A NOVEL CHARACTERIZATION OF THE IWASAWA DECOMPOSITION OF A SIMPLE LIE GROUP

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This appendix is about (essential) uniqueness of the *Iwasawa (or horospherical) decomposition* $G = KAN$of a semisimple Lie group G. This means:

Theorem 0.1. *Assume that G is a connected Lie group with simple Lie algebra $\mathfrak{g}$. Assume that $G = KL$ for some closed subgroups $K, L < G$ with $K \cap L$ discrete. Then up to order, the Lie algebra $\mathfrak{k}$ of K is maximally compact, and the Lie algebra $\mathfrak{l}$ of L is isomorphic to $\mathfrak{a} + \mathfrak{n}$, the Lie algebra of AN.*

1. General facts on decompositions of Lie groups

For a group G, a subgroup $H < G$ and an element $g \in G$ we define $H^g = gHg^{-1}$.

Lemma 1.1. *Let G be a group and $H, L < G$ subgroups. Then the following statements are equivalent:*

(i) $G = HL$ *and* $H \cap L = \{\mathbf{1}\}$.
(ii) $G = HL^g$ *and* $H \cap L^g = \{\mathbf{1}\}$ *for all* $g \in G$.

Proof. Clearly, we only have to show that $(ii) \Rightarrow (i)$. Suppose that $G = HL$ with $H \cap L = \{\mathbf{1}\}$. Then we can write $g \in G$ as $g = hl$ for some $h \in H$ and $l \in L$. Observe that $L^g = L^h$ and so

$$H \cap L^g = H \cap L^h = H^h \cap L^h = (H \cap L)^h = \{\mathbf{1}\}\ .$$

Moreover we record

$$HL^g = HL^h = HLh = Gh = G\,.$$

□

In the sequel, capital Latin letters will denote real Lie groups and the corresponding lower case fractur letters will denote the associated Lie algebra, i.e. G is a Lie group with Lie algebra $\mathfrak{g}$.

Lemma 1.2. *Let G be a Lie group and $H, L < G$ closed subgroups. Then the following statements are equivalent:*

(i) $G = HL$ *with* $H \cap L = \{\mathbf{1}\}$.
(ii) *The multiplication map*

$$H \times L \to G, \qquad (h, l) \mapsto hl$$

is an analytic diffeomorphism.

Proof. Standard structure theory. □

If G is a Lie group with closed subgroups $H, L < G$ such that $G = HL$ with $H \cap L = \{\mathbf{1}\}$, then we refer to (G, H, L) as a *decomposition triple*.

Lemma 1.3. *Let (G, H, L) be a decomposition triple. Then:*

$$(\forall g \in G) \qquad \mathfrak{g} = \mathfrak{h} + \mathrm{Ad}(g)\mathfrak{l} \qquad \textit{and} \qquad \mathfrak{h} \cap \mathrm{Ad}(g)\mathfrak{l} = \{0\}\,. \tag{1.1}$$

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Proof. In view of Lemma 1.2, the map $H \times L \to G, \quad (h,l) \mapsto hl$ is a diffeomorphism. In particular, the differential at $(\mathbf{1}, \mathbf{1})$ is a diffeomorphism which means that $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$, $\mathfrak{h} \cap \mathfrak{l} = \{0\}$. As we may replace L by L^g, e.g. Lemma 1.1, the assertion follows. □

Question 1. Assume that G is connected. Is it then true that (G, H, L) is a decomposition triple if and only if the algebraic condition (1.1). is satisfied.

Remark 1.4. If the Lie algebra $\mathfrak{g}$ splits into a direct sum of subalgebras $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$, then we cannot conclude in general that $G = HL$ holds. For example, let $\mathfrak{h} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ is a minimal parabolic subalgebra and $\mathfrak{l} = \overline{\mathfrak{n}}$ is the opposite of $\mathfrak{n}$. Then $HL = MAN\overline{N}$ is the open Bruhat cell in G. A similar example is when $\mathfrak{g} = \mathrm{sl}(n, \mathbb{R})$ with $\mathfrak{h}$ the upper triangular matrices and $l = \mathfrak{so}(p, n-p)$ for $0 < p < n$. In this case $HL \subset G$ is a proper open subset. Notice that in both examples condition (1.1) is violated as $\mathfrak{h} \cap \mathrm{Ad}(g)\mathfrak{l} \neq \{0\}$ for appropriate $g \in G$.

