Zariski's Multiplicity Question — A survey

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Abstract

In 1971, Zariski asked the following question: *is the multiplicity of a reduced analytic hypersurface singularity in $\mathbb{C}^n$ depends only on its embedded topological type?* More precisely, if $f,g\colon(\mathbb{C}^n,0)\to(\mathbb{C},0)$ are reduced germs (at the origin) of holomorphic functions such that the corresponding germs of hypersurfaces in $\mathbb{C}^n$ have the same embedded topological type, then is it true that f and g have the same multiplicity at 0 ? Instead of dealing with a pair (f,g), one can also consider a similar question for a family $(F_t)_t$. More precisely, if $f\colon(\mathbb{C}^n,0)\to(\mathbb{C},0)$ is a reduced germ of holomorphic function and $(F_t)_t$ a topologically V–constant (holomorphic) deformation of it, then is it true that $(F_t)_t$ is equimultiple ? More than thirty years later, these problems are, in general, still unsettled (even for hypersurfaces with isolated singularities). The answer to the first question is, nevertheless, known to be positive in the special case of plane curve singularities (i.e., when $n=2$). Concerning families, equimultiplicity is known for topologically V–constant deformations of isolated quasihomogeneous and semiquasihomogeneous singularities and topologically V–constant deformations within a family of convenient Newton nondegenerate isolated singularities. Equimultiplicity is also known for topologically V–constant deformations within some very special families of nonisolated singularities. Moreover, it is known that the multiplicity is an embedded C^1 invariant as well as an embedded topological right–left bilipschitz invariant. Several other (more specific) results are also known and will be mentioned in this survey. Our aim here is to provide in a short exposition a general overview of Zariski's problem which is one of the most fascinating (but also very difficult!) problem in equisingularity theory.

Introduction

Let $f,g\colon(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be reduced germs (at the origin) of holomorphic functions, $V_f:=f^{-1}(0)$, $V_g:=g^{-1}(0)$ the corresponding germs of hypersurfaces in $\mathbb{C}^n$, and ν_f,ν_g the multiplicities at 0 of V_f and V_g respectively. Zariski's multiplicity question is as follows (cf. [Z1]).

Question 0.1. If f and g are topologically V–equivalent, then is it true that $\nu_f=\nu_g$?

One says that f and g are topologically V–equivalent if there is a germ of homeomorphism $\varphi\colon(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ such that $\varphi(V_f)=V_g$.

The question is, in general, still unsettled. Nevertheless, the answer is known to be *yes* in the following special cases:

(i) if $n=2$ (Zariski [Z2]);
(ii) if $\nu_f=1$, that is, if 0 is *not* a critical point of f (A'Campo [A'Ca1], Lê [L1]);

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(iii) if $n=3$ and $\nu_f=2$ (Navarro Aznar [Na1,2]);
(iv) if $n=3$ and f and g are quasihomogeneous with an isolated critical point at the origin (Xu–Yau [Y1] and [XY]);
(v) if $n=3$, f and g have an isolated critical point at 0 and the arithmetic genus of V_f at 0 is ≤ 2 (Yau [Y2]);
(vi) if φ is a C^1–diffeomorphism (Ephraim [Ep], Trotman [T1,2,4]);
(vii) if f and g are topologically right–left equivalent by bilipschitz homeomorphisms (Risler-Trotman [RT]).

One says that f and g are topologically right–left equivalent by bilipschitz homeomorphisms if there are germs of bilipschitz homeomorphisms $\varphi\colon (\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ and $\phi\colon(\mathbb{C},0)\to(\mathbb{C},0)$ (i.e., φ, φ^{-1}, ϕ and ϕ^{-1} are *lipschitz* maps) such that $f=\phi\circ g\circ\varphi$.

Also, instead of dealing with a pair (f,g), one can consider a similar question for a family $(F_t)_t$. More precisely, let $f\colon(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a reduced germ of holomorphic function, V_f the corresponding germ of hypersurface in $\mathbb{C}^n$, and ν_f the multiplicity of V_f at 0. Let

$$\begin{aligned} F\colon (\mathbb{C}^n\times\mathbb{C},\{0\}\times\mathbb{C}) &\to (\mathbb{C},0)\\ (z,t) &\mapsto F(z,t)=F_t(z), \end{aligned}$$

be a *deformation* of f, that is, F is a germ of holomorphic function such that $F_0=f$ and, for all t near 0, the germ F_t is reduced. Let ν_{F_t} be the multiplicity of $V_{F_t}:=F_t^{-1}(0)$ at 0.

Question 0.2. If $F=(F_t)_t$ is topologically V–constant (i.e., if, for all t near 0, F_t is topologically V–equivalent to $F_0=f$), then is it true that $\nu_{F_t}=\nu_f$ for all t near 0 ?

By Lê–Ramanujam's theorem [LR], any μ–constant deformation of an isolated hypersurface singularity in $\mathbb{C}^n$, with $n\neq 3$, is topologically V–constant [(1)]. On the other hand, it is known that if two germs of holomorphic functions are topologically V–equivalent, then they have necessarily the same Milnor number at 0 [(2)]. Hence, in the special case where f has an *isolated* singularity at 0, and provided $n\neq 3$, Question 0.2 can be reformulated as follows.

Question 0.3. We suppose that $n\neq 3$ and that f has an *isolated* critical point at 0. If the deformation $F=(F_t)_t$ is μ–constant, then is it true that $\nu_{F_t}=\nu_f$ for all t near 0 ?

Of course Question 0.3 makes sense even in the case $n=3$ and the answer (including this case) is known to be *yes* if f is quasihomogeneous or semiquasihomogeneous (Greuel [Gr] and O'Shea [O'Sh] — see also Trotman [T3,4]), or if f, together with the F_t, for any t close enough to 0, are convenient and have a nondegenerate Newton principal part (Abderrahmane [Ab] and Saia–Tomazella [ST]).

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(1) A deformation $(F_t)_t$ of an isolated hypersurface singularity f is said to be μ–constant if, for all t near 0, the Milnor number of F_t at 0 is equal to the Milnor number of $F_0=f$ at 0. We recall that the *Milnor number* μ_g at 0 of a germ of holomorphic function $g\colon(\mathbb{C}^n,0)\to(\mathbb{C},0)$ is given by $\mu_g=\dim_{\mathbb{C}}(\mathbb{C}\{z_1,\ldots,z_n\}/(\frac{\partial g}{\partial z_1},\ldots,\frac{\partial g}{\partial z_n}))$, where $\mathbb{C}\{z_1,\ldots,z_n\}$ is the ring of convergent power series centered at 0, which is identified with the ring $\mathcal{O}_n$ of germs of holomorphic functions at 0, and where $(\frac{\partial g}{\partial z_1},\ldots,\frac{\partial g}{\partial z_n})$ is the *Jacobian ideal* of g, generated by all the partial derivatives of g. Notice that $\mu_g<\infty$ if and only if 0 is, at worst, an *isolated* critical point of g.

(2) Let $f,g\colon(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be two germs of holomorphic functions. If μ_f is finite, so is μ_g (cf. Theorem 10.3); now, if μ_f and μ_g are finite, then they are equal (cf. [L2], [Te1]).

There are several other partial positive answers to Zariski's questions: for example, see the papers by Comte–Milman–Trotman [CMT], Greuel–Pfister [GP1], A'Campo [A'Ca2], Mendris–Néméthi [MN], Eyral [Ey], etc... We also point out the *reduction* theorem by Massey [M1,2] connecting isolated and aligned singularities.

Note also that a positive answer to Question 0.1 would automatically imply a positive answer to Question 0.2 (and 0.3), while the reverse is not true.

Also, although Zariski's questions make sense only in the *hypersurface* case (cf. Section 12), Gau–Lipman, in [GL1,2], generalized to high–codimensional (complex) closed analytic subsets in $\mathbb{C}^n$ the *differential* type result of Ephraim and Trotman, and Comte, in [C], proved a *bilipschitz* type result in such a high–codimensional situation.

In this survey article we review in detail all these results.

Throughout, we consider the complex space $\mathbb{C}^n$ with fixed coordinates $z = (z_1, \ldots, z_n)$, unless the contrary is said explicitly. We suppose $n \geq 2$. We recall that the ring $\mathcal{O}_n$ of germs of holomorphic functions at 0 is naturally isomorphic to the ring $\mathbb{C}\{z_1, \ldots, z_n\}$ of convergent power series centered at 0, so that we may identify a germ $f \in \mathcal{O}_n$ with its power series $f = \sum_\alpha a_\alpha \, z^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$, and $a_\alpha \in \mathbb{C}$. We consider only germs f at the *origin* and such that $f(0) = 0$. We note $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The notation $F\colon (\mathbb{C}^n \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0)$ implies $F(\{0\} \times \mathbb{C}) = \{0\}$. We assume that our germs are not identically zero. By abuse of language, we may also call *germ* any arbitrarily small *representative* of it.

Content

1. Multiplicity and order

Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function and let V_f be the corresponding germ of hypersurface in $\mathbb{C}^n$. The decomposition of f into its homogeneous components is written as $f(z) = \sum_{j=1}^{+\infty} f^j(z)$, with

$$f^j(z) = \sum_{\alpha_1+\ldots+\alpha_n=j} a_\alpha\, z^\alpha \tag{1.1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$, and $a_\alpha \in \mathbb{C}$ (f^j is either zero or a homogeneous polynomial of degree j). The *order* of f at 0 is the smallest integer j such that $f^j \not\equiv 0$. The multiplicity of V_f at 0 is the number of points of intersection, near 0, of V_f with a generic (complex) line in $\mathbb{C}^n$ passing arbitrarily close to 0 but not through 0. If the germ f is *reduced*, then the order of f at 0 is equal to the multiplicity of V_f at 0. In this case, we say equally 'order of f at 0' or 'multiplicity of V_f at 0' (or even 'multiplicity of f at 0') and we denote this common number by ν_f. Notice that, since $n \geq 2$, if f is regular or has an isolated critical point at 0, then it is automatically reduced.

The following lemma reduces Zariski's Questions 0.1 and 0.2 to *irreducible* germs.

