Almost–Periodic Attractors for a Class of Nonautonomous Reaction–Diffusion Equations on $\mathbb{R}^N$ I. Global Stabilization Processes.+

by

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Abstract. In this paper, we investigate global stabilization phenomena of certain classical solutions to nonautonomous semilinear parabolic partial differential equations with Neumann boundary conditions on bounded domains in $\mathbb{R}^N$. Given any such classical solution u, we prove in particular that there exists a spatially homogeneous, time almost–periodic classical solution $\hat{u}$ which captures u with respect to an appropriate Sobolev norm as the time variable goes to infinity. The class of equations which we analyze here contains in particular Fisher's type reaction–diffusion equations of population genetics. Our method of investigation mainly rests upon a combination of some geometric arguments with parabolic comparison principles.

1. Introduction and Outline.

Consider the class of real semilinear parabolic Neumann boundary value problems of the form

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + s(t)g(u(x,t)), & (x,t) \in \Omega \times \mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (u_0,u_1) & \\ \frac{\partial u}{\partial \underset{\sim}{n}}(x,t) = 0 & , (x,t) \in \partial\Omega \times \mathbb{R}^+ \end{cases} \tag{1.1}$$

In equations (1.1), Ω denotes an open bounded connected subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $N \in [2,\infty) \cap \mathbb{N}^+$, while Δ stands for Laplace's operator in the x–variables. Furthermore, $s : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is the restriction to $\mathbb{R}^+$ of a Bohr almost–periodic function on $\mathbb{R}$ which we shall also denote by s, while $g : \mathbb{R} \longrightarrow \mathbb{R}$ is sufficiently smooth and possesses at least two zeroes u_0 and u_1 such that $g(u) > 0$ for every $u \in (u_0,u_1)$. Finally, $\mathrm{Ran}(u)$ denotes the range of u and $\underset{\sim}{n}$ stands for the normalized outer normal vector to $\partial\Omega$. Consider also the initial value problem

$$\begin{cases} \hat{u}'(t) = s(t)g(\hat{u}(t)), \; t \in \mathbb{R} \\ \mathrm{Ran}(\hat{u}) \subseteq [u_0,u_1] \\ \hat{u}(0) = \hat{\nu} \in [u_0,u_1] \end{cases} \tag{1.2}$$

The central theme of this article is devoted to the analysis of global stabilization phenomena of certain classical solutions to problem (1.1) toward classical Bohr almost–periodic solutions of (1.2) as $t \longrightarrow \infty$. This problem is motivated by several important questions of population genetics, such as the space–time evolution of gene frequencies in the population of a migrating diploid species when the selection function s

takes almost–periodic seasonal variations into account ([7], [9], [16]). Following the works of [12], [14], [19] and [26], the problem of stabilization for solutions to reaction–diffusion equations within the context of population genetics has been intensively investigated recently, often for equations in one space dimension and for selection functions depending on $x \in \Omega$ but independent of time ([3], [5], [10], [21], [22]). One notable exception is the work by Henry who, following earlier considerations of Fleming [13], gave a thorough analysis of the set of all equilibria for some parabolic equations with Neumann boundary conditions in N space dimensions, using variational and bifurcation–theoretical arguments [15].

The considerations and the results discussed in the following sections differ from those of the above articles in at least two respects. On the one hand, equations (1.1) are in N space dimensions and the selection function depends explicitely and almost–periodically on time, but remains spatially homogeneous over Ω along with g. On the other hand, we are not looking for time independent asymptotic solutions but rather for spatially homogeneous, time almost–periodic solutions such as those satisfying equations (1.2), in such a way that the vibration spectra of those asymptotic solutions be determined by the vibration spectrum of s.

The rest of this article will accordingly be organized as follows. In Section 2, we first prove the existence of a one–parameter family $\{\hat{u}\}_{\hat{\nu} \in [u_0, u_1]}$ of classical almost–periodic solutions to (1.2), under appropriate restrictions on s and g; in addition, we also prove that every Fourier exponent of $\hat{u}$ is a finite linear combination with integer coefficients of the Fourier exponents of s, and that each such $\hat{u}$ remains uniformly bounded away from the two equilibria u_0 and u_1 when $\hat{\nu} \in (u_0, u_1)$. These facts are then used to prove that for every (suitably defined) classical solution to problem (1.1) which exists globally in $t \in \mathbb{R}_0^+$, there exists a $\hat{u} \in \{\hat{u}\}_{\hat{\nu} \in (u_0, u_1)}$ which captures u with respect to an appropriate Sobolev norm as $t \longrightarrow \infty$. However, this is not to say that $\hat{u}$ is a global attractor, for $\hat{u}$ depends on u in general. With $p \in (N,\infty)$, the central idea of the proof amounts to combining some geometric arguments based on the existence of exponential dichotomies for the diffusion semigroup on $L^p(\Omega;\mathbb{R})$ with parabolic comparison principles. As an immediate consequence of the above result, we also prove in Section 2 that the solutions $\{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$ represent all of the classical time almost–periodic solutions to problem (1.1) within the admissible class. While the results of Section 2 hold exclusively for the case where s has an almost–periodic primitive and hence a time average equal to zero, we devote Section 3 to the proof of a theorem concerning the case where s has a strictly negative (resp. strictly positive) time average. In this case, we can prove that u_0 (resp. u_1) becomes a global attractor for the solutions to (1.1) as $t \longrightarrow \infty$, again with respect to an appropriate Sobolev topology. The method of proof of Section 3 is similar to that of Section 2, though the corresponding stabilization phenomena have different physical origins. In Section 4 we apply the results of Sections 2 and 3 to Fisher's type equations of population genetics. Finally, Section 5 is devoted to the discussion of some generalizations and some open problems concerning Neumann boundary value problems of the form

$$\left.\begin{cases} u_t(x,t) = \Delta u(x,t)+s(x,t)g(u(x,t)), & (x,t) \in \Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (u_0,u_1) & \\ \dfrac{\partial u}{\partial \underline{n}}(x,t) = 0\ , & (x,t) \in \partial\Omega\times\mathbb{R}^+ \end{cases}\right\} \tag{1.3}$$

where $t \longrightarrow s(x,t)$ is Bohr almost–periodic for each $x \in \bar{\Omega}$. For a short announcement of our results we refer the reader to [27]. In the second part of this work [28], we develop a local geometric stability theory through the construction of codimension–one stable manifolds and of one–dimensional center (inertial) manifolds around each $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in[u_0,u_1]}$. The combination of that theory with the results of this paper then lead to explicit decay rates concerning the stabilization processes discussed here.

2. Asymptotic Almost–Periodicity of Certain Classical Solutions to Problem (1.1).

We begin this section with the analysis of equation (1.2), and we refer the reader to [6], [8], [11], [18] and [25] for a presentation of all of the basic facts concerning almost–periodic functions. Let $\mathbb{R}_B$ be the Bohr compactification of the real line endowed with its usual structure of compact topological group [23]. We identify the algebra of all complex–valued Bohr almost–periodic functions on $\mathbb{R}$ with $\mathscr{C}(\mathbb{R}_B,\mathbb{C})$, the commutative Banach algebra of all complex continuous functions on $\mathbb{R}_B$ with respect to the usual operations and the uniform norm. Let μ_B be the Haar measure on $\mathbb{R}_B$ normalized in such a way that $\mu_B(\mathbb{R}_B) = 1$. For every $s \in \mathscr{C}(\mathbb{R}_B,\mathbb{C})$, we have the Fourier series expansion

$$s(t) \sim \sum_{k\in\mathbb{N}^+} s_k \exp[i\Lambda_k t] \tag{2.1}$$

In expression (2.1) we have defined

$$s_k = \mu_B(s\chi_k) = \lim_{\ell\longrightarrow\infty} \ell^{-1} \int_0^\ell dt s(t)\chi_k(t) \tag{2.2}$$

for each k, with $t \longrightarrow \chi_k(t) = \exp[-i\Lambda_k t]$ for every $t \in \mathbb{R}$. Following [8] and [18], we define the module of s as the set of all finite linear combinations with integer coefficients of the Fourier exponents $\{\Lambda_k\}_{k\in\mathbb{N}^+}$ of s, and we denote it by Mod(s). Our first result is the following

Proposition 2.1. Let $s \in \mathscr{C}(\mathbb{R}_B,\mathbb{R})$ be such that $t \longrightarrow \int_0^t d\xi\, s(\xi) = 0(1)$ as $|t| \longrightarrow \infty$. Let $g \in \mathscr{C}^{(1)}(\mathbb{R},\mathbb{R})$, assume that there exist $u_{0,1} \in \mathbb{R}$ with $g(u_0) = g(u_1) = 0$ and $g(u) > 0$ for each $u \in (u_0,u_1)$, in such a way that $g'(u_0) > 0$ and $g'(u_1) < 0$. Furthermore, let G be any primitive of $1/g$ over the open interval (u_0,u_1). Assume that

$$\lim_{u \longrightarrow u_0} G(u) = -\infty \tag{2.3}$$

$$\lim_{u \longrightarrow u_1} G(u) = +\infty \tag{2.4}$$

and let G^{-1} be the monotone inverse of G. Then the following statements hold:

(A) For every $\hat{\nu} \in [u_0,u_1]$, problem (1.2) possesses a Bohr almost–periodic solution $\hat{u} \in \mathscr{C}^{(1)}(\mathbb{R},\mathbb{R})$. Specifically, if $\hat{\nu} = u_0$ (resp. $\hat{\nu} = u_1$), we may take $\hat{u} = u_0$ (resp. $\hat{u} = u_1$) identically on $\mathbb{R}$; if $\hat{\nu} \in (u_0,u_1)$, the solution is uniquely determined by

$$\hat{u}(t) = G^{-1}\left\{ \int_0^t d\xi s(\xi) + G(\hat{\nu}) \right\} \tag{2.5}$$

for each $t \in \mathbb{R}$. In the latter case we have the module containment

$$\mathrm{Mod}(\hat{u}) \subseteq \mathrm{Mod}(s) \tag{2.6}$$

(B) Every solution $\hat{u}$ given by (2.5) remains uniformly bounded away from the two equilibria u_0 and u_1; that is

$$\inf_{t \in \mathbb{R}} (u_1 - \hat{u}(t)) > 0 \tag{2.7}$$

and

$$\inf_{t \in \mathbb{R}} (\hat{u}(t) - u_0) > 0 \tag{2.8}$$

(C) For every solution $\hat{u}$ given by (2.5), the function $t \longrightarrow 1/g(\hat{u}(t))$ is Bohr almost–periodic.

