Bazhanov-Stroganov model from 3D approach

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Abstract. We apply a 3-dimensional approach to describe a new parametrization of the L-operators for the 2-dimensional Bazhanov-Stroganov (BS) integrable spin model related to the chiral Potts model. This parametrization is based on the solution of the associated classical discrete integrable system. Using a 3-dimensional vertex satisfying a modified tetrahedron equation, we construct an operator which generalizes the BS quantum intertwining matrix S. This operator describes the isospectral deformations of the integrable BS model.

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Introduction

The aim of this paper is to describe in detail the interrelation between specific 3-dimensional (3D) and 2-dimensional (2D) integrable lattice spin models. In particular, we shall make use of 3D techniques in order to derive the isospectral transformations of a 2D model, which would be difficult to find directly. The 3D model we shall exploit is the generalized Zamolodchikov-Bazhanov-Baxter (ZBB) model [1, 2, 3] in the vertex formulation [4, 5, 6, 7]. This model describes chirally interacting $\mathbb{Z}_N$-spins placed on the links of a 3D cubic lattice. The corresponding 2D model will be the integrable Bazhanov-Stroganov (BS) model [8], which is related to the integrable Chiral Potts (CP) model [9, 10, 11].

We shall use the approach to the generalized ZBB model developed in [5, 6]. The dynamical variables are affine Weyl pairs $\mathbf{u}_j, \mathbf{w}_j$ which live on the links j of an oriented 3D lattice. The cornerstone of the approach is the explicit construction of a canonical map of the triplet of Weyl pairs on the three incoming links of a vertex to the Weyl pair triplet on the three outgoing links. This map defines the Boltzmann weights and by construction satisfies the tetrahedron equation. The canonical map is uniquely determined postulating a Baxter Z-invariance and a specific linear problem for the Weyl variables.

The generalized ZBB model with $\mathbb{Z}_N$-spins is obtained taking the Weyl variables to commute to the N-th root of unity. Then the Weyl centers $\mathbf{u}_j^N$ and $\mathbf{w}_j^N$ are classical dynamical variables of a classical discrete integrable system of Hirota form which is determined by the canonical mapping and boundary conditions. This system can be solved using standard tools of algebraic geometry, i.e. Θ functions and Fay-identities. Using the rational limit of the Θ-functions this is handled in a practical explicit way.

The Weyl operators at N-th root of unity are represented by $N \times N$ matrices. Then the canonical mapping decomposes into a functional mapping of the centers and a finite-dimensional transformation given by $N^3 \times N^3$ matrix. It was shown in [5] that the matrix element of this matrix coincide with the Boltzmann weights of ZBB model. In this approach the ZBB model appears in the special case that the functional mapping is trivial, i.e. that trivial solutions to the Hirota-type equations are chosen. Non-trivial solutions for the functional mapping mean non-trivial classical dynamics of the Weyl centers, in particular, solitonic solutions. The final aim is to find separated variables.

It is well-known that the few-layer ZBB model, with quasiperiodic boundary conditions in the direction orthogonal to the layers, is related to the integrable chiral Potts model. The main aim of this paper is the explicit construction of various properties of the Bazhanov-Stroganov quantum chain directly from the linear problem and the canonical mapping operator of the generalized ZBB model. The linear problem leads to the BS L-operator. The quantum intertwining operator of the BS model is obtained as the product of two 3D canonical mapping operators. In case of the trivial functional mapping this gives the well-known BS S-matrix. However, this is generalized if non-trivial classical dynamics is taken into account. Intertwining through the whole BS chain leads to isospectrality transforms of the transfer matrix. A special case of the BS model is the relativistic Toda chain, for which isospectral transforms have been constructed already in [14]. An important advantage of the 3D approach to 2D problems is the flexibility regarding the choice of parametrization. The CP parametrization turns out to be less convenient for the dynamical case than a parametrization using simple cross-ratios and rational Θ-functions.

The paper is organized as follows: In section 1 we summarize the main features and formulae of the models considered. Then in section 2 the L-operator and the quantum intertwining relation for the BS-model will be derived using the canonical mapping approach to the ZBB-model. A new parametrization of the BS-intertwining matrix in terms of cross-ratios is introduced. In the following section 3 we introduce a classical counterpart of the BS-model and find the transformation realizing the intertwining of two Lax-operators, using the functional mapping of the 3D vertex ZBB-model. Section 4 starts stating the main result of the paper, the explicit formula for the isospectral transformation of the BS-model. The proof of this proposition is given in the following subsections. Section 5 summarizes the results.

1. The 3D and 2D models considered

We start with a summary of some basic features of the models considered in the later sections. This will also serve to establish the notation.

1.1. Vertex formulation of the generalized ZBB-model

In the vertex formulation of the ZBB-model [4] the quantum variables are attached to the links j of a 3D oriented lattice. They are taken to be elements $(\mathbf{u}_j, \mathbf{w}_j)$ of an ultra-local affine Weyl algebra at root of unity:

$$\mathbf{u}_j \cdot \mathbf{w}_j = \omega\, \mathbf{w}_j \cdot \mathbf{u}_j; \qquad \omega^N = 1; \qquad N \in \mathbb{Z}; \qquad N \geq 2 \tag{1}$$

and $\mathbf{u}_i \cdot \mathbf{w}_j = \mathbf{w}_j \cdot \mathbf{u}_i$ for $i \neq j$. We also attach a scalar variable κ_j to each link j and denote these variables together as

$$\mathfrak{w}_j \;=\; (\mathbf{u}_j, \mathbf{w}_j, \kappa_j)\,. \tag{2}$$

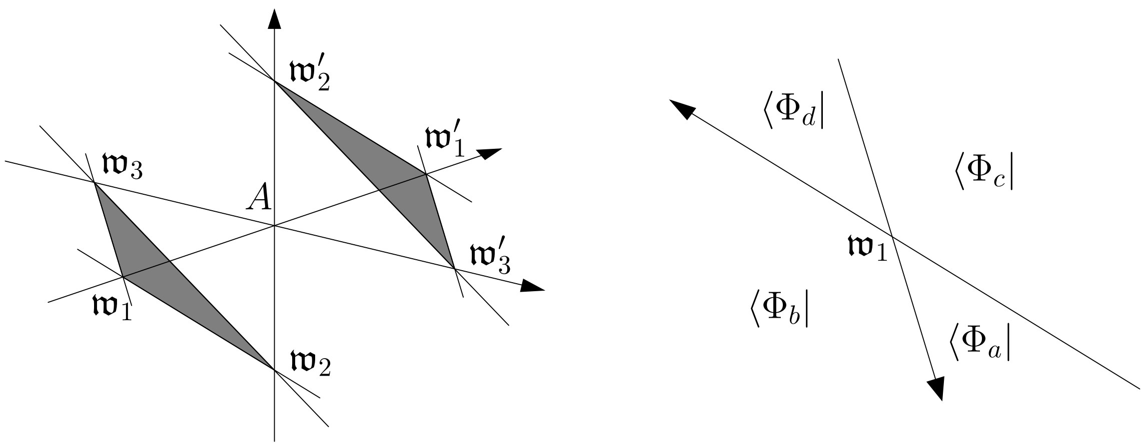

Figure 1. Left: The six links of the basic lattice intersecting in the vertex A, intersected by auxiliary planes (shaded) in two different positions: first passing through $\mathfrak{w}_1,\ \mathfrak{w}_2,\ \mathfrak{w}_3$ and second through $\mathfrak{w}'_1,\ \mathfrak{w}'_2,\ \mathfrak{w}'_3$. The second position is obtained from the first by moving the auxiliary plane parallel through the vertex A. The Weyl variables, elements of $\mathfrak{w}_i,\ \mathfrak{w}'_i$, live on the links of the basic lattice. $\mathcal{R}_{123}$ can be considered to be attached to the vertex A, it maps the left auxiliary triangle onto the upper right one. Right: the auxiliary plane in the neighborhood of $\mathfrak{w}_1$, showing the "co-currents" $\langle\Phi_a|,\ \ldots\ ,\langle\Phi_d|$ in the four sectors cut out by the directed lines $\overrightarrow{\mathbf{w}_2\mathbf{w}_1}$ and $\overrightarrow{\mathbf{w}_3\mathbf{w}_1}$. The Linear Problem relates these four adjacent co-currents according to the values of $\mathfrak{w}_1 = (\mathbf{u}_1, \mathbf{w}_1, \kappa_1)$.

Fig. 1 shows on the left the three Weyl variables $\mathfrak{w}_j$ on the ingoing links of a vertex A and the three variables $\mathfrak{w}'_j$ on the corresponding outgoing links.

In the approach of [5] the basic object of the generalized vertex ZBB-model is the operator $\mathcal{R}_{123}$ mapping canonically the triple affine Weyl algebra on the ingoing links to the corresponding triple Weyl algebra on the outgoing links. This mapping is an invertible rational mapping: For any rational function Ψ of the $\mathbf{u}_1, \ldots, \mathbf{w}_3$, we define

$$(\mathcal{R}_{123} \circ \Psi)\,(\mathbf{u}_1, \mathbf{w}_1, \mathbf{u}_2, \ldots, \mathbf{w}_3) \;\stackrel{\text{def}}{=}\; \Psi(\mathbf{u}'_1, \mathbf{w}'_1, \mathbf{u}'_2, \ldots, \mathbf{w}'_3). \tag{3}$$

In [5, 6], $\mathcal{R}_{123}$ has been determined uniquely postulating a Baxter Z-invariance (lines may be shifted with respect to each other) and a linear relation ("Linear Problem") between the four "co-currents" attached to the four adjacent sectors around each $\mathfrak{w}_j$ in the auxiliary plane. The right hand part of Fig. 1 shows the auxiliary plane in the neighborhood of $\mathfrak{w}_1$ and the corresponding four co-currents. The "Linear Problem" at $\mathfrak{w}_1$ is taken to be

$$0 \;=\; \langle\Phi_a| \;+\omega^{1/2}\langle\Phi_b|\,\mathbf{u}_1 \;+\langle\Phi_c|\,\mathbf{w}_1 \;+\kappa_1\langle\Phi_d|\,\mathbf{u}_1\mathbf{w}_1\,, \tag{4}$$

analogously at all links j. The lines in the auxiliary plane have a direction (we shall not go into the rule here) so that e.g. the co-current appearing in (4) multiplied by $\omega^{1/2}\mathbf{u}_1$ is the one *between* the arrows.

For the 2D auxiliary plane the relation (4) contains analog information as does in the standard 1D quantum chain case the QIS L-operator relation

$$\langle\Phi^{(k)}|\,L^{(k)}(x) \;=\; \langle\Phi^{(k+1)}|\,\ell^{(k)}.$$

Before giving the explicit formula for the mapping operator, we use the fact that for ω a N-th root of unity, the affine Weyl operators (1) can be represented by $N\times N$ matrices. Omitting for a moment the index j (because of the ultralocality the full representation space is just a direct product), we write

$$\mathbf{u}\,\equiv u\,\mathbf{X}\,;\qquad \mathbf{w}\equiv\,w\,\mathbf{Z}\,;\qquad u,\,w\,\in\mathbb{C};\qquad \mathbf{X}\,\mathbf{Z}\;=\;\omega\,\mathbf{Z}\,\mathbf{X}, \tag{5}$$

and shall use the natural basis

$$\mathbf{X}\,|\beta\,\rangle \;=\; \omega^{\beta}\,|\beta\,\rangle\,;\qquad \mathbf{Z}\,|\beta\,\rangle \;=\; |\beta+1\,\rangle\,;\qquad \langle\,\alpha|\beta\,\rangle \;=\; \delta_{\alpha,\beta}\,. \tag{6}$$

Clearly $\mathbf{X}^N \;=\; \mathbf{Z}^N \;=\; 1$. The N-th powers of the Weyl elements are centers and we shall denote them by $U_j,\ W_j$:

$$\mathbf{u}_j^N \;=\; u_j^N \;\equiv\; U_j;\qquad \mathbf{w}_j^N \;=\; w_j^N \;\equiv\; W_j. \tag{7}$$

Now $(\mathbf{u}_j{+}\mathbf{w}_j\,)^N \;= U_j + W_j$, and the mapping $\mathcal{R}_{123}$ implies a purely *functional* mapping $\mathcal{R}^{(f)}_{123}$ of the centers $U_j,\ W_j$, or taking N-th roots, of the $u_j,\ w_j$:

$$\left(\mathcal{R}^{(f)}_{123}\circ\psi\right)(u_1,\,w_1,\,u_2,\,\ldots,\,w_3) \;\stackrel{def}{=}\; \psi(u_1',\,w_1',\,u_2',\,\ldots\,,w_3')\,. \tag{8}$$

The remarkable feature (observed in [22]) arises that $\mathcal{R}_{123}$ decomposes into a matrix conjugation $\mathbf{R}_{123}$ (this is a $N^3\times N^3$-matix) and a purely functional mapping $\mathcal{R}^{(f)}_{123}$:

$$\mathcal{R}_{123}\,\circ\,\Psi \;=\; \mathbf{R}_{123}\,\left(\mathcal{R}^{(f)}_{123}\circ\Psi\right)\,\mathbf{R}^{-1}_{123}\,. \tag{9}$$

The matrix $\mathbf{R}_{123}$ can be given compactly in terms of the Bazhanov-Baxter cyclic functions $w_p(n)$ (not to be confused with w in (5)) which depend on the $\mathbb{Z}_N$-variable n and on a point $p=(x,\,y)$ restricted to a Fermat curve:

$$\frac{w_p(n)}{w_p(n-1)} \;=\; \frac{y}{1\,-\,\omega^n\,x}\,;\qquad x^N\,+\,y^N\;=\;1\,;\qquad n\in\mathbb{Z}_N;\quad n\geq 1;\quad w_p(0)=1\,. \tag{10}$$

The cyclic property $w_p(n+N)=w_p(n)$ is guaranteed by the Fermat curve restriction in (10). The functions $w_p(n)$ are root of unity analogs of q-gamma functions and can be used to develop the theory of the corresponding q-hypergeometric functions (see e.g. [4, 12]). One can show [5, 6, 7, 24] that $\mathbf{R}_{123}$ can be written as a weighted cross-ratio of four of these cyclic functions. In components:

$$\mathbf{R}_{i_1 i_2 i_3}^{j_1 j_2 j_3} \;=\; \delta_{i_2+i_3,j_2+j_3}\omega^{(j_1-i_1)j_3}\frac{w_{p_1}(i_2-i_1)w_{p_2}(j_2-j_1)}{w_{p_3}(j_2-i_1)w_{p_4}(i_2-j_1)}\,. \tag{11}$$

Here $p_i \;=\; (x_i, y_i),\; i \;=\; 1,2,3,4$ are four points on the Fermat curve (10) which are related by the constraint [4]

$$x_1\, x_2 \;=\; \omega\, x_3\, x_4\,. \tag{12}$$

These Fermat points can be expressed in terms of the parameters $u_j,\, w_j,\, \kappa_j,\, u'_j,\, w'_j$ of the initial and final Weyl pairs $\mathfrak{w}_j,\; \mathfrak{w}'_j\; (j=1,2,3)$:

$$\begin{aligned} &x_1 = \omega^{-1/2}\frac{u_2}{\kappa_1 u_1}\;;\quad x_2 = \omega^{-1/2}\frac{\kappa_2 u'_2}{u'_1}\;;\quad x_3 = \omega^{-1}\frac{u'_2}{u_1}\;;\\ &\frac{y_3}{y_1} = \frac{\kappa_1 w_1}{u'_3}\;;\quad \frac{y_4}{y_1} = \omega^{-1/2}\frac{\kappa_3 w_3}{w_2}\,. \end{aligned} \tag{13}$$

${u'_1}^N,\, {u'_2}^N,\, {u'_3}^N$ are related to the initial variables by the functional transformation (8). Defining $K_j \equiv \kappa_j^N$, the mapping $\mathcal{R}_{123}^{(f)}$ is explicitly given by

$$\begin{aligned} \frac{U'_1}{U_1} = \frac{W'_3}{W_3} \;&=\; \frac{K_2 U_2 W_2}{K_1 U_1 W_2 + K_3 U_2 W_3 + K_1 K_3 U_1 W_3}\,;\\ \frac{W_1}{W'_1} = \frac{W'_2}{W_2} \;&=\; \frac{W_1 W_3}{W_1 W_2 + U_3 W_2 + K_3 U_3 W_3}\,;\\ \frac{U'_2}{U_2} = \frac{U_3}{U'_3} \;&=\; \frac{U_1 U_3}{U_2 U_3 + U_2 W_1 + K_1 U_1 W_1}\,. \end{aligned} \tag{14}$$

If we need the u'_j rather than the U'_j, which is the case when we calculate the Fermat points via (13), we have to take N-th roots. The choice of phases is restricted by the fact that the complete mapping $\mathcal{R}_{123}$ leaves the following three products invariant [24]:

$$\mathbf{w}_1\,\mathbf{w}_2 = \mathbf{w}'_1\,\mathbf{w}'_2; \qquad \mathbf{u}_2\,\mathbf{u}_3 = \mathbf{u}'_2\,\mathbf{u}'_3; \qquad \mathbf{u}_1\,\mathbf{w}_3^{-1} = \mathbf{u}'_1{\mathbf{w}'_3}^{-1}. \tag{15}$$

Considering a 3D model of $\mathbb{Z}_N$ spins on the links of the lattice, the $\mathbf{R}_{i_1 i_2 i_3}^{j_1 j_2 j_3}$ can be taken to be the (generally not positive) Boltzmann weights of the vertices. Via the Fermat parameters, these depend on the scalar parameters $u_1,\, u_2, \ldots, \kappa_3$. Each solution of the functional equations gives rise to an integrable 3D model.

It can be seen, that by construction, the mapping $\mathcal{R}_{123}$ satisfies the tetrahedron equation and that $\mathbf{R}_{123}$ solves the modified tetrahedron equation, see e.g. [25].

