Analytic Continuation of Fundamental Solutions to Elliptic Equations

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MPI/95-77

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August 3, 1995

Abstract

The theorem on existence of endlessly continuable (that is, analytic everywhere in $\mathbf{C}^n$ except for an analytic set of codimension 1) fundamental solution for a differential equation with polynomial coefficients is proved.

Introduction

The problem of analytic continuation of solutions to differential equations to complex domains arises in a lot of problems of asymptotic theory of differential equations and mathematical physics. Among them we mention the problem of continuation of wave fields outside their initial domain of definition (and, consequently, the investigation of stability of computational algorythms and minimizing of antenna size) in the electrodynamics, the so-called "mother body" problem in gravity theory and geophysics, the problem of constructing exact semi-classical asymptotics in quantum mechanics and others.

It seems that the most natural way of investigation of such problems is to construct an analytic continuation for a fundamental solutions to the corresponding differential equations. Such an investigation is exactly the main goal of the present paper.

*Until September 30, 1995.

As a tool of investigating fundamental solutions we have chosen a notion of an *elementary solution* for differential operators in complex domains introduced by the authors (see [1], [2]). We shall show that a fundamental solution for differential operators can be expressed in terms of the corresponding elementary solution and, hence, to investigate the analytic continuation of a fundamental solution, one can use the information about the analytic properties of the corresponding elementary solution.

Thus, the outline of the paper is as follows:

1. In Section 1, we construct an analytic continuation of an elementary solution for partial differential operators with polynomial coefficients. The main result of this section is the theorem on *endless continuability*[1] *of the elementary solutions* for such operators, provided that the velocity of propagation of singularities of solutions to the corresponding differential equation is finite (see Condition 1 below). Here we describe also the singularity set of the constructed elementary solution.

2. In Section 2, we investigate *analytic continuations of fundamental solutions* to partial differential equations. The case of differential equations with constant coefficients is considered separately, since the formulas for a fundamental solution, in this case, are more explicit. We present here also an example of computation of a fundamental solution for the Laplace operator in $\mathbf{R}^3$ with the help of our formulas. The last subsection of this section contains the investigation of the general case fo equations with polynomial coefficients.

1 Construction of elementary solution

In this section, we shall prove the existence of an endlessly continuable elementary solution for a differential operator with polynomial coefficients in the complex space $\mathbf{C}^n_x$. This result will be used in the next section for constructing analytic continuations of fundamental solutions to elliptic differential equations in the real space.

1.1 Statement of the problem

Consider a differential operator

$$H\left(x, -\frac{\partial}{\partial x}\right) = \sum_{|\alpha|\leq m} P_\alpha(x)\left(-\frac{\partial}{\partial x}\right)^\alpha \tag{1}$$

[1]The definition of this notion the reader can find below.

with polynomial coefficients $P_\alpha(x)$ in variables $x = (x^1, \ldots, x^n) \in \mathbf{C}^n_x$. Denote by

$$H(x,p) = \sum_{k=0}^{m} H_k(x,p)$$

the (full) symbol of operator (1), where $H_k(x,p)$ are homogeneous components of this symbol:

$$H_k(x,p) = \sum_{|\alpha|=k} P_\alpha(x)\, p^\alpha.$$

The function $H_m(x,p)$ is called, as usual, the *principal symbol* of operator (1).

First of all we recall the definition of the notion of elementary solution (see [1], [2]).

Definition 1 A function $G = G(x, q_0, q, t)$, $q_0 \in C$, $q = (q_1, \ldots, q_n)$ is called an *elementary solution* for operator (1) if this function is a solution to the following Cauchy problem:

$$\begin{cases} \left[\frac{\partial}{\partial t}\left(\frac{\partial}{\partial q_0}\right)^{m-1} + H\left(x, -\frac{\partial}{\partial x}\right)\right] G(x, q_0, q, t) = 0, \\ G(x, q_0, q, 0) = \ln\,(q_0 + qx), \end{cases} \tag{2}$$

$qx = q_1x^1 + \ldots + q_nx^n$, having the form

$$G(x, q_0, q, t) = \ln\,[q_0 + S(x,q,t)] \sum_{k=0}^{\infty} \frac{(q_0 + S(x,q,t))^k}{k!} a_k(x,q,t) \tag{3}$$

with some regular coefficients $a_k(x,q,t)$, for small values of t. Here the function $S(x,q,t)$ is defined as a solution to the following Cauchy problem for the Hamilton-Jacobi equation:

$$\begin{cases} \frac{\partial S}{\partial t} + H_m\left(x, \frac{\partial S}{\partial x}\right) = 0, \\ S(x,q,0) = qx. \end{cases} \tag{4}$$

Remark 1 Problem (2) has, in general, nonunique solution. However, if we require that the solution to this problem is searched in the form (3), then the unique solution is selected. To understand what information needed for the selection of a unique solution is contained in representaion (3), one can rewrite (2) in the form of a Cauchy problem

$$\begin{cases} \left[\frac{\partial}{\partial t} + \left(\frac{\partial}{\partial q_0}\right)^{1-m} H\left(x, -i\frac{\partial}{\partial x}\right)\right] G(x, q_0, q, t) = 0, \\ G(x, q_0, q, 0) = \ln\,(q_0 + qx), \end{cases} \tag{5}$$

where the function G is supposed to belong to the space $\mathcal{A}_\alpha(\Sigma_G)$, $-1 < \alpha < 0$,

$$\Sigma_G = \{q_0 + S(x,q,t) = 0\}. \tag{6}$$

(We recall [2] that the space $\mathcal{A}_\alpha(\Sigma_G)$ consists of functions $f(x, q_0, q, t)$ such that

$$|f(x, q_0, q, t)| \le C\,|q_0 + S(x, q, t)|^\alpha.)$$

Problem (5) has a unique solution, unlike problem (2). One shall easily verify that the operator $(\partial/\partial q_0)^{-1}$ is well-defined in the space $\mathcal{A}_\alpha(\Sigma_G)$ with $\alpha > -1$ as an operator

$$\left(\frac{\partial}{\partial q_0}\right)^{-1} : \mathcal{A}_\alpha(\Sigma_G) \to \mathcal{A}_{\alpha+1}(\Sigma_G).$$

Clearly, set (6) is a singularity set of the function G.