<u>Proof</u>. It follows from a classic theorem of Bohr that the boundedness of $t \longrightarrow \int_0^t d\xi s(\xi)$ implies its almost–periodicity [18], so that the first part of statement (A) is immediate by separation of variables. Moreover, for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that every $\delta(\epsilon)$–almost period of $t \longrightarrow \int_0^t d\xi s(\xi)$ is an ϵ–almost–period of $\hat{u}$; this and a classic criterion of Favard imply relation (2.6) ([8], [11]). As for statement (B), assume that relation (2.8) does not hold. Then there exists a sequence $(t_n) \subset \mathbb{R}$ with $\hat{u}(t_n) \longrightarrow u_0$ as $n \longrightarrow \infty$; because of relation (2.3), this implies that $G(\hat{u}(t_n)) \longrightarrow -\infty$ as $n \longrightarrow \infty$. But this is impossible, for it follows from relation (2.5) that

$$G(\hat{u}(t_n)) = \int_0^{t_n} d\xi s(\xi) + G(\hat{\nu}) \tag{2.9}$$

and hence that $G(\hat{u}(t_n)) = 0(1)$ as $n \longrightarrow \infty$, by the boundedness of $t \longrightarrow \int_0^t d\xi s(\xi)$.
The proof of relation (2.7) is of course similar, with (2.4) replacing (2.3). Finally, we prove

statement (C). We already know that $t \longrightarrow g(\hat{u}(t)) \in \mathscr{C}(\mathbb{R}_B,\mathbb{R})$, so that it remains to prove that $\inf_{t\in\mathbb{R}} g(\hat{u}(t)) > 0$; if this is not true, then there exists a sequence $(t_n) \subset \mathbb{R}$ such that $g(\hat{u}(t_n)) \longrightarrow 0$ as $n \longrightarrow \infty$. By the hypotheses concerning g, there exists a neighborhood $\mathscr{N}_{u_0}$ of u_0 such that $g'(u) > 0$ for every $u \in \mathscr{N}_{u_0}$; similarly, there exists a neighborhood $\mathscr{N}_{u_1}$ of u_1 such that $g'(u) < 0$ for every $u \in \mathscr{N}_{u_1}$. From this, the properties of g and the fact that $g(\hat{u}(t_n)) \longrightarrow 0$, we easily infer the existence of subsequence $(t_{\hat{m}})$ such that $(\hat{u}(t_{\hat{m}})) \subseteq \mathscr{N}_{u_0}$ is monotone decreasing, or the existence of a subsequence $(t_{\hat{n}})$ such that $(\hat{u}(t_{\hat{n}})) \subseteq \mathscr{N}_{u_1}$ is monotone increasing. By the smoothness of g and the fact that $g(u) > 0$ for each $u \in (u_0,u_1)$, we conclude that $\hat{u}(t_{\hat{m}}) \longrightarrow u_0$ as $\hat{m} \longrightarrow \infty$, or that $\hat{u}(t_{\hat{n}}) \longrightarrow u_1$ as $\hat{n} \longrightarrow \infty$. In either case this contradicts the conclusion of statement (B). We conclude that $\inf_{t\in\mathbb{R}} g(\hat{u}(t)) > 0$ which, together with the almost–periodicity of $t \longrightarrow g(\hat{u}(t))$, implies that $t \longrightarrow 1/g(\hat{u}(t)) \in \mathscr{C}(\mathbb{R}_B,\mathbb{R})$. ■

We shall denote by $\{\hat{u}\}_{\hat{\nu}\in[u_0,u_1]}$ the one–parameter family of solutions of Proposition 2.1, and by $\{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$ those solutions determined by relation (2.5). Let $[N/2]$ be the integer part of $N/2$; throughout the remaining part of this paper, we shall assume that Ω has a $\mathscr{C}^{3+[N/2]}$–boundary in the sense of [1], in such a way that Ω lies only on one side of $\partial\Omega$, and that it satisfies the interior ball condition for every $x \in \partial\Omega$. We shall also write $\bar{\Omega}$ for the compact closure of Ω, and denote by $\mathscr{C}^{2,1}(\Omega\times\mathbb{R}^+,\mathbb{R})$ the set consisting of all functions $z \in \mathscr{C}(\Omega\times\mathbb{R}^+,\mathbb{R})$ such that $(x,t) \longrightarrow \partial_t^\gamma D^\alpha z(x,t) \in \mathscr{C}(\Omega\times\mathbb{R}^+,\mathbb{R})$ for all $\alpha = (\alpha_1,\dots,\alpha_N) \in \mathbf{N}^N$, $\gamma \in \mathbf{N}$, satisfying $\sum_{j=1}^{N} \alpha_j + 2\gamma \leq 2$. In a similar way we define $\mathscr{C}^{1,0}(\bar{\Omega}\times\mathbb{R}^+,\mathbb{R})$ as the set consisting of all $z \in \mathscr{C}(\bar{\Omega} \times \mathbb{R}^+,\mathbb{R})$ with the property that

$D^\alpha z \in \mathscr{C}(\overline{\Omega}\times\mathbb{R}^+,\mathbb{R})$ for all $\alpha \in \mathbb{N}^N$ such that $\sum_{j=1}^{N} \alpha_j \leq 1$. Now fix $p \in (N,\infty)$; we shall call a classical solution to problem (1.1) any function $u \in \mathscr{C}^{2,1}(\Omega\times\mathbb{R}^+,\mathbb{R}) \cap \mathscr{C}(\overline{\Omega}\times\mathbb{R}_0^+,\mathbb{R}) \cap \mathscr{C}^{1,0}(\overline{\Omega}\times\mathbb{R}^+,\mathbb{R})$ which in addition satisfies the following conditions:

(C_1) $|u(x,t)-u(x,t')| \leq c(x)|t-t'|$ for every $t,t' \in \mathbb{R}_0^+$ and some $c \in L^p(\Omega,\mathbb{R})$.

(C_2) $x \longrightarrow u(x,t) \in \mathscr{C}^{(2)}(\overline{\Omega},\mathbb{R})$ for every $t \in \mathbb{R}_0^+$.

(C_3) $(x,t) \longrightarrow u_t(x,t) \in \mathscr{C}(\overline{\Omega}\times\mathbb{R}^+,\mathbb{R})$ and in fact $t \longrightarrow u_t(x,t) \in \mathscr{C}(\mathbb{R}^+,\mathbb{R})$ uniformly in $x \in \overline{\Omega}$.

(C_4) u satisfies relations (1.1) identically.

The main result of this section is then the following

Theorem 2.1. Assume that s and g satisfy the same hypotheses as in Proposition 2.1. Assume in addition that s is Hölder continuous on $\mathbb{R}^+$ and let u be a classical solution to (1.1) for some $p \in (N,\infty)$. Then there exists a $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$ such that

$$\lim_{t\longrightarrow\infty} \sup_{x\in\overline{\Omega}} |u(x,t) - \hat{u}(t)| = 0 \tag{2.10}$$

Moreover, the given classical solution satisfies the relations

$$\lim_{t\to\infty} \sup_{x\in\bar{\Omega}} |\nabla u(x,t)| = 0 \tag{2.11}$$

and

$$\lim_{t\to\infty} \sup_{\substack{x,y\in\Omega \\ x\neq y}} |x-y|^{-\beta}|u(x,t)-u(y,t)| = 0 \tag{2.12}$$

$$\lim_{t\to\infty} \sup_{\substack{x,y\in\Omega \\ x\neq y}} |x-y|^{-\beta}|\nabla u(x,t)-\nabla u(y,t)| = 0 \tag{2.13}$$

for every $\beta \in (0,1-p^{-1}N]$. Finally, every Fourier exponent of $\hat{u}$ is a finite linear combination with integer coefficients of the Fourier exponents of s .

