5.1. *Let $B = B^{k_L,1} \otimes \cdots \otimes B^{k_1,1}$ be a crystal of type $D_n^{(1)}$ and λ a dominant integral weight. The bijection $\tilde{\Phi} : \mathrm{RC}(\lambda, B) \to \mathcal{P}(\lambda, B)$ preserves the statistics, that is $cc(\nu^\bullet, J^\bullet) = D(\tilde{\Phi}(\nu^\bullet, J^\bullet))$ for all $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B)$.*

A proof of this theorem is given in Section 5.4 using the results of Sections 5.2 and 5.3.

Example 5.2. For the rigged configuration of Example 4.3 we have

$$\tilde{\Phi}(\nu^\bullet, J^\bullet) = \boxed{3} \otimes \boxed{\bar{4}} \otimes \begin{array}{|c|}\hline \bar{1} \\ \hline 1 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}$$

and $cc(\nu^\bullet, J^\bullet) = D(\tilde{\Phi}(\nu^\bullet, J^\bullet)) = 10$.

5.1. **Fermionic formulas.** An immediate corollary of Theorems 4.1 and 5.1 is the equality

$$\sum_{b\in\mathcal{P}(\lambda,B)} q^{D(b)} = \sum_{(\nu^\bullet,J^\bullet)\in\mathrm{RC}(\lambda,B)} q^{cc(\nu^\bullet,J^\bullet)}.$$

The left-hand side is in fact the one dimensional configuration sum

$$X(\lambda, B) = \sum_{b\in\mathcal{P}(\lambda,B)} q^{D(b)}.$$

The right-hand side can be simplified slightly by observing that the generating function of partitions in a box of width p and height m is the q-binomial coefficient

$$\begin{bmatrix} m+p \\ m \end{bmatrix} = \frac{(q)_{p+m}}{(q)_m (q)_p},$$

where $(q)_m = (1-q)(1-q^2)\cdots(1-q^m)$. Hence the right-hand side becomes the fermionic formula

$$M(\lambda, B) = \sum_{\nu^\bullet\in C(\lambda,B)} q^{cc(\nu^\bullet)} \prod_{\substack{i\geq 1\\ a\in J}} \begin{bmatrix} m_i^{(a)} + p_i^{(a)} \\ m_i^{(a)} \end{bmatrix}$$

where $m_i^{(a)}$ and $p_i^{(a)}$ are as defined in Section 4.1. This proves the following result conjectured in [5, 4].

Theorem 5.3. *For $\mathfrak{g}$ of type $D_n^{(1)}$, λ a dominant weight and $B = B^{k_L,1} \otimes \cdots \otimes B^{k_1,1}$*

$$X(\lambda, B) = M(\lambda, B).$$

5.2. **Relations between the various maps.** The maps $\mathrm{ts}, \mathrm{tj}, \mathrm{lh}, \delta$ were defined in Sections 4.3 and 4.4. We may define similar maps using the dualities $*$ and θ.

For crystals let $\mathrm{rh} = * \circ \mathrm{lh} \circ *$ and $\mathrm{bs} = * \circ \mathrm{ts} \circ *$. Explicitly, $\mathrm{rh}(b' \otimes b) = b'$ for $b' \otimes b \in B' \otimes B^{1,1}$ and $\mathrm{bs}(b' \otimes m_k \cdots m_2 m_1) = b' \otimes m_k \cdots m_2 \otimes m_1$ for $b' \otimes m_k \cdots m_1 \in B' \otimes \hat{B}^{k,1}$ for $2 \leq k \leq n-1$. Similarly $\mathrm{bs}(b' \otimes m_n \cdots m_2 m_1) = b' \otimes m_n \cdots m_2 \otimes m_1$ for $b' \otimes m_n \cdots m_1 \in B' \otimes E^{n-1,1}$ or $E^{n,1}$.

On the rigged configuration side define $\tilde{\delta} = \theta \circ \delta \circ \theta$ and $\mathrm{bj} = \theta \circ \mathrm{tj} \circ \theta$. Explicitly, bj adds parts of lengths one to the same rigged partitions as tj; however in this case they are not singular, but of label 0.

In the following $[\cdot, \cdot]$ denotes the commutator.

Lemma 5.4. *We have $[\mathrm{bj}, \delta] = 0$ and $[\mathrm{tj}, \tilde{\delta}] = 0$.*

Proof. The relation $[\mathrm{tj}, \tilde{\delta}] = 0$ follows from $[\mathrm{bj}, \delta] = 0$ by conjugation by θ. Hence it suffices to prove $[\mathrm{bj}, \delta] = 0$.

Let $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B)$ where $B = \hat{B}^{1,1} \otimes \hat{B}^{k,1} \otimes B'$. We assume throughout the proof that $1 \leq k < n$. The cases $B = \hat{B}^{1,1} \otimes E^{n,1} \otimes B'$ and $B = \hat{B}^{1,1} \otimes E^{n-1,1} \otimes B'$ can be proven in a similar fashion. Let $\ell^{(a)}$ and $\overline{\ell}^{(a)}$ (resp. $s^{(a)}$ and $\overline{s}^{(a)}$) be the singular strings selected by δ in $(\nu^\bullet, J^\bullet)$ (resp. $\mathrm{bj}(\nu^\bullet, J^\bullet)$). Clearly, $s^{(a)} \leq \ell^{(a)}$ and $\overline{s}^{(a)} \leq \overline{\ell}^{(a)}$ since bj changes neither vacancy numbers nor riggings by Lemma 4.2 and its definition $\mathrm{bj} = \theta \circ \mathrm{tj} \circ \theta$. If $s^{(a)} = \ell^{(a)}$ and $\overline{s}^{(a)} = \overline{\ell}^{(a)}$ for all a, the lemma holds.

Assume that b is minimal such that $s^{(b)} < \ell^{(b)}$. Certainly we must have $1 \leq b < k$, $s^{(a)} = \ell^{(a)} = 1$ for $1 \leq a < b$, $s^{(b)} = 1 < \ell^{(b)}$, $m_1^{(b)} = 0$ and $p_1^{(b)} = 0$. One may easily verify that (see also [17, Eq. (3.10)])

$$\begin{aligned} -p_{i-1}^{(a)} + 2p_i^{(a)} - p_{i+1}^{(a)} &= -\sum_{b \in J} (\alpha_a \mid \alpha_b)\, m_i^{(b)} + \delta_{i,1} L^a \\ &\geq -\sum_{b \in J} (\alpha_a \mid \alpha_b)\, m_i^{(b)}, \end{aligned}$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 else. Using $m_1^{(b)} = 0$ and $p_1^{(b)} = 0$ this reads for $i = 1$ and $a = b$

$$-p_2^{(b)} = m_1^{(b-1)} + m_1^{(b+1)} + L^b.$$

Since $p_i^{(a)} \geq 0$ for all admissible rigged configurations, this implies that all quantities in this equation must be zero. For $b > 1$ the condition $m_1^{(b-1)} = 0$ contradicts $\ell^{(b-1)} = 1$. For $b = 1$ we have $L^1 \geq 1$ which makes $p_2^{(1)} \geq 0$ impossible. Hence $s^{(a)} = \ell^{(a)}$ for all a as desired. The proof that $\overline{s}^{(a)} = \overline{\ell}^{(a)}$ is analogous. $\square$

Lemma 5.5. *We have* $[\delta, \tilde{\delta}] = 0$.

The proof of Lemma 5.5 is outlined in Appendix C.

We say that f corresponds to g under Φ if the following diagram commutes:

$$\begin{array}{ccc} \mathrm{RC}(\lambda, B) & \xrightarrow{\Phi_B} & \mathcal{P}(\lambda, B) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathrm{RC}(\lambda', B') & \xrightarrow[\Phi_{B'}]{} & \mathcal{P}(\lambda', B'). \end{array}$$

Proposition 5.6. *The following maps correspond under* Φ*:*

(1) bs *and* bj
(2) ts *and* tj
(3) lh *and* δ
(4) rh *and* $\tilde{\delta}$
(5) $*$ *and* θ
(6) R *and* id

Proof. Points (2) and (3) hold by definition.

We prove (1) by induction on B. For simplicity we drop the dependency on λ in all diagrams to follow. Also for brevity we denote $\hat{B}^{k,1} \otimes B \otimes \hat{B}^{k',1}$ symbolically by $\substack{\square\\\square}B\substack{\square\\\square}$. The splitting $\hat{B}^{k,1} \to \hat{B}^{k-1,1} \otimes \hat{B}^{1,1}$ will be denoted by $\substack{\square\\\square} \to \square\ \square$ and so on. Consider the following diagram:

$$
\begin{tikzcd}
\mathrm{RC}(\begin{smallmatrix}\square\\ \square\end{smallmatrix}B\begin{smallmatrix}\square\\ \square\end{smallmatrix}) \arrow[rrr, "\Phi"] \arrow[dr, "\mathrm{tj}"] \arrow[ddd, "\mathrm{bj}"'] & & & \mathcal{P}(\begin{smallmatrix}\square\\ \square\end{smallmatrix}B\begin{smallmatrix}\square\\ \square\end{smallmatrix}) \arrow[dl, "\mathrm{ts}"'] \arrow[ddd, "\mathrm{bs}"] \\
& \mathrm{RC}(\square\ \square B\begin{smallmatrix}\square\\ \square\end{smallmatrix}) \arrow[r, "\Phi"] \arrow[d, "\mathrm{bj}"'] & \mathcal{P}(\square\ \square B\begin{smallmatrix}\square\\ \square\end{smallmatrix}) \arrow[d, "\mathrm{bs}"] & \\
& \mathrm{RC}(\square\ \square B\square\ \square) \arrow[r, "\Phi"'] & \mathcal{P}(\square\ \square B\square\ \square) & \\
\mathrm{RC}(\begin{smallmatrix}\square\\ \square\end{smallmatrix}B\square\ \square) \arrow[ur, "\mathrm{tj}"'] \arrow[rrr, "\Phi"'] & & & \mathcal{P}(\begin{smallmatrix}\square\\ \square\end{smallmatrix}B\square\ \square) \arrow[ul, "\mathrm{ts}"]
\end{tikzcd}
\tag{5.3}
$$

This diagram can be viewed as a cube with the front face given by the large square. The back face commutes by induction. It is easy to check that the left and right faces commute. The top and bottom face commute by (2). Hence it follows from [17, Lemma 5.3] that the front face commutes since ts is injective.

Next consider the diagram:

$$
\begin{tikzcd}
\mathrm{RC}(\square B\begin{smallmatrix}\square\\ \square\end{smallmatrix}) \arrow[rrr, "\Phi"] \arrow[dr, "\delta"] \arrow[ddd, "\mathrm{bj}"'] & & & \mathcal{P}(\square B\begin{smallmatrix}\square\\ \square\end{smallmatrix}) \arrow[dl, "\mathrm{lh}"'] \arrow[ddd, "\mathrm{bs}"] \\
& \mathrm{RC}(B\begin{smallmatrix}\square\\ \square\end{smallmatrix}) \arrow[r, "\Phi"] \arrow[d, "\mathrm{bj}"'] & \mathcal{P}(B\begin{smallmatrix}\square\\ \square\end{smallmatrix}) \arrow[d, "\mathrm{bs}"] & \\
& \mathrm{RC}(B\square\ \square) \arrow[r, "\Phi"'] & \mathcal{P}(B\square\ \square) & \\
\mathrm{RC}(\square B\square\ \square) \arrow[ur, "\delta"'] \arrow[rrr, "\Phi"'] & & & \mathcal{P}(\square B\square\ \square) \arrow[ul, "\mathrm{lh}"]
\end{tikzcd}
\tag{5.4}
$$

The back face commutes by induction, the top and bottom faces commute by the definition of Φ. It is easy to check that the right face commutes. The left face commutes by Lemma 5.4. Hence the front face commutes by [17, Lemma 5.3]. Diagrams (5.3) and (5.4) together prove (1).

For the proof of (4) we use the same abreviated notation as before. $\hat{B}^{1,1}$ is symbolically denoted by $\square$. Consider the following diagram:

(5.5)
$$\begin{tikzcd}
\mathrm{RC}(\square B\square) \arrow[rrr, "\Phi"] \arrow[dr, "\delta"] \arrow[ddd, "\tilde{\delta}"'] & & & \mathcal{P}(\square B\square) \arrow[dl, "\mathrm{lh}"'] \arrow[ddd, "\mathrm{rh}"] \\
& \mathrm{RC}(B\square) \arrow[r, "\Phi"] \arrow[d, "\tilde{\delta}"'] & \mathcal{P}(B\square) \arrow[d, "\mathrm{rh}"] & \\
& \mathrm{RC}(B) \arrow[r, "\Phi"'] & \mathcal{P}(B) & \\
\mathrm{RC}(\square B) \arrow[rrr, "\Phi"'] \arrow[ur, "\delta"] & & & \mathcal{P}(\square B) \arrow[ul, "\mathrm{lh}"']
\end{tikzcd}$$

The back face commutes by induction and the top and bottom face commute by (3). It is easy to verify that the right face commutes. The left face commutes by Lemma 5.5. Hence the front face commutes, since lh is injective for fixed initial weight λ.

Next consider the diagram:

(5.6)
$$\begin{tikzcd}
\mathrm{RC}(\begin{smallmatrix}\square\\\square\end{smallmatrix} B\square) \arrow[rrr, "\Phi"] \arrow[dr, "\mathrm{tj}"] \arrow[ddd, "\tilde{\delta}"'] & & & \mathcal{P}(\begin{smallmatrix}\square\\\square\end{smallmatrix} B\square) \arrow[dl, "\mathrm{ts}"'] \arrow[ddd, "\mathrm{rh}"] \\
& \mathrm{RC}(\square\ \square B\square) \arrow[r, "\Phi"] \arrow[d, "\tilde{\delta}"'] & \mathcal{P}(\square\ \square B\square) \arrow[d, "\mathrm{rh}"] & \\
& \mathrm{RC}(\square\ \square B) \arrow[r, "\Phi"'] & \mathcal{P}(\square\ \square B) & \\
\mathrm{RC}(\begin{smallmatrix}\square\\\square\end{smallmatrix} B) \arrow[rrr, "\Phi"'] \arrow[ur, "\mathrm{tj}"] & & & \mathcal{P}(\begin{smallmatrix}\square\\\square\end{smallmatrix} B) \arrow[ul, "\mathrm{ts}"']
\end{tikzcd}$$

The back face commutes by induction, the top and bottom face commute by (2). It is easy to verify that the right face commutes. The left face commutes by Lemma 5.4. Hence the front face commutes, since ts is injective. Diagrams (5.5) and (5.6) together prove that (4) holds.

For the proof of point (5) consider the following diagram:

(5.7)
$$\begin{tikzcd}
\mathrm{RC}(\square B) \arrow[rrr, "\Phi"] \arrow[dr, "\delta"] \arrow[ddd, "\theta"'] & & & \mathcal{P}(\square B) \arrow[dl, "\mathrm{lh}"'] \arrow[ddd, "*"] \\
& \mathrm{RC}(B) \arrow[r, "\Phi"] \arrow[d, "\theta"'] & \mathcal{P}(B) \arrow[d, "*"] & \\
& \mathrm{RC}(B^*) \arrow[r, "\Phi"'] & \mathcal{P}(B^*) & \\
\mathrm{RC}(B^*\square) \arrow[rrr, "\Phi"'] \arrow[ur, "\tilde{\delta}"] & & & \mathcal{P}(B^*\square) \arrow[ul, "\mathrm{rh}"']
\end{tikzcd}$$

The back face commutes by induction. The left and right faces commute by definition. The top and bottom faces commute by (3) and (4), respectively. Since $\tilde{\delta}$ is injective, the front face commutes by [17, Lemma 5.3].