We require the following condition on the trajectories of the Hamiltonian system corresponding to Hamilton-Jacobi equation (4):

Condition 1 For any positive number ε and any compact set $K \subset \mathbf{C}_x^n$ there exist a compact set $K' \subset \mathbf{C}_x^n$ such that any trajectory of the Hamiltonian system

$$\begin{cases} \dot{x} = \dfrac{\partial H_m(x,p)}{\partial p}, \\ \dot{p} = -\dfrac{\partial H_m(x,p)}{\partial x}, \end{cases} \quad \begin{cases} x|_{t=0} = x_0, \\ p|_{t=0} = q \end{cases} \tag{7}$$

with the origin in the complement of K' does not intersect K for $|\tau| < \varepsilon$ (here τ is a parameter along the trajectory).

Remark 2 This condition makes sense since the solution

$$\begin{cases} x = x(x_0, q, t), \\ p = p(x_0, q, t) \end{cases}$$

determines homogeneous functions $x(x_0, q, t)$ and $p(x_0, q, t)$ fo orders 0 and 1, correspondingly, with respect to the following action of the group $\mathbf{C}_*$:

$$\lambda(x_0, q, t) = \left(x_0, \lambda q, \lambda^{1-m} t\right).$$

In Subsection 1.2 below, we prove the existence of an endlessly continuable elementary solution, and, in Subsection 1.3, we investigate singularities of the constructed solution.

1.2 Existence of an elementary solution

First of all, we recall the definition of an endlessly continuable function.

Definition 2 The function $F(x)$ defined in some domain of the complex plane $\mathbf{C}_x^n$ is called to be an *endlessly continuable* function if for any given positive number L there exists an analytic set Y such that $F(x)$ can be analytically continued along any path of the length less than L avoiding the set Y.

Roughly speeking, this definition means that the function $F(x)$ in question can be analytically continued up to a (ramifying) analytic function with analytic set of singularities on its Riemannian surface.

The proof of the existence of an endlessly continuable elementary solution will be divided into the two following steps:

1. First, we shall prove that there exists a positive number ε such that the solution to (2) of the form (3) exists for $|t| < \varepsilon$, and arbitrary x and q.

2. Second, we shall globalize the obtained solution in the variable t using the so-called *step-by-step method.*

Let us proceed with the first of the above steps. We denote by r_α the degree of the polynomial $P_\alpha(x)$ in (1). Enlarging, if required, the numbers r_α, one can assume that there exists a number k such that

$$r_\alpha = k + |\alpha|$$

for any multiindex α with $|\alpha| \leq m$. These enlarged numbers r_α will be, as above, referred as the degrees of the polynomials $P_\alpha(x)$; certainly, for such a treatment to be possible, one should consider polynomials of degree r_α with possibly vanishing principal part.

Denote

$$\mathcal{P}_\alpha\left(x^0, x\right) = \left(x^0\right)^{r_\alpha} P_\alpha\left(\frac{x}{x^0}\right).$$

Under the above assumptions, the functions $\mathcal{P}_\alpha\left(x^0, x\right)$ are homogeneous polynomials in $\left(x^0, x\right)$ of degree r_α. Let $\mathcal{G}\left(x^0, x, q_0, q, t\right)$ be the solution to the following Cauchy problem:

$$\begin{cases} \left[\dfrac{\partial}{\partial t}\left(\dfrac{\partial}{\partial q_0}\right)^{m-1} + \sum\limits_{|\alpha|\leq m} \mathcal{P}_\alpha\left(x^0, x\right)\left(-\dfrac{\partial}{\partial x}\right)^\alpha\right] \mathcal{G}\left(x^0, x, q_0, q, t\right) = 0, \\ \mathcal{G}\left(x^0, x, q_0, q, 0\right) = \ln\left(q_0 x^0 + q x\right), \end{cases} \tag{8}$$

having the form

$$\mathcal{G}\left(x^0, x, q_0, q, t\right) = \ln\left[q_0 + S\left(x^0, x, q, t\right)\right] \sum_{k=0}^{\infty} \frac{\left(q_0 + S\left(x^0, x, q, t\right)\right)^k}{k!} a_k\left(x^0, x, q, t\right)$$

with an appropriate choice of the function $S\left(x^0, x, q, t\right)$. The following affirmation is valid:

Proposition 1 *The solution $\mathcal{G}(x^0, x, q_0, q, t)$ to system* (8) *exists in a conical neighborhood of any point (x^0, x) for $|t| < \varepsilon$ and arbitrary (q_0, q). This solution is a homogeneous function of order* 0 *with respect to the following action of the group $\mathbf{C}_*$*[2]*:*

$$\lambda\left(q_0, q, x^0, x\right) = \left(\lambda^{-\frac{k}{m-1}} q_0, \lambda^{-\frac{k}{m-1}} q, \lambda x^0, \lambda x\right) \tag{9}$$

for any fixed value of t.

Proof. First of all we notice, that the existence of the solution to (8) in a (non conical) neighborhood of any point (x^0, x) is proved in the book [2] (see Subsection 5.2.3 there). The homogeneity properties with respect to the action (9) of $\mathbf{C}^*$ of the solution to problem (8) can be verified by the straightforward computations, and we leave the corresponding verification to the reader. Therefore, this solution can be continued, due to the homogeneity, into a conical neighborhood of the point (x^0, x). The proof is complete.

The globalization of the obtained solution in variables (x^0, x) is quite simple. It follows from the Heine-Borel lemma if one takes into account the uniqueness of solution to problem (8). The latter follows from the fact that this problem can be rewritten as a Cauchy problem similar to problem (5) (see Remark 1 above).

Later on, it is quite evident that the solution to (2) can be obtained from the solution to (8) by putting $x^0 = 1$. Thus, we arrive at the following result:

Proposition 2 *Under the above assumptions there exists a positive number ε such that the solution $G(x, q_0, q, t)$ to* (2) *of the form* (3) *exists for $|t| < \varepsilon$ globally in the variables (x, q_0, q).*

This affirmation completes the first step of the construction of the elementary solution.