The proof of this result will require several steps. First, let $H^{2,p}(\mathbb{C}) = H^{2,p}(\Omega,\mathbb{C})$ be the usual Sobolev space consisting of all complex L^p–functions z with L^p–distributional derivatives $D^\alpha z$ for $|\alpha| \in [0,2]$, equipped with the norm

$$z \longrightarrow \|z\|_{2,p} = \left\{ \sum_{|\alpha|\in[0,2]} \|D^\alpha z\|_p^p \right\}^{1/p} \tag{2.14}$$

where $\|\cdot\|_p$ denotes the usual L^p–norm. For $\beta \in (0,1-p^{-1}N]$, let $\mathscr{C}^{1,\beta}(\mathbb{C}) = \mathscr{C}^{1,\beta}(\bar{\Omega},\mathbb{C})$ be the Banach space of all complex Hölder continuous functions on Ω with Hölderian derivatives $D^\alpha z$ of exponent β for $|\alpha| \in [0,1]$ and the norm

$$\|z\|_{1,\beta} = \|z\|_{1,\infty} + \max_{|\alpha| \in [0,1]} \sup_{\substack{x,y \in \bar{\Omega} \\ x \neq y}} |x-y|^{-\beta} |D^{\alpha}z(x) - D^{\alpha}z(y)| =$$

$$= \max_{|\alpha| \in [0,1]} \sup_{x \in \bar{\Omega}} |D^{\alpha}z(x)| + \max_{|\alpha| \in [0,1]} \sup_{\substack{x,y \in \bar{\Omega} \\ x \neq y}} |x-y|^{-\beta} |D^{\alpha}z(x) - D^{\alpha}z(y)| \tag{2.15}$$

It is well known that there exists the continuous embedding

$$H^{2,p}(\mathbb{C}) \longrightarrow \mathscr{C}^{1,\beta}(\mathbb{C}) \tag{2.16}$$

and that $H^{2,p}(\mathbb{C})$ is a commutative Banach algebra with respect to the usual pointwise operations and a norm equivalent to (2.14) ([1]). Now let $H^{2,p}(\mathbb{R}) = H^{2,p}(\Omega,\mathbb{R})$ be the real component of $H^{2,p}(\mathbb{C})$ and let u be a classical solution of Theorem 2.1; for every $t \in \mathbb{R}_0^+$, define $u(t) : \bar{\Omega} \longrightarrow \mathbb{R}$ by $u(t)(x) = u(x,t)$. It is then clear that $u(t) \in H^{2,p}_{\mathscr{N}}(\mathbb{R})$ because of conditions (C_2) and (C_4), where

$$H^{2,p}_{\mathscr{N}}(\mathbb{R}) = \left\{ z \in H^{2,p}(\mathbb{R}) : \frac{\partial z}{\partial \nu}(x) = 0, \, x \in \partial\Omega \right\} \tag{2.17}$$

Because of embedding (2.16) and relations (2.15), the proof of Theorem 2.1 is thus reduced to proving the existence of a $\hat{u} \in \{\hat{u}\}_{\hat{\nu} \in (u_0,u_1)}$ such that $\|u(t)-\hat{u}(t)\|_{2,p} \longrightarrow 0$ as $t \longrightarrow \infty$. Our first step toward this amounts to showing that every classical solution to problem (1.1) remains uniformly bounded away from the two equilibria u_0 and u_1. In that we generalize statement (B) of Proposition 2.1. The precise result is the following

<u>Proposition 2.2</u>. Let s and g satisfy the same hypotheses as in Proposition 2.1. Let u

be any classical solution to problem (1.1). Then u remains uniformly bounded away from the two equilibria u_0 and u_1 ; that is

$$\inf_{(x,t)\in\bar{\Omega}\times\mathbb{R}_0^+} (u(x,t)-u_0) > 0 \tag{2.18}$$

and

$$\inf_{(x,t)\in\bar{\Omega}\times\mathbb{R}_0^+} (u_1-u(x,t)) > 0 \tag{2.19}$$

Moreover,

$$\inf_{(x,t)\in\bar{\Omega}\times\mathbb{R}_0^+} g(u(x,t)) > 0 \tag{2.20}$$

<u>Proof</u>. We first note that it is sufficient to prove relation (2.19), for (2.19) implies (2.18). In order to see this define $\tilde{u}(x,t) = u_0 + u_1 - u(x,t)$; it is then clear that $\tilde{u}$ is a classical solution to the Neumann boundary–value problem

$$\left\{\begin{array}{ll} \tilde{u}_t(x,t) = \Delta\tilde{u}(x,t) + \tilde{s}(t)g_{u_0,u_1}(\tilde{u}(x,t)), & (x,t) \in \Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(\tilde{u}) \subseteq (u_0,u_1) & \\ \frac{\partial\tilde{u}}{\partial\underset{\sim}{n}}(x,t) = 0\ , & (x,t) \in \partial\Omega\times\mathbb{R}^+ \end{array}\right\} \tag{2.21}$$

where $\tilde{s} = -s$ and $g_{u_0,u_1}(\tilde{u}) = g(u_0+u_1-\tilde{u})$. In addition, we observe that $\tilde{s}$ and g_{u_0,u_1} satisfy exactly the same hypotheses as s and g in Proposition 2.1, and that

$$\inf_{(x,t)\in\bar{\Omega}\times\mathbb{R}_0^+} (u(x,t)-u_0) = \inf_{(x,t)\in\bar{\Omega}\times\mathbb{R}_0^+} (u_1-\tilde{u}(x,t)) \tag{2.22}$$

Thus relation (2.19) implies relation (2.18). In order to prove inequality (2.19), we first note that the estimate

$$s = \inf_{x\in\bar{\Omega}} (u_1-u(x,0)) > 0 \tag{2.23}$$

holds. Indeed, inequality (2.23) is an immediate consequence of the range condition in (1.1) and of the continuity of $x \longrightarrow u(x,0)$ on the compact set $\bar{\Omega}$. In order to derive relation (2.19) from relation (2.23), we now invoke the parabolic maximum principle along with the argument given in the proof of statement (B) of Proposition 2.1. To this end, define $v : \bar{\Omega} \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ by

$$v(x,t) = \exp\left[\alpha\left\{\int_{\hat{\mu}}^{u(x,t)} \frac{d\xi}{g(\xi)} - \int_0^t d\xi s(\xi)\right\}\right] \tag{2.24}$$

where $\hat{\mu} \in (u_0,u_1)$ and $\alpha \in \mathbb{R}^+ \cap (\max_{\xi\in[u_0,u_1]} g'(\xi),\infty)$. A direct calculation gives

$$\nabla v(x,t) = \frac{\alpha\, v(x,t)}{g(u(x,t))} \nabla u(x,t) \tag{2.25}$$

and

$$\Delta v(x,t) =$$

$$\frac{\alpha v(x,t)}{g(u(x,t))}\Delta u(x,t) + \frac{\alpha}{g(u(x,t))}\nabla u(x,t)\cdot\nabla v(x,t) - \frac{\alpha v(x,t)g'(u(x,t))}{g^2(u(x,t))}|\nabla u(x,t)|^2$$

$$= v_t(x,t) + \frac{\alpha|\nabla u(x,t)|^2}{g^2(u(x,t))}(\alpha - g'(u(x,t)))v(x,t) \qquad (2.26)$$

upon using (2.24) and (2.25). From relations (2.24), (2.26), the above choice of α and the first hypothesis concerning u , we conclude that
$v \in \mathscr{C}^{2,1}(\Omega\times\mathbb{R}^+,\mathbb{R}) \cap \mathscr{C}(\bar{\Omega}\times\mathbb{R}_0^+,\mathbb{R}) \cap \mathscr{C}^{1,0}(\bar{\Omega}\times\mathbb{R}^+,\mathbb{R})$ and satisfies the parabolic boundary value problem

$$\left\{\begin{array}{ll} v_t(x,t) \le \Delta v(x,t), & (x,t) \in \Omega\times\mathbb{R}^+ \\ \frac{\partial v}{\partial n}(x,t) = 0 , & (x,t) \in \partial\Omega\times\mathbb{R}^+ \end{array}\right\} \qquad (2.27)$$

We then infer from relations (2.24) and (2.23) that

$$v(x,0) = \exp\left[\alpha\int_{\hat{\mu}}^{u(x,0)}\frac{d\xi}{g(\xi)}\right] \le \exp\left[\alpha\int_{\hat{\mu}}^{u_1-\delta}\frac{d\xi}{g(\xi)}\right] \equiv c_1 \qquad (2.28)$$

uniformly in $x \in \bar{\Omega}$, and consequently that $v(x,t) \le c_1$ uniformly in $(x,t) \in \bar{\Omega}\times\mathbb{R}_0^+$ by the parabolic maximum principle applied to problem (2.27). From relation (2.24) it now follows that

$$G(u(x,t)) = \int_0^t d\xi s(\xi) + \alpha^{-1} Ln(v(x,t)) + G(\hat{\mu}) \tag{2.29}$$

where G is as in Proposition 2.1. From the boundedness of $t \longrightarrow \int_0^t d\xi s(\xi)$ and the above uniform bound for $v(x,t)$, we conclude from relation (2.29) that there exists $c_2 \in \mathbb{R}^+$ such that

$$G(u(x,t)) \leq c_2 \tag{2.30}$$

for every $(x,t) \in \bar{\Omega} \times \mathbb{R}_0^+$. Assume now that relation (2.19) does not hold; then there exists a sequence $(x_n,t_n)_{n=1}^{\infty} \subset \bar{\Omega} \times \mathbb{R}_0^+$ such that $\lim_{n \to \infty} u(x_n,t_n) = u_1$. Upon using relation (2.4) we then conclude that $G(u(x_n,t_n)) \longrightarrow \infty$ as $n \longrightarrow \infty$, in contradiction to (2.30). In order to prove relation (2.20), we may now argue exactly as in the proof of statement (C) of Proposition 2.1. ∎

<u>Remark</u>. It is clear that all four conditions (C_1)–(C_4) were not used in the proof of Proposition 2.2.

