Catalog of dessins d'enfants with ≤ 4 edges

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Abstract

In this work all the dessins d'enfant with no more than 4 edges are listed and their Belyi pairs are computed. In order to enumerate all dessins the technique of matrix model computations was used. The total number of dessins is 134; among them 77 are spherical, 53 of genus 1 and 4 of genus 2. The automorphism groups of all the dessins are also found.

Dessins are listed by the number of edges. Dessins with the same number of edges are ordered lexicographically by their lists of 0-valencies. The corresponding matrix model for any list of 0-valencies is given and computed. Complex matrix models for dessins with $1-3$ edges are used. For the dessins with 4 edges we use Hermitian matrix model, correlators for which are computed in [1].

Contents

1 Introduction

Dessin d'enfant is a compact connected smooth oriented surface S together with a graph Γ on it such that the complement $S \setminus \Gamma$ is homeomorphic to a disjoint union of open discs. The theory of Dessins d'enfants was initiated by A. Grothendieck in [6, 5] and actively developed thereafter, see [7] and references therein. Dessins d'enfants became rather popular within the last decades; they provide a possibility to describe in the easy and visually effective combinatorial language of graphs on surfaces many difficult and deep concepts and results of Inverse Galois theory, Teichmüller and moduli spaces, Maps and hypermaps, Matrix models, Quantum gravity, String theory, etc.

Dessins d'enfants appear naturally in different branches of mathematics. A smooth irreducible complete complex algebraic curve, defined over the field $\overline{\mathbb{Q}}$, provides a dessin d'enfant in the following way. On such a curve X according to the famous Belyi theorem there exists a nonconstant rational function β having at most 3 critical values. Denote $X_{\mathbb{C}}$ its complexification and $\beta_{\mathbb{C}}$ the natural lift of β to $X_{\mathbb{C}}$. By definition, such β's and $\beta_{\mathbb{C}}$'s are *Belyi functions.* Without loss of generality we assume that the critical values of β are in $\{0,\ 1,\ \infty\}$. Moreover, replacing β by $4\beta(1-\beta)$, if needed, we can assume that $1-\beta$ *has only double zeros*; such Belyi functions are called *clean.* Then $\beta_{\mathbb{C}}^{-1\circ}([0,1])$ is a dessin d'enfant on the topological model of $X_{\mathbb{C}}$ whose edges are $\{\beta_{\mathbb{C}}^{-1\circ}([0,1])\}$ and vertices are $\{\beta_{\mathbb{C}}^{-1\circ}(0)\}$. In the main text we omit the complexification subscripts.

In this work, using the matrix model approach, (see [1, 3, 4]) we listed all the dessins d'enfants with no more than 4 edges. There are two 1-edge dessins, both of them are of genus zero, fifteen 2-edge dessins, among them only one is of genus 1, twenty 3-edge dessins: 14 sperical and 6 of genus 1, and one hundred seven 4-edge dessins: 57 spherical dessins, 46 dessins of genus 1, and 4 dessins of genus 2. The total number of dessins is 134. The main result is the calculation of the corresponding Belyi *pairs* (in the case of positive genus it means finding the curve and the Belyi function on it). This catalog is the analog of the well-known Betrema-Peré-Zvonkine catalog [2], where the trees with no more than 8 edges with their Belyi functions are collected.

Acknowledgments

We would like to thank all participants of the scientific seminar "Graphs on surfaces and curves over number fields" for the encouragement and interesting discussions. The 2nd, 5th, 7th, and 8th authors are grateful to Russian Federal Nuclear Energy Agency for financial support. Also the 2nd author would like to thank the grants NWO-RFBR 047.011.2004.026 (RFBR 05-02-89000-NWO-a), NSh-8004.2006.2, RFBR 07-01-00441-a, the 3rd, 4th, 5th and 8th authors would like to thank the grants RFBR 07-01-00441-a and NSH-5666.2006.1, the 5th author would like to thank the grant MK-2687.2007.1, the 7th author would like to thank the grants RFBR 07-01-00441-a and NSh-8065.2006.2. The 5th author would like to express her deep gratitude to Max-Plank Institute of Mathematics in Bonn, where some part of this work was done, for warm scientific atmosphere and financial support.

2 Comments on Belyi functions

For genus 0 dessins it is natural to write a Belyi function as a fraction of two polynomials. For the simplification of checking we factorize these polynomials.

Since all the curves of genus 1 and 2 are hyperelliptic, we always write their equations in the form $y^2 = F(x)$ and denote by τ the hyperelliptic involution $\tau : (x, y) \mapsto (x, -y)$.

For the curves of positive genus, there are two different cases:

- Let β be invariant with respect to a hyperelliptic involution τ (i.e. $\beta = \beta \circ \tau$), then β can be written as a quotient of two polynomials depending on the coordinate in a quotient space (by this involution), which is a projective line. In this case we write $(X : y^2 = F(x), \beta = \frac{P(x)}{Q(x)})$.
- If Belyi function is not invariant with respect to τ, it is convenient to use symmetric functions in β and β^τ:

$$n_0 = \beta \cdot \beta^\tau,\ n_1 = (\beta - 1)(\beta^\tau - 1).$$

It is easy to reconstruct the Belyi function from this pair and the equation of the curve. Indeed,

$$\beta = \frac{n_0 - n_1 + 1}{2} + y\sqrt{\frac{(n_0 - n_1)^2 - 2(n_0 + n_1) + 1}{4F}} \tag{1}$$

For the convenience of the reader we give all above objects in these cases, namely:
$X : \{y^2 = F(x)\}$, $\beta = \frac{P(x)+Q(x)y}{R(x)}$, $n_0 = \frac{P(x)^2 - Q(x)^2F(x)}{R(x)^2}$, $n_1 = \frac{(P(x)-R(x))^2 - Q(x)^2F(x)}{R(x)^2}$.

It is also useful to factorize numerators and denominators of functions n_0 and n_1. Note that the degrees of factors in the numerator of n_0 are related to the valencies of the vertices of the dessin, and degrees of factors in denominator are related to the valencies of faces.

Listing automorphisms, we denote Z_d the cyclic group of order d.

3 On the matrix models and their use to list the dessins

In order to enumerate the dessins we use matrix model methods, see for example the monograph [7, Chapter 3] and references therein.

For the dessins with 1, 2 and 3 edges we use the complex matrix model. Let

$$\langle f(Z,Z^+)\rangle := \frac{\int\limits_{Z\in M_N(\mathbb{C})} f(Z,Z^+)e^{-Tr(ZZ^+)}dZdZ^+}{\int\limits_{Z\in M_N(\mathbb{C})} e^{-Tr(ZZ^+)}dZdZ^+},$$

where $M_N(\mathbb{C})$ is the set of all complex $N\times N$-matrices, Z^+ denotes the transposed complex conjugated matrix to Z, and Tr is the trace function, $dZdZ^+ = \prod\limits_{l=1}^{n}\prod\limits_{j=1}^{n} dx_{lj}dy_{lj}$ for $Z = (x_{lj} + i\,y_{lj})$. For dessins with 4 edges we use the Hermitian matrix model. Let

$$\langle g(H)\rangle := \frac{\int\limits_{H\in\mathcal{H}_N(\mathbb{C})} g(H)e^{-Tr(H^2)}dH}{\int\limits_{H\in\mathcal{H}_N(\mathbb{C})} e^{-Tr(H^2)}dH},$$

where $\mathcal{H}_N(\mathbb{C})$ is the set of all Hermitian $N\times N$-matrices, i.e. $H = H^+$, and $dH = \prod\limits_{l=1}^{n} h_{ll}\prod\limits_{l=1}^{n-1}\prod\limits_{j=l+1}^{n} du_{lj}dv_{lj}$ for $H = (h_{lj}) = (u_{lj} + i\,v_{lj})$. The functions $\langle f(Z,Z^+)\rangle$ and $\langle f(H)\rangle$ are called *complex* and *Hermitian correlators*, respectively.

Let $\langle a_1,\ldots,a_\alpha\rangle$ be the list of valencies of vertices of a certain graph Γ and $n=\frac{1}{2}\sum\limits_{i=1}^{\alpha} a_i$ be the number of edges of this graph. We denote

$$f_{a_1,\ldots,a_\alpha} := Tr(Z^{a_1})\cdots Tr(Z^{a_\alpha})Tr^n((Z^+)^2)$$

in the case of complex matrix models, and

$$g_{a_1,\ldots,a_\alpha} := Tr(H^{a_1})\cdots Tr(H^{a_\alpha})$$

in the case of Hermitian matrix models. Note that in both cases: for complex or Hermitian matrix models, the correlators $\langle f_{a_1,\ldots,a_\alpha}(Z,Z^+)\rangle$, correspondingly $\langle g_{a_1,\ldots,a_\alpha}(H)\rangle$ are polynomials in N (the size of the matrices). Then by the Wick formula, see [7, Theorem 3.2.5], it follows that after an appropriate normalization the coefficient under N^i is the sum of $\dfrac{1}{|Aut(D)|}$, where the summation goes over all the dessins D with i faces obtained from Γ. By presenting the corresponding dessins we conclude the enumeration of dessins with 1, 2, 3, and 4 edges.

The connected (corresponding to the connected dessins) correlators for any list of valencies are presented in the text. Correlators for complex matrix models were computed directly. For Hermitian matrix models we use the values of correlators found in [1].

4 1-edge dessins

The following matrix model corresponds to the dessins with one vertex:

$$\langle Tr(Z^2)Tr((Z^+)^2)\rangle = 2\cdot 2\left(\frac{1}{2}N^2\right).$$

4.1 Valencies $\langle 2|1,1\rangle$

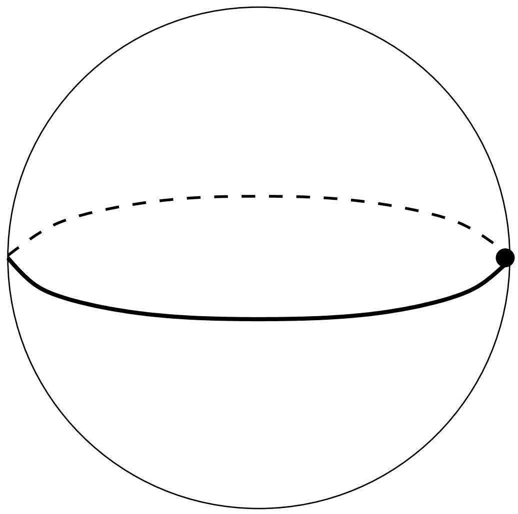


Figure 1: $S(2|11)$. The automorphism group is Z_2. Dual dessin $S(11|2)$, see Figure 2 on the page 8. Belyi function is $\beta = \frac{z^2}{z^2-1}$.

The following matrix model corresponds to the dessins with two vertices:

$$\langle Tr^2(Z)Tr((Z^+)^2)\rangle = 2! \cdot 2 \left(\frac{1}{2}N\right).$$

4.2 Valencies $\langle 1, 1|2\rangle$

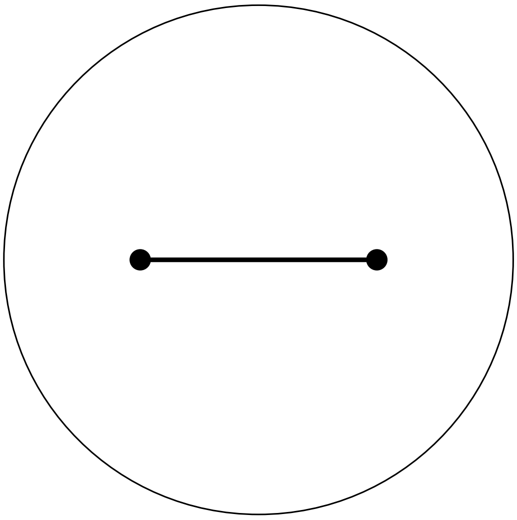


Figure 2: $S(11|2)$. The automorphism group is Z_2. Dual dessin $S(2|11)$, see Figure 1 on the page 7. Belyi function is $\beta = 1 - z^2$.

5 2-edge dessins

We start with the dessins with unique vertex, thus its valency is 4. The following matrix model corresponds to these dessins:

$$\langle Tr(Z^4)Tr^2((Z^+)^2)\rangle = 4 \cdot 2! \cdot 2^2 \left(\frac{1}{2}N^3 + \frac{1}{4}N\right).$$

5.1 Valencies $\langle 4|2,1,1\rangle$

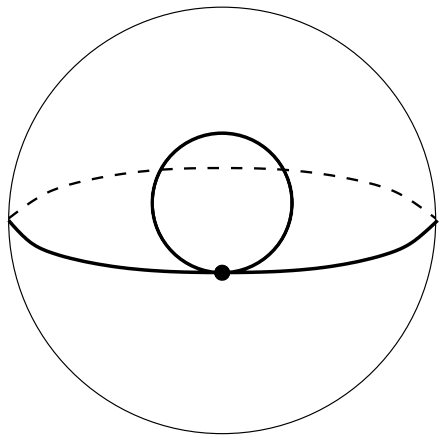


Figure 3: $S(4|211)$. The automorphism group is Z_2. Dual dessin $S(211|4)$, see Figure 7 on the page 12. Belyi function is $\beta = \frac{1}{-4z^2(z^2-1)}$.

5.2 Valencies $\langle 4|4\rangle$

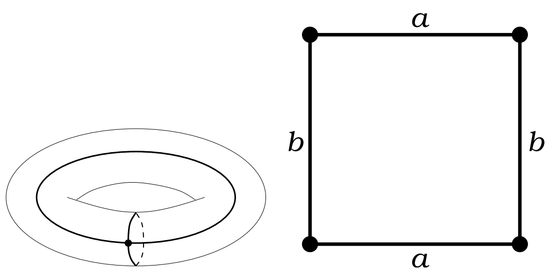


Figure 4: $T(4,4)$. The automorphism group is $Z_2 \oplus Z_2$. This dessin is selfdual. Belyi function is $(X : y^2 = x^3 - x, \beta = x^2)$.

Let us consider now the dessins with 2 vertices. For the partition $4 = 3 + 1$ we have:

$$\langle Tr(Z^3)Tr(Z)Tr^2((Z^+)^2)\rangle = 3 \cdot 2! \cdot 2^2 \left(N^2\right).$$

5.3 Valencies $\langle 3, 1|3, 1\rangle$

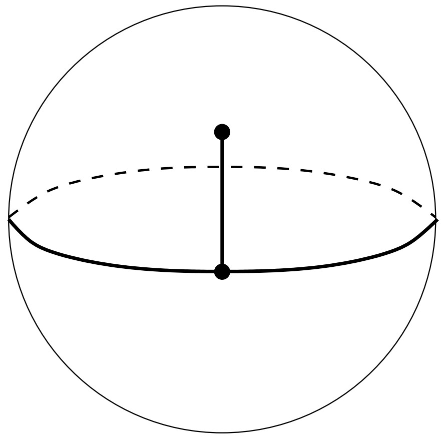


Figure 5: $S(31|31)$. There are no automorphisms of this dessin. This dessin is selfdual. Belyi function is $\beta = -64\frac{z^3(z-1)}{8z+1}$.

For the partition $4 = 2 + 2$ we have:

$$\langle Tr^2(Z^2)Tr^2((Z^+)^2)\rangle = 2! \cdot 2^2 \cdot 2! \cdot 2^2 \left(\frac{1}{4}N^2\right).$$

5.4 Valencies $\langle 2, 2|2, 2\rangle$

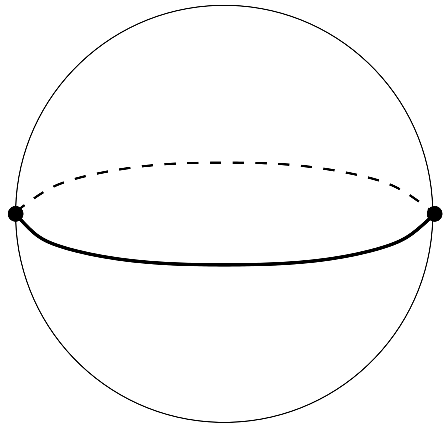


Figure 6: $S(22|22)$. The automorphism group is $Z_2 \oplus Z_2$. This dessin is selfdual. Belyi function is $\beta = \frac{(z^2-1)^2}{-4z^2}$.

$$\langle Tr(Z^2)Tr^2(Z)Tr^2((Z^+)^2)\rangle = 2 \cdot 2! \cdot 2! \cdot 2^2 \left(\frac{1}{2}N\right).$$

5.5 Valencies $\langle 2, 1, 1|4\rangle$

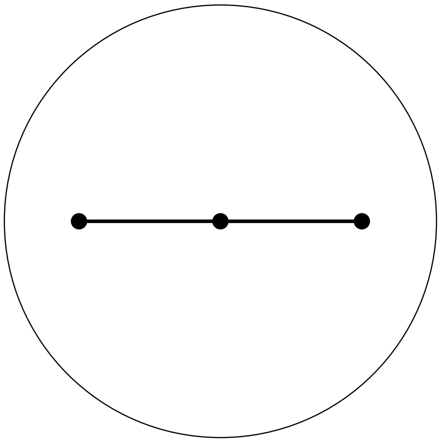


Figure 7: $S(211|4)$. The automorphism group is Z_2. Dual dessin $S(4|211)$, see Figure 3 on the page 9. Belyi function is $\beta = -4z^2(z^2 - 1)$.

6 3-edge dessins

$$\langle Tr(Z^6)Tr^3((Z^+)^2)\rangle = 6\cdot 3!\cdot 2^3\left(\frac{5}{6}N^4+\frac{5}{3}N^2\right) =$$
$$= 6\cdot 3!\cdot 2^3\left(\left(\frac{1}{3}+\frac{1}{2}\right)N^4+\left(1+\frac{1}{2}+\frac{1}{6}\right)N^2\right)$$

6.1 Valencies $\langle 6|3,1,1,1\rangle$

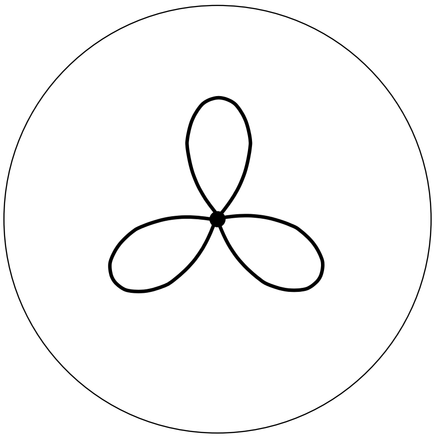


Figure 8: $S(6|3111)$. The automorphism group is Z_3. Dual dessin $S(3111|6)$, see Figure 25 on the page 23. Belyi function is $\beta = \frac{1}{-4z^3(z^3-1)}$.

6.2 Valencies $\langle 6|2,2,1,1\rangle$

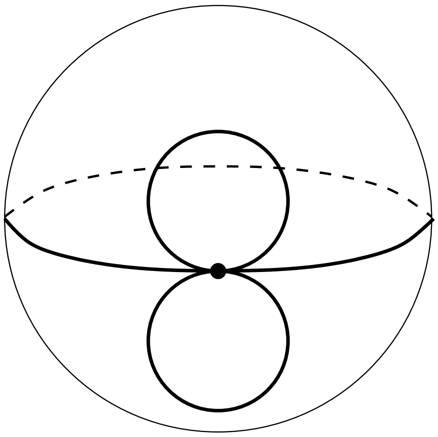
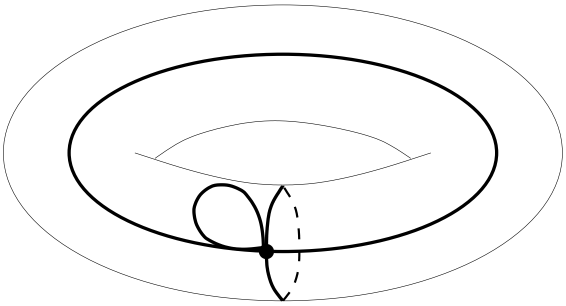


Figure 9: $S(6|2211)$. The automorphism group is Z_2. Dual dessin $S(2211|6)$, see Figure 27 on the page 25. Belyi function is $\beta = \frac{-4}{(z^2-1)^2(z^2-4)}$.