1.2. Integrable Chiral Potts model

The integrable chiral Potts model (CP) is defined on a 2D lattice with $\mathbb{Z}_N$ spins σ_j attached to the vertices. There are two sets of directed rapidity lines $p,\, p'$, and $q,\, q'$ which cross on the links of the lattice, see Fig. 2. If the edge linking the spins is between the rapidity directions, the pair interaction Boltzmann weight (which depends on both rapidities crossing on the link) is $\overline{W}_{pq}(\sigma_j - \sigma_{j'})$. If the edge is to the right of the rapidity directions, it is $W_{pq}(\sigma_j - \sigma_{j'})$. Such Boltzmann weights which satisfy the star-triangle relation (for the explicit form of R_{pqr} see e.g. [15])

$$\sum_d \overline{W}_{qr}(d-b)W_{pr}(d-a)\overline{W}_{pq}(c-d) \;=\; R_{pqr}\, W_{pq}(b-a)\overline{W}_{pr}(c-b)W_{qr}(c-a) \qquad (16)$$

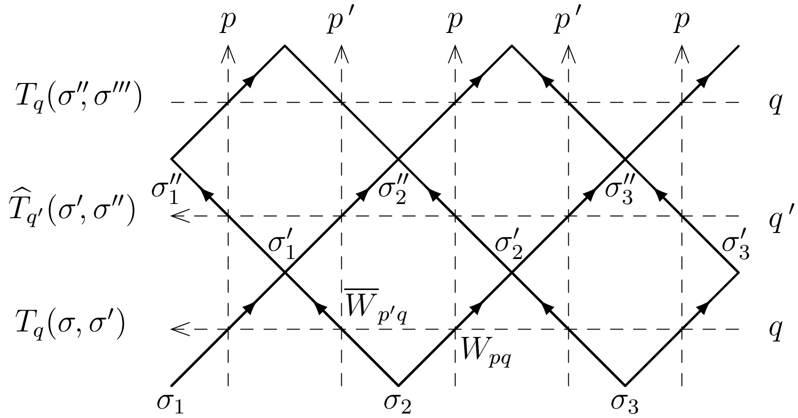

Figure 2. The square diagonal directed lattice with the $\mathbb{Z}_N$-variables at the vertices and the Boltzmann weights W_{pq} on the right pointing links and $\overline{W}_{pq}$ on the left pointing links. The weight corresponding to the line from σ_2 to σ_2' is $W_{pq}(\sigma_2 - \sigma_2')$, analogous for the link from σ_2 to σ_1' it is $\overline{W}_{p'q}(\sigma_2 - \sigma_1')$. Dashed lines are the rapidity lines which indicate the parameter dependence of the Boltzmann weights. For simplicity, we show only the special case of alternating rapidities p, p' and q, q'. There are two different transfer matrices T and $\widehat{T}$.

have been constructed in [9] and are defined by the difference relations ($n \in \mathbb{Z}_N$)

$$\frac{W_{pq}(n)}{W_{pq}(n-1)} \;=\; \left(\frac{\mu_p}{\mu_q}\right)\,\frac{y_q \;-\; \omega^n\, x_p}{y_p \;-\; \omega^n\, x_q}\,; \qquad \frac{\overline{W}_{pq}(n)}{\overline{W}_{pq}(n-1)} \;=\; (\mu_p\,\mu_q)\,\frac{\omega\, x_p \;-\; \omega^n\, x_q}{y_q \;-\; \omega^n\, y_p}\,. \qquad (17)$$

We shall use the normalization $W_{pq}(0) \;=\; \overline{W}_{pq}(0) \;=\; 1$. The parameters appearing in (17) are constrained to the high-genus curve

$$x_q^N + y_q^N \;=\; k\,(x_q^N\, y_q^N \,+\, 1); \qquad k\, x_q^N = 1 - k'\mu_q^{-N}; \qquad k\, y_q^N = 1 - k'\mu_q^N, \qquad (18)$$

(same for x_p, y_p, μ_p) where k and k' are fixed temperature-like parameters related by $k^2 + k'^2 = 1$. The constraints (18) guarantee the cyclic property

$$W_{pq}(n+N) = W_{pq}(n); \qquad \overline{W}_{pq}(n+N) = \overline{W}_{pq}(n)\,. \qquad (19)$$

The star-triangle relations (16) are quite special since the three rapidities involved appear not only as differences as usual, but each separately. Due to this feature, many standard techniques cannot be applied straightforwardly to the CP model. Functional relations involving the BS-model discussed in the following have been crucial for obtaining analytic solutions for the CP-model, see e.g. [17, 18, 19, 20, 21] and references therein.

1.3. Bazhanov-Stroganov model

The CP model has been found to be intimately related to the six-vertex model in a seminal paper by Bazhanov and Stroganov [8]. They first noticed that the twisted six-vertex R-matrix

$$R(\lambda,\nu) \;=\; \begin{pmatrix} \lambda-\omega\nu & 0 & 0 & 0 \\ 0 & \omega(\lambda-\nu) & \lambda(1-\omega) & 0 \\ 0 & \nu(1-\omega) & \lambda-\nu & 0 \\ 0 & 0 & 0 & \lambda-\omega\nu \end{pmatrix}. \tag{20}$$

at root of unity $\omega=e^{2\pi i/N}$ intertwines not only the six-vertex L-operator, but also the following L-operators containing Weyl elements $\mathbf{X}$, $\mathbf{Z}$:

$$L(\lambda,\mathbf{a}) \;=\; \begin{pmatrix} 1+\lambda\, b_1\mathbf{Z}; & \lambda\,\mathbf{X}^{-1}(a_1-b_2\mathbf{Z}) \\ \mathbf{X}(a_2-b_3\mathbf{Z}); & \lambda\, a_1a_2+b_2b_3b_1^{-1}\mathbf{Z} \end{pmatrix}; \quad \mathbf{X}\,\mathbf{Z}=\omega\,\mathbf{Z}\,\mathbf{X}\,, \tag{21}$$

where $\lambda,\ a_1,\ \ldots\ ,\ b_3\ \in\mathbb{C}$. We collectively denote the parameters $a_1,\ldots,\ b_3$ by $\mathbf{a}$. In the representation (6) the intertwining relation is

$$\sum_{j_1,j_2,\beta} R_{i_1j_1,i_2j_2}(\lambda,\nu)L^{\alpha_1\beta}_{j_1k_1}(\lambda,\mathbf{a})\;L^{\beta\alpha_2}_{j_2k_2}(\nu,\mathbf{a}) \;=\; \sum_{j_1,j_2,\beta} L^{\alpha_1\beta}_{i_2j_2}(\nu,\mathbf{a})\;L^{\beta\alpha_2}_{i_1j_1}(\lambda,\mathbf{a})R_{j_1k_1,j_2k_2}(\lambda,\nu). \tag{22}$$

where greek indices run over the values $0,1,\ldots,N-1$ and the latin indices take the values $0,1$.

Moreover, Bazhanov and Stroganov found that there is also an intertwining relation with respect to the $\mathbb{Z}_N$ (greek) indices, i.e. in the Weyl quantum space, if the parameters $\mathbf{a}$ are chosen as

$$a_1=x_q;\qquad a_2=\omega x_{q'};\qquad b_1=\frac{y_qy_{q'}}{\mu_q\mu_{q'}};\qquad b_2=\frac{y_{q'}}{\mu_q\mu_{q'}};\qquad b_3=\frac{y_q}{\mu_q\mu_{q'}}\,, \tag{23}$$

where the x_q, y_q, μ_q etc. satisfy the CP conditions (18) with fixed k. Writing (21) with the parameters (23) as $L(\lambda;\,q,q')$, we get

$$L(\lambda;\,q,q')=\begin{pmatrix} 1\,+\,\lambda\dfrac{y_qy_{q'}}{\mu_q\mu_{q'}}\,\mathbf{Z} & \lambda\,\mathbf{X}^{-1}\left(x_q-\dfrac{y_{q'}}{\mu_q\mu_{q'}}\,\mathbf{Z}\right) \\ \mathbf{X}\left(\omega x_{q'}-\dfrac{y_q}{\mu_q\mu_{q'}}\,\mathbf{Z}\right) & \lambda\,\omega\,x_qx_{q'}+\dfrac{1}{\mu_q\mu_{q'}}\,\mathbf{Z} \end{pmatrix}. \tag{24}$$

Apart from the spectral parameter λ, this $L(\lambda;\,q,q')$ depends on three independent continous variables, e.g. x_q, $x_{q'}$ and the modulus k. We shall not write the latter explicitly as an argument. The quantum space intertwining relation is

$$\begin{aligned}\sum_{\beta_1\beta_2,k} \mathsf{S}_{\alpha_1\alpha_2;\beta_1\beta_2}(p,p',q,q')\;L^{\beta_1\gamma_1}_{i_1k}(\lambda;p,p')\;L^{\beta_2,\gamma_2}_{k\,i_2}(\lambda;q,q')\\ =\sum_{\beta_1\beta_2,k} L^{\alpha_2\beta_2}_{i_1k}(\lambda;q,q')\;L^{\alpha_1\beta_1}_{k\,i_2}(\lambda;p,p')\;\;\mathsf{S}_{\beta_1,\beta_2;\gamma_1,\gamma_2}(p,p',q,q').\end{aligned} \tag{25}$$

The matrix S turns out to be the product of four CP-Boltzmann weights (17):

$$\mathsf{S}_{\alpha_1\alpha_2,\beta_1\beta_2}(p,p',q,q') \;=\; W_{pq'}(\alpha_1-\alpha_2)W_{p'q}(\beta_2-\beta_1)\overline{W}_{pq}(\beta_2-\alpha_1)\overline{W}_{p'q'}(\beta_1-\alpha_2). \tag{26}$$

One can verify the relations (25), (26) by explicit calculations, using (6) and (17) several times, e.g.

$$\sum_{\beta_1}\mathbf{Z}_{\alpha_1\beta_1}\mathsf{S}_{\beta_1,\beta_2;\gamma_1,\gamma_2}=\mathsf{S}_{\alpha_1-1,\beta_2;\gamma_1,\gamma_2}=\mu_q\mu_{q'}\,\frac{y_p-\omega^{\alpha_1-\alpha_2}x_{q'}}{y_{q'}-\omega^{\alpha_1-\alpha_2}x_p}\;\frac{\omega x_p-\omega^{\beta_2-\alpha_1+1}x_q}{y_q-\omega^{\beta_2-\alpha_1+1}y_p}\;\mathsf{S}_{\alpha_1,\beta_2;\gamma_1,\gamma_2}\,.$$

The Bazhanov-Stroganov periodic quantum chain of length Q is defined via its L-operator $L_{ik}^{\alpha\beta}(\lambda;\,q,q')$ given in (24) and the corresponding monodromy matrix

$$\mathsf{M}\left(\lambda,\{q_i,q_i'\}_{i=0}^{Q-1}\right)\;=\;L(\lambda;q_0,q_0')\,L(\lambda;q_1,q_1')\,L(\lambda;q_2,q_2')\;\ldots\;L(\lambda;q_{Q-1},q_{Q-1}'), \tag{27}$$

where each L has its pair of rapidities q,q' and all these rapidities may be different while keeping the Baxter modulus k to be the same for all L. The transfer matrix is

$$\mathbf{t}\;=\;\mathrm{Tr}_{\mathbb{C}^2}\,\mathsf{M}\,. \tag{28}$$

Baxter [18, 19] calls this model the $\tau_2(t_q)$-model after the notation τ_2 introduced for the transfer matrix in equation (5.33) of [8]. Mostly in Baxter's work not the fully inhomogenous model is used, but the rapidities take two alternating values. In (A.7) we give the relation to Baxter's notation. Fusion of the BS-transfer matrices leads to the functional relations mentioned in the last subsection.

2. 3D approach to the BS model

2.1. L-operator

We now show how the BS-L-operator can be obtained from the Linear Problem (4) of the 3D approach, imposing a periodicity condition. We follow the general argument of [23], which also gives a quantum group background of the construction.

Consider the domain of the auxiliary plane containing the four variables $\mathfrak{w}_1,\ \tilde{\mathfrak{w}}_1,\ \mathfrak{w}_2,\ \tilde{\mathfrak{w}}_2$, see Fig. 3. The co-currents around each Weyl variable are taken to be related by the Linear Problem (4). We impose the periodicity condition

$$\langle\psi_{-1}|=\xi\,\langle\psi_1|\,;\qquad\qquad\langle\phi_{-1}|=\xi\langle\phi_1|\,;\qquad\qquad\langle\chi_{-1}|=\xi\langle\chi_1| \tag{29}$$

in the vertical direction, with ξ a quasi-momentum. Then the conditions (4) at $\mathfrak{w}_1$ and $\tilde{\mathfrak{w}}_1$ (those in the left hand dotted box of Fig. 3 denoted L_1) are

$$\begin{aligned}0\;&=\;\langle\,\psi_0|+\xi\omega^{1/2}\langle\,\psi_1|\tilde{\mathbf{u}}_1+\langle\,\phi_0|\tilde{\mathbf{w}}_1+\xi\tilde{\kappa}_1\langle\,\phi_1|\tilde{\mathbf{u}}_1\tilde{\mathbf{w}}_1\;,\\0\;&=\;\langle\,\psi_1|+\;\omega^{1/2}\langle\,\psi_0|\mathbf{u}_1+\langle\,\phi_1|\mathbf{w}_1\;+\;\kappa_1\langle\,\phi_0|\mathbf{u}_1\mathbf{w}_1\;.\end{aligned} \tag{30}$$

These linear relations can be rewritten in matrix form as follows

$$\langle\,\psi|\,(\omega\,\xi\,\mathbf{u}_1\,\tilde{\mathbf{u}}_1\;-\;1)\tilde{\mathbf{w}}_1^{-1}=\langle\,\phi|\cdot L_1(\xi)\,, \tag{31}$$

where $\langle\,\phi|$ and $\langle\,\psi|$ denote the rows $(\langle\,\phi_0|,\langle\,\phi_1|)$ and $(\langle\,\psi_0|,\langle\,\psi_1|)$, respectively, and $L_1(\xi)$ is the following 2×2 matrix with operator valued elements

$$L_1(\xi)\;=\;\begin{pmatrix}1-\omega^{1/2}\xi\mathbf{u}_1\tilde{\mathbf{u}}_1\kappa_1\mathbf{w}_1\tilde{\mathbf{w}}_1^{-1} & -\mathbf{u}_1\left(\omega^{1/2}-\kappa_1\mathbf{w}_1\tilde{\mathbf{w}}_1^{-1}\right)\\ & \\ \xi\tilde{\mathbf{u}}_1\left(\tilde{\kappa}_1-\omega^{1/2}\mathbf{w}_1\tilde{\mathbf{w}}_1^{-1}\right) & -\omega^{1/2}\xi\mathbf{u}_1\tilde{\mathbf{u}}_1\tilde{\kappa}_1+\mathbf{w}_1\tilde{\mathbf{w}}_1^{-1}\end{pmatrix}. \tag{32}$$

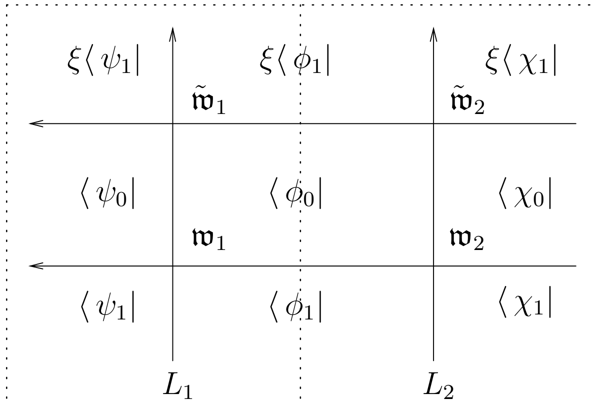

Figure 3. Piece of the auxiliary plane which corresponds to the product of two L-operators. The Weyl pairs together with parameters κ_1, κ_2, $\tilde{\kappa}_1$, $\tilde{\kappa}_2$ are associated with the corresponding vertices in this plane.

We want to use a matrix representation for the Weyl elements. Observing that only the three elements $\mathbf{w}_1\tilde{\mathbf{w}}_1^{-1}$, $\mathbf{u}_1$, $\tilde{\mathbf{u}}_1$ appear, we may use a N-dimensional representation, writing

$$\mathbf{w}_1\tilde{\mathbf{w}}_1^{-1} = \frac{w_1}{\tilde{w}_1}\,\mathbf{Z}\,; \qquad \mathbf{u}_1 = u_1\,\mathbf{X}\,; \qquad \tilde{\mathbf{u}}_1 = \tilde{u}_1\,\mathbf{X}^{-1} \tag{33}$$

in the basis (6). In order to bring this into a form comparable with (24) we put

$$\kappa_1 = \omega^{1/2}\frac{x_q}{y_{q'}}; \qquad \tilde{\kappa}_1 = \omega^{-1/2}\frac{y_q}{x_{q'}}; \qquad \frac{w_1}{\tilde{w}_1} = \omega^{-1}\frac{y_q y_{q'}}{x_q x_{q'}\mu_q\mu_{q'}}\,, \tag{34}$$

so that

$$L_1(\xi) \;=\; \begin{pmatrix} 1-\xi u_1\tilde{u}_1\dfrac{y_q}{x_{q'}\mu_q\mu_{q'}}\mathbf{Z}\,; & -u_1\mathbf{X}\left(\omega^{1/2}-\omega^{-1/2}\dfrac{y_q}{x_{q'}\mu_q\mu_{q'}}\mathbf{Z}\right) \\ \xi\,\tilde{u}_1\omega^{-1/2}\dfrac{y_q}{x_{q'}}\mathbf{X}^{-1}\left(1-\dfrac{y_{q'}}{x_q\mu_q\mu_{q'}}\mathbf{Z}\right); & -\xi u_1\tilde{u}_1\dfrac{y_q}{x_{q'}}+\dfrac{y_q y_{q'}}{\omega x_q x_{q'}\mu_q\mu_{q'}}\mathbf{Z} \end{pmatrix}. \tag{35}$$

Conjugating with the Pauli matrix σ_2 and introducing a new spectral parameter λ by

$$\lambda \;=\; -\,\frac{1}{\omega\,u_1\tilde{u}_1 x_q\,y_q\,\xi} \tag{36}$$

we obtain

$$\sigma_2\,L_1(\xi)\,\sigma_2 = \frac{1}{\lambda\,\omega\,x_q x_{q'}}\begin{pmatrix} 1\,+\,\lambda\dfrac{y_q y_{q'}}{\mu_q\mu_{q'}}\mathbf{Z}\,; & \dfrac{\mathbf{X}^{-1}}{\omega^{1/2}u_1 x_q}\left(x_q-\dfrac{y_{q'}}{\mu_q\mu_{q'}}\mathbf{Z}\right) \\ \lambda\,\omega^{1/2}u_1x_q\mathbf{X}\left(\omega\,x_{q'}-\dfrac{y_q}{\mu_q\mu_{q'}}\mathbf{Z}\right)\;; & \lambda\,\omega\,x_q x_{q'}+\dfrac{1}{\mu_q\mu_{q'}}\mathbf{Z} \end{pmatrix}. \tag{37}$$

A gauge transformation with

$$P_1 \;=\; \begin{pmatrix} \sqrt{\omega^{1/2}u_1x_q\lambda} & 0 \\ 0 & 1/\sqrt{\omega^{1/2}u_1x_q\lambda} \end{pmatrix} \tag{38}$$

leads to

$$P_1\,\sigma_2\,L_1(\xi)\,\sigma_2\,P_1^{-1} \;=\; \frac{1}{\lambda\,\omega\,x_q\,y_q}\,L(\lambda;\,q,\,q') \tag{39}$$

with $L(\lambda;\,q,\,q')$ defined in (24). We shall see in (58) that the identification (34) will also appear when we express the BS intertwining matrix S within the 3D framework. Using the parametrization introduced later in section 2.4 we will show in (70) that considering the monodromy matrix, the factor $u_1\tilde{u}_1 x_q y_q$ relating the spectral parameters ξ and λ does not depend on the site considered. The same holds for the combination $u_1 x_q$ appearing in (38). So the BS-monodromy (27) can be written as

$$\mathsf{M}\left(\lambda,\{q_i,q_i'\}_{i=0}^{Q-1}\right) \;=\; P_1\sigma_2\,L_0(\xi)\,L_1(\xi)\,\ldots\,L_{Q-1}(\xi)\,\sigma_2 P_1^{-1}\,\prod_{i=0}^{Q-1}(\lambda\,\omega\,x_{q_i}y_{q_i})\,. \tag{40}$$

We finally remark that demanding periodicity (29) not after *two* vertical steps as done here, but after N steps, one obtains $N\times N$ L-matrices, see [23].

2.2. 3D interpretation of the relation $\mathsf{S}LL = LL\mathsf{S}$.