Lemma 1.2 (Ephraim [Ep]). *If the answer to Question 0.1 (respectively Question 0.2) is positive for irreducible germs, then it is positive in general.*

If $f = f_1 \ldots f_s$ is the factorization of a reduced germ f of holomorphic function into its irreducible factors, then $\nu_f = \sum_{i=1}^{s} \nu_{f_i}$. Lemma 1.2 thus follows from the following result.

Lemma 1.3 (Ephraim [Ep]). *Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be reduced germs of holomorphic functions and let V_f, V_g be the corresponding germs of hypersurfaces in $\mathbb{C}^n$. If there is a germ of homeomorphism $\varphi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\varphi(V_f) = V_g$, then necessarily φ maps each irreducible component of V_f onto an irreducible component of V_g.*

By [W, Chapter 3, Theorem 2B], the irreducible components of V_f are the V_f–closures of the connected components of the regular part of V_f. The lemma is thus a consequence of Theorem 10.3 below which asserts that if a reduced germ of holomorphic function is topologically V–equivalent to a regular (reduced) germ then it is itself regular.

2. Multiplicity and first homology

In [Ep], Ephraim gave two interpretations of the multiplicity in terms of the first homology. This section is about these characterizations.

Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a reduced germ of holomorphic function, V_f the corresponding germ of hypersurface in $\mathbb{C}^n$, and ν_f the multiplicity of V_f at 0. Then, with notation (1.1), $f^{\nu_f} \not\equiv 0$ and the decomposition of f into its homogeneous components is written as

$$f(z) = f^{\nu_f}(z) + \sum_{j=\nu_f+1}^{+\infty} f^j(z)$$

(f^{ν_f} is called the *initial polynomial* of f at 0). Let $C(V_f)$ be the tangent cone of V_f at 0 (cf. [W, Chapter 7, Definition 1G]). By [W, Chapter 7, Theorem 4A], $C(V_f)$ is nothing but the zero set $(f^{\nu_f})^{-1}(0)$ of the initial polynomial f^{ν_f} of f. Thus a (complex) line $L \subset \mathbb{C}^n$ through 0, determined by a vector $z_0 \in \mathbb{C}^n - \{0\}$, is contained in $C(V_f)$ if and only if $f^{\nu_f}(z_0) = 0$. So, if L is not contained in the tangent cone $C(V_f)$ (equivalently, if $L \cap C(V_f) = \{0\}$), then the multiplicity ν_f of f at 0 is equal to the order of $f_{|L}$ at 0.

First interpretation. Let $L \subset \mathbb{C}^n$ be a line through 0 such that $L \cap C(V_f) = \{0\}$. Then 0 is an isolated point of $L \cap V_f$ and $\nu_f =$ order of $f_{|L}$ at 0. Hence if $D \subset L$ is a closed 2–disc at 0 so small that $D \cap V_f = \{0\}$, and if γ is a generator of the first homology group (3) $H_1(D - \{0\})$, then, choosing an isomorphism $H_1(\mathbb{C} - \{0\}) \simeq \mathbb{Z}$, $(f_{|L})_*(\gamma) \in H_1(\mathbb{C} - \{0\})$ is, up to sign, nothing but the order of $f_{|L}$ at 0, that is, according to the discussion above, $\pm\nu_f$ (the homomorphism $(f_{|L})_*\colon H_1(D - \{0\}) \to H_1(\mathbb{C} - \{0\})$ is induced by f between the mentioned homology groups).

Let $\mathbb{S}_\varepsilon^{2n-1}$ be the boundary of the closed ball $\overline{\mathbb{B}}_\varepsilon^{2n} := \{z \in \mathbb{C}^n\,;\, |z| \leq \varepsilon\}$. The Local Conic Structure Lemma of Burghelea–Verona [BV] (see also Ephraim [Ep]) says that for any ε small enough

$$(2.1) \qquad (\overline{\mathbb{B}}_\varepsilon^{2n-1}, \overline{\mathbb{B}}_\varepsilon^{2n-1} \cap V_f) \overset{\text{homeo}}{\simeq} (\mathbb{S}_\varepsilon^{2n-1}, \mathbb{S}_\varepsilon^{2n-1} \cap V_f) \times [0,\varepsilon] \Big/ (\mathbb{S}_\varepsilon^{2n-1}, \mathbb{S}_\varepsilon^{2n-1} \cap V_f) \times \{0\}.$$

Second interpretation. Here we suppose that f is *irreducible*. Let B be an open ball in $\mathbb{C}^n$ at 0 such that its closure is contained in the open ball $\mathbb{B}_\varepsilon^{2n}$ for an ε so small that (2.1) holds. Then the homomorphism $f_*\colon H_1(B - V_f) \to H_1(\mathbb{C} - \{0\})$ is an isomorphism. Indeed, it is not difficult to see that it is onto. On the other hand, since f is irreducible, by Lefschetz duality, $H_1(B - V_f) \simeq \mathbb{Z}$. It follows that we have an isomorphism. Let L and D as in the *first interpretation*. By shrinking D (if necessary) we can assume that $D - \{0\} \subset B - V_f$. Factor $f\colon D - \{0\} \to \mathbb{C} - \{0\}$ as the composite of $i\colon D - \{0\} \to B - V_f$ and $f\colon B - V_f \to \mathbb{C} - \{0\}$, where i is the inclusion. By the first interpretation, if γ is a generator of $H_1(D - \{0\})$, then $(f_{|L})_*(\gamma) = f_*(i_*(\gamma)) \in H_1(\mathbb{C} - \{0\})$ represents, up to sign, the multiplicity ν_f. Since $f_*\colon H_1(B - V_f) \simeq \mathbb{Z} \to H_1(\mathbb{C} - \{0\}) \simeq \mathbb{Z}$ is an isomorphism, $i_*(\gamma) \in H_1(B - V_f) \simeq \mathbb{Z}$ is also, up to sign, the multiplicity ν_f.

These two interpretations of the multiplicity are used by Ephraim in [Ep] to show that the multiplicity is an embedded C^1 invariant (cf. Section 5).

3. Multiplicity and Łojasiewicz exponent

In [RT], Risler–Trotman gave an interpretation of the multiplicity in terms of a certain Łojasiewicz exponent. This section is about this characterization.

Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a reduced germ of holomorphic function, V_f the corresponding germ of hypersurface in $\mathbb{C}^n$, and ν_f the multiplicity of V_f at 0. The *Łojasiewicz exponent* λ_f of f at 0 is defined by

$$\lambda_f := \inf\{\lambda > 0\,;\, \exists K > 0,\ \mathrm{dist}(z, V_f)^\lambda \leq K\,|f(z)|,\ \forall z \text{ near } 0\},$$

where $\mathrm{dist}(z, V_f)$ is the usual distance between z and V_f.

(3) Throughout, homology group means singular homology group with integer coefficients.

Theorem 3.1 (Risler-Trotman [RT]). *The following equality holds: $\lambda_f = \nu_f$.*

The idea of the proof is as follows. Without loss of generality, one can assume that the z_n–axis Oz_n is not contained in the tangent cone $C(V_f)$ of V_f at 0, so that $f(0,\dots,0,z_n) \neq 0$ for any $z_n \neq 0$ close enough to 0 and the multiplicity of f at 0 coincide with the order of $f_{|Oz_n}$ at 0. Hence, by the Weierstrass Preparation Theorem, for z near 0, the germ $f(z)$ can be represented as a product $f(z) = g(z)\,h(z)$, where $g(z)$ is a germ of holomorphic function which does not vanish around 0 and where $h(z)$ is of the form

$$h(z) = z_n^{\nu_f} + z_n^{\nu_f - 1} f_1(z_1,\dots,z_{n-1}) + \ldots + f_{\nu_f}(z_1,\dots,z_{n-1}),$$

with, for $1 \leq i \leq \nu_f$, $f_i \in \mathbb{C}\{z_1,\dots,z_{n-1}\}$, $f_i(0) = 0$ and the order of f_i at 0 is $\geq i$. Since $\lambda_f = \lambda_h$ and $\nu_f = \nu_h$, we can assume that $f = h$. By considering elements z in the z_n–axis, it is easy to see that $\nu_f \leq \lambda_f$. To show the other inequality, $\lambda_f \leq \nu_f$, write $f(z) = (z_n - a_1)\dots(z_n - a_{\nu_f})$, where $a_i := a_i(z_1,\dots z_{n-1})$, $1 \leq i \leq \nu_f$, are the roots (not necessarily distinct) of the polynomial $z_n \mapsto f(z_1,\dots,z_{n-1},z_n)$. Then, since the point $(z_1,\dots,z_{n-1},a_i) \in V_f$, we have $\operatorname{dist}((z_1,\dots,z_n),V_f) \leq |z_n - a_i|$, and by taking the product over all i one gets $\lambda_f \leq \nu_f$ as desired.

This interpretation of the multiplicity is used by Risler–Trotman in [RT] to show that the multiplicity is an embedded topological right–left bilipschitz invariant (cf. Section 6).

4. Topological types of isolated singularities

Let $f, g\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ be reduced germs of holomorphic functions and let V_f, V_g be the corresponding germs of hypersurfaces in $\mathbb{C}^n$.

Definitions 4.1.

(i) One says that f and g are *topologically V–equivalent* [(4)] if there is a germ of homeomorphism $\varphi\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ such that $\varphi(V_f) = V_g$.

(ii) One says that f and g are *topologically right–left equivalent* if there are germs of homeomorphisms $\varphi\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ and $\phi\colon (\mathbb{C},0) \to (\mathbb{C},0)$ such that $f = \phi \circ g \circ \varphi$.

(iii) One says that f and g are *topologically right equivalent* if there is a germ of homeomorphism $\varphi\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ such that $f = g \circ \varphi$.

The topological right equivalence implies the topological right–left equivalence which in turn implies the topological V–equivalence.

The proof of the Local Conic Structure Lemma of Burghelea–Verona [BV] (cf. (2.1)) shows that the pairs $(\mathbb{S}^{2n-1}_\varepsilon, \mathbb{S}^{2n-1}_\varepsilon \cap V_f)$ for any ε small enough are homeomorphic. The topological type of these pairs is called the *link* of V_f at 0.