Next consider the following diagram:

$$
\begin{tikzcd}
\mathrm{RC}(\boxminus B) \arrow[rrr, "\Phi"] \arrow[dr, "\mathrm{tj}"] \arrow[ddd, "\theta"'] & & & \mathcal{P}(\boxminus B) \arrow[dl, "\mathrm{ts}"'] \arrow[ddd, "*"] \\
& \mathrm{RC}(\Box\,\Box B) \arrow[r, "\Phi"] \arrow[d, "\theta"'] & \mathcal{P}(\Box\,\Box B) \arrow[d, "*"] & \\
& \mathrm{RC}(B^*\Box\,\Box) \arrow[r, "\Phi"'] & \mathcal{P}(B^*\Box\,\Box) & \\
\mathrm{RC}(B^*\boxminus) \arrow[ur, "\mathrm{bj}"] \arrow[rrr, "\Phi"'] & & & \mathcal{P}(B^*\boxminus) \arrow[ul, "\mathrm{bs}"']
\end{tikzcd}
\tag{5.8}
$$

The back face commutes by induction. The left and right faces commute by definition. The bottom and top faces commute by (1) and (2), respectively. Since bj is injective, the front face commutes by [17, Lemma 5.3]. Hence (5) holds by diagrams (5.7) and (5.8).

For the proof of (6) we may assume that $R = R_i$ is the combinatorial R-matrix acting on the i-th and $(i+1)$-th tensor factor. By induction we may assume that $i = L-1$ where L is the number of tensor factors in B. By part (5) we may assume that $i = 1$. Again by induction we may assume that $L = 2$, which means that $B = B^{k_2,1} \otimes B^{k_1,1}$. Hence, since R is a crystal isomorphism, it suffices to check (6) for highest weight elements in $B = B^{k',1} \otimes B^{k,1}$. The combinatorial R-matrix $R : B^{k',1} \otimes B^{k,1} \to B^{k,1} \otimes B^{k',1}$ for all highest weight elements is given explicitly in Appendix D. It is tedious but straightforward to check (6) for these. $\Box$

5.3. **Properties of the statistics.** In this section we show how the statistics cc and D change under tj and bs, respectively. These results will be used in the next section in the proof of Theorem 5.1. For the next two lemmas we assume that B is the tensor product of crystals $\hat{B}^{k,1}$ for $1 \le k < n$, $E^{n-1,1}$ or $E^{n,1}$.

Lemma 5.7. *For $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B)$ we have*

$$
cc(\mathrm{tj}(\nu^\bullet, J^\bullet)) - cc(\nu^\bullet, J^\bullet) = \begin{cases} (\epsilon_1 - \epsilon_k \mid \boldsymbol{L}) + 1 & \text{if } B = \hat{B}^{k,1} \otimes B' \text{ for } 1 \le k < n \\ (\epsilon_1 - \epsilon_n \mid \boldsymbol{L}) + 1 & \text{if } B = E^{n,1} \otimes B' \\ (\epsilon_1 + \epsilon_n \mid \boldsymbol{L}) + 1 & \text{if } B = E^{n-1,1} \otimes B'. \end{cases}
$$

Proof. First consider the case $B = \hat{B}^{k,1} \otimes B'$. Then by (5.2)

$$
\begin{aligned}
cc(\mathrm{tj}(\nu^\bullet)) =& \frac{1}{2} \sum_{a,b} \sum_{i,j} (\alpha_a \mid \alpha_b) \min(i,j) \\
& \times \Big(m_i^{(a)} + \chi(i=1)\chi(a<k) \Big) \Big(m_j^{(b)} + \chi(j=1)\chi(b<k) \Big) \\
=& cc(\nu^\bullet) + \sum_{b,j} (\alpha_1 + \cdots + \alpha_{k-1} \mid \alpha_b) m_j^{(b)} + 1.
\end{aligned}
$$

Since tj adds a singular string of length one to the first $k-1$ rigged partitions, the change from the labels is, using (4.3),

$$p_1^{(1)} + p_1^{(2)} + \cdots + p_1^{(k-1)} = (\alpha_1 + \cdots + \alpha_{k-1} \mid \boldsymbol{L} - \sum_{b,j} \alpha_b m_j^{(b)}).$$

Hence

$$cc(\mathrm{tj}(\nu^\bullet, J^\bullet)) - cc(\nu^\bullet, J^\bullet) = (\epsilon_1 - \epsilon_k \mid \boldsymbol{L}) + 1.$$

The cases $B = E^{n,1} \otimes B'$ and $B = E^{n-1,1} \otimes B'$ are similar. □

Lemma 5.8. *For $p \in \mathcal{P}(\lambda, B)$ we have*

$$D(\mathrm{bs}(p)) - D(p) = \begin{cases} (\epsilon_1 - \epsilon_k \mid \boldsymbol{L}) + 1 & \text{if } B = B' \otimes \hat{B}^{k,1} \text{ for } 1 \le k < n \\ (\epsilon_1 - \epsilon_n \mid \boldsymbol{L}) + 1 & \text{if } B = B' \otimes E^{n,1} \\ (\epsilon_1 + \epsilon_n \mid \boldsymbol{L}) + 1 & \text{if } B = B' \otimes E^{n-1,1}. \end{cases}$$

Proof. It follows from the structure of (5.1) that we can restrict ourselves to the case when B consists of one or two tensor factors. Let $b \in \hat{B}^{k,1}$ ($E^{n-1,1}$ or $E^{n,1}$) and $b' \in \hat{B}^{k',1}$ ($E^{n-1,1}$ or $E^{n,1}$) such that $e_i(b) = 0$ and $e_i(b' \otimes b) = 0$ for $1 \le i \le n$. Furthermore, write $\mathrm{bs}(b) = b_2 \otimes b_1 \in \hat{B}^{k-1,1} \otimes B^{1,1}$ ($\hat{B}^{n-1,1} \otimes B^{1,1}$) and

$$R(b' \otimes b) = \hat{b} \otimes \hat{b}'$$
$$R(b' \otimes b_2) = \bullet \otimes \tilde{b}'$$
$$R(\tilde{b}' \otimes b_1) = \bullet \otimes \tilde{\tilde{b}}',$$

where the crystal elements denoted $\bullet$ are not of interest to us here. Let $\boldsymbol{L}'$ be the weight corresponding to $\hat{B}^{k',1}$ (resp. $E^{n-1,1}$ or $E^{n,1}$) (see (4.1)). Then it suffices to prove the following two equations:

(5.9) $$D(b_2 \otimes b_1) - D(b) = 1$$

(5.10) $$H(b' \otimes b_2) + H(\tilde{b}' \otimes b_1) + D(\tilde{\tilde{b}}') - H(b' \otimes b) - D(\hat{b}')$$
$$= \begin{cases} (\epsilon_1 - \epsilon_k \mid \boldsymbol{L}') & \text{if } b \in \hat{B}^{k,1} \\ (\epsilon_1 - \epsilon_n \mid \boldsymbol{L}') & \text{if } b \in E^{n,1} \\ (\epsilon_1 + \epsilon_n \mid \boldsymbol{L}') & \text{if } b \in E^{n-1,1}. \end{cases}$$

To prove (5.9) and (5.10) we use the same notation as in Appendix D. Namely, a column with entries $\cdots a_4 \cdots a_3 a_2 \cdots a_1$, where $a_2 \cdots a_1$ denotes a sequence of consecutive integers $a_2\, a_2 - 1\, a_2 - 2\, \ldots a_1 + 1\, a_1$, is compactly written as $[a_1..a_2 \mid a_3..a_4 \mid \cdots]$.

Let us first concentrate on (5.9) for $b \in \hat{B}^{k,1}$. Since $e_i(b) = 0$ for $1 \le i \le n$, b is of the form $b = [1..p \mid \overline{p}..\overline{\ell}]$. Then $b_2 \otimes b_1 = [2..p \mid \overline{p}..\overline{\ell}] \otimes [1]$ and $R(b_2 \otimes b_1) = \tilde{b}_1 \otimes \tilde{b}_2 = [\ell - 1] \otimes [1..p-1 \mid \overline{p-1}..\overline{\ell-1}]$. Hence $D(b) = p - \ell + 1$ and $D(b_2 \otimes b_1) = H(b_2 \otimes b_1) + D(b_1) + D(\tilde{b}_2) = 1 + 0 + (p - \ell + 1) = p - \ell + 2$ which proves (5.9). The highest weight vector in $E^{n,1}$ is $b = [1..n]$ and $b_2 \otimes b_1 = [2..n] \otimes [1]$, so that $R(b_2 \otimes b_1) = \tilde{b}_1 \otimes \tilde{b}_2 = [n] \otimes [1..n-1]$. In this case $D(b) = 0$ and $D(b_2 \otimes b_1) = H(b_2 \otimes b_1) + D(b_1) + D(\tilde{b}_2) = 1 + 0 + 0 = 1$. The highest weight vector in $E^{n-1,1}$ is $b = [1..n-1\, \overline{n}]$ and the calculation is analogous to the case $E^{n,1}$.

Equation (5.10) can be proven for each case listed in Appendix D, where it should be noted that the results for $B^{k',1} \otimes B^{k,1}$ with $1 \le k' \le k \le n-2$ can be extended to $\hat{B}^{k',1} \otimes \hat{B}^{k,1}$ with $1 \le k' \le k \le n-1$ and $\hat{B}^{k',1} \otimes \hat{B}^{n,1}$ ($\hat{B}^{k',1} \otimes \hat{\hat{B}}^{n,1}$) with $k' < n$.

Here we show the details for Case 1 for $k' < k \le n$ and $r \le \ell$. In this case $b' \otimes b = [1..\ell \mid k+1..p \mid \overline{p}..\overline{q} \mid \overline{\ell}..\overline{r}] \otimes [1..k]$, $b_2 = [2..k]$ and $b_1 = [1]$. Also by the results of Appendix D, $H(b' \otimes b) = k' - \ell$ and $D(\hat{b}') = 0$. To determine $H(b' \otimes b_2)$, $H(\tilde{b}' \otimes b_1)$ and $D(\tilde{\tilde{b}}')$, note that

$$e_q e_{q+1} \cdots e_{n-2}\, e_n e_{n-1} \cdots e_1 (b' \otimes b_2) = [1..\ell \mid k..p \mid \overline{p}..\overline{q+1} \mid \overline{\ell}..\overline{r}] \otimes [1..k-1].$$

Acting on this by the R-matrix one obtains

$$\begin{cases} [1..\ell+1 \mid k'+1..k-2 \mid \overline{k-2}..\overline{q+1} \mid \overline{\ell+1}..\overline{r}] \otimes [1..k'] & \text{if } p = k-1,\ k-q \text{ even} \\ [1..\ell \mid k'+1..p \mid \overline{p}..\overline{q+1} \mid \overline{\ell}..\overline{r}] \otimes [1..k'] & \text{else} \end{cases}$$

so that in both cases $\tilde{b}' = [2..k'+1]$ and $\tilde{\tilde{b}}' = [1..k']$. Since H is constant on classical components, $H(b' \otimes b_2) = k' - \ell$, $H(\tilde{b}' \otimes b_1) = 1$ and $D(\tilde{\tilde{b}}') = 0$. Hence the result of (5.10) is 1 as claimed for $k' < k \le n$.

In Appendix D, the R-matrix is only worked out for the case $k' \le k$. To prove (5.10) for $k \le k'$, consider the equation under the duality operation $*$ which swaps tensor factors. Since $\mathrm{ts} = * \circ \mathrm{bs} \circ *$, (5.10) becomes

$$(5.11) \qquad \begin{aligned} &H(b_2' \otimes b) + H(b_1' \otimes \tilde{b}) + D(\tilde{\tilde{b}}) - H(b' \otimes b) - D(\hat{b}) \\ &= \begin{cases} (\epsilon_1 - \epsilon_{k'} \mid \boldsymbol{L}) & \text{if } b' \in \hat{B}^{k',1} \\ (\epsilon_1 - \epsilon_n \mid \boldsymbol{L}) & \text{if } b' \in E^{n,1} \\ (\epsilon_1 + \epsilon_n \mid \boldsymbol{L}) & \text{if } b' \in E^{n-1,1} \end{cases} \end{aligned}$$

where $\mathrm{ts}(b') = b_1' \otimes b_2'$, $R(b_2' \otimes b) = \tilde{b} \otimes \bullet$, $R(b_1' \otimes \tilde{b}) = \tilde{\tilde{b}} \otimes \bullet$, and $\boldsymbol{L}$ is the weight corresponding to $\hat{B}^{k,1}$ (resp. $E^{n,1}$, $E^{n-1,1}$). Again (5.11) can be verified for all cases of Appendix D. Here we demonstrate that it holds for Case 1 with $k' \le k < n$ and $r \le \ell$, so that $b' \otimes b = [1..\ell \mid k+1..p \mid \overline{p}..\overline{q} \mid \overline{\ell}..\overline{r}] \otimes [1..k]$, $b_1' = [\overline{r}]$ and $b_2' = [1..\ell \mid k+1..p \mid \overline{p}..\overline{q} \mid \overline{\ell}..\overline{r+1}]$. Hence $H(b' \otimes b) = k' - \ell$ and $H(b_2' \otimes b) = k' - \ell - 1$. Next let us calculate $D(\hat{b})$. By Appendix D, $\hat{b} = [1..\ell \mid k'+1..p \mid \overline{p}..\overline{q} \mid \overline{\ell}..\overline{r}]$ if $p \neq k$ or $k-q$ odd. Acting with a product of e_i with $i \neq 0$ to make $\hat{b}$ into a highest weight vector, one finds that $D(\hat{b}) = d + \max\{0, p-q+1-k'+\ell\}$. However, since $k' = 2\ell + 2p - k - q - r + 2$ we have $p - q + 1 - k' + \ell = (k-p) + (r-\ell) - 1 < 0$ because $k - p \le 0$ and $r - \ell \le 0$. This implies that $D(\hat{b}) = d$. Similarly, for $p = k$ and $k - q$ even one finds that $D(\hat{b}) = d+1$. By Appendix D

$$\tilde{b} = \begin{cases} [1..\ell+1 \mid k'..k-1 \mid \overline{k-1}..\overline{q} \mid \overline{\ell+1}..\overline{r+1}] & \text{if } p = k,\ k-q \text{ even} \\ [1..\ell \mid k'..p \mid \overline{p}..\overline{q} \mid \overline{\ell}..\overline{r+1}] & \text{else.} \end{cases}$$

In both cases the highest weight vector obtained from $b_1' \otimes \tilde{b}$ by the application of a product of e_i's with $i \neq 0$ is of the form $[\overline{y}] \otimes [1..x \mid \overline{x}..\overline{y+1}]$ where $x = k - d$ when $p = k$ and $k - q$ even and $x = k - d + 1$ otherwise. The variable y is fixed by the requirement that $k = 2x - y$. By Case 5a of Appendix D we have

$$R([\overline{y}] \otimes [1..x \mid \overline{x}..\overline{y+1}]) = [1..x-1 \mid \overline{x-1}..\overline{y} \mid \overline{1}] \otimes [1]$$

so that $D(\tilde{\tilde{b}}) = x - y + 1 = k + 1 - x$ which is $d+1$ if $p = k$ and $k - q$ even and d otherwise. Hence altogether the left-hand side of (5.11) yields 0 as expected.