2. The case of one factor being maximal compact

Throughout this section G denotes a semi-simple connected Lie group with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Set $K = \exp \mathfrak{k}$ and note that $\mathrm{Ad}(K)$ is maximal compact subgroup in $\mathrm{Ad}(G)$.

For what follows we have to recall some results of Mostow on maximal solvable subalgebras in $\mathfrak{g}$. Let $\mathfrak{c} \subset \mathfrak{g}$ be a Cartan subalgebra. Replacing $\mathfrak{c}$ by an appropriate $\mathrm{Ad}(G)$-conjugate we may assume that $\mathfrak{c} = \mathfrak{t}_0 + \mathfrak{a}_0$ with $\mathfrak{t}_0 \subset \mathfrak{k}$ and $\mathfrak{a}_0 \subset \mathfrak{p}$. Write $\Sigma = \Sigma(\mathfrak{a}, \mathfrak{g}) \subset \mathfrak{a}^* \backslash \{0\}$ for the non-zero $\mathrm{ad}\mathfrak{a}_0$-spectrum on $\mathfrak{g}$. For $\alpha \in \Sigma$ write $\mathfrak{g}^\alpha$ for the associated eigenspcae. Call $X \in \mathfrak{a}_0$ *regular* if $\alpha(X) \neq 0$ for all $\alpha \in \Sigma$. Associated to a regular element $X \in \mathfrak{a}$ we associate a nilpotent subalgebra

$$\mathfrak{n}_X = \bigoplus_{\substack{\alpha \in \Sigma \\ \alpha(X) > 0}} \mathfrak{g}^\alpha .$$

If $\mathfrak{a} \subset \mathfrak{p}$ happens to be maximal abelian, then we will write $\mathfrak{n}$ instead of $\mathfrak{n}_X$.

With this notation we have:

Theorem 2.1. *Let $\mathfrak{g}$ be a semi-simple Lie algebra. Then the following assertions hold:*

(i) *Every maximal solvable subalgebra $\mathfrak{r}$ of $\mathfrak{g}$ contains a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$.*

(ii) *Up to conjugation with an element of $\mathrm{Ad}(G)$ every maximal solvable subalgebra of $\mathfrak{g}$ is of the form*

$$\mathfrak{r} = \mathfrak{c} + \mathfrak{n}_X$$

for some regular element $X \in \mathfrak{a}_0$.

Proof. [3], Theorem 4.1. □

We choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and write $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ for the associated root system. For a choice of positive roots we obtain a unipotent subalgebra $\mathfrak{n}$. Write $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ and fix a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{m}$. Write A, N, T for the analytic subgroups of G corresponding to $\mathfrak{a}$, $\mathfrak{n}$, $\mathfrak{t}$. Notice that $\mathfrak{t} + \mathfrak{a} + \mathfrak{n}$ is a maximal solvable subalgebra by Theorem 2.1.

Lemma 2.2. *Let $L < G$ be a closed subgroup such that $G = KL$ with $K \cap L = \{\mathbf{1}\}$. Then there is an Iwasawa decomposition $G = NAK$ such that*

$$N \subset L \subset TAN \qquad \text{and} \qquad L \simeq LT/T \simeq AN\,. \tag{2.1}$$

Conversely, if L is a closed subgroup of G satisfying (2.1), then $G = KL$ with $K \cap L = \{\mathbf{1}\}$.

Proof. Our first claim is that L contains no non-trivial compact subgroups. In fact, let $L_K \subset L$ be a compact subgroup. As all maximal compact subgroups of G are conjugate, we find a $g \in G$ such that $L_K^g \subset K$. But $L_K^g \cap K \subset L^g \cap K = \{\mathbf{1}\}$ by Lemma 1.1. This establishes our claim.