Definition 4.2. One says that f and g are *link equivalent* if $(\mathbb{S}^{2n-1}_\varepsilon, \mathbb{S}^{2n-1}_\varepsilon \cap V_f)$ is homeomorphic to $(\mathbb{S}^{2n-1}_{\varepsilon'}, \mathbb{S}^{2n-1}_{\varepsilon'} \cap V_g)$ for all $\varepsilon, \varepsilon'$ small enough.

(4) One also says *topologically equisingular* (cf. [Z1]).

By (2.1), the link equivalence implies the topological V–equivalence.

On the other hand, if f and g are link equivalent, we can always find a germ of homeomorphism φ: $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\varphi(V_f) = V_g$ and $|\varphi(z)| = |z|$ for all z near 0 (use the proof of the Local Conic Structure Lemma of [BV]).

If f and g have an *isolated* critical point at 0 and if they are topologically right–left equivalent, then g is topologically right equivalent either to f or to $\bar{f}$, the conjugate of f, which has the same multiplicity as f (King [Ki1]) (5), so that, under these hypotheses, there always exists a germ of homeomorphism φ: $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\varphi(V_f) = V_g$ and $|g \circ \varphi(z)| = |f(z)|$ for all z near 0. See the discussion in [CMT].

We also have the following result.

Theorem 4.3 (King [Ki1], Perron [P], Saeki [Sae]). *Suppose that f and g have an isolated critical point at* 0*. Then the following conditions are equivalent:*

(i) *f and g are topologically right–left equivalent;*

(ii) *f and g are topologically V–equivalent;*

(iii) *f and g are link equivalent.*

(i) $\Rightarrow$ (ii) is obvious; (ii) $\Rightarrow$ (iii) is due to Saeki [Sae]; (iii) $\Leftrightarrow$ (i) is due to King [Ki1] for $n \neq 3$ and to Perron [P] for $n = 3$.

Concerning families, King (6) also proved the following theorem.

Theorem 4.4 (King [Ki2]). *Let f: $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function, with $n \neq 3$, and let F: $(\mathbb{C}^n \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0)$, $(z, t) \mapsto F(z, t) = F_t(z)$, be a germ of holomorphic function such that $F_0 = f$. We suppose that f has an isolated critical point at the origin (in particular f is reduced). If $F = (F_t)_t$ is topologically V–constant or μ–constant (in particular this implies that, for all t near* 0*, F_t has an isolated critical point at* 0 *and, consequently, F_t is reduced), then it is topologically right constant.*

We recall that $(F_t)_t$ is said to be topologically V–constant (respectively topologically right constant) if, for all t near 0, F_t is topologically V–equivalent (respectively topologically right equivalent) to $F_0 = f$; $(F_t)_t$ is said to be μ–constant if, for all t near 0, the Milnor number of F_t at 0 is equal to the Milnor number of $F_0 = f$ at 0.

Remark. Definitions 4.1 and 4.2 make sense only for *reduced* germs (consider $z_1^2 z_2^2$ and $z_1 z_2$).

5. C^1 invariance of the multiplicity

In [Ep] and [T1,2,4], Ephraim and Trotman proved (independently) that if two reduced hypersurface singularities are C^1–diffeomorphic (as embedded germs), then they necessarily have the same multiplicity. This gives a positive answer to Question 0.1 in the special case

(5) If, moreover, $n = 2$ or f has a nondegenerate Newton principal part (cf. Section 10), then f is topologically right equivalent to $\bar{f}$ (Nishimura [Ni]; see also [Sae]).

(6) In fact, King proved a more general result (for details, see [Ki2, Corollary 3]).

where φ is a C^1–diffeomorphism. This section is about this result.

Let $f, g: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be reduced germs of holomorphic functions, V_f, V_g the corresponding germs of hypersurfaces in $\mathbb{C}^n$, and ν_f, ν_g the multiplicities at 0 of V_f, V_g respectively.

Theorem 5.1 (Ephraim [Ep] and Trotman [T1,2,4]). *Suppose there is a germ of C^1–diffeomorphism $\varphi: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\varphi(V_f) = V_g$. Then $\nu_f = \nu_g$.*

The idea of Ephraim's proof is as follows. By Lemma 1.2, one can suppose that f and g are irreducible. Choose L, D and B for the germ f as in Section 2, and consider a generator γ of $H_1(D - \{0\})$. Thus $i_*(\gamma) \in H_1(B - V_f)$ represents (up to sign) the multiplicity ν_f, where $i: D - \{0\} \to B - V_f$ is the inclusion. Assume that ε is small enough so that (2.1) is also true with g instead of f and choose another open ball $B' \subset \mathbb{C}^n$, centered at 0, the closure of which is contained in $\mathbb{B}^{2n}_\varepsilon$ and such that $\varphi_*: H_1(B - V_f) \to H_1(B' - V_g)$ is an isomorphism (shrink B if necessary). As above (cf. Section 2), one shows that the homomorphism $g_*: H_1(B' - V_g) \to H_1(\mathbb{C} - \{0\})$ is an isomorphism. Thus $g_*(\varphi_*(i_*(\gamma))) \in H_1(\mathbb{C} - \{0\}) \simeq \mathbb{Z}$ represents ν_f too. Now, since φ is a C^1–diffeomorphism, we can write $\varphi(z) = A(z) + o(|z|)$, where $A: \mathbb{C}^n \to \mathbb{C}^n$ is a *real* linear isomorphism. One checks easily that $A(L) \cap C(V_g) = \{0\}$. Using this fact, one can prove that, if D is chosen small enough, then the maps $g \circ \varphi \circ i$ and $g^{\nu_g} \circ A_{|L} \circ j$, where $j: D - \{0\} \to L - \{0\}$ is the inclusion, are homotopic in $\mathbb{C} - \{0\}$, so that $g^{\nu_g}_*((A_{|L})_*(j_*(\gamma))) = g_*(\varphi_*(i_*(\gamma))) \in H_1(\mathbb{C} - \{0\})$ also represents ν_f (we recall that g^{ν_g} is the initial polynomial of g). Since $(A_{|L})_*: H_1(L - \{0\}) \to H_1(A(L) - \{0\})$ and $j_*: H_1(D - \{0\}) \to H_1(L - \{0\})$ are isomorphisms, $(A_{|L})_*(j_*(\gamma))$ is a generator of $H_1(A(L) - \{0\})$, and it thus follows from Lemma 5.2 below (applied to $g^{\nu_g}_*: H_1(A(L) - \{0\}) \to H_1(\mathbb{C} - \{0\})$) that $\nu_f \leq \nu_g$. Since φ^{-1} is also a C^1–diffeomorphism, by symmetry, $\nu_g \leq \nu_f$.

Lemma 5.2 (Ephraim [Ep]). *Let $P \ni 0$ be a real linear subspace in $\mathbb{C}^n$ with real dimension 2, and let h be a (complex) homogeneous polynomial of degree k. Suppose that $h_{|P}$ vanishes only at 0. Then $h_*: H_1(P - \{0\}) \simeq \mathbb{Z} \to H_1(\mathbb{C} - \{0\}) \simeq \mathbb{Z}$ is the multiplication by k' with $|k'| \leq k$.*

Remark (Ephraim [Ep]). The proof of Theorem 5.1 in fact shows that it is enough for φ and φ^{-1} to be homeomorphisms which are differentiable at 0.

In [T1,2,4], Trotman gave another proof of Theorem 5.1. In particular, he uses a different interpretation of the multiplicity: briefly, if L is a line in $\mathbb{C}^n$ such that $L \cap C(V_f) = \{0\}$ (as in Section 2) then, by [W, Chapter 7, Theorem 7P], the multiplicity ν_f is equal to the intersection number at 0 of V_f with L as it is defined by Lefschetz in [Lef, Chapter IV].

6. Topological right–left bilipschitz invariance of the multiplicity

In [RT], Risler–Trotman gave a positive answer to Question 0.1 in the special case where f and g are topologically right–left equivalent by bilipschitz homeomorphisms. This section concerns this result. The main tool in the proof is Theorem 3.1 connecting multiplicity and Łojasiewicz exponent. A generalization due to Comte–Milman–Trotman (cf. [CMT]) of the Risler–Trotman's result is also discussed.

Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be reduced germs of holomorphic functions, V_f, V_g the corresponding germs of hypersurfaces in $\mathbb{C}^n$, ν_f, ν_g the multiplicities at 0 of V_f, V_g respectively, and λ_f, λ_g the Łojasiewicz exponents at 0 of f, g respectively.

Theorem 6.1 (Risler–Trotman [RT]). *Suppose that f and g are topologically right–left equivalent by bilipschitz homeomorphisms, that is, suppose there are germs of bilipschitz homeomorphisms $\varphi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and $\phi\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $f = \phi \circ g \circ \varphi$. Then $\nu_f = \nu_g$.*

We recall that a germ of continuous map $\Phi\colon (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ is said to be *lipschitz* if there is a constant $K > 0$ such that $|\Phi(z) - \Phi(z')| \leq K\,|z - z'|$ for any z, z' near 0. A germ of homeomorphism $\Phi\colon (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ is said to be *bilipschitz* if Φ and Φ^{-1} are lipschitz.

Theorem 6.1 is an immediate corollary of Theorem 3.1 and the following result.

Theorem 6.2 (Risler–Trotman [RT]). *Suppose there is a germ of homeomorphism $\varphi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ sending V_f onto V_g and satisfying the following four conditions:*

(i) $\exists A > 0$ *such that* $|\varphi(z) - \varphi(z')| \leq A\,|z - z'|$, $\forall z \notin V_f$, $\forall z' \in V_f$ *near* 0*;*

(ii) $\exists B > 0$ *such that* $|\varphi^{-1}(z) - \varphi^{-1}(z')| \leq B\,|z - z'|$, $\forall z \notin V_g$, $\forall z' \in V_g$ *near* 0*;*

(iii) $\exists C > 0$ *such that* $|g \circ \varphi(z)/f(z)| \leq C$, $\forall z \notin V_f$ *near* 0*;*

(iv) $\exists D > 0$ *such that* $|f \circ \varphi^{-1}(z)/g(z)| \leq D$, $\forall z \notin V_g$ *near* 0*.*

Then $\lambda_f = \lambda_g$.