6.3 Valencies $\langle 6|5,1\rangle$



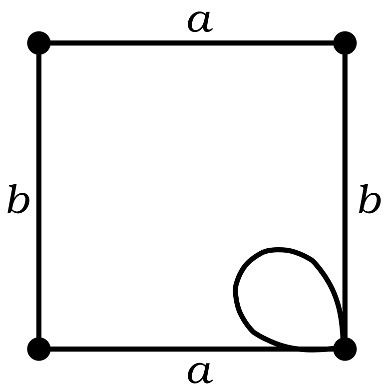


Figure 10: $T(6|51)$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(51|6)$, see Figure 15 on the page 17. Belyi function is $\beta = -\frac{1}{216}\frac{\left(2\,x^3+2\,yx-29\,x^2-15\,y+85\,x-50\right)x^3}{2\,x+1}$ on the curve $X\,:\,y^2 = x^4-14\,x^3+29\,x^2-60\,x$. $n_0 = \frac{625}{11664}\frac{x^6}{2\,x+1}$, $n_1 = \frac{1}{11664}\frac{\left(29\,x^3-54\,x^2+108\,x+108\right)^2}{2\,x+1}$

6.4 Valencies $\langle 6|4,2\rangle$

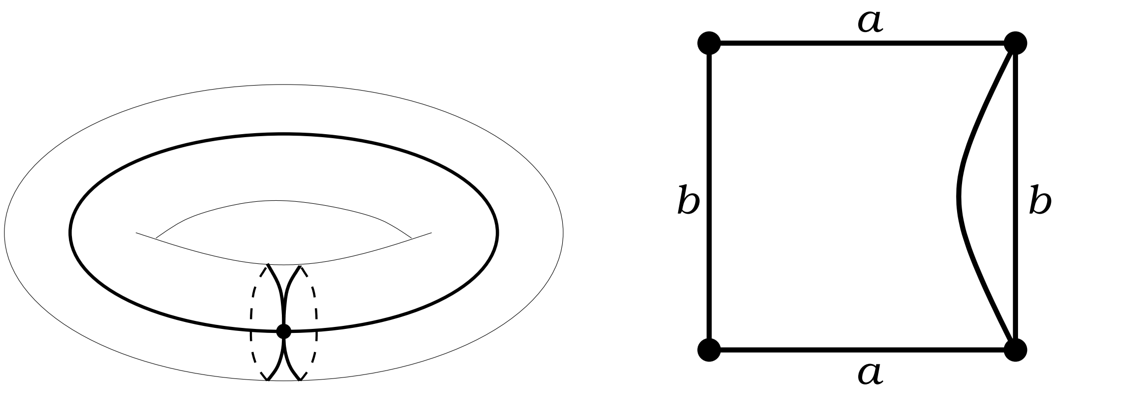


Figure 11: $T(6|42)$. The automorphism group is Z_2. Dual dessin $T(42|6)$, see Figure 17 on the page 18. Belyi function is $(X : y^2 = x(x+3)(x-1), \beta = \frac{4x^3}{27(x-1)})$.

6.5 Valencies $\langle 6|3,3\rangle$

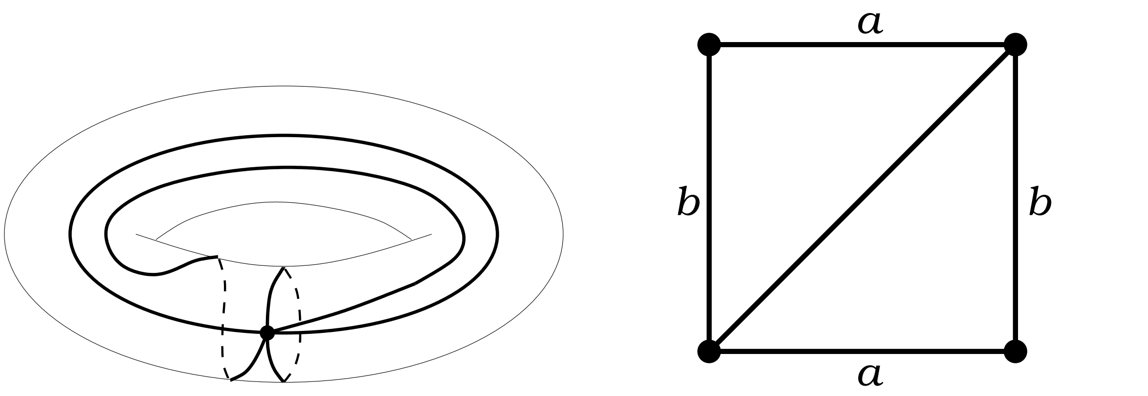


Figure 12: $T(6|33)$. Valencies $(6|3,3)$. The automorphism group is Z_6. Dual dessin $T(33|6)$, see Figure 22 on the page 21. Belyi function is $(X : y^2 = x^4 - x, \beta = x^3)$.

$$\langle Tr(Z^5)Tr(Z)Tr^3((Z^+)^2)\rangle = 5\cdot 3!\cdot 2^3\left(2N^3+N\right) =$$
$$= 5\cdot 3!\cdot 2^3\left((1+1)N^3+N\right).$$

6.6 Valencies $\langle 5,1|4,1,1\rangle$

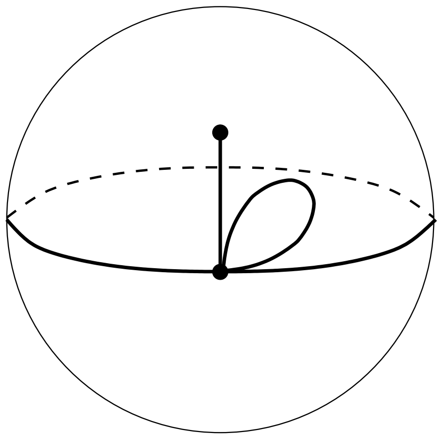


Figure 13: $S(51|411)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(411|51)$, see Figure 18 on the page 19. Belyi function is $\beta = \frac{256(z-1)}{z^4(z^2+4z+20)}$.

6.7 Valencies $\langle 5,1|3,2,1\rangle$

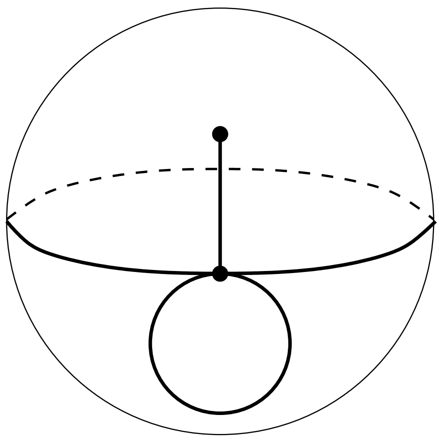
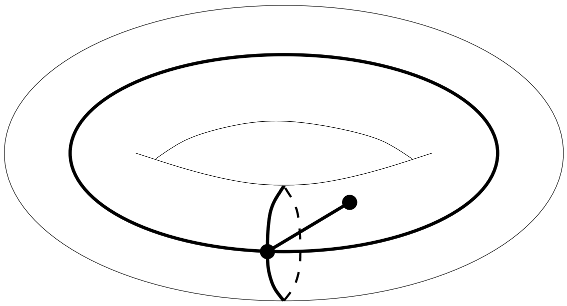


Figure 14: $S(51|321)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(321|51)$, see Figure 23 on the page 22. Belyi function is $\beta = \frac{-64(15z+1)}{3125z^3(z-1)^2(5z-8)}$.

6.8 Valencies $\langle 5,1|6\rangle$



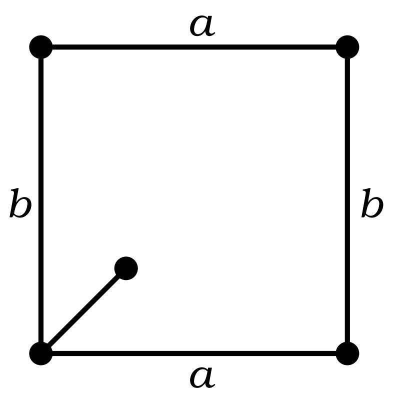


Figure 15: $T(51|6)$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(6|51)$, see Figure 10 on the page 14. Belyi function is $\beta = \frac{25\,x^3-270\,yx-255\,x^2+216\,y+522\,x-216}{1250}$ on the curve $X\,:\,y^2 = -\frac{5}{18}\,x^3+\frac{29}{36}\,x^2-\frac{7}{3}x+1$. $n_0 = \frac{x^5(12+x)}{2500}$, $n_1 = \frac{(x^3+6x^2-18x+58)^2}{2500}$.

$$\langle Tr(Z^4)Tr(Z^2)Tr^3((Z^+)^2)\rangle = 4\cdot 2\cdot 3!\cdot 2^3\left(N^3+\frac{1}{2}N\right).$$

6.9 Valencies $\langle 4,2|3,2,1\rangle$

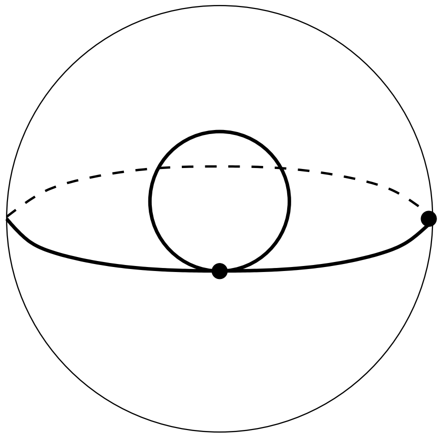


Figure 16: $S(42|321)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(321|42)$, see Figure 24 on the page 22. Belyi function is $\beta = \frac{-(3z-2)^2}{4z^3(z-1)^2(z+2)}$.

6.10 Valencies $\langle 4,2|6\rangle$

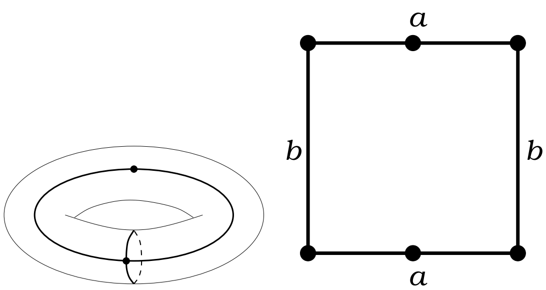


Figure 17: $T(42|6)$. The automorphism group is Z_2. Dual dessin $T(6|42)$, see Figure 11 on the page 15. Belyi function is $(X : y^2 = 4x^3 - 39x + 35, \beta = \frac{(x-1)^2(2x+7)}{27})$.

$$\langle Tr(Z^4)Tr(Z)Tr(Z)Tr^3((Z^+)^2)\rangle = 2!\cdot 4\cdot 3!\cdot 2^3\left(\frac{3}{2}N^2\right) =$$
$$= 2!\cdot 4\cdot 3!\cdot 2^3\left(\left(1+\frac{1}{2}\right)N^2\right).$$

6.11 Valencies $\langle 4,1,1|5,1\rangle$

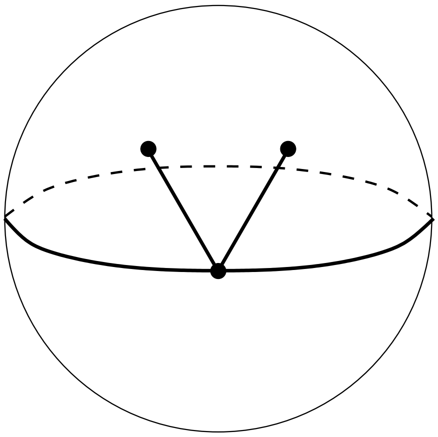


Figure 18: $S(411|51)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(51|411)$, see Figure 13 on the page 16. Belyi function is $\beta = \frac{z^4(z^2+4z+20)}{256(z-1)}$

6.12 Valencies $\langle 4,1,1|3,3\rangle$

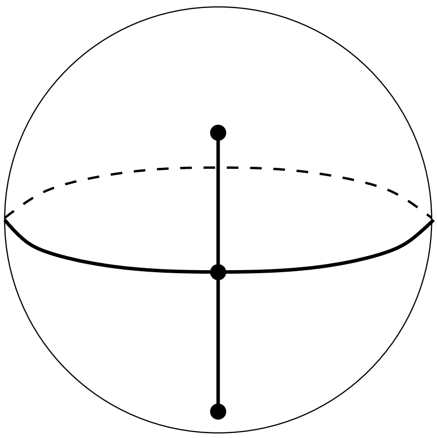


Figure 19: $S(411|33)$. The automorphism group is Z_2. Dual dessin $S(33|411)$, see Figure 20 on the page 20. Belyi function is $\beta = \frac{z^4(4z^2-3)}{4(z^2-1)^3}$.

$$\langle Tr(Z^3)Tr(Z^3)Tr^3((Z^+)^2)\rangle =$$

$$= 2! \cdot 3^2 \cdot 3! \cdot 2^3 \left(\frac{2}{3}N^3 + \frac{1}{6}N\right) = 2! \cdot 3^2 \cdot 3! \cdot 2^3 \left(\left(\frac{1}{2} + \frac{1}{6}\right) N^3 + \frac{1}{6}N\right).$$

6.13 Valencies $\langle 3, 3|4, 1, 1\rangle$

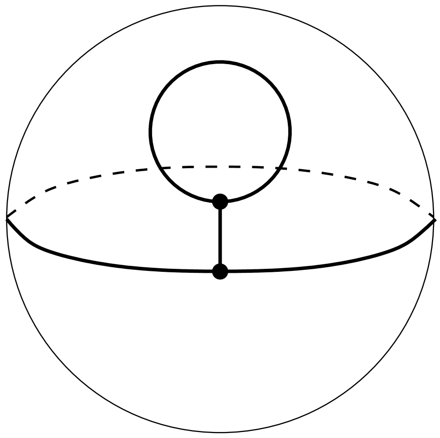


Figure 20: $S(33|411)$. The automorphism group is Z_2. Dual dessin $S(411|33)$, see Figure 19 on the page 19. Belyi function is $\beta = \frac{4(z^2-1)^3}{z^4(4z^2-3)}$.

6.14 Valencies $\langle 3, 3|2, 2, 2\rangle$

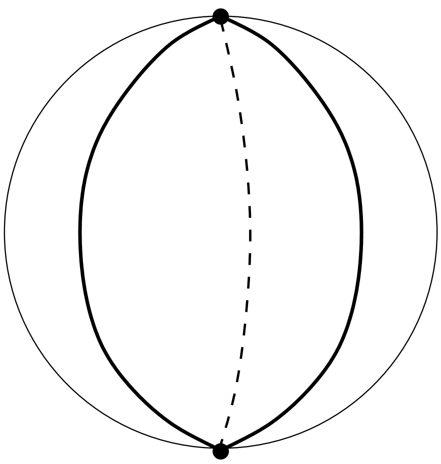


Figure 21: $S(33|222)$. The automorphism group is Z_6. Dual dessin $S(222|33)$, see Figure 26 on the page 24. Belyi function is $\beta = \frac{-4z^3}{(z^3-1)^2}$.

6.15 Valencies $\langle 3, 3|6\rangle$

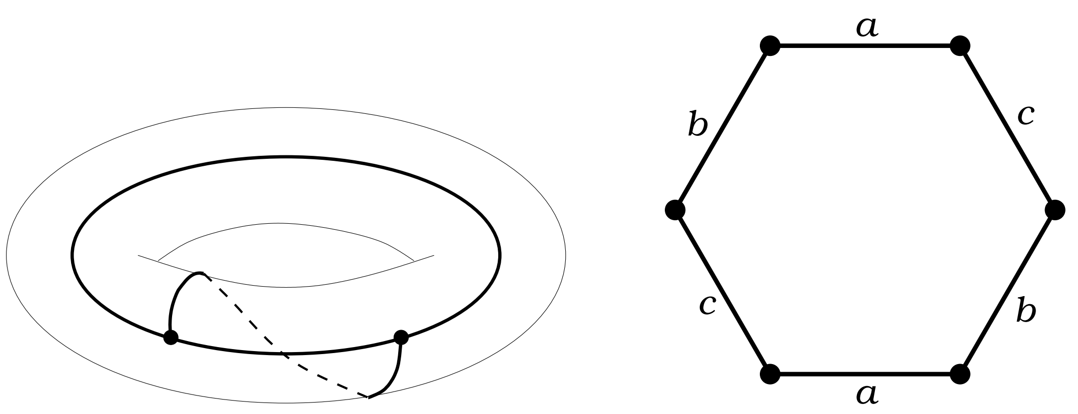


Figure 22: $T(33|6)$. The automorphism group is Z_6. Dual dessin $T(6|33)$, see Figure 12 on the page 15. Belyi function is $(X : y^2 = x^3 - 1, \beta = x^3)$.

$$\langle Tr(Z^3)Tr(Z^2)Tr(Z)Tr^3((Z^+)^2)\rangle =$$
$$= 3 \cdot 2 \cdot 3! \cdot 2^3 \left(2N^2\right) = 3 \cdot 2 \cdot 3! \cdot 2^3 \left((1+1)N^2\right).$$

6.16 Valencies $\langle 3, 2, 1|5, 1\rangle$

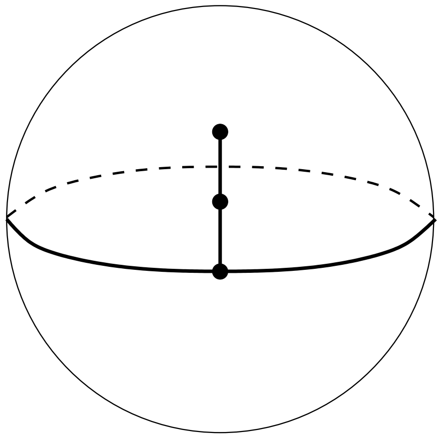


Figure 23: $S(321|51)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(51|321)$, see Figure 14 on the page 16. Belyi function is $\beta = \frac{-z^3(z-5)^2(z-8)}{64(3z+1)}$

6.17 Valencies $\langle 3, 2, 1|4, 2\rangle$

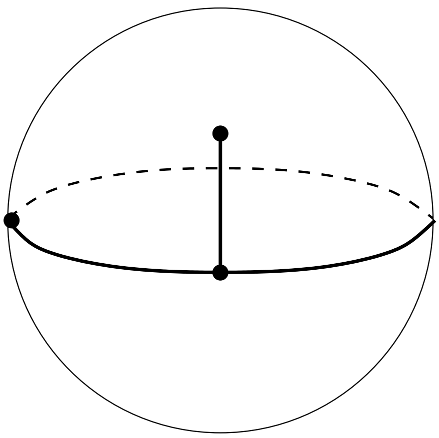


Figure 24: $S(321|42)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(42|321)$, see Figure 16 on the page 18. Belyi function is $\beta = \frac{-4z^3(z-1)^2(z+2)}{(3z-2)^2}$.

$$\langle Tr(Z^3)Tr^3(Z)Tr^3((Z^+)^2)\rangle = 3 \cdot 3! \cdot 2^3 \left(\frac{1}{3}N\right).$$

6.18 Valencies $\langle 3,1,1,1|6\rangle$

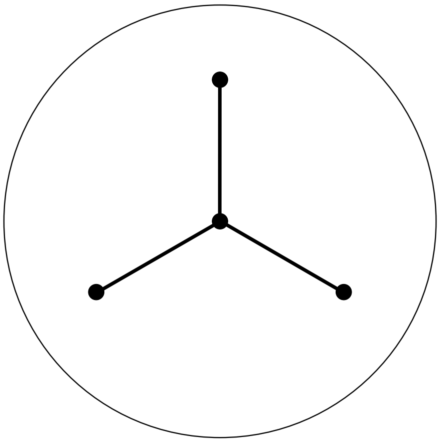


Figure 25: $S(3111|6)$. The automorphism group is Z_3. Dual dessin $S(6|3111)$, see Figure 8 on the page 13. Belyi function is $\beta = -4z^3(z^3-1)$.

$$\langle Tr^3(Z^2)Tr^3((Z^+)^2)\rangle = 3! \cdot 2^3 \cdot 3! \cdot 2^3 \left(\frac{1}{6}N^2\right).$$

6.19 Valencies $\langle 2,2,2|3,3\rangle$

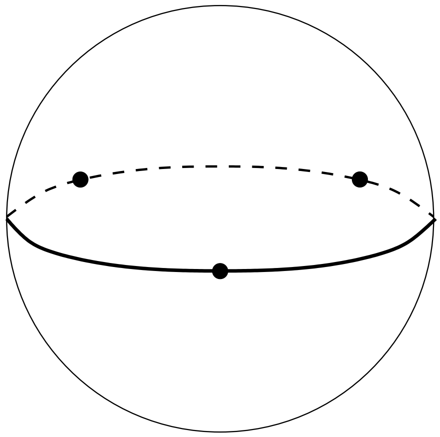


Figure 26: $S(222|33)$. The automorphism group is Z_3. Dual dessin $S(33|222)$, see Figure 21 on the page 20. Belyi function is $\beta = \frac{-(z^3-1)^2}{4z^3}$.

$$\langle Tr^2(Z^2)Tr^2(Z)Tr^3((Z^+)^2)\rangle = 2!\cdot 2^2\cdot 2!\cdot 3!\cdot 2^3\left(\frac{1}{2}N\right).$$

6.20 Valencies $\langle 2,2,1,1|6\rangle$

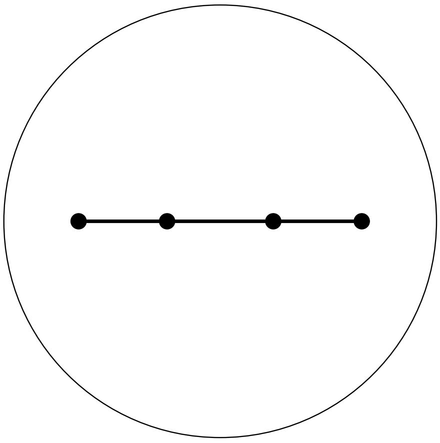


Figure 27: $S(2211|6)$. The automorphism group is Z_2. Dual dessin $S(6|2211)$, see Figure 9 on the page 13. Belyi function is $\beta = -(4z^2-1)^2(z^2-1)$.