Finally let us show that (5.11) yields 2 for the tensor product $E^{n-1,1} \otimes E^{n,1}$ and 0 for $E^{n,1} \otimes E^{n,1}$. The highest weight elements in $E^{n-1,1} \otimes E^{n,1}$ are of the form $b' \otimes b =$

$[1..n-2\ell-1 \mid \overline{n}..\overline{n-2\ell}] \otimes [1..n]$. By Appendix D, $H(b' \otimes b) = 2\ell$ and $D(\hat{b}) = 0$. By the definition of ts, $b'_1 = [\overline{n-2\ell}]$ and $b'_2 = [1..n-2\ell-1 \mid \overline{n}..\overline{n-2\ell+1}]$. Furthermore

$$\tilde{b} \otimes \bullet = R(b'_2 \otimes b) = [1..n-2\ell-1 \mid n\overline{n}..\overline{n-2\ell+1}] \otimes [1..n-1]$$

with $H(b'_2 \otimes b) = 2\ell$. The highest weight vector corresponding to $b'_1 \otimes \tilde{b}$ is $[\overline{n}] \otimes [1..n]$ with $R([\overline{n}] \otimes [1..n]) = [1..n-1 \mid \overline{1}] \otimes [1]$. Hence $H(b'_1 \otimes \tilde{b}) = 1$ and $D(\tilde{\tilde{b}}) = 1$, so that (5.11) yields indeed 2.

The highest weight elements in $E^{n,1} \otimes E^{n,1}$ are of the form $b' \otimes b = [1..n-2\ell \mid \overline{n}..\overline{n-2\ell+1}] \otimes [1..n]$. By Appendix D, $H(b' \otimes b) = 2\ell$ and $D(\hat{b}) = 0$. By the definition of ts, $b'_1 = [\overline{n-2\ell+1}]$ and $b'_2 = [1..n-2\ell \mid \overline{n}..\overline{n-2\ell+2}]$. Furthermore

$$\tilde{b} \otimes \bullet = R(b'_2 \otimes b) = [1..n-2\ell+1 \mid \overline{n-1}..\overline{n-2\ell+1}] \otimes [1..n-1]$$

with $H(b'_2 \otimes b) = 2\ell-1$. The highest weight vector corresponding to $b'_1 \otimes \tilde{b}$ is $[1] \otimes [1..n-1 \mid \overline{n-1}]$ with $R([1] \otimes [1..n-1 \mid \overline{n-1}]) = [1..n-1 \mid \overline{n-1}] \otimes [1]$. Hence $H(b'_1 \otimes \tilde{b}) = 0$ and $D(\tilde{\tilde{b}}) = 1$, so that (5.11) yields indeed 0.

All other cases can be shown in a similar fashion. □

5.4. **Proof of Theorem 5.1.** By Proposition 5.6 the following diagram commutes

$$\begin{array}{ccccc}
\mathrm{RC}(\lambda, B) & \xrightarrow{\Phi} & \mathcal{P}(\lambda, B) & \xrightarrow{*} & \mathcal{P}(\lambda, B) \\
{\scriptstyle \mathrm{tj}}\downarrow & & \downarrow{\scriptstyle \mathrm{ts}} & & \downarrow{\scriptstyle \mathrm{bs}} \\
\mathrm{RC}(\lambda, \widetilde{B}) & \xrightarrow[\Phi]{} & \mathcal{P}(\lambda, \widetilde{B}) & \xrightarrow[*]{} & \mathcal{P}(\lambda, \widetilde{B}).
\end{array}$$

Since $\tilde{\Phi} = * \circ \Phi$, this implies that tj corresponds to bs under $\tilde{\Phi}$.

Let $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B)$ and $b = \tilde{\Phi}(\nu^\bullet, J^\bullet) \in \mathcal{P}(\lambda, B)$. Using (4.4) and (4.5) define $(\hat{\nu}^\bullet, \hat{J}^\bullet) = \mathrm{emb}_{\mathrm{RC}}(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\hat{\lambda}, \hat{B})$ and $\hat{b} = \mathrm{emb}_{\mathcal{P}}(b) \in \mathcal{P}(\hat{\lambda}, \hat{B})$ with $\hat{\lambda}$ and $\hat{B}$ as defined in Section 4.5. It is easy to see that $cc(\hat{\nu}^\bullet, \hat{J}^\bullet) = 2cc(\nu^\bullet, J^\bullet)$ since all parts of the partitions and all riggings double. Similarly, by [18, Eq. (6.23)] it follows that $D_{\hat{B}}(\hat{b}) = 2D_B(b)$. Since the embedding map commutes with the bijection $\tilde{\Phi}$ by Theorem 4.4 and the definition of the spinor case, we may restrict ourselves to the image of emb.

There exists a sequence $S_{\mathcal{P}}$ of maps bs and R which transforms $\hat{b}$ into a path of single boxes, that is $S_{\mathcal{P}} : \mathcal{P}(\hat{\lambda}, \hat{B}) \to \mathcal{P}(\hat{\lambda}, (B^{1,1})^N)$ where $N = \sum_{a=1}^n a\hat{L}^a + n\hat{\hat{L}}^n$. By Proposition 5.6 there is a corresponding sequence $S_{\mathrm{RC}} : \mathrm{RC}(\hat{\lambda}, \hat{B}) \to \mathrm{RC}(\hat{\lambda}, (B^{1,1})^N)$ under $\tilde{\Phi}$ consisting of maps tj and id.

By Lemmas 5.7 and 5.8 and the fact that R and id do not change the statistics, it follows that for $\hat{b} = \tilde{\Phi}(\hat{\nu}^\bullet, \hat{J}^\bullet)$

$$D(S_{\mathcal{P}}(\hat{b})) = cc(S_{\mathrm{RC}}(\hat{\nu}^\bullet, \hat{J}^\bullet)) \qquad \text{implies that} \qquad D(\hat{b}) = cc(\hat{\nu}^\bullet, \hat{J}^\bullet),$$

which in turn implies that $D(b) = cc(\nu^\bullet, J^\bullet)$. But Theorem 5.1 for the case $B = (B^{1,1})^N$ has already been proven in [19].

Appendix A. Proof of Theorem 4.1

The proof proceeds by induction on B. Given a tensor product $B = B^{k_L,1} \otimes B^{k_{L-1},1} \otimes \cdots \otimes B^{k_1,1}$, there is a unique sequence of transformations $- : B = B^{1,1} \otimes B' \to B'$, $\mathrm{ts} : B = B^{k,1} \otimes B' \to B^{1,1} \otimes B^{k-1,1} \otimes B'$ for $2 \le k \le n-2$ and $-_s : B = B^{k,1} \otimes B' \to B'$ for $k = n-1, n$ resulting in the empty crystal. If B is empty then both $\mathrm{RC}(\lambda, B)$ and $\mathcal{P}(\lambda, B)$ are the empty set unless $\lambda = \varnothing$, in which case $\mathcal{P}(\lambda, B)$ is the singleton consisting of the empty path, $\mathrm{RC}(\lambda, B)$ is the singleton consisting of the empty rigged configuration, and $\Phi_\varnothing$ is the unique bijection $\mathrm{RC}(\varnothing,\varnothing) \to \mathcal{P}(\varnothing,\varnothing)$.

Proof of (1)**.** Any map Φ_B satisfying (1) is injective by definition and unique by induction. It was proven in [19] that $-$ and δ are bijections.

Proof of (2)**.** To prove the existence and uniqueness of a bijection satisfying (2), it must be shown that $\Phi_{\widetilde{B}}$ restricts to a bijection $\Phi_{\widetilde{B}} : \mathrm{tj}(\mathrm{RC}(\lambda, B)) \to \mathrm{ts}(\mathcal{P}(\lambda, B))$. The proof relies on two lemmas for which we need the following notation. Let $(\nu^\bullet, J^\bullet) \in \mathrm{tj}(\mathrm{RC}(\lambda, B))$ and $(\tilde{\nu}^\bullet, \tilde{J}^\bullet) = \mathrm{tj} \circ \delta(\nu^\bullet, J^\bullet)$. Let $\ell^{(a)}, \overline{\ell}^{(a)}$ be the length of the selected singular strings in $(\nu^\bullet, J^\bullet)$ by δ, and $\tilde{\ell}^{(a)}, \tilde{\overline{\ell}}^{(a)}$ the ones selected in $(\tilde{\nu}^\bullet, \tilde{J}^\bullet)$ by δ.

Lemma A.1. *The following inequalities hold:*

(1) $\tilde{\ell}^{(a)} \geq \ell^{(a+1)}$ *for* $1 \leq a \leq n-3$
(2) $\tilde{\ell}^{(n-2)} \geq \min(\ell^{(n-1)}, \ell^{(n)})$
(3) $\tilde{\ell}^{(n-1)} \geq \overline{\ell}^{(n-2)}$ *if* $\ell^{(n-1)} = \min(\ell^{(n-1)}, \ell^{(n)})$
$\tilde{\ell}^{(n)} \geq \overline{\ell}^{(n-2)}$ *if* $\ell^{(n)} = \min(\ell^{(n-1)}, \ell^{(n)})$
(4) $\tilde{\overline{\ell}}^{(a)} \geq \overline{\ell}^{(a-1)}$ *for* $2 \leq a \leq n-1$.

Similarly, let $b \in \mathcal{P}(\lambda, B)$, $\tilde{b} = - \circ \mathrm{ts}(b)$ and $\tilde{\tilde{b}} = - \circ \mathrm{ts}(\tilde{b})$. More precisely, if $b = m_k m_{k-1} \cdots m_1 \otimes b'$, then $\tilde{b} = m_{k-1} m_{k-2} \cdots m_1 \otimes b'$ and $\tilde{\tilde{b}} = m_{k-2} \cdots m_1 \otimes b'$. By induction $(\tilde{\nu}^\bullet, \tilde{J}^\bullet) = \Phi^{-1}(\tilde{b})$ and $(\tilde{\tilde{\nu}}^\bullet, \tilde{\tilde{J}}^\bullet) = \Phi^{-1}(\tilde{\tilde{b}})$ exist. Let $\tilde{s}^{(a)}, \tilde{\overline{s}}^{(a)}$ be the lengths of the selected strings by δ^{-1} in $(\tilde{\tilde{\nu}}^\bullet, \tilde{\tilde{J}}^\bullet)$ adding $m_{k-1} \in B^{1,1}$ to the path, and $s^{(a)}, \overline{s}^{(a)}$ the length of the selected strings in $(\tilde{\nu}^\bullet, \tilde{J}^\bullet)$ adding the step $m_k \in B^{1,1}$ to the path.

Lemma A.2. *The following inequalities hold:*

(1) $\overline{s}^{(a)} \leq \tilde{\overline{s}}^{(a+1)}$ *for* $1 \leq a \leq n-3$
(2) $\overline{s}^{(n-2)} \leq \max(\tilde{s}^{(n-1)}, \tilde{s}^{(n)})$
(3) $s^{(n-1)} \leq \tilde{s}^{(n-2)}$ *if* $\tilde{s}^{(n-1)} = \max(\tilde{s}^{(n-1)}, \tilde{s}^{(n)})$
$s^{(n)} \leq \tilde{s}^{(n-2)}$ *if* $\tilde{s}^{(n)} = \max(\tilde{s}^{(n-1)}, \tilde{s}^{(n)})$
(4) $s^{(a+1)} \leq \tilde{s}^{(a)}$ *for* $0 \leq a \leq n-3$.

Proof of Lemma A.1. Writing $p_i^{(a)}(\nu^\bullet)$ to indicate the dependence of the vacancy numbers on the configuration, one obtains from the algorithm for δ and (4.3) that for $1 \leq a \leq n-3$

(A.1)
$$\begin{aligned}
p_i^{(a)}(\tilde{\nu}^\bullet) &= p_i^{(a)}(\nu^\bullet) - \chi(\ell^{(a-1)} \leq i < \ell^{(a)}) + \chi(\ell^{(a)} \leq i < \ell^{(a+1)}) \\
&\quad + \chi(\overline{\ell}^{(a)} \leq i < \overline{\ell}^{(a-1)}) - \chi(\overline{\ell}^{(a+1)} \leq i < \overline{\ell}^{(a)}) \\
p_i^{(n-2)}(\tilde{\nu}^\bullet) &= p_i^{(n-2)}(\nu^\bullet) - \chi(\ell^{(n-3)} \leq i < \ell^{(n-2)}) + \chi(\ell^{(n-2)} \leq i < \min(\ell^{(n-1)}, \ell^{(n)})) \\
&\quad + \chi(\overline{\ell}^{(n-2)} \leq i < \overline{\ell}^{(n-3)}) - \chi(\max(\ell^{(n-1)}, \ell^{(n)}) \leq i < \overline{\ell}^{(n-2)}) \\
p_i^{(n-1)}(\tilde{\nu}^\bullet) &= p_i^{(n-1)}(\nu^\bullet) - \chi(\ell^{(n-2)} \leq i < \ell^{(n-1)}) + \chi(\ell^{(n-1)} \leq i < \overline{\ell}^{(n-2)}) \\
p_i^{(n)}(\tilde{\nu}^\bullet) &= p_i^{(n)}(\nu^\bullet) - \chi(\ell^{(n-2)} \leq i < \ell^{(n)}) + \chi(\ell^{(n)} \leq i < \overline{\ell}^{(n-2)}).
\end{aligned}$$

Since $(\nu^\bullet, J^\bullet) \in \mathrm{tj}(\mathrm{RC}(\lambda, B))$, there is a singular string of length one in the first $k-1$ rigged partitions, so that $\ell^{(a)} = 1$ for $0 \leq a \leq k-1$ (recall that $\ell^{(0)} = 1$ by definition).

Similarly, $\tilde{\ell}^{(a)} = 1$ for $0 \le a \le k-2$. Hence $\tilde{\ell}^{(a)} = \ell^{(a+1)} = 1$ for $0 \le a \le k-2$ so that (1) holds in this case.

We proceed by induction on a. Fix $k-1 \le a \le n-3$. By induction hypothesis $\tilde{\ell}^{(a-1)} \ge \ell^{(a)}$. By definition $\tilde{\ell}^{(a)} \ge \tilde{\ell}^{(a-1)}$, so that $\tilde{\ell}^{(a)} \ge \ell^{(a)}$. By (A.1), $(\tilde{\nu}^\bullet, \tilde{J}^\bullet)^{(a)}$ has no singular string of length i in the range $\ell^{(a)} \le i < \ell^{(a+1)}$ which implies that $\tilde{\ell}^{(a)} \ge \ell^{(a+1)}$ for $1 \le a \le n-3$. Hence (1) holds.

Next consider $a = n-2$. By construction, $\tilde{\ell}^{(n-2)} \ge \tilde{\ell}^{(n-3)}$ and by (1), $\tilde{\ell}^{(n-3)} \ge \ell^{(n-2)}$. Hence $\tilde{\ell}^{(n-2)} \ge \ell^{(n-2)}$. By (A.1), $(\tilde{\nu}^\bullet, \tilde{J}^\bullet)^{(n-2)}$ has no singular string of length i in the range $\ell^{(n-2)} \le i < \min(\ell^{(n-1)}, \ell^{(n)})$. Hence $\tilde{\ell}^{(n-2)} \ge \min(\ell^{(n-1)}, \ell^{(n)})$ which proves (2).

Consider $(\tilde{\nu}^\bullet, \tilde{J}^\bullet)^{(n-1)}$ and $(\tilde{\nu}^\bullet, \tilde{J}^\bullet)^{(n)}$. By construction, $\tilde{\ell}^{(n-1)} \ge \tilde{\ell}^{(n-2)}$ and $\tilde{\ell}^{(n)} \ge \tilde{\ell}^{(n-2)}$. Using (2) we get $\tilde{\ell}^{(n-1)} \ge \min(\ell^{(n-1)}, \ell^{(n)})$ and $\tilde{\ell}^{(n)} \ge \min(\ell^{(n-1)}, \ell^{(n)})$. If $\min(\ell^{(n-1)}, \ell^{(n)}) = \ell^{(n-1)}$, then $\tilde{\ell}^{(n-1)} \ge \ell^{(n-1)}$. Now looking at the change in vacancy number (A.1) for $p_i^{(n-1)}$, we conclude that $\tilde{\ell}^{(n-1)} \ge \overline{\ell}^{(n-2)}$. If on the other hand $\min(\ell^{(n-1)}, \ell^{(n)}) = \ell^{(n)}$, then $\tilde{\ell}^{(n)} \ge \ell^{(n)}$. Now, by the change in vacancy number (A.1) for $p_i^{(n)}$, $\tilde{\ell}^{(n)} \ge \overline{\ell}^{(n-2)}$. This proves (3).