The idea of the proof of Theorem 6.2 is as follows. By (ii), $\mathrm{dist}(z, V_f) \leq B\,\mathrm{dist}(\varphi(z), V_g)$ for any z near 0. On the other hand, there exists a constant $K > 0$ such that $\mathrm{dist}(\varphi(z), V_g)^{\lambda_g} \leq K\,|g{\circ}\varphi(z)|$ for any z near 0. Combined with (iii), these two observations show that $\mathrm{dist}(z, V_f)^{\lambda_g} \leq B^{\lambda_g} KC\,|f(z)|$ for any z near 0. The inequality $\lambda_f \leq \lambda_g$ follows. By symmetry, Conditions (i) and (iv) imply $\lambda_g \leq \lambda_f$.

In [CMT], Comte–Milman–Trotman generalized Theorem 6.1 as follows.

Theorem 6.3 (Comte–Milman–Trotman [CMT]). *Suppose there is a germ of homeomorphism $\varphi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ sending V_f onto V_g and satisfying the following two conditions:*

(i) $\exists A, B > 0$ *such that* $A|z| \leq |\varphi(z)| \leq B|z|$ *for all* z *near* 0*;*

(ii) $\exists C, D > 0$ *such that* $C|f(z)| \leq |g \circ \varphi(z)| \leq D|f(z)|$ *for all* z *near* 0*.*

Then $\nu_f = \nu_g$.

By [W, Chapter 1, Lemma 4E], ν_f (respectively ν_g) is the largest number ν such that $|f(z)|/|z|^\nu$ (respectively $|g(z)|/|z|^\nu$) is bounded near 0. So, by Conditions (i) and (ii),

$$\frac{|f(z)|}{|z|^{\nu_g}} \leq \frac{1}{C}\frac{|g(\varphi(z))|}{|\varphi(z)|^{\nu_g}} B^{\nu_g}$$

is bounded near 0, so that $\nu_g \leq \nu_f$. By symmetry, $\nu_g = \nu_f$.

Remark 1. Conditions (iii) and (iv) of Theorem 6.2 imply Condition (ii) of Theorem 6.3. Conditions (i) and (ii) of Theorem 6.2 imply: $\exists A, B > 0$ such that $A|z| \leq |\varphi(z)| \leq B|z|$ for all z near 0, $z \notin V_f$, which is almost Condition (i) of Theorem 6.3.

Remark 2 (Comte–Milman–Trotman [CMT]). If f and g have an *isolated* critical point at the origin and if they are topologically V–equivalent, then there always exists a germ of homeomorphism φ_1: $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ sending V_f onto V_g and satisfying Condition (i) of Theorem 6.3. Indeed, under these hypotheses, the germs f and g are link equivalent (cf. Theorem 4.3). Therefore there exists a germ of homeomorphism φ_1: $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\varphi_1(V_f) = V_g$ and $|\varphi_1(z)| = |z|$ for all z near 0 (cf. Section 4). This implies Condition (i) of Theorem 6.3.

Also, under these hypotheses, we can always find a germ of homeomorphism φ_2: $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ sending V_f onto V_g and satisfying Condition (ii) of Theorem 6.3. Indeed, if f and g have an isolated critical point at 0 and if they are topologically V–equivalent, then g is topologically right equivalent either to f or to $\bar{f}$ (cf. Section 4). Hence there exists a germ of homeomorphism φ_2 with $\varphi_2(V_f) = V_g$ and $|g \circ \varphi_2(z)| = |f(z)|$ for all z near 0. This implies Condition (ii) of Theorem 6.3.

But, given two topologically V–equivalent germs f and g with an isolated critical point at 0, it is still unknown whether there exists a germ of homeomorphism φ: $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ sending V_f onto V_g and satisfying *both* Conditions (i) and (ii) of Theorem 6.3.

7. Plane curve singularities

Let f, g: $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be reduced germs of holomorphic functions, V_f, V_g the corresponding germs of curves in $\mathbb{C}^2$, and ν_f, ν_g the multiplicities at 0 of V_f, V_g respectively.

Theorem 7.1 (Zariski [Z2]). *If f and g are topologically V–equivalent, then $\nu_f = \nu_g$.*

Since the germs are reduced, they have, at worst, an isolated critical point at 0. So, by [D1, Proposition (6.39)] (see also Samuel [Sam]), we can assume that f and g are germs of polynomials. Moreover, by Lemma 1.2, we can assume that these germs of polynomials are (analytically) irreducible. Now, it is well known (cf. [L3]) that the topological V–type [(7)] of such germs is completely characterized by the multiplicity and the *Puiseux exponents* (equivalently, by the sequence of *Puiseux pairs*).

Compare with Teissier [Te2].

Remark. In [A'Ca2], A'Campo gave another proof of Theorem 7.1 by different methods. See also Karras [Ka]. Precisely, Theorem 7.1 is deduced from Theorem 10.1 below as follows. By Lemma 1.2, we can assume that f and g are irreducible. Since the projectivized tangent cone of an irreducible plane curve singularity is a one–point space, the Euler–Poincaré charasteristic of its complement in $\mathbb{CP}^1$ is equal to 1.

8. Semiquasihomogeneous and quasihomogeneous isolated singularities

We start with some basic definitions.

Definition 8.1. Let $w = (w_1, \ldots, w_n)$ be a weight on the coordinates $(z_1, \ldots, z_n)$ by

(7) One says that two germs of holomorphic functions have the same *topological V–type* if they are topologically V–equivalent.

strictly positive integers w_i, $1 \leq i \leq n$. One says that a monomial $a_\alpha\, z^\alpha = a_\alpha\, z_1^{\alpha_1} \dots z_n^{\alpha_n}$, $a_\alpha \in \mathbb{C}$, has *w–degree* d if $\sum_i w_i \alpha_i = d$. Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a polynomial. One says that f is *quasihomogeneous* (or weighted homogeneous), with respect to the weight w, and has w–degree d, if it can be written as a $\mathbb{C}$–linear combination of monomials of w–degree d. In other words, f is quasihomogeneous with weight w and w–degree d if and only if for any $\lambda \in \mathbb{C}$ and any $(z_1, \dots, z_n) \in \mathbb{C}^n$ one has:

$$f(\lambda^{w_1} z_1, \dots, \lambda^{w_n} z_n) = \lambda^d f(z_1, \dots, z_n).$$

One says that f is *semiquasihomogeneous*, with respect to the weight w, and has w–degree d, if f is of the form $f = f' + f''$, where f' is a quasihomogeneous polynomial with weight w and w–degree d having an *isolated* critical point at the origin, and where f'' is a polynomial of *w–order* strictly greater than d (one says that a polynomial f'' has w–order d'' if all its monomials have w–degree greater than or equal to d'' and at least one of them has (exactly) w–degree d'').

Of course, one says that f is *quasihomogeneous* (respectively *semiquasihomogeneous*) if there exists a weight $w = (w_1, \dots, w_n)$ such that f is quasihomogeneous (respectively semiquasihomogeneous) with respect to w. One says that f is *homogeneous* if it is quasihomogeneous with respect to the weight $w = (1, \dots, 1)$.

Concerning germs of holomorphic functions with an isolated singularity at 0, we have the following definition.

Definition 8.2. Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function. We suppose that f has an *isolated* critical point at 0. One says that f is *quasihomogeneous* if it is in the Jacobian ideal of f, that is, $f \in (\partial f/\partial z_1, \dots, \partial f/\partial z_n) \subset \mathbb{C}\{z_1, \dots, z_n\}$. By a theorem of Saito [Sai] (see also [D1, Theorem (7.42)]), this is equivalent to say that, after a biholomorphic change of coordinates, f becomes the germ of a quasihomogeneous polynomial; in other words, f is *analytically* right equivalent to the germ of a quasihomogeneous polynomial ('analytically' means that the homeomorphism φ occuring in Definition 4.1 (iii) is here an *analytic* isomorphism). One says that f is *semiquasihomogeneous* if it is analytically right equivalent to the germ of a semiquasihomogeneous polynomial.

Remark (Lê–Ramanujam [LR] — see also Greuel–Pfister [GP2]). If f is semiquasihomogeneous, then it has an isolated critical point at 0.

The next result due (independently) to Greuel [G] and O'Shea [O'Sh] (see also Trotman [T3,4]) gives equimultiplicity for any μ–constant deformations of a quasihomogeneous isolated hypersurface singularity.

Theorem 8.3 (Greuel [G] and O'Shea [O'Sh] — see also Trotman [T3,4]). *Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a (reduced) germ of holomorphic function with an isolated critical point at the origin.* [8] *We suppose that f is quasihomogeneous. If $F\colon (\mathbb{C}^n \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0)$, $(z, t) \mapsto F(z, t) = F_t(z)$, is a μ–constant deformation of f, then it is equimultiple, that is, $\nu_{F_t} = \nu_{F_0} = \nu_f$ for all t near 0.*

[8] As it is already mentioned in Section 1, notice again that, since $n \geq 2$, the hypothesis of having an isolated critical point at 0 already implies that the germ f is reduced.

Note that, since f has an isolated singularity at 0, the μ–constancy implies that, for any t sufficiently close to 0, the germ F_t has also an isolated singularity at 0, and, consequently, it is automatically reduced.

We recall that a germ F is said to be a *deformation* of f if F is holomorphic, $F_0 = f$ and, for all t near 0, the germ F_t is reduced. A deformation F is said to be *μ–constant* if, for all t near 0, the Milnor number of F_t at 0 is equal to the Milnor number of $F_0 = f$ at 0. A deformation F is called *equimultiple* if $\nu_{F_t} = \nu_f$ for all t near 0. As usual ν_f (respectively ν_{F_t}) is the multiplicity at 0 of $V_f = f^{-1}(0)$ (respectively $V_{F_t} = F_t^{-1}(0)$).

Since the Milnor number is an invariant of the topological V–type (see Introduction, footnote (2)), Theorem 8.3 has the following immediate corollary.