Our second step toward proving Theorem 2.1 amounts to constructing a $\hat{u} \in \{\hat{u}\}_{\hat{\nu} \in (u_0,u_1)}$ such that $u(t) - \hat{u}(t) \longrightarrow 0$ strongly in $L^p(\mathbb{R})$. To this end, some preparatory remarks are in order. Let $v_0 \in \mathbb{R}^+$; in complete formal analogy with relation (2.24), it is possible to define $\hat{u}$ by

$$v_0 = \exp\left[\alpha\left\{\int_{\hat{\mu}}^{\hat{u}(t)} \frac{d\xi}{g(\xi)} - \int_0^t d\xi s(\xi)\right\}\right] \tag{2.31}$$

where α and $\hat{\mu}$ are as in relation (2.24). It follows immediately from relation (2.31) that

$$G\circ\hat{u}(t) = \int_0^t d\xi s(\xi) + \alpha^{-1}\mathrm{Ln}(v_0) + G(\hat{\mu}) \tag{2.32}$$

and hence that $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$ with $\hat{\nu} = G^{-1}(\alpha^{-1}\mathrm{Ln}(v_0) + G(\hat{\mu}))$. Now for every $t \in \mathbb{R}_0^+$ define $v(t) : \bar{\Omega} \longrightarrow \mathbb{R}$ by $v(t)(x) = v(x,t)$ where $v(x,t)$ is given by relation (2.24). Our strategy to construct the appropriate $\hat{u}$ is then the following: we first invoke the existence of exponential dichotomies for the diffusion semigroup generated by Laplace's operator in (1.1) to construct a $v_0 \in \mathbb{R}^+$ in such a way that $v(t) \longrightarrow v_0$ in measure on Ω as $t \longrightarrow \infty$. We then prove that

$$G\circ u(t) - G\circ\hat{u}(t) = \alpha^{-1}(\mathrm{Ln}(v(t)) - \mathrm{Ln}(v_0)) \tag{2.33}$$

converges in measure on Ω as $t \longrightarrow \infty$ which, together with the uniform estimates of Proposition 2.2 and the properties of G , implies that $\|u(t)-\hat{u}(t)\|_p \longrightarrow 0$. To carry out this program, we denote by $\Delta_{p,\mathscr{N}}$ the $L^p(\mathbb{C})$–realization of Laplace's operator on the domain $\mathrm{Dom}(\Delta_{p,\mathscr{N}}) = H^{2,p}_{\mathscr{N}}(\mathbb{C})$, where $H^{2,p}_{\mathscr{N}}(\mathbb{C})$ is given by (2.17) with $H^{2,p}(\mathbb{C})$ replacing $H^{2,p}(\mathbb{R})$. It follows from the standard methods of [20] that $\Delta_{p,\mathscr{N}}$ is the infinitesimal generator of a compact holomorphic contraction semigroup on $L^p(\mathbb{C})$; in addition, $\Delta_{p,\mathscr{N}}$ has a discrete point spectrum, namely $\sigma(\Delta_{p,\mathscr{N}}) = \{\lambda_k\}_{k\in N^+} \cup \{0\}$ where $\{\lambda_k\}_{k\in N^+} \subset \mathbb{R}^-$, the λ_k's have finite multiplicities and $\lambda_k \longrightarrow -\infty$ as $k \longrightarrow \infty$. This follows from the fact that Ω has a $\mathscr{C}^{3+[N/2]}$–boundary, which implies that the above spectral properties of $\Delta_{p,\mathscr{N}}$ are directly inherited from those of the corresponding L^2–theory. We write $\left\{W_{\Delta_{p,\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ for the restriction of the corresponding semigroup to $L^p(\mathbb{R})$. Since $\Delta_{p,\mathcal{N}}$ satisfies Neumann boundary conditions and since $\exp\left[\sigma(\Delta_{p,\mathcal{N}})t\right] \subseteq \sigma\left[W_{\Delta_{p,\mathcal{N}}}(t)\right]$ for every $t \in \mathbb{R}_0^+$, it is clear that $\left\{W_{\Delta_{p,\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ does not enjoy exponential decay properties on the whole of $L^p(\mathbb{R})$. We next identify a codimension–one subspace of $L^p(\mathbb{R})$ on which $\left\{W_{\Delta_{p,\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ decays exponentially rapidly with a rate determined by the largest negative eigenvalue of $\Delta_{p,\mathcal{N}}$. Let I_p denote the identity operator on $L^p(\mathbb{R})$; on this space define the operators P and Q by

$$P = I_p - Q$$

$$(Qz)(x) = |\Omega|^{-1} \int_\Omega dx z(x) \tag{2.34}$$

where $|\Omega|$ stands for Lebesgue's measure of Ω. It is easily verified that P and Q are projection operators on $L^p(\mathbb{R})$. Our next result states in particular that $\left\{W_{\Delta_{p,\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ decays exponentially on Ran P.

Proposition 2.3. The diffusion semigroup $\left\{W_{\Delta_{p,\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ leaves Ran P globally invariant; moreover, if λ_1 denotes the largest negative eigenvalue of $\Delta_{p,\mathcal{N}}$, there exist positive constants $c_{3,4}$ depending on N, p, λ_1 and the geometry of Ω, such that the estimates

$$\|W_{\Delta_{p,\mathscr{N}}}(t)Pz\|_p \leq c_3 \exp[\lambda_1 t]\,\|z\|_p \tag{2.35}$$

and

$$\|\Delta_{p,\mathscr{N}} W_{\Delta_{p,\mathscr{N}}}(t)Pz\|_p \leq c_4 t^{-1} \exp[\lambda_1 t]\,\|z\|_p \tag{2.36}$$

hold for every $t \in \mathbb{R}^+$ and every $z \in L^p(\mathbb{R})$. Finally, $\left\{W_{\Delta_{p,\mathscr{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ leaves Ran Q pointwise invariant; that is

$$W_{\Delta_{p,\mathscr{N}}}(t)z = z \tag{2.37}$$

for every $t \in \mathbb{R}^+$ and every $z \in$ Ran Q.

Proof. By Gauss' divergence theorem and relations (2.34), we see that $\Delta_{p,\mathscr{N}}Q = Q\Delta_{p,\mathscr{N}} = 0$, and hence that $\Delta_{p,\mathscr{N}}P = P\Delta_{p,\mathscr{N}}$ on $\mathrm{Dom}(\Delta_{p,\mathscr{N}})$, which proves the global invariance of Ran P and Ran Q under $\left\{W_{\Delta_{p,\mathscr{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$. Since Ω satisfies the interior ball condition for every $x \in \partial\Omega$ we now observe that Ran Q $= E(0)$, the one–dimensional eigenspace of $\Delta_{p,\mathscr{N}}$ corresponding to the zero eigenvalue; relation (2.37) then follows from the fact that $\exp\left[\sigma(\Delta_{p,\mathscr{N}})t\right] \subseteq \sigma\left[W_{\Delta_{p,\mathscr{N}}}(t)\right]$ for every $t \in \mathbb{R}_0^+$. It remains to prove estimates (2.35) and (2.36). Let $\Delta_{2,\mathscr{N}}$ be the $L^2(\mathbb{C})$–realization of Laplace's operator on $\mathrm{Dom}(\Delta_{2,\mathscr{N}}) = H^{2,2}_{\mathscr{N}}(\mathbb{C})$ and let $\left\{W_{\Delta_{2,\mathscr{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ be the restriction of the corresponding semigroup on $L^2(\mathbb{R})$. Standard Hilbert space methods then lead to the estimate

$$\|W_{\Delta_{2,\mathcal{N}}}(t)Pz\|_2 \leq \exp[\lambda_1 t]\,\|z\|_2 \tag{2.38}$$

for every $t \in \mathbb{R}^+$ and every $z \in L^2(\mathbb{R})$. In order to prove estimate (2.35) from (2.38), we now use a smoothing argument of [4]; that is, for each $t \in \mathbb{R}^+$, there exists $c_t(N,p,\Omega) \in \mathbb{R}^+$ such that $W_{\Delta_{2,\mathcal{N}}}(t)$ maps $L^2(\mathbb{R})$ into $L^p(\mathbb{R})$ in such a way that

$$\|W_{\Delta_{2,\mathcal{N}}}(t)z\|_p \leq c_t(N,p,\Omega)\|z\|_2 \tag{2.39}$$

for every $z \in L^2(\mathbb{R})$. We also note that there exists the continuous embedding $L^p(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$, and consequently that $\left\{W_{\Delta_{p,\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ is the restriction of $\left\{W_{\Delta_{2,\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ to $L^p(\mathbb{R})$. We may consequently write

$$\|W_{\Delta_{p,\mathcal{N}}}(t)Pz\|_p = \|W_{\Delta_{p,\mathcal{N}}}(1)W_{\Delta_{p,\mathcal{N}}}(t-1)Pz\|_p =$$

$$= \|W_{\Delta_{2,\mathcal{N}}}(1)W_{\Delta_{2,\mathcal{N}}}(t-1)Pz\|_p \leq c_{t=1}(N,p,\Omega)\|W_{\Delta_{2,\mathcal{N}}}(t-1)Pz\|_2 \leq$$

$$\leq c_{t=1}(N,p,\Omega)\exp[\lambda_1(t-1)]\,\|z\|_2 \leq$$

$$\leq c_{t=1}(N,p,\Omega)\exp[-\lambda_1]\,|\Omega|^{\frac{p-2}{2p}}\exp[\lambda_1 t]\,\|z\|_p \tag{2.40}$$

which proves estimate (2.35) for every $t \in (1,\infty)$. As for the case $t \in (0,1]$, pick any

$c \in [\exp[-\lambda_1],\infty)$. Since $\left\{W_{\Delta_{p,\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ is a contraction semigroup on $L^p(\mathbb{R})$, we then obtain the inequalities

$$\|W_{\Delta_{p,\mathcal{N}}}(t)z\|_p \leq \|z\|_p \leq c\exp[\lambda_1]\,\|z\|_p \leq$$

$$\leq c\exp[\lambda_1 t]\,\|z\|_p \tag{2.41}$$

valid for every $t \in (0,1]$ and every $z \in L^p(\mathbb{R})$. From relations (2.34) and (2.41) we then infer that

$$\|W_{\Delta_{p,\mathcal{N}}}(t)Pz\|_p \leq c\exp[\lambda_1 t]\left\{\|z\|_p + \|Qz\|_p\right\}$$

$$\leq 2\,c\exp[\lambda_1 t]\,\|z\|_p \tag{2.42}$$

The desired estimate (2.35) then follows from inequalities (2.40) and (2.42) for a suitably chosen $c_3 \in \mathbb{R}^+$. We can prove estimate (2.36) in a similar way. ■

Remark. The above proof of estimate (2.35) offers an alternative to the method of [2], from which inequalities of the same kind can also be deduced. The strategy of [2] is, however, entirely different from ours: the author first proves the basic estimates within an appropriate L^1–theory, from which the corresponding L^p–inequalities can be obtained through the Riesz–Thorin interpolation method. Proposition 2.3 will also play an important role in the second part of this work [28].