7 4-edges dessins

7.1 Valencies $\langle 8|*\rangle$

$$\langle Tr(H^8)\rangle = 8\left(\frac{7}{4}N^5+\frac{35}{4}N^3+\frac{21}{8}N\right)=$$
$$=8\left(\left(1+\frac{1}{2}+\frac{1}{4}\right)N^5+\left(7+3\cdot\frac{1}{2}+\frac{1}{4}\right)N^3+\left(2+\frac{1}{2}+\frac{1}{8}\right)N\right).$$

Valencies $\langle 8|4,1,1,1,1\rangle$

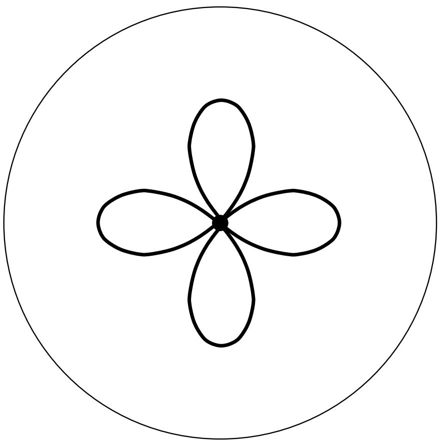


Figure 28: $S(8|41111)$. The automorphism group is Z_4. Dual dessin $S(41111|8)$, see Figure 121 on the page 70. Belyi function is $\beta = \frac{z^8}{4(z-1)(z+1)(z^2+1)}$.

Valencies $\langle 8|3,2,1,1,1\rangle$

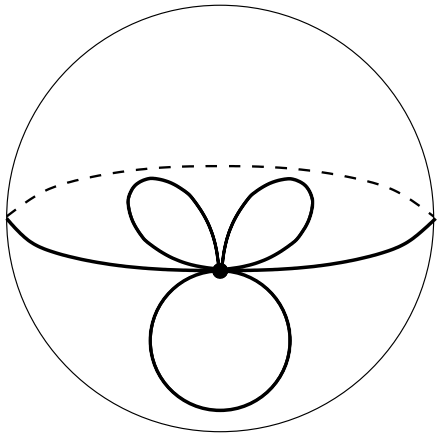


Figure 29: $S(8|32111)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(32111|8)$, see Figure 132 on the page 77. Belyi function is $\beta = \frac{729}{1024}\frac{z^8}{(z-1)(3z^2+8z+16)(3z-4)^2}$.

Valencies $\langle 8|2,2,2,1,1\rangle$

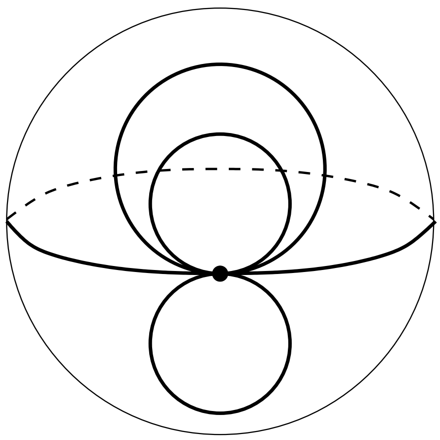


Figure 30: $S(8|22211)$. The automorphism group is Z_2. Dual dessin $S(22211|8)$, see Figure 134 on the page 79. Belyi function is $\beta = \frac{1}{-4z^2(z^2-2)(z-1)^2(z+1)^2}$.

Valencies $\langle 8|6,1,1\rangle$

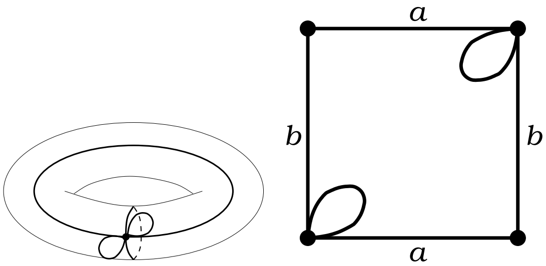


Figure 31: $T(8|611)\_2$. The automorphism group is Z_2. Dual dessin $T(611|8)\_2$, see Figure 76 on the page 50. Belyi function is $(X : y^2 = 3\,x^3 + 8\,x^2 + 16\,x, \beta = \frac{27}{256}\,\frac{x^4}{x-1})$.

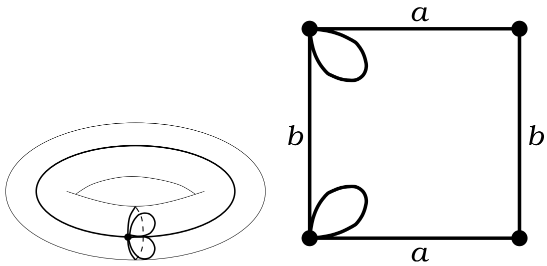


Figure 32: $T(8|611)\_1A$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(611|8)\_1A$, see Figure 77 on the page 50.

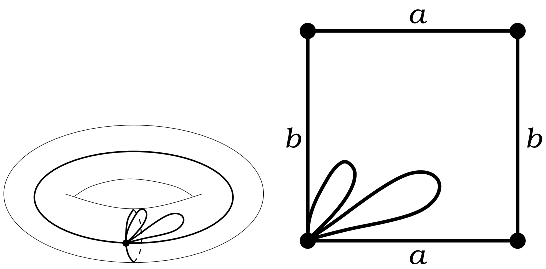


Figure 33: $T(8|611)\_1B$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(611|8)\_1B$, see Figure 78 on the page 50.

For the dessin $T(8|611)\_1A$ Belyi function β can be found from the equality $\frac{1}{\beta} = -\frac{1}{22769316864}\left(-835 - 872\sqrt{2}\right)(x^4 - 16\,yx^2 + 24\sqrt{2}yx^2 - 648\,x^3 + 272\sqrt{2}x^3 - 4032\sqrt{2}yx + 4224\,yx - 19296\sqrt{2}x^2 + 34200\,x^2 + 30240\sqrt{2}y - 50112\,y - 818208\,x + 706752\sqrt{2}x - 1088640\sqrt{2} + 1708560)$ on the curve $X : y^2 = -x^3 + 20\,x^2 - 8\sqrt{2}x^2 + 1104\sqrt{2}x - 132\,x$.
For the dessin $T(8|611)\_1B$ Belyi function β can be found from the equality $\frac{1}{\beta} = -\frac{1}{22769316864}\left(-835 + 872\sqrt{2}\right)(x^4 - 16\,yx^2 - 24\sqrt{2}yx^2 - 648\,x^3 - 272\sqrt{2}x^3 + 4032\sqrt{2}yx + 4224\,yx + 19296\sqrt{2}x^2 + 34200\,x^2 - 30240\sqrt{2}y - 50112\,y - 818208\,x - 706752\sqrt{2}x + 1088640\sqrt{2} + 1708560)$ on the curve $X : y^2 = -x^3 + 20\,x^2 + 8\sqrt{2}x^2 - 1104\sqrt{2}x - 132\,x$.

Valencies $\langle 8|5,2,1\rangle$

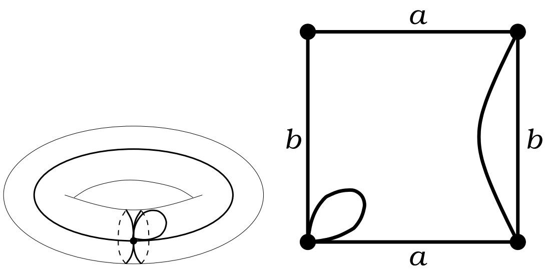


Figure 34: $T(8|521)A_+$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(521|8)A_+$, see Figure 92 on the page 58.

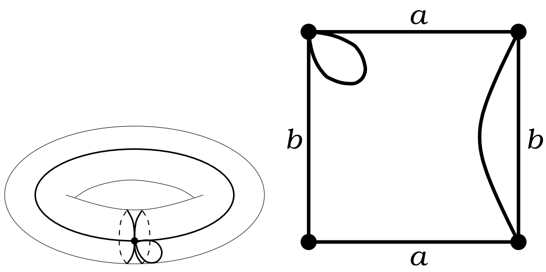


Figure 35: $T(8|521)A_-$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(521|8)A_-$, see Figure 93 on the page 58.

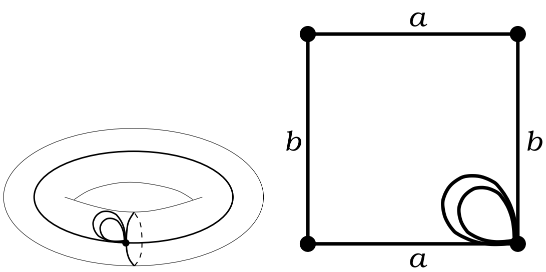


Figure 36: $T(8|521)B$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(521|8)B$, see Figure 94 on the page 58.

For the dessins $T(8|521)A_+$, $T(8|521)A_-$ and $T(8|521)B$ Belyi function β can be found from $\frac{1}{\beta} = \frac{1}{735306250}\,(552\,\nu^2 - 617\,\nu + 68)\,(42875\,x^4 + 1756160\,\nu\,yx^2 - 860160\,\nu^2yx^2 - 4543840\,yx^2 + 3959200\,\nu\,x^3 - 10346175\,x^3 - 1926400\,\nu^2x^3 + 31782912\,\nu^2yx - 63438592\,\nu\,yx + 168996968\,yx + 18916352\,\nu^2x^2 - 37781632\,\nu\,x^2 + 100206428\,x^2 - 257512128\,y - 48381952\,\nu^2y + 96684032\,\nu\,y - 62101504\,\nu^2x - 330259656\,x + 123960064\,\nu\,x + 48381952\,\nu^2 - 96684032\,\nu + 257512128)$ on the curve $X\,:\,y^2 = -\frac{\left(17\,\nu^2+8-42\,\nu\right)}{960400}\,(19600\,x^2 + 55552\,\nu\,x - 18432\,\nu^2x - 88408\,x + 338963 - 130592\,\nu + 65792\,\nu^2)(x-1)$.

Here $256\nu^3 - 544\nu^2 + 1427\nu - 172 = 0$, and real ν corresponds to the case B.

Valencies $\langle 8|4,3,1\rangle$

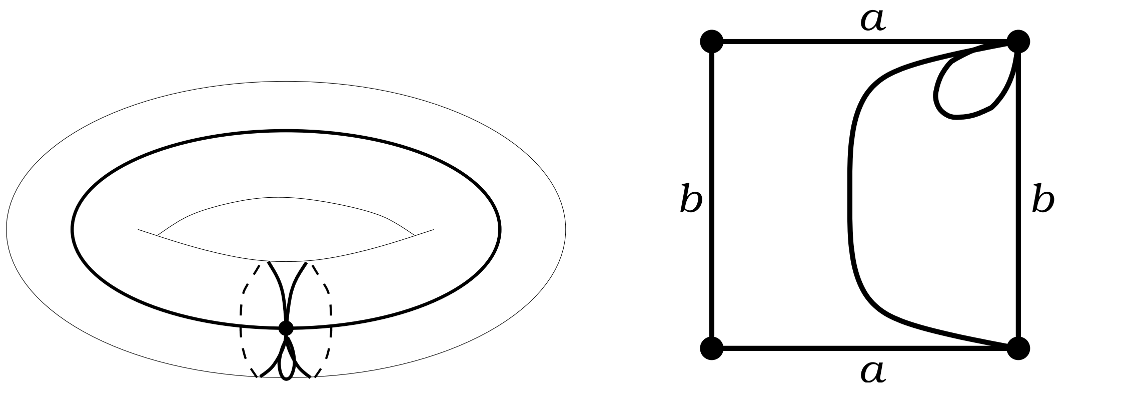


Figure 37: $T(8|431)A$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(431|8)A$, see Figure 110 on the page 65. Belyi function is $\beta = -\frac{85766121}{256}(5488\,x^4 + 14112\,yx^2 - 26264\,x^3 + 37548\,yx - 202741\,x^2 + 3240\,y - 73368\,x - 3240)^{-1}$ on the curve $X\ :\ y^2 = \frac{1}{81}(1-x)(448\,x^2 + 1872\,x + 81)$. $n_0 = \frac{62523502209}{65536}\frac{1}{x^4(4\,x+45)(4\,x+21)^3}$, $n_1 = \frac{\left(4096\,x^4+55296\,x^3+158976\,x^2+55296\,x-247617\right)^2}{65536x^4(4\,x+45)(4\,x+21)^3}$.

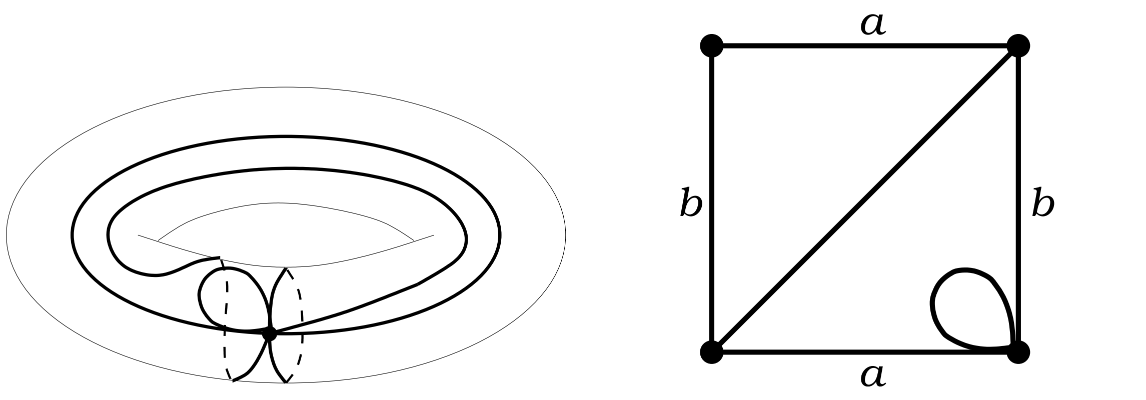


Figure 38: $T(8|431)B$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(431|8)B$, see Figure 111 on the page 65. Belyi function is $\beta = -36\,(81\,x^4 - 108\,yx^2 + 288\,x^3 - 96\,yx + 308\,x^2 - 24\,y + 160\,x + 32)^{-1}$ on the curve $X\ :\ y^2 = \frac{4}{9}\,(x+1)\,(9\,x^2 + 4\,x + 4)$. $n_0 = \frac{16}{3}\frac{1}{(x-2)(3\,x+2)^3x^4}$, $n_1 = \frac{1}{27}\frac{\left(27\,x^4-36\,x^2-32\,x-20\right)^2}{(x-2)(3\,x+2)^3x^4}$.

Valencies $\langle 8|4,2,2\rangle$

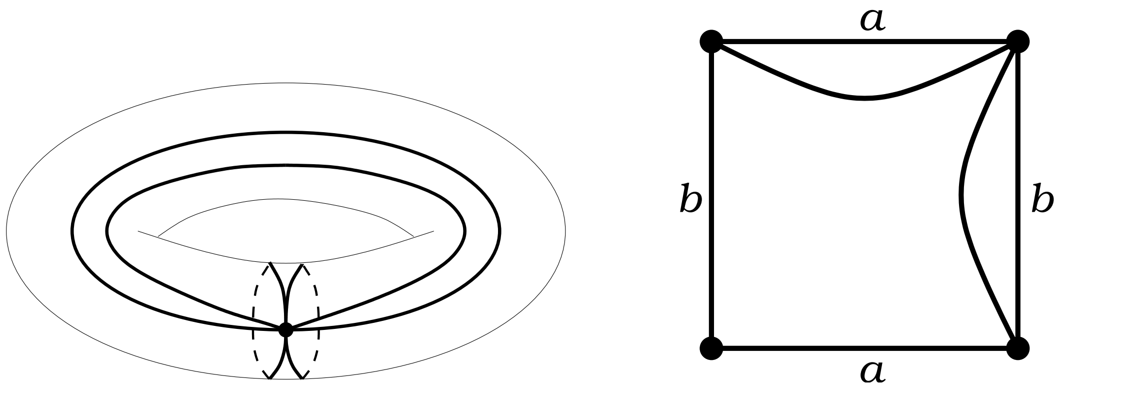


Figure 39: $T(8|422)\_4$. The automorphism group is $Z_2 \oplus Z_2$. Dual dessin $T(422|8)\_4$, see Figure 114 on the page 67. Belyi function is $(X : y^2 = x^3 - x, \beta = \frac{1}{4}\frac{x^4}{(x-1)(x+1)})$.

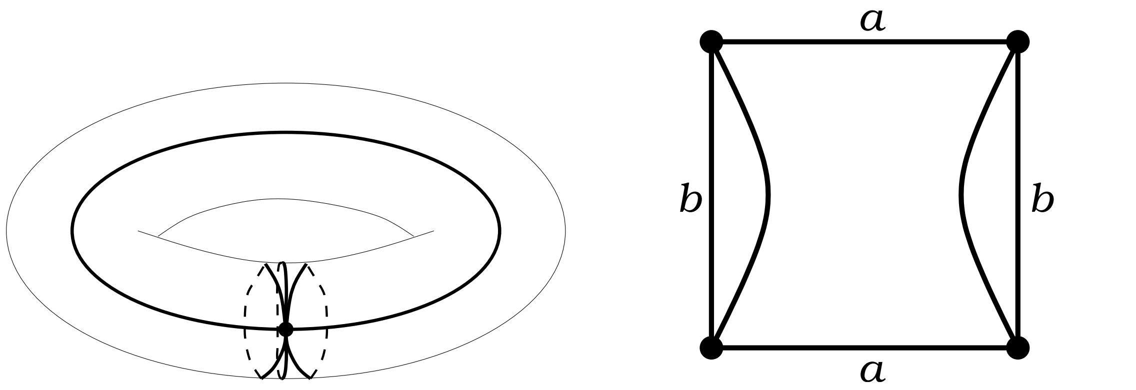


Figure 40: $T(8|422)\_2$. The automorphism group is Z_2. Dual dessin $T(422|8)\_2$, see Figure 115 on the page 67. Belyi function is $X : y^2 = 4x^3 - 4x^2 - x, \beta = \frac{1}{16x^2(x-1)^2}$.