Consider the product of the successive action two L-operators (the simplest monodromy)

$$\langle\,\psi|\,(\omega\,\xi\,\mathbf{u}_2\,\tilde{\mathbf{u}}_2\,-1)\,(\omega\,\xi\,\mathbf{u}_1\,\tilde{\mathbf{u}}_1\,-1)\,\tilde{\mathbf{w}}_2^{-1}\tilde{\mathbf{w}}_1^{-1} \;=\langle\,\chi|\cdot L_2(\xi)\,L_1(\xi)\,. \tag{41}$$

We are interested in the relation of the action of $L_2(\xi)\,L_1(\xi)$ to the action of $L_1(\xi)\,L_2(\xi)$. In the 3D approach, $\mathcal{R}_{123}$ maps a triangle into a reflected one, recall the left hand picture of Fig. 1. Let us consider the four Weyl operators in the auxiliary plane as in Fig. 3 (the scalar κ_j is included in $\mathfrak{w}_j$, see (2)) and a further variable $\mathfrak{w}_3$, see the bottom auxiliary plane in Fig. 4.

We consider the special case of $\mathcal{R}_{123}$ mapping the triple Weyl algebra $(\,\mathfrak{w}_1,\,\mathfrak{w}_2,\,\mathfrak{w}_3)$ into $(\,\mathfrak{w}_1',\,\mathfrak{w}_2',\,\mathfrak{w}_3')$ with the functional part of the transformation $\mathcal{R}_{123}^{(f)}$ taken to be trivial (Bazhanov-Baxter case): $u_i'=u_i,\;\; w_i'=w_i,\;\; i=1,2,3$. We act with a similar mapping $\widetilde{\mathcal{R}}_{\widetilde{1}\widetilde{2}3}$ on the initial triple Weyl algebra $(\,\widetilde{\mathfrak{w}}_1,\,\widetilde{\mathfrak{w}}_2,\,\mathfrak{w}_3')$ and obtain the triple algebra $(\,\widetilde{\mathfrak{w}}_1',\,\widetilde{\mathfrak{w}}_2',\,\mathfrak{w}_3'')$. As in Fig. 3 we demand periodicity after the second step in the third direction: $\mathfrak{w}_3''=\mathfrak{w}_3$. Fig. 4 illustrates these mappings.

In the upper auxiliary plane the action of the operators L_1 and L_2 appears in the reversed order if we keep track of the direction of the lines.

So, taking into account the property (9) we expect an intertwining relation of the form (25), with S being a bilinear expression in $\mathbf{R}_{123}$ and $\widetilde{\mathbf{R}}_{\widetilde{1}\widetilde{2}3}$.

2.3. BS matrix S *from ZBB model*

In order to derive the precise relation, let us calculate the trace of the product of two matrices $\mathbf{R}$ and $\widetilde{\mathbf{R}}$, both of the form (11), but the first with the Fermat parameters $p_i=(x_i,y_i)$, the second with $\widetilde{p}_i=(\widetilde{x}_i,\widetilde{y}_i)\,,\;\; i=1,2,3,4\,,$ each satisfying the restriction (12):

$$\mathsf{S}_{i_1i_2,\ell_1\ell_2}^{j_1j_2,k_1k_2} = \sum_{m,n\in\mathbb{Z}_N}\mathbf{R}_{i_1i_2m}^{j_1j_2n}\;\;\widetilde{\mathbf{R}}_{\ell_1\ell_2n}^{k_1k_2m}\,. \tag{42}$$

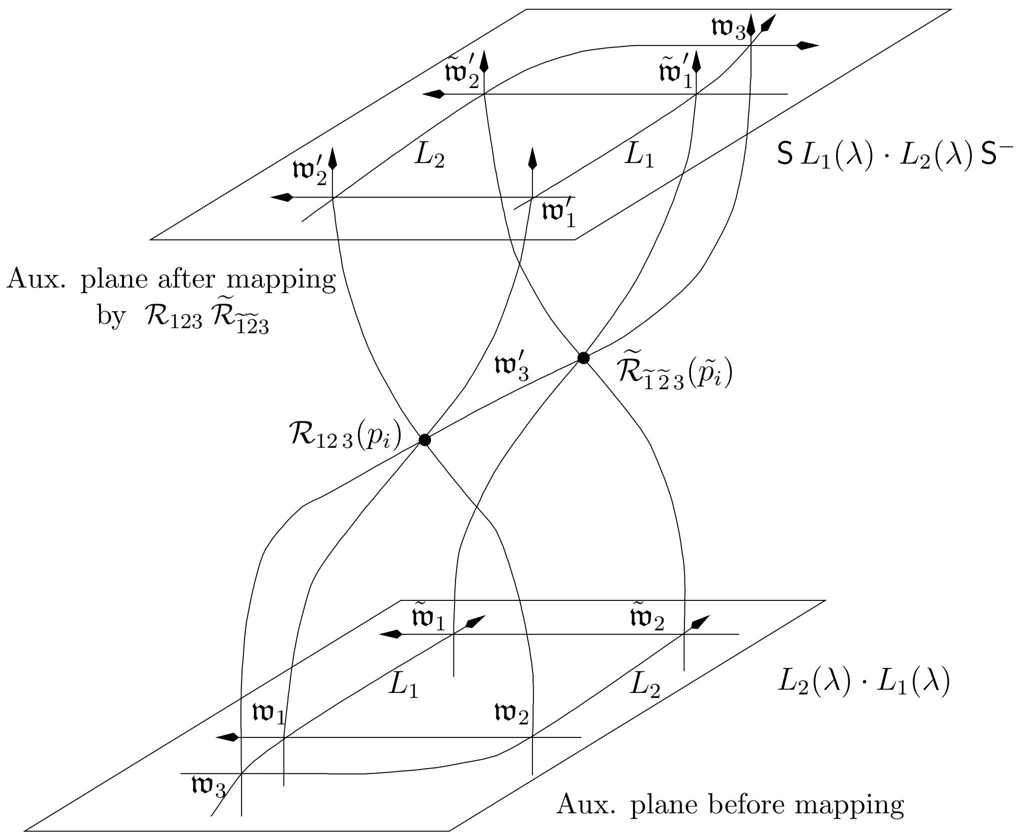

Figure 4. Graphical image of the intertwining relation for the BS model showing the origin of the Bazhanov-Stroganov intertwining matrix S. The two elements $\mathfrak{w}_1$ and $\tilde{\mathfrak{w}}_1$ form the operator $L_1(\lambda)$, the elements $\mathfrak{w}_2$ and $\tilde{\mathfrak{w}}_2$ the operator $L_2(\lambda)$, see the earlier Fig. 3. Periodicity in the third direction gives that the two $\mathfrak{w}_3$ at the top right and bottom left are the same. The two points where $\mathcal{R}_{123}$ and $\widetilde{\mathcal{R}}_{\tilde{1}\tilde{2}3}$ act are vertices of the physical 3-dim lattice.

Actually, we shall not need the full $N^4 \times N^4$ matrix (42) but only those matrix elements in (42) with

$$i_1 + \ell_1 = i_2 + \ell_2 = j_1 + k_1 = j_2 + k_2 = 0\,, \tag{43}$$

forming a $N^2 \times N^2$ matrix. Inserting the explicit expressions (11) and renaming the discrete variables we obtain

$$\mathsf{S}_{\alpha_1\alpha_2}^{\beta_1\beta_2} = \omega^{(\alpha_1-\beta_1)(\beta_2-\alpha_2)}\; \frac{w_{p_1}(\alpha_2-\alpha_1)w_{\tilde{p}_1}(\alpha_1-\alpha_2)}{w_{p_3}(\beta_2-\alpha_1)w_{\tilde{p}_3}(\alpha_1-\beta_2)}\; \frac{w_{p_2}(\beta_2-\beta_1)w_{\tilde{p}_2}(\beta_1-\beta_2)}{w_{p_4}(\alpha_2-\beta_1)w_{\tilde{p}_4}(\beta_1-\alpha_2)}\,. \tag{44}$$

In order to rewrite (44) in the form (26) we use the following property of the cyclic functions $w_p(n)$‡:

$$w_p(n) = \frac{1}{w_{Op}(-n)\;\Phi(n)}\,, \tag{45}$$

where O is an automorphism of the Fermat curve such that

$$p = (x, y) \mapsto Op = (\omega^{-1}x^{-1}, \omega^{-1/2}x^{-1}y) \tag{46}$$

‡ Relation (45) is an analog of well known relation $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$, see e.g. [4, 12].

and

$$\Phi(n) = (-)^n \omega^{n^2/2}\,. \tag{47}$$

So we get

$$\begin{aligned} \mathsf{S}_{\alpha_1\alpha_2}^{\beta_1\beta_2} &= \omega^{(\alpha_1-\beta_1)(\beta_2-\alpha_2)} \frac{\Phi(\alpha_1-\beta_2)\Phi(\alpha_2-\beta_1)}{\Phi(\alpha_1-\alpha_2)\Phi(\beta_2-\beta_1)} \\ &\times \frac{w_{\tilde{p}_1}(\alpha_1-\alpha_2)}{w_{Op_1}(\alpha_1-\alpha_2)} \frac{w_{p_2}(\beta_2-\beta_1)}{w_{O\tilde{p}_2}(\beta_2-\beta_1)} \frac{w_{O\tilde{p}_3}(\beta_2-\alpha_1)}{w_{p_3}(\beta_2-\alpha_1)} \frac{w_{Op_4}(\beta_1-\alpha_2)}{w_{\tilde{p}_4}(\beta_1-\alpha_2)} \end{aligned} \tag{48}$$

Now we can identify

$$\begin{aligned} W_{pq'}(\alpha_1-\alpha_2) &\equiv \frac{w_{\tilde{p}_1}(\alpha_1-\alpha_2)}{w_{Op_1}(\alpha_1-\alpha_2)}\,; \quad W_{p'q}(\beta_2-\beta_1) \equiv \frac{w_{p_2}(\beta_2-\beta_1)}{w_{O\tilde{p}_2}(\beta_2-\beta_1)}\,; \\ \overline{W}_{pq}(\beta_2-\alpha_1) &\equiv \frac{w_{O\tilde{p}_3}(\beta_2-\alpha_1)}{w_{p_3}(\beta_2-\alpha_1)}\,; \quad \overline{W}_{p'q'}(\beta_1-\alpha_2) \equiv \frac{w_{Op_4}(\beta_1-\alpha_2)}{w_{\tilde{p}_4}(\beta_1-\alpha_2)}\,, \end{aligned} \tag{49}$$

because of the trivial identity

$$\omega^{(\alpha_1-\beta_1)(\beta_2-\alpha_2)} \frac{\Phi(\alpha_1-\beta_2)\Phi(\alpha_2-\beta_1)}{\Phi(\alpha_1-\alpha_2)\Phi(\beta_2-\beta_1)} \;=\; 1. \tag{50}$$

The notation for the rapidities p, q, p', q' which parameterize the point on the Baxter curve (18) should not be mixed up with that of the eight Fermat curve points $p_i,\ \tilde{p}_i\ (i = 1, 2, 3, 4)$. Using (49) and (17), (10) we can write

$$\begin{aligned} \frac{W_{p,q'}(n)}{W_{p,q'}(n-1)} &= \frac{\mu_p y_{q'}}{\mu_{q'} y_p} \frac{1-\omega^n (x_p/y_{q'})}{1-\omega^n (x_{q'}/y_p)} \equiv \frac{\omega^{1/2}\tilde{y}_1 x_1}{y_1} \frac{1-\omega^n(\omega x_1)^{-1}}{1-\omega^n \tilde{x}_1} \\ \frac{W_{p',q}(n)}{W_{p',q}(n-1)} &= \frac{\mu_{p'} y_q}{\mu_q y_{p'}} \frac{1-\omega^n (x_{p'}/y_q)}{1-\omega^n (x_q/y_{p'})} \equiv \frac{\omega^{1/2} y_2 \tilde{x}_2}{\tilde{y}_2} \frac{1-\omega^n(\omega \tilde{x}_2)^{-1}}{1-\omega^n x_2} \end{aligned} \tag{51}$$

and

$$\begin{aligned} \frac{\overline{W}_{p,q}(n)}{\overline{W}_{p,q}(n-1)} &= \frac{\omega\mu_p\mu_q}{x_p^{-1} y_q} \frac{1-\omega^{n-1}(x_q/x_p)}{1-\omega^n (y_p/y_q)} \equiv \frac{\omega^{-1/2}\tilde{y}_3}{y_3\tilde{x}_3} \frac{1-\omega^{n-1}(\omega x_3)}{1-\omega^n(\tilde{x}_3\omega)^{-1}} \\ \frac{\overline{W}_{p',q'}(n)}{\overline{W}_{p',q'}(n-1)} &= \frac{\omega\mu_{p'}\mu_{q'}}{x_{p'}^{-1} y_{q'}} \frac{1-\omega^{n-1}(x_{q'}/x_{p'})}{1-\omega^n (y_{p'}/y_{q'})} \equiv \frac{\omega^{-1/2} y_4}{\tilde{y}_4 x_4} \frac{1-\omega^{n-1}(\omega\tilde{x}_4)}{1-\omega^n(\omega x_4)^{-1}}\,. \end{aligned} \tag{52}$$

From (51) and (52) we can read off the relation of the Fermat parameters to the CP-variables on the Baxter curve:

$$\begin{aligned} x_1 &= \frac{y_{q'}}{\omega x_p}\,; \quad x_2 = \frac{x_q}{y_{p'}}\,; \quad x_3 = \frac{x_q}{\omega x_p}\,; \quad x_4 = \frac{y_{q'}}{\omega y_{p'}}\,; \\ \tilde{x}_1 &= \frac{x_{q'}}{y_p}\,; \quad \tilde{x}_2 = \frac{y_q}{\omega x_{p'}}\,; \quad \tilde{x}_3 = \frac{y_q}{\omega y_p}\,; \quad \tilde{x}_4 = \frac{x_{q'}}{\omega x_{p'}}\,. \end{aligned} \tag{53}$$

Note that both restrictions (12) are valid automatically for the Fermat points p_i and $\tilde{p}_i$. Comparing the coefficients in front of the ratios in (51) and (52) we can identify the ratios $\tilde{y}_i/y_i$ with ratios of the Baxter curve parameters

$$\begin{aligned} \frac{\tilde{y}_1}{y_1} &= \omega^{1/2}\frac{\mu_p x_p}{\mu_{q'} y_p}\,; \quad \frac{\tilde{y}_2}{y_2} = \omega^{-1/2}\frac{\mu_q y_{p'}}{\mu_{p'} x_{p'}}\,; \\ \frac{\tilde{y}_3}{y_3} &= \omega^{1/2}\frac{\mu_p\mu_q x_p}{y_p}\,; \quad \frac{\tilde{y}_4}{y_4} = \omega^{-1/2}\frac{y_{p'}}{\mu_{p'}\mu_{q'} x_{p'}}\,. \end{aligned} \tag{54}$$

From (54) we obtain an important relation which connects the Fermat points p_i and $\tilde{p}_i$:

$$\frac{y_1 y_2}{y_3 y_4} = \frac{\tilde{y}_1 \tilde{y}_2}{\tilde{y}_3 \tilde{y}_4}\,. \tag{55}$$

By taking the N-th powers of the formulas (54) one can see that the cyclic property of the functions $w_p(n)$ (10) implies the cyclic property (19) of the Boltzmann weights (17).

We conclude this discussion of the emergence of the BS-S-matrix within the ZBB-model with showing that the parametrization of the L-operator postulated in (34) agrees with the identifications (53),(54) made here. First we pass from the parameters $u_i,\ \tilde{u}_i,\ w_i,\ \tilde{w}_i,\ \kappa_i,\ \tilde{\kappa}_i$ used in (32) to the Fermat parameters by (13). Since the functional transformation is taken to be trivial, we can omit all primes on the u_i and w_i.

$$\begin{aligned} &x_1 = \omega^{-1/2}\frac{u_2}{\kappa_1 u_1}\,; \quad x_2 = \omega^{-1/2}\frac{\kappa_2 u_2}{u_1}\,; \quad x_3 = \omega^{-1}\frac{u_2}{u_1}\,; \\ &\frac{y_3}{y_1} = \frac{\kappa_1 w_1}{u_3}\,; \quad \frac{y_4}{y_1} = \omega^{-1/2}\frac{\kappa_3 w_3}{w_2}\,. \end{aligned} \tag{56}$$

The counterpart with tildes is:

$$\begin{aligned} &\tilde{x}_1 = \omega^{-1/2}\frac{\tilde{u}_2}{\tilde{\kappa}_1 \tilde{u}_1}\,; \quad \tilde{x}_2 = \omega^{-1/2}\frac{\tilde{\kappa}_2 \tilde{u}_2}{\tilde{u}_1}\,; \quad \tilde{x}_3 = \omega^{-1}\frac{\tilde{u}_2}{\tilde{u}_1}\,; \\ &\frac{\tilde{y}_3}{\tilde{y}_1} = \frac{\tilde{\kappa}_1 \tilde{w}_1}{u_3}\,; \quad \frac{\tilde{y}_4}{\tilde{y}_1} = \omega^{-1/2}\frac{\kappa_3 w_3}{\tilde{w}_2}\,. \end{aligned} \tag{57}$$

Observe that there are no tildes on κ_3, u_3 and w_3. (56) and (57) immediately give

$$\begin{aligned} &\kappa_1 = \omega^{1/2}\frac{x_3}{x_1}\,; \quad \tilde{\kappa}_1 = \omega^{1/2}\frac{\tilde{x}_3}{\tilde{x}_1}\,; \quad \frac{w_1}{\tilde{w}_1} = \frac{\tilde{\kappa}_1\, \tilde{y}_1\, y_3}{\kappa_1\, y_1\, \tilde{y}_3}\,; \\ &\frac{u_2}{u_1} = \omega x_3\,; \quad \frac{\tilde{u}_2}{\tilde{u}_1} = \omega \tilde{x}_3. \end{aligned} \tag{58}$$

The first three equations of (34) follow by simply inserting from (53) into the first three equations of (58). The last two equations of (58) with (53) lead to

$$x_p u_2 = x_q u_1 \quad \text{and} \quad y_p \tilde{u}_2 = y_q \tilde{u}_1 \qquad \text{or} \quad u_1 \tilde{u}_1 x_q y_q = u_2 \tilde{u}_2 x_p y_p\,. \tag{59}$$

(59) shows that the rescaling of the spectral parameter $\xi^{-1} = -\omega u_1 \tilde{u}_1 x_q y_q\, \lambda$ is the same for $L_2(\lambda; p, p')$ as it is for $L_1(\lambda; q, q')$.

Summarizing, the relation of the parameters $\kappa_1,\ \tilde{\kappa}_1,\ \kappa_2,\ \tilde{\kappa}_2,\ \frac{w_1}{\tilde{w}_1},\ \frac{w_2}{\tilde{w}_2}$ to the CP-parameters is:

$$\begin{aligned} &\kappa_1 = \omega^{1/2}\frac{x_q}{y_{q'}}; \quad \tilde{\kappa}_1 = \omega^{-1/2}\frac{y_q}{x_{q'}}; \quad \kappa_2 = \omega^{1/2}\frac{x_p}{y_{p'}}; \quad \tilde{\kappa}_2 = \omega^{-1/2}\frac{y_p}{x_{p'}}; \\ &\frac{w_1}{\tilde{w}_1} = \omega^{-1}\frac{y_q y_{q'}}{x_q x_{q'} \mu_q \mu_{q'}}; \qquad \frac{w_2}{\tilde{w}_2} = \omega^{-1}\frac{y_p y_{p'}}{x_p x_{p'} \mu_p \mu_{p'}}\,. \end{aligned} \tag{60}$$

2.4. Parametrization in terms of cross-ratios

The intertwining matrix S defined by (26) depends on 5 independent continuous parameters. One may use several equivalent parameterizations:

(i) CP-parametrization: $\quad q,\ q',\ p,\ p',\ k\,.$

(ii) Fermat parametrization: $x_1,\ x_2,\ x_3,\ \tilde{x}_1,\ \tilde{x}_2,\ \tilde{x}_3$ with the constraint (55).