The next geometric result is an easy consequence of Proposition 2.3 and of the parabolic maximum principle; it plays a crucial role in the construction of the appropriate attractor

$\hat{u} \in \{\hat{u}\}_{\hat{\nu} \in (u_0, u_1)}$.

Proposition 2.4. Let s and g satisfy the same hypotheses as in Proposition 2.1 and let $v(t) : \bar{\Omega} \longrightarrow \mathbb{R}$ be defined by $v(t)(x) = v(x,t)$ where $v(x,t)$ is given by relation (2.24). Fix $t_0 \in \mathbb{R}^+$ and define $\tilde{v}(t) = W_{\Delta_{p,\mathscr{N}}}(t-t_0)v(t_0)$ for every $t \in [t_0,\infty)$, where $\left\{W_{\Delta_{p,\mathscr{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ is the diffusion semigroup of Proposition 2.3. Then the inequality $v(t) \leq \tilde{v}(t)$ holds for every $t \in [t_0,\infty)$, pointwise everywhere on Ω . Moreover, $\tilde{v}(t) - Qv(t_0) \longrightarrow 0$ strongly in $L^p(\mathbb{R})$ as $t \longrightarrow \infty$, where Q is the projection operator defined in relation (2.34).

Proof. Since $v(t_0) \in H^{2,p}_{\mathscr{N}}(\mathbb{R})$, it follows from the fact that $\left\{W_{\Delta_{p,\mathscr{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ is the restriction of a holomorphic semigroup and from elliptic regularity theory that $\tilde{v}(t) \in H^{4,p}_{\mathscr{N}}(\mathbb{R})$ for every $t \in (t_0,\infty)$; in addition, $\tilde{v} \in \mathscr{C}([t_0,\infty),H^{2,p}_{\mathscr{N}}(\mathbb{R})) \cap \mathscr{C}^{(1)}((t_0,\infty),H^{2,p}_{\mathscr{N}}(\mathbb{R}))$ and satisfies the Neumann initial value problem

$$\left\{\begin{array}{lll} \tilde{v}'(t) = \Delta_{p,\mathscr{N}}\tilde{v}(t) & \text{on } \Omega & , t \in (t_0,\infty) \\ \tilde{v}(t_0) = v(t_0) & & \\ \dfrac{\partial\tilde{v}(t)}{\partial\underline{n}} = 0 & \text{on } \partial\Omega & , t \in (t_0,\infty) \end{array}\right\} \tag{2.43}$$

Now, define $\tilde{v} : \bar{\Omega} \times [t_0,\infty)$ by $\tilde{v}(x,t) = \tilde{v}(t)(x)$; it follows from the above remarks, embedding (2.16) and relations (2.15) that $\tilde{v} \in \mathscr{C}^{(1)}(\Omega \times (t_0,\infty),\mathbb{R})$ and that $x \longrightarrow \tilde{v}(x,t) \in \mathscr{C}^{(2)}(\bar{\Omega},\mathbb{R})$ for every $t \in [t_0,\infty)$; this along with relations (2.43) leads to

$$\left\{\begin{array}{lll} \tilde{v}_t(x,t) = & \Delta\tilde{v}(x,t)\,, & (x,t) \in \Omega\times(t_0,\infty) \\ \tilde{v}(x,t_0) = & v(x,t_0)\,, & x \in \Omega \\ \frac{\partial\tilde{v}}{\partial\nu}(x,t) = 0\,, & & (x,t) \in \partial\Omega\times(t_0,\infty) \end{array}\right\} \tag{2.44}$$

We furthermore notice that $\tilde{v} \in \mathscr{C}^{2,1}(\Omega\times(t_0,\infty),\mathbb{R}) \cap \mathscr{C}(\bar{\Omega}\times[t_0,\infty),\mathbb{R}) \cap \mathscr{C}^{1,0}(\bar{\Omega}\times(t_0,\infty),\mathbb{R})$, as a consequence of the corresponding property for v. The first statement of the lemma then follows from relations (2.27), (2.44) and the parabolic minimum principle applied to the function $\tilde{v}(t) - v(t)$. As for the second statement, we note that $Q\tilde{v}(t) = Qv(t_0)$ for every $t \in [t_0,\infty)$, from the definition of $\tilde{v}(t)$ and relation (2.37). Hence

$$\|\tilde{v}(t) - Qv(t_0)\|_p = \|\tilde{v}(t) - Q\tilde{v}(t)\|_p =$$

$$= \|W_{\Delta_{p,\mathscr{N}}}(t-t_0)Pv(t_0)\|_p \longrightarrow 0$$

as $t \longrightarrow \infty$, by relation (2.35) of Proposition 2.3. ∎

We now combine the above results to prove the following

Proposition 2.5. Let s and g satisfy the same hypotheses as in Proposition 2.1; for $t \in \mathbb{R}_0^+$, let $v(t)$ be as in Proposition 2.4. Then there exists $v_0 \in \mathbb{R}^+$ such that $v(t) \longrightarrow v_0$ in measure on Ω as $t \longrightarrow \infty$.

Proof. In fact we prove the stronger result that there exists $v_0 \in \mathbb{R}^+$ such that $v(t) \longrightarrow v_0$ strongly in $L^1(\mathbb{R})$. To this end, consider the function $t \longrightarrow Qv(t) : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$; it is clearly differentiable on $\mathbb{R}^+$ with a non positive derivative, for the inequality

$$\frac{Qv(t)}{dt} \leq |\Omega|^{-1} \int_{\Omega} dx(\Delta_{p,\mathcal{N}} v(t))(x) = 0 \tag{2.45}$$

holds as a consequence of (2.27), and for $(x,t) \longrightarrow v_t(x,t) \in \mathscr{C}(\bar{\Omega}\times\mathbb{R}^+,\mathbb{R})$ holds as a consequence of the first part of (C_3). Hence

$$\lim_{t\to\infty} Qv(t) = \inf_{t\in\mathbb{R}_0^+} Qv(t) \geq 0 \tag{2.46}$$

Define $v_0 = \inf_{t\in\mathbb{R}_0^+} Qv(t)$; we first observe that $v_0 \in \mathbb{R}^+$. In order to see this we rewrite expression (2.24) as

$$v(x,t) = \varphi(t)\exp[\alpha\, G(u(x,t))] \tag{2.47}$$

where

$$\varphi(t) = \exp\left[-\alpha\left\{G(\hat{\mu}) + \int_0^t d\xi s(\xi)\right\}\right] \tag{2.48}$$

with G as in Proposition 2.1. It then follows from the boundedness of $t \longrightarrow \int_0^t d\xi s(\xi)$ that $\varphi_0 = \inf_{t\in\mathbb{R}_0^+} \varphi(t) > 0$. It also follows from estimate (2.18) that there exists $c_5 \in \mathbb{R}^+$ such that $\exp[\alpha\, G(u(x,t))] \geq c_5$ for every $(x,t) \in \bar{\Omega} \times \mathbb{R}_0^+$, since G is monotone increasing on (u_0,u_1). Combining these facts with relation (2.47) then gives

$$v_0 = \inf_{t \in \mathbb{R}_0^+} Qv(t) \geq c_5 \varphi_0 > 0 \tag{2.49}$$

In order to show that $v(t) \longrightarrow v_0$ strongly in $L^1(\mathbb{R})$, let $(v(t) - Qv(t))^+ = \max(v(t) - Qv(t),0)$, $(v(t) - Qv(t))^- = \min(v(t) - Qv(t),0)$ be the positive and negative part of $v(t) - Qv(t)$, respectively. Let $\|\cdot\|_1$ be the usual L^1–norm; as a consequence of relation (2.46) and of the definition of v_0, we first notice that for every $\epsilon > 0$, there exists $t_\epsilon > 0$ such that the sequence of inequalities

$$\begin{aligned} \|v(t)-v_0\|_1 &\leq \|v(t)-Qv(t)\|_1 + |\Omega|\,\epsilon = \\ &= 2\int_\Omega dx(v(t)(x) - Qv(t)(x))^+ + |\Omega|\,\epsilon \end{aligned} \tag{2.50}$$

holds for every $t \geq t_\epsilon$, since

$$\int_\Omega dx(v(t)-Qv(t))(x) = \int_\Omega dx(v(t)-Qv(t))^+(x) + \int_\Omega dx(v(t)-Qv(t))^-(x) = 0$$

by definition of the operator Q. Now define

$$\Omega_t^+ = \{x \in \Omega : v(t)(x) - QV(t)(x) > 0\} \tag{2.51}$$

Upon using the definition of v_0 and the results of Proposition 2.4 with $t_0 = t_\epsilon$, we then see that there exists $\tilde{t}_\epsilon \geq t_\epsilon$ such that the sequence of estimates

$$
\begin{aligned}
&\int_{\Omega} dx(v(t)(x) - Qv(t)(x))^{+} = \int_{\Omega_t^{+}} dx(v(t)(x) - Qv(t)(x)) \leq \\
&\leq \int_{\Omega_t^{+}} dx(\tilde{v}(t)(x) - Qv(t_\epsilon)(x)) + \int_{\Omega_t^{+}} dx(Qv(t_\epsilon)(x) - Qv(t)(x)) \leq \\
&\leq \int_{\Omega} dx\,|\tilde{v}(t)(x) - Qv(t_\epsilon)(x)| + |\Omega|\,|Qv(t_\epsilon) - Qv(t)| \leq \\
&\leq \|\tilde{v}(t) - Qv(t_\epsilon)\|_1 + |\Omega|\,|Qv(t_\epsilon) - v_0| \leq (1+|\Omega|)\epsilon
\end{aligned}
\tag{2.52}
$$

holds for every $t \geq \tilde{t}_\epsilon$. The combination of estimates (2.50) and (2.52) then shows that $v(t) \longrightarrow v_0$ strongly in $L^1(\mathbb{R})$ as $t \longrightarrow \infty$, and hence in measure on Ω. ■

We conclude our second step toward proving theorem 2.1 by the following result.