Valencies $\langle 8|3,3,2\rangle$

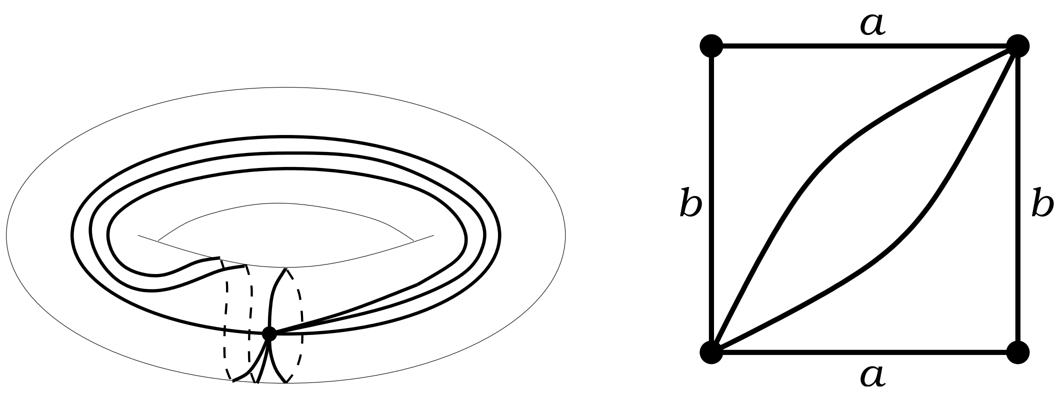


Figure 41: $T(8|332)$. The automorphism group is Z_2. Dual dessin $T(332|8)$, see Figure 125 on the page 72. Belyi function is $X : y^2 = x\,(x-1)\,(3\,x^2+8\,x+16)\,, \beta = \frac{27}{256}\,\frac{x^4}{x-1}$.

Valencies $\langle 8|8 \rangle$

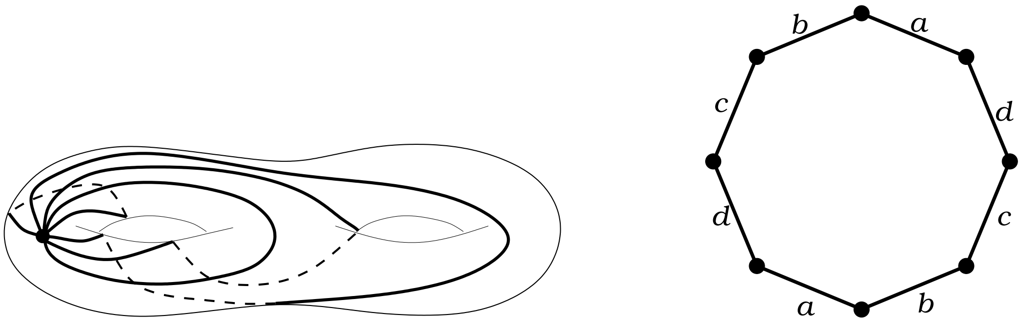


Figure 42: $P(8|8)$_8. The automorphism group is Z_8. Dual dessin $P(8|8)$_8, see Figure 42 on the page 33. Belyi function is $(X : y^2 = x^5 - x, \beta = x^4)$.

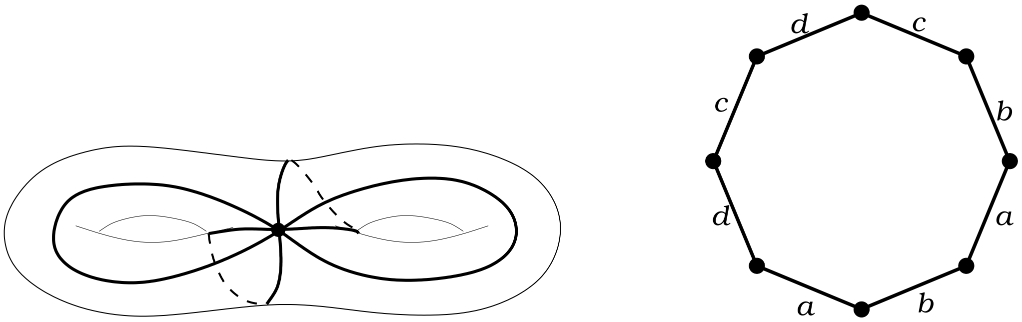


Figure 43: $P(8|8)$_2. The automorphism group is Z_2. Dual dessin $P(8|8)$_2, see Figure 43 on the page 33. Belyi function is $\beta = -(-xy + x^4 - 2x^2 + 1)^2$ on the curve $X : y^2 = (x^2 - 2)(x^4 - 2x^2 + 2)$. $n_0 = 1$, $n_1 = 4(x-1)^4(x+1)^4$.

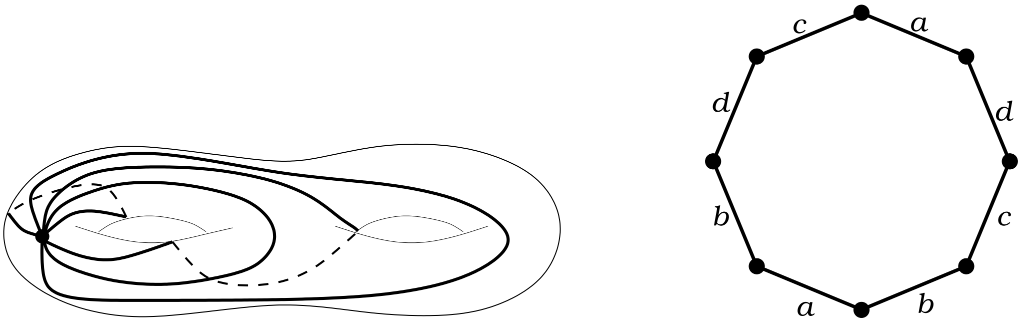


Figure 44: $P(8|8)\_1A$. There are no nontrivial automorphisms of this dessin. Dual dessin $P(8|8)\_1A$, see Figure 44 on the page 34.

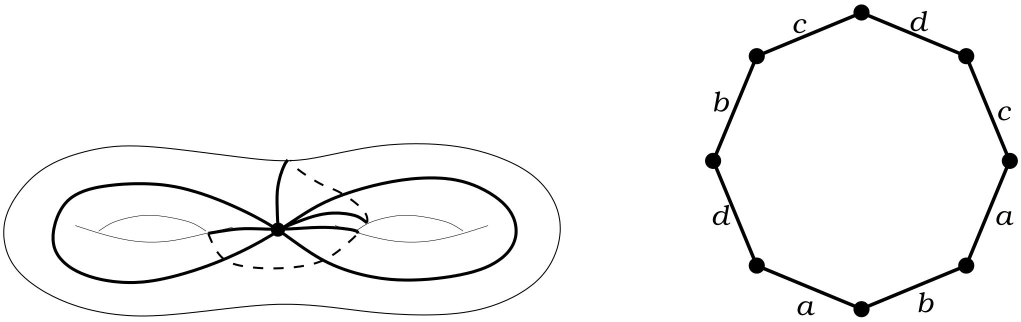


Figure 45: $P(8|8)\_1B$. There are no nontrivial automorphisms of this dessin. Dual dessin $P(8|8)\_1B$, see Figure 45 on the page 34.

Pairwise conjugate Belyi function of dessins $P(8|8)\_1A$ and $P(8|8)\_1B$:
$\beta = -1/8\,(-5+4\,\mu)\,(x-1-\mu)\,(x+1-\mu)\,(x+1)\,(x^2-2\,x-2\,\mu\,x+1)y + \frac{1}{56}\,(-5+4\,\mu)\,(7\,x^8-8\,x^7-40\,\mu\,x^7+140\,x^6-56\,\mu\,x^5+168\,x^5-14\,x^4-224\,\mu\,x^4+168\,x^3-56\,\mu\,x^3+140\,x^2-8\,x-40\,\mu\,x+7)$ on the curve $X:\{y^2=x^6-\frac{24}{7}\,\mu\,x^5-2/7\,x^5+\frac{107}{49}\,x^4-\frac{200}{49}\,\mu\,x^4+\frac{500}{49}\,x^3-\frac{48}{49}\,\mu\,x^3-\frac{200}{49}\,\mu\,x^2+\frac{107}{49}\,x^2-\frac{24}{7}\,\mu\,x-2/7\,x+1\}$.
$n_0=x^8$, $n_1=-\frac{1}{196}\,(-9+4\,\mu)(7\,x^4+12\,x^3-4\,\mu\,x^3+6\,x^2-16\,\mu\,x^2+12\,x-4\,\mu\,x+7)^2$, where $\mu=\pm\sqrt{2}$.

7.2 Valencies $\langle 7, 1|*\rangle$

$$\langle Tr(H^7)Tr(H)\rangle = 7\left(5N^4 + 10N^2\right).$$

Valencies $\langle 7, 1|5, 1, 1, 1\rangle$

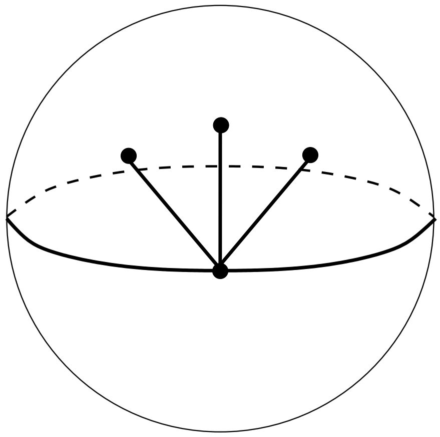


Figure 46: $S(71|5111)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(5111|71)$, see Figure 95 on the page 59. Belyi function is $\beta = \frac{16384z(z-1)^7}{896z^3-2912z^2+3216z-1225}$.

Valencies $\langle 7, 1|4, 2, 1, 1\rangle$

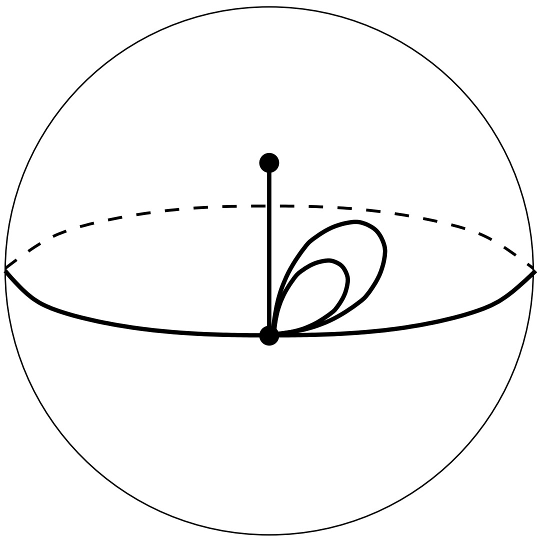


Figure 47: $S(71|4211)_+$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(4211|71)_+$, see Figure 116 on the page 68. Belyi function is $\beta = \frac{-7340032(i\sqrt{7}+21)z(z-1)^7}{(896z^2-1904z+48iz\sqrt{7}+1029-49i\sqrt{7})(112z-119-5i\sqrt{7})^2}$

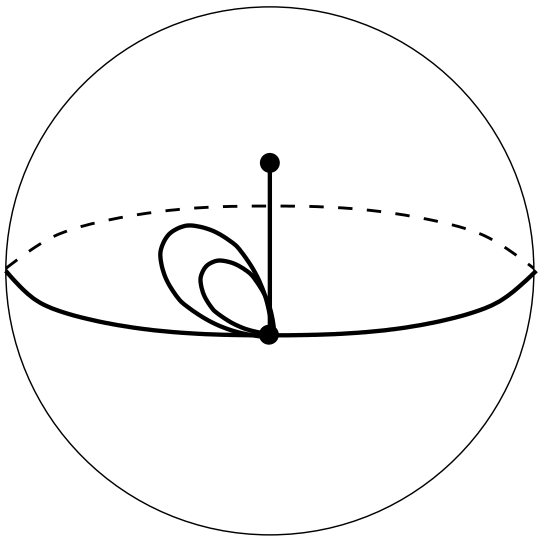


Figure 48: $S(71|4211)_-$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(4211|71)_-$, see Figure 117 on the page 68. Belyi function is $\beta = \frac{-7340032(-i\sqrt{7}+21)z(z-1)^7}{(896z^2-1904z-48iz\sqrt{7}+1029+49i\sqrt{7})(112z-119+5i\sqrt{7})^2}$

Valencies $\langle 7, 1|3, 3, 1, 1\rangle$

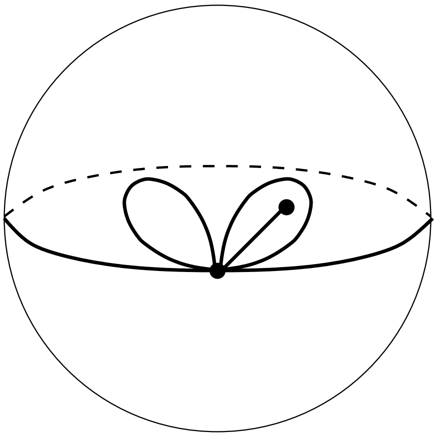


Figure 49: $S(71|3311)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(3311|71)$, see Figure 126 on the page 73. Belyi function is $\beta = -1728\,\frac{z}{(1+z^2-5\,z)^3(49-13\,z+z^2)}$.

Valencies $\langle 7, 1|3, 2, 2, 1\rangle$

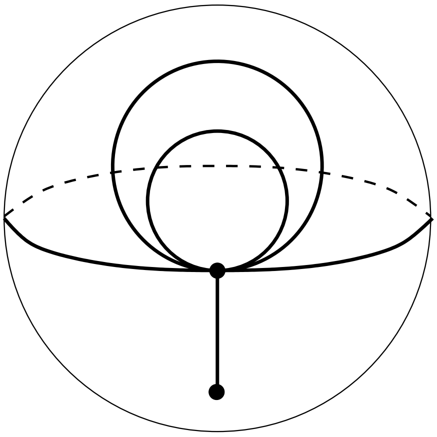


Figure 50: $S(71|3221)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(3221|71)$, see Figure 129 on the page 75. Belyi function is $\beta = -\frac{1}{256}\,\frac{z^7(-48+z)}{(z+1)(7\,z^2+28\,z+24)^2}$.

Valencies $\langle 7,1|7,1\rangle$

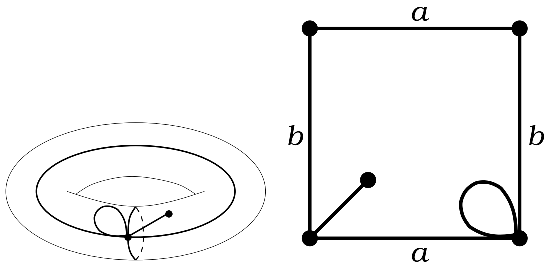


Figure 51: $T(71|71)A_+$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(71|71)A_-$, see Figure 52 on the page 37.

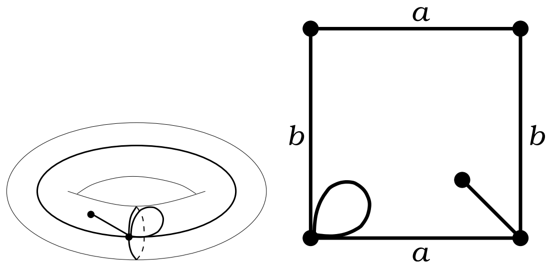


Figure 52: $T(71|71)A_-$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(71|71)A_+$, see Figure 51 on the page 37.

For the dessins $T(71|71)A_+$ $T(71|71)A_-$ Belyi function is $\beta = -\frac{(\nu+3)}{8(32\,x-51-7\,\nu)}(-2\,\nu\,yx^3 + 10\,yx^3 + 56\,x^4 - 64\,yx^2 - 14\,\nu\,x^3 - 266\,x^3 + 116\,yx + 12\,\nu\,yx + 504\,x^2 + 56\,\nu\,x^2 - 11\,\nu\,y - 73\,y - 454\,x - 82\,\nu\,x + 171 + 41\,\nu)$ on the curve of the form $X : y^2 = -\frac{1}{32}\,(5+\nu)\,(16\,x^3 + 4\,\nu\,x^2 - 52\,x^2 - 4\,\nu\,x + 68\,x + \nu - 37)$, where $\nu = \pm i\sqrt{7}$.

In this case $n_0 = \frac{(13+7\,\nu)(x+3)(x-1)^7}{8(32\,x-51-7\,\nu)}$, $n_1 = \frac{(13+7\,\nu)\left(8\,x^4-16\,x^3-16\,x^2+80\,x-59-7\,\nu\right)^2}{512(32\,x-51-7\,\nu)}$.

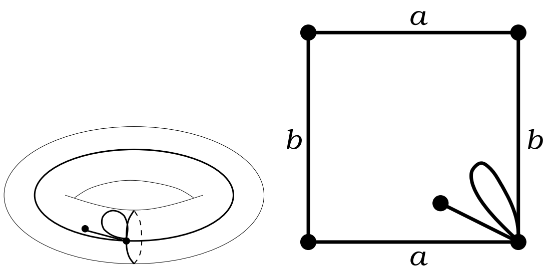


Figure 53: $T(71|71)B_+$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(71|71)B_+$, see Figure 53 on the page 38. Belyi function is: ..

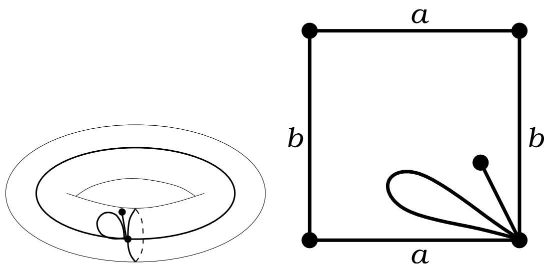


Figure 54: $T(71|71)B_-$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(71|71)B_-$, see Figure 54 on the page 38. Belyi function can be seen below.

For the dessin $T(71|71)B_+$ Belyi function is $\beta = -\frac{343}{33554432}\frac{\left(-91\,i\sqrt{7}+87\right)\left(98\,x^2+21\,x+21\,i\sqrt{7}x+2\right)\left(14\,x^2+7\,x+7\,i\sqrt{7}x-2\right)^2y}{x} + \frac{1}{8388608x}(-91\,i\sqrt{7}+87)(1647086\,x^8+3294172\,x^7+3294172\,i\sqrt{7}x^7-15764966\,x^6+4941258\,i\sqrt{7}x^6-4705960\,i\sqrt{7}x^5-17882648\,x^5-3882417\,i\sqrt{7}x^4+5260591\,x^4+672280\,i\sqrt{7}x^3+2554664\,x^3-321734\,x^2+100842\,i\sqrt{7}x^2+2044\,i\sqrt{7}x+1532\,x+686)$ on the curve $X: -x^4-\frac{11}{7}\,i\sqrt{7}x^3-\frac{11}{7}\,x^3+\frac{519}{98}\,x^2-\frac{153}{98}\,i\sqrt{7}x^2+\frac{51}{49}\,i\sqrt{7}x+\frac{869}{343}\,x+y^2-1=0$. $n_0=1$, $n_1=-\frac{343}{33554432}\frac{\left(-91\,i\sqrt{7}+87\right)\left(14\,x^2+7\,x+7\,i\sqrt{7}x-2\right)^4}{x}$.