We prove (4) by downward induction on a. Recall that $\overline{\tilde{\ell}}^{(n-1)} = \max(\tilde{\ell}^{(n-1)}, \tilde{\ell}^{(n)})$ by definition. Using (3) we obtain $\overline{\tilde{\ell}}^{(n-1)} \ge \overline{\ell}^{(n-2)}$. This starts the induction at $a = n-1$. Fix $2 \le a \le n-2$ and assume that $\overline{\tilde{\ell}}^{(a+1)} \ge \overline{\ell}^{(a)}$ by induction hypothesis. By construction, $\overline{\tilde{\ell}}^{(a)} \ge \overline{\tilde{\ell}}^{(a+1)}$, so that $\overline{\tilde{\ell}}^{(a)} \ge \overline{\ell}^{(a)}$. By the change in vacancy number (A.1), $(\tilde{\nu}^\bullet, \tilde{J}^\bullet)^{(a)}$ has no singular string of length i in the range $\overline{\ell}^{(a)} \le i < \overline{\ell}^{(a-1)}$. This implies that $\overline{\tilde{\ell}}^{(a)} \ge \overline{\ell}^{(a-1)}$ as desired. □

Proof of Lemma A.2. The proof is similar to the proof of Lemma A.1. □

To show that $\Phi_{\widetilde{B}} \circ \mathrm{tj}(\mathrm{RC}(\lambda, B)) \subset \mathrm{ts}(\mathcal{P}(\lambda, B))$, let $(\nu^\bullet, J^\bullet) \in \mathrm{tj}(\mathrm{RC}(\lambda, B))$ and $m_k \otimes m_{k-1} m_{k-2} \cdots m_1 \otimes b' = \Phi_{\widetilde{B}}(\nu^\bullet, J^\bullet)$. We need to show that either $m_k > m_{k-1}$ or $(m_k, m_{k-1}) = (n, \overline{n}), (\overline{n}, n)$.

If $m_k = p$ with $p < n$, then $\ell^{(a)} = \infty$ for $a \ge p$ and by Lemma A.1 $\tilde{\ell}^{(a)} = \infty$ for $a \ge p-1$. This implies that $m_k > m_{k-1}$.

If $m_k = n$, then $\ell^{(n-1)} < \infty$, $\ell^{(n)} = \infty$ and $\overline{\ell}^{(a)} = \infty$ for $1 \le a \le n-2$. By Lemma A.1 this implies $\tilde{\ell}^{(n-1)} = \infty$ and $\overline{\tilde{\ell}}^{(a)} = \infty$ for $1 \le a \le n-2$. Hence $m_{k-1} = \overline{n}$ or $m_{k-1} < m_k$. The case $m_k = \overline{n}$ is analogous.

If $m_k = \overline{n-1}$, then $\ell^{(n-1)}, \ell^{(n)} < \infty$ and $\overline{\ell}^{(a)} = \infty$ for $1 \le a \le n-2$. By Lemma A.1 it follows that either $\tilde{\ell}^{(n-1)}$ or $\tilde{\ell}^{(n)}$ is ∞ so that indeed $m_{k-1} < m_k$.

Finally, if $m_k = \overline{p}$ with $1 \le p \le n-2$, then $\overline{\ell}^{(a)} = \infty$ for $1 \le a < p$. By Lemma A.1 it follows that $\overline{\tilde{\ell}}^{(p)} = \infty$ so that again $m_k > m_{k-1}$.

Next it needs to be shown that $\Phi_{\widetilde{B}} : \mathrm{tj}(\mathrm{RC}(\lambda, B)) \to \mathrm{ts}(\mathcal{P}(\lambda, B))$ is surjective. First consider the case $B = \hat{B}^{k,1} \otimes B'$ for $2 \le k \le n-1$. Let $b = m_k \otimes m_{k-1} m_{k-2} \cdots m_1 \otimes b' \in \mathrm{ts}(\mathcal{P}(\lambda, B))$. Let $(\nu^\bullet, J^\bullet) = \Phi_{\widetilde{B}}^{-1}(b)$. We need to show that $(\nu^\bullet, J^\bullet) \in \mathrm{tj}(\mathrm{RC}(\lambda, B))$. In other words, it needs to be shown that $(\nu^\bullet, J^\bullet)^{(a)}$ has a singular string of length one for $1 \le a \le k-1$. By induction on k it follows that $\tilde{s}^{(a)} = 0$ for $0 \le a \le k-2$ (remember that $s^{(0)} = 0$ by definition). Hence by Lemma A.2, $s^{(a)} = 0$ for $0 \le a \le k-1$ which implies that there are singular strings of lengths one in $(\nu^\bullet, J^\bullet)^{(a)}$ for $1 \le a \le k-1$.

This completes the proof of case (2).

Proof of (3)**.** To prove the existence and uniqueness of a bijection satisfying (3) we need to show that (a) δ_s is well-defined and (b) $\Phi_{B'}$ restricts to a bijection $\delta_s(\mathrm{RC}(\lambda, B)) \to -_s(\mathcal{P}(\lambda, B))$.

Let $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B)$ and $(\nu_0^\bullet, J_0^\bullet) = \mathrm{emb}_{\mathrm{RC}}(\nu^\bullet, J^\bullet)$. Inductively define $(\nu_j^\bullet, J_j^\bullet) = \delta \circ \mathrm{tj}(\nu_{j-1}^\bullet, J_{j-1}^\bullet)$ for $1 \le j \le n$ and let $\ell_j^{(a)}$, $\overline{\ell}_j^{(a)}$ be the lengths of the singular strings selected by δ in $\mathrm{tj}(\nu_j^\bullet, J_j^\bullet)$ for $0 \le j < n$. For (a) it needs to be shown that $(\nu_n^\bullet, J_n^\bullet)$ is in the image of $\mathrm{emb}_{\mathrm{RC}}$. This follows from the following lemma:

Lemma A.3. *For all $0 \le j < n$ we have:*

(1) *For $1 \le a \le n-2$, $\ell_j^{(a)}$ is odd, and*

$$\ell_j^{(a)} = \begin{cases} 1 & \text{if } a < n-j \\ \overline{\ell}_{j-(n-a)}^{(a)} - 1 & \text{if } a \ge n-j. \end{cases}$$

In addition for $i = \ell_j^{(a)} - 1$, $p_i^{(a)}(\nu_{j+1}^\bullet)$ is even.

(2) *For $k = n$ and j even, or $k = n-1$ and j odd we have*

$$\begin{aligned} &\ell_j^{(n-1)} = \ell_{j-1}^{(n-1)} - 1 \quad \text{and } \ell_j^{(n)} \text{ is even} \\ &\min(\ell_j^{(n-1)}, \ell_j^{(n)}) = \ell_j^{(n-1)} \\ &p_i^{(n-1)}(\nu_{j+1}^\bullet) \text{ is even for } i = \ell_j^{(n-1)} - 1. \end{aligned}$$

Similarly, for $k = n$ and j odd, or $k = n-1$ and j even we have

$$\begin{aligned} &\ell_j^{(n)} = \ell_{j-1}^{(n)} - 1 \quad \text{and } \ell_j^{(n-1)} \text{ is even} \\ &\min(\ell_j^{(n-1)}, \ell_j^{(n)}) = \ell_j^{(n)} \\ &p_i^{(n)}(\nu_{j+1}^\bullet) \text{ is even for } i = \ell_j^{(n)} - 1. \end{aligned}$$

(3) *For $1 \le a \le n-2$, $\overline{\ell}_j^{(a)}$ is even.*

Lemma A.3 implies that all partitions in $(\nu_n^\bullet, J_n^\bullet)$ have only parts of even lengths and even riggings so that $(\nu_n^\bullet, J_n^\bullet)$ is indeed in the image of $\mathrm{emb}_{\mathrm{RC}}$. Hence δ_s is well-defined.

Proof of Lemma A.3. For concreteness we assume throughout that $k = n$. The case $k = n-1$ is entirely analogous. For $j = 0$, by definition $\mathrm{tj}(\nu_0^\bullet, J_0^\bullet)$ has singular strings of lengths 1 in the first $n-1$ rigged partitions. Hence $\ell_0^{(a)} = 1$ for $1 \le a \le n-1$. This proves (1). Furthermore $\mathrm{tj}(\nu_0^\bullet, J_0^\bullet)^{(n)}$ contains only even strings so that $\ell_0^{(n)} > \ell_0^{(n-1)} = 1$ is even which proves (2). All strings of length greater than one in $(\nu_0^\bullet, J_0^\bullet)^{(a)}$ for $1 \le a \le n-2$ are even so that $\overline{\ell}_0^{(a)}$ are even which verifies (3).

Suppose the lemma is true for $0 \le j' < j$. We will prove it for j. Set $b = n-j$. Clearly $\ell_j^{(a)} = 1$ for $1 \le a < b$ by the definition of tj. We proceed by induction on $b \le a < n-2$.

Assume that (1) holds for $p < a$. In particular $\ell_j^{(p)} = \overline{\ell}_{j-(n-p)}^{(p)} - 1$ for $b \le p < a$. We want to show that $\ell_j^{(a)} = \overline{\ell}_{j-(n-a)}^{(a)} - 1$. For $a = b$, we have $\overline{\ell}_{j-(n-a)}^{(a)} - 1 \ge 1 = \ell_j^{(a-1)}$. Similarly for $a > b$ note that by Lemma A.1 $\overline{\ell}_{j-(n-a)}^{(a)} - 1 \ge \overline{\ell}_{j-(n-a+1)}^{(a-1)} - 1 = \ell_j^{(a-1)}$.

We will show that for $\ell_j^{(a-1)} \le i < \overline{\ell}_{j-(n-a)}^{(a)} - 1$ there is no singular string in $(\nu_j^\bullet, J_j^\bullet)^{(a)}$. For this range of i the change in vacancy number is

$$\begin{aligned} p_i^{(a)}(\nu_j^\bullet) = p_i^{(a)}(\nu_0^\bullet) &- \sum_{k=0}^{j-1} \chi(\ell_k^{(a-1)} \le i < \ell_k^{(a)}) + \sum_{k=0}^{j-1} \chi(\ell_k^{(a)} \le i < \ell_k^{(a+1)}) \\ &- \sum_{k=0}^{j-1} \chi(\overline{\ell}_k^{(a+1)} \le i < \overline{\ell}_k^{(a)}) + \sum_{k=0}^{j-1} \chi(\overline{\ell}_k^{(a)} \le i < \overline{\ell}_k^{(a-1)}) \\ = p_i^{(a)}(\nu_0^\bullet) &+ \chi(\ell_{j-1}^{(a)} \le i < \ell_{j-1}^{(a+1)}) - \chi(\overline{\ell}_{a-b}^{(a+1)} \le i < \overline{\ell}_{a-b}^{(a)}). \end{aligned}$$

By induction, $\ell_{j-1}^{(a+1)} = \overline{\ell}_{a-b}^{(a+1)} - 1$, so that the vacancy numbers in this range change by ± 1 except for $i = \ell_{j-1}^{(a+1)}$. By induction $\ell_{j-1}^{(a+1)}$ is odd and $\nu_j^{(a)}$ has no odd parts of length $\overline{\ell}_{a-b-1}^{(a)} \le i < \overline{\ell}_{a-b}^{(a)} - 1$. In addition the riggings in this range are even, so that there are no singular strings for $\ell_j^{(a-1)} \le i < \overline{\ell}_{a-b}^{(a)} - 1$ as claimed. Hence $\ell_j^{(a)} = \overline{\ell}_{a-b}^{(a)} - 1$.

To see that $p_i^{(a)}(\nu_{j+1}^\bullet)$ is even for $i = \ell_j^{(a)} - 1$ note that $\overline{\ell}_{a-b+m}^{(a-1)} \ge \overline{\ell}_{a-b}^{(a)}$ and $\overline{\ell}_{a-b+m+1}^{(a+1)} \ge \overline{\ell}_{a-b}^{(a)}$ for $m \ge 0$. Furthermore by induction $\ell_{j-m}^{(a-1)} = \overline{\ell}_{a-b-m-1}^{(a-1)} - 1$ for $m \ge 0$ and $\ell_{j-m}^{(a+1)} = \overline{\ell}_{a-b-m+1}^{(a+1)} - 1$ for $m \ge 1$, so that all parts of length less than $\ell_j^{(a)}$ in $\nu_{j+1}^{(a-1)}$ and $\nu_{j+1}^{(a+1)}$ are even. Hence $p_i^{(a)}(\nu_{j+1}^\bullet)$ is even.

For $a = n-2$, we will also show that there is no singular string in $(\nu_j^\bullet, J_j^\bullet)^{(n-2)}$ of length $\ell_j^{(n-3)} \le i < \overline{\ell}_{j-2}^{(n-2)} - 1$. Similarly to before we find that in this range of i the change in vacancy number is

$$\begin{aligned} p_i^{(n-2)}(\nu_j^\bullet) &= p_i^{(n-2)}(\nu_0^\bullet) \\ &\quad + \chi(\ell_{j-1}^{(n-2)} \le i < \min(\ell_{j-1}^{(n-1)}, \ell_{j-1}^{(n)})) - \chi(\max(\ell_{j-2}^{(n-1)}, \ell_{j-2}^{(n)}) \le i < \overline{\ell}_{j-2}^{(n-2)}). \end{aligned}$$

If j is even, by induction $\min(\ell_{j-1}^{(n-1)}, \ell_{j-1}^{(n)}) = \ell_{j-1}^{(n)}$, $\max(\ell_{j-2}^{(n-1)}, \ell_{j-2}^{(n)}) = \ell_{j-2}^{(n)}$ and $\ell_{j-1}^{(n)} = \ell_{j-2}^{(n)} - 1$ with $\ell_{j-2}^{(n)}$ is even, so that the vacancy number changes by ± 1 except for $i = \ell_{j-2}^{(n)} - 1$. However, there is no odd string in $\nu_j^{(n-2)}$ for $\overline{\ell}_{j-3}^{(n-3)} \le i < \overline{\ell}_{j-2}^{(n-2)} - 1$ and all riggings in this range are even. Hence $\ell_j^{(n-2)} = \overline{\ell}_{j-2}^{(n-2)} - 1$ as desired. The case j odd is analogous.

To verify that $p_i^{(n-2)}(\nu_{j+1}^\bullet)$ is even for $i = \ell_j^{(n-2)} - 1$ let us again assume that j is even. Note that $\ell_{j-1}^{(n-1)} \ge \overline{\ell}_{j-2}^{(n-2)}$ by Lemma A.1 and by induction $\ell_{j-k+1}^{(n)} = \ell_{j-k}^{(n)} - 1$ for $k > 0$ even and $\ell_{j-k+1}^{(n-1)} = \ell_{j-k}^{(n-1)} - 1$ for $k > 1$ odd. Hence $\nu_{j+1}^{(n-1)}$ and $\nu_{j+1}^{(n)}$ contain only even strings of length less than $\ell_j^{(n-2)}$. Similarly, $\nu_{j+1}^{(n-3)}$ contains only even strings of length less than $\ell_j^{(n-2)}$ which proves the assertion. The case j odd is again analogous. This concludes the proof of (1).