Proposition 2.6. Let s and g satisfy the same hypotheses as in Proposition 2.1. Let u be any classical solution to problem (1.1). Then there exists a $\hat{u} \in \{\hat{u}\}_{\hat{\nu} \in (u_0, u_1)}$ such that $u(t) - \hat{u}(t) \longrightarrow 0$ strongly in $L^p(\mathbb{R})$ as $t \longrightarrow \infty$.

Proof. We define $\hat{u}$ by relation (2.31), where v_0 is the positive number constructed in Proposition 2.5. We first conclude from relation (2.47) and the proof of Proposition 2.5 that

$$v(x,t) \geq c_5\, \varphi_0 > 0 \tag{2.53}$$

uniformly in $(x,t) \in \bar{\Omega} \times \mathbb{R}_0^+$, so that the estimates

$$|\mathrm{Ln}(v(x,t))-\mathrm{Ln}(v_0)| \le \max(v^{-1}(x,t);v_0^{-1})\,|v(x,t)-v_0| \le$$

$$\le \max((c_5\,\varphi_0)^{-1};v_0^{-1})\,|v(x,t)-v_0| \tag{2.54}$$

hold for every $(x,t) \in \bar{\Omega} \times \mathbb{R}_0^+$. Letting $c_6 = \max((c_5\,\varphi_0)^{-1};v_0^{-1})$, it then follows from relation (2.33) that

$$|Gou(t) - Go\hat{u}(t)| \le \alpha^{-1} c_6\,|v(t)-v_0| \tag{2.55}$$

Hence $Gou(t) - Go\hat{u}(t) \longrightarrow 0$ in measure on Ω as $t \longrightarrow \infty$ by the result of Proposition 2.5. In order to conclude that $Gou(t) - Go\hat{u}(t) \longrightarrow 0$ strongly in $L^p(\mathbb{R})$, it is then sufficient to prove that there exists $c_7 \in \mathbb{R}^+$ such that

$$\|Gou(t)-Go\hat{u}(t)\|_{\infty,\bar{\Omega}} = \max_{x\in\bar{\Omega}} |G(u(x,t))-G(\hat{u}(t))| \le c_7 \tag{2.56}$$

But estimate (2.56) follows immediately from the uniform estimates (2.7), (2.19) and from the fact that G is monotone increasing on (u_0,u_1). It remains to prove that $u(t)-\hat{u}(t) \longrightarrow 0$ in $L^p(\mathbb{R})$. To this end, write

$$w(t) = Gou(t) \tag{2.57}$$

$$\hat{w}(t) = Go\hat{u}(t) \tag{2.58}$$

for every $t \in \mathbb{R}^+$. Since $\mathrm{Ran}\, G = \mathbb{R}$, it follows that there exists $y(x,t,s) \in (u_0,u_1)$ such that $G(y(x,t,s)) = \hat{w}(t) + s(w(x,t) - \hat{w}(t))$ for every $(x,t) \in \bar{\Omega} \times \mathbb{R}_0^+$ and every

$s \in (0,1)$. Upon using the mean–value theorem and the inverse function theorem, we then obtain

$$u(x,t) - \hat{u}(t) = G^{-1}(w(x,t)) - G^{-1}(\hat{w}(t)) =$$

$$= \int_0^1 ds (G^{-1})'(\hat{w}(t) + s(w(x,t) - \hat{w}(t)))(w(t,x) - \hat{w}(t)) =$$

$$= \int_0^1 ds\, g(y(x,t,s))(w(x,t) - \hat{w}(t)) \tag{2.59}$$

We conclude from this and from the smoothness of g on the compact interval $[u_0,u_1]$ that there exists $c_8 \in \mathbb{R}^+$ such that

$$|u(x,t)-\hat{u}(t)| \leq c_8 |w(x,t) - \hat{w}(t)| \tag{2.60}$$

for every $(x,t) \in \bar{\Omega} \times \mathbb{R}_0^+$. But inequality (2.60), relations (2.57) and (2.58) and the first part of the proof then imply that

$$\|u(t) - \hat{u}(t)\|_p \leq c_8 \|G \circ u(t) - G \circ \hat{u}(t)\|_p \longrightarrow 0$$

strongly in $L^p(\mathbb{R})$ as $t \longrightarrow \infty$. ■

We can now complete the proof of Theorem 2.1. It is here that conditions (C_1)–(C_4) preceding the statement of Theorem 2.1 all play their crucial role.

<u>Proof of Theorem 2.1</u>. It remains to prove that $u(t) - \hat{u}(t) \longrightarrow 0$ strongly in $H^{2,p}_{\mathcal{N}}(\mathbb{R})$. Let $\rho(\Delta_{p,\mathcal{N}})$ be the resolvent set of Laplace's operator $\Delta_{p,\mathcal{N}}$ and fix $\lambda_0 \in \rho(\Delta_{p,\mathcal{N}}) \cap \mathbb{R}$. We first renorm $H^{2,p}_{\mathcal{N}}(\mathbb{R})$ by

$$z \longrightarrow \|z\|_{\lambda_0,2,p} = \|(\lambda_0 - \Delta_{p,\mathcal{N}})z\|_p \tag{2.61}$$

It follows from the closed graph theorem and from standard elliptic theory that the norm (2.61) is equivalent to that defined by (2.14). Since we already know that $u(t) - \hat{u}(t) \longrightarrow 0$ strongly in $L^p(\mathbb{R})$ as $t \longrightarrow \infty$, it remains to prove that $\Delta_{p,\mathcal{N}}u(t) \longrightarrow 0$ strongly in $L^p(\mathbb{R})$ as $t \longrightarrow \infty$. We first notice that if u is a classical solution to problem (1.1), then it follows from conditions (C_1)–(C_4) that $t \longrightarrow u(t) \in \mathscr{C}(\mathbb{R}_0^+, L^p(\mathbb{R})) \cap \mathscr{C}^{(1)}(\mathbb{R}^+, L^p(\mathbb{R}))$, and that this function satisfies the ordinary differential equation

$$\begin{cases} u'(t) = \Delta_{p,\mathcal{N}}u(t) + s(t)g\circ u(t)\,, & t \in \mathbb{R}^+ \\ \operatorname{Ran}(u(t)) \subseteq (u_0,u_1)\,, & t \in \mathbb{R}_0^+ \end{cases} \tag{2.62}$$

on $L^p(\mathbb{R})$. Now define $y \in \mathscr{C}(\mathbb{R}_0^+, L^p(\mathbb{R})) \cap \mathscr{C}^{(1)}(\mathbb{R}^+, L^p(\mathbb{R}))$ by $y(t) = u(t) - \hat{u}(t)$; it follows from this definition and from relations (2.62) and (1.2) that y satisfies the differential equation

$$y'(t) = \Delta_{p,\mathcal{N}}y(t) + s(t)\{g\circ u(t) - g\circ \hat{u}(t)\} \tag{2.63}$$

on $L^p(\mathbb{R})$, and that $\Delta_{p,\mathcal{N}}y(t) = \Delta_{p,\mathcal{N}}u(t)$ for every $t \in \mathbb{R}_0^+$. Thus it remains to prove that

$$f(t) = s(t)(g\circ u(t) - g\circ \hat{u}(t)) \longrightarrow 0 \tag{2.64}$$

$$y'(t) \longrightarrow 0 \tag{2.65}$$

strongly in $L^p(\mathbb{R})$. Since s is bounded, the former statement is an immediate consequence of Proposition 2.6 and of the smoothness of g since

$$g(u(x,t))-g(\hat{u}(t)) = \int_0^1 d s\, g'(\hat{u}(t)+s(u(x,t)-\hat{u}(t)))(u(x,t)-\hat{u}(t))$$

for every $(x,t) \in \bar{\Omega} \times \mathbb{R}_0^+$; this implies the inequality

$$|g(u(x,t))-g(\hat{u}(t))| \leq c_9 |u(x,t)-\hat{u}(t)| \tag{2.66}$$

for some $c_9 \in \mathbb{R}^+$ since $\hat{u}(t)+s(u(x,t)-\hat{u}(t)) \in (u_0,u_1)$. This leads to the desired conclusion

$$\|g\circ u(t)-g\circ\hat{u}(t)\|_p \leq c_9\|u(t)-\hat{u}(t)\|_p \longrightarrow 0 \tag{2.67}$$

as $t \longrightarrow \infty$. We conclude the proof in showing that statement (2.65) holds. In order to accomplish this we first project equation (2.63) onto the subspaces Ran P and Ran Q , where P and Q are the projection operators defined by relations (2.34). We obtain

$$Py'(t) = \Delta_{p,\mathcal{N}} Py(t) + Pf(t) \tag{2.68}$$

$$Qy'(t) = Qf(t) \tag{2.69}$$

since $\Delta_{p,\mathcal{N}} Q = Q\Delta_{p,\mathcal{N}} = 0$ on $H^{2,p}_{\mathcal{N}}(\mathbb{R})$, and it follows immediately from relations (2.69) and (2.64) that $Qy'(t) \longrightarrow 0$ strongly in $L^p(\mathbb{R})$. It remains to show that $Py'(t) \longrightarrow 0$ strongly in $L^p(\mathbb{R})$. In order to accomplish this we first pick $t_0 \in \mathbb{R}^+$ and we invoke the variation of constants formula to rewrite relation (2.68) as