Corollary 8.4. *Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a (reduced) germ of holomorphic function with an isolated critical point at the origin. We suppose that f is quasihomogeneous. If $F\colon (\mathbb{C}^n \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0)$, $(z,t) \mapsto F(z,t) = F_t(z)$, is a topologically V–constant deformation of f, then it is equimultiple.*

Note that, since f has an isolated singularity at 0, the topological V–constancy implies that, for any t sufficiently close to 0, the germ F_t has an isolated singularity at 0 too.

We recall that a deformation F is said to be *topologically V–constant* if, for each t near 0, the germ F_t is topologically V–equivalent to the germ $F_0 = f$.

Remark. By Lê–Ramanujam's theorem [LR], a μ–constant deformation of an isolated hypersurface singularity in $\mathbb{C}^n$ is topologically V–constant, provided $n \neq 3$. Hence, if $n \neq 3$, Corollary 8.4 is equivalent to Theorem 8.3.

The idea of O'Shea's proof of Theorem 8.3 is as follows. By the theorem of Saito [Sai] mentioned in Definition 8.2, we can assume that the germ f is the germ of a polynomial. By hypothesis, there is a weight $w = (w_1, \ldots, w_n)$ such that f is quasihomogeneous with respect to w. Let d be the w–degree of f. By the theory of semiuniversal unfoldings and by a theorem of Varchenko [V], we can assume that the deformation F is of the form

$$F(z,t) = f(z) + \sum_{j=1}^{l} \theta_j(t)\, g_j(z),$$

where $\theta_j\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ and $g_j\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are germs of holomorphic functions such that, for each j, $\theta_j \not\equiv 0$ and all the monomials of g_j have w–degree greater than or equal to d.

For each $1 \leq i \leq n$, let k_i be the least integer such that (up to coefficient) the monomial $z_i^{\alpha_i} z_{k_i}$ appears in the expansion $f = \sum_\alpha a_\alpha\, z^\alpha$ ($k_i = i$ is allowed). Such an integer k_i exists since otherwise every point in the z_i–axis would be a critical point (cf. [Ar]). By renumbering, we can assume $\alpha_1 \leq \ldots \leq \alpha_n$. Let $z_1^{\beta_1} \ldots z_n^{\beta_n}$ be a monomial of $F_t(z)$ and let $d + d'$ be its w–degree, $d' \geq 0$. To prove the theorem, it suffices to show that $\sum_i \beta_i \geq \alpha_1 + 1$. Indeed, if this inequality holds, then $\nu_f = \nu_{F_t} = \alpha_1 + 1$ for all t near 0.

Write $\alpha_i = \alpha_1 + \gamma_i$, $\gamma_i \geq 0$ ($1 \leq i \leq n$). Since $z_i^{\alpha_i} z_{k_i}$ has w–degree d, $w_i\alpha_i + w_{k_i} = d$, that is, $w_i(\alpha_1 + \gamma_i) + w_{k_i} = d$, and thus (multiplying by β_i) $\beta_i w_i(\alpha_1 + \gamma_i) + \beta_i w_{k_i} = \beta_i d$. By taking the sum over all i, one gets:

$$d(\alpha_1 - \sum_i \beta_i) + \alpha_1 d' + \sum_i \beta_i w_i \gamma_i + \sum_i \beta_i w_{k_i} = 0.$$

Since the term $\sum_i \beta_i w_{k_i}$ is > 0 while both the terms $\sum_i \beta_i w_i \gamma_i$ and $\alpha_1 d'$ are ≥ 0, the above equation can hold only if $\sum_i \beta_i > \alpha_1$.

Greuel also uses the theory of semiuniversal unfoldings and Varchenko's theorem, but otherwise his argument is different. In particular, a key point of his proof is the following result which is interesting itself.

Theorem 8.5 (Greuel [G] and Trotman [T3,4]). *Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a (reduced) germ of holomorphic function with an isolated critical point at the origin and let $F\colon (\mathbb{C}^n \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0)$, $(z, t) \mapsto F(z, t) = F_t(z)$, be a μ–constant deformation of f. If F is of the form*

$$F(z, t) = f(z) + \theta(t)\, g(z),$$

where $\theta\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ and $g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are germs of holomorphic functions such that $\theta \not\equiv 0$, then $\nu_g \geq \nu_f$; in particular, $(F_t)_t$ is equimultiple.

More generally, for any μ–constant deformation $(F_t)_t$ of an isolated singularity f, the family $(F_t^{\nu_F})_t$ is equimultiple, where $F_t^{\nu_F}(z) := F^{\nu_F}(z, t) =$ the initial polynomial *in z and t* of $F(z, t)$ (cf. [G], [T3,4]). One must not confuse $F^{\nu_F}(z, t)$ with the initial polynomial *in z* of $F(z, t)$. The proof is based on the valuation test for μ–constant deformations of Lê–Saito [LS] and Teissier [Te1]. Notice that there exist μ–constant families which are not of the form $f(z) + \theta(t)\, g(z)$ (consider the following example by Arnol'd: $z_1^3 + z_2^3 + z_3^3 + z_4^3 + (az_1 + bz_2 + cz_3 + dz_4)^3 + ez_1 z_2 z_3 z_4$).

Remark (Greuel [G] — see also Trotman [T3,4]). In Theorem 8.3 (or Corollary 8.4), we can replace the word 'quasihomogeneous' by 'semiquasihomogeneous'.

In [La], Laufer explained Theorem 8.3 from a different viewpoint in the special case of *surface* singularities in $\mathbb{C}^3$.

Concerning surface singularities, Xu–Yau proved in [Y1] and [XY], the following result for pairs (f, g).

Theorem 8.6 (Xu–Yau [Y1] and [XY]). *Let $f, g\colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be (reduced) germs of holomorphic functions, V_f, V_g the corresponding germs of surfaces in $\mathbb{C}^3$, and ν_f, ν_g the multiplicities at 0 of V_f, V_g respectively. We assume that f and g have an isolated critical point at 0 and that they are quasihomogeneous. If f and g are topologically V–equivalent, then $\nu_f = \nu_g$.*

The fact here is that the topolocical V–types of isolated quasihomogeneous surface singularities in $\mathbb{C}^3$ are well known.

We conclude this section with the following result by Greuel–Pfister [GP1].

Theorem 8.7 (Greuel–Pfister [GP1]). *Consider the germ of holomorphic function $F\colon (\mathbb{C}^n \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0)$, $(z_1, \ldots, z_n, t) \mapsto F(z_1, \ldots, z_n, t) = F_t(z_1, \ldots, z_n)$, given by*

$$F_t(z_1, \ldots, z_n) = G_t(z_1, \ldots, z_{n-1}) + z_n^2\, H_t(z_1, \ldots, z_n),$$

where $G\colon(\mathbb{C}^{n-1}\times\mathbb{C},\{0\}\times\mathbb{C})\to(\mathbb{C},0)$*,* $(z_1,\dots,z_{n-1},t)\mapsto G(z_1,\dots,z_{n-1},t)=G_t(z_1,\dots,z_{n-1})$*, and* $H\colon(\mathbb{C}^n\times\mathbb{C},\{0\}\times\mathbb{C})\to(\mathbb{C},0)$*,* $(z_1,\dots,z_n,t)\mapsto H(z_1,\dots,z_n,t)=H_t(z_1,\dots,z_n)$*, are germs of holomorphic functions, with* $n\geq 3$*. Suppose that, for all* t *near* 0*, the germ* G_t *is reduced and the germ* F_t *has an isolated critical point at* 0 *(in particular* F_t *is reduced and* G_t *has an isolated critical point at* 0*). Also, assume that* G_0 *is semiquasihomogeneous or that* $n=3$*. Finally, suppose that* $F=(F_t)_t$ *is topologically V–constant. Then* $G=(G_t)_t$ *is equimultiple. In particular, if, moreover, for all* t *near* 0*, the multiplicity at* 0 *of the germ* G_t *is less than or equal to the order at* 0 *of the (nonreduced) germ* $(z_1,\dots,z_n)\mapsto z_n^2\,H_t(z_1,\dots,z_n)$*, then* $F=(F_t)_t$ *is equimultiple.*

Theorem 8.7 is a corollary of Theorems 8.3 and 7.1 combined with Lemma 8.8 below. Indeed, by applying Lemma 8.8 to the family $(F_t)_t$, with $L=\{(z_1,\dots,z_n)\in\mathbb{C}^n\ ;\ z_n=0\}$, one gets that $(G_t)_t$ is μ–constant. Now apply Theorem 8.3 or 7.1 according to the case under consideration.

Lemma 8.8 (Greuel–Pfister [GP1]). *Let* $F\colon(\mathbb{C}^n\times\mathbb{C},\{0\}\times\mathbb{C})\to(\mathbb{C},0)$*,* $(z_1,\dots,z_n,t)\mapsto F(z_1,\dots,z_n,t)=F_t(z_1,\dots,z_n)$ *be a germ of holomorphic function such that, for all* t *near* 0*, the germ* F_t *has an isolated critical point at* 0*. Let* $L\simeq\mathbb{C}^{n-1}$ *be a hyperplane in* $\mathbb{C}^n$ *passing through the origin such that, for all* t *near* 0*, the germ* $F_{t|L}$ *has an isolated critical point at* 0 *and such that the polar curve of* $(F_{t|L})_t$ *is equal to the intersection of* $L\times\mathbb{C}$ *with the polar curve of* $(F_t)_t$*. Then, if* $(F_t)_t$ *is* μ*–constant, so is* $(F_{t|L})_t$*.*

We recall that the *polar curve* of a familly $F=(F_t)_t$ of *isolated* hypersurface singularities is the curve singularity in $\mathbb{C}^n\times\mathbb{C}$ defined by the ideal $(\partial F/\partial z_1,\dots,\partial F/\partial z_n)\subset\mathbb{C}\{z_1,\dots,z_n,t\}$.

The proof of the lemma is based again on the valuation test for μ–constant deformations of Lê–Saito [LS] and Teissier [Te1].