For the dessin $T(71|71)B_-$ Belyi function is $\beta = -\frac{343}{33554432}\frac{\left(91\,i\sqrt{7}+87\right)\left(98\,x^2+21\,x-21\,i\sqrt{7}x+2\right)\left(14\,x^2+7\,x-7\,i\sqrt{7}x-2\right)^2y}{x} + \frac{1}{8388608x}\left(91\,i\sqrt{7}+87\right)(1647086\,x^8+3294172\,x^7-3294172\,i\sqrt{7}x^7-15764966\,x^6-4941258\,i\sqrt{7}x^6+4705960\,i\sqrt{7}x^5-17882648\,x^5+3882417\,i\sqrt{7}x^4+5260591\,x^4-672280\,i\sqrt{7}x^3+2554664\,x^3-321734\,x^2-100842\,i\sqrt{7}x^2-2044\,i\sqrt{7}x+1532\,x+686)$ on the curve $X: -x^4+\frac{11}{7}\,i\sqrt{7}x^3-\frac{11}{7}\,x^3+\frac{519}{98}\,x^2+\frac{153}{98}\,i\sqrt{7}x^2-\frac{51}{49}\,i\sqrt{7}x+\frac{869}{343}\,x+y^2-1=0$. $n_0=1$, $n_1=\frac{343}{33554432}\frac{\left(91\,i\sqrt{7}+87\right)\left(14\,x^2+7\,x-7\,i\sqrt{7}x-2\right)^4}{x}$.

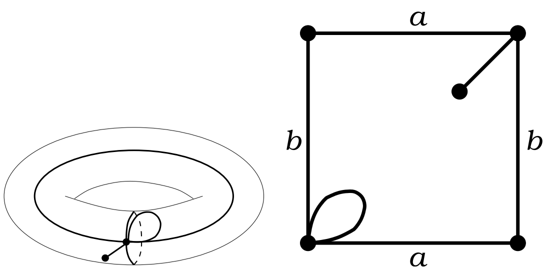


Figure 55: $T(71|71)C$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(71|71)C$, see Figure 55 on the page 39. Belyi function is $\beta = \frac{1}{128}\frac{x^3(7x+y)}{x-1}$ on the curve $X : y^2 = 4x^3 + 13x^2 + 32x$. $n_0 = -\frac{1}{4096}\frac{x^7(x-8)}{x-1}$, $n_1 = -\frac{1}{4096}\frac{\left(x^4-4x^3-8x^2-32x+64\right)^2}{x-1}$.

Valencies $\langle 7,1|6,2\rangle$

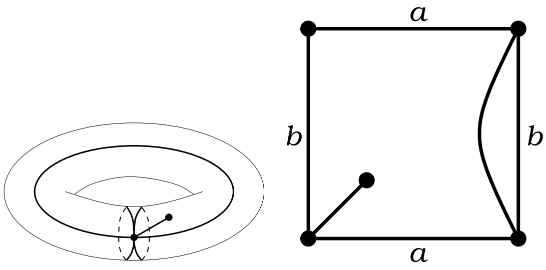


Figure 56: $T(71|62)_+$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(62|71)_+$, see Figure 64 on the page 44.

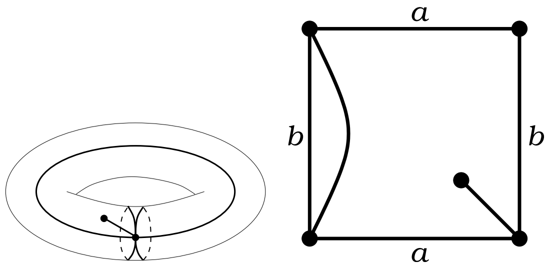


Figure 57: $T(71|62)_-$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(62|71)_-$, see Figure 65 on the page 44.

For the dessins $T(71|62)_+$ and $T(71|62)_-$ Belyi function is $\beta = -\frac{2}{49(-112\,x-147+81\,\nu)^2}(784\,x^5 + 19404\,\nu\,yx^3 + 12348\,yx^3 + 15435\,x^4 + 567\,\nu\,x^4 + 54684\,\nu\,yx^2 + 188748\,yx^2 + 80892\,x^3 - 6804\,\nu\,x^3 + 48384\,\nu\,yx + 338688\,yx - 40824\,\nu\,x^2 + 146664\,x^2 + 13608\,\nu\,y + 140616\,y + 104976\,x - 45360\,\nu\,x - 13608\,\nu + 25272)$ on the curve of the form $X : y^2 = -\frac{(35+9\,\nu)}{87808}(2\,x+1)(14\,x^2 + 18\,\nu\,x + 42\,x + 9\,\nu + 189)$, where $\nu = \pm i\sqrt{7}$.

In this case $n_0 = \frac{4}{49}\frac{(x-24)x^7}{(-112\,x-147+81\,\nu)^2}$, $n_1 = \frac{1}{49}\frac{\left(2\,x^4-24\,x^3-144\,x^2-944\,x-1077+567\,\nu\right)^2}{(-112\,x-147+81\,\nu)^2}$.

Valencies $\langle 7,1|5,3\rangle$

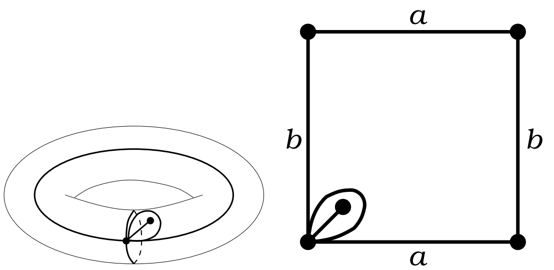


Figure 58: $T(71|53)A$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(53|71)A$, see Figure 82 on the page 52.

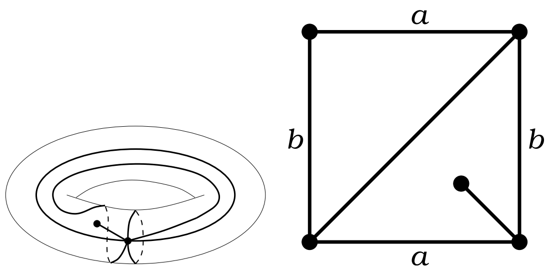
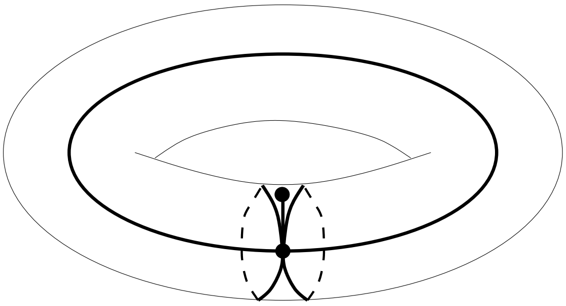


Figure 59: $T(71|53)B$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(53|71)B$, see Figure 83 on the page 52.

For the dessins $T(71|53)A$, $T(71|53)B$ Belyi function has the form $\beta = \frac{(78133+7625\,\nu)}{8232}\,(169344\,x^5+17640\,yx^3-47250\,\nu\,x^4-489510\,x^4-5250\,\nu\,yx^2-7350\,yx^2+1269324\,x^3+94500\,\nu\,x^3+12250\,\nu\,yx-54950\,yx-3537716\,x^2+94500\,\nu\,x^2+45885\,y-7125\,\nu\,y-301000\,\nu\,x+4558440\,x-1978857+160125\,\nu)$ on the curve of the form $X:$ $y^2=\frac{1}{350}\,(2688\,x^3-10388\,x^2+100\,\nu\,x^2-500\,\nu\,x+15972\,x+1275\,\nu-17247)(12x-13)$. Here $\nu=\pm\sqrt{105}$
and $n_0 = \frac{1944(78133+7625\,\nu)}{343}\,(x+3)\,(x-1)^7$, $n_1 = \frac{78133+7625\,\nu}{131712}\,(864\,x^4-1728\,x^3-1728\,x^2+5504\,x-11895+875\,\nu)^2$.

Valencies $\langle 7, 1|4, 4\rangle$



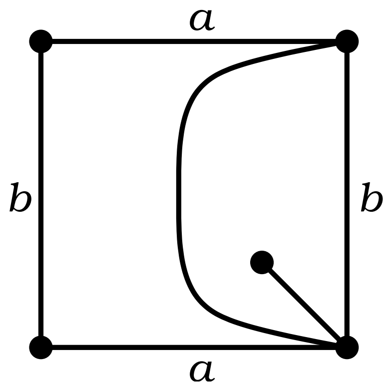


Figure 60: $T(71|44)$. There is no automorphism of this dessin. Dual dessin $T(44|71)$, see Figure 100 on the page 61. Belyi function is $\beta = \frac{128}{343}(294\,x^4 - 28\,yx^2 + 154\,x^3 - 7\,yx + 77\,x^2 - 2\,y + 13\,x + 2)$ on the curve $X : y^2 = 112\,x^4 + 56\,x^3 + 37\,x^2 + 6\,x + 1$. $n_0 = -\frac{65536}{343}\,(x-2)\,x^7$, $n_1 = -\frac{1}{343}\,(256\,x^4 - 256\,x^3 - 128\,x^2 - 128\,x - 13)^2$.

7.3 Valencies $\langle 6,2|*\rangle$

$$\langle Tr(H^6)Tr(H^2)\rangle = 6\cdot 2\left(\frac{5}{2}N^4+5N^2\right) =$$
$$= 6\cdot 2\left(\left(2+\frac{1}{2}\right)N^4+\left(4+2\left(\frac{1}{2}\right)\right)N^2\right).$$

Valencies $\langle 6,2|4,2,1,1\rangle$

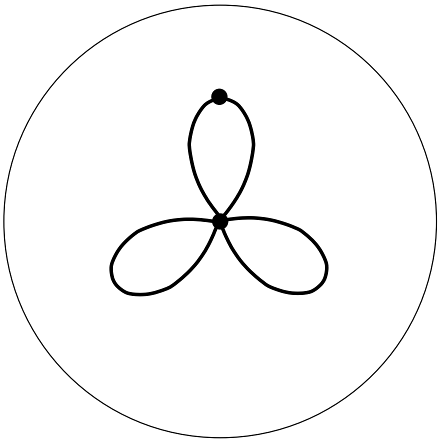


Figure 61: $S(62|4211)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(4211|62)$, see Figure 118 on the page 69. Belyi function is $\beta = -\frac{1}{108}\frac{z^6(z-4)^2}{(z^2+2z+3)(z-3)^2}$.

Valencies $\langle 6,2|3,3,1,1\rangle$

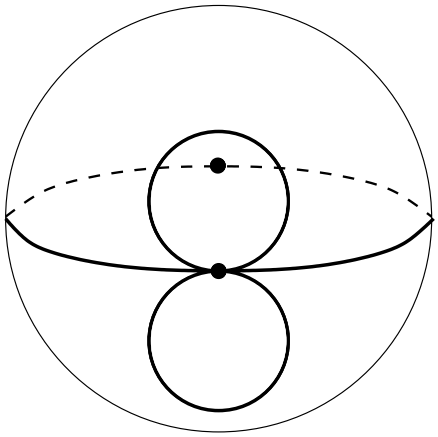


Figure 62: $S(62|3311)$. The automorphism group is Z_2. Dual dessin $S(3311|62)$, see Figure 127 on the page 73. Belyi function is $\beta = -64\frac{z^6}{(-1+3z)(3z+1)(z-1)^3(z+1)^3}$.

Valencies $\langle 6, 2|3, 2, 2, 1\rangle$

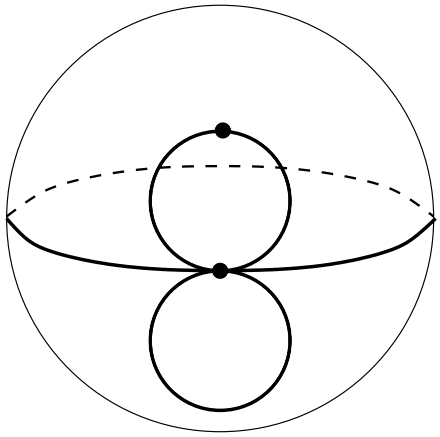


Figure 63: $S(62|3221)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(3221|62)$, see Figure 130 on the page 75. Belyi function is $\beta = -4\,\frac{z^6(z-2)^2}{(4\,z+1)(-1-2\,z+2\,z^2)^2}$.

Valencies $\langle 6, 2|7, 1\rangle$

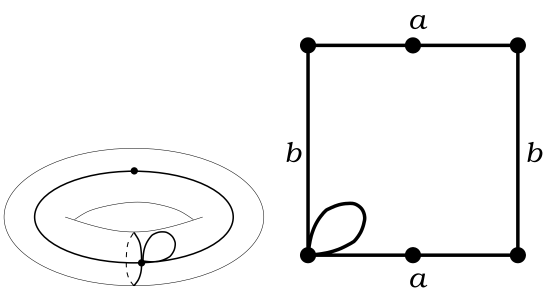


Figure 64: $T(62|71)_+$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(71|62)_+$, see Figure 56 on the page 40.

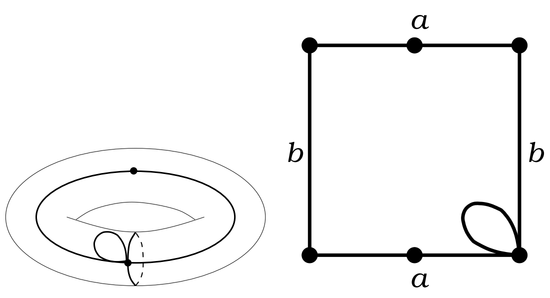


Figure 65: $T(62|71)_-$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(71|62)_-$, see Figure 57 on the page 40.

For the dessins $T(62|71)_+$ and $T(62|71)_-$ Belyi function has the form $\beta = \frac{(11\,\nu+7)}{224(x-24)x^7}(-686\,x^5 + 1078\,\nu\,x^5 + 197568\,yx^3 + 20727\,\nu\,x^4 - 18963\,x^4 + 691488\,yx^2 - 211680\,\nu\,yx^2 + 117180\,\nu\,x^3 - 5292\,x^3 + 762048\,yx - 423360\,\nu\,yx + 237384\,\nu\,x^2 + 264600\,x^2 - 181440\,\nu\,y + 254016\,y + 344736\,x + 184032\,\nu\,x + 46656\,\nu + 108864)$ on the curve of the form $X : y^2 = -\frac{(35+9\,\nu)}{87808}\,(2\,x+1)(14\,x^2 + 18\,\nu\,x + 42\,x + 9\,\nu + 189)$, where $\nu = \pm i\sqrt{7}$.
For these dessins $n_0 = \frac{49}{4}\,\frac{(-112\,x-147+81\,\nu)^2}{(x-24)x^7}$,
$n_1 = \frac{1}{4}\,\frac{\left(2\,x^4-24\,x^3-144\,x^2-944\,x-1077+567\,\nu\right)^2}{(x-24)x^7}$.

Valencies $\langle 6, 2|6, 2\rangle$

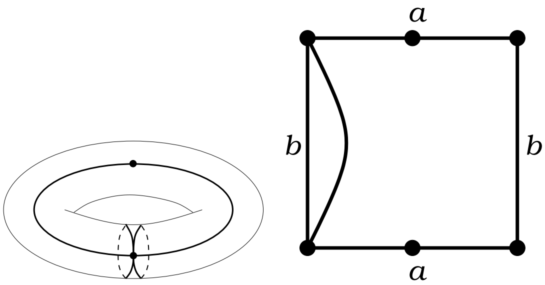


Figure 66: $T(62|62)\_2$. The automorphism group is Z_2. This dessin is selfdual. Belyi function is $(X : y^2 = x\,(x+8)\,(x-1)\,, \beta = \frac{1}{64}\,\frac{x^3(x+8)}{x-1})$.

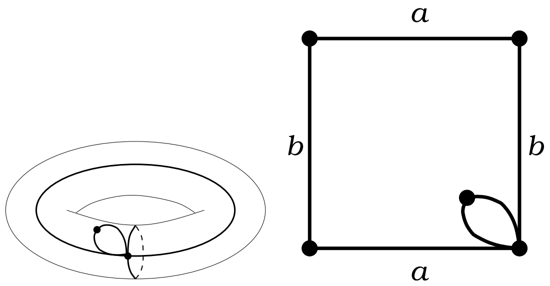


Figure 67: $T(62|62)\_1$. There are no nontrivial automorphisms of this dessin. This dessin is selfdual. Belyi function is $\beta = -\frac{1}{432}\,\frac{\left(18\,x^3-810\,x^2+9342\,x-5382\right)y}{(4\,x-89)^2} - \frac{1}{432}\,\frac{4\,x^5-103\,x^4-1172\,x^3+28030\,x^2+126536\,x-295247}{(4\,x-89)^2}$ on the curve $X : y^2 = 4\,(x-2)(x^2+2\,x+73)$. $n_0 = \frac{1}{186624}\,\frac{(x-29)^2(x-5)^6}{(4\,x-89)^2}$, $n_1 = \frac{1}{186624}\,\frac{\left(x^4-44\,x^3+510\,x^2-572\,x+35161\right)^2}{(4\,x-89)^2}$.

Valencies $\langle 6, 2|5, 3\rangle$

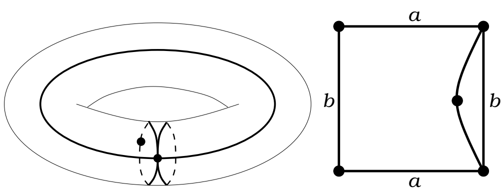


Figure 68: $T(62|53)$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(53|62)$, see Figure 84 on the page 54. Belyi function is $\beta = \frac{1}{162}(x^5 + yx^3 + \frac{65}{8}\,x^4 + 6\,yx^2 + x^3 - 64\,x^2 - 16\,y - 16\,x + 80)$ on the curve

Valencies $\langle 6, 2|4, 4\rangle$

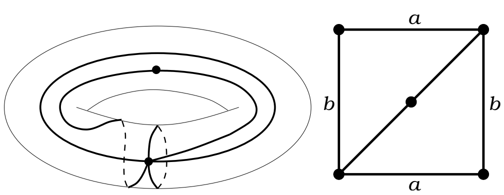


Figure 69: $T(62|44)$. The automorphism group is Z_2. Dual dessin $T(44|62)$, see Figure 101 on the page 61. Belyi function is $(X : y^2 = (x-2)(4x^2+4x+3)x, \beta = -\frac{16}{27}x^3(x-2))$.

7.4 Valencies $\langle 6,1,1|*\rangle$

$$\langle Tr(H^6)Tr^2(H)\rangle = 6\cdot 2!\left(5N^3+\frac{5}{2}N\right) =$$
$$= 6\cdot 2!\left(\left(4+2\left(\frac{1}{2}\right)\right)N^3+\left(2+\frac{1}{2}\right)N\right).$$

Valencies $\langle 6,1,1|6,1,1\rangle$

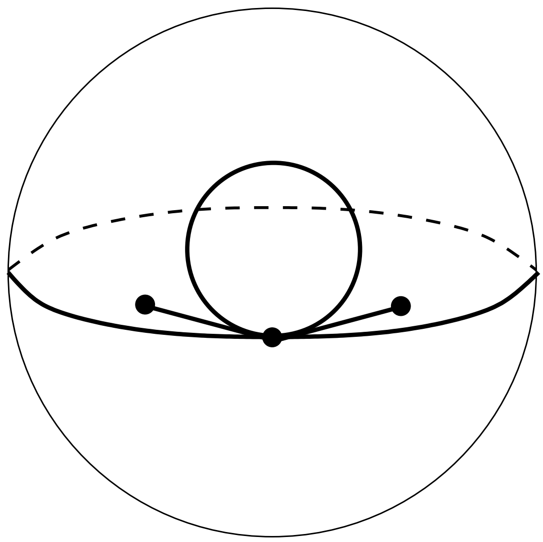


Figure 70: $S(611|611)\_2$. The automorphism group is Z_2. This dessin is selfdual. Belyi function is $\beta = -4\,\frac{(z^2-2)z^6}{4\,z^2+1}$.