For the proof of (2) assume that j is even. To prove $\ell_j^{(n-1)} = \ell_{j-1}^{(n-1)} - 1$ we show that there are no singular strings of length $\ell_j^{(n-2)} \le i < \ell_{j-1}^{(n-1)} - 1$ in $(\nu_j^\bullet, J_j^\bullet)^{(n-1)}$. By induction $\min(\ell_{j-2}^{(n-1)}, \ell_{j-2}^{(n)}) = \ell_{j-2}^{(n-1)}$, so that by Lemma A.1 $\ell_{j-1}^{(n-1)} \ge \overline{\ell}_{j-2}^{(n-2)} = \ell_j^{(n-2)} + 1$. Furthermore, $\ell_j^{(n-2)} \ge \ell_{j-1}^{(n-2)}$. Hence for $\ell_j^{(n-2)} \le i < \ell_{j-1}^{(n-1)} - 1$ the vacancy number changes by

$$\begin{aligned}p_i^{(n-1)}(\nu_j^\bullet) =&p_i^{(n-1)}(\nu_0^\bullet) - \sum_{k=0}^{j-1} \chi(\ell_k^{(n-2)} \le i < \ell_k^{(n-1)}) + \sum_{k=0}^{j-1} \chi(\ell_k^{(n-1)} \le i < \overline{\ell}_k^{(n-2)})\\ =&p_i^{(n-1)}(\nu_0^\bullet) - \chi(\ell_{j-1}^{(n-2)} \le i < \ell_{j-1}^{(n-1)}).\end{aligned}$$

Since $p_i^{(n-1)}(\nu_0^\bullet)$ is even and all riggings in this range are even, this implies that there are no singular strings. Hence $\ell_j^{(n-1)} = \ell_{j-1}^{(n-1)} - 1$ as claimed.

By induction $\min(\ell_{j-1}^{(n-1)}, \ell_{j-1}^{(n)}) = \ell_{j-1}^{(n)}$, so that by Lemma A.1 $\ell_j^{(n)} \ge \overline{\ell}_{j-1}^{(n-2)}$. On the other hand $\ell_j^{(n-1)} = \ell_{j-1}^{(n-1)} - 1 < \overline{\ell}_{j-1}^{(n-2)}$ which proves that $\min(\ell_j^{(n-1)}, \ell_j^{(n)}) = \ell_j^{(n-1)}$. Note that $\ell_{j'}^{(n)} \le \overline{\ell}_{j-1}^{(n-2)}$ for $j' < j$ so that there are only even strings of length greater or equal to $\overline{\ell}_{j-1}^{(n-2)}$ in $\nu_j^{(n)}$. Hence $\ell_j^{(n)}$ must be even. All strings of length less than $\ell_j^{(n-1)}$ in $\nu_{j+1}^{(n-2)}$ are even by induction. Hence $p_i^{(n-1)}(\nu_{j+1}^\bullet)$ is even for $i = \ell_j^{(n-1)} - 1$.

The case j is odd is analogous. This concludes the proof of (2).

For (3) note that $\overline{\ell}_j^{(a)} \ge \overline{\ell}_{j-1}^{(a-1)}$ by Lemma A.1. There are only even strings of length greater or equal to $\overline{\ell}_{j-1}^{(a-1)}$ in $\nu_j^{(a)}$ which proves that $\overline{\ell}_j^{(a)}$ is even. □

To prove (b) we need to show that the crystal element defined by δ_s is indeed an element of $B^{n,1}$ (resp. $B^{n-1,1}$) and that δ_s^{-1} exists and is well-defined. The proof of the latter is similar to the proof of Lemma A.3 and is omitted here. To show that the step defined by δ_s is in $B^{n,1}$ (resp. $B^{n-1,1}$) we need to show that (3.3) holds. For concreteness let us assume that the crystal is $B^{n,1}$. The condition $m_h < m_{h+1}$ follows from (2) of Theorem 4.1 by the definition of δ_s.

Let j be maximal such that both $\ell_{j-1}^{(n-1)}, \ell_{j-1}^{(n)} < \infty$. If j is even, $\ell_j^{(n-1)} = \ell_{j-1}^{(n-1)} - 1 < \infty$ by Lemma A.3, so that $\ell_j^{(n)} = \infty$. This implies that $m_{n-j} = n$. Similarly, if j is odd, $\ell_j^{(n)} = \ell_{j-1}^{(n)} - 1 < \infty$ and $\ell_j^{(n-1)} = \infty$, so that $m_{n-j} = \overline{n}$. This proves point 3 of (3.3). It also shows that either n or $\overline{n}$ occurs in $b = m_n \dots m_1$. To see that not both n and $\overline{n}$ can occur, we note that if j is even then $\ell_{j+1}^{(n)} = \ell_j^{(n)} - 1 = \infty$ and $\min(\ell_{j+1}^{(n-1)}, \ell_{j+1}^{(n)}) = \ell_{j+1}^{(n)}$ by Lemma A.3, so that $\ell_{j+1}^{(n-1)} = \infty$ as well. Hence $m_{n-j-1} \le n-1$. The case j odd is analogous.

It remains to show that precisely one of p and $\overline{p}$ occurs in b for $1 \le p \le n-1$. Suppose $m_{n-j} = \overline{p}$. Then $\overline{\ell}_j^{(p-1)} = \infty$ and $\overline{\ell}_j^{(p)} < \infty$. By Lemma A.1 $\overline{\ell}_{j-1}^{(p-1)} \le \overline{\ell}_j^{(p)} < \infty$ and $\overline{\ell}_{j+1}^{(p)} \ge \overline{\ell}_j^{(p-1)} = \infty$. Setting $a = n-p+j$ it follows from Lemma A.3 that

$$\begin{aligned}\ell_a^{(p-1)} &= \overline{\ell}_{j-1}^{(p-1)} - 1 < \infty & \ell_a^{(p)} &= \overline{\ell}_j^{(p)} - 1 < \infty\\ \ell_{a+1}^{(p-1)} &= \overline{\ell}_j^{(p-1)} - 1 = \infty & \ell_{a+1}^{(p)} &= \overline{\ell}_{j+1}^{(p)} - 1 = \infty.\end{aligned}$$

Hence p does not occur in b.

Now suppose $\overline{p} \notin b$. We show that then $p \in b$. If $\overline{p} \notin b$ there must be some j such that $\overline{\ell}_{j-1}^{(p)}, \overline{\ell}_{j-1}^{(p-1)} < \infty$ and $\overline{\ell}_j^{(p)}, \overline{\ell}_j^{(p-1)} = \infty$. Setting as before $a = n-p+j$ this implies by Lemma A.3 that $\ell_a^{(p-1)} = \overline{\ell}_{j-1}^{(p-1)} - 1 < \infty$ and $\ell_a^{(p)} = \overline{\ell}_j^{(p)} - 1 = \infty$. Hence $m_{n-a} = p$. This proves point 2 of (3.3).

This concludes the proof of Theorem 4.1.

Appendix B. Proof of Theorem 4.4

In the following let us suppress the weight λ and write $\mathrm{RC}(B)$ for $\mathrm{RC}(\lambda, B)$. It is straightforward to check that the diagram

$$\begin{tikzcd}
\mathrm{RC}(B^{k,1}\otimes B) \arrow[r, "\mathrm{emb}_{RC}"] \arrow[ddd, "\delta\circ\mathrm{tj}"'] & \mathrm{RC}(B^{k,1}\otimes B^{k,1}\otimes \hat{B}) \arrow[d, "\delta\circ\mathrm{tj}"] \\
 & \mathrm{RC}(B^{k-1,1}\otimes B^{k,1}\otimes \hat{B}) \arrow[d, "\mathrm{id}"] \\
 & \mathrm{RC}(B^{k,1}\otimes B^{k-1,1}\otimes \hat{B}) \arrow[d, "\delta\circ\mathrm{tj}"] \\
\mathrm{RC}(B^{k-1,1}\otimes B) \arrow[r, "\mathrm{emb}_{RC}"] & \mathrm{RC}(B^{k-1,1}\otimes B^{k-1,1}\otimes \hat{B})
\end{tikzcd} \tag{B.1}$$

commutes.

We will show here that the crystal analogon to this diagram commutes:

$$\begin{tikzcd}
B^{k,1} \arrow[r, "\mathrm{emb}_B"] \arrow[ddd, "-\circ\mathrm{ts}"'] & B^{k,1}\otimes B^{k,1} \arrow[d, "-\circ\mathrm{ts}"] \\
 & B^{k-1,1}\otimes B^{k,1} \arrow[d, "R"] \\
 & B^{k,1}\otimes B^{k-1,1} \arrow[d, "-\circ\mathrm{ts}"] \\
B^{k-1,1} \arrow[r, "\mathrm{emb}_B"] & B^{k-1,1}\otimes B^{k-1,1}.
\end{tikzcd} \tag{B.2}$$

By the definition of Φ and Proposition 5.6, the commutative diagrams (B.1) and (B.2) imply Theorem 4.4.

First we verify (B.2) for classically highest weight elements. All classically highest weight elements $b \in B^{k,1}$ are of the form $b = k\cdots 21$ or $b = \overline{\ell}\cdots\overline{p}p\cdots 21$ for some $1 \le \ell \le p$ such that $2p - \ell + 1 = k$. For $b = k\cdots 21$ everything is determined by weight considerations and it is easy to see that both paths in the diagram (B.2) yield $(k-1)\cdots 21\otimes(k-1)\cdots 21$. For $b = \overline{\ell}\cdots\overline{p}p\cdots 21$ explicit calculations show that $\mathrm{emb}_B(b) = \overline{\ell}\cdots\overline{k}(\ell-1)\cdots 21\otimes k\cdots 21$. Hence

$$\begin{aligned}
-\circ\mathrm{ts}(b) &= \overline{\ell+1}\cdots\overline{p}\,p\cdots 21\\
\mathrm{emb}_B\circ-\circ\,\mathrm{ts}(b) &= \overline{\ell+1}\cdots\overline{k-1}\,\ell\cdots 21\otimes(k-1)\cdots 21\\
-\circ\mathrm{ts}\circ\mathrm{emb}_B(b) &= \overline{\ell+1}\cdots\overline{k}\;(\ell-1)\cdots 21\otimes k\cdots 21.
\end{aligned}$$

From its definition, that R commutes with all crystal operators, one may check that

$$R(\overline{\ell+1}\cdots\overline{k}\;(\ell-1)\cdots 21\otimes k\cdots 21) = \overline{\ell}\cdots\overline{k-1}\,\ell\cdots 21\otimes(k-1)\cdots 21.$$

Hence indeed $\mathrm{emb}_B\circ-\circ\,\mathrm{ts}(b) = -\circ\mathrm{ts}\circ R\circ-\circ\,\mathrm{ts}\circ\mathrm{emb}_B(b)$.

Next we show that if the diagram commutes for $b \in B^{k,1}$ then it commutes for $f_i b$ for $1 \le i \le n$ provided that $f_i b \neq 0$.

Suppose that $b = \beta b'$ where β is a letter in the alphabet $\mathcal{A} = \{1, 2, \ldots, n, \overline{n}, \ldots, \overline{2}, \overline{1}\}$ and b' is a column of height $k-1$. We claim that

$$\mathrm{emb}_B(b) = \beta a' \otimes a \tag{B.3}$$

$$R(a' \otimes a) = \beta\tilde{a} \otimes \tilde{a}' \tag{B.4}$$

for some column a' of height $k-1$, a of height k, and columns $\tilde{a}$ and $\tilde{a}'$ of height $k-1$. Equations (B.3) and (B.4) are certainly true for classically highest weight elements b as is clear from the previous explicit calculations. Now suppose (B.3) and (B.4) hold for b. Then they hold for $f_i b$ for $1 \le i \le n$ if $f_i b \neq 0$ since $\mathrm{emb}_B(B^{k,1})$ is aligned and R is a crystal isomorphism so that $f_i R = R f_i$.

If f_i does not act on β in b, then $\mathrm{emb}_B \circ - \circ \mathrm{ts}(f_i b) = f_i^2 \circ \mathrm{emb}_B \circ - \circ \mathrm{ts}(b)$ and

$$\begin{aligned}
&- \circ\mathrm{ts} \circ R \circ - \circ \mathrm{ts} \circ \mathrm{emb}_B(f_i b)\\
=& - \circ\mathrm{ts} \circ R \circ - \circ \mathrm{ts} \circ f_i^2 \circ \mathrm{emb}_B(b)\\
=& - \circ\mathrm{ts} \circ R \circ f_i^2 \circ - \circ \mathrm{ts} \circ \mathrm{emb}_B(b)\\
=& - \circ\mathrm{ts} \circ f_i^2 \circ R \circ - \circ \mathrm{ts} \circ \mathrm{emb}_B(b)\\
=& f_i^2 \circ - \circ \mathrm{ts} \circ R \circ - \circ \mathrm{ts} \circ \mathrm{emb}_B(b)
\end{aligned}$$

by (B.3) and (B.4) so that the diagram still commutes. Similarly, if f_i acts on β in b, then $\mathrm{emb}_B \circ - \circ \mathrm{ts}(f_i b) = \mathrm{emb}_B \circ - \circ \mathrm{ts}(b)$ and also

$$\begin{aligned}
&- \circ\mathrm{ts} \circ R \circ - \circ \mathrm{ts} \circ \mathrm{emb}_B(f_i b)\\
=& - \circ\mathrm{ts} \circ R \circ - \circ \mathrm{ts} \circ f_i^2 \circ \mathrm{emb}_B(b)\\
=& - \circ\mathrm{ts} \circ R \circ f_i \circ - \circ \mathrm{ts} \circ \mathrm{emb}_B(b)\\
=& - \circ\mathrm{ts} \circ f_i \circ R \circ - \circ \mathrm{ts} \circ \mathrm{emb}_B(b)\\
=& - \circ\mathrm{ts} \circ R \circ - \circ \mathrm{ts} \circ \mathrm{emb}_B(b)
\end{aligned}$$

by (B.3) and (B.4) so that the diagram still commutes. This concludes the proof of Theorem 4.4.

Appendix C. Proof of Lemma 5.5

One may easily verify that (see also [17, Eq. (3.10)])

$$\begin{aligned}
-p_{i-1}^{(a)} + 2p_i^{(a)} - p_{i+1}^{(a)} &= -\sum_{b \in J} (\alpha_a \mid \alpha_b)\, m_i^{(b)} + \delta_{i,1} L^a\\
&\ge -\sum_{b \in J} (\alpha_a \mid \alpha_b)\, m_i^{(b)},
\end{aligned} \tag{C.1}$$

The proof of $[\delta, \tilde{\delta}] = 0$ follows from Lemmas C.1 and C.2 below. The proof of these lemmas is very technical and follows similar arguments as in [17, Appendix A]. Details are available upon request.