9. Convenient Newton nondegenerate (isolated) singularities

Let $f\colon(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a germ of holomorphic function defined by a convergent power series $\sum_\alpha a_\alpha\,z^\alpha$, where $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, $a_\alpha\in\mathbb{C}$, and $z^\alpha=z_1^{\alpha_1}\dots z_n^{\alpha_n}$. The *Newton polyhedron* $\Gamma_+(f;z)$ of f at 0 with respect to the coordinates $z=(z_1,\dots,z_n)$ is the convex hull in $\mathbb{R}^n_+$ of the set

$$\bigcup_{a_\alpha\neq 0}(\alpha+\mathbb{R}^n_+),$$

where $\mathbb{R}^n_+=\{(x_1,\dots,x_n)\in\mathbb{R}^n\ ;\ x_i\geq 0 \text{ for } 1\leq i\leq n\}$. The *Newton boundary* $\Gamma(f;z)$ of f at 0 (with respect to z) is the union of the *compact* faces of the boundary of $\Gamma_+(f;z)$. The polynomial $\sum_{\alpha\in\Gamma(f;z)}a_\alpha\,z^\alpha$ is called the *Newton principal part* of f at 0 (with respect to z). For a face Δ of $\Gamma(f;z)$, one defines the *face function* f_Δ by $f_\Delta(z):=\sum_{\alpha\in\Delta}a_\alpha\,z^\alpha$. One says that f is *nondegenerate* on Δ if the equations

$$\frac{\partial f_\Delta}{\partial z_1}(z)=\dots=\frac{\partial f_\Delta}{\partial z_n}(z)=0$$

have no common solution on $z_1\dots z_n\neq 0$. When f is nondegenerate on every face Δ of $\Gamma(f;z)$, one says that f has a *nondegenerate Newton principal part* (with respect to z). One says that

f is *convenient* (with respect to z) if the intersection of $\Gamma(f;z)$ with each coordinate axis is nonempty, that is, if, for $1 \leq i \leq n$, the monomial $z_i^{\alpha_i}$, $\alpha_i \geq 1$, appears in the expression $\sum_\alpha a_\alpha \, z^\alpha$ with a non–zero coefficient.

For more details about this theory, we refer to Kouchnirenko [Ko] and Oka [O1,2].

Remark (Oka [O2] and Greuel–Pfister [GP2]). If f is convenient and has a nondegenerate Newton principal part, then f has at most an isolated singularity at 0.

The following result by Abderrahmane [Ab] and Saia–Tomazella [ST] gives equimultiplicity for any μ–constant deformation within a family of convenient Newton nondegenerate isolated singularities.

Theorem 9.1 (Abderrahmane [Ab] and Saia–Tomazella [ST]). *Suppose that f has an isolated critical point at 0 (in particular f is reduced) and that it is convenient (with respect to the coordinates z). If $F\colon(\mathbb{C}^n \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0)$, $(z,t) \mapsto F(z,t) = F_t(z)$, is a μ–constant deformation of f such that, for all t near 0, the germ F_t has a nondegenerate Newton principal part (with respect to z), then $F = (F_t)_t$ is equimultiple.*

Remark. As above, since the Milnor number is an invariant of the topological V–type, one can replace 'μ–constant' by 'topologically V–constant'.

The idea of Abderrahmane's proof is as follows. By the theory of semiuniversal unfoldings, we can assume that the deformation F is of the form

$$F(z,t) = f(z) + \sum_{j=1}^{l} \theta_j(t)\, g_j(z),$$

where $\theta_j\colon (\mathbb{C},0) \to (\mathbb{C},0)$ and $g_j\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ are germs of holomorphic functions, $\theta_j \not\equiv 0$. For all $t \neq 0$ close enough to 0, the germs F_t have the same Newton boundary $\Gamma(F_t;z)$. On the other hand, since f is convenient, so is F_t. Let $\alpha^i = (\alpha_1^i, \ldots, \alpha_n^i) \in \mathbb{Z}^n$, $1 \leq i \leq k$, be the vertices of $\Gamma(F_t;z)$, $t \neq 0$. Consider the germ of holomorphic function $G\colon (\mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^k, \{0\} \times \mathbb{C} \times \mathbb{C}^k) \to (\mathbb{C},0)$ defined by

$$G(z,t,s) = G_{t,s}(z) = F(z,t) + \sum_{i=1}^{k} s_i \, z_1^{\alpha_1^i} \ldots z_n^{\alpha_n^i}.$$

Fix t_0 sufficiently close to 0. We are going to see that the family $(G_{t_0,s})_s$ is μ–constant. First, note that it follows from the upper semicontinuity of the Milnor number that $\mu_f \geq \mu_{G_{t,s}}$ for all t, s near 0, where μ_f and $\mu_{G_{t,s}}$ are the Milnor numbers at 0 of f and $G_{t,s}$ respectively. In particular, $G_{t,s}$ has an isolated singularity at 0. On the other hand, since F_{t_0} is convenient and has an isolated singularity at 0, by a theorem of Kouchnirenko [Ko], $\mu_{F_{t_0}} \geq \eta_{F_{t_0}}$, where $\mu_{F_{t_0}}$ is the Milnor number of F_{t_0} at 0 and $\eta_{F_{t_0}}$ is the *Newton number* of F_{t_0} at 0 (9). In fact, since F_{t_0} has

(9) The *Newton number* $\eta_{F_{t_0}}$ of the *convenient* germ F_{t_0} at 0 is defined by $\eta_{F_{t_0}} = \sum_{i=0}^{n} (-1)^{n-i}\, i!\, \mathcal{V}_i$, where $\mathcal{V}_0 = 1$, $\mathcal{V}_n$ is the n–dimensional volume of the compact polyhedron $\Gamma_-(F_{t_0};z)$ which is the cone over $\Gamma(F_{t_0};z)$ with the origin as vertex, and where, for $1 \leq i \leq n-1$, $\mathcal{V}_i$ is the sum of the i–dimensional volumes of the intersections of $\Gamma_-(F_{t_0};z)$ with the coordinates planes of dimension i.

a nondegenerate Newton principal part, one has $\mu_{F_{t_0}} = \eta_{F_{t_0}}$ (cf. [Ko]). Since $(F_t)_t$ is μ–constant, one deduces $\eta_{F_{t_0}} = \mu_f$. Since, for all s close enough to 0, the germ $G_{t_0,s}$ is convenient and has an isolated singularity at 0, the theorem of Kouchnirenko also implies $\mu_{G_{t_0,s}} \geq \eta_{G_{t_0,s}}$. On the other hand, if $t_0 \neq 0$, then $\Gamma_+(G_{t_0,s}; z) = \Gamma_+(F_{t_0}; z)$ for all s near 0. Therefore, by a theorem of Furuya (cf. [F]), $\eta_{G_{t_0,s}} = \eta_{F_{t_0}} = \mu_f$ for all s near 0. In other words, if $t_0 \neq 0$, then $(G_{t_0,s})_s$ is μ–constant. Concerning $(G_{0,s})_s$, since both $G_{t_0,s}$ and $G_{0,s}$ are convenient and $\Gamma_+(G_{0,s}; z) \subset \Gamma_+(G_{t_0,s}; z)$ for all s sufficiently close to 0, the theorem of Furuya says $\eta_{G_{0,s}} \geq \eta_{G_{t_0,s}}$. All together we have $\mu_f \geq \eta_{G_{0,s}} \geq \eta_{G_{t_0,s}} = \mu_f$, that is, $(G_{0,s})_s$ is μ–constant too.

Now, from the μ–constancy of $(G_{0,s})_s$ and the Massey's multiparameter version of the Lê–Saito–Teissier's valuation test for μ–constancy (cf. [M2, Theorem 6.8]), one gets:

$$\sum_{i=1}^{k} |z^{\alpha^i}| \ll \sum_{j=1}^{n} \Big| \frac{\partial f}{\partial z_j}(z) + \sum_{i=1}^{k} s_i \alpha_j^i z_j^{\alpha_j^i - 1} z_1^{\alpha_1^i} \ldots z_{j-1}^{\alpha_{j-1}^i} z_{j+1}^{\alpha_{j+1}^i} \ldots z_n^{\alpha_n^i} \Big|$$

as $(z, s) \to (0, 0)$. Putting $s = 0$ in this relation gives, in particular,

$$|z^{\alpha^i}| \ll \sum_{j=1}^{n} \Big| \frac{\partial f}{\partial z_j}(z) \Big|$$

for each vertex α^i ($1 \leq i \leq k$). Therefore the multiplicity at 0 of $z \mapsto z^{\alpha^i}$ is greater than or equal to ν_f. One deduces $\nu_{F_t} = \nu_f$ for any t sufficiently close to 0.

10. Aligned singularities

The notion of aligned singularities was introduced by Massey in [M2]. Aligned singularities generalize isolated singularities and smooth one–dimensional singularities (in particular, line singularities). Let us recall the definition.

Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function. A *good stratification* for f at 0 is an analytic stratification of the germ V_f such that the smooth part of V_f is a stratum and so that the stratification satisfies Thom's a_f condition with respect to the complement of V_f, that is, if $(p_k)_k$ is a sequence of points in the complement of V_f such that $p_k \to p \in S$, where S is a stratum, and the tangent space $T_{p_k} V_{f - f(p_k)} \to T$, then $T_p S \subset T$. Notice that good stratifications always exist (cf. Hamm-Lê [HL]). An *aligned* good stratification for f at 0 is a good stratification for f at 0 in which the closure of each stratum of the singular set of f is smooth. If such an aligned good stratification exists, and if the dimension (at 0) of the singular locus of f is s, one says that f has an *s–dimensional aligned singularity* at 0. If $\mathcal{S}$ is an aligned good stratification for f at 0, one says that a linear choice of coordinates $z = (z_1, \ldots, z_n)$ is an *aligning set of coordinates* for $\mathcal{S}$ if, for each $1 \leq i \leq n-1$, the $(n-i)$-plane in $\mathbb{C}^n$ defined by $z_1 = \ldots = z_i = 0$ intersects transversely the closure of each stratum of $\mathcal{S}$ of dimension $\geq i$ at the origin. One says that a set of coordinates $z = (z_1, \ldots, z_n)$ is aligning for f at 0 if there exists an aligned good stratification for f at 0 with respect to which z is aligning. Notice that, given an aligned singularity, aligning sets of coordinates are generic (in the inductive pseudo–Zariski topology).

Regarding this class of singularities, Massey proved the following *reduction* theorem.