Next we show that L is solvable. For that let $L = S_L \times R_L$ be a Levi decomposition, where S is semi-simple and R is reductive. If $S \neq \mathbf{1}$, then there is a non-trivial maximal compact subgroup $S_K \subset S$. Hence $S = \mathbf{1}$ by our previous claim and $L = R_L$ is solvable.

Next we turn to the specific structure of $\mathfrak{l}$, the Lie algebra of $\mathfrak{l}$. Let $\mathfrak{r} = \mathfrak{c} + \mathfrak{n}_X$ be a maximal solvable subalgebra of $\mathfrak{g}$ which contains $\mathfrak{l}$. As before we write $\mathfrak{c} = \mathfrak{t}_0 + \mathfrak{a}_0$ for the Cartan subalgebra of $\mathfrak{r}$. . We claim that $\mathfrak{a}_0 = \mathfrak{a}$ is maximal abelian in $\mathfrak{p}$. In fact, notice that $\mathfrak{l} \cap \mathfrak{t}_0 = \{0\}$ and so $\mathfrak{l} \hookrightarrow \mathfrak{r}/\mathfrak{t}_0 \simeq \mathfrak{a}_0 + \mathfrak{n}_X$ injects as vector spaces. Hence

$$\dim \mathfrak{l} = \dim \mathfrak{a} + \dim \mathfrak{n} \leq \dim \mathfrak{a}_0 + \dim \mathfrak{n}_X .$$

But $\dim \mathfrak{a}_0 \leq \dim \mathfrak{a}$ and $\dim \mathfrak{n}_X \leq \dim \mathfrak{n}$ and therefore $\mathfrak{a} = \mathfrak{a}_0$. Hence $\mathfrak{r} = \mathfrak{t} + \mathfrak{a} + \mathfrak{n}$. As $\mathfrak{l} \simeq \mathfrak{r}/\mathfrak{t}$ as vector spaces we thus get hat $L \simeq LT/T \simeq R/T \simeq AN$ as homogeneous spaces. We now show that $N \subset L$ which will follow from $\mathfrak{n} \subset [\mathfrak{l}, \mathfrak{l}]$. For that choose a regular element $X \in \mathfrak{a}$. By what we know already, we then find an element $Y \in \mathfrak{t}$ such that $X + Y \in \mathfrak{l}$. Notice that $\mathrm{ad}(X + Y)$ is invertible on $\mathfrak{n}$ and hence $\mathfrak{n} \subset [X + Y, \mathfrak{n}]$. Finally, observe that

$$[X + Y, \mathfrak{n}] = [X + Y, \mathfrak{r}] = [X + Y, \mathfrak{l} + \mathfrak{t}] = [X + Y, \mathfrak{l}]$$

which concludes the proof of the first assertion of the lemma.

Finally, the second assertion of the lemma is immediate from the Iwasawa decomposition of G. □

3. Manifold decompositions for decomposition triples

Throughout this section G denotes a connected Lie group.

Let (G, H, L) be a decomposition triple and let us fix maximal compact subgroups K_H and K_L of H and L respectively. We choose a maximal compact subgroup K of G such that $K_H \subset K$. As we are free to replace L by any conjugate L^g, we may assume in addition that $K_L \subset K$.

We then have the following fact, see also [4], Lemma 1.2.

Lemma 3.1. *Let (G, H, L) be a decomposition triple. Then (K, K_H, K_L) is a decomposition triple, i.e. the map*

$$K_H \times K_L \to K, \quad (h, l) \mapsto hl$$

is a diffeomorphism.

Before we proof the Lemma we recall a fundamental result of Mostow concernig the topology of a connected Lie group G, cf. [2] If $K < G$ is a maximal compact subgroup of G, then there exists a a vector space V and a homeomorphism $G \simeq K \times V$. In particular G is a deformation retract of K and thus $H_\bullet(G, \mathbb{R}) = H_\bullet(K, \mathbb{R})$.