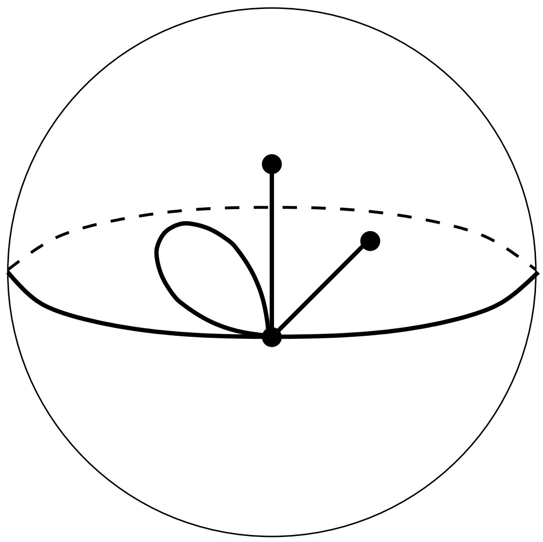


Figure 71: $S(611|611)\_1$. There are no nontrivial automorphisms of this dessin. This dessin is selfdual. Belyi function is $\beta = -\frac{27}{4}\,\frac{(3\,z^2+6\,z+7)z^6}{21\,z^2-12\,z+4}$.

Valencies $\langle 6,1,1|5,2,1\rangle$

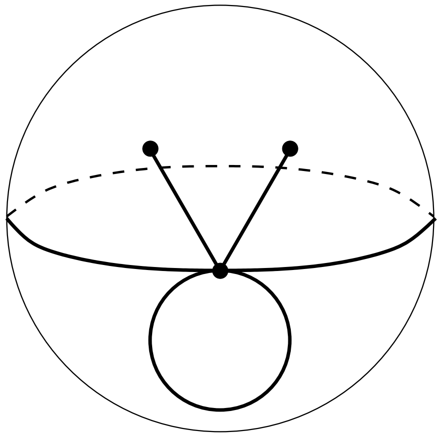


Figure 72: $S(611|521)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(521|611)$, see Figure 86 on the page 55. Belyi function is $\beta = 4\,\frac{z^6\left(9\,z^2+24\,z+70\right)}{(4\,z-1)(14\,z-5)^2}$.

Valencies $\langle 6,1,1|4,3,1\rangle$

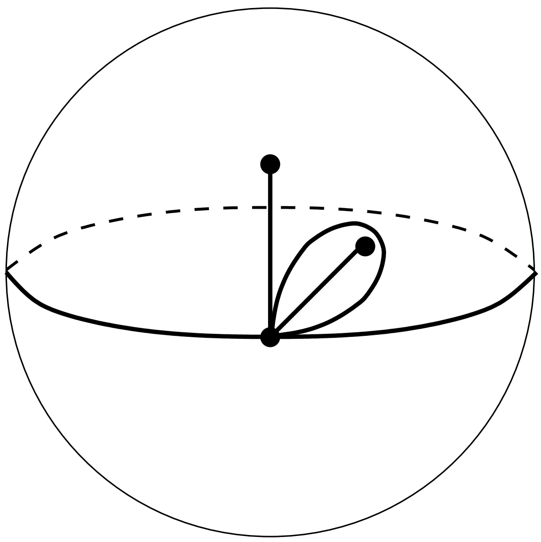


Figure 73: $(S(611|431)_+$. There are no nontrivial automorphisms of this dessin. Dual dessin $(S(431|611)_+$, see Figure 104 on the page 63. Belyi function is $\beta = 12\,\frac{\left(-6\,z^2+20\,i\sqrt{3}z-12\,z-19\,i\sqrt{3}+17\right)z^6}{\left(-747+1763\,i\sqrt{3}\right)\left(3\,z+2\,i\sqrt{3}+3\right)(z-1)^3}$.

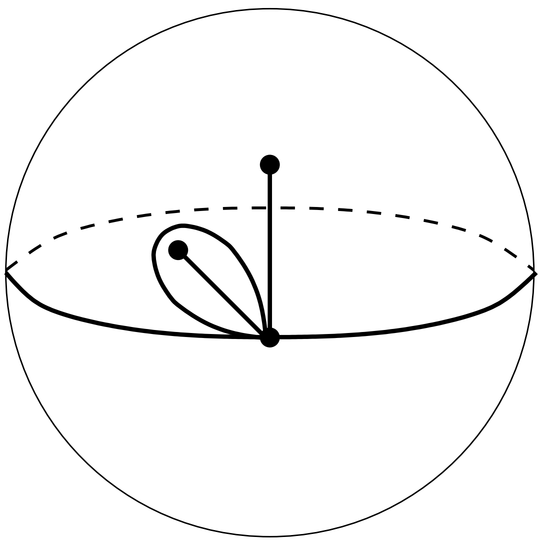


Figure 74: $(S(611|431)_-$. There are no nontrivial automorphisms of this dessin. Dual dessin $(S(431|611)_-$, see Figure 105 on the page 63. Belyi function is $\beta = -12\,\frac{\left(6\,z^2+20\,i\sqrt{3}z+12\,z-19\,i\sqrt{3}-17\right)z^6}{\left(1763\,i\sqrt{3}+747\right)\left(-3\,z-3+2\,i\sqrt{3}\right)(z-1)^3}$.

Valencies $\langle 6,1,1|3,3,2\rangle$

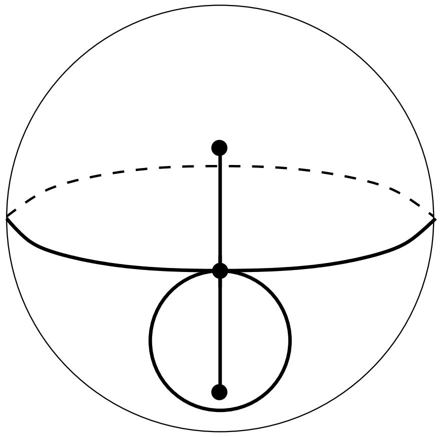


Figure 75: $S(611|332)$. The automorphism group is Z_2. Dual dessin $S(332|611)$, see Figure 122 on the page 71. Belyi function is $\beta = -4\,\frac{z^6\left(9\,z^2+2\right)}{\left(4\,z^2+1\right)^3}$.

Valencies $\langle 6,1,1|8\rangle$

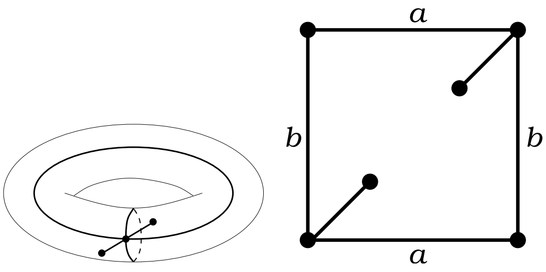


Figure 76: $T(611|8)\_2$. The automorphism group is Z_2. Dual dessin $T(8|611)\_2$, see Figure 31 on the page 27. Belyi function is $(X : y^2 = x\,(x^2 + 1/2\,x + 3/16)\,, \beta = -\frac{256}{27}\,x^3\,(x-1))$.

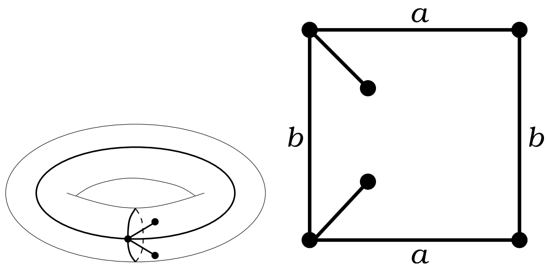


Figure 77: $T(611|8)\_1A$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(8|611)\_1A$, see Figure 32 on the page 27.

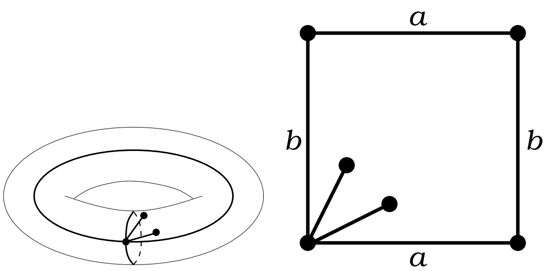


Figure 78: $T(611|8)\_1B$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(8|611)\_1B$, see Figure 33 on the page 27.

For the dessin $T(611|8)\_1A$ Belyi function is $\beta = -\frac{1}{22769316864}\left(-835 - 872\sqrt{2}\right)(x^4 - 16\,yx^2 + 24\sqrt{2}yx^2 - 648\,x^3 + 272\sqrt{2}x^3 - 4032\sqrt{2}yx + 4224\,yx - 19296\sqrt{2}x^2 + 34200\,x^2 + 30240\sqrt{2}y - 50112\,y - 818208\,x + 706752\sqrt{2}x - 1088640\sqrt{2} + 1708560)$ on the curve $X : y^2 = -x^3 + 20\,x^2 - 8\sqrt{2}x^2 + 1104\sqrt{2}x - 132\,x$.
For the dessin $T(611|8)\_1B$ Belyi function is $\beta = -\frac{1}{22769316864}\left(-835 + 872\sqrt{2}\right)(x^4 - 16\,yx^2 - 24\sqrt{2}yx^2 - 648\,x^3 - 272\sqrt{2}x^3 + 4032\sqrt{2}yx + 4224\,yx + 19296\sqrt{2}x^2 + 34200\,x^2 - 30240\sqrt{2}y - 50112\,y - 818208\,x - 706752\sqrt{2}x + 1088640\sqrt{2} + 1708560)$ on the curve $X : y^2 = -x^3 + 20\,x^2 + 8\sqrt{2}x^2 - 1104\sqrt{2}x - 132\,x$.
Where $n_0 = -\frac{(-2217993 - 1456240\,\nu)}{518441790453234794496}\,(x^2 + 76\,x - 152\,\nu\,x + 44100 - 19600\,\nu)\,(x + 6 - 12\,\nu)^6$, $n_1 = -\frac{(-2217993 - 1456240\,\nu)}{518441790453234794496}\,(x^4 + 56\,x^3 - 112\,\nu\,x^3 + 22680\,x^2 - 10080\,\nu\,x^2 - 756000\,x + 665280\,\nu\,x + 24794640 - 25197696\,\nu)^2$. Here $\nu^2 = 2$, $\nu > 0$ corresponds to the case A and $\nu < 0$ corresponds to the case B.

7.5 Valencies $\langle 5,3|*\rangle$

$$\langle Tr(H^5)Tr(H^3)\rangle = 5\cdot 3\left(3N^4+4N^2\right).$$

Valencies $\langle 5,3|5,1,1,1\rangle$

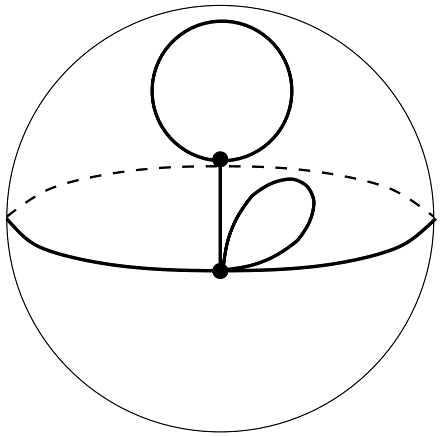


Figure 79: $S(53|5111)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(5111|53)$, see Figure 96 on the page 59. Belyi function is $\beta = \frac{1}{4}\,\frac{(z-4)^3 z^5}{6\,z^3-22\,z^2-12\,z-9}$.

Valencies $\langle 5,3|4,2,1,1\rangle$

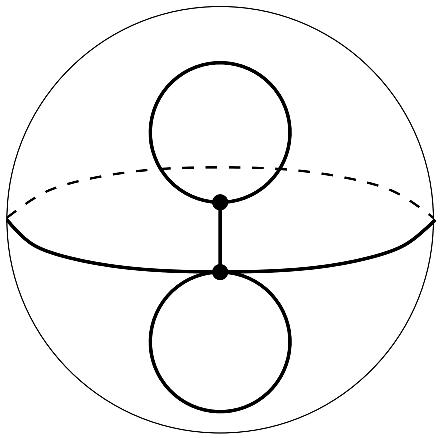


Figure 80: $S(53|4211)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(4211|53)$, see Figure 119 on the page 69. Belyi function is $\beta = \frac{1}{4}\,\frac{(-1+8\,z)^3}{(3+z)^2 z^4 (9\,z^2+42\,z-5)}$.

Valencies $\langle 5, 3|3, 2, 2, 1\rangle$

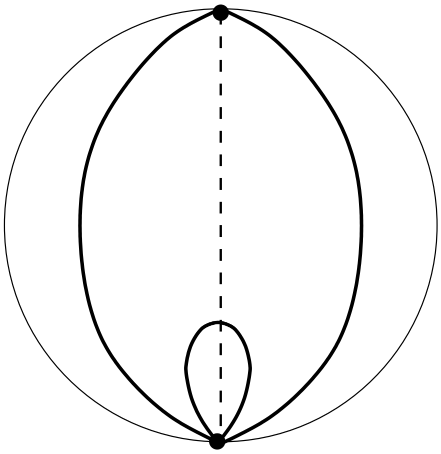


Figure 81: $S(53|3221)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(3221|53)$, see Figure 131 on the page 76. Belyi function is $\beta = 4\,\frac{(3\,z+4)^3 z^5}{(1+7\,z)(3\,z+1)^3(z^2+1)^2}$.

Valencies $\langle 5, 3|7, 1\rangle$

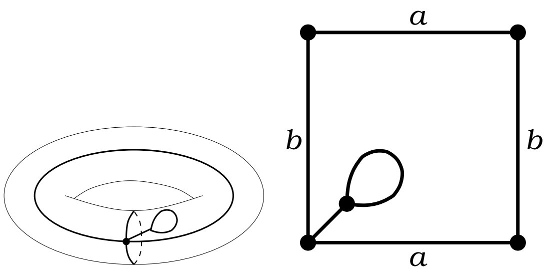


Figure 82: $T(53|71)A$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(71|53)A$, see Figure 58 on the page 41.

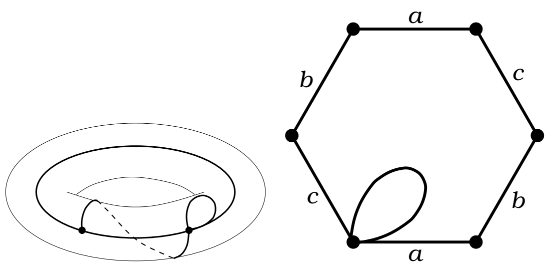


Figure 83: $T(53|71)B$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(71|53)B$, see Figure 59 on the page 41.

For the dessins $T(53|71)A$ and $T(53|71)B$ Belyi function has the form $\beta = \frac{(135\,\nu+1379)}{2730105000(64\,x-105+45\,\nu}(297675\,\nu\,yx^3 - 4456305\,yx^3 + 49546350\,x^4 - 201285\,\nu\,yx^2 + 25806879\,yx^2 - 151587135\,x^3 - 14586075\,\nu\,x^3 - 8511615\,\nu\,yx + 173009165\,yx + 24310125\,\nu\,x^2 - 1233745275\,x^2 - 54618075\,\nu\,y + 59676225\,y - 4281377625\,x + 1348801875\,\nu\,x + 972759375\,\nu - 25761054375)$ on the curve of the form $X : y^2 = \frac{225(675\,\nu+5033)}{45019072}(-420\,x^3 + 119\,x^2 + 405\,\nu\,x^2 - 21210\,x + 630\,\nu\,x - 176400 + 18900\,\nu)$.
Here $\nu^2 = 105$.
Now $n_0 = \frac{343(17983+1755\,\nu)}{15552000}\,\frac{(x+5)^3(x-3)^5}{64\,x-105+45\,\nu}$,
$n_1 = \frac{7(17983+1755\,\nu)}{62208000}\,\frac{\left(14\,x^4-420\,x^2+560\,x-60825+6075\,\nu\right)^2}{64\,x-105+45\,\nu}$.

Valencies $\langle 5,3|6,2\rangle$

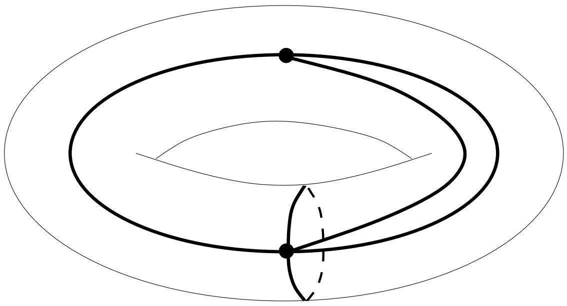
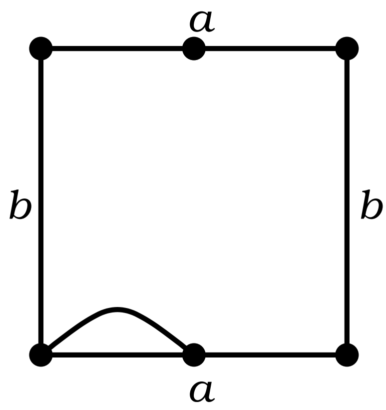



Figure 84: $T(53|62)$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(62|53)$, see Figure 68 on the page 46. Belyi function is $\beta = 1296\,(8\,yx^3 + 48\,yx^2 - 128\,y + 8\,x^5 + 65\,x^4 + 8\,x^3 - 512\,x^2 - 128\,x + 640)^{-1}$ on the curve $X : y^2 = \frac{1}{4}(x-1)(4x+5)(x^2+4x-20)$. $n_0 = 20736\,\frac{1}{(x+8)^2x^6}$, $n_1 = \frac{\left(x^4+8\,x^3-128\,x-16\right)^2}{(x+8)^2x^6}$.

Valencies $\langle 5,3|5,3\rangle$

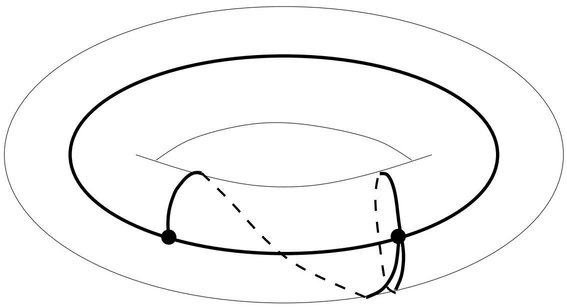
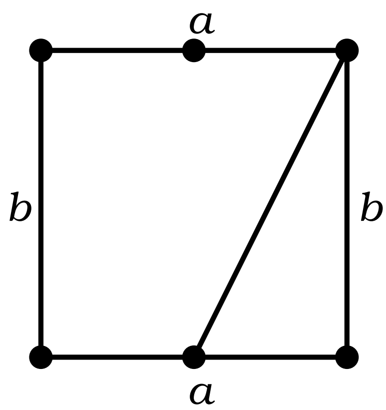



Figure 85: $T(53|53)$. There are no nontrivial automorphisms of this dessin. This dessin is selfdual. Belyi function is $\beta = -\frac{26}{25}\,x^2 + 1/2\,x^5 - \frac{24}{25}\,x - \frac{8}{25}\,y + \frac{59}{50}\,x^3 + \frac{25}{16}\,x^4 + \frac{8}{25}\,yx + yx^2 + 1/2\,yx^3 + \frac{8}{25}$ on the curve $X : y^2 = \frac{1}{4}\,(x-1)\,(13\,x^2 + 12\,x + 4\,x^3 - 4)$. $n_0 = \frac{1}{6400}\,(9\,x+16)^3\,x^5$, $n_1 = \frac{1}{6400}\,(27\,x^4 + 72\,x^3 + 32\,x^2 - 128\,x - 48)^2$.