To perform the second step one should first construct a formula which gives a solution to the problem

$$\begin{cases} \left[\dfrac{\partial}{\partial t}\left(\dfrac{\partial}{\partial q_0}\right)^{m-1} + H\left(x, -\dfrac{\partial}{\partial x}\right)\right] U(x, q_0, t) = 0, \\ U(x, q_0, t)\big|_{t=\tau} = U_0(x, q_0), \end{cases} \tag{10}$$

similar to (2) but with arbitrary Cauchy data $U_0(x, q_0)$ given at arbitrary value τ of the variable t (at least for sufficiently small values of $|t-\tau|$). This can be done with the help of the above constructed elementary solution. Namely, consider the integral

$$\begin{aligned} U(x, q_0, t) &= \left(-\frac{i}{2\pi}\right)^n \left(\frac{\partial}{\partial q_0}\right)^{n+1} \int_h G(x, q_0', q, t-\tau)\big|_{q_0' = q_0 - \widetilde{q}_0 - qy} \\ &\times \; U_0(y, \widetilde{q}_0)\, dy \wedge d\widetilde{q}_0 \wedge dq \end{aligned} \tag{11}$$

[2]One should take into account that the solution to problem (8) must be considered modulo holomorphic functions (this means that $\mathcal{G}$ is a hyperfunction). Under this treatment, for example, the function $\ln\left[q_0 + S(x^0, x, q, t)\right]$ is a homogeneous function of order zero.

over a ramifying homology class

$$h = h(x, q_0, q, t) \tag{12}$$

which will be defined below (here by qy we denote the sum $q_1 y^1 + \ldots + q_n y^n$). Then the following affirmation is valid:

Proposition 3 *For some concrete choice of ramifying class* (12), *formula* (11) *gives a solution to* (10) *for* $|t - \tau| < \varepsilon$.

Proof. The fact that the function given by (11) satisfies the equation involved into problem (10) is a direct consequence of the equation involved into problem (2) for $G(x, q_0, q, t)$; this affirmation is valid for any choice of the ramifying class $h(x, q_0, q, t)$. Let us substitute function (11) into the left-hand part of the initial condition of problem (10). We obtain

$$\begin{aligned} U(x, q_0, t)|_{t=\tau} &= \left(-\frac{i}{2\pi}\right)^n \left(\frac{\partial}{\partial q_0}\right)^{n+1} \int_h G(x, q_0 - \widetilde{q}_0 - qy, q, 0) \\ &\times \; U_0(y, \widetilde{q}_0)\, dy \wedge d\widetilde{q}_0 \wedge dq = n! \left(-\frac{i}{2\pi}\right)^n \int_h \frac{U_0(y, \widetilde{q}_0)\, dy \wedge d\widetilde{q}_0 \wedge dq}{(q_0 - \widetilde{q}_0 + q(x - y))^{n+1}}. \end{aligned}$$

Suppose that $h = \delta h_1$ where δ is the Leray coboundary homomorphism and h_1 is a standard homology class involved into the integral representation

$$U_0(x, q_0) = n! \left(-\frac{i}{2\pi}\right)^n \int_h \operatorname{Res} \frac{U_0(y, \widetilde{q}_0)\, dy \wedge d\widetilde{q}_0 \wedge dq}{(q_0 - \widetilde{q}_0 + q(x - y))^{n+1}} \tag{13}$$

(see [2], Subsection 3.1.2). Then the initial conditions in (10) are fulfilled.

Unfortunately, the class $h = \delta h_1$ cannot be lifted onto the Riemannian surface of the function G since any representative of this class must encircle the singularities

$$q_0 - \widetilde{q}_0 + q(x - y) = 0$$

of the integrand in (13) whereas the function G has the singularity of logarithmic type on this latter set. So, we must cut a representative of δh_1 along a contour which is a shift of some representative of h_1 onto δh_1 and then glue to this cut two chains with boundaries in the set Σ_{U_0} of singularities of the function $U_0(y, \widetilde{q}_0)$ (more detailed description of classes of this kind the reader can find in the above cited book [2], pp 385 - 388). Thus, we obtain the class

$$h(x, q_0, q, t) \in H_{2n+1}\left(\mathbf{C}^{2n+1}, \Sigma_{U_0}\right),$$

for which the initial conditions for Cauchy problem (10) are fulfilled.

Up to the moment, our considerations were of the formal character, that is, we have not paid attention to the domain of definition of integral (11) determining the required solution to Cauchy problem (10). To complete the proof, one should show, that this integral determines an analytic function in variables (x, q_0, t) having only analytic singularities for $|t| < \varepsilon$. This follows from Condition 1 above.

Actually, let K be a compact set in the space $\mathbf{C}^n_x$. Then, due to the Condition 1, there exists a ball B_R of some positive radius R in the space $\mathbf{C}^n_x$ such that any trajectory originated at any point outside B_R do not intersect K for $|t| < \varepsilon$. Now let us consider the function $U_0(y, \widetilde{q}_0)$ involved into the integrand of (11) as a function defined in the set $B_R \setminus \Sigma_{U_0}$ (or, more exactly, on the Riemannian surface over this set. Then, the considerations similar to those used in the proof of Proposition 5.14 of the book [2] (see pp. 391 - 392 there) show that for the point (x, q_0, t) to be a singularity of integral (11) it is nesessary that there exists some solution to the Hamiltonian system (7) coming to the point (x, q_0) from some point of any strata of $\partial B_R \cup \Sigma_{U_0}$ with the value of the parameter along the trajectory equal to t. From the other hand, the choice of the ball B_R guarantees that the trajectories of system (7) emanated from points of the boundary strata of the set $\partial B_R \cup \Sigma_{U_0}$ (that is, of the strata which are contained in ∂B_R) do not reach the compact K. Hence, the intersection of the set of singularity of integral (11) with the set K equals the set of endpoints of the trajectories emanated from analytic strata of the set Σ lying inside the ball B_R (that is, of the points of these trajectories corresponding to the value of the parameter equal to t), and, therefore, this intersection is an analytic set. Since our considerations are valid for any compact set $K \subset \mathbf{C}^n_x$, it is clear that the set of singularities of integral (11) is an analytic set for $|t| < \varepsilon$ and for all values of (x, q_0). This completes the proof of the Proposition.