$$Py(t) = W_{\Delta_{p,\mathcal{N}}}(t-t_0)Py(t_0) + \int_{t_0}^{t} d\xi W_{\Delta_{p,\mathcal{N}}}(t-\xi)Pf(\xi) \qquad (2.70)$$

for every $t \in (t_0,\infty)$. We next notice that f is Hölder continuous on $\mathbb{R}^+$ as an $L^p(\mathbb{R})$–valued function. In fact, this is an immediate consequence of the Hölder continuity of s and that of the functions $t \longrightarrow g \circ u(t)$ and $t \longrightarrow g \circ \hat{u}(t)$, since the latter functions have L^p–norms uniformly bounded in t from the basic estimates (2.7), (2.8), (2.18) and (2.19) (The Hölder continuity of $t \longrightarrow g \circ u(t)$ follows immediately from relation (2.67) and condition (C_1), while that of $t \longrightarrow g \circ \hat{u}(t)$ follows from the fact that $\hat{u}(t)$ has a uniformly bounded derivative according to equation (1.2)). It then follows from relation (2.70) and the standard arguments of [20] and [17] that

$$Py'(t) = \Delta_{p,\mathcal{N}} W_{\Delta_{p,\mathcal{N}}}(t-t_0)Py(t_0) + \int_{t_0}^{t} d\xi \Delta_{p,\mathcal{N}} W_{\Delta_{p,\mathcal{N}}}(t-\xi)P\{f(\xi)-f(t)\}$$

$$+ W_{\Delta_{p,\mathcal{N}}}(t-t_0)Pf(t) \qquad (2.71)$$

for every $t \in (t_0,\infty)$. Since $\left\{W_{\Delta_{p,\mathcal{N}}}(t)\right\}_{t \in \mathbb{R}_0^+}$ is the restriction of a holomorphic semigroup to $L^p(\mathbb{R})$, we now can estimate the first term of relation (2.71) as

$$\|\Delta_{p,\mathcal{N}} W_{\Delta_{p,\mathcal{N}}}(t-t_0)Py(t_0)\|_p \leq 0((t-t_0)^{-1})\|Py(t_0)\|_p \qquad (2.72)$$

so that $\Delta_{p,\mathcal{N}} W_{\Delta_{p,\mathcal{N}}}(t-t_0)Py(t_0) \longrightarrow 0$ strongly in $L^p(\mathbb{R})$ as $t \longrightarrow \infty$. We conclude in a similar way that $W_{\Delta_{p,\mathcal{N}}}(t-t_0)Pf(t) \longrightarrow 0$ strongly in $L^p(\mathbb{R})$ as $t \longrightarrow \infty$, as a consequence of relations (2.35) and (2.64). Finally, a similar statement holds for the second term of (2.71) when t_0 is sufficiently large, upon using estimate (2.36), the Hölder continuity of f along with variations on the theme of the proof of Theorem 5.8.2 of [20]. We conclude that $Py'(t) \longrightarrow 0$ strongly in $L^p(\mathbb{R})$ as $t \longrightarrow \infty$, and hence that relation (2.65) holds. Finally, the very last statement of Theorem 2.1 follows immediately from relation (2.6). ■

Remarks. (1) Upon exploiting the Banach algebra properties of $H^{2,p}_{\mathcal{N}}(\mathbb{R})$ along with the basic estimates (2.7), (2.8), (2.18), (2.19) and statement (C) of Proposition 2.1, it is possible to show that $u(t) - \hat{u}(t) \longrightarrow 0$ strongly in $H^{2,p}_{\mathcal{N}}(\mathbb{R})$ if, and only if, $v(t) - v_0 \longrightarrow 0$ strongly in $H^{2,p}_{\mathcal{N}}(\mathbb{R})$ as $t \longrightarrow \infty$. Thus, the stabilization property of the function (2.24) toward v_0 is in fact not merely limited to the convergence in measure on Ω as in Proposition 2.5.

(2) As an immediate consequence of Proposition 2.2 and of Theorem 2.1, we note that the two equilibria u_0 and u_1 can never be attractors under the hypotheses of Theorem 2.1. This is intuitively understandable since, with $s \in \mathscr{C}(\mathbb{R}_B,\mathbb{R})$, the hypothesis

$$t \longrightarrow \int_0^t d\xi s(\xi) = O(1) \text{ as } |t| \longrightarrow \infty \text{ means that } t \longrightarrow \int_0^t d\xi s(\xi) \text{ is almost-periodic,}$$

and hence that $\mu_B(s) = 0$ (take relation (2.2) with $k = 0$). Therefore, every classical solution u to (1.1) stabilizes around a spatially homogeneous solutions $\hat{u}$ which oscillates almost-periodically between the equilibria u_0 and u_1. But $\hat{u}$ depends on u in general.

(3) Because of relation (2.6), every Fourier exponent of the attractors $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$ is a finite linear combination with integer coefficients of the Fourier exponents of s. The oscillation properties of the $\hat{u}$'s are thereby completely controlled by those of the selection function s.

(4) In the second part of this work [28], we shall in fact prove that $u(t) - \hat{u}(t) \longrightarrow 0$ in $H^{2,p}_{\mathcal{N}}(\mathbb{R})$ with a polynomial rate of decay. The proof of such a result lies beyond the scope of the method used in this article, and requires a more subtle construction of local invariant manifolds around each $\hat{u}$ following the methods of [24], [29], [30], [31] and [32].

While Theorem 2.1 establishes the asymptotic almost–periodicity of every classical solution to problem (1.1), it also implies that the spatially homogeneous solutions $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$ are the only time almost–periodic classical solutions to problem (1.1) within the admissible class. The precise result is the following

Corollary 2.1. Let s and g satisfy the same hypotheses as in Theorem 2.1 and let u be a classical solution to problem (1.1). Then $t \longrightarrow u(x,t)$ is the restriction to $\mathbb{R}_0^+$ of a Bohr almost–periodic function for each $x \in \bar{\Omega}$ if, and only if, $u(x,t) = \hat{u}(t)$ for every $(x,t) \in \bar{\Omega} \times \mathbb{R}_0^+$, for some $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$.

Proof. The statement follows immediately from relation (2.10) and from elementary properties of almost–periodic functions. ■

In the next section, we investigate the stabilization properties of the classical solutions to problem (1.1) when $\mu_B(s) \neq 0$.

3. The Two Equilibria u_0 and u_1 as Global Attractors.

If the selection function s has a non zero time average $\mu_B(s)$, the classical solutions to problem (1.1) are no longer uniformly bounded away from the two equilibria u_0 and u_1. In fact, if $\mu_B(s) < 0$ (resp. $\mu_B(s) > 0$), and if g satisfies the same hypotheses as in the preceding sections, we can show that u_0 (resp. u_1) becomes a global attractor. The notion of classical solution used in this section is exactly the same as that of Section 2, and the precise result is the following

Theorem 3.1. Let $s \in \mathscr{C}(\mathbb{R}_B,\mathbb{R})$ be such that $\mu_B(s) \neq 0$ and assume that g satisfies the same hypotheses as in Proposition 2.1. Assume in addition that s is Hölder continuous on $\mathbb{R}^+$ and let u be a classical solution to problem (1.1). Then either $\mu_B(s) < 0$ and we have

$$\lim_{t \to \infty} \sup_{x \in \overline{\Omega}} |u(x,t) - u_0| = 0 \tag{3.1}$$

or $\mu_B(s) > 0$ and we have

$$\lim_{t \to \infty} \sup_{x \in \overline{\Omega}} |u(x,t) - u_1| = 0 \tag{3.2}$$

Moreover, relations (2.12) and (2.13) still hold for every $\beta \in (0,1-p^{-1}N]$.

Proof. Let $u(t)$, $t \in \mathbb{R}_0^+$, be as in the preceding section; it is then sufficient to prove that $u(t) - u_0 \longrightarrow 0$ strongly in $H^{2,p}_{\mathscr{N}}(\mathbb{R})$ as $t \longrightarrow \infty$ in the first case, and that $u(t) - u_1 \longrightarrow 0$ strongly in $H^{2,p}_{\mathscr{N}}(\mathbb{R})$ in the second case. Let $v(t) : \overline{\Omega} \longrightarrow \mathbb{R}$ be as in Proposition 2.4, that is

$$v(t) = \varphi(t)\exp[\alpha G o u(t)] \tag{3.3}$$

according to relation (2.47), where

$$\varphi(t) = \exp\left[-\alpha\{G(\hat{\mu}) + \int_0^t d\xi s(\xi)\}\right] \tag{3.4}$$

We first notice that estimates (2.23), (2.27) and (2.28) of Proposition 2.2 remain unchanged in this case, for they are independent of the hypothesis concerning $t \longrightarrow \int_0^t d\xi s(\xi)$. Therefore, if c_1 is as in relation (2.28), we still have the estimate

$$\|v(t)\|_{\infty,\overline{\Omega}} = \max_{x \in \overline{\Omega}} |v(x,t)| \leq c_1 \tag{3.5}$$

uniformly in $t \in \mathbb{R}_0^+$. We then conclude from relations (3.3) and (3.5) that the inequality

$$\|\exp[\alpha G o u(t)]\|_{\infty,\overline{\Omega}} \leq \frac{c_1}{\varphi(t)} \tag{3.6}$$

holds for every $t \in \mathbb{R}_0^+$. Now if $\mu_B(s) < 0$, it then follows from relation (3.4) that $\varphi(t) \longrightarrow \infty$ as $t \longrightarrow \infty$, so that $\exp[\alpha G o u(t)] \longrightarrow 0$ uniformly on $\overline{\Omega}$ as $t \longrightarrow \infty$ as a consequence of relation (3.6). Hence $G o u(t) \longrightarrow -\infty$ uniformly on $\overline{\Omega}$ which, together with relation (2.3), implies statement (3.1). In order to get relation (3.2), we first perform a transformation identical to that described at the beginning of the proof of Proposition 2.2. It is then clear that relation (3.2) is a consequence of (3.1). It remains to prove that $\Delta_{p,\mathcal{N}} u(t) \longrightarrow 0$ strongly in $L^p(\mathbb{R})$ as $t \longrightarrow \infty$. But this can be done in exactly the same way as in the proof of Theorem 2.1, upon replacing $y(t) = u(t) - \hat{u}(t)$ by $y(t) = u(t) - u_{0,1}$. ∎

<u>Remark</u>. In the second part of this work [28], we shall in fact prove that $u(t) - u_{0,1} \longrightarrow 0$ in $H^{2,p}_{\mathcal{N}}(\mathbb{R})$ exponentially rapidly. Again, the proof of this will require

a more elaborate construction of local stable manifolds around u_0 and u_1.