Theorem 10.1 (Massey [M2]). *The following are equivalent:*

(i) *for all $n \geq 4$, the answer to Question 0.2 is positive for every family $(F_t)_t$ of (reduced) analytic hypersurfaces with isolated singularities (i.e., for all t near 0, F_t has an isolated singularity at 0);*

(ii) *for all $n \geq 4$, there exists an integer s such that the answer to Question 0.2 is positive for every family $(F_t)_t$ of reduced analytic hypersurfaces with s–dimensional aligned singularities (i.e., for all t near 0, F_t has an s–dimensional aligned singularity at 0);*

(iii) *for all $n \geq 4$, for all integer s, the answer to Question 0.2 is positive for every family $(F_t)_t$ of reduced analytic hypersurfaces with s–dimensional aligned singularities.*

In the special case of smooth one–dimensional singularities (in particular line singularities), the result already appears in [M1].

The idea of the proof is as follows.

(iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i) is not difficult. Suppose (ii) is true for some integer $s \geq 1$ and consider a topologically V–constant family $(F_t)_t$ such that each $F_t\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ has an isolated singularity at 0. Then the family $(\tilde{F}_t)_t$, defined by $\tilde{F}_t(z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+s}) = F_t(z_1, \ldots, z_n)$, is topologically V–constant and such that each $\tilde{F}_t$ has an s–dimensional aligned singularity at 0. By hypothesis, it is thus equimultiple. But of course $\nu_{F_t} = \nu_{\tilde{F}_t}$.

To see (i) $\Rightarrow$ (iii), proceed as follows. Suppose (i) is true and consider a topologically V–constant family $(F_t)_t$ such that each $F_t\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ has an s–dimensional aligned singularity at 0, for some integer s. Let $(t_k)_{k \in \mathbb{N}}$ be an infinite sequence of points in $\mathbb{C}$ tending to 0. Let $z = (z_1, \ldots, z_n)$ be an aligning set of coordinates, at 0, for F_0 and for F_{t_k}, for all $k \in \mathbb{N}$ (such a coordinates system always exists by the Baire Category Theorem). Then, since $(F_t)_t$ is a topologically V–constant family of aligned singularities, the Lê numbers (cf. [M2, Definition 1.11]) $\lambda^i_{F_0,z}$ $(0 \leq i \leq n-1)$ of F_0 at 0 with respect to z are equal to the Lê numbers $\lambda^i_{F_{t_k},z}$ of F_{t_k} at 0 with respect to z, for all k large enough (cf. [M2, Corollary 7.8]). Hence, by an inductive application of the Massey's generalized Iomdine–Lê formula (cf. [M2, Theorem 4.5 and Corollary 4.6]), for all integers $j_1, \ldots, j_s$ such that $0 \ll j_1 \ll j_2 \ll \ldots \ll j_s$, the germs $F_0 + z_1^{j_1} + \ldots + z_s^{j_s}$ and $F_{t_k} + z_1^{j_1} + \ldots + z_s^{j_s}$ have an isolated singularity at 0 and the same Milnor number at 0, provided k is large enough(10). In particular, by the upper semicontinuity of the Milnor number, this implies that, for all t sufficiently close to 0, the germ $F_t + z_1^{j_1} + \ldots + z_s^{j_s}$ has an isolated singularity at 0 and the same Milnor number at 0 as $F_0 + z_1^{j_1} + \ldots + z_s^{j_s}$. By Lê–Ramanujam's theorem [LR], since $n \neq 3$, this implies that the family $(F_t + z_1^{j_1} + \ldots + z_s^{j_s})_t$ is topologically V–constant. By hypothesis it is thus equimultiple. As the j_i's are arbitrarily large, $(F_t)_t$ is equimultiple too.

The next result by the author [Ey] extends Greuel–Pfister's Theorem 8.7 (concerning *isolated* singularities) to *aligned* singularities. In addition, the result also answers Zariski's Question 0.2

(10) According to [M2], if we are using the coordinates $(z_1,\ldots,z_n)$ for the germ F_t, we use the rotated coordinates $\tilde{z}=(z_{s+1},z_{s+2},\ldots,z_n,z_1,\ldots,z_s)$ for the germ $F_t+z_1^{j_1}+\ldots+z_s^{j_s}$.

for two other cases.

Theorem 10.2 (Eyral [Ey]). *Consider the germ of holomorphic function $F\colon (\mathbb{C}^n\times\mathbb{C},\{0\}\times\mathbb{C})\to(\mathbb{C},0)$, $(z_1,\dots,z_n,t)\mapsto F(z_1,\dots,z_n,t)=F_t(z_1,\dots,z_n)$, given by*

$$F_t(z_1,\dots,z_n)=G_t(z_1,\dots,z_{n-1})+z_n^2\,H_t(z_1,\dots,z_n),$$

where $G\colon(\mathbb{C}^{n-1}\times\mathbb{C},\{0\}\times\mathbb{C})\to(\mathbb{C},0)$, $(z_1,\dots,z_{n-1},t)\mapsto G(z_1,\dots,z_{n-1},t)=G_t(z_1,\dots,z_{n-1})$, and $H\colon(\mathbb{C}^n\times\mathbb{C},\{0\}\times\mathbb{C})\to(\mathbb{C},0)$, $(z_1,\dots,z_n,t)\mapsto H(z_1,\dots,z_n,t)=H_t(z_1,\dots,z_n)$, are germs of holomorphic functions, with $n\geq 3$. Assume that, for all t near 0, the germs F_t and G_t are reduced and F_t has an s–dimensional aligned singularity at 0. Also suppose that $(F_t)_t$ is topologically V–constant. Let $(t_k)_k$ be an infinite sequence of points in $\mathbb{C}$ tending to 0. Assume that the coordinates $z=(z_1,\dots,z_n)$, or some circular permutation of them, form an aligning set of coordinates at 0 for F_0 and for F_{t_k}, for all $k\in\mathbb{N}$. Finally suppose that at least one of the following four conditions is satisfied:

- (i) *for all t near 0, the germ G_t is convenient and has a nondegenerate Newton principal part with respect to the coordinates $z'=(z_1,\dots,z_{n-1})$;*
- (ii) *for all t near 0, the germ G_t is of the form $G_t(z')=\alpha(z')+\theta(t)\,\beta(z')$, where $\alpha,\beta\colon(\mathbb{C}^{n-1},0)\to(\mathbb{C},0)$ and $\theta\colon(\mathbb{C},0)\to(\mathbb{C},0)$, $\theta\not\equiv 0$, are germs of holomorphic functions;*
- (iii) *G_0 is the germ of a semiquasihomogeneous polynomial with respect to z';*
- (iv) *$n=3$.*

Then $(G_t)_t$ is equimultiple. In particular, if, moreover, for all t near 0, the multiplicity at 0 of the germ G_t is less than or equal to the order at 0 of the (nonreduced) germ $(z_1,\dots,z_n)\mapsto z_n^2\,H_t(z_1,\dots,z_n)$, then $(F_t)_t$ is equimultiple.

The proof of Theorem 10.2 is a combination of Massey's proof of Theorem 10.1 and Greuel–Pfister's proof of Theorem 8.7, together combined with the results of Abderrahmane and Saia–Tomazella (Theorem 9.1) in case (i), Greuel and Trotman (Theorem 8.5) in case (ii), Greuel and O'Shea (Theorem 8.3) in case (iii), and Zariski (Theorem 7.1) in case (iv). Indeed, let $\zeta=(\zeta_1,\dots,\zeta_n)$ be a circular permutation of the coordinates $z=(z_1,\dots,z_n)$. We use the notation $\zeta_p:=z_n$ for the 'special' coordinate z_n. Suppose that ζ is aligning for F_0 and for F_{t_k} at 0, all k. Then, as in the proof of Theorem 10.1, since $(F_t)_t$ is a topologically V–constant family of aligned singularities, one shows that, for all integers $j_1,\dots,j_s$ such that $0\ll j_1\ll j_2\ll\dots\ll j_s$, the family $(F_t+\zeta_1^{j_1}+\dots+\zeta_s^{j_s})_t$ is a μ–constant family of isolated singularities(11). This implies, in particular, that $G_t+\zeta_1^{j_1}+\dots+\zeta_s^{j_s}$, where, if $1\leq p\leq s$, the term $\zeta_p^{j_p}$ is omitted, has an isolated singularity at 0(12) for all small t. Hence, as in the proof of Theorem 8.7, by applying Lemma 8.8 to the family $(F_t+\zeta_1^{j_1}+\dots+\zeta_s^{j_s})_t$, with the hyperplane L in $\mathbb{C}^n$ defined by $\zeta_p=0$, one gets that $(G_t+\zeta_1^{j_1}+\dots+\zeta_s^{j_s})_t$, where, again, if $1\leq p\leq s$, the term $\zeta_p^{j_p}$ is omitted, is also a μ–constant family of isolated singularities. Since the j_i's can be chosen arbitrarily large, we conclude with Theorems 9.1, 8.5, 8.3 and 7.1 according to the case under consideration.

(11) Since we are using the coordinates $\zeta=(\zeta_1,\dots,\zeta_n)$ for the germ F_t, we use the coordinates $\tilde\zeta=(\zeta_{s+1},\zeta_{s+2},\dots,\zeta_n,\zeta_1,\dots,\zeta_s)$ for the germ $F_t+\zeta_1^{j_1}+\dots+\zeta_s^{j_s}$.

(12) For G_t, we use the coordinates $\zeta'=(\zeta_1,\dots,\zeta_n)$, where ζ_p is omitted; for $G_t+\zeta_1^{j_1}+\dots+\zeta_s^{j_s}$, where, if $1\leq p\leq s$, the term $\zeta_p^{j_p}$ is omitted, we use the coordinates $\tilde\zeta'=(\zeta_{s+1},\zeta_{s+2},\dots,\zeta_n,\zeta_1,\dots,\zeta_s)$, where ζ_p is omitted.

In the special case where $s = 0$ (i.e., for isolated singularities), Parts (iii) and (iv) of Theorem 10.2 reduce to Greuel–Pfister's Theorem 8.7 (at least when G_0 is a polynomial and modulo the hypothesis for the coordinates z, or some circular permutation of them, of being aligning).