Let $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B)$ where $B = (B^{1,1})^{\otimes 2} \otimes B'$. The following notation is used:

$$\begin{aligned}
\delta(\nu^\bullet, J^\bullet) &= (\dot{\nu}^\bullet, \dot{J}^\bullet)\\
\tilde{\delta}(\nu^\bullet, J^\bullet) &= (\tilde{\nu}^\bullet, \tilde{J}^\bullet)\\
\tilde{\delta} \circ \delta(\nu^\bullet, J^\bullet) &= (\tilde{\dot{\nu}}^\bullet, \tilde{\dot{J}}^\bullet)\\
\delta \circ \tilde{\delta}(\nu^\bullet, J^\bullet) &= (\dot{\tilde{\nu}}^\bullet, \dot{\tilde{J}}^\bullet).
\end{aligned}$$

Furthermore, let $\{\dot{\ell}^{(k)}, \dot{s}^{(k)}\}$, $\{\tilde{\ell}^{(k)}, \tilde{s}^{(k)}\}$, $\{\tilde{\dot{\ell}}^{(k)}, \tilde{\dot{s}}^{(k)}\}$ and $\{\dot{\tilde{\ell}}^{(k)}, \dot{\tilde{s}}^{(k)}\}$ be the lengths of the strings that are shortened in the transformations $(\nu^\bullet, J^\bullet) \mapsto (\dot{\nu}^\bullet, \dot{J}^\bullet)$, $(\nu^\bullet, J^\bullet) \mapsto (\tilde{\nu}^\bullet, \tilde{J}^\bullet)$, $(\dot{\nu}^\bullet, \dot{J}^\bullet) \mapsto (\tilde{\dot{\nu}}^\bullet, \tilde{\dot{J}}^\bullet)$ and $(\tilde{\nu}^\bullet, \tilde{J}^\bullet) \mapsto (\dot{\tilde{\nu}}^\bullet, \dot{\tilde{J}}^\bullet)$, respectively. (In Section 4.4 the lengths of the s-strings was denoted by $\overline{\ell}$). We call the strings, whose lengths are labeled by an ℓ, ℓ-strings and those labeled by an s, s-strings.

Lemma C.1. *The following cases occur at $(\nu^\bullet, J^\bullet)^{(k)}$:*

I. ***Nontwisted case.*** *In this case the ℓ-string selected by δ (resp. $\tilde{\delta}$) in $(\nu^\bullet, J^\bullet)^{(k)}$ is different from the s-string selected by $\tilde{\delta}$ (resp. δ) in $(\nu^\bullet, J^\bullet)^{(k)}$. For the ℓ-strings one of the following must hold:*

- (ℓa) ***Generic case.*** *If δ and $\tilde{\delta}$ do not select the same ℓ-string, then $\dot{\tilde{\ell}}^{(k)} = \dot{\ell}^{(k)}$ and $\tilde{\dot{\ell}}^{(k)} = \tilde{\ell}^{(k)}$.*
- (ℓb) ***Doubly singular case.*** *In this case δ and $\tilde{\delta}$ select the same ℓ-string, so that $\dot{\ell}^{(k)} = \tilde{\ell}^{(k)} =: \ell$. Then*
  - (1) *If $\tilde{\dot{\ell}}^{(k)} < \ell$ (or $\dot{\tilde{\ell}}^{(k)} < \ell$) then $\tilde{\dot{\ell}}^{(k)} = \dot{\tilde{\ell}}^{(k)} = \ell - 1$ and $m_{\ell-1}^{(k+1)} = 0$ for $k < n-2$, $m_{\ell-1}^{(n-1)} = m_{\ell-1}^{(n)} = 0$ for $k = n-2$ and $m_{\ell-1}^{(n-2)} = 0$ for $k = n-1, n$.*
  - (2) *If $\tilde{\dot{\ell}}^{(k)} = \ell$ (or $\dot{\tilde{\ell}}^{(k)} = \ell$) then case I.($\ell s$)(1') (or I.($\ell s$)(1)) holds or $\tilde{\dot{\ell}}^{(k)} = \dot{\tilde{\ell}}^{(k)} = \ell$.*
  - (3) *If $\tilde{\dot{\ell}}^{(k)} > \ell$ (or $\dot{\tilde{\ell}}^{(k)} > \ell$) then case I.($\ell s$)(1',2) (or I.($\ell s$)(1,2)) holds or $\tilde{\dot{\ell}}^{(k)} = \dot{\tilde{\ell}}^{(k)}$ and $\tilde{\dot{\ell}}^{(k)} \le \tilde{\ell}^{(k+1)}$, $\dot{\tilde{\ell}}^{(k)} \le \dot{\ell}^{(k+1)}$ for $k < n-2$, $\tilde{\dot{\ell}}^{(n-2)} \le \min\{\tilde{\ell}^{(n-1)}, \tilde{\ell}^{(n)}\}$, $\dot{\tilde{\ell}}^{(n-2)} \le \min\{\dot{\ell}^{(n-1)}, \dot{\ell}^{(n)}\}$ for $k = n-2$, and $\tilde{\dot{\ell}}^{(k)} \le \tilde{s}^{(n-2)}$, $\dot{\tilde{\ell}}^{(k)} \le \dot{s}^{(n-2)}$ for $k = n-1, n$.*

For the s-strings, case I.(ℓs) holds or one the following must hold:

- (sa) ***Generic case.*** *If δ and $\tilde{\delta}$ do not select the same s-string, then $\dot{\tilde{s}}^{(k)} = \dot{s}^{(k)}$ and $\tilde{\dot{s}}^{(k)} = \tilde{s}^{(k)}$.*
- (sb) ***Doubly singular case.*** *In this case δ and $\tilde{\delta}$ select the same s-string, so that $\dot{s}^{(k)} = \tilde{s}^{(k)} =: s$. Then*
  - (1) *If $\tilde{\dot{s}}^{(k)} < s$ (or $\dot{\tilde{s}}^{(k)} < s$) then $\tilde{\dot{s}}^{(k)} = \dot{\tilde{s}}^{(k)} = s - 1$ and $m_{s-1}^{(k-1)} = 0$.*
  - (2) *If $\tilde{\dot{s}}^{(k)} = s$ (or $\dot{\tilde{s}}^{(k)} = s$) then $\tilde{\dot{s}}^{(k)} = \dot{\tilde{s}}^{(k)} = s$.*
  - (3) *If $\tilde{\dot{s}}^{(k)} > s$ (or $\dot{\tilde{s}}^{(k)} > s$) then $\tilde{\dot{s}}^{(k)} = \dot{\tilde{s}}^{(k)}$, $\tilde{\dot{s}}^{(k)} \le \tilde{s}^{(k-1)}$ and $\dot{\tilde{s}}^{(k)} \le \dot{s}^{(k-1)}$.*
- (ℓs) ***Mixed case.*** *One of the following holds:*
  - (1) *$\dot{\ell}^{(k)} = \tilde{\ell}^{(k)} =: \ell$, $\dot{\tilde{\ell}}^{(k)} = \dot{s}^{(k)} = \dot{\ell}^{(k+1)} =: \ell'$, $\tilde{\dot{\ell}}^{(k)} = \dot{\tilde{s}}^{(k)} =: \ell''$, $\tilde{s}^{(k)} = \tilde{\dot{s}}^{(k)}$ or possibly the same conditions for ℓ and ℓ', $\tilde{\dot{\ell}}^{(k)} = \tilde{s}^{(k)} = \tilde{s}^{(k+1)} = \ell''$, $\dot{\tilde{s}}^{(k)} = \tilde{\dot{s}}^{(k)} = \ell'''$, $m_{\ell''}^{(k-1)} = 0$, $m_{\ell''}^{(k)} = 1$, $m_{\ell''}^{(k+1)} = 2$ if case I.(ℓs)(1) does not hold at $k-1$. Furthermore, either $\tilde{\dot{\ell}}^{(k)} \le \tilde{\ell}^{(k+1)}$ or case I.(ℓs)(1) holds at $k+1$ with the same values of ℓ' and ℓ''. Similarly, either $\dot{\tilde{s}}^{(k)} \le \dot{s}^{(k-1)}$ or case I.(ℓs)(1) holds at $k-1$ with the same values of ℓ' and ℓ''.*
  - (1') *$\dot{\ell}^{(k)} = \tilde{\ell}^{(k)} =: \ell$, $\tilde{\dot{\ell}}^{(k)} = \tilde{s}^{(k)} = \tilde{\ell}^{(k+1)} =: \ell'$, $\dot{\tilde{\ell}}^{(k)} = \tilde{\dot{s}}^{(k)} =: \ell''$, $\dot{s}^{(k)} = \dot{\tilde{s}}^{(k)}$ or possibly the same conditions for ℓ and ℓ', $\dot{\tilde{\ell}}^{(k)} = \dot{s}^{(k)} = \dot{s}^{(k+1)} = \ell''$, $\tilde{\dot{s}}^{(k)} = \dot{\tilde{s}}^{(k)} = \ell'''$, $m_{\ell''}^{(k-1)} = 0$, $m_{\ell''}^{(k)} = 1$, $m_{\ell''}^{(k+1)} = 2$ if case I.(ℓs)(1') does not hold at $k-1$. Furthermore, either $\dot{\tilde{\ell}}^{(k)} \le \dot{\ell}^{(k+1)}$*

or case I.(ℓs)(1') holds at $k+1$ with the same values of ℓ' and ℓ''. Similarly, either $\tilde{\dot{s}}^{(k)} \le \tilde{s}^{(k-1)}$ or case I.(ℓs)(1') holds at $k-1$ with the same values of ℓ' and ℓ''.

(2) *For $k < n-2$ (resp. $k = n-2$) $\dot{\ell}^{(k)} = \tilde{\ell}^{(k)} =: \ell$, $\dot{s}^{(k)} = \tilde{s}^{(k)} = \dot{s}^{(k+1)} = \tilde{s}^{(k+1)} =: \ell'$ (resp. $\dot{s}^{(k)} = \tilde{s}^{(k)} = \dot{\ell}^{(n-1)} = \dot{\ell}^{(n)} = \tilde{\ell}^{(n-1)} = \tilde{\ell}^{(n)} = \ell'$), $\dot{\tilde{\ell}}^{(k)} = \tilde{\dot{\ell}}^{(k)} = \ell''$, $\dot{\tilde{s}}^{(k)} = \tilde{\dot{s}}^{(k)} := \ell'''$ and case I.(ℓs)(2) holds at $k+1$ (resp. $n-1$ and n) with the same values of ℓ' and ℓ'' and $\ell = \ell'$, $\ell''' = \ell''$. Also, either $\tilde{\dot{s}}^{(k)} \le \tilde{s}^{(k-1)}$ and $\dot{\tilde{s}}^{(k)} \le \dot{s}^{(k-1)}$ or case I.(ℓs)(2) holds at $k-1$ with the same values of ℓ' and ℓ''. For $k = n-1, n$, $\dot{\ell}^{(k)} = \tilde{\ell}^{(k)} = \dot{s}^{(n-2)} = \tilde{s}^{(n-2)} = \ell'$ and $\tilde{\dot{\ell}}^{(k)} = \dot{\tilde{\ell}}^{(k)} = \ell''$. In addition case I.($\ell s$)(2) holds at $n-2$ with the same values of ℓ' and ℓ''.*

II. ***Twisted case.*** *In this case the ℓ-string in $(\nu^\bullet, J^\bullet)^{(k)}$ selected by δ is the same as the s-string selected by $\tilde{\delta}$ or vice versa. In the first case $\dot{\ell}^{(k)} = \tilde{s}^{(k)} =: \ell$. Then $\tilde{\ell}^{(k)} = \tilde{\dot{\ell}}^{(k)}$ and one of the following holds:*

(1) *If $\dot{\tilde{\ell}}^{(k)} < \ell$, then $\dot{\tilde{\ell}}^{(k)} = \tilde{\dot{s}}^{(k)} = \ell - 1$, $m_{\ell-1}^{(k+1)} = 0$ or $m_{\ell-1}^{(k+1)}(\tilde{\nu}^\bullet) = 0$, and $m_{\ell-1}^{(k-1)} = 0$ or $m_{\ell-1}^{(k-1)}(\dot{\nu}^\bullet) = 0$. Furthermore $\dot{s}^{(k)} = \dot{\tilde{s}}^{(k)}$.*

(2) *If $\dot{\tilde{\ell}}^{(k)} = \ell$, then $\dot{\tilde{\ell}}^{(k)} = \tilde{\dot{s}}^{(k)} = \ell$ and $\dot{s}^{(k)} = \dot{\tilde{s}}^{(k)}$.*

(3) *If $\dot{\tilde{\ell}}^{(k)} > \ell$, then*
   - (i) $\dot{\tilde{\ell}}^{(k)} = \tilde{\dot{s}}^{(k)}$ *and* $\dot{s}^{(k)} = \dot{\tilde{s}}^{(k)}$, *or*
   - (ii) $\dot{\tilde{\ell}}^{(k)} = \dot{s}^{(k)}$ *and* $\tilde{\dot{s}}^{(k)} = \dot{\tilde{s}}^{(k)} \le \dot{s}^{(k-1)}$.

   Furthermore, either $\dot{\tilde{\ell}}^{(k)} \le \dot{\ell}^{(k+1)}$ or $\dot{\ell}^{(k)} = \dot{\ell}^{(k+1)}$, $\dot{\tilde{\ell}}^{(k)} = \dot{\tilde{\ell}}^{(k+1)}$, $m_\ell^{(k+1)} = 1$ and Case II.(3)(i) holds at $k+1$. Similarly, either $\tilde{\dot{s}}^{(k)} \le \tilde{s}^{(k-1)}$ or $\dot{\ell}^{(k)} = \dot{\ell}^{(k-1)}$, $\dot{\tilde{\ell}}^{(k)} = \dot{\tilde{\ell}}^{(k-1)}$, $m_\ell^{(k-1)} = 1$ and Case II.(3) holds at $k-1$.

If the ℓ-string in $(\nu^\bullet, J^\bullet)^{(k)}$ selected by $\tilde{\delta}$ is the same as the s-string selected by δ, then $\tilde{\ell}^{(k)} = \dot{s}^{(k)} =: \ell$. In this case $\dot{\ell}^{(k)} = \dot{\tilde{\ell}}^{(k)}$ and one of the following holds:

(1') *If $\tilde{\dot{\ell}}^{(k)} < \ell$, then $\tilde{\dot{\ell}}^{(k)} = \dot{\tilde{s}}^{(k)} = \ell - 1$, $m_{\ell-1}^{(k+1)} = 0$ or $m_{\ell-1}^{(k+1)}(\dot{\nu}^\bullet) = 0$, and $m_{\ell-1}^{(k-1)} = 0$ or $m_{\ell-1}^{(k-1)}(\tilde{\nu}^\bullet) = 0$. Furthermore $\tilde{s}^{(k)} = \tilde{\dot{s}}^{(k)}$.*

(2') *If $\tilde{\dot{\ell}}^{(k)} = \ell$, then $\tilde{\dot{\ell}}^{(k)} = \dot{\tilde{s}}^{(k)} = \ell$ and $\tilde{s}^{(k)} = \tilde{\dot{s}}^{(k)}$.*

(3') *If $\tilde{\dot{\ell}}^{(k)} > \ell$, then*
   - (i) $\tilde{\dot{\ell}}^{(k)} = \dot{\tilde{s}}^{(k)}$ *and* $\tilde{s}^{(k)} = \tilde{\dot{s}}^{(k)}$, *or*
   - (ii) $\tilde{\dot{\ell}}^{(k)} = \tilde{s}^{(k)}$ *and* $\dot{\tilde{s}}^{(k)} = \tilde{\dot{s}}^{(k)} \le \tilde{\ell}^{(k-1)}$.

   Furthermore, either $\tilde{\dot{\ell}}^{(k)} \le \tilde{\ell}^{(k+1)}$ or $\tilde{\ell}^{(k)} = \tilde{\ell}^{(k+1)}$, $\tilde{\dot{\ell}}^{(k)} = \tilde{\dot{\ell}}^{(k+1)}$, $m_\ell^{(k+1)} = 1$ and Case II.(3')(i) holds at $k+1$. Similarly, either $\dot{\tilde{s}}^{(k)} \le \dot{s}^{(k-1)}$ or $\tilde{\ell}^{(k)} = \tilde{\ell}^{(k-1)}$, $\tilde{\dot{\ell}}^{(k)} = \tilde{\dot{\ell}}^{(k-1)}$, $m_\ell^{(k-1)} = 1$ and Case II.(3') holds at $k-1$.

Lemma C.2. $\tilde{\dot{J}}^\bullet = \dot{\tilde{J}}^\bullet$.