(iii) Weyl-parametrization: $\kappa_1,\ \tilde{\kappa}_1,\ \kappa_2,\ \tilde{\kappa}_2,\ w_1/\tilde{w}_1,\ w_2/\tilde{w}_2,$ again with one constraint.

One obtains (ii) from (i) by (53) and (54), while (iii) is obtained from (i) by (60). In (iii) one may choose $u_2/u_1,\ \tilde{u}_2/\tilde{u}_1$ instead of $w_1/\tilde{w}_1,\ w_2/\tilde{w}_2$ as is seen from (56), (57).

In the following let us concentrate on the *functional* transformations, which deal with only the N-th powers of the variables. We start considering the Weyl parametrization (iii). The functional transformations have been solved in [26], first rewriting them in Hirota form and then using standard methods of algebraic geometry. The results can be read off from (65), (66) of [24] and Table 1 of [26]. In the moment we are dealing with only the trivial version of the functional transformations. Their solutions can be obtained either by specializing Table 1 of [26], or directly from (14) solving the system

$$(K_2U_2 - K_1U_1)\,W_2 \;=(K_1U_1+U_2)\,K_3W_3\,;$$
$$(W_1-K_3U_3)\,W_3 \;=\; (W_1+U_3)\,W_2\,; \qquad (U_1-U_2)\,U_3 \;=\; (K_1U_1+U_2)\,W_1 \tag{61}$$

in terms of three pairs of complex points which we shall call $X',\ X,\ \ Y',\ Y;\ \ Z_0',\ Z_0$. The solution is

$$\begin{aligned}
U_1 &= -\varepsilon\,\frac{Y-Z_0'}{Y-Z_0}; &\quad U_2 &= -\varepsilon\,\frac{X-Z_0'}{X-Z_0}; &\quad U_3 &= -\varepsilon\,\frac{X-Y'}{X-Y};\\
W_1 &= \varepsilon\,\frac{Y'-Z_0}{Y-Z_0}; & W_2 &= \varepsilon\,\frac{X'-Z_0}{X-Z_0}; & W_3 &= \varepsilon\,\frac{X'-Y}{X-Y};\\
K_1 &= -\begin{bmatrix} Y' & Y\\ Z_0' & Z_0\end{bmatrix}; & K_2 &= -\begin{bmatrix} X' & X\\ Z_0' & Z_0\end{bmatrix}; & K_3 &= -\begin{bmatrix} X' & X\\ Y' & Y\end{bmatrix},
\end{aligned} \tag{62}$$

where

$$\varepsilon \;=\; (-1)^N\,; \qquad K_i \;=\; \kappa_i^N\,, \qquad i=1,2,3, \tag{63}$$

and for cross-ratios we use the notation

$$\begin{bmatrix} A & B\\ C & D\end{bmatrix} \;\equiv\; \frac{(A-C)\,(B-D)}{(A-D)\,(B-C)}\,. \tag{64}$$

There are analogous equations for the variables with tilde. We solve these introducing another pair of complex points $Z_1',\ Z_1$. The expressions are the same as (62), only everywhere $Z_0',\ Z_0'$ replaced by $Z_1',\ Z_1$, e.g. $\widetilde{U}_1 \;=\; -\varepsilon\,(Y-Z_1')/(Y-Z_1)$, *etc.* Observe that the ratios appearing in (iii) can be written as cross-ratios:

$$\frac{W_1}{\widetilde{W}_1} \;=\; \begin{bmatrix} Y' & Y\\ Z_0 & Z_1\end{bmatrix}; \qquad \frac{W_2}{\widetilde{W}_2} \;=\; \begin{bmatrix} X' & X\\ Z_0 & Z_1\end{bmatrix}, \tag{65}$$

so that all six variables in (iii) are expressible as cross-ratios of eight complex points. Frequently, we shall use calligrafic letters to denote *pairs* of points:

$$\mathcal{X} \;\equiv\; (X',X); \quad \mathcal{Y} \;\equiv\; (Y',Y); \quad \mathcal{Z}_0 \;\equiv\; (Z_0',Z_0), \quad \mathcal{Z}_1 \;\equiv\; (Z_1',Z_1). \tag{66}$$

Due to projective invariance, from these eight points five independent cross ratios can be formed. Observe that to the index 1 the pairs $\mathcal{Y}$, $\mathcal{Z}$ are associated, to the index 2 the pairs $\mathcal{Z}$, $\mathcal{X}$ etc.

The Fermat parametrization (ii) is obtained from (62) by (56), (57):

$$x_1^N=\begin{bmatrix} X & Y' \\ Z_0' & Z_0 \end{bmatrix};\quad x_2^N=\begin{bmatrix} Y & X' \\ Z_0 & Z_0' \end{bmatrix};\quad x_3^N=\begin{bmatrix} Y & X \\ Z_0 & Z_0' \end{bmatrix};\quad x_4^N=\begin{bmatrix} X' & Y' \\ Z_0' & Z_0 \end{bmatrix};$$
$$y_1^N=\begin{bmatrix} X & Z_0' \\ Y' & Z_0 \end{bmatrix};\quad y_2^N=\begin{bmatrix} Y & Z_0 \\ X' & Z_0' \end{bmatrix};\quad y_3^N=\begin{bmatrix} Y & Z_0 \\ X & Z_0' \end{bmatrix};\quad y_4^N=\begin{bmatrix} X' & Z_0' \\ Y' & Z_0 \end{bmatrix}. \quad (67)$$

and analogous formulas for $\tilde{x}_i$ and $\tilde{y}_i$ where the pair $\mathcal{Z}_0$ is replaced by $\mathcal{Z}_1$. Observe that Fermat curve conditions (10) and (12) are trivially satisfied.

We shall see now that also Baxter's modules k^2, k'^2 and the N-th powers of the CP parameters x_p, y_p, etc. have similar good expressions in terms of points X, X', ... Since (53) and (54) involve only ratios of the CP variables, to solve e.g. for x_p^N and y_p^N we have to use the Baxter curve relation (18).
One may proceed as follows: From (60) and (62) get

$$\left(\frac{\tilde{\kappa}_2\,\tilde{w}_2}{\kappa_2\,w_2}\right)^N=(\mu_p\mu_{p'})^N=\begin{bmatrix} X' & X \\ Z_1' & Z_0' \end{bmatrix}.$$

Then use (18) to write

$$k\,y_p^N\;=\;1-\frac{k'}{\mu_{p'}^N}\,\mu_p^N\,\mu_{p'}^N\;=\;1\;-\;(\mu_p\mu_{p'})^N\left(1-k\,x_{p'}^N\right).$$

Substitute here from (60) $x_{p'}=\omega^{-1/2}y_p/\tilde{\kappa}_2$ and solve for y_p^N :

$$k\,y_p^N\;=\;\frac{1\;-\;(\mu_p\,\mu_{p'})^N}{1\;+\;(\mu_p\,\mu_{p'})^N\tilde{K}_2^{-1}}.$$

and just insert, using also (62). To obtain x_p^N the same procedure works. So we find all CP-variables:

$$\begin{aligned}
k\,x_p^N&=\begin{bmatrix} X & Z_1' \\ Z_0 & Z_0' \end{bmatrix};\quad k\,y_p^N=\begin{bmatrix} X & Z_0' \\ Z_1 & Z_1' \end{bmatrix};\quad k'\mu_p^N=\begin{bmatrix} X & Z_1 \\ Z_0' & Z_1' \end{bmatrix};\\
k\,x_{p'}^N&=\begin{bmatrix} X' & Z_0' \\ Z_1 & Z_1' \end{bmatrix};\quad k\,y_{p'}^N=\begin{bmatrix} X' & Z_1' \\ Z_0 & Z_0' \end{bmatrix};\quad k'\mu_{p'}^N=\begin{bmatrix} X' & Z_0 \\ Z_1' & Z_0' \end{bmatrix};\\
k\,x_q^N&=\begin{bmatrix} Y & Z_1' \\ Z_0 & Z_0' \end{bmatrix};\quad k\,y_q^N=\begin{bmatrix} Y & Z_0' \\ Z_1 & Z_1' \end{bmatrix};\quad k'\mu_q^N=\begin{bmatrix} Y & Z_1 \\ Z_0' & Z_1' \end{bmatrix};\\
k\,x_{q'}^N&=\begin{bmatrix} Y' & Z_0' \\ Z_1 & Z_1' \end{bmatrix};\quad k\,y_{q'}^N=\begin{bmatrix} Y' & Z_1' \\ Z_0 & Z_0' \end{bmatrix};\quad k'\mu_{q'}^N=\begin{bmatrix} Y' & Z_0 \\ Z_1' & Z_0' \end{bmatrix}.
\end{aligned} \quad (68)$$

Each line of (68) yields the same expression for $k^2 = k\,x_p^N + k\,y_p^N - k^2 x_p^N y_p^N$:

$$k^2 = \left[\begin{array}{cc} Z_0 & Z_0' \\ Z_1 & Z_1' \end{array}\right] \qquad \text{or} \qquad k'^2 = \left[\begin{array}{cc} Z_0' & Z_1' \\ Z_0 & Z_1 \end{array}\right]. \tag{69}$$

We also note that Baxter's rapidities p, p', q, q' correspond directly to the points X, X', Y, Y', respectively, while Baxter's module k does not depend on these points.

Returning to the parametrization of L-operator (32) we found in (33) that it depends on three parameters κ_1, $\tilde{\kappa}_1$ and $w_1/\tilde{w}_1$. These are cross-ratios of six points Y, Y', Z_0, Z_0', Z_1 and Z_1'. Due to projective invariance six points give rise to only three independent cross-ratios. These correspond to the rapidities q, q' and the module k in the Bazhanov-Stroganov parametrization (24).

Finally note that in this parametrization the combinations encountered in (36) and (38) depend only on $\mathcal{Z}_0$ and $\mathcal{Z}_1$:

$$U_1\,\widetilde{U}_1\,x_q^N\,y_q^N \;=\; \frac{Z_1' - Z_0'}{Z_0 - Z_1}\,; \qquad U_1\,x_q^N \;=\; -\epsilon\,\frac{Z_1' - Z_0'}{k\,(Z_1' - Z_0)}\,. \tag{70}$$

3. Classical BS-model and intertwining of its L-operators

3.1. Functional mapping on N-powers of Weyl operators

As has been mentioned before, for $\omega^N = 1$ the Nth powers of the Weyl operators are central and can be considered classical variables. As in (7) we use $(i = 1, 2, 3)$:

$$U_i = \mathbf{u}_i^N\,; \quad W_i = \mathbf{w}_i^N\,; \quad \tilde{U}_i = \tilde{\mathbf{u}}_i^N\,; \quad \tilde{W}_i = \tilde{\mathbf{w}}_i^N\,; \quad K_i = \kappa_i^N\,; \quad \tilde{K}_i = \tilde{\kappa}_i^N\,, \tag{71}$$

and $\Lambda = \xi^N$.

According to the approach developed in [5, 27] we introduce a classical analog of the linear problem (30)

$$\begin{aligned} 0 &= \Psi_0 - \Psi_1 \Lambda \tilde{U}_1 + \Phi_0 \tilde{W}_1 + \Phi_1 \Lambda \tilde{K}_1 \tilde{U}_1 \tilde{W}_1 \\ 0 &= \Psi_1 - \Psi_0 U_1 + \Phi_1 W_1 + \Phi_0 K_1 U_1 W_1 \end{aligned} \tag{72}$$

which can be rewritten in the matrix form

$$\Psi\,(\Lambda\,U_1\,\tilde{U}_1 - 1) = \Phi \cdot \mathcal{L}_1(\Lambda) \tag{73}$$

and defines the classical L-operator

$$\mathcal{L}_1(\Lambda) = \begin{pmatrix} \tilde{W}_1 + \Lambda U_1 \tilde{U}_1 K_1 W_1 & U_1\left(\tilde{W}_1 + K_1 W_1\right) \\ \Lambda \tilde{U}_1\left(W_1 + \tilde{K}_1 \tilde{W}_1\right) & W_1 + \Lambda \tilde{K}_1 U_1 \tilde{U}_1 \tilde{W}_1 \end{pmatrix} \tag{74}$$

acting in the space of the linear variables $\Psi = (\Psi_0, \Psi_1)$ and $\Phi = (\Phi_0, \Phi_1)$. We take (74) to define a discrete classical analog of the Bazhanov-Stroganov model. We can also define (74) by an averaging prescription from (32): Define [16]

$$\langle A(\,\xi^N\,)\,\rangle \;=\; \prod_{i\,\in\,\mathbb{Z}_N} A(\,\xi\,\omega^i\,)\,; \quad \left\langle \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\rangle = \begin{pmatrix} \langle A\rangle & \langle B\rangle \\ \langle C\rangle & \langle D\rangle \end{pmatrix}. \tag{75}$$

Then

$$\mathcal{L}_1(\Lambda) \;=\; W_1 \,\cdot\, \langle\, L_1(\xi)\,\rangle\,. \tag{76}$$

Analogously, we introduce an operator $\mathcal{L}_2(\Lambda)$ such that the classical variables and parameters $U_1,\ \tilde{U}_1,\ W_1,\ \tilde{W}_1,\ K_1,\ \tilde{K}_1$ are replaced by $U_2,\ \tilde{U}_2,\ W_2,\ \tilde{W}_2,\ K_2,\ \tilde{K}_2$. Let $\mathcal{L}_1^\star(\Lambda)$ and $\mathcal{L}_2^\star(\Lambda)$ again be L-operators of the form (74), but with the variables

$$U_i^\star\,,\quad W_i^\star\,,\quad \tilde{U}_i^\star\,,\quad \tilde{W}_i^\star\,,\quad K_i\,,\quad \tilde{K}_i\,,\qquad i=1,2. \tag{77}$$

Our aim is now to find the transformation $(U_1,\tilde{U}_1,W_1,\ldots,\tilde{W}_2)\mapsto(U_1^\star,\,\tilde{U}_1^\star,W_1^\star,\ldots,\tilde{W}_2^\star)$ which solves the intertwining relation

$$\mathcal{L}_2(\Lambda)\,\mathcal{L}_1(\Lambda) \;=\; \mathcal{L}_1^\star(\Lambda)\,\mathcal{L}_2^\star(\Lambda)\,. \tag{78}$$

However, trying to find the nonlinear 8-variable mapping by direct calculation without further guidance looks quite hopeless. Fortunately, the 3D approach will provide a solution to this problem [5].

3.2. Solving the classical BS-intertwining relation via the 3D functional transformation

The mapping (78) we are looking for can be found using the functional mapping of the vertex ZBB-model given in (14) [5]. We introduce two additional variables U_3, W_3 and the additional parameter K_3 and consider the rational mapping $\mathcal{R}_{123}^{(f)}$

$$\mathcal{R}_{123}^{(f)}\;:\quad U_1,W_1,U_2,W_2,U_3,W_3\;\;\mapsto\;\;U_1',W_1',U_2',W_2',U_3',W_3' \tag{79}$$

given explicitly in (14). We define the composition of two of these rational transformations (79)

$$\mathcal{R}_{123}^{(f)}\;:\quad U_1,W_1,U_2,W_2,U_3,W_3\;\;\mapsto\;\;U_1^\star,W_1^\star,U_2^\star,W_2^\star,U_3',W_3'\,, \tag{80}$$

$$\mathcal{R}_{\tilde{1}\tilde{2}3}^{(f)}\;:\quad \tilde{U}_1,\tilde{W}_1,\tilde{U}_2,\tilde{W}_2,U_3',W_3'\;\;\mapsto\;\;\tilde{U}_1^\star,\tilde{W}_1^\star,\tilde{U}_2^\star,\tilde{W}_2^\star,U_3^\star,W_3^\star\,, \tag{81}$$

together with a periodic condition

$$U_3^\star=U_3\,,\quad W_3^\star=W_3 \tag{82}$$

and denote this composition by

$$\begin{aligned} S_{12}^{(f)}\;:\quad & U_1,W_1,U_2,W_2,\tilde{U}_1,\tilde{W}_1,\tilde{U}_2,\tilde{W}_2\\ & \mapsto\;\;U_1^\star,W_1^\star,U_2^\star,W_2^\star,\tilde{U}_1^\star,\tilde{W}_1^\star,\tilde{U}_2^\star,\tilde{W}_2^\star\,. \end{aligned} \tag{83}$$

In (80) the constants K_1,K_2,K_3 have to be used, while in (81) the constants are $\tilde{K}_1,\tilde{K}_2$ and K_3. With these definitions we have

Proposition 1 *The rational transformation $S_{12}^{(f)}$ (83) solves the mapping defined by the intertwining relations (78).*

The *Proof* is provided by straightforward calculation. We first determine $U_3^\star$, $W_3^\star$ in terms of the variables U_3, W_3 and other variables from the successive application of first (81) and then (80). Imposing the periodicity condition (82) gives two equations which can be solved easily for the auxiliary variables U_3 and W_3, leading to

$$\begin{aligned} U_3 &= \frac{U_1(\tilde{K}_1\tilde{U}_1\tilde{W}_1 + K_1\tilde{U}_2W_1) + \tilde{U}_2(U_2W_1 + U_1\tilde{W}_1)}{U_1\tilde{U}_1 - U_2\tilde{U}_2} \\ W_3 &= \frac{W_2\tilde{W}_2(\tilde{K}_2K_2\tilde{U}_2U_2 - \tilde{K}_1K_1\tilde{U}_1U_1)}{K_3(U_2(\tilde{K}_1\tilde{U}_1\tilde{W}_2 + K_2\tilde{U}_2W_2) + \tilde{K}_1\tilde{U}_1(K_1U_1\tilde{W}_2 + K_2U_2W_2))} \end{aligned} \qquad (84)$$

Then (84) is used to eliminate U_3 and W_3 and we get

$$\frac{U_1^\star}{U_1} = \frac{\tilde{U}_1}{\tilde{U}_1^\star} = \frac{\tilde{K}_1\tilde{U}_1(K_1U_1\tilde{W}_2 + K_2U_2W_2) + U_2(\tilde{K}_1\tilde{U}_1\tilde{W}_2 + K_2\tilde{U}_2W_2)}{K_1U_1(\tilde{K}_1\tilde{U}_1W_2 + \tilde{K}_2\tilde{U}_2\tilde{W}_2) + \tilde{U}_2(K_1U_1W_2 + \tilde{K}_2U_2\tilde{W}_2)};$$

$$\frac{U_2^\star}{U_2} = \frac{\tilde{U}_2}{\tilde{U}_2^\star} = \frac{U_1(\tilde{K}_1\tilde{U}_1\tilde{W}_1 + K_1\tilde{U}_2W_1) + \tilde{U}_2(U_2W_1 + U_1\tilde{W}_1)}{U_2\tilde{W}_1(\tilde{K}_1\tilde{U}_1 + \tilde{U}_2) + \tilde{U}_1W_1(K_1U_1 + U_2)};$$

$$\frac{W_1^\star}{W_1} = \frac{W_2}{W_2^\star} = -\frac{K_3}{W_1\tilde{W}_2}\frac{V_1}{V_0}; \qquad \frac{\tilde{W}_1^\star}{\tilde{W}_1} = \frac{\tilde{W}_2}{\tilde{W}_2^\star} = -\frac{K_3}{\tilde{W}_1W_2}\frac{V_2}{V_0}; \qquad (85)$$

where

$$V_0 = (K_1\tilde{K}_1U_1\tilde{U}_1 - K_2\tilde{K}_2U_2\tilde{U}_2)\,(U_1\tilde{U}_1 - U_2\tilde{U}_2);$$

$$\begin{aligned} V_1 = {} & \tilde{K}_1U_1\tilde{U}_1^2\left[U_2(W_1 + \tilde{K}_1\tilde{W}_1)(\tilde{W}_2 + K_2W_2) + K_1U_1W_1\tilde{W}_2\right] \\ & + K_2U_2\tilde{U}_2^2\left[U_1(\tilde{W}_1 + K_1W_1)(W_2 + \tilde{K}_2\tilde{W}_2) + \tilde{K}_2U_2W_1\tilde{W}_2\right] \\ & + U_1\tilde{U}_1U_2\tilde{U}_2\left[K_2W_1W_2(1 + K_1\tilde{K}_1) + \tilde{K}_1\tilde{W}_1\tilde{W}_2(1 + K_2\tilde{K}_2) + 2\,\tilde{K}_1K_2\tilde{W}_1W_2\right], \end{aligned} \qquad (86)$$

and V_2 is obtained from V_1 interchanging variables with tildes and without tildes.