7.6 Valencies $\langle 5,2,1|*\rangle$

$$\langle Tr(H^5)Tr(H^2)Tr(H)\rangle = 5\cdot 2\left(6N^3+3N\right).$$

Valencies $\langle 5,2,1|6,1,1\rangle$

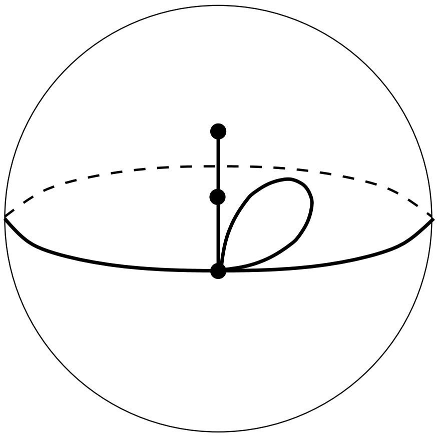


Figure 86: $S(521|611)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(611|521)$, see Figure 72 on the page 48. Belyi function is $\beta = -1/4\,\frac{(5\,z-14)^2(z-4)z^5}{70\,z^2+24\,z+9}$.

Valencies $\langle 5,2,1|5,2,1\rangle$

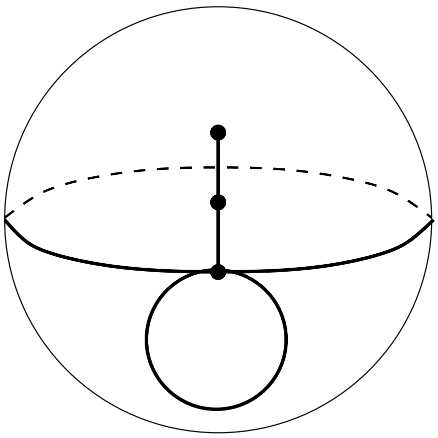


Figure 87: $S(521|521)A$. There are no nontrivial automorphisms of this dessin. This dessin is selfdual. Belyi function is $\beta = \{16\,\frac{\left(391+550\,\nu+455\,\nu^2\right)(z+2\,\nu)(z+1)^2z^5}{(16\,z-\nu+7\,\nu^2-4)(-8\,z+3\,\nu+3\,\nu^2-4)^2}, 7\,\nu^3+2\,\nu^2-\nu-4=0, \nu>0\}$.

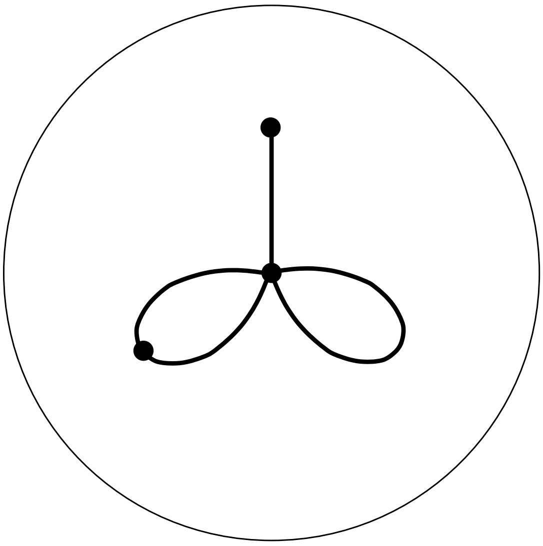


Figure 88: $S(521|521)B_+$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(521|521)B_+$, see Figure 88 on the page 56. Belyi function is $\beta = \{16\,\frac{\left(391+550\,\nu+455\,\nu^2\right)(z+2\,\nu)(z+1)^2 z^5}{(16\,z-\nu+7\,\nu^2-4)(-8\,z+3\,\nu+3\,\nu^2-4)^2}, 7\,\nu^3 + 2\,\nu^2 - \nu - 4 = 0, Im\,\nu < 0\}$.

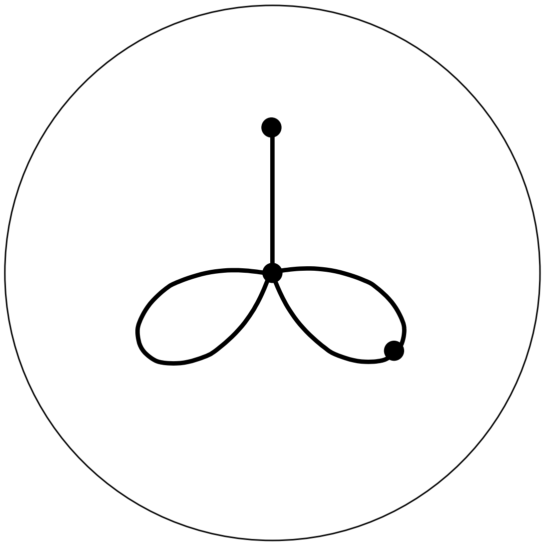


Figure 89: $S(521|521)B_-$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(521|521)B_-$, see Figure 89 on the page 56. Belyi function is $\beta = \{16\,\frac{\left(391+550\,\nu+455\,\nu^2\right)(z+2\,\nu)(z+1)^2 z^5}{(16\,z-\nu+7\,\nu^2-4)(-8\,z+3\,\nu+3\,\nu^2-4)^2}, 7\,\nu^3 + 2\,\nu^2 - \nu - 4 = 0, Im\,\nu > 0\}$.

Valencies $\langle 5, 2, 1 | 4, 3, 1 \rangle$

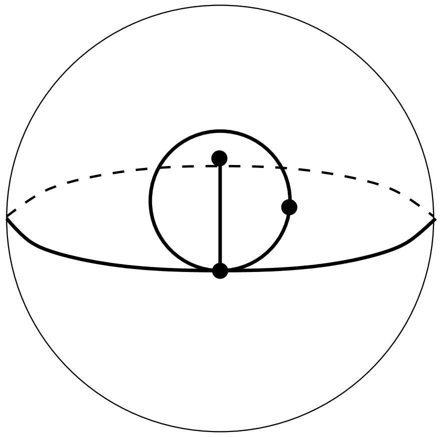


Figure 90: $S(521|431)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(431|521)$, see Figure 106 on the page 63. Belyi function is $\beta = -16\,\frac{z^5(z+3)(6\,z-7)^2}{(15\,z-4)(7\,z-4)^3}$.

Valencies $\langle 5, 2, 1 | 3, 3, 2 \rangle$

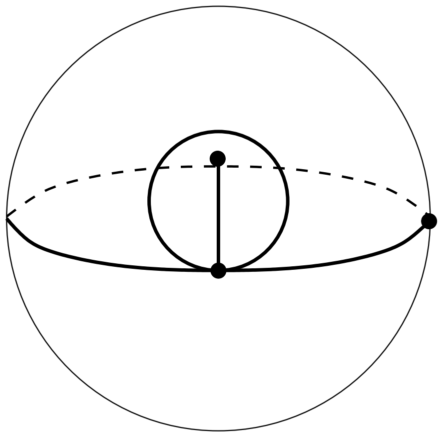


Figure 91: $S(521|332)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(332|521)$, see Figure 123 on the page 71. Belyi function is $\beta = \frac{27}{4}\,\frac{z^5(7\,z+2)^2(11\,z-4)}{(6\,z^2-1)^3(4\,z+1)^2}$.

Valencies $\langle 5, 2, 1 | 8 \rangle$

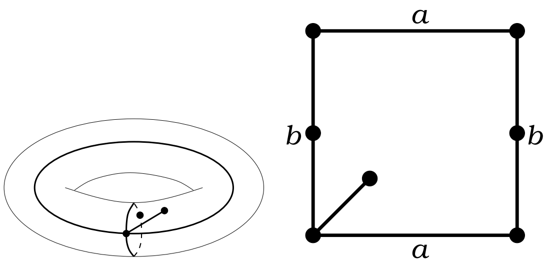


Figure 92: $T(521|8)A_+$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(8|521)A_+$, see Figure 34 on the page 28.

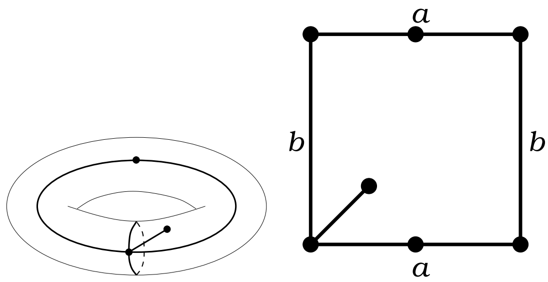


Figure 93: $T(521|8)A_-$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(8|521)A_-$, see Figure 35 on the page 28.

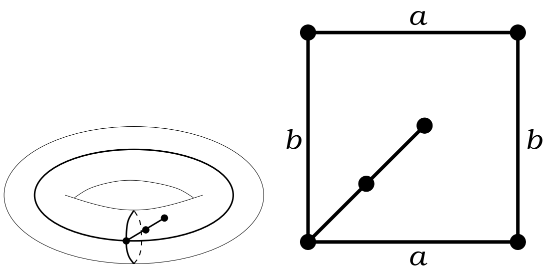


Figure 94: $T(521|8)B$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(8|521)B$, see Figure 36 on the page 28.

For the dessins $T(521|8)A_+$, $T(521|8)A_-$ $T(521|8)B$ Belyi function is $\beta = \frac{1}{735306250}\,(552\,\nu^2 - 617\,\nu + 68)\,(42875\,x^4 + 1756160\,\nu\,yx^2 - 860160\,\nu^2yx^2 - 4543840\,yx^2 + 3959200\,\nu\,x^3 - 10346175\,x^3 - 1926400\,\nu^2x^3 + 31782912\,\nu^2yx - 63438592\,\nu\,yx + 168996968\,yx + 18916352\,\nu^2x^2 - 37781632\,\nu\,x^2 + 100206428\,x^2 - 257512128\,y - 48381952\,\nu^2y + 96684032\,\nu\,y - 62101504\,\nu^2x - 330259656\,x + 123960064\,\nu\,x + 48381952\,\nu^2 - 96684032\,\nu + 257512128)$ on the curve of the form $X : y^2 = -\frac{\left(17\,\nu^2+8-42\,\nu\right)}{960400}\,(19600\,x^2 + 55552\,\nu\,x - 18432\,\nu^2x - 88408\,x + 338963 - 130592\,\nu + 65792\,\nu^2)(x-1)$.

Here $256\nu^3 - 544\nu^2 + 1427\nu - 172 = 0$, and the real root corresponds to the case B. $n_0 = -\frac{\left(-1974439\,\nu+8411384\,\nu^2+115196\right)}{3460321800250000000}\,(1225\,x + 90376 - 34944\,\nu + 16384\,\nu^2)(1225\,x + 56519 - 19936\,\nu + 10496\,\nu^2)^2x^5$, $n_1 = -\frac{\left(-1974439\,\nu+8411384\,\nu^2+115196\right)}{185122979184640000000}\,(313600\,x^4 + 26036992\,x^3 - 9576448\,\nu\,x^3 + 4784128\,\nu^2x^3 + 307426304\,x^2 - 114917376\,\nu\,x^2 + 57409536\,\nu^2x^2 - 1834522624\,x + 689504256\,\nu\,x - 344457216\,\nu^2x + 6757769763 - 2539197472\,\nu + 1270132992\,\nu^2)^2$.

7.7 Valencies $\langle 5,1,1,1|*\rangle$

$$\langle Tr(H^5)Tr^3(H)\rangle = 5\cdot 3!\left(2N^2\right).$$

Valencies $\langle 5,1,1,1|7,1\rangle$

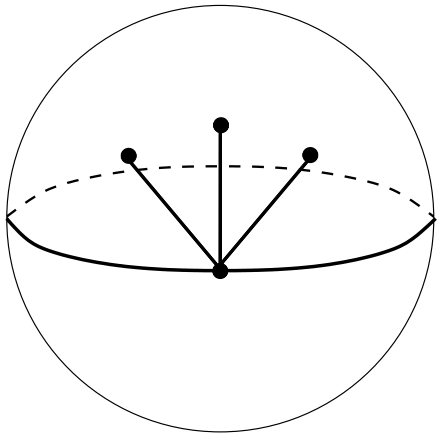


Figure 95: $S(5111|71)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(71|5111)$, see Figure 46 on the page 35. Belyi function is $\beta = \frac{1}{16384}\frac{z^5(1225z^3-3216z^2+2912z-896)}{(z-1)^7}$.

Valencies $\langle 5,1,1,1|5,3\rangle$

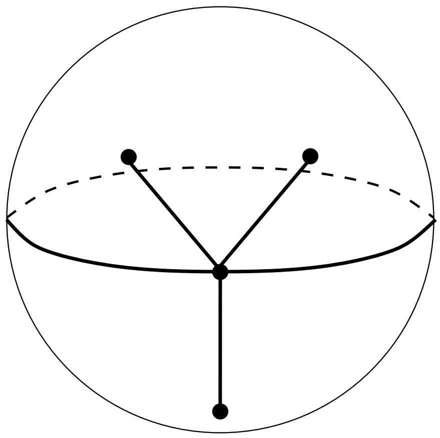


Figure 96: $S(5111|53)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(53|5111)$, see Figure 79 on the page 51. Belyi function is $\beta = 4\,\frac{z^5\left(9\,z^3+12\,z^2+22\,z-6\right)}{(4\,z-1)^3}$.

7.8 Valencies $\langle 4,4|*\rangle$

$$\langle Tr^2(H^4)\rangle = 4^2\cdot 2!\left(\frac{9}{8}N^4+\frac{15}{8}N^2\right) =$$
$$= 4^2\cdot 2!\left(\left(2\cdot\frac{1}{2}+\frac{1}{8}\right)N^4+\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)N^2\right).$$

Valencies $\langle 4,4|4,2,1,1\rangle$

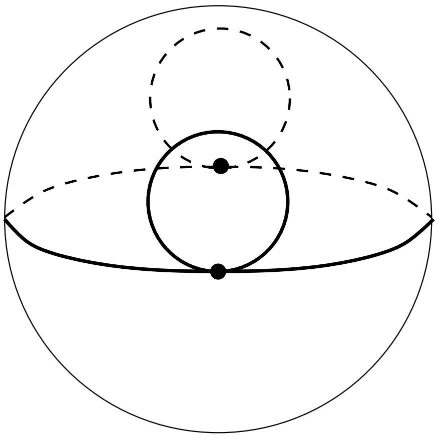


Figure 97: $S(44|4211)$. The automorphism group is Z_2. Dual dessin $S(4211|44)$, see Figure 120 on the page 69. Belyi function is $\beta = 1/4\,\frac{\left(z^2+1\right)^4}{z^4(2\,z^2+1)}$.

Valencies $\langle 4,4|3,3,1,1\rangle$

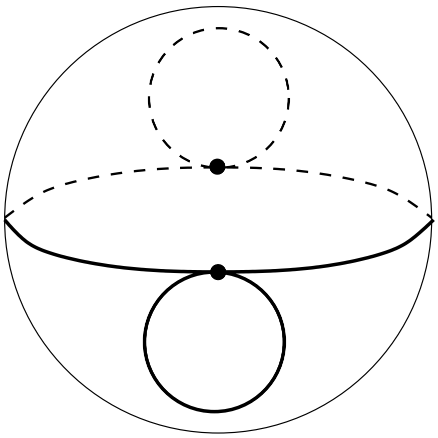


Figure 98: $S(44|3311)$. The automorphism group is Z_2. Dual dessin $S(3311|44)$, see Figure 128 on the page 74. Belyi function is $\beta = -432\,\frac{z^4}{(2\,z^2+10\,z-1)(2\,z^2+2\,z-1)^3}$.

Valencies $\langle 4, 4|2, 2, 2, 2\rangle$

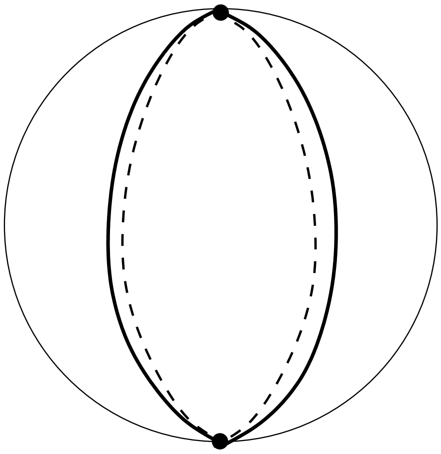


Figure 99: $S(44|2222)$. The automorphism group is $Z_4 \oplus Z_2$. Dual dessin $S(2222|44)$, see Figure 133 on the page 78. Belyi function is $\beta = 4\frac{z^4}{(z^4+1)^2}$.

Valencies $\langle 4, 4|7, 1\rangle$

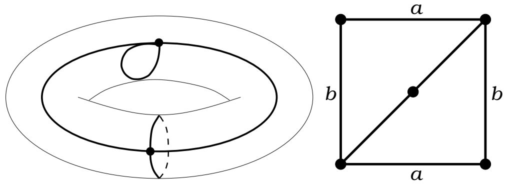


Figure 100: $T(44|71)$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(71|44)$, see Figure 60 on the page 42. Belyi function is $\beta = -\frac{1}{512x}(-343\,yx^3 + 2401\,x^4 - 931\,yx^2 + 9604\,x^3 - 581\,yx + 11662\,x^2 - y + 3860\,x - 7)$ on the curve $X : y^2 = (4\,x+7)\,(7\,x^2 + 18\,x + 7)$. $n_0 = -\frac{343}{65536}\,\frac{\left(7\,x^2+14\,x+3\right)^4}{x}$, $n_1 = -\frac{7}{65536}\,\frac{\left(343\,x^4+1372\,x^3+1666\,x^2+588\,x-65\right)^2}{x}$.

Valencies $\langle 4, 4|6, 2\rangle$

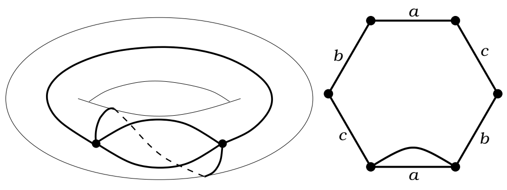


Figure 101: $T(44|62)$. The automorphism group is Z_2. Dual dessin $T(62|44)$, see Figure 69 on the page 46. Belyi function is $(X : y^2 = (x-1)\,(3\,x^2 + 8\,x + 16)\,, \beta = \frac{27}{256}\,\frac{x^4}{x-1})$.

Valencies $\langle 4, 4|4, 4\rangle$

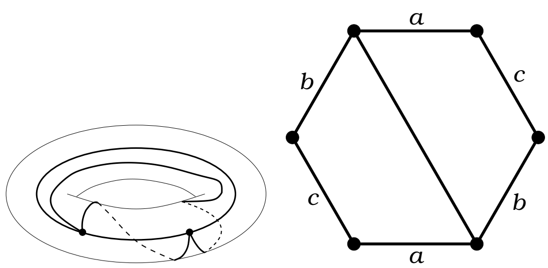


Figure 102: $T(44|44)\_8$. The automorphism group is $Z_4 \oplus Z_2$. This dessin is selfdual. Belyi function is $(X : y^2 = x^4 - 1, \beta = x^4)$.

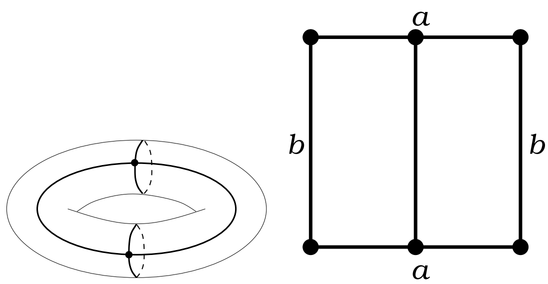


Figure 103: $T(44|44)\_4$. The automorphism group is $Z_2 \oplus Z_2$. This dessin is selfdual. Belyi function is $(X : y^2 = (x^2-1)(x^2-2), \beta = (x^2-1)^2)$.