In the next section, we discuss several examples which illustrate the use of Theorems 2.1 and 3.1.

4. Application to Fisher's Type Equations of Population Genetics.

We are primarily concerned with the applications of the preceding results to problems of the form (1.1) which occur in population genetics, such as Fisher's equations and its variations. We begin with the following

Example 4.1. Consider the problem

$$\begin{cases} u_t(x,t)=\Delta u(x,t)+(\cos(\omega_1 t)+\cos(\omega_2 t))u(x,t)(1-u(x,t))(\alpha u(x,t)+(1-\alpha)(1-u(x,t))) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , \ (x,t) \in \Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (0,1) \\ \frac{\partial u}{\partial \underset{\sim}{n}}(x,t) = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad , \ (x,t) \in \partial\Omega\times\mathbb{R}^+ \end{cases} \tag{4.1}$$

where $\alpha \in (0,1)$, and where $\{\omega_1,\omega_2\} \subset \mathbb{R}/\{0\}$ is rationally independent. Here we have $g(u) = u(1-u)(\alpha u+(1-\alpha)(1-u))$ with $u_0 = 0$, $u_1 = 1$ and $s(t) = \cos(\omega_1 t) + \cos(\omega_2 t)$. We can easily verify that all of the hypotheses of Theorem 2.1 are satisfied. We conclude that every classical solution to problem (4.1) remains uniformly bounded away from $u_0 = 0$ and $u_1 = 1$, and that it stabilizes around a spatially homogeneous, time quasiperiodic attractor $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(0,1)}$. In addition, we note that every Fourier exponent Λ of $\hat{u}$ is of the form $\Lambda = k_1\omega_1 + k_2\omega_2$ where $k_{1,2} \in \mathbb{Z}$. The set $\{\omega_1,\omega_2\}$ is thereby a possible integer basis for the set of all Fourier exponents of $\hat{u}$.

Remark. Equation (4.1) often occurs in certain problems of population genetics, such as the description of the space time evolution of the fraction of one of two alleles in the population of a migrating diploid species when the selection function s takes quasiperiodic seasonal variations into account ([9], [19], [26]). In this context, the result of Example 4.1 means that both alleles will persist in the population for all times, and that the fractions of the two alleles will eventually evolve nearly quasiperiodically in time with oscillation properties entirely controlled by those of the seasonal variations.
In contrast to Example 4.1, we now consider the following

Example 4.2. Consider the problem

$$\left\{\begin{array}{ll} u_t(x,t)=\Delta u(x,t)+(\cos(\omega_1 t)+\cos(\omega_2 t)-1)u(x,t)(1-u(x,t))\exp[-u(x,t)] & \\ & ,\ (x,t) \in \Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (0,1) & \\ \frac{\partial u}{\partial n}(x,t) = 0 & ,\ (x,t) \in \partial\Omega\times\mathbb{R}^+ \end{array}\right\} \tag{4.2}$$

where $\{\omega_1,\omega_2\}$ is as in Example 4.1. Here $g(u) = u(1-u)\exp[-u]$ so that $u_0 = 0$ and $u_1 = 1$. Moreover, $s(t) = \cos(\omega_1 t) + \cos(\omega_2 t) - 1$ with $\mu_B(s) = -1$, and it is easily checked that all of the hypotheses of Theorem 3.1 hold. We conclude that every classical solution to problem (4.2) converges to $u_0 = 0$, irrespective of the actual values of ω_1 and ω_2.

Remark. In the context of population genetics, the result of Example 4.2 means that only one of the alleles will eventually survive in the population.

Example 4.3. Conclusions entirely similar to those of Example 4.1 hold for the boundary value problem

$$\begin{cases} u_t(x,t)=\Delta u(x,t)+\sin(\omega t)\sin(\pi u(x,t)) \,, & (x,t) \in \Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (0,1) & \\ \frac{\partial u}{\partial \underset{\sim}{n}}(x,t) = 0 & ,\ (x,t) \in \partial\Omega\times\mathbb{R} \end{cases} \tag{4.3}$$

where $g(u) = \sin(\pi u)$, $u_0 = 0$ and $u_1 = 1$; here $s(t) = \sin(\omega t)$ with $\omega \in \mathbb{R}/\{0\}$, and all of the attractors $\{\hat{u}\}_{\hat{\nu}\in(0,1)}$ are of course time periodic with period $\tau = 2\pi|\omega|^{-1}$.

Example 4.4. Conclusions similar to those of Example 4.2 hold for the boundary value problem

$$\begin{cases} u_t(x,t)=\Delta u(x,t)+(\sin(\omega t)+1)u(x,t)(1-u(x,t))(\alpha u(x,t)+(1-\alpha)(1-u(x,t))) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ,\ (x,t) \in \Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (0,1) \\ \frac{\partial u}{\partial \underset{\sim}{n}}(x,t) = 0 \qquad\qquad\qquad\qquad\qquad ,\ (x,t) \in \partial\Omega\times\mathbb{R}^+ \end{cases} \tag{4.4}$$

where $\alpha \in (0,1)$, with the exception of the fact that the global attractor is now $u_1 = 1$ instead of $u_0 = 0$.

We conclude this article with some remarks and with the discussion of some open problems.

5. Some Remarks and Some Open Problems.

It is first natural to ask whether the results of the preceding sections remain valid in the case of Neumann boundary value problems of the form

$$\left\{\begin{array}{ll} u_t(x,t)=\Delta u(x,t)+s(x,t)g(u(x,t)), & (x,t) \in \Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (u_0,u_1) & \\ \frac{\partial u}{\partial \underline{n}}(x,t) = 0 & ,(x,t) \in \partial\Omega\times\mathbb{R}^+ \end{array}\right\} \quad (5.1)$$

where the selection function depends explicitely on $x \in \overline{\Omega}$ in such a way that $t \longrightarrow s(x,t)$ is Bohr almost–periodic for each $x \in \overline{\Omega}$, and where g satisfies the same hypotheses as above. If s is sufficiently smooth on $\overline{\Omega} \times \mathbb{R}_0^+$, let $\bar{s}(t) = \max_{x\in\overline{\Omega}} s(x,t)$; then $\bar{s} \in \mathscr{C}(\mathbb{R}_B,\mathbb{R})$ and, upon using essentially the same arguments as above and the condition $t \longrightarrow \int_0^t d\xi\bar{s}(\xi) = 0(1)$ as $|t| \longrightarrow \infty$, we can prove that every classical solution u to (5.1) stabilizes around a spatially homogeneous, time almost–periodic solution $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in[u_0,u_1)}$ of the initial value problem

$$\left\{\begin{array}{l} \dot{\hat{u}}(t) = \bar{s}(t)g(\hat{u}(t)),\ t \in \mathbb{R} \\ \mathrm{Ran}(u) \subseteq [u_0,u_1) \\ \hat{u}(0) = \hat{\nu} \in [u_0,u_1) \end{array}\right\} \quad (5.2)$$

in the sense of relations (2.10), (2.11), (2.12) and (2.13) of Theorem 2.1.

On the other hand, let $\underline{s}(t) = \min_{x\in\overline{\Omega}} s(x,t)$; we can then also prove that conclusion (3.1) (resp. conclusion (3.2)) of Theorem 3.1 remains valid in the case of a classical solution to problem (5.1), provided that the condition $\mu_B(s) < 0$ be replaced by $\mu_B(\bar{s}) < 0$ (resp. that $\mu_B(s) > 0$ be replaced by $\mu_B(\underline{s}) > 0$).

It is, however, also worth mentioning that there exists in relation with problem (5.1) the additional possibility of having $\mu_B(\underline{s}) < 0 < \mu_B(\overline{s})$. This, of course, does not occur for problem (1.1) where $s(x,t) = s(t)$ for every $x \in \overline{\Omega}$. In this case, it is tempting to conjecture that there exists a unique time almost–periodic solution to problem (1.1) which is neither identically equal to u_0 non identically equal to u_1 , and which is a global attractor for all classical solutions to (1.1). This was in fact recently proved in [16] when $t \longrightarrow s(x,t)$ is periodic, but remains an open problem in the general almost–periodic case. The source of this difficulty lies primarily in the fact that there is no natural substitute for the notion of Poincaré time–map in the almost–periodic case.

Finally, if $s(x,t) = s(t)$ for every $x \in \overline{\Omega}$ and if s is Bohr almost–periodic but not periodic, there exists the additional possibility of having $\mu_B(s) = 0$ without having $t \longrightarrow \int_0^t d\xi s(\xi) = 0(1)$ as $|t| \longrightarrow \infty$. This is for instance the case for the function $s(t) = \sum_{k=1}^{\infty} k^{-2} \exp[ik^{-2}t]$, because of the notorious difficulty of small divisors (in the preceding exemple, the sequence $(\Lambda_k = k^{-2})_{k=1}^{\infty}$ of all Fourier exponents of s converges to zero as $k \longrightarrow \infty$, which precludes the primitive of s from being almost–periodic). We did not investigate this case any further.

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