Proof. As $H \cap L = \{\mathbf{1}\}$, it follows that $K_H \cap K_L = \{\mathbf{1}\}$. Thus compactness of K_L and K_H implies that the map

$$K_H \times K_L \to K, \quad (h, l) \mapsto hl$$

has closed image. It remains to show that the image is open. This will follow from $\dim K_H + \dim K_L = \dim K$. In fact $G \simeq H \times L$ implies that G is homeomorphic to $K_H \times K_L \times V_H \times V_L$ for vector spaces V_H and V_L. Thus

$$H_\bullet(K, \mathbb{R}) = H_\bullet(G, \mathbb{R}) = H_\bullet(K_H \times K_L, \mathbb{R})$$

and Künneth implies for any $n \in \mathbb{N}_0$ that

$$H_n(K, \mathbb{R}) \simeq \sum_{j=0}^{n} H_j(K_H, \mathbb{R}) \otimes H_{n-j}(K_L, \mathbb{R}).$$

Now, for an orientable connected compact manifold M we recall that $H_{\dim M}(M,\mathbb{R}) = \mathbb{R}$ and $H_n(M,\mathbb{R}) = \{0\}$ for $n > \dim M$. Next Lie groups are orientable and we deduce from the Künneth identity from above that $\dim K_H + \dim K_L = \dim K$. This concludes the proof of the lemma. $\square$

Let us write $\mathfrak{k}_\mathfrak{h}$ and $\mathfrak{k}_\mathfrak{l}$ for the Lie algebras of K_H and K_L respectively. Then, as (K, K_H, K_L) is a decomposition triple, it follows from Lemma 1.3 that

$$\mathfrak{k} = \mathfrak{k}_\mathfrak{h} + \mathrm{Ad}(k)\mathfrak{k}_\mathfrak{l} \qquad \text{and} \qquad \mathfrak{h} \cap \mathrm{Ad}(k)\mathfrak{l} = \{0\}\,.$$

Let now $\mathfrak{t}_h \subset \mathfrak{k}_\mathfrak{h}$ be a maximal toral subalgebra and extend it to a maximal torus $\mathfrak{t}$, i.e. $\mathfrak{t}_\mathfrak{h} \subset \mathfrak{t}$. Now pick a maximal toral subalgebra $\mathfrak{t}_\mathfrak{l}$. Replacing $\mathfrak{l}$ by an appropriate $\mathrm{Ad}(K)$-conjugate, we may assume that $\mathfrak{t}_\mathfrak{l} \subset \mathfrak{t}$ (all maximal toral subalgebras in $\mathfrak{k}$ are conjugate). Finally write T, T_H, T_L for the corresponding tori in T.

Lemma 3.2. *If (K, K_H, K_L) is a decomposition triple for a compact Lie group K, then (T, T_H, T_L) is a decomposition triple for the maximal torus T. In particular*

$$\operatorname{rank} K = \operatorname{rank} K_H + \operatorname{rank} K_L\,. \tag{3.1}$$

Proof. We already know that $\mathfrak{t}_\mathfrak{h} + \mathfrak{t}_\mathfrak{l} \subset \mathfrak{t}$ with $\mathfrak{t}_\mathfrak{h} \cap \mathfrak{t}_\mathfrak{l} = \{0\}$. It remain s to verify that $\mathfrak{t}_\mathfrak{h} + \mathfrak{t}_\mathfrak{l} = \mathfrak{t}$. We argue by contradiction. Let $X \in \mathfrak{t}, X \not\in \mathfrak{t}_\mathfrak{h} + \mathfrak{h}_\mathfrak{l}$. As $\mathfrak{k} = \mathfrak{k}_\mathfrak{h} + \mathfrak{k}_\mathfrak{l}$, we can write $X = X_\mathfrak{h} + X_\mathfrak{l}$ for some $X_\mathfrak{h} \in \mathfrak{k}_\mathfrak{h}$ and $X_\mathfrak{l} \in \mathfrak{k}_\mathfrak{l}$.