APPENDIX D. COMBINATORIAL R-MATRIX ON HIGHEST WEIGHT ELEMENTS

Here we give the combinatorial R-matrix $R : B^{k',1} \otimes B^{k,1} \to B^{k,1} \otimes B^{k',1}$ explicitly for highest weight elements. The following notation is employed. A column with entries $\cdots a_4 \cdots a_3 a_2 \cdots a_1$, where $a_2 \cdots a_1$ denotes a sequence of consecutive integers $a_2\, a_2 -$

$1\, a_2-2\, \ldots a_1+1\, a_1$, is compactly written as $[a_1..a_2 \mid a_3..a_4 \mid \cdots]$. We use the convention that a segment $[.. \mid a..b \mid ..]$ is empty if $b < a$.

Since R is an involution, it suffices to state R for the case $k' \leq k$. For $k' = k$ we have $R = \mathrm{id}$. Hence for $R : B^{n,1} \otimes B^{n,1} \to B^{n,1} \otimes B^{n,1}$ the R-matrix is the identity. The highest weight elements are given by $[1..n-2\ell \mid \overline{n}..\overline{n-2\ell+1}] \otimes [1..n]$ for $0 \leq \ell \leq n/2$ with local energy $H = \ell$. The same results hold for $R : B^{n-1,1} \otimes B^{n-1,1} \to B^{n-1,1} \otimes B^{n-1,1}$ with n and $\overline{n}$ interchanged.

For $R : B^{n-1,1} \otimes B^{n,1} \to B^{n,1} \otimes B^{n-1,1}$ the highest weight elements get mapped as follows:

$$\begin{aligned} &[1..n-2\ell-1 \mid \overline{n}..\overline{n-2\ell}] \otimes [1..n] \\ \stackrel{R}{\longleftrightarrow} &[1..n-2\ell-1 \mid n \mid \overline{n-1}..\overline{n-2\ell}] \otimes [1..n-1 \mid \overline{n}] \end{aligned}$$

where $0 \leq \ell < n/2$. The local energy is $H = \ell$.

Next consider the case $R : B^{k',1} \otimes B^{n,1} \to B^{n,1} \otimes B^{k',1}$ where $k' \leq n-2$. The highest weight elements are $[1..\ell \mid \overline{n}n...\overline{n}n \mid \overline{\ell}..\overline{r}] \otimes [1..n]$ and $[1..\ell \mid \overline{n}n...\overline{n} \mid \overline{\ell}..\overline{r}] \otimes [1..n]$ where $r \leq \ell+1$ and the pieces $\overline{n}n..\overline{n}n$ and $\overline{n}n..\overline{n}$ could have length zero. Then

$$\begin{aligned} [1..\ell \mid \overline{n}n...\overline{n}n \mid \overline{\ell}..\overline{r}] \otimes [1..n] \stackrel{R}{\longleftrightarrow} &[1..r-1 \mid \ell+d+1..n \mid \overline{\ell+d}..\overline{r}] \\ &\otimes \left[1..\frac{k'+\ell+d}{2} \mid \overline{\frac{k'+\ell+d}{2}}..\overline{\ell+d+1}\right] \end{aligned}$$

with $H = \frac{k'-\ell+d}{2}$ and

$$\begin{aligned} [1..\ell \mid \overline{n}n...\overline{n} \mid \overline{\ell}..\overline{r}] \otimes [1..n] \stackrel{R}{\longleftrightarrow} &[1..r-1 \mid \ell+d+2..n-1 \mid \overline{n} \mid \overline{\ell+d+1}..\overline{r}] \\ &\otimes \left[1..\frac{k'+\ell+d+1}{2} \mid \overline{\frac{k'+\ell+d+1}{2}}..\overline{\ell+d+2}\right] \end{aligned}$$

with $H = \frac{k'-\ell+d+1}{2}$.

Similarly, the case $R : B^{k',1} \otimes B^{n-1,1} \to B^{n-1,1} \otimes B^{k',1}$ with $k' \leq n-2$ is obtained from the above by interchanging n and $\overline{n}$ everywhere.

The remaining cases are all of the form $R : B^{k',1} \otimes B^{k,1} \to B^{k,1} \otimes B^{k',1}$ with $1 \leq k' \leq k \leq n-2$. The highest weight elements $u' \otimes u \in B^{k',1} \otimes B^{k,1}$ are summarized in the following table:

Case	$u' \otimes u$	parameter ranges
1	$\left[1..\ell \mid k+1..p \mid \overline{p}..\overline{q} \mid \overline{\ell}..\overline{r}\right] \otimes [1..k]$	$\ell \geq 0; p \geq k; q \leq p+1; r \leq \ell+1$
2a	$[1..\ell \mid \overline{\ell}..\overline{r}] \otimes [1..a \mid \overline{a}..\overline{b}]$	$a \geq b; \ell \geq 1; r \leq \ell+1$
2b	$[1..\ell \mid \overline{\ell}..\overline{r} \mid \overline{b-1}..\overline{q}] \otimes [1..a \mid \overline{a}..\overline{b}]$	$a \geq b; \ell \geq r > b > q \geq 1$
3	$[1..\ell \mid b..p \mid \overline{\ell}..\overline{r}] \otimes [1..a \mid \overline{a}..\overline{b}]$	$a \geq b; \ell \geq 0; p \geq b > \ell+1 \geq r$
4	$[1..\ell \mid b..p \mid \overline{p}..\overline{q} \mid \overline{\ell}..\overline{r}] \otimes [1..a \mid \overline{a}..\overline{b}]$	$a \geq b; \ell \geq 0; p \geq b, q; r \leq \ell+1$
5a	$[1..\ell \mid \overline{b-1}..\overline{q}] \otimes [1..a \mid \overline{a}..\overline{b}]$	$a \geq b > q; \ell \geq 0$
5b	$[1..\ell \mid \overline{b-1}..\overline{q} \mid \overline{\ell}..\overline{r}] \otimes [1..a \mid \overline{a}..\overline{b}]$	$a \geq b > q > \ell+1; \ell \geq r \geq 1$

In cases 1 and 4 $..p \mid \overline{p}..$ may also be replaced by $..n\overline{n} \cdots n\overline{n}..$ or $..\overline{n}n \cdots \overline{n}n...$

The answers for $R(u' \otimes u)$ in the various cases are given in the tables below. For brevity, we use the definition $d = \ell - r + 1$. By weight considerations $R([1..k'] \otimes [1..k]) =$

$[1..k] \otimes [1..k']$. All answers are proven by induction. As an example, we demonstrate how to derive Case 2a with $r \le \ell$ and $k' < b \le a-d$ by induction on d. We assume that Case 2a holds for $d=0$ (which can also be proven by induction).

First consider $d=1$ so that $u = [1..\ell \mid \overline{\ell}] \otimes [1..a \mid \overline{a}..\overline{b}]$. Then $k' < b$ implies that $\ell < b-1$ so that

$$\begin{aligned} u' &= f_1 f_2 \cdots f_{\ell-1}(u) = [1..\ell \mid \overline{1}] \otimes [1..a \mid \overline{a}..\overline{b}] \\ u'' &= f_2 f_3 \cdots f_\ell(u') = [1 \mid 3..\ell+1 \mid \overline{1}] \otimes [1..a \mid \overline{a}..\overline{b}] \\ u''' &= f_0(u'') = [1..\ell+1] \otimes [1..a \mid \overline{a}..\overline{b}]. \end{aligned}$$

By induction hypothesis, $v''' = R(u''') = [1..a \mid \overline{a}..\overline{b}] \otimes [1..\ell+1]$ and

$$\begin{aligned} v'' &= e_0(v''') = [1..a-1 \mid \overline{a-1}..\overline{b} \mid \overline{21}] \otimes [1..\ell+1] \\ v' &= e_\ell e_{\ell-1} \cdots e_2(v'') = [1..a-1 \mid \overline{a-1}..\overline{b} \mid \overline{\ell+1} \mid \overline{1}] \otimes [1..\ell+1] \\ v &= e_{\ell-1} e_{\ell-2} \cdots e_1(v') = [1..a-1 \mid \overline{a-1}..\overline{b} \mid \overline{\ell+1}\, \overline{\ell}] \otimes [1..\ell+1] \end{aligned}$$

which by (2.7) and (2.1) indeed yields $v = R(u) = [1..a-d \mid \overline{a-d}..\overline{b} \mid \overline{k'}..\overline{r}] \otimes [1..k']$. Note also that $H(u''') = 0$ by induction hypothesis. Since f_0 acts on the left tensor factor in both u'' and v'', we have $H(u) = 1$.

Next consider $d > 1$ so that $u = [1..\ell \mid \overline{\ell}..\overline{r}] \otimes [1..a \mid \overline{a}..\overline{b}]$ with $r < \ell$. Then $r \le \ell - 1 \le k' - 2 < b - 2$ so that

$$\begin{aligned} u' &= f_2 f_3 \cdots f_r f_1 f_2 \cdots f_{r-1}(u) = [1..\ell \mid \overline{\ell}..\overline{r+2} \mid \overline{2}\,\overline{1}] \otimes [1..a \mid \overline{a}..\overline{b}] \\ u'' &= f_0(u') = [1..\ell+1 \mid \overline{\ell+1}..\overline{r+2}] \otimes [1..a \mid \overline{a}..\overline{b}]. \end{aligned}$$

By induction hypothesis, $v'' = R(u'') = [1..a-d+1 \mid \overline{a-d+1}..\overline{b} \mid \overline{k'}..\overline{r+2}] \otimes [1..k']$ and

$$\begin{aligned} v' &= e_0(v'') = [1..a-d \mid \overline{a-d}..\overline{b} \mid \overline{k'}..\overline{r+2} \mid \overline{2}\,\overline{1}] \otimes [1..k'] \\ v &= e_{r-1} e_{r-2} \cdots e_1 e_r e_{r-1} \cdots e_2(v') = [1..a-d \mid \overline{a-d}..\overline{b} \mid \overline{k'}..\overline{r}] \otimes [1..k'] \end{aligned}$$

which yields $v = R(u) = [1..a-d \mid \overline{a-d}..\overline{b} \mid \overline{k'}..\overline{r}] \otimes [1..k']$ by (2.7) and (2.1). By induction hypothesis $H(u'') = k' - \ell - 1$. Since f_0 acts on the left tensor factor in both u' and v', we have $H(u) = k' - \ell$ as claimed.

Case	$R(u' \otimes u)$	conditions	$H(u' \otimes u)$
1	$\big[1..\ell+1 \mid k'+1..k-1 \mid \overline{k-1}..\overline{q} \mid \overline{\ell+1}..\overline{r}\big] \otimes [1..k']$	$p=k$ and $k-q$ even	$k'-\ell$
	$\big[1..\ell \mid k'+1..p \mid \overline{p}..\overline{q} \mid \overline{\ell}..\overline{r}\big] \otimes [1..k']$	otherwise	$k'-\ell$
$2a$	$[1..a-d \mid \overline{a-d}..\overline{b} \mid \overline{k'}..\overline{r}] \otimes [1..k']$	$k' < b \leq a-d$	$k'-\ell$
	$[1..a-b+\ell+1 \mid k'+1..b-1 \mid \overline{a-b+\ell+1}..\overline{r}] \otimes [1..k']$	$k' < b; a-d < b$	$k'-\ell$
	$[1..\frac{k-k'+2\ell}{2} \mid \overline{\frac{k-k'+2\ell}{2}}..\overline{r}] \otimes [1..\frac{k'+b-1}{2} \mid \overline{\frac{k'+b-1}{2}}..\overline{b}]$	$b \leq k'$, $k+k'$ even	$k' - \frac{b+r-2}{2}$
	$[1..b-1 \mid b+1..\frac{k-k'+2\ell+1}{2} \mid \overline{\frac{k-k'+2\ell+1}{2}}..\overline{r}] \otimes [1..\frac{k'+b}{2} \mid \overline{\frac{k'+b}{2}}..\overline{b+1}]$	$b \leq k'$, $k+k'$ odd	$k' - \frac{b+r-3}{2}$
$2b$	$[1..\frac{k-k'+2\ell}{2} \mid \overline{\frac{k-k'+2\ell}{2}}..\overline{r} \mid \overline{b-1}..\overline{q}] \otimes [1..\frac{k'+b-1}{2} \mid \overline{\frac{k'+b-1}{2}}..\overline{b}]$	$k+k'$ even	$k' - \frac{r+q-2}{2}$
	$[1..b-1 \mid b+1..\frac{k-k'+2\ell+1}{2} \mid \overline{\frac{k-k'+2\ell+1}{2}}..\overline{r} \mid \overline{b-1}..\overline{q}] \otimes [1..\frac{k'+b}{2} \mid \overline{\frac{k'+b}{2}}..\overline{b}]$	$k+k'$ odd	$k' - \frac{r+q-3}{2}$
3	$[1..\ell \mid k'-k+b..p \mid \overline{\ell}..\overline{r}] \otimes [1..k'-k+a \mid \overline{k'-k+a}..\overline{k'-k+b}]$	$k < p$	$k'-k+a-\ell$
	$[1..\frac{k+p-2d}{2} \mid \overline{\frac{k+p-2d}{2}}..\overline{p+1} \mid \overline{\ell+d}..\overline{r}] \otimes [1..\frac{k'+\ell+d}{2} \mid \overline{\frac{k'+\ell+d}{2}}..\overline{\ell+d+1}]$	$k' \leq p \leq k$, $k+p$ even	$\frac{k'-r+1}{2}$
	$[1..\ell+d \mid \ell+d+2..\frac{k+p-2d+1}{2} \mid \overline{\frac{k+p-2d+1}{2}}..\overline{p+1} \mid \overline{\ell+d}..\overline{r}]$ $\otimes[1..\frac{k'+\ell+d+1}{2} \mid \overline{\frac{k'+\ell+d+1}{2}}..\overline{\ell+d+2}]$	$k' \leq p \leq k$, $k+p$ odd	$\frac{k'-r+2}{2}$
	$[1..\frac{k+k'-2d}{2} \mid \overline{\frac{k+k'-2d}{2}}..\overline{p+1} \mid \overline{b-1}..\overline{r}] \otimes [1..\frac{k'+b-1}{2} \mid \overline{\frac{k'+b-1}{2}}..\overline{b}]$	$p < k'$, $k+k'$ even	$\frac{k'-b+2d+1}{2}$
	$[1..b-1 \mid b+1..\frac{k+k'-2d+1}{2} \mid \overline{\frac{k+k'-2d+1}{2}}..\overline{p+1} \mid \overline{b-1}..\overline{r}] \otimes [1..\frac{k'+b}{2} \mid \overline{\frac{k'+b}{2}}..\overline{b+1}]$	$p < k'$, $k+k'$ odd	$\frac{k'-b+2d+2}{2}$