Finally, we simply insert into (78). These are 2×2 matrices with four entries and we compare the coefficients of Λ^0, Λ^1, Λ^2 each, giving 12 equations which turn out to be correct. □

Because of the periodic boundary conditions (82) the mapping (83) has the invariants

$$U_1\tilde{U}_1 = U_1^\star\tilde{U}_1^\star; \quad U_2\tilde{U}_2 = U_2^\star\tilde{U}_2^\star; \quad W_1W_2 = W_1^\star W_2^\star; \quad \tilde{W}_1\tilde{W}_2 = \tilde{W}_1^\star\tilde{W}_2^\star. \qquad (87)$$

Note that the first two invariants in (87) reflect the fact that the combinations $U_i\tilde{U}_i$ are scale factors of the spectral parameter, see (59). The last two invariants show that a difference in the normalization of the classical and quantum L-operators (74) and (32) is not important.

4. Main result: Isospectral transform of the BS transfer matrix.

We consider the BS-quantum chain of length Q defined by the monodromy

$$\mathbf{M}(\xi) = L_0(\xi, u_0, \ldots, \tilde{\kappa}_0)\, L_1(\xi, u_1, \ldots, \tilde{\kappa}_1)\, \ldots\, L_{Q-1}(\xi, u_{Q-1}, \ldots, \tilde{\kappa}_{Q-1})$$

and the transfer matrix

$$\mathbf{T}(\xi) \;=\; \mathrm{tr}_{\mathbb{C}^2}\, \mathbf{M}(\xi) \tag{88}$$

where the Lax operators are defined by (32), (33)

$$L_q(\xi; u_q, \tilde{u}_q, \frac{w_q}{\tilde{w}_q}, \kappa_q, \tilde{\kappa}_q) = \begin{pmatrix} 1-\omega^{1/2}\xi\, u_q\, \tilde{u}_q\, \kappa_q\, \dfrac{w_q}{\tilde{w}_q}\, \mathbf{Z}_q; & -u_q \mathbf{X}_q \left(\omega^{1/2} - \kappa_q\, \dfrac{w_q}{\tilde{w}_q}\, \mathbf{Z}_q\right) \\ \xi\, \tilde{u}_q\, \mathbf{X}_1^{-1}\!\left(\tilde{\kappa}_1 - \omega^{1/2}\dfrac{w_q}{\tilde{w}_q}\, \mathbf{Z}_q\right); & -\omega^{1/2}\xi\, u_q\, \tilde{u}_q\, \tilde{\kappa}_q + \dfrac{w_q}{\tilde{w}_q}\, \mathbf{Z}_q \end{pmatrix}. \tag{89}$$

The arguments $u_q, \tilde{u}_q, w_q/\tilde{w}_q, \kappa_q, \tilde{\kappa}_q$ of all L_q $(q=0,\ldots,Q-1)$ may be different and shall be parameterized following (62) in terms of $Q+2$ pairs of points $\mathcal{Y}_q$ and $\mathcal{Z}_0$, $\mathcal{Z}_1$:

$$u_q^N \;=\; -\,\varepsilon\frac{Y_q - Z_0'}{Y_q - Z_0}; \qquad \tilde{u}_q^N \;=\; -\varepsilon\frac{Y_q - Z_1'}{Y_q - Z_1}; \qquad \frac{w_q^N}{\tilde{w}_q^N} \;=\; \begin{bmatrix} Y_q' & Y_q \\ Z_0 & Z_1 \end{bmatrix};$$

$$\kappa_q^N \;=\; -\begin{bmatrix} Y_q' & Y_q \\ Z_0' & Z_0 \end{bmatrix}; \quad \tilde{\kappa}_q^N \;=\; -\begin{bmatrix} Y_q' & Y_q \\ Z_1' & Z_1 \end{bmatrix}. \tag{90}$$

In the equivalent formulation of the transfer matrix in (28) in terms of CP variables $x_q,\ x_{q'}, \ldots, k^2$ this means that we take the Baxter modulus k^2 to be the same for all L_q but the two rapidities in each L_q may be different. The normalization adopted in (88) differs from (27), (28), see (40).

The main result of this paper, which will be proven in the remaining part of this section, is the following:

Proposition 2 *For a given k and a fixed set of $2Q$ rapidities, there exists a $Q-1$ parametric family of transfer matrices with the same spectrum as the initial one. This family of transfer matrices is defined by the same formulas (88), (89), (90) where the $\mathbb{C}^N$ matrices satisfy as usual $\mathbf{X}_{q_1}\mathbf{Z}_{q_2} \;=\; \omega^{\delta_{q_1 q_2}}\mathbf{Z}_{q_2}\mathbf{X}_{q_1}$, but without being normalized to $\mathbf{X}_q^N \;=\; \mathbf{Z}_q^N \;=\; 1$. The form of the inhomogeneous centers is given by (96).*

In order to state the results for $\mathbf{X}_q^N$, $\mathbf{Z}_q^N$, we have to consider the following system of algebraic equations for the unknowns P, P', with the pairs $\mathcal{Y}_q$, $\mathcal{Z}_0$, $\mathcal{Z}_1$ being given:

$$\prod_{q=0}^{Q-1} \begin{bmatrix} P' & P \\ Y_q' & Y_q \end{bmatrix} = 1\,; \tag{91}$$

$$\prod_{j=0,1} \begin{bmatrix} P' & P \\ Z_j' & Z_j \end{bmatrix} = 1\,. \tag{92}$$

The system (91), (92) has exactly $g \;=\; Q-1$ nonequivalent solutions $\{(P_k', P_k)\}$, $k=0,...,g-1$, the pair (P_k', P_k) taken to be equivalent to the pair (P_k, P_k').

Now, given a fixed set of g pairs of complex numbers (P_0', P_0), ..., (P_{g-1}', P_{g-1}), we define the function H of a g-dimensional vector $(f_0,\, f_1, ..., f_{g-1}) \equiv \{f_k\}$, denoted

$H(\{f_k\})$ (this arises as the rational limit of the Θ-function on a genus g generic algebraic curve, see the Appendix of [14]) § by

$$H(\{f_k\}) \;=\; \frac{\det|\,P_j^k \;-\; f_j P_j'^{\,k}\,|_{j,k=0}^{g-1}}{\prod_{k>j}(P_k - P_j)}\;, \tag{93}$$

This is seen to be normalized to $H(\{0\}) = 1$. Let further

$$\mathbf{f}_k(Y) \;=\; \frac{P_k - Y}{P_k' - Y}\; f_k; \qquad \sigma_k(\mathcal{Y}) \;\equiv\; \sigma_k(Y', Y) \;=\; \begin{bmatrix} P_k' & P_k \\ Y' & Y \end{bmatrix}, \tag{94}$$

recall (66) $\mathcal{Y} \;=\; (Y', Y)$, and define

$$I_k(q) \;=\; \prod_{j=0}^{q-1} \begin{bmatrix} P_k' & P_k \\ Y_j & Y_j' \end{bmatrix} \;=\; \prod_{j=0}^{q-1} \sigma_k^{-1}(\mathcal{Y}_j)\,; \qquad I_k(0) \;=\; I_k(Q) \;=\; 1\,. \tag{95}$$

Then, the spectrum of the transfer matrix defined by (88), (89), (90) with

$$\begin{aligned} \mathbf{X}_q^N \;&=\; \frac{H\,(\,\{\mathbf{f}_k(Y_q)\;I_k(q)\}\,)}{H\,(\,\{\mathbf{f}_k(Y_q)\;I_k(q)\;\sigma_k(\mathcal{Z}_0)\}\,)}\;; \\ \mathbf{Z}_q^N \;&=\; \frac{H\,(\,\{\mathbf{f}_k(Z_0)\;I_k(q+1)\}\,)}{H\,(\,\{\mathbf{f}_k(Z_1)\;I_k(q+1)\}\,)} \cdot \frac{H\,(\,\{\mathbf{f}_k(Z_1)\;I_k(q)\}\,)}{H\,(\,\{\mathbf{f}_k(Z_0)\;I_k(q)\}\,)} \end{aligned} \tag{96}$$

does not depend on the set $\{f_k\}$. If $\{f_k\} = \{0\}$, then (96) reduces to $\mathbf{X}_q^N = \mathbf{Z}_q^N = 1$, and so the spectrum is the same as for the initial transfer matrix.

In order to prove these statements we shall make extensive use of the 3D-formalism which will allow an easy description of the necessary intertwining operations. We shall derive a generalization of the parametrization (90) which will involve functions H depending of the parameters $\{f_k\}$. A key point will be that the product $u_q\tilde{u}_q$ and the κ_q, $\tilde{\kappa}_q$ will not involve the $\{f_k\}$, see eqs.(108), and that in (88) $\mathbf{Z}_q$ is always multiplied by (the $\{f_k\}$-dependent) factor $w_q/\tilde{w}_q$ and $\mathbf{X}_q$ is multiplied by the $\{f_k\}$-dependent u_q. Details should become clear as we proceed.

4.1. Uniformization of the classical maps

The map (83) describes the explicit relation between the final "star" variables and the initial "non-star" variables. According to [5, 26] this map, as well as map (14), can be parameterized in terms of algebraic geometry data. In this paper we will consider uniformization of these maps using a specific set of the rational functions and identities between them. This construction was exploited previously in [14].

Consider a general three-dimensional lattice with classical variables placed on its edges. Let $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$ be marks for the vertices in this 3D lattice with

§ These rational functions appear in soliton theory (see, for example, [13], g: number of solitons), and in 3D integrable models [29]:

$H(\,) = 1;\qquad H(f_0) = 1 - f_0;\qquad H(f_0,f_1) = 1 - \frac{P_1-P_0'}{P_1-P_0} f_0 - \frac{P_0-P_1'}{P_1-P_0} f_1 + \frac{P_1'-P_0'}{P_1-P_0}\, f_0 f_1;$

$H(f_0,f_1,f_2) = 1 - \frac{(P_0'-P_2)(P_0'-P_1)}{(P_0-P_1)(P_0-P_2)}\, f_0 + \ldots + \frac{(P_1'-P_2)(P_0'-P_2)(P_1'-P_0')}{(P_2-P_1)(P_2-P_0)(P_1-P_0)}\, f_0 f_1 + \ldots \frac{(P_0'-P_1')(P_1'-P_2')(P_2'-P_0)}{(P_2-P_1)(P_2-P_0)(P_1-P_0)}\, f_0 f_1 f_2.$

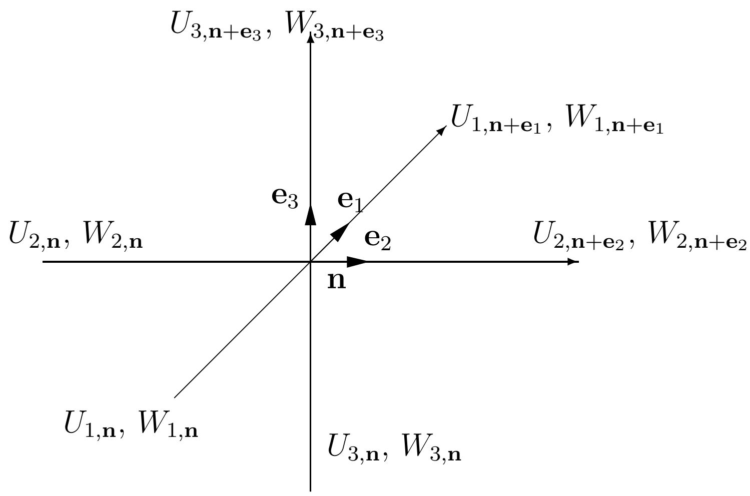

Figure 5. Association of the dynamical variables to the edges of the cubic lattice. The vector $\mathbf{e}_1$ is pointing into the paper plane.

discrete coordinates n_1, n_2, n_3. The assignment of the classical variables to the links around a given vertex $\mathbf{n}$ is shown in Fig. 5. In this notation the map (14) relating the neighboring classical variables is

$$\begin{aligned}
&\frac{U_{1,\mathbf{n}+\mathbf{e}_1}}{U_{1,\mathbf{n}}} = \frac{W_{3,\mathbf{n}+\mathbf{e}_3}}{W_{3,\mathbf{n}}} \\
&\quad = \frac{K_{2:n_1,n_3} U_{2,\mathbf{n}} W_{2,\mathbf{n}}}{K_{1:n_2,n_3} U_{1,\mathbf{n}} W_{2,\mathbf{n}} + K_{3:n_1,n_2} U_{2,\mathbf{n}} W_{3,\mathbf{n}} + K_{1:n_2,n_3} K_{3:n_1,n_2} U_{1,\mathbf{n}} W_{3,\mathbf{n}}}\,; \\
&\frac{W_{1,\mathbf{n}}}{W_{1,\mathbf{n}+\mathbf{e}_1}} = \frac{W_{2,\mathbf{n}+\mathbf{e}_2}}{W_{2,\mathbf{n}}} = \frac{W_{1,\mathbf{n}} W_{3,\mathbf{n}}}{W_{1,\mathbf{n}} W_{2,\mathbf{n}} + U_{3,\mathbf{n}} W_{2,\mathbf{n}} + K_{3:n_1,n_2} U_{3,\mathbf{n}} W_{3,\mathbf{n}}}\,; \\
&\frac{U_{2,\mathbf{n}+\mathbf{e}_2}}{U_{2,\mathbf{n}}} = \frac{U_{3,\mathbf{n}}}{U_{3,\mathbf{n}+\mathbf{e}_3}} = \frac{U_{1,\mathbf{n}} U_{3,\mathbf{n}}}{U_{2,\mathbf{n}} U_{3,\mathbf{n}} + U_{2,\mathbf{n}} W_{1,\mathbf{n}} + K_{1:n_2,n_3} U_{1,\mathbf{n}} W_{1,\mathbf{n}}}\,.
\end{aligned} \tag{97}$$

One may think about these relations as discrete equations which describe the interrelation of the classical variables along the 3D lattice.

The next step is to observe that after the change of variables

$$\begin{aligned}
U_{1,\mathbf{n}} &= -\,\varepsilon \frac{Y_{n_2} - Z'_{n_3}}{Y_{n_2} - Z_{n_3}} \frac{\tau_{2,\mathbf{n}}}{\tau_{2,\mathbf{n}+\mathbf{e}_3}}\,; \quad & W_{1,\mathbf{n}} &= \varepsilon \frac{Z_{n_3} - Y'_{n_2}}{Z_{n_3} - Y_{n_2}} \frac{\tau_{3,\mathbf{n}+\mathbf{e}_2}}{\tau_{3,\mathbf{n}}}\,; \\
U_{2,\mathbf{n}} &= -\,\varepsilon \frac{X_{n_1} - Z'_{n_3}}{X_{n_1} - Z_{n_3}} \frac{\tau_{1,\mathbf{n}}}{\tau_{1,\mathbf{n}+\mathbf{e}_3}}\,; \quad & W_{2,\mathbf{n}} &= \varepsilon \frac{Z_{n_3} - X'_{n_1}}{Z_{n_3} - X_{n_1}} \frac{\tau_{3,\mathbf{n}}}{\tau_{3,\mathbf{n}+\mathbf{e}_1}}\,; \\
U_{3,\mathbf{n}} &= -\,\varepsilon \frac{X_{n_1} - Y'_{n_2}}{X_{n_1} - Y_{n_2}} \frac{\tau_{1,\mathbf{n}+\mathbf{e}_2}}{\tau_{1,\mathbf{n}}}\,; \quad & W_{3,\mathbf{n}} &= \varepsilon \frac{Y_{n_2} - X'_{n_1}}{Y_{n_2} - X_{n_1}} \frac{\tau_{2,\mathbf{n}}}{\tau_{2,\mathbf{n}+\mathbf{e}_1}}\,,
\end{aligned} \tag{98}$$

together with the cross-ratio parametrization of the $K_{i:n_j n_k}$

$$K_{1:n_2,n_3} = -\left[\begin{array}{cc} Y'_{n_2} & Y_{n_2} \\ Z'_{n_3} & Z_{n_3} \end{array}\right];\; K_{2:n_1,n_3} = -\left[\begin{array}{cc} Z'_{n_3} & Z_{n_3} \\ X'_{n_1} & X_{n_1} \end{array}\right];\; K_{3:n_1,n_2} = -\left[\begin{array}{cc} X'_{n_1} & X_{n_1} \\ Y'_{n_2} & Y_{n_2} \end{array}\right], \tag{99}$$

each of the relations in (97) can be written in the form of a three-linear Hirota-type equation for the triple of unknown functions $\tau_{\alpha,\mathbf{n}}$, $\alpha = 1, 2, 3$:

$$\begin{aligned} &(X_\alpha - X_\beta)(X'_\beta - X'_\gamma)(X_\gamma - X_\alpha)\tau_{\alpha,\mathbf{n}+\mathbf{e}_\beta+\mathbf{e}_\gamma}\tau_{\beta,\mathbf{n}}\tau_{\gamma,\mathbf{n}} \\ &\quad + (X_\alpha - X'_\beta)(X_\beta - X_\gamma)(X'_\gamma - X_\alpha)\tau_{\alpha,\mathbf{n}}\tau_{\beta,\mathbf{n}+\mathbf{e}_\gamma}\tau_{\gamma,\mathbf{n}+\mathbf{e}_\beta} \\ =\ &(X_\alpha - X_\beta)(X'_\beta - X_\gamma)(X'_\gamma - X_\alpha)\tau_{\alpha,\mathbf{n}+\mathbf{e}_\beta}\tau_{\beta,\mathbf{n}+\mathbf{e}_\gamma}\tau_{\gamma,\mathbf{n}} \\ &\quad + (X_\alpha - X'_\beta)(X_\beta - X'_\gamma)(X_\gamma - X_\alpha)\tau_{\alpha,\mathbf{n}+\mathbf{e}_\gamma}\tau_{\beta,\mathbf{n}}\tau_{\gamma,\mathbf{n}+\mathbf{e}_\beta}\ , \end{aligned} \tag{100}$$

where $\{\alpha, \beta, \gamma\}$ is any even permutation of the set $\{1, 2, 3\}$. The notations in (100) are related to those in (98) as follows:

$$\mathcal{X}_1 = \mathcal{X}_{n_1},\ \ \mathcal{X}_2 = \mathcal{Y}'_{n_2},\ \ \mathcal{X}_3 = \mathcal{Z}_{n_3}\,, \qquad \text{where} \qquad \mathcal{Y}' = (Y, Y')\,. \tag{101}$$

Now our goal is to describe the general solution to (100) in the ring of the "rational Θ-functions" H defined in (93). The main tool to be used will be an identity, which is the rational limit of the Fay identity ("rational Fay-identity"):

Let A, B, C, D be any pair-wise different complex parameters. Using the definitions given in (93),(94), the following identity is valid:

$$\begin{aligned} H(\{f_k\})\ H(\{f_k\,\sigma_k(A,B)\,\sigma_k(C,D)\}) &= \begin{bmatrix} A & B \\ D & C \end{bmatrix} H(\{f_k\sigma_k(A,B)\})\,H(\{f_k\sigma_k(C,D)\}) \\ &+ \begin{bmatrix} A & D \\ B & C \end{bmatrix} H(\{f_k\sigma_k(A,D)\})\ H(\{f_k\sigma_k(C,B)\}) \end{aligned} \tag{102}$$

For a proof of this identity see the Appendix to the paper [14]. In order to get the form of (100), we combine two such identities and obtain the more complicated "rational double-Fay identity"

$$\begin{aligned} &(X-Y)(Y'-Z')(Z-X)\ H(\{\mathbf{f}_k(X)\,\sigma_k(\mathcal{Y})\,\sigma_k(\mathcal{Z})\})\ H(\{\mathbf{f}_k(Y)\})\ H(\{\mathbf{f}_k(Z)\}) \\ &+\ (X-Y')(Y-Z)(Z'-X)\ H(\{\mathbf{f}_k(X)\})\ H(\{\mathbf{f}_k(Y)\sigma_k(\mathcal{Z})\})\ H(\{\mathbf{f}_k(Z)\sigma_k(\mathcal{Y})\}) \\ &= (X-Y)(Y'-Z)(Z'-X)\ H(\{\mathbf{f}_k(X)\sigma_k(\mathcal{Y})\})\ H(\{\mathbf{f}_k(Y)\sigma_k(\mathcal{Z})\})\ H(\{\mathbf{f}_k(Z)\}) \\ &+\ (X-Y')(Y-Z')(Z-X)\ H(\{\mathbf{f}_k(Z)\sigma_k(\mathcal{Y})\})\ H(\{\mathbf{f}_k(Y)\})\ H(\{\mathbf{f}_k(X)\sigma_k(\mathcal{Z})\}). \end{aligned} \tag{103}$$

This identity involves five complex points X, $\mathcal{Y}$, $\mathcal{Z}$ and the sets $\{f_k\}$ and $\{P_k,\ P'_k\}$. Dividing by $(X-Z')(Y-Z)(X-Y')(Z'-Y)/(Y-Z')$ the factors multiplying the functions H can be written as cross-ratios. The structure of both (102) and (103) is precisely the same as that of the corresponding identities for Θ-functions, see e.g. equations (21)–(24) of [26].