Now we can state and prove the main assertion of this section.

Theorem 1 *Under the above formulated conditions, the elementary solution for operator* (1) *is an endlessly continuable analytic function of the variables* (x, q_0, q, t).

Proof. We must check that for any given positive number L there exists an analytic set Y such that for any path of length less than L in the space with variables (x, q_0, q, t) avoiding the set Y and originated from some point of regularity of the fundamental solution $G(x, q_0, q, t)$ with $t = 0$, this fundamental solution can be analytically continued along this path. However, it is evident that, when constructing the analytic continuation of the fundamental solution along such a path, one requires to provide the finite number of steps in the variable t of the length ε, and, therefore, the set of singularities of this continuation, being a union of the finite number of analytic sets, is, in turn, an analytic set. This proves the theorem.

1.3 Singularities of the elementary solution

Here we give the more detailed investigation of the set of singularities of the above constructed endlessly continuable elementary solution. To do this, we introduce the following geometric object connected with Hamilton-Jacobi equation (4).

Consider the space $\mathbf{C}^{2n+1}_{(q_0,x,p,y,q)}$ as a contact space with the contact structure defined by the following differential form:

$$dq_0 + p\,dx + y\,dq.$$

Let $\mathcal{L}_0$ be a Legendre manifold in this space given by the equations

$$\begin{cases} p = q, \\ x = y, \\ q_0 = -qx. \end{cases}$$

This Legendre manifold can be lifted to the contact space

$$\mathbf{C}^{2n+3}_{(q_0,x,p,y,q,t,E)} \tag{14}$$

(with the structure form $dq_0 + p\,dx + y\,dq + E\,dt$) by the relations

$$\begin{cases} E = -H_m(x,p), \\ t = 0. \end{cases}$$

This lifting will be denoted by the same letter $\mathcal{L}_0$; clearly, it lyes on the zero level of the Hamilton function

$$\mathcal{H}(x,p,t,E) = E + H_m(x,p). \tag{15}$$

Hence, the Hamiltonian flow $\mathcal{L}$ of the manifold $\mathcal{L}_0$ with Hamilton function (15) is a Legendre manifold in contact space (14). The solution $S(x,q,t)$ to system (4) is, evidently, a generating function for the manifold $\mathcal{L}$ for small values of t. Expansion (3) for the elementary solution $G(x,q_0,q,t)$ show that, at least for small values of t, the singularities of the elementary solution coincide with the projection of the manifold $\mathcal{L}$ to the space $\mathbf{C}^{2n+2}_{(x,q_0,q,t)}$. The following statement shows that this fact is a global one.

Theorem 2 *The set Σ_G of singularities of the above constructed elementary solution lye in the projection of the manifold $\mathcal{L}$ to the space $\mathbf{C}^{2n+2}_{(x,q_0,q,t)}$.*

Proof. Since this fact takes place in a neighborhood of $t = 0$, we must prove the assertion for iterations of the step-by-step procedure used for the construction of the elementary solution in the previous subsection. To be short, we shall consider here only the second iteration; all other iterations can be considered in the same way, though the computations are more complicated.

So, let us consider the expression for the elementary solution arising on the second step of the step-by-step procedure. This expression reads

$$\begin{aligned} G(x,q_0,q,t) &= \left(-\frac{i}{2\pi}\right)^n \left(\frac{\partial}{\partial q_0}\right)^{n+1} \int\limits_h G(x, q_0-\widetilde{q}_0-\widetilde{q}y, \widetilde{q}, t-\tau) \\ &\times \; G(y,\widetilde{q}_0,q,\tau)\, dy \wedge d\widetilde{q}_0 \wedge d\widetilde{q}. \end{aligned} \tag{16}$$

The singularities of the factors under the integral sign are given by

$$q_0 - \widetilde{q}_0 - \widetilde{q}y + S(x,\widetilde{q},t-\tau) = 0, \tag{17}$$

and

$$\widetilde{q}_0 + S(y,q,\tau) = 0, \tag{18}$$

respectively. As it was shown above, the singularities of integral (16) are originated by the analytic strata of the singularity set of the integrand. Since equations (17) and (18) determine regular analytic manifolds, the singularities of (16) can occur only at those values of variables (x,q_0,q,t) for which manifolds (17) and (18) are tangent to each other. The conditions of tangency between these two manifolds are

$$\begin{cases} \widetilde{q}_0 = q_0 - \widetilde{q}y + S(x,\widetilde{q},t-\tau), \\ \widetilde{q}_0 = -S(y,q,\tau), \\ y = \dfrac{\partial S}{\partial q}(x,\widetilde{q},t-\tau), \\ \widetilde{q} = \dfrac{\partial S}{\partial y}(y,q,\tau). \end{cases} \tag{19}$$

Since the equations of the Legendre manifold $\mathcal{L}$ can be written in terms of the function S in the form

$$\begin{cases} q_0 = -S(x,q,t), \\ y = \dfrac{\partial S}{\partial q}(x,q,t), \\ p = \dfrac{\partial S}{\partial x}(x,q,t), \end{cases} \tag{20}$$

one can see that system (19) is solvable only for the values of (x,q_0,q,t) lying in the projection of the Legendre manifold $\mathcal{L}$ on the space $\mathbf{C}^{2n+2}_{(x,q_0,q,t)}$. This proves the theorem.