Theorem 10.2 answers positively Zariski's Question 0.2 for special classes of high–dimensional singularities without any assumption on the topological constancy, that is, without any hypothesis of the type 'embedded C^1 differentiability' (as in Theorem 5.1) or 'embedded right–left bilipschitz property' (as in Theorem 6.1) for example.

Remark. If one replaces the word *semiquasihomogeneous* by *quasihomogeneous* in Theorem 10.2 Part (iii), the argument above does *not* work. Indeed, in this case, $G_0 + \zeta_1^{j_1} + \ldots + \zeta_s^{j_s}$ ($\zeta_p^{j_p}$ is omitted) is neither quasihomogeneous with an isolated singularity nor semiquasihomogeneous, so that we cannot apply Theorem 8.3 of Greuel and O'Shea (we recall that a quasihomogeneous polynomial is *not* semiquasihomogeneous if it has a nonisolated critical point at 0). By contrast, one can replace *semiquasihomogeneous* by *quasihomogeneous* in Theorem 8.7. Indeed, as it is mentioned in this theorem, the hypothesis for the F_t's of having an isolated critical point at 0 automatically implies a similar property for the G_t's and, consequently, if G_0 is quasihomogeneous, then it is necessarily semiquasihomogeneous too. This shows that Theorem 10.2 is not an immediate consequence of Theorems 10.1 and 8.7. Note that we can replace *semiquasihomogeneous* by *quasihomogeneous with an isolated singularity* in Theorem 10.2 Part (iii).

Example (cf. [Ey]). One checks easily that Theorem 10.2 applies to the case $G_t(z_1, z_2) = z_1^2+z_2^2+(1-t)\,z_1^3$ and $H_t(z_1, z_2, z_3) = t\,z_2^2$, so that $F_t(z_1, z_2, z_3) = z_1^2+z_2^2+(1-t)\,z_1^3+z_3^2\,t\,z_2^2$. Note this example also shows that the special classes of high–dimensional singularities we consider in Theorem 10.2 are not empty.

11. Further results

The following result is a corollary of A'Campo's work [A'Ca2].

Theorem 11.1 (A'Campo [A'Ca2], Lê–Teissier [LT], Karras [Ka]). *Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be (reduced) germs of holomorphic functions with an isolated critical point at* 0*, let V_f, V_g be the corresponding germs of hypersurfaces in $\mathbb{C}^n$ and let $C(V_f)$, $C(V_g)$ be the tangent cones at* 0 *of V_f and V_g respectively. Let $\mathbb{P}C(V_f)$, $\mathbb{P}C(V_g) \subset \mathbb{CP}^{n-1}$ be the projectivized tangent cones at* 0 *of V_f and V_g respectively ($\mathbb{P}C(V_f)$ (respectively $\mathbb{P}C(V_g)$) is the hypersurface in the complex projective space $\mathbb{CP}^{n-1}$ over which $C(V_f)$ (respectively $C(V_g)$) is a cone). Finally, let ν_f and ν_g be the multiplicities at* 0 *of f and g respectively. If f and g are topologically V–equivalent and if the Euler–Poincaré characteristics of $\mathbb{CP}^{n-1} - \mathbb{P}C(V_f)$ and $\mathbb{CP}^{n-1} - \mathbb{P}C(V_g)$ are non–zero, then $\nu_f = \nu_g$.*

It follows from the work by A'Campo [A'Ca2] that if the Euler–Poincaré characteristic of $\mathbb{CP}^{n-1} - \mathbb{P}C(V_f)$ is non–zero then

$$\nu_f = \inf\{m \in \mathbb{N}\ ;\ \Lambda(h^m) \neq 0\},$$

where h^m is the m-th power of the local monodromy h of V_f at 0 and $\Lambda(h^m)$ is the *Lefschetz* *number* of h^m. Since the Lefschetz numbers $\Lambda(h^m)$ are invariants of the topological V–type of the singularity, one gets Theorem 11.1.

Remark (cf. [Ka]). In general, it is even not known whether the Euler–Poincaré characteristic of $\mathbb{CP}^{n-1} - \mathbb{P}C(V_f)$ is an invariant of the topological V–type.

Another corollary of A'Campo's work [A'Ca2] is the following result by Navarro Aznard [Na1,2] concerning multiplicity two surface singularities in $\mathbb{C}^3$.

Theorem 11.2 (Navarro Aznar [Na1,2]). *Let f, g: $(\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be reduced germs of holomorphic functions and let ν_f and ν_g be the multiplicities at the origin of f and g respectively. We suppose that f and g are topologically V–equivalent. If $\nu_f = 2$, so is ν_g.*

Combining [A'Ca1, Théorème 3] with [L1, Proposition], one also gets the following result about nonsingular germs (cf. [Ep] and [GL1,2]).

Theorem 11.3 (A'Campo [A'Ca1] and Lê [L1]). *Let f, g: $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be reduced germs of holomorphic functions and let ν_f and ν_g be the multiplicities at the origin of f and g respectively. We suppose that f and g are topologically V–equivalent. If $\nu_f = 1$, so is ν_g.*

In other words, if a reduced germ of holomorphic function is topologically V–equivalent to a regular (reduced) germ, then it is itself regular.

In [MN], Mendris–Némethi studied isolated surface singularities in $\mathbb{C}^3$ of type $\tilde{f}(z_1, z_2, z_3) = f(z_1, z_2) + z_3^k$, where f is an irreducible plane curve singularity. In particular, they answer positively Zariski's question for such germs. Here is the precise statement of their result.

Theorem 11.4 (Mendris–Némethi [MN]). *Let f: $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an irreducible germ of holomorphic function and let $\tilde{f}$: $(\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a suspension of f in $\mathbb{C}^3$, that is, $\tilde{f}(z_1, z_2, z_3) = f(z_1, z_2) + z_3^k$ for some integer $k \geq 2$ (this implies that $\tilde{f}$ has, at most, an isolated singularity at 0 and, consequently, it is reduced). If g: $(\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is a reduced germ of holomorphic function topologically V–equivalent to $\tilde{f}$, then $\nu_g = \nu_{\tilde{f}}$.*

Concerning isolated surface singularities in $\mathbb{C}^3$, Yau [Y2] proved the following result.

Theorem 11.5 (Yau [Y2]). *Let f, g: $(\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be (reduced) germs of holomorphic functions with an isolated critical point at the origin, V_f, V_g the corresponding germs of surfaces in $\mathbb{C}^3$, and ν_f, ν_g the multiplicities at 0 of V_f, V_g respectively. We suppose that f and g are topologically V–equivalent. If the arithmetic genus of V_f at 0 (cf. [Wa] or [Y2]) is less than or equal to 2, then $\nu_f = \nu_g$.*

Notice that the arithmetic genus of a two–dimensional isolated hypersurface singularity is an invariant of the topological V–type (cf. [Y2]).

12. Multiplicity on high–codimensional analytic sets

Let X be a germ (at the origin) of (pure) d–dimensional closed (complex) analytic subset in $\mathbb{C}^n$. The *multiplicity* of X at 0, denoted by ν_X, is the number of points of intersection, near 0, of X with a generic $(n-d)$–dimensional affine subspace in $\mathbb{C}^n$ passing arbitrarily close to 0 but not through 0.

Already in [Z1] Zariski observed that if X and Y are germs of closed analytic subsets in $\mathbb{C}^n$ of codimension > 1 and φ: $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a germ of homeomorphism sending X onto Y, then the multiplicities ν_X and ν_Y are *not* necessarily the same. Indeed, one can show easily that, given two germs of irreducible curves C, C' in $\mathbb{C}^3$, there always exists a germ of homeomorphism φ: $(\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ sending C onto C' (cf. [Z1], [Te3], [GL1,2]). Nevertheless, in [GL1,2], Gau–Lipman proved the following differential type result which generalizes the theorem by Ephraim and Trotman (cf. Theorem 5.1).

Theorem 12.1 (Gau–Lipman [GL1,2]). *Let X and Y be germs of closed analytic subsets in $\mathbb{C}^n$. Suppose there is a germ of homeomorphism φ: $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, sending X onto Y, such that both φ and φ^{-1} are differentiable at 0. Then $\nu_X = \nu_Y$.*

We also have the following non–differential type result due to Comte concerning bilipschitz homeomorphisms.

Theorem 12.2 (Comte [C]). *Let X and Y be germs of d–dimensional closed analytic subsets in $\mathbb{C}^n$. Suppose there is a germ of homeomorphism φ: $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, sending X onto Y, such that both φ and φ^{-1} are lipschitz maps with lipschitz constants A and B, respectively, satisfying*

$$(1 \leq)\, AB \leq \Big(1 + \frac{1}{\sup(\nu_X, \nu_Y)}\Big)^{1/2d}.$$

Then $\nu_X = \nu_Y$.

We recall that the *lipschitz constant* K of a lipschitz germ Φ: $(\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ is given by $K = \sup\{|\Phi(z) - \Phi(z')|/|z - z'|\ ;\ z \neq z' \text{ near } 0\}$.

In fact, Comte proved a more general result. He does not assume that φ is defined on a neighbourhood of 0 in the ambient space $\mathbb{C}^n$ but only in a neighbourhood of 0 in X. Comte also proved a related result about 'small bilipschitz isotopies' (for details see [C]).

We conclude with the following result by Hironaka [H].

Theorem 12.3 (Hironaka [H]). *Let X be a germ of closed analytic subset in $\mathbb{C}^n$ and let $\mathcal{S}$ be a Whitney stratification of X. Then X is equimultiple along every stratum of $\mathcal{S}$. In particular (cf. [RT]), if F: $(\mathbb{C}^n \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0)$, $(z,t) \mapsto F(z,t) = F_t(z)$, is a deformation of a reduced holomorphic germ f: $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that $F^{-1}(0)$ can be endowed with a Whitney stratification having $\{0\} \times \mathbb{C}$ as a stratum, then $(F_t)_t$ is equimultiple.*

Notice that there exist μ–constant families $F = (F_t)_t$ of isolated hypersurface singularities for which $F^{-1}(0)$ cannot be endowed with a Whitney stratification having $\{0\} \times \mathbb{C}$ as a stratum (cf. Briançon–Speder [BS]).

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