Comparing (100) and (103), we conclude that for arbitrary $\{f_k\}$ the discrete equations (100) on the 3D cubic lattice are solved by (we have to use $\mathcal{Y}'$ instead of $\mathcal{Y}$ in (103) which explains the inverse in the middle term of (105)):

$$\begin{aligned} &\tau_{1,\mathbf{n}} = H(\{\mathbf{f}_k\,(X_{n_1})\,\sigma_k(\mathcal{Y}_{n_2})I_{k:\mathbf{n}}\})\,; \quad \tau_{3,\mathbf{n}} = H(\{\mathbf{f}_k\,(Z_{n_3})\,\sigma_k(\mathcal{Y}_{n_2})I_{k:\mathbf{n}}\})\,, \\ &\qquad \tau_{2,\mathbf{n}} = H\left(\left\{\mathbf{f}_k\left(Y'_{n_2}\right)\sigma_k(\mathcal{Y}_{n_2})I_{k:\mathbf{n}}\right\}\right) = H(\{\mathbf{f}_k\,(Y_{n_2})\,I_{k:\mathbf{n}}\})\ , \end{aligned} \tag{104}$$

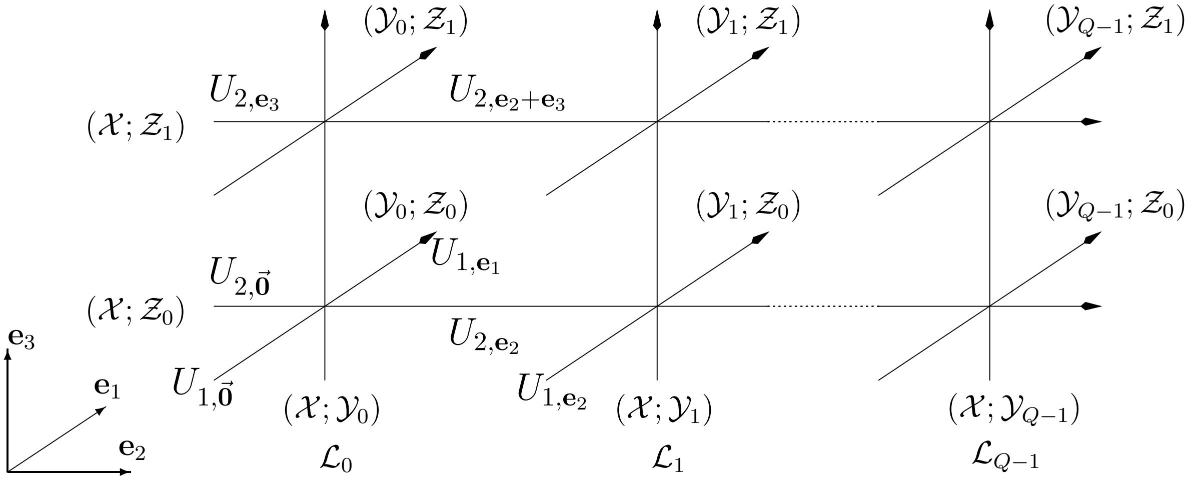

Figure 6. Slice of the 3-dimensional lattice associated with the BS chain. The orientation of $\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$ is shown in the left lower corner. The lattice is taken periodic after two steps in the $\mathbf{e}_3$-direction. The classical variables are assigned to the links as indicated in Fig. 5. In the Figure we show a few of these variables where we just indicate the $U_{i,\mathbf{n}}$ variables, the $W_{i,\mathbf{n}}$ at the same links are not shown. Observe that $\tilde{U}_{i,\mathbf{n}} \equiv U_{i,\mathbf{n}+\mathbf{e}_3},\ \tilde{W}_{i,\mathbf{n}} \equiv W_{i,\mathbf{n}+\mathbf{e}_3}$.

where we introduced

$$I_{k:\mathbf{n}} \;=\; \prod_{m_1=0}^{n_1-1} \sigma_k(\mathcal{X}_{m_1}) \prod_{m_2=0}^{n_2-1} \sigma_k^{-1}(\mathcal{Y}_{m_2}) \prod_{m_3=0}^{n_3-1} \sigma_k(\mathcal{Z}_{m_3})\,; \qquad I_{k:\vec{\mathbf{0}}} \;=\; 1\,. \tag{105}$$

This is the rational analog to solving trilinear Hirota equations by use of the double Fay identity for Θ-functions [5, 26].

Inserting (104) and (105) into (98) we get the parametrization for all variables on the lattice:

$$\begin{aligned}
U_{1,\mathbf{n}} &= -\,\varepsilon\frac{Y_{n_2}-Z'_{n_3}}{Y_{n_2}-Z_{n_3}}\,\frac{H\left(\{\mathbf{f}_k\left(Y_{n_2}\right) I_{k:\mathbf{n}}\}\right)}{H\left(\{\mathbf{f}_k\left(Y_{n_2}\right) I_{k:\mathbf{n}}\sigma_k(\mathcal{Z}_{n_3})\}\right)}\,;\\
U_{2,\mathbf{n}} &= -\,\varepsilon\frac{X_{n_1}-Z'_{n_3}}{X_{n_1}-Z_{n_3}}\,\frac{H\left(\{\mathbf{f}_k\left(X_{n_1}\right) I_{k:\mathbf{n}}\sigma_k(\mathcal{Y}_{n_2})\}\right)}{H\left(\{\mathbf{f}_k\left(X_{n_1}\right) I_{k:\mathbf{n}}\sigma_k(\mathcal{Y}_{n_2})\sigma_k(\mathcal{Z}_{n_3})\}\right)}\,;\\
U_{3,\mathbf{n}} &= -\,\varepsilon\frac{X_{n_1}-Y'_{n_2}}{X_{n_1}-Y_{n_2}}\,\frac{H\left(\{\mathbf{f}_k\left(X_{n_1}\right) I_{k:\mathbf{n}}\}\right)}{H\left(\{\mathbf{f}_k\left(X_{n_1}\right) I_{k:\mathbf{n}}\sigma_k(\mathcal{Y}_{n_2})\}\right)}\,.
\end{aligned} \tag{106}$$

The $W_{i:\mathbf{n}}$ are obtained from the $U_{i:\mathbf{n}}$ interchanging the arguments and putting an overall minus sign.

4.2. The classical BS chain

In the last subsection we considered the whole 3D lattice. In the following, we shall be interested in applying this 3D formalism to the BS chain defined by the product of operators $\mathcal{L}_q$. The chain will be taken in the $\mathbf{e}_2$-direction, see Fig. 6. As in Fig. 3, for each $\mathcal{L}_q$ we have to consider a pair of vertices, here on top of each other in the $\mathbf{e}_3$-direction, with periodic boundary conditions after two steps. Just one layer with open b.c. is needed in the $\mathbf{e}_1$-direction.

To each line of the 3D lattice we associate two pairs of points as shown in Fig. 6: a line in direction $\mathbf{e}_1$ is labeled by the two pairs $\mathcal{Y}_i$, $\mathcal{Z}_i$. Classical variables are associated to the links as in Fig. 5. In the notation of (74) we have $\tilde{U}_{i,\mathbf{n}} \equiv U_{i,\mathbf{n}+\mathbf{e}_3}$, $\tilde{W}_{i,\mathbf{n}} \equiv W_{i,\mathbf{n}+\mathbf{e}_3}$. The indices take the values $n_1 = 0$, $n_2 = 0, 1, \ldots, Q-1$, $n_3 = 0, 1$ starting from the left bottom.

Looking to the allocation of the variables to the links of the lattice, the Lax operators associated to the lines $\mathcal{Y}_q$ will be taken to be of the form (74) and depending on the index "1"-variables $U_{1,q\mathbf{e}_2}$, $W_{1,q\mathbf{e}_2}$, $K_{1,q}$ and their tilde counterparts. We indicate this by writing $\mathcal{L}_{1,q}$. Explicitly:

$$\mathcal{L}_{1,q}(\Lambda) = \begin{pmatrix} \tilde{W}_{1,q\mathbf{e}_2} + \Lambda U_{1,q\mathbf{e}_2}\tilde{U}_{1,q\mathbf{e}_2}K_{1:q}W_{1,q\mathbf{e}_2} & U_{1,q\mathbf{e}_2}\left(\tilde{W}_{1,q\mathbf{e}_2} + K_{1:q}W_{1,q\mathbf{e}_2}\right) \\ \Lambda\tilde{U}_{1,q\mathbf{e}_2}\left(W_{1,q\mathbf{e}_2} + \tilde{K}_{1:q}\tilde{W}_{1,q\mathbf{e}_2}\right) & W_{1,q\mathbf{e}_2} + \Lambda\tilde{K}_{1:q}U_{1,q\mathbf{e}_2}\tilde{U}_{1,q\mathbf{e}_2}\tilde{W}_{1,q\mathbf{e}_2} \end{pmatrix}. \quad (107)$$

Since these classical variables are solutions of (100), according to (106) we can write them in terms of the rational Θ-functions H defined in (93):

$$U_{1,q\mathbf{e}_2} = -\varepsilon\frac{Y_q - Z'_0}{Y_q - Z_0}\frac{H(\{\mathbf{f}_k(Y_q)\, I_k(q)\})}{H(\{\mathbf{f}_k(Y_q)\, I_k(q)\sigma_k(\mathcal{Z}_0)\})}; \qquad \tilde{U}_{1,q\mathbf{e}_2} = \frac{(Y_q - Z'_0)(Y_q - Z'_1)}{(Y_q - Z_0)(Y_q - Z_1)}\frac{1}{U_{1,q\mathbf{e}_2}};$$

$$\frac{W_{1,q\mathbf{e}_2}}{\tilde{W}_{1,q\mathbf{e}_2}} = \left[\begin{array}{cc} Y'_q & Y_q \\ Z_0 & Z_1 \end{array}\right]\frac{H(\{\mathbf{f}_k(Z_0)I_k(q+1)\})}{H(\{\mathbf{f}_k(Z_1)I_k(q)\})}\frac{H(\{\mathbf{f}_k(Z_0)I_k(q)\})}{H(\{\mathbf{f}_k(Z_1)I_k(q+1)\})};$$

$$K_{1:q} = -\left[\begin{array}{cc} Y'_q & Y_q \\ Z'_0 & Z_0 \end{array}\right]; \qquad \tilde{K}_{1:q} = -\left[\begin{array}{cc} Y'_q & Y_q \\ Z'_1 & Z_1 \end{array}\right]. \quad (108)$$

Here $\{f_k\}$ is an arbitrary set of Q complex parameters. The $I_k(q)$ are given in (95) and are related to the $\mathbf{I}_{k:\mathbf{n}}$ of (105) by $I_k(q) = \mathbf{I}_{k:q\mathbf{e}_2}$.

We define the classical monodromy matrix, the classical counterpart of (88), as:

$$\mathcal{M}_1(\Lambda) = \prod_{q'=0}^{Q-1}\tilde{W}^{-1}_{1,q'\mathbf{e}_2}\;\mathcal{L}_{1,0}(\Lambda)\,\mathcal{L}_{1,1}(\Lambda)\,\cdots\,\mathcal{L}_{1,q}(\Lambda)\,\cdots\,\mathcal{L}_{1,Q-1}(\Lambda). \quad (109)$$

The simplest choice, still compatible with the functional mapping relating neighboring variables, is to take all $\{f_k\} = \{0\}$, resulting in all H being unity. Then $\mathcal{M}_1$ still depends on the Q pairs $\mathcal{Y}_q$, the two pairs $\mathcal{Z}_0$, $\mathcal{Z}_1$ and the spectral parameter Λ. Since all $\mathcal{Y}_q$ may be chosen differently, in general also for $\{f_k\} = \{0\}$ the chain will be inhomogenous.

4.3. Uniformization of the classical BS intertwining mapping.

We like to establish an isospectrality transformation by commuting an auxiliary Lax operator through the monodromy $\mathcal{M}_1$ (109). For this we shall first parameterize the intertwining of *two* Lax operators considered in Sec. 3.1, maps (79) and (83), in terms of cross ratios and H-functions. We use the results (106) specialized to a single site, e.g. $\mathbf{n} = \vec{0}$. In this subsection this index will be suppressed.

For a compact notation, we introduce the following functions

$$U\left(\{f_k\}, \mathcal{A}, \mathcal{B}\right) = -\varepsilon \frac{A-B'}{A-B} \frac{H(\{\mathbf{f}_k(A)\})}{H(\{\mathbf{f}_k(A)\,\sigma_k(\mathcal{B})\})} ;$$
$$W(\{f_k\}, \mathcal{A}, \mathcal{B}) = - U\left(\{f_k\}, \mathcal{B}, \mathcal{A}\right) . \tag{110}$$

For convenience we write $\mathcal{A}$ as the argument of U although it does not depend on A'.

Using (110), equations (106) uniformize the mapping (79) as follows:

$$U_1, W_1 = U, W(\{f_k\}, \mathcal{Y}, \mathcal{Z}) ; \qquad U_2, W_2 = U, W(\{f_k\,\sigma_k(\mathcal{Y})\}, \mathcal{X}, \mathcal{Z}) ;$$
$$U_3, W_3 = U, W(\{f_k\}, \mathcal{X}, \mathcal{Y}) ;$$
$$U_1', W_1' = U, W(\{f_k\,\sigma_k(\mathcal{X})\}, \mathcal{Y}, \mathcal{Z}) ; \qquad U_2', W_2' = U, W(\{f_k\}, \mathcal{X}, \mathcal{Z}) ;$$
$$U_3', W_3' = U, W(\{f_k\,\sigma_k(\mathcal{Z})\}, \mathcal{X}, \mathcal{Y}) , \tag{111}$$

with

$$K_1 = -\begin{bmatrix} Y' & Y \\ Z' & Z \end{bmatrix} ; \quad K_2 = -\begin{bmatrix} X' & X \\ Z' & Z \end{bmatrix} ; \quad K_3 = -\begin{bmatrix} X' & X \\ Y' & Y \end{bmatrix} . \tag{112}$$

To do the same for the map (83) which describes the intertwining of the $\mathcal{L}$, we go back to (80),(81),(82) and express these in terms of (111). We shall write only the equations for the U-variables, the W are analogous, see (110).

We describe the mapping $\mathcal{R}_{123}^{(f)}$ (80) by (111), (112) with the parameters $\{f_k\}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}_0$:

$$U_1 = U(\{f_k\}, \mathcal{Y}, \mathcal{Z}_0) ; \quad U_2 = U(\{f_k\,\sigma_k(\mathcal{Y})\}, \mathcal{X}, \mathcal{Z}_0) ; \quad U_3 = U(\{f_k\}, \mathcal{X}, \mathcal{Y}) ;$$
$$U_1^\star = U(\{f_k\,\sigma_k(\mathcal{X})\}, \mathcal{Y}, \mathcal{Z}_0) ; \quad U_2^\star = U(\{f_k\}, \mathcal{X}, \mathcal{Z}_0) ; \quad U_3' = U(\{f_k\,\sigma_k(\mathcal{Z}_0)\}, \mathcal{X}, \mathcal{Y});$$
$$K_1 = -\begin{bmatrix} Y' & Y \\ Z_0' & Z_0 \end{bmatrix} ; \quad K_2 = -\begin{bmatrix} X' & X \\ Z_0' & Z_0 \end{bmatrix} ; \quad K_3 = -\begin{bmatrix} X' & X \\ Y' & Y \end{bmatrix} . \tag{113}$$

The corresponding parameters for $\mathcal{R}_{\tilde{1}\tilde{2}3}^{(f)}$ will be called $\{g_k\}, \mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}_1$:

$$\tilde{U}_1 = U\left(\{g_k\}, \mathcal{Y}, \mathcal{Z}_1\right) ; \quad \tilde{U}_2 = U\left(\{g_k\,\sigma_k(\mathcal{Y})\}, \mathcal{X}, \mathcal{Z}_1\right) ; \quad U_3' = U\left(\{g_k\}, \mathcal{X}, \mathcal{Y}\right) ;$$
$$\tilde{U}_1^\star = U\left(\{g_k\,\sigma_k(\mathcal{X})\}, \mathcal{Y}, \mathcal{Z}_1\right) ; \quad \tilde{U}_2^\star = U\left(\{g_k\}, \mathcal{X}, \mathcal{Z}_1\right); \quad U_3^\star = U\left(\{g_k\,\sigma_k(\mathcal{Z}_1)\}, \mathcal{X}, \mathcal{Y}\right) ;$$
$$\tilde{K}_1 = -\begin{bmatrix} Y' & Y \\ Z_1' & Z_1 \end{bmatrix} ; \quad \tilde{K}_2 = -\begin{bmatrix} X' & X \\ Z_1' & Z_1 \end{bmatrix} ; \quad \tilde{K}_3 = K_3. \tag{114}$$

Since U_3' of the first map (113) is the initial variable in the second map (114) we must take

$$g_k = f_k\,\sigma_k(\mathcal{Z}_0) . \tag{115}$$

Now the periodic condition (82) $U_3^\star = U_3$ requires that

$$g_k\,\sigma_k(\mathcal{Z}_1) \;=\; f_k \qquad \text{or} \qquad \sigma_k(\mathcal{Z}_0)\,\sigma_k(\mathcal{Z}_1) \;=\; 1\,, \tag{116}$$

which is equation (92). Inserting (115) into (114) we get the uniformization of the composite map $S_{12}^{(f)}$ (83) which solves the classical intertwining relations (78)

$$\mathcal{L}_2(\Lambda)\,\mathcal{L}_1(\Lambda) \;=\; \mathcal{L}_1^\star(\Lambda)\,\mathcal{L}_2^\star(\Lambda)\,.$$

4.4. Auxiliary classical L-operator, Isospectrality

We now introduce an auxiliary classical operator $\mathcal{L}_0^{aux}(\Lambda)$ which by successive intertwining through the chain and imposing a periodic condition will lead to an isospectral transformation of the classical transfer matrix tr $\mathcal{M}_1(\Lambda)$. The monodromy $\mathcal{M}_1$ is given by (109). Using the notation (110) the arguments of its operators $\mathcal{L}_{1,q}$ are

$$U_{1,q} \;=\; U\left(\left\{\,f_k\,\textstyle\prod_{j=1}^{q}\sigma_k^{-1}(\mathcal{Y}_j)\right\},\,\mathcal{Y}_q,\,\mathcal{Z}_0\right), \tag{117}$$

analogously $W_{1,q}$, see (110). $\tilde{U}_{1,q}$, $\tilde{W}_{1,q}$ are obtained replacing $\mathcal{Z}_0 \to \mathcal{Z}_1$ and $f_k \to f_k^\star = f_k\sigma_k(\mathcal{Z}_0)$.