2 Fundamental solutions for elliptic operators

In this Section, we shall show, that for any elliptic differential operator (1) with polynomial coefficients in the real space $\mathbf{R}^n_x$, there exists a fundamental solution which can be analytically continued to the complex space $\mathbf{C}^n_x$ up to an endlessly continuable function. In spite of the fact that this affirmation can be proved for general operators with polynomial coefficients, we present first the construction of an analytic continuation of the required type for the operators with constant coefficients since the proof for this case gives more explicit expressions for the continuation of a fundamental solution than that in the general case of operators with polynomial coefficients. For the similar reason, we divide the investigation of the case of operators with constant coefficients into two parts, investigating first the case of homogeneous operators with constant coefficients, that is, the operators which contain only derivatives of higher order.

2.1 Homogeneous operators with constant coefficients

Let

$$H\left(-\frac{\partial}{\partial x}\right) = \sum_{|\alpha|=m} a_\alpha \left(-\frac{\partial}{\partial x}\right)^\alpha \tag{21}$$

be an elliptic homogeneous (in order of derivatives) operator with constant coefficients P_α in the real space $\mathbf{R}^n_x$. Due to the definition, this operator can be considered also as the operator in the complex space $\mathbf{C}^n_x$, and we shall consider below this complexification. We denote by

$$H(p) = \sum_{|\alpha|=m} a_\alpha p^\alpha$$

the symbol of this operator (which in this case is a homogeneous function in the variables p of degree m and, hence, coincides with its principal symbol H_m). To construct the analytic continuation of a fundamental solution for this operator we need to introduce the following auxiliary objects:

1. The submanifold

$$\Sigma_x = \{p \mid px = 0\}$$

in the complex projective space $\mathbf{CP}_{n-1,p}$.

2. The characteristic set

$$\operatorname{char} H = \{p \mid H(p) = 0\} \subset \mathbf{CP}_{n-1,p}.$$

Here $p = (p_1, \ldots, p_n)$ are considered as homogeneous coordinates in the space $\mathbf{CP}_{n-1,p}$. To be short, we require that $\operatorname{char} H$ is a regular submanifold in $\mathbf{CP}_{n-1,p}$. This requirement is not nesessary (see Remark 3 below).

The following statement is valid:

Proposition 4 *The manifold Σ_x is tangent to* $\operatorname{char} H$ *if and only if the point x lyes on the characteristic cone of the origin $x = 0$. This means that*

$$x = \lambda H_p\left(p^0\right),$$

for some $p^0 \in \operatorname{char} H$ and some $\lambda \in \mathbf{C}_*$.

Proof. Let us prove first the "if" part of the proposition. If $x = \lambda H_p\left(p^0\right)$ for some $p^0 \in \operatorname{char} H$ and some $\lambda \in \mathbf{C}_*$, then

$$p^0 x = \lambda p^0 H_p\left(p^0\right) = m\lambda H\left(p^0\right) = 0,$$

where we have used the Euler equality for the homogeneous function $H(p)$ of order m. Hence, the point p^0 belongs to the set Σ_x. Later on, the conormal vectors to the manifolds Σ_x and $\operatorname{char} H$ are given by the equalities

$$\left(x^1, \ldots, x^n\right) = \lambda\left(H_{p_1}\left(p^0\right), \ldots, H_{p_n}\left(p^0\right)\right)$$

and

$$\left(H_{p_1}\left(p^0\right), \ldots, H_{p_n}\left(p^0\right)\right),$$

correspondingly. Since these two vectors are, clearly, linearly dependent, the point p^0 is a point of tangency between Σ_x and $\operatorname{char} H$.

The proof of the "only if" part of this proposition goes in a similar way and is left to the reader. The proof is complete.

Now we suppose that there exists a regular point $p^0 \in \operatorname{char} H$ such that the tangency between Σ_x and $\operatorname{char} H$ is quadratic (this follows from the ellipticity condition, at least, for equations with real coefficients). Let us fix this value of p^0 and denote by x_0 the corresponding value of x (so that $x_0 = \lambda H_p\left(p^0\right)$ for some $\lambda \neq 0$). Then, for any x close to x_0 but lying outside the characteristic cone $\mathcal{K}_0$ of the origin, the intersection

$$\Sigma_x \cap \operatorname{char} H$$

is biholomorphic to a complex quadrics in some neighborhood of the point p_0. Let $h(x)$ be the vanishing cycle of this quadrics:

$$h(x) \in H_{n-3}\left(\Sigma_x \cap \operatorname{char} H\right).$$

Denote

$$h_1(x) = \delta h(x) \in H_{n-2}\left(\Sigma_x \setminus \operatorname{char} H\right),$$

where δ is the Leray coboundary (see [3], [4], [2]). Below we shall use also the Leray form $\omega(p)$ on the projective space $\mathbf{CP}_{n-1,p}$ given by the relation

$$\omega(p) = \sum_{k=1}^{n} (-1)^{k-1} p_k \, dp_1 \wedge \ldots \wedge \widehat{dp_k} \wedge \ldots \wedge dp_n, \tag{22}$$

where the hat over the differential dp_k means that this differential must be omitted in the outer product.

Now we are able to write down the expression for the fundamental solution $K(x)$:

$$H\left(-\frac{\partial}{\partial x}\right) K(x) = \delta(x),$$

which allows to continue it to complex values of x. Namely, let us consider the function given by

$$\begin{aligned} K(x) &= (-1)^{n-m-1}(n-m-1)! \left(\frac{i}{2\pi}\right)^{n-1} \int\limits_{h_1(x)} \operatorname*{Res}_{\Sigma_x} \frac{\omega(p)}{H(p)(px)^{n-m}} \\ &= 2\pi i (-1)^{n-m-1}(n-m-1)! \left(\frac{i}{2\pi}\right)^{n-1} \int\limits_{h(x)} \operatorname*{Res}_{\Sigma_x} \operatorname*{Res}_{\operatorname{char} H} \frac{\omega(p)}{H(p)(px)^{n-m}} \end{aligned} \tag{23}$$

for $n > m$ and

$$\begin{aligned} K(x) &= \frac{1}{(m-n)!} \left(\frac{i}{2\pi}\right)^{n-1} \int\limits_{h_3(x)} \frac{(px)^{m-n}\,\omega(p)}{H(p)} \\ &= 2\pi i \frac{1}{(m-n)!} \left(\frac{i}{2\pi}\right)^{n-1} \int\limits_{h_2(x)} \operatorname*{Res}_{\operatorname{char} H} \frac{(px)^{m-n}\,\omega(p)}{H(p)} \end{aligned} \tag{24}$$

for $n \le m$. Here the classes $h_2(x)$ and $h_3(x)$ are defined as follows:

Consider the exact triangle[3]

$$\begin{tikzcd} & H_*(\operatorname{char} H) \arrow[dl, "p"'] & \\ H_*(\operatorname{char} H, \Sigma_x) \arrow[rr, "\partial"] & & H_*(\operatorname{char} H \cap \Sigma_x) \arrow[ul, "i"'] \end{tikzcd}$$

[3]All homology is considered in a neighborhood of the point p_0.