For a compact Lie algebra $\mathfrak{k}$ with maximal toral subalgebra $\mathfrak{t} \subset \mathfrak{k}$ we recall the direct vector space decomposition $\mathfrak{k} = \mathfrak{t} \oplus [\mathfrak{t}, \mathfrak{k}]$. As $\mathfrak{t}_\mathfrak{h} + \mathfrak{t}_\mathfrak{l} \subset \mathfrak{t}$ we hence may assume that $X_\mathfrak{h} \in [\mathfrak{t}_\mathfrak{h}, \mathfrak{k}_\mathfrak{h}]$ and $X_\mathfrak{l} \in [\mathfrak{t}_\mathfrak{l}, \mathfrak{k}_\mathfrak{l}]$. But then we get

$$X = X_\mathfrak{h} + X_\mathfrak{l} \in [\mathfrak{t}_\mathfrak{h}, \mathfrak{k}_\mathfrak{h}] + [\mathfrak{t}_\mathfrak{l}, \mathfrak{k}_\mathfrak{l}] \subset [\mathfrak{t}, \mathfrak{k}]$$

and therefore $X \in \mathfrak{t} \cap [\mathfrak{t}, \mathfrak{k}] = \{0\}$, a contradiction. $\square$

4. Decompositions of compact Lie groups

Decompositions of compact Lie groups is an algebraic feature as the following Lemma, essentially due to Oniščik, shows.

Lemma 4.1. *Let $\mathfrak{k}$ be a compact Lie algebra and $\mathfrak{k}_1, \mathfrak{k}_2 < \mathfrak{k}$ be two subalgebras. Then the following statements are equivalent:*

(i) $\mathfrak{k} = \mathfrak{k}_1 + \mathfrak{k}_2$ *with* $\mathfrak{k}_1 \cap \mathfrak{k}_2 = \{0\}$

(ii) *Let K, K_1, K_2 be simply connected Lie groups with Lie algebras $\mathfrak{k}, \mathfrak{k}_1$ and $\mathfrak{k}_2$. Write $\iota_i : K_i \to K$, $i = 1, 2$ for the natural homomorphisms sitting over the inclusions $\mathfrak{k}_i \hookrightarrow \mathfrak{k}$. Then the map*

$$m : K_1 \times K_2 \to K, \quad (k_1, k_2) \mapsto \iota_1(k_1)\iota_2(k_2)$$

is a homeomorphism.

Proof. The implication $(ii) \Rightarrow (i)$ is clear. We establish $(i) \Rightarrow (ii)$. We need that m is onto and deduce this from [4], Th. 3.1. Then K becomes a homogeneous space for the left-right action of $K_1 \times K_2$. The stabilizer of $\mathbf{1}$ is given by the discrete subgroup $F = \{(k_1, k_2) : \iota_1(k_1) = \iota_2(k_2)^{-1}$, i.e. $K \simeq K_1 \times K_2/F$. As K_1 and K_2 are simply connected, we conclude that $\pi_1(K) = F$ and thus $F = \{\mathbf{1}\}$ as K is simply connected. $\square$

We now show the main result of this section.

Lemma 4.2. *Let (K, K_1, K_2) be a decomposition triple of a connected compact simple Lie group. Then $K_1 = \mathbf{1}$ or $K_2 = \mathbf{1}$.*

Before we prove this, a few remarks are in order.

Remark 4.3. (a) If K is of exceptional type, then the result can be easily deduced from $\dim K = \dim K_1 + \dim K_2$ and the rank equality $\operatorname{rank} K = \operatorname{rank} K_1 + \operatorname{rank} K_2$, cf. Lemma 3.2. For example if K is of type G_2. Then a non-trivial decomposition $K = K_1 K_2$ must have $\operatorname{rank} K_i = 1$, i.e. $\mathfrak{k}_i = \mathfrak{su}(2)$. But

$$14 = \dim K \neq \dim K_1 + \dim K_2 = 6\,.$$

(b) The assertion of the lemma is not true if we only require $K = K_1 K_2$ and drop $K_1 \cap K_2 = \{\mathbf{1}\}$. For example if K is of type G_2. then $K = K_1 K_2$ with K_i locally SU(3) and $K_1 \cap K_2 = T$ a maximal torus.

Proof. The proof is short, but uses a powerful tool, namely the structure of the cohomology ring of the compact group K. See for instance [5] or [1]. □

Putting matters together this concludes the proof of Theorem 0.1.

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