We start writing (omitting the argument Λ which will always remain the same)

$$\mathcal{L}_0^{aux}\;\mathcal{L}_{1,0}\,\mathcal{L}_{1,1}\,\ldots\,\mathcal{L}_{1,Q-1} \;=\; \mathcal{L}_{1,0}^\star\,\mathcal{L}_1^{aux}\,\mathcal{L}_{1,1}\ldots\,\mathcal{L}_{1,Q-1}\,. \tag{118}$$

We shall achieve our goal using as auxiliary operator the operator $\mathcal{L}_{2,q}$, which is $\mathcal{L}_{1,q}$ of (107) with just the first indices "1" replaced by "2" (the second index $q\mathbf{e}_2$ is not modified). Then the intertwining in (118) is the same as that considered in the previous subsection and the arguments of the various Lax operators are as follows (again, as in (117), we write here only the $U_{i,j}$):

$$\begin{aligned} &\mathcal{L}_{1,0}: \quad U_{1,0} \;=\; U\left(\{f_k\},\mathcal{Y}_0,\mathcal{Z}_0\right); && \mathcal{L}_{1,0}^\star: \quad U_{1,0}^\star \;=\; U\left(\{f_k\,\sigma_k(\mathcal{X})\},\mathcal{Y}_0,\mathcal{Z}_0\right);\\ &\mathcal{L}_0^{aux}: \quad U_{2,0} \;=\; U\left(\{f_k\sigma_k(\mathcal{Y}_0)\},\mathcal{X},\mathcal{Z}_0\right); && \mathcal{L}_1^{aux}: \quad U_{2,0}^\star \;=\; U\left(\{f_k\},\mathcal{X},\mathcal{Z}_0\right). \end{aligned} \tag{119}$$

Moving $\mathcal{L}_1^{aux}(\Lambda)$ one step further through the monodromy matrix (109) we write

$$\mathcal{L}_{1,0}^\star\;\mathcal{L}_1^{aux}\;\mathcal{L}_{1,1}\,\mathcal{L}_{1,2}\,\ldots\,\mathcal{L}_{1,Q-1} \;=\; \mathcal{L}_{1,0}^\star\;\mathcal{L}_{1,1}^\star\;\mathcal{L}_2^{aux}\;\mathcal{L}_{1,2}\ldots\,\mathcal{L}_{1,Q-1}\,.$$

Again using (114) and (115), the variables of the new operators are:

$$\mathcal{L}_{1,1}^\star:\; U_{1,\mathbf{e}_2}^\star = U\left(\{f_k\sigma_k(\mathcal{X})/\sigma_k(\mathcal{Y}_1)\},\mathcal{Y}_1,\mathcal{Z}_0\right); \qquad \mathcal{L}_2^{aux}:\; U_{2,\mathbf{e}_2}^\star = U\left(\{f_k/\sigma_k(\mathcal{Y}_1)\},\mathcal{X},\mathcal{Z}_0\right).$$

Continuing this way, we finally arrive at

$$\mathcal{L}_0^{aux}(\Lambda)\,\mathcal{M}_1(\Lambda) \;=\; \mathcal{M}_1^\star(\Lambda)\,\mathcal{L}_Q^{aux}(\Lambda)\,, \tag{120}$$

where

$$\mathcal{M}_1^\star \;=\; \prod_{q'=0}^{Q-1}\tilde{W}_{1,q'\mathbf{e}_2}^{-1}\,\mathcal{L}_{1,0}^\star\;\mathcal{L}_{1,2}^\star\,\ldots\,\mathcal{L}_{1,q}^\star\,\ldots\,\mathcal{L}_{1,Q-1}^\star \tag{121}$$

and the operators in (120), (121) have the following arguments:

$$\mathcal{L}_{1,q}^\star: \qquad U_{1,q\mathbf{e}_2}^\star = U\left(\left\{f_k\,\sigma_k(\mathcal{X})\textstyle\prod_{j=1}^{q}\sigma_k^{-1}(\mathcal{Y}_j)\right\},\,\mathcal{Y}_q,\,\mathcal{Z}_0\right); \tag{122}$$

$$\mathcal{L}_q^{aux}: \qquad U_{2,q\mathbf{e}_2}^\star = U\left(\left\{f_k\textstyle\prod_{j=0}^{q-1}\sigma_k^{-1}(\mathcal{Y}_j)\right\},\,\mathcal{X},\,\mathcal{Z}_0\right). \tag{123}$$

We demand a periodic boundary condition for the auxiliary operator $\mathcal{L}_q^{aux}$:

$$\mathcal{L}_Q^{aux}(\Lambda) \;=\; \mathcal{L}_0^{aux}(\Lambda). \tag{124}$$

This is fulfilled if the constraint (91) is imposed, i.e. if

$$I_k(Q-1) \;\equiv\; \prod_{j=0}^{Q-1} \sigma_k^{-1}(\mathcal{Y}_j) \;=\; 1\,.$$

Since the $\mathcal{L}_q^{aux}$ depend on the points $\mathcal{Y}_q$ only via $\{f_k/\sigma_k(\mathcal{Y}_{q-1})\}$ and have the two other arguments fixed to be $\mathcal{X},\ \mathcal{Z}_0$ resp. $\mathcal{X},\ \mathcal{Z}_1$, the parameters $K_2,\,\tilde{K}_2$ in the $\mathcal{L}_q^{aux}$ are constant along the chain and are the same as in (113) and (114).

Now taking the trace of (120) and using (124), we see that the transfer matrices

$$\mathrm{tr}\,\mathcal{M}_1(\Lambda) \qquad \text{and} \qquad \mathrm{tr}\,\mathcal{M}_1^{\star}(\Lambda) \qquad \text{are isospectral.} \tag{125}$$

$\mathcal{M}_1$ is composed of the Lax operators with the arguments (117), the arguments of $\mathcal{M}_1^{\star}$ are given in (122). The classical integrals of motion T_k given by the generating series

$$\sum_{k=0}^{Q} \mathsf{T}_k\,\Lambda^k = \mathrm{tr}\;\mathcal{M}_1(\Lambda) \;=\; \mathrm{tr}\;\mathcal{M}_1^{\star}(\Lambda) \tag{126}$$

are invariants of the isospectral transformation.

Summarizing: Eq. (92) results from the periodicity in the $\mathbf{e}_3$-direction, while (91) guarantees the periodicity of the auxiliary operator in the $\mathbf{e}_2$-direction. Both together determine the parameters P_k and P'_k. Due to the substitution $f_k \;\to\; f_k^{\star} \;=\; f_k\sigma_k(\mathcal{X})$ in the variables, compare (117) with (122), there is a non-trivial isospectrality if some $f_k \neq 0$.

4.5. Isospectral transformation of the BS quantum chain

Finally, we like to generalize the isospectrality (125) of the classical BS-model to the quantum BS-model which is defined by the monodromy $\mathbf{M}(\xi)$ and transfer matrix $\mathbf{T}(\xi)$ of (88). For this we have to find explicit expressions for the operators $\mathbf{Q}$, L_{aux} and $\mathbf{M}^{\star}$ in

$$\mathbf{Q}\,L_{\mathrm{aux}}(\xi)\cdot\mathbf{M}(\xi) = \mathbf{M}^{\star}(\xi)\cdot L_{\mathrm{aux}}^{\star}(\xi)\,\mathbf{Q}\,. \tag{127}$$

Once this has been found, imposing periodicity $L_{\mathrm{aux}}(\xi) \;=\; L_{\mathrm{aux}}^{\star}(\xi)$ and taking the trace over both spaces $\mathbb{C}^2$ and $\mathbb{C}^N$, we will have the intertwining relation

$$\mathbf{K}\,\mathbf{T}(\xi) \;=\; \mathbf{T}^{\star}(\xi)\,\mathbf{K}\,; \qquad \mathbf{T} \;=\; \mathrm{Tr}_{\mathbb{C}^2}\,\mathbf{M}\,; \qquad \mathbf{K} \;=\; \mathrm{Tr}_{\mathbb{C}^N}\,\mathbf{Q}\,. \tag{128}$$

We start considering the intertwining of *two* quantum Lax operators (89):

$$\mathbf{S}_{p\,q}\,L_p(\xi;\,u_p,\,\tilde{u}_p,\ldots,\,\tilde{\kappa}_p)\cdot L_q(\xi;\,u_q,\,\tilde{u}_q,\ldots,\,\tilde{\kappa}_q)$$
$$=\; L_q(\xi;\,u_q^{\star},\,\tilde{u}_q^{\star},\ldots,\,\tilde{\kappa}_q)\cdot L_p(\xi;\,u_p^{\star},\,\tilde{u}_p^{\star},\ldots,\,\tilde{\kappa}_p)\,\mathbf{S}_{p\,q}. \tag{129}$$

The Lax operators are matrices both in $\mathbb{C}^2$ and $\mathbb{C}^N$. Written in components (129) takes the same form as (25). We also remark that in (129), instead of L_q of (89) we could also use $L(\lambda;\, q, q')$ of (24) since according to (59) the gauge transformations of (39) cancel: $P_q^{-1} P_p \;=\; 1$.

We want to find an explicit expression for $\mathbf{S}_{p\,q}$ for the case that the variables $u_p,\, u_q,\, etc.$ on the left of (129) are related to those on the right, $u_p^\star,\, u_q^\star,\, etc.$, by the functional functional mapping (83), i.e. by (113)–(116), where we take $p \leftrightarrow 2,\; q \leftrightarrow 1$.

If the functional mapping is chosen to be trivial, (129) reduces to the Bazhanov-Stroganov intertwining relation (25) with $\mathbf{S}_{p\,q}$ equal to S as given in (26), (42), (43). Then in the cross-section parametrization the relation depends on the four pairs $\mathcal{X},\, \mathcal{Y},\, \mathcal{Z}_0,\, \mathcal{Z}_1$. If the CP-parametrization is chosen, one has to use (68). It appears that the parametrization of the intertwining operator $\mathbf{S}$ in terms of the CP-functions x_p, x_q, $x_{p'}$ and $x_{q'}$ becomes inconvenient in the dynamical case since (68) and (69) seem to have no simple generalization to the case $\{f_k\} \neq \{0\}$.

However, the formulas for the Fermat points (67) have a nice dynamical extension, and this can be used if we construct the quantum intertwiner $\mathbf{S}$ from the 3D-operator $\mathcal{R}_{123}$ imposing periodical boundary conditions.

Now we can generalize the derivation of section 3.2 to the quantum case. In (9) we have seen that since $\omega^N = 1$, the map $\mathcal{R}_{123}$ acting in the space of rational functions of Weyl operators Ψ decomposes into the functional mapping $\mathcal{R}_{123}^{(f)}$ and a similarity transformation by the matrix $\mathbf{R}_{123}$. The composition of two such maps is

$$\begin{aligned}\mathcal{R}_{\tilde{1}\tilde{2}3}\,\mathcal{R}_{123}\circ\Psi \;&=\; \mathcal{R}_{\tilde{1}\tilde{2}3}^{(f)}\circ\Big(\mathbf{R}_{\tilde{1}\tilde{2}3}\cdot\mathcal{R}_{123}^{(f)}\circ\big(\mathbf{R}_{123}\cdot\Psi\cdot\mathbf{R}_{123}^{-1}\big)\cdot\mathbf{R}_{\tilde{1}\tilde{2}3}^{-1}\Big)\\ &= \mathcal{R}_{\tilde{1}\tilde{2}3}^{(f)}\,\mathcal{R}_{123}^{(f)}\circ\Big(\Big(\mathcal{R}_{123}^{(f)}\circ\mathbf{R}_{\tilde{1}\tilde{2}3}\Big)\cdot\mathbf{R}_{123}\cdot\Psi\cdot\mathbf{R}_{123}^{-1}\cdot\Big(\mathcal{R}_{123}^{(f)}\circ\mathbf{R}_{\tilde{1}\tilde{2}3}^{-1}\Big)\Big)\end{aligned} \tag{130}$$

where the conjugation matrices map

$$\mathbf{R}_{123}\;:\;\mathbf{u}_1,\mathbf{w}_1,\mathbf{u}_2,\mathbf{w}_2,\mathbf{u}_3,\mathbf{w}_3\;\;\mapsto\;\;\mathbf{u}_1^\star,\mathbf{w}_1^\star,\mathbf{u}_2^\star,\mathbf{w}_2^\star,\mathbf{u}_3',\mathbf{w}_3'\,, \tag{131}$$

and

$$\mathcal{R}_{123}^{(f)}\circ\mathbf{R}_{\tilde{1}\tilde{2}3}\;:\;\tilde{\mathbf{u}}_1,\tilde{\mathbf{w}}_1,\tilde{\mathbf{u}}_2,\tilde{\mathbf{w}}_2,\mathbf{u}_3',\mathbf{w}_3'\;\;\mapsto\;\;\tilde{\mathbf{u}}_1^\star,\tilde{\mathbf{w}}_1^\star,\tilde{\mathbf{u}}_2^\star,\tilde{\mathbf{w}}_2^\star,\mathbf{u}_3^\star,\mathbf{w}_3^\star\,. \tag{132}$$

Imposing the periodic boundary conditions

$$\mathbf{u}_3^\star = \mathbf{u}_3\,;\qquad\qquad \mathbf{w}_3^\star = \mathbf{w}_3\,, \tag{133}$$

which imply the classical ones (82), we define the quantum analog of the classical map (83) by

$$\mathbf{S}_{12} = \mathrm{tr}_3\;\Big(\mathcal{R}_{123}^{(f)}\circ\mathbf{R}_{\tilde{1}\tilde{2}3}\Big)\cdot\mathbf{R}_{123}\,. \tag{134}$$

An expression $\Big(\mathcal{R}_{123}^{(f)}\circ\mathbf{R}_{\tilde{1}\tilde{2}3}\Big)$ means that the Fermat point parameters which enter the matrix elements of the matrix $\mathbf{R}_{\tilde{1}\tilde{2}3}$ in the third quantum space should be taken after the functional mapping $\mathcal{R}_{123}^{(f)}$ of (14) has been applied.

Formula (134) provides a generalization of the Bazhanov-Stroganov intertwining operator (42). It performs a canonical map of normalized Weyl operators

$$\frac{\mathbf{u}_1^\star}{u_1^\star}\;=\;\mathbf{S}_{12}\cdot\frac{\mathbf{u}_1}{u_1}\cdot\mathbf{S}_{12}^{-1}\,;\qquad\quad \frac{\tilde{\mathbf{u}}_1^\star}{\tilde{u}_1^\star}\;=\;\mathbf{S}_{12}\cdot\frac{\tilde{\mathbf{u}}_1}{\tilde{u}_1}\cdot\mathbf{S}_{12}^{-1}\,;\quad \text{etc.} \tag{135}$$

The Fermat points x_1, x_2, x_3 determining the matrix $\mathbf{R}_{123}$ in (134) are calculated from (56) raised to the N-th power and then using (113). Analogously for $\mathbf{R}_{\tilde{1}\tilde{2}3}$ take the N-th power of (57) and use (114), (115). The result is

$$x_1^N = \begin{bmatrix} X & Y' \\ Z_0' & Z_0 \end{bmatrix} \frac{H(\{\mathbf{f}_k(X)\sigma_k(\mathcal{Y})\})\ H(\{\mathbf{f}_k(Y)\sigma_k(\mathcal{Z}_0)\})}{H(\{\mathbf{f}_k(Y)\}\}\ H(\{\mathbf{f}_k(X)\sigma_k(\mathcal{Y})\sigma_k(\mathcal{Z}_0)\})};$$

$$\tilde{x}_1^N = \begin{bmatrix} Y' & X \\ Z_0 & Z_0' \end{bmatrix} \begin{bmatrix} Y' & X \\ Z_1 & Z_1' \end{bmatrix} x_1^{-N}, \qquad etc. \tag{136}$$

The $\mathbf{R}$ are determined by the x_j rather than by the x_j^N. So we must take N-th roots. The possible choices of phases in this step have been discussed in [24].

The cancellation of the H-functions in the product $x_1^N \tilde{x}_1^N$ arises as follows: The matrix $\mathcal{R}_{123}^{(f)} \circ \mathbf{R}_{\tilde{1}\tilde{2}3}$ has its Fermat parameters given by the same formulas as $\mathbf{R}_{123}$, just with $\mathcal{Z}_0$ replaced by $\mathcal{Z}_1$, and the vector f_k replaced by $\tilde{f}_k = f_k\,\sigma_k(\mathcal{Z}_0)$. Then take into account the periodicity $\sigma_k(\mathcal{Z}_0)\,\sigma_k(\mathcal{Z}_1) = 1$, equation (92).