Since $H_*(\operatorname{char} H) = 0$, there exists a unique homology class

$$h_2(x) \in H_{n-2}(\operatorname{char} H, \Sigma_x)$$

such that $\partial h_2(x) = h(x)$. Finally, we put

$$h_3(x) = \delta h_2(x) \in H_{n-1}(\mathbf{CP}_{n-1,p} \setminus \operatorname{char} H, \Sigma_x).$$

We shall prove that the function $K(x)$ given by (23), (24) is a fundamental solution for operator (21). First of all, the following affirmation is valid:

Proposition 5 *The above defined function $K(x)$ is a solution to the homogeneous equation*

$$H\left(-\frac{\partial}{\partial x}\right) K(x) = 0 \tag{25}$$

outside the characteristic cone $\mathcal{K}_0$ of the origin. The latter characteristic cone is exactly the singularity set of the function $K(x)$.

Proof. The fact that $K(x)$ is a solution to (25) can be proved by straightforward computations. The description of singularities of the function $K(x)$ follows directly from the Thom theorem (see, for example, [2], [4]) if one takes into account the result of Proposition 4. The proof is complete.

Remark 3 If the set $\operatorname{char} H$ is not a regular submanifold of the space $\mathbf{CP}_{n-1,p}$ but only a stratified set, then, as it is shown in the book [2], Section 5.1.8, each strata of the set $\operatorname{char} H$ originates its own notion of characteristic leaves. The result of Proposition 5 remains valid in this situation as well if one takes into account that the notion of the characteristic cone, in this case, must be modified as described in the above cited book.

Later on, the asymptotic behavior of the function $K(x)$ near points of the characteristic cone $\mathcal{K}_0$ can be investigated quite similar to the investigation of the asymptotic behavior of the R-transform performed in [2], Section 3.3.2. The result is

$$K(x) \simeq f_{m-\frac{n}{2}-1}(k(x)), \tag{26}$$

where $k(x) = 0$ is the equation of $\mathcal{K}_0$, and the functions $f_j(z)$ are given by the relations

$$f_j(z) = \begin{cases} (-1)^{j-1}(-j-1)!\, z^j, \ j < 0, \\ \frac{z^j}{j!} \ln z, \ j \geq 0, \end{cases}$$

for integer values of j, and by

$$f_j(z) = \frac{z^j}{\Gamma(j+1)},$$

for noninteger values of j.

Summarizing all the above considerations, we arrive at the following statement:

Theorem 3 *The function $K(x-y)$ is a fundamental solution for the operator* (21) *for real values of* (x,y):

$$H\left(-\frac{\partial}{\partial x}\right) K(x-y) = \delta(x-y).$$

This solution can be analytically continued up to the function in the space $\mathbf{C}^n_x \times \mathbf{C}^n_y$ as an endlessly continuable ramifying function with singularities on the characteristic cone $\mathcal{K}_y$ of the point y for each fixed value of the latter variable. The asymptotics of this continuation is given by formula (26).

2.2 Example: the Laplace operator in $\mathbf{R}^3$

For the Laplace operator in the space $\mathbf{R}^3$ one has

$$H(p) = p_1^2 + p_2^2 + p_3^2$$

and, hence, the function $K(x)$ is given by the integral of the form (23):

$$K(x) = 2\pi i \left(\frac{i}{2\pi}\right)^{n-1} \int_{h(x)} \operatorname*{Res}_{p_1^2+p_2^2+p_3^2=0} \operatorname*{Res}_{\Sigma_x} \frac{\omega(p)}{H(p)(px)} \tag{27}$$

with the integrand

$$\operatorname*{Res}_{p_1^2+p_2^2+p_3^2=0} \operatorname*{Res}_{\Sigma_x} \frac{p_1 dp_2 \wedge dp_3 - p_2 dp_1 \wedge dp_3 + p_3 dp_1 \wedge dp_2}{(p_1^2+p_2^2+p_3^2)(p_1 x^1 + p_2 x^2 + p_3 x^3)}.$$

We shall compute integral (27) in terms of the chart $p_1 = 1$ of the projective space $\mathbf{CP}_{2,p}$. Thus, we must compute the expression

$$\operatorname*{Res}_{1+p_2^2+p_3^2=0} \operatorname*{Res}_{\Sigma_x} \frac{dp_2 \wedge dp_3}{(1+p_2^2+p_3^2)(x^1+p_2x^2+p_3x^3)}.$$

Later on, it is evident that the function $K(x)$ is a homogeneous function of order -1 with respect to the variables x. Hence, it is sufficient to compute this function, say, for $x^2 = 1$. Now we have

$$\operatorname*{Res}_{\Sigma_x} \frac{dp_2 \wedge dp_3}{(1+p_2^2+p_3^2)(x^1+p_2+p_3x^3)} = \frac{dp_3}{p_3^2\left(1+(x^3)^2\right) + 2x^1x^3p_3 + \left(1+(x^1)^2\right)}, \tag{28}$$

where p_3 is used as a coordinate on the manifold Σ_x. Computing the residue on the manifold char H, we arrive at the expression

$$\operatorname*{Res}_{p_3=p_3^{\pm}(x)} \operatorname*{Res}_{\Sigma_x} \frac{dp_2 \wedge dp_3}{(1+p_2^2+p_3^2)(x^1+p_2x^2+p_3x^3)} = \pm \frac{1}{2i\sqrt{1+(x^1)^2+(x^3)^2}},$$

where

$$p_3 = p_3^{\pm}(x) = \frac{x^1x^3 \pm i\sqrt{1+(x^1)^2+(x^3)^2}}{\left(1+(x^3)^2\right)}$$

are zeroes of the denominator on the right in (28). Since $h(x) = p_3^-(x) - p_3^+(x)$, we have

$$K(x)|_{x^2=1} = 2\pi i \left(\frac{i}{2\pi}\right)^2 \frac{i}{\sqrt{1+(x^1)^2+(x^3)^2}} = \frac{1}{2\pi\sqrt{1+(x^1)^2+(x^3)^2}}.$$