Case	$R(u' \otimes u)$	conditions	$H(u' \otimes u)$
4	$[1..\ell \mid k'-k+b..p \mid \overline{p}..\overline{q} \mid \overline{\ell}..\overline{r}] \otimes [1..k'-k+a \mid \overline{k'-k+a}..\overline{k'-k+b}]$	$k<p$	$k'-k+a-\ell$
	$[1..\frac{2\ell+k-p+2}{2} \mid k'+b-p..p-1 \mid \overline{p-1}..\overline{q} \mid \overline{\frac{2\ell+k-p+2}{2}}..\overline{r}]$ $\otimes[1..\frac{2k'+b-p-1}{2} \mid \overline{\frac{2k'+b-p-1}{2}}..\overline{k'+b-p}]$	$k-2d<p\le k$, $k'<p$ $k+p$ even, $p-q$ even	$k'-\ell+\frac{b-p-1}{2}$
	$[1..\frac{2\ell+k-p+1}{2} \mid k'+b-p+1..p \mid \overline{p}..\overline{q} \mid \overline{\frac{2\ell+k-p+1}{2}}..\overline{r}]$ $\otimes[1..\frac{2k'+b-p}{2} \mid \overline{\frac{2k'+b-p}{2}}..\overline{k'+b-p+1}]$	$k-2d<p\le k$, $k'<p$ $k+p$ odd, $p-q$ even	$k'-\ell+\frac{b-p}{2}$
	$[1..\frac{2\ell+k-p}{2} \mid k'+b-p..p \mid \overline{p}..\overline{q} \mid \overline{\frac{2\ell+k-p}{2}}..\overline{r}]$ $\otimes[1..\frac{2k'+b-p-1}{2} \mid \overline{\frac{2k'+b-p-1}{2}}..\overline{k'+b-p}]$	$k-2d<p\le k$, $k'<p$ $k+p$ even, $p-q$ odd	$k'-\ell+\frac{b-p-1}{2}$
	$[1..\frac{2\ell+k-p-1}{2} \mid k'+b-p+1..p+1 \mid \overline{p+1}..\overline{q} \mid \overline{\frac{2\ell+k-p-1}{2}}..\overline{r}]$ $\otimes[1..\frac{2k'+b-p}{2} \mid \overline{\frac{2k'+b-p}{2}}..\overline{k'+b-p+1}]$	$k-2d<p\le k$, $k'<p$ $k+p$ odd, $p-q$ odd	$k'-\ell+\frac{b-p}{2}$
	$[1..\ell+1+d \mid k'+b-p..\frac{k+p-2d-2}{2} \mid \overline{\frac{k+p-2d-2}{2}}..\overline{q} \mid \overline{\ell+1+d}..\overline{r}]$ $\otimes[1..\frac{2k'+b-p-1}{2} \mid \overline{\frac{2k'+b-p-1}{2}}..\overline{k'+b-p}]$	$k'<p\le k-2d$ $k+p$ even, $p-q$ even	$k'-\ell+\frac{b-p-1}{2}$
	$[1..\ell+1+d \mid k'+b-p+1..\frac{k+p-2d-1}{2} \mid \overline{\frac{k+p-2d-1}{2}}..\overline{q} \mid \overline{\ell+1+d}..\overline{r}]$ $\otimes[1..\frac{2k'+b-p}{2} \mid \overline{\frac{2k'+b-p}{2}}..\overline{k'+b-p+1}]$	$k'<p\le k-2d$ $k+p$ odd, $p-q$ even	$k'-\ell+\frac{b-p}{2}$
	$[1..\ell+d \mid k'+b-p..\frac{k+p-2d}{2} \mid \overline{\frac{k+p-2d}{2}}..\overline{q} \mid \overline{\ell+d}..\overline{r}]$ $\otimes[1..\frac{2k'+b-p-1}{2} \mid \overline{\frac{2k'+b-p-1}{2}}..\overline{k'+b-p}]$	$k'<p\le k-2d$ $k+p$ even, $p-q$ odd	$k'-\ell+\frac{b-p-1}{2}$
	$[1..\ell+d \mid k'+b-p+1..\frac{k+p-2d+1}{2} \mid \overline{\frac{k+p-2d+1}{2}}..\overline{q} \mid \overline{\ell+d}..\overline{r}]$ $\otimes[1..\frac{2k'+b-p}{2} \mid \overline{\frac{2k'+b-p}{2}}..\overline{k'+b-p+1}]$	$k'<p\le k-2d$ $k+p$ odd, $p-q$ odd	$k'-\ell+\frac{b-p}{2}$

Case	$R(u' \otimes u)$	conditions	$H(u' \otimes u)$
4	$[1..\frac{2\ell+k-k'}{2} \mid b..p \mid \overline{p}..\overline{q} \mid \overline{\frac{2\ell+k-k'}{2}}..\overline{r}] \otimes [1..\frac{k'+b-1}{2} \mid \overline{\frac{k'+b-1}{2}}..\overline{b}]$	$\frac{k+k'-2d}{2} < p \leq k'$, $k+k'$ even	$\begin{cases} \frac{2k'-r-q+4}{2} & p-q \text{ even} \\ \frac{2k'-r-q+2}{2} & p-q \text{ odd} \end{cases}$
cont.	$[1..\frac{2\ell+k-k'+1}{2} \mid b-1..p-1 \mid \overline{p-1}..\overline{q} \mid \overline{\frac{2\ell+k-k'+1}{2}}..\overline{r}]$ $\otimes [1..\frac{k'+b-2}{2} \mid \overline{\frac{k'+b-2}{2}}..\overline{b-1}]$	$\frac{k+k'-2d+1}{2} < p \leq k'$ $k+k'$ odd, $p-q$ even	$\frac{2k'-r-q+3}{2}$
	$[1..\frac{2\ell+k-k'-1}{2} \mid b+1..p+1 \mid \overline{p+1}..\overline{q} \mid \overline{\frac{2\ell+k-k'-1}{2}}..\overline{r}]$ $\otimes [1..\frac{k'+b}{2} \mid \overline{\frac{k'+b}{2}}..\overline{b+1}]$	$\frac{k+k'-2d-1}{2} < p \leq k'$ $k+k'$ odd, $p-q$ odd	$\frac{2k'-r-q+3}{2}$
	$[1..\ell+p-k'+d \mid b..\frac{k+k'-2d}{2} \mid \overline{\frac{k+k'-2d}{2}}..\overline{q} \mid \overline{\ell+p-k'+d}..\overline{r}]$ $\otimes [1..\frac{k'+b-1}{2} \mid \overline{\frac{k'+b-1}{2}}..\overline{b}]$	$k' < p+d \leq \frac{k+k'}{2}$, $p \leq k'$ $k+k'$ even	$\begin{cases} \frac{2k'-r-q+4}{2} & p-q \text{ even} \\ \frac{2k'-r-q+2}{2} & p-q \text{ odd} \end{cases}$
	$[1..\ell+p-k'+d \mid b-1..\frac{k+k'-2d-1}{2} \mid \overline{\frac{k+k'-2d-1}{2}}..\overline{q} \mid \overline{\ell+p-k'+d}..\overline{r}]$ $\otimes [1..\frac{k'+b-2}{2} \mid \overline{\frac{k'+b-2}{2}}..\overline{b-1}]$	$k' < p+d \leq \frac{k+k'+1}{2}$, $p \leq k'$ $k+k'$ odd, $p-q$ even	$\frac{2k'-r-q+3}{2}$
	$[1..\ell+p-k'+d \mid b+1..\frac{k+k'-2d+1}{2} \mid \overline{\frac{k+k'-2d+1}{2}}..\overline{q} \mid \overline{\ell+p-k'+d}..\overline{r}]$ $\otimes [1..\frac{k'+b}{2} \mid \overline{\frac{k'+b}{2}}..\overline{b+1}]$	$k' < p+d \leq \frac{k+k'-1}{2}$, $p \leq k'$ $k+k'$ odd, $p-q$ odd	$\frac{2k'-r-q+3}{2}$
	$[1..\ell \mid b..\frac{k+2p-k'}{2} \mid \overline{\frac{k+2p-k'}{2}}..\overline{q} \mid \overline{\ell}..\overline{r}] \otimes [1..\frac{k'+b-1}{2} \mid \overline{\frac{k'+b-1}{2}}..\overline{b}]$	$p \leq k'-d$, $k+k'$ even	$\begin{cases} \frac{2k'-r-q+4}{2} & b-\ell \text{ even} \\ \frac{2k'-r-q+2}{2} & b-\ell \text{ odd} \end{cases}$
	$[1..\ell \mid b-1..\frac{k+2p-k'-1}{2} \mid \overline{\frac{k+2p-k'-1}{2}}..\overline{q} \mid \overline{\ell}..\overline{r}]$ $\otimes [1..\frac{k'+b-2}{2} \mid \overline{\frac{k'+b-2}{2}}..\overline{b-1}]$	$p \leq k'-d$ $k+k'$ odd, $b-\ell$ even	$\frac{2k'-r-q+3}{2}$
	$[1..\ell \mid b+1..\frac{k+2p-k'+1}{2} \mid \overline{\frac{k+2p-k'+1}{2}}..\overline{q} \mid \overline{\ell}..\overline{r}]$ $\otimes [1..\frac{k'+b}{2} \mid \overline{\frac{k'+b}{2}}..\overline{b+1}]$	$p \leq k'-d$ $k+k'$ odd, $b-\ell$ odd	$\frac{2k'-r-q+3}{2}$

Case	$R(u' \otimes u)$	conditions	$H(u' \otimes u)$
$5a$	$[1..\ell \mid k'+1..a \mid \overline{a}..\overline{q}] \otimes [1..k']$	$q > \ell+1$, $k'-\ell$ even	$k'-\ell$
	$[1..\ell+1 \mid k'+1..a-1 \mid \overline{a-1}..\overline{q} \mid \overline{\ell+1}] \otimes [1..k']$	$q > \ell+1$, $k'-\ell$ odd	$k'-\ell$
	$[1..\ell \mid b..\frac{k+2b-k'-2}{2} \mid \overline{\frac{k+2b-k'-2}{2}}..\overline{q}] \otimes [1..\frac{k'+b-1}{2} \mid \overline{\frac{k'+b-1}{2}}..\overline{b}]$	$q \le \ell+1 < b$, $k+k'$ even	$\begin{cases} \frac{2k'-\ell-q+3}{2} & b-\ell \text{ even} \\ \frac{2k'-\ell-q+1}{2} & b-\ell \text{ odd} \end{cases}$
	$[1..\ell \mid b-1..\frac{k+2b-k'-3}{2} \mid \overline{\frac{k+2b-k'-3}{2}}..\overline{q}] \otimes [1..\frac{k'+b-2}{2} \mid \overline{\frac{k'+b-2}{2}}..\overline{b-1}]$	$q \le \ell+1 < b$, $k+k'$ odd $b-\ell$ even	$\frac{2k'-\ell-q+2}{2}$
	$[1..\ell \mid b+1..\frac{k+2b-k'-1}{2} \mid \overline{\frac{k+2b-k'-1}{2}}..\overline{q}] \otimes [1..\frac{k'+b}{2} \mid \overline{\frac{k'+b}{2}}..\overline{b+1}]$	$q \le \ell+1 < b$, $k+k'$ odd $b-\ell$ odd	$\frac{2k'-\ell-q+2}{2}$
	$[1..\frac{k+2\ell-k'}{2} \mid \overline{\frac{k+2\ell-k'}{2}}..\overline{\ell+1} \mid \overline{b-1}..\overline{q}] \otimes [1..\frac{k'+b-1}{2} \mid \overline{\frac{k'+b-1}{2}}..\overline{b}]$	$q, b \le \ell+1$, $k+k'$ even	$\frac{k'+b-2q+1}{2}$
	$[1..b-1 \mid b+1..\frac{k+2\ell-k'+1}{2} \mid \overline{\frac{k+2\ell-k'+1}{2}}..\overline{\ell+1} \mid \overline{b-1}..\overline{q}]$ $\otimes[1..\frac{k'+b}{2} \mid \overline{\frac{k'+b}{2}}..\overline{b+1}]$	$q, b \le \ell+1$, $k+k'$ odd	$\frac{k'+b-2q+2}{2}$

Case	$R(u' \otimes u)$	conditions	$H(u' \otimes u)$
$5b$	$[1..\ell+d+1 \mid k'+1..a-1-d \mid \overline{a-1-d}..\overline{q} \mid \overline{\ell+d+1}..\overline{r}]$ $\otimes[1..k']$	$q \geq \ell+d+2$, $k'-r$ even $a-b \geq d-1$	$k'-\ell$
	$[1..\ell+d \mid k'+1..a-d \mid \overline{a-d}..\overline{q} \mid \overline{\ell+d}..\overline{r}]$ $\otimes[1..k']$	$q \geq \ell+d+2$, $k'-r$ odd $a-b \geq d-1$	$k'-\ell$
	$[1..\ell+a-b+2 \mid k'+1..b-2 \mid \overline{b-2}..\overline{q} \mid \overline{\ell+a-b+2}..\overline{r}]$ $\otimes[1..k']$	$q \geq \ell+d+2$, $k'-r$ even $a-b < d-1$	$k'-\ell$
	$[1..\ell+a-b+1 \mid k'+1..b-1 \mid \overline{b-1}..\overline{q} \mid \overline{\ell+a-b+1}..\overline{r}]$ $\otimes[1..k']$	$q \geq \ell+d+2$, $k'-r$ odd $a-b < d-1$	$k'-\ell$
	$[1..q-1 \mid b..a-\frac{q-r}{2} \mid \overline{a-\frac{q-r}{2}}..\overline{r}] \otimes [1..\frac{k'+b-1}{2} \mid \overline{\frac{k'+b-1}{2}}..\overline{b}]$	$\ell+d+1 \geq q > \ell+1$ $r-q$ even, $a-b \geq \frac{q-r-2}{2}$	$\begin{cases} \frac{2k'-r-q+4}{2} & k'-r \text{ even} \\ \frac{2k'-r-q+2}{2} & k'-r \text{ odd} \end{cases}$
	$[1..a-b+\frac{q+r}{2} \mid \overline{b-1}..\overline{q} \mid \overline{a-b+\frac{q+r}{2}}..\overline{r}] \otimes [1..\frac{k'+b-1}{2} \mid \overline{\frac{k'+b-1}{2}}..\overline{b}]$	$\ell+d+1 \geq q > \ell+1$ $r-q$ even, $a-b < \frac{q-r-2}{2}$	$\begin{cases} \frac{2k'-r-q+4}{2} & k'-r \text{ even} \\ \frac{2k'-r-q+2}{2} & k'-r \text{ odd} \end{cases}$
	$[1..q-1 \mid b-1..a-\frac{q-r+1}{2} \mid \overline{a-\frac{q-r+1}{2}}..\overline{r}]$ $\otimes[1..\frac{k'+b-2}{2} \mid \overline{\frac{k'+b-2}{2}}..\overline{b-1}]$	$\ell+d+1 \geq q > \ell+1$, $r-q$ odd $k'-r$ even, $a-b \geq \frac{q-r-3}{2}$	$\frac{2k'-r-q+3}{2}$
	$[1..a-b+\frac{q+r+1}{2} \mid \overline{b-2}..\overline{q} \mid \overline{a-b+\frac{q+r+1}{2}}..\overline{r}]$ $\otimes[1..\frac{k'+b-2}{2} \mid \overline{\frac{k'+b-2}{2}}..\overline{b-1}]$	$\ell+d+1 \geq q > \ell+1$, $r-q$ odd $k'-r$ even, $a-b < \frac{q-r-3}{2}$	$\frac{2k'-r-q+3}{2}$
	$[1..q-1 \mid b+1..a-\frac{q-r-1}{2} \mid \overline{a-\frac{q-r-1}{2}}..\overline{r}]$ $\otimes[1..\frac{k'+b}{2} \mid \overline{\frac{k'+b}{2}}..\overline{b+1}]$	$\ell+d+1 \geq q > \ell+1$, $r-q$ odd $k'-r$ odd, $a-b \geq \frac{q-r-1}{2}$	$\frac{2k'-r-q+3}{2}$
	$[1..a-b+\frac{q+r-1}{2} \mid \overline{b}..\overline{q} \mid \overline{a-b+\frac{q+r-1}{2}}..\overline{r}]$ $\otimes[1..\frac{k'+b}{2} \mid \overline{\frac{k'+b}{2}}..\overline{b+1}]$	$\ell+d+1 \geq q > \ell+1$, $r-q$ odd $k'-r$ odd, $a-b < \frac{q-r-1}{2}$	$\frac{2k'-r-q+3}{2}$

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