Given g, $\{f_k\}$ and the four pairs $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}_0$, $\mathcal{Z}_1$, from (136) we obtain the Fermat points which are then used to calculate the operator $\mathbf{S}_{12}$ from (48) with (50). Suppressing to indicate g, $\mathcal{Z}_0$, $\mathcal{Z}_1$ explicity, we shall write

$$\mathbf{S}_{12} \equiv \mathbf{S}(\{f_k\}, \mathcal{X}, \mathcal{Y}). \tag{137}$$

Finally, we consider the BS-quantum chain of length Q. The Lax operators forming the monodromy are taken to be of the same form as the operator L_q in (129), with the scalar variables as in (117). So, indicating the parameters we write, recall (95):

$$L_q(\xi;\, u_q, \tilde{u}_q, \frac{w_q}{\tilde{w}_q}, \kappa_q, \tilde{\kappa}_q) \equiv L(\xi;\, \{f_k\, I_k(q-1)\}, \mathcal{Y}_q, \mathcal{Z}_0, \mathcal{Z}_1). \tag{138}$$

so that the monodromy is

$$\mathbf{M}(\xi) = L(\xi;\{f_k\}, \mathcal{Y}_0, \mathcal{Z}_0, \mathcal{Z}_1) \cdot L(\xi;\{f_k/\sigma_k(\mathcal{Y}_1)\}, \mathcal{Y}_1, \mathcal{Z}_0, \mathcal{Z}_1) \cdot \ldots$$
$$\ldots \cdot L(\xi;\{f_k\, I_k(Q-1)\}, \mathcal{Y}_{Q-1}, \mathcal{Z}_0, \mathcal{Z}_1). \tag{139}$$

The auxiliary operator L_{aux} is chosen to be of the same form, like L_p in (129), with the scalar variables taken as for $\mathcal{L}_0^{aux}$ in (119):

$$L_{aux}(\xi) \equiv L(\xi;\{f_k\sigma_k(\mathcal{Y}_0)\}, \mathcal{X}, \mathcal{Z}_0, \mathcal{Z}_1).$$

Commuting the auxiliary operator through the chain proceeds in analogy to the classical case of section 4.4, only in each step there is also the conjugation by a matrix $\mathbf{S}(\{f_k I(q-1)\}, \mathcal{X}, \mathcal{Y}_q)$. So $\mathbf{M}^\star(\xi)$ is given substituting on the right hand side of (139) $f_k \to f_k\,\sigma_k(\mathcal{X})$ and we have

$$\mathbf{Q} = \mathbf{S}(\{f_k\}, \mathcal{X}, \mathcal{Y}_0) \cdot \mathbf{S}(\{f_k/\sigma_k(\mathcal{Y}_1)\}, \mathcal{X}, \mathcal{Y}_1) \cdot \ldots \cdot \mathbf{S}(\{f_k I(Q-1)\}, \mathcal{X}, \mathcal{Y}_{Q-1}). \tag{140}$$

Now all information necessary for the proof of Proposition 2 has been collected: The matrix conjugation by $\mathbf{K}$ does not change the spectrum so that we have to consider just the functional transform. Indeed, as seen in (108) $u_1^N \tilde{u}_1^N$ is independent of the $\{f_k\}$ and $u_1\mathbf{X}$ has the same $\{f_k\}$-dependence as $\tilde{u}_1\mathbf{X}^{-1}$ and this is written in the first equation of (96). Also, since $\mathbf{Z}_q$ appears only in the combination $(w_q/\tilde{w}_q)\,\mathbf{Z}_q$, from (108) we confirm the second line of (96). The two conditions (91) and (92) fixing the P_q, P'_q have already been encountered in (124) and (116).

The operator $\mathbf{K}$ explicitly given by (128),(140),(134) performs the isospectral transformation of the BS quantum transfer matrix since according to (128) the spectrum of the transfer matrices $\mathbf{T}(\xi;\{\,f_k\})$ and $\mathbf{T}(\xi;\{\,f_k\,\sigma_k(\mathcal{X})\,\})$ is the same.

5. Conclusion

This paper has been devoted to describe the Bazhanov-Stroganov quantum chain using the tools of the 3D integrable generalized Zamolodchikov-Baxter-Bazhanov model in the vertex formulation of [5]. The BS-Lax operator is constructed from the Linear Problem (4) imposing periodicity after two layers. The BS quantum intertwiner S, which is a product of four chiral Potts Boltzmann weights, is obtained applying twice the matrix conjugation part $\mathbf{R}_{123}$ of the 3D mapping operator $\mathcal{R}_{123}$. The corresponding functional operator $\mathcal{R}^{(f)}_{123}$ is used to solve the intertwining of two classical BS Lax-operator, a relation which would be difficult to obtain without the insight from 3D.

There are many possible parametrization of the 3D mapping operator $\mathcal{R}_{123}$. These give rise to different more and less convenient parameterizations of the BS-transfer matrix. The parametrization in terms of cross-ratios and rational Θ-functions H turns out to be the most useful and is adopted for the explicit description of the isospectral transformations of the classical and quantum BS-transfer matrices. Whether our results form a sufficient preparation for solving the problem of separation of variables for the BS-model, similarly to what has been tried for other models in the papers [14, 28], is subject of further work in progress.

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Appendix: Alternative definitions of the Bazhanov-Stroganov L-operator.

In this appendix we give the relation of two alternative formulations of the BS model to our definitions (24).

Appendix A.1. Bazhanov-Stroganov model

In the Bazhanov-Stroganov original paper [8] the L-operator has been defined by

$$L^{BS}(p;q,q')=\rho_1\begin{pmatrix} -c_pd_pb_qb_{q'}\mathbf{Z}^\rho+\omega a_pb_pd_qd_{q'}\mathbf{Z}^{-\rho} & b_pd_p\left(-\omega\, a_qd_{q'}\mathbf{Z}^{-\rho}+c_qb_{q'}\mathbf{Z}^\rho\right)\mathbf{X} \\ \omega\, a_pc_p\left(d_qa_{q'}\mathbf{Z}^{-\rho}-b_qc_{q'}\mathbf{Z}^\rho\right)\mathbf{X}^{-1} & -\omega\left(c_pd_pa_qa_{q'}\mathbf{Z}^{-\rho}+a_pb_pc_qc_{q'}\mathbf{Z}^\rho\right)\end{pmatrix}$$

where the chiral Potts variables $a_p,\ b_p,\ c_p,\ d_p,\ etc.$ are used, from which we obtain the variables of (17), (18) by

$$x_p=a_p/d_p;\quad y_p=b_p/c_p;\quad \mu_p=d_p/c_p\,,\quad \text{analogously with indices}\quad p',\ q,\ q'\,.$$

The operators $\mathbf{X}$ and $\mathbf{Z}$ satisfy (5), i.e. $\mathbf{X}\,\mathbf{Z}\,=\,\omega\,\mathbf{Z}\,\mathbf{X}$ and the representation with $\mathbf{X}$ diagonal, given in (6), is used. Furthermore, $\rho=(N-1)/2\,,$ and ρ_1 is a constant. Extracting some factors, we arrive at

$$L^{BS}(p;\,q,q')=\rho_1\omega a_pb_pd_qd_{q'}\;\mathbf{Z}^{-\rho}\begin{pmatrix} 1\,+\,\lambda_1\lambda_2\frac{y_qy_{q'}}{\mu_q\mu_{q'}}\mathbf{Z} & \lambda_1\mathbf{X}^{-1}\left(x_q-\frac{y_{q'}}{\mu_q\mu_{q'}}\,\mathbf{Z}\right) \\ \lambda_2\mathbf{X}\left(\omega x_{q'}-\frac{y_q}{\mu_q\mu_{q'}}\,\mathbf{Z}\right) & \lambda_1\lambda_2\,\omega\,x_qx_{q'}\,+\frac{1}{\mu_q\mu_{q'}}\,\mathbf{Z}\end{pmatrix}, \tag{A.1}$$

where

$$\lambda_1\;=\;-\frac{1}{x_p};\qquad \lambda_2\;=\;\frac{1}{\omega\,y_p}\,. \tag{A.2}$$

A simple gauge transformation by $P=diag(\lambda_2^{1/2},\lambda_2^{-1/2})$ gives us the relation

$$L^{BS}(p;\,q,q')\;=\;\rho_1\,\omega a_pb_pd_qd_{q'}\;\mathbf{Z}^{-\rho}P\;\;L(\lambda_1\lambda_2;\,q,q')\;\;P^{-1} \tag{A.3}$$

with our $L(\lambda;\,q,q')$ of (24).

Appendix A.2. $\tau^{(2)}(t_q)$-model

In equation (5.33) of [8] a monodromy matrix $\tau_k^{(2)}$ is introduced as the starting object for a fusion procedure. $\tau_k^{(2)}$, which essentially is the monodromy of the BS model (27), has played a major role in solving the Chiral Potts model [17, 18, 19, 20]. Since in many papers the definitions of [17] have been used, we give the relation to our (27).
Equation (3.44) of [17] is (as in (27), we denote the length of the system by Q):

$$\left[\tau_{k,q}^{(j)}\right]_{\sigma,\sigma'}\;=\;\sum_{m_0,\ldots,m_{Q-1}=0}^{j-1}\;\prod_{J=1}^{L}\left\{\omega^{-m_J\left(\sigma_{J+1}-\sigma'_J+k\right)}\;\frac{\eta_{q,j,\sigma_J-\sigma'_J+k}}{\eta_{q,j,m_J}}\times\right.$$
$$\left.\times\;\;F_{pq}(j,\sigma_J-\sigma'_J+k,m_J)\;F_{p'q}(j,\sigma_{J+1}-\sigma'_{J+1}+k,m_J)\right\}\;;$$
$$\left[\tau_{k,q}^{(j)}\right]_{\sigma,\sigma'}=0\qquad\text{if}\quad j-k\le\sigma_J-\sigma'_J<N-k\qquad\text{for any}\;\;J. \tag{A.4}$$

The $\tau_{k,q}^{(j)}$ are transfer matrices leading from the initial Z_N spins $\sigma\,=\,\{\sigma_0,\ldots,\sigma_{Q-1}\}$ to the final indices $\sigma'\,=\,\{\sigma'_0,\ldots,\sigma'_{Q-1}\}$. There is a fusion hierarchy $j\,=\,0,\ldots,N$ of the $\tau_{k,q}^{(j)}$ which is exploited in the applications to the CPM. Of direct interest here is the lowest non-trivial case $j = 2$. The index k labels a redundancy, see (3.51) of [17], we shall take $k = 0$. Then the second part of (A.4) tells us that the $\left[\tau^{(j)}_{0,q}\right]_{\sigma,\sigma'}$ are non-vanishing only if for all J we have $\sigma_J - \sigma'_J = 0$ or 1. So we can express the spin dependence in terms of the unit matrix at J and the two standard matrices $\mathbf{X}_J$ and $\mathbf{Z}_J$.

Equation (3.37) of [17] defines

$$\eta_{q,j,\alpha} \;=\; \omega^{j\,\alpha}\, t_q^{\alpha} \prod_{\ell=\alpha+1}^{\alpha+N-j} (1-\omega^{\ell}) \qquad \text{with} \qquad t_q = x_q\, y_q$$

from which here we need only $\eta_{q,2,1}/\eta_{q,2,0} \;=\; -\omega\, x_q\, y_q$. Equation (3.38) of [17] gives

$$F_{pq}(j,\alpha,m) \;=\; \mu_p^{\alpha}\, y_p^{\alpha-m}\, \Phi(t_p,\omega^{\alpha} t_q)_m^{\alpha,\,j-\alpha-1} \quad \text{for} \quad 0\le \alpha,\; m<j\le\; N,$$

where $\Phi(x,y)_i^{m,n}$ is a polynomial in x, y which is expressible in terms of Gauss polynomials. We need only (equation (3.48) of [17]):

$$F_{pq}(2,0,0)=1;\;\; F_{pq}(2,0,1)=-\omega\frac{t_q}{y_p};\;\; F_{pq}(2,1,0)=\frac{\mu_p}{y_p};\;\; F_{pq}(2,1,1)=-\omega\frac{x_p\mu_p}{y_p}\,.$$

Let us write

$$s_J=\sigma_J-\sigma'_J; \qquad \eta_i=\begin{cases} 1 & i=0\\ -\omega\, t_q & i=1.\end{cases}$$

Then, omitting the first argument $j=2$ of $F(j,\ldots,\ldots)$ and the index $k=0$ of $\tau^{(2)}_{k,q}$:

$$\left[\tau^{(2)}_q\right]_{\sigma,\sigma'} \;=\; \sum_{m_0,m_1,m_2,\ldots} \omega^{m_0(\sigma'_0-\sigma_1)}\frac{\eta_{s_0}}{\eta_{m_0}}\, F_{pq}(s_0,m_0)\, F_{p'q}(s_1,m_0)\, \omega^{m_1(\sigma'_1-\sigma_2)}\frac{\eta_{s_1}}{\eta_{m_1}}\, F_{pq}(s_1,m_1)\,\times$$
$$\times\;\; F_{p'q}(s_2,m_1)\,\omega^{m_2(\sigma'_2-\sigma_3)}\,\frac{\eta_{s_2}}{\eta_{m_2}}\, F_{pq}(s_2,m_2)\, F_{p'q}(s_3,m_2)\;\times\ldots$$

We collect all terms containing the spins $\sigma_J,\ \sigma'_J$ into a 2×2-matrix τ_J:

$$(\tau_J)_{m_{J-1},m_J} = \omega^{m_J\sigma'_J - m_{J-1}\sigma_J}\,\frac{\eta_{s_J}}{\eta_{m_J}} F_{p'q}(s_J,m_{J-1})F_{pq}(s_J,m_J)\,. \tag{A.5}$$

So that the transfer matrix is

$$\left[\tau^{(2)}_q\right]_{\sigma,\sigma'} = Tr\;\tau_0\,\tau_1\,\tau_2\,\ldots\,\tau_{Q-1}. \tag{A.6}$$

We now write (A.5) in matrix form:

$$(\tau_J)_{m_{J-1},m_J} \;=\; \begin{pmatrix} 1 & \omega^{\sigma'_J}/y_p \\ -\omega^{-\sigma_J}\omega t_q/y_{p'} & -\omega t_q/(y_p y_{p'}) \end{pmatrix}_{m_{J-1},m_J} \delta_{\sigma_J,\sigma'_J}$$
$$+\;\omega\,\frac{\mu_p\mu_{p'}}{y_p y_{p'}} \begin{pmatrix} -t_q & -\omega^{\sigma'_J}x_p \\ \omega^{-\sigma_J}\omega x_{p'} t_q & \omega^{-s_J}\omega x_p x_{p'} \end{pmatrix}_{m_{J-1},m_J} \delta_{\sigma'_J,\sigma_J-1}$$
$$=\;\frac{1}{\lambda}\,\frac{\mu_p\mu_{p'}}{y_p y_{p'}} \begin{pmatrix} 1+\lambda\frac{y_p y_{p'}}{\mu_p\mu_{p'}}\mathbf{Z}_J; & \lambda\,\mathbf{X}_J\left(\omega x_p - \frac{y_{p'}}{\mu_p\mu_{p'}}\mathbf{Z}_J\right) \\ \mathbf{X}_J^{-1}\left(x_{p'} - \frac{y_{p'}}{\mu_p\mu_{p'}}\mathbf{Z}_J\right); & \lambda\,\omega\, x_p x_{p'} + \frac{1}{\mu_p\mu_{p'}}\mathbf{Z}_J \end{pmatrix}_{m_{J-1},m_J} \mathbf{Z}_J^{-1},$$

where we put $\lambda = -1/(\omega t_q)$, compare this with $\lambda_1\lambda_2$ in (A.2). So (L^T denotes the 2×2 matrix transpose of L)

$$(\tau_J)_{m_{J-1},m_J} = \frac{1}{\lambda}\frac{\mu_p\mu_{p'}}{y_p y_{p'}}\, P\, L_J^T\left(\frac{-1}{\omega t_q};\, p',\, p\right)\, P^{-1}\mathbf{Z}_J^{-1}; \qquad P=\begin{pmatrix}\lambda^{1/2} & 0\\ 0 & \lambda^{-1/2}\end{pmatrix}. \quad \text{(A.7)}$$

Observe that Baxter's variable t_q corresponds to our spectral parameter, hence the polynomial dependence of the transfer matrix on t_q.

References

[1] Zamolodchikov A B 1981 *Tetrahedron equations and the relativistic S-matrix of straight strings in 2+1 dimensions*, *Commun. Math. Phys.* **79** 489–505

[2] Bazhanov V V and Baxter R J 1992 *New solvable lattice models in three dimensions*, *J. Stat. Phys.* **69** 453–485

[3] ——1993 *J. Stat. Phys.* **71** 839

[4] Sergeev S M, Mangazeev V V and Stroganov Yu G 1996 *The vertex formulation of the Bazhanov-Baxter Model*, *J. Stat. Phys.* **82** 31

[5] Sergeev S M 1999 *Quantum 2+1 evolution model*, *J. Phys. A: Math. Gen.* **32** 5693–5714

[6] ——2002 Second PhD thesis

[7] ——*Quantum integrable models in discrete 2+1-dimensional space-time*, *Phys. of Particles and Nuclei* **35** (2004) 1–31

[8] Bazhanov V and Stroganov Yu 1990 *Chiral Potts model as a descendant of the six-vertex model*, *J. Stat. Phys.* **59** 799–817

[9] Baxter R J, Perk J H H and Au-Yang H 1988 *New solutions of the star-triangle relations for the chiral Potts model*, *Phys. Lett.* **A128** 138–142

[10] Au-Yang H and Perk J H H 1995 *The chiral Potts models revisited*, *J. Stat. Phys.* **78** 17–78

[11] ——1997 *The many faces of the chiral Potts model*, *Int. J. Mod. Phys.* **B11** 11–26

[12] ——2002 *Star-Triangle Equations and Identities in Hypergeometric Series*, *Int. J. Mod. Phys.* **B16** 1853-1865.

[13] Hirota R *The Direct Method in Soliton Theory*, *Cambridge tracts in Mathematics* **155**, Cambridge University Press, 2004

[14] Pakuliak S and Sergeev S 2002 *Quantum relativistic Toda chain at root of unity: isospectrality, modified Q-operator and functional Bethe ansatz*, *Intern. J. of Math. and Math. Sc.* **31** 513–554

[15] Matveev V B and Smirnov A O 1990 *Some comments on the solvable chiral Potts model*, *Lett. Math. Phys.* **19** 179–185

[16] Tarasov V O 1992 *Cyclic monodromy matrices for the R-matrix of the six-vertex model and the chiral Potts model with fixed spin boundary conditions*, *Int. J. Mod. Phys.* **A7** Suppl. **1B**, 963–975

[17] Baxter R J, Bazhanov V V and Perk J H H 1990 *Functional relations for the transfer matrices of the Chiral Potts model*, *Int. J. Mod. Phys.* **B** 4 803 – 869

[18] Baxter R J 2004 *Transfer matrix functional relations for the functional relations for the generalized $\tau_2(t_q)$ model*, *J. Stat. Phys.* **117** 1–25 *The six and eight-vertex models revisited*, *J. Stat. Phys.* **114** 43–66

[19] ——2005 *The order parameter of the chiral Potts model*, *Phys. Rev. Lett.* **94** 130602

[20] Au-Yang H, Jin Bai-Qi and Perk J H H 2001 *Baxter's solution for the free energy of the chiral Potts model*, *J. Stat. Phys.* **102** 471-499

[21] Roan S S *Chiral Potts Rapidity Curve Descended from Six-vertex Model and Symmetry Group of Rapidities*, cond-mat/0410011

[22] Bazhanov V, Bobenko A and Reshetikhin N 1996 *Quantum discrete sine-Gordon model at roots of 1*, *Commun. Math. Phys.* **175** 377–400

[23] Isaev A P and Sergeev S M 2003 *Quantum Lax operators and discrete 2+1 dimensional integrable models*, *Lett. Math. Phys.* **64** 57–64

[24] von Gehlen G, Pakuliak S and Sergeev S 2003 *Explicit free parameterization of the Modified Tetrahedron Equation*, *J. Phys. A: Math. Gen.* **36** 975–998

[25] ——2004 *The modified tetrahedron equation and its solutions*, *Int. J. Mod. Phys.* **A19** Suppl. 179-204

[26] ——2004 *Theta function parametrization and fusion for 3D integrable Boltzmann weights*, *J. Phys. A: Math. Gen.* **37** 1159–1179

[27] Pakuliak S and Sergeev S 2003 *Spectral curves and parametrization of a discrete integrable three-dimensional model*, *Theor. and Math. Phys.* **136**(1) 917–935

[28] Sergeev S 2002 *Functional equations and quantum separation of variables for 3D spin models*, Preprint MPI 02-46

[29] ——1999 *Solitons in a 3d integrable model*, solv-int/9904013