Finally, taking into account the homogeneity property of the function $K(x)$, we arrive at the expression

$$K(x) = \frac{1}{2\pi\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}}$$

which is in accordance with the usual expression of the fundamental solution for the Laplace operator in the space $\mathbf{R}^3$.

2.3 Inhomogeneous operators with constant coefficients

Here we consider operators of the form

$$H\left(-\frac{\partial}{\partial x}\right) = \sum_{|\alpha|\leq m} a_\alpha \left(-\frac{\partial}{\partial x}\right)^\alpha \tag{29}$$

which are not homogeneous with respect of the order of differentiation but still have constant coefficients a_α. In this case, the fundamental solution for the operator (29) can be written down in terms of the Green function for the dual (with respect to the R-transform [2]) operator $H\left(p\frac{d}{dp_0}\right)$. We recall (see [2], p.292) that the function $G(p_0, p)$ is called a *Green function* for the operator $H\left(p\frac{d}{dp_0}\right)$ if it is a solution to the following Cauchy problem:

$$\begin{cases} H\left(p\frac{d}{dp_0}\right) G(p_0, p) = 0, \\ G\big|_{p_0=0} = \ldots = \left.\frac{d^{m-2}G}{dp_0}\right|_{p_0=0} = 0, \\ \left.\frac{d^{m-1}G}{dp_0}\right|_{p_0=0} = \frac{1}{H_m(p)}, \end{cases} \tag{30}$$

where $H_m(p)$ is, as above, the homogeneous component of the symbol $H(p)$ of operator (29) of maximal degree m.

Now, we shall prove that a fundamental solution for the operator (29) is given by the formula

$$K(x) = \left(\frac{i}{2\pi}\right)^{n-1} \int\limits_{h_3(x)} \frac{\partial^{n-1}G}{\partial p_0^{n-1}}(px, p)\,\omega(p), \tag{31}$$

where $h_3(p)$ is the above defined homology class and $\omega(p)$ is the Leray form (see (22)). First, the following affirmation is valid:

Proposition 6 *The function $K(x)$ is a solution to the homogeneous equation*

$$H\left(-\frac{\partial}{\partial x}\right) K(x) = 0 \tag{32}$$

outside the characteristic cone $\mathcal{K}_0$ of the origin. The latter characteristic cone is exactly the singularity set of the function $K(x)$.

Proof. The fact that the function $K(x)$ is a solution to equation (32) can be proved with the help of the straightforward computations. Namely, we have

$$H\left(-\frac{\partial}{\partial x}\right)K(x)=\left(\frac{i}{2\pi}\right)^{n-1}\int\limits_{h_2(x)}\frac{\partial^{n-1}}{\partial p_0^{n-1}}\left\{H\left(p\frac{\partial}{\partial p_0}\right)G\right\}\Big|_{p_0=px}\omega(p)=0,$$

due to the equation involved into Cauchy problem (30). Later on, one can investigate the singularities of the function $G(p_0,p)$ using the representation of this function of the form

$$G(p_0,p)=\frac{1}{2\pi i}\int\limits_{C(p)}\frac{e^{\lambda p_0}\,d\lambda}{H(\lambda p)},\tag{33}$$

where the contour $C(p)$ encircles all the zeroes of the denominator of the integrand. In particular, from this formula it follows that the function $G(p_0,p)$ has singularities of univalued character only. These singularities lye exactly on the set $\operatorname{char} H$, so the definition (31) is correct.

Actually, singular points of the integrand in (33) can be found from the following algebraic equation of the m-th order:

$$\sum_{k=0}^{m}\lambda^k H_k(p)=0.\tag{34}$$

(Here $H_k(p)$ are homogeneous components of order k of the symbol $H(p)$.) The singularities of (33) can occur only for those values of p for which one of the roots of equation (34) tends to infinity. This happens for $H_m(p)=0$ only, so that the singularities of $G(p_0,p)$ exactly coincide with $\operatorname{char} H=\{H_m(p)=0\}$. Later on, when p encircles $\operatorname{char} H$, all roots of (34) are contained uniformly in some compact set in $\mathbf{C}$ and, hence, the contour $C(p)$ remains unchanged. Therefore, all singularities of $G(p_0,p)$ are of the univalued type.

Further, the description of the singularity set of the function $K(x)$ given in Proposition 6 follows now from the Thom theorem. This complete the proof of the proposition.

Now we shall try to show that the function $K(x-y)$ is a fundamental solution for the operator $H(-\partial/\partial x)$. This fact can be verified, as above, by means of computation of asimptotic expansion of the function $K(x)$ at points of the characteristic cone $\mathcal{K}_0$. Below we present one more method of proving this assertion.

To do this, we write down the formula which gives the solution to the Cauchy problem for the operator $H(-\partial/\partial x)$ with zero Cauchy data on some (arbitrary) manifold X which is not everywhere characteristic in terms of the fundamental solution $K(x-y)$:

$$u(x)=\int\limits_{H(x)}K(x-y)f(y)\,dy,\tag{35}$$

where $H(x)$ is a relative homology class of the complement of the characteristic cone $\mathcal{K}_x$ modulo X. Substituting the function $K(x-y)$ in the form (31) into the formula (35), we come to the expression for $u(x)$ in the form

$$u(x) = \left(\frac{i}{2\pi}\right)^{n-1} \int\limits_{H_1(x)} \frac{\partial^{n-1} G}{\partial p_0^{n-1}} (p(x-y), p) f(y)\, dy \wedge \omega(p). \tag{36}$$

This formula coincides with formula (5.50) of the book [2] since in a neighborhood of any nonsingular point of the initial manifold X the homology group

$$H_{2n-1}\left(\mathbf{CP}_{n-1,p} \times \mathbf{C}_x^n, X \cup \Sigma_x\right)$$

has one generator, and, hence, the class $H_1(x)$ is uniquely defined. From the other hand, formula (36) gives a solution to the above mentioned Cauchy problem for any right-hand part $f(x)$. Therefore, for any $f(x)$ the function given by (36) satisfies the equation

$$H\left(-\frac{\partial}{\partial x}\right) u = f$$

and, hence, $K(x-y)$ is a fundamental solution for operator (29). This proves the following statement:

Theorem 4 *The function $K(x-y)$ is a fundamental solution for the operator* (29) *for real values of (x,y):*

$$H\left(-\frac{\partial}{\partial x}\right) K(x-y) = \delta(x-y).$$

This solution can be analytically continued up to the function in the space $\mathbf{C}_x^n \times \mathbf{C}_y^n$ as an endlessly continuable ramifying function with singularities on the characteristic cone $\mathcal{K}_y$ of the point y for each fixed value of the latter variable.

2.4 Operators with polynomial coefficients

In this subsection, we shall give the proof of the existence of an endlessly continuable fundamental solution for the general case of operators with polynomial coefficients of the form (1). The formula for a fundamental solution in this case reads

$$K(x,y) = 2\pi i \left(\frac{i}{2\pi}\right)^{n-1} \int\limits_{h(x,y)} \left(\frac{\partial}{\partial q_0}\right)^{n-m} G(x, q_0, q, t)\big|_{q_0+qy=0}\, \widetilde{\omega}(q,t), \tag{37}$$

where

$$\widetilde{\omega}(q,t) = (1-m)\, t\, dq_1 \wedge \ldots \wedge dq_n - dt \wedge \omega(q)$$

is a modified Leray form (the form $\omega(q)$ is given by relation (22) above) and the function $G(x, q_0, q, t)$ is the elementary solution constructed in Section 1 above. To complete the definition of the function $K(x, y)$ one should give the description of the homology class $h(x, y)$ involved in the definition (37) of this function. We shall first define this class for values of x lying close to the vertex of the characteristic cone $\mathcal{K}_y$. We recall that the singualrity set Σ_G of the elementary solution for operator (1), at least for small values of t, is given by

$$\Sigma_G = \{q_0 + S(x, q, t) = 0\}.$$

(see formula (6) above). The straightforward computations using equations (20) of the Hamilton flow $\mathcal{L}$ show that, if x lyes on the characteristic cone $\mathcal{K}_y$, then the manifold Σ_G is tangent to char H at some point. Moreover, in this case, the point x lyes on the trajectory of the Hamiltonian vector field emenated from the point y with natural parameter along this trajectory equal to t. The set char $H \cap \Sigma_G \cap \{t = 0\}$ is in this case biholomorphic to the complex quadrics, and we denote by

$$\widetilde{h}(x, y) \in H_{n-3}(\text{char}\, H \cap \Sigma_G \cap \{t = 0\})$$

the vanishing class of this quadrics. Later on, moving the variable t from zero to a value such that char H is tangent to $\Sigma_G \cap \{t = \text{const}\}$, we shall construct a relative homology class

$$\widetilde{h}_1(x, y) \in H_{n-2}(\text{char}\, H \cap \Sigma_G,\ \{t = 0\})$$

such that $\partial \widetilde{h}_1(x, y) = \widetilde{h}(x, y)$. Now, two successive applications of the Leray coboundary homomorphism δ lead us to the homology class

$$h(x, y) \in H_n(\Omega \setminus (\text{char}\, H \cup \Sigma_G),\ \{t = 0\}),$$

where Ω is the quotient space of the space $\mathbf{C}^{n+1}_{(q,t)}$ modulo the action of the group $\mathbf{C}_*$ given by

$$\lambda(q, t) = \left(\lambda q, \lambda^{1-m} t\right).$$

This is exactly the homology class used in definition (37) of the function $K(x, y)$.

The proof of the fact that the function $K(x, y)$ is a fundamental solution for the operator $H(x, -\partial/\partial x)$ goes quite similar to the corresponding proof of the previous subsection. The only thing rest is to prove the endless continuability of the obtained fundamental solution.

To do this, we introduce one more condition to the differential operator in question.

Condition 2 The projection of the set Σ_G of singularities of the function $G(x, q_0, q, t)$ on the space Ω is a proper mapping.

Remark 4 This is, in essence, the exact formulation of the condition that all trajectories of Hamiltonian system (7) are coming to infinity as $t \to \infty$. This follows from the description of the singularity set Σ_G of the elementary solution G given in Subsection 1.3.

Under this condition the proof of endless continuability of the function $K(x, y)$ goes quite similar to the proof of the endless continuability of the elementary solution. Thus, we arrive at the following result:

Theorem 5 *Let $H(x, -\partial/\partial x)$ be a differential operator satisfying Conditions* 1 *and* 2 *above. Then there exists a fundamental solution for this operator which can be analytically continued up to an endlessly continuable function in the complex space $\mathbb{C}^{2n}_{(x,y)}$.*

References

[1] B. Sternin and V. Shatalov. On a notion of elementary solution in complex theory. *Soviet Math. Dokl.*, **44**, No. 2, 1992, 567 - 571.

[2] B. Sternin and V. Shatalov. *Differential Equations on Complex Manifolds.* Kluwer Academic Publishers, Dordrecht, 1994.

[3] J. Leray. Le calcul différentiel et intégral sur une variété analytique complexe (probléme de Cauchy III). *Bul. Soc. Math. de France*, **3**, No. 87, 1959, 81 - 180.

[4] F. Pham. *Introduction a L'Étude Topologique des Singularités de Landau.* Gauthier-Villars, Paris, 1967.

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