7.9 Valencies $\langle 4,3,1|*\rangle$

$$\langle Tr(H^4)Tr(H^3)Tr(H)\rangle = 4\cdot 3\left(6N^3+2N\right).$$

Valencies $\langle 4,3,1|6,1,1\rangle$

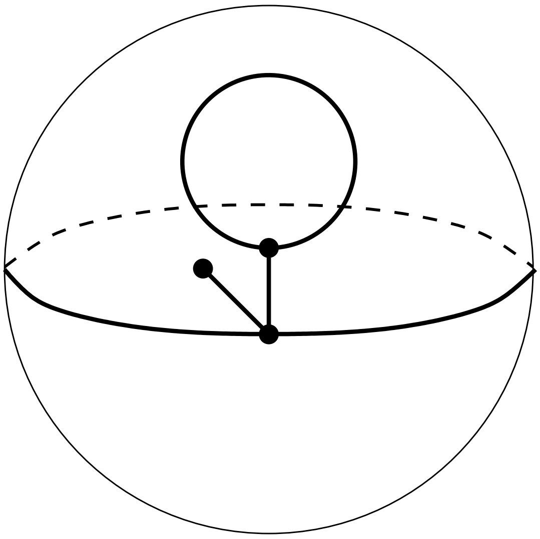


Figure 104: $S(431|611)_+$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(611|431)_+$, see Figure 73 on the page 48. Belyi function is $\beta = -\frac{49}{4}\,\frac{\left(87\,i\sqrt{3}+211\right)(z-1)^3\left(-7\,z-3+2\,i\sqrt{3}\right)z^4}{686\,z^2-672\,z+56\,i\sqrt{3}z-57\,i\sqrt{3}-51}$.

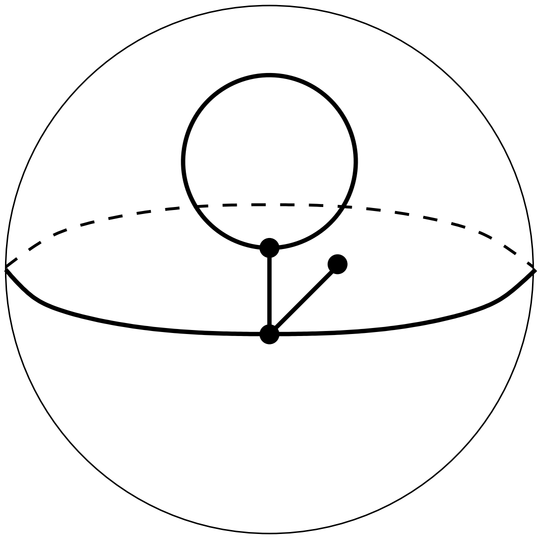


Figure 105: $S(431|611)_-$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(611|431)_-$, see Figure 74 on the page 48. Belyi function is $\beta = \frac{49}{4}\,\frac{\left(87\,i\sqrt{3}-211\right)(z-1)^3\left(7\,z+3+2\,i\sqrt{3}\right)z^4}{-686\,z^2+56\,i\sqrt{3}z+672\,z+51-57\,i\sqrt{3}}$.

Valencies $\langle 4,3,1|5,2,1\rangle$

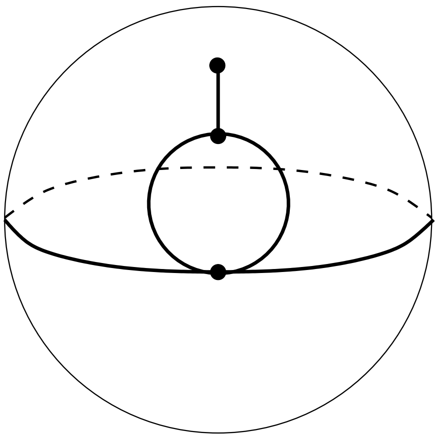


Figure 106: $S(431|521)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(521|431)$, see Figure 90 on the page 57. Belyi function is $\beta = -1/16\,\frac{(4\,z-7)^3(4\,z-15)z^4}{(3\,z+1)(7\,z-6)^2}$.

Valencies $\langle 4,3,1|4,3,1\rangle$

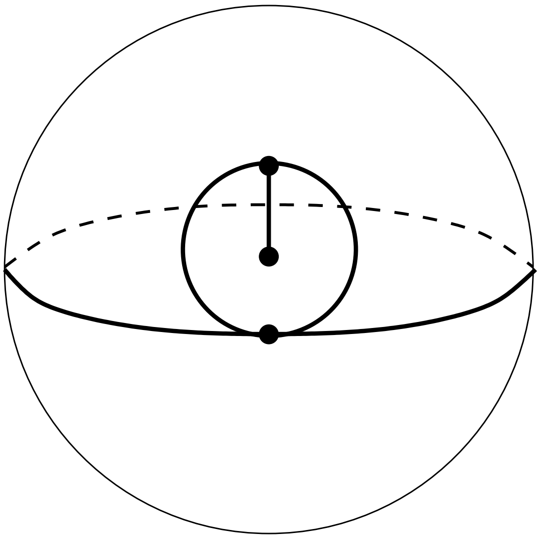


Figure 107: $S(431|431)A$. There are no nontrivial automorphisms of this dessin. This dessin is selfdual. Belyi function is $\beta = \frac{1}{3294172}\frac{\left(835+872\sqrt{2}\right)\left(-z+8+5\sqrt{2}\right)\left(-z-8+9\sqrt{2}\right)^3 z^4}{\left(-z-11+8\sqrt{2}\right)(z-1)^3}$.

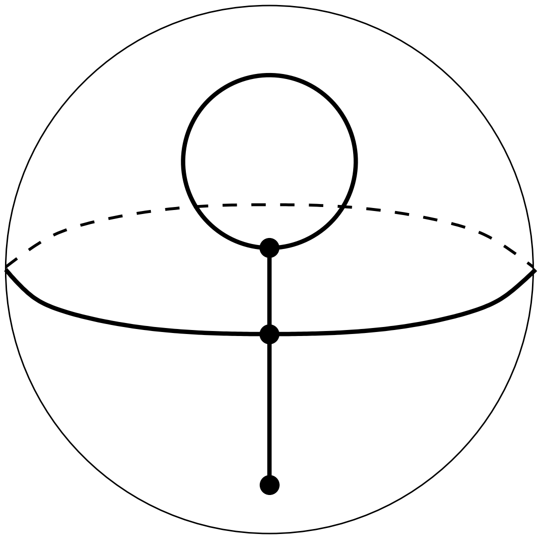


Figure 108: $S(431|431)B$. There are no nontrivial automorphisms of this dessin. This dessin is selfdual. Belyi function is $\beta = \frac{1}{3294172}\frac{\left(-835+872\sqrt{2}\right)\left(z-8+5\sqrt{2}\right)\left(z+8+9\sqrt{2}\right)^3 z^4}{\left(z+11+8\sqrt{2}\right)(z-1)^3}$.

Valencies $\langle 4,3,1|4,2,2\rangle$

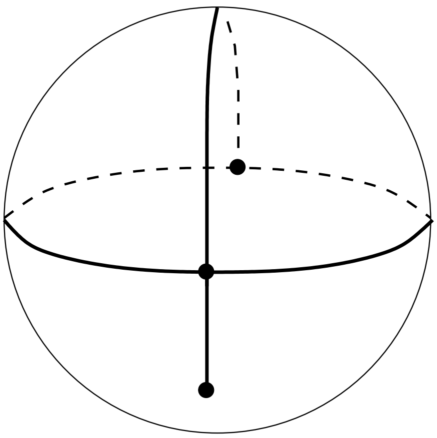


Figure 109: $S(431|422)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(422|431)$, see Figure 112 on the page 66. Belyi function is $\beta = -4\,\frac{(z+1)^4 z^3 (z+4)}{(6\,z^2+4\,z+1)^2}$.

Valencies $\langle 4,3,1|8\rangle$

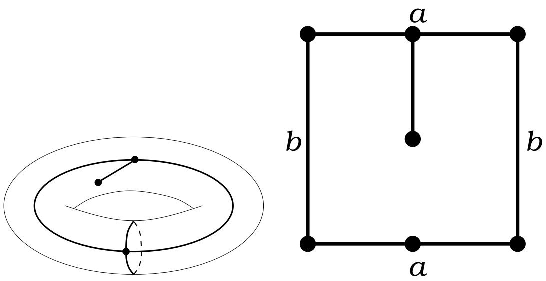


Figure 110: $T(431|8)A$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(8|431)A$, see Figure 37 on the page 30. Belyi function is $\beta = -\frac{256}{85766121}(5488\,x^4 + 14112\,yx^2 - 26264\,x^3 + 37548\,yx - 202741\,x^2 + 3240\,y - 73368\,x - 3240)$ on the curve $X : y^2 = \frac{1}{81}(1 - x)(448\,x^2 + 1872\,x + 81)$. $n_0 = \frac{65536}{62523502209}\,(4\,x+45)\,(4\,x+21)^3\,x^4$, $n_1 = \frac{(4096\,x^4+55296\,x^3+158976\,x^2+55296\,x-247617)^2}{62523502209}$.

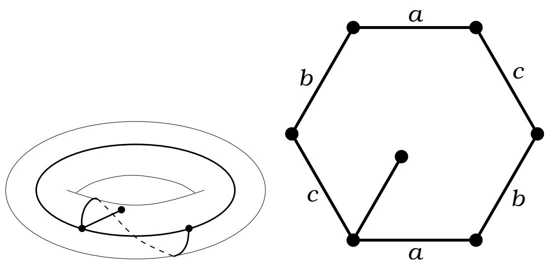


Figure 111: $T(431|8)B$. There are no nontrivial automorphisms of this dessin. Dual dessin $T(8|431)B$, see Figure 38 on the page 30. Belyi function is $\beta = -9/4\,x^4 + 3\,yx^2 - 8\,x^3 + 8/3\,yx - \frac{77}{9}\,x^2 + 2/3\,y - \frac{40}{9}\,x - \frac{8}{9}$ on the curve $X : y^2 = \frac{4}{9}\,(x+1)\,(9\,x^2+4\,x+4)$. $n_0 = \frac{3}{16}\,(x-2)\,(3x+2)^3x^4$, $n_1 = \frac{1}{144}\,(27\,x^4 - 36\,x^2 - 32\,x - 20)^2$.

7.10 Valencies $\langle 4,2,2|*\rangle$

$$\langle Tr(H^4)Tr^2(H^2)\rangle = 4\cdot 2^2\cdot 2!\left(\frac{3}{2}N^3+\frac{3}{4}N\right) =$$
$$= 4\cdot 2^2\cdot 2!\left(\left(1+\frac{1}{2}\right)N^3+\left(\frac{1}{2}+\frac{1}{4}\right)N\right).$$

Valencies $\langle 4,2,2|4,3,1\rangle$

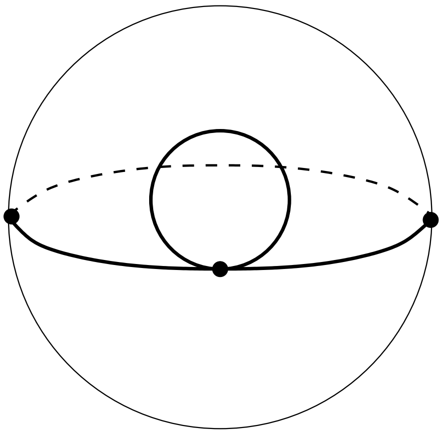


Figure 112: $S(422|431)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(431|422)$, see Figure 109 on the page 64. Belyi function is $\beta = -1/4\,\frac{\left(z^2+4\,z+6\right)^2 z^4}{(z+1)^4(4\,z+1)}$.

Valencies $\langle 4,2,2|4,2,2\rangle$

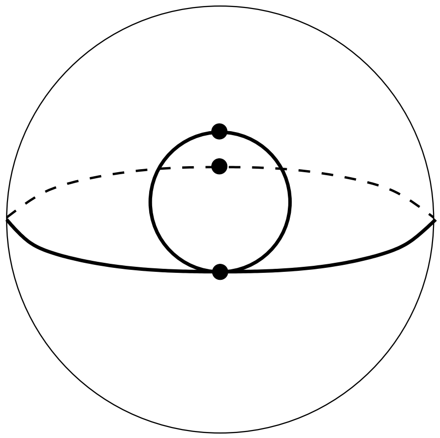


Figure 113: $S(422|422)$. The automorphism group is Z_2. This dessin is selfdual. Belyi function is $\beta = -1/4\,\frac{\left(z^2-2\right)^2 z^4}{(z-1)^2(z+1)^2}$.

Valencies $\langle 4,2,2|8\rangle$

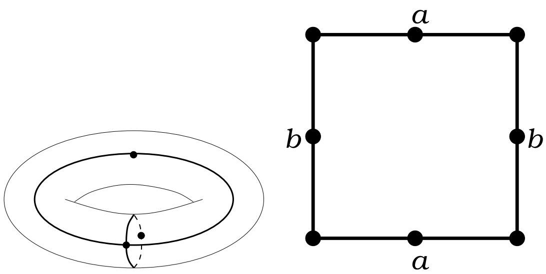


Figure 114: $T(422|8)\_4$. The automorphism group is Z_4. This dessin is self-dual. Belyi function is $(X : y^2 = x(x-1)(x+1), \beta = -4\,x^2(x-1)(x+1))$.

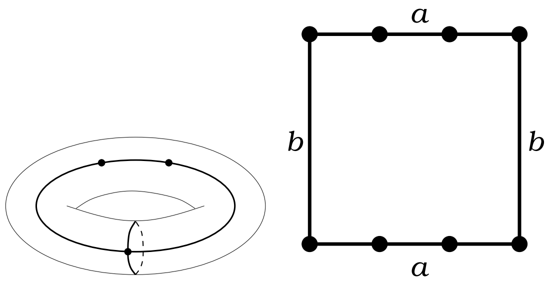


Figure 115: $T(422|8)\_2$. The automorphism group is Z_2. This dessin is self-dual. Belyi function is $(X : y^2 = (x^2 - 2\,x - 1)\,x, \beta = (x-2)^2\,x^2)$.

7.11 Valencies $\langle 4,2,1,1|*\rangle$

$$\langle Tr(H^4)Tr(H^2)Tr^2(H)\rangle = 4\cdot 2\cdot 2!\left(\frac{9}{2}N^2\right) =$$
$$= 4\cdot 2\cdot 2!\left(\left(4+\frac{1}{2}\right)N^2\right).$$

Valencies $\langle 4,2,1,1|7,1\rangle$

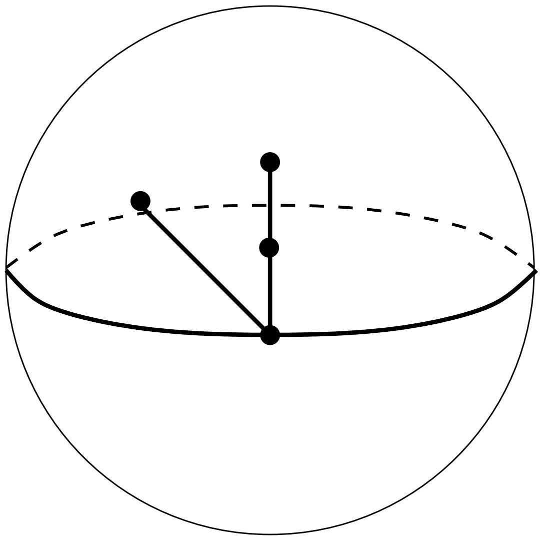


Figure 116: $S(4211|71)_+$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(71|4211)_+$, see Figure 47 on the page 35. Belyi function is $\beta = \frac{(49z^2-90z-2i\sqrt{7}z+42+2i\sqrt{7})(128z+5i\sqrt{7}-119)^2z^4}{512(16377+181i\sqrt{7})(z-1)^7}$

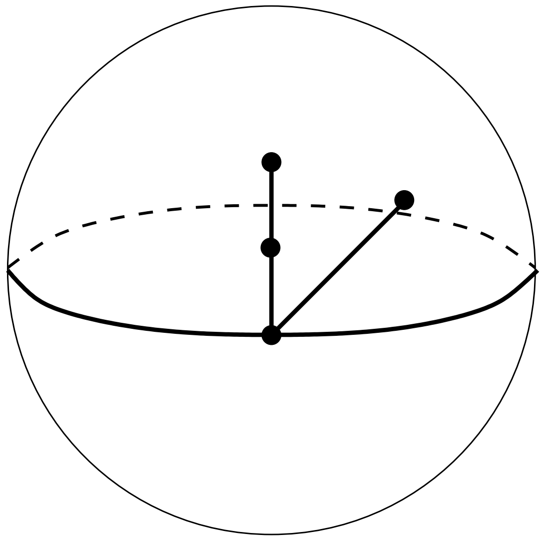


Figure 117: $S(4211|71)_-$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(71|4211)_-$, see Figure 48 on the page 35. Belyi function is $\beta = \frac{(49z^2-90z+2i\sqrt{7}z+42-2i\sqrt{7})(128z-5i\sqrt{7}-119)^2z^4}{512(16377-181i\sqrt{7})(z-1)^7}$

Valencies $\langle 4,2,1,1|6,2\rangle$

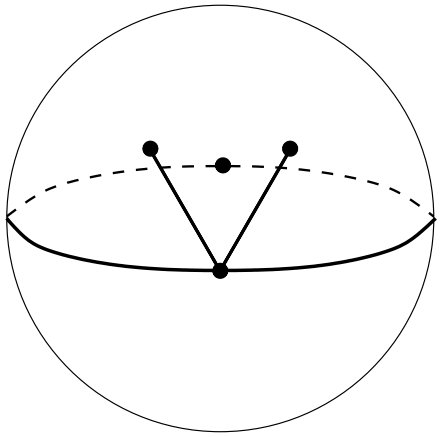


Figure 118: $S(4211|62)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(62|4211)$, see Figure 61 on the page 43. Belyi function is $\beta = -108\,\frac{z^4\left(1+2\,z+3\,z^2\right)(-1+3\,z)^2}{(-1+4\,z)^2}$

Valencies $\langle 4,2,1,1|5,3\rangle$

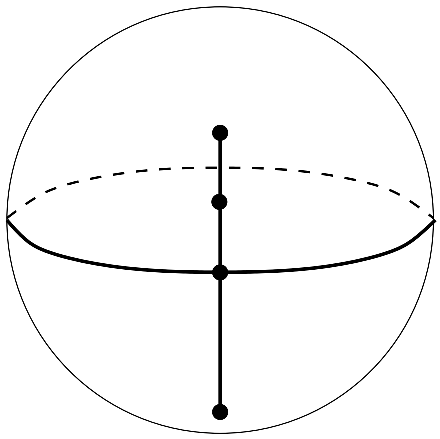


Figure 119: $S(4211|53)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(53|4211)$, see Figure 80 on the page 51. Belyi function is $\beta = 4\,\frac{(3\,z+1)^2\left(-9-42\,z+5\,z^2\right)}{(z-8)^3 z^5}$.

Valencies $\langle 4,2,1,1|4,4\rangle$

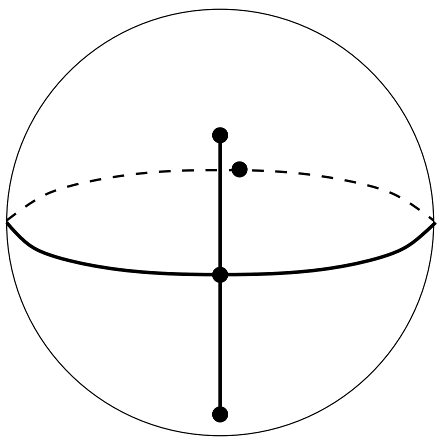


Figure 120: $S(4211|44)$. The automorphism group is Z_2. Dual dessin $S(44|4211)$, see Figure 97 on the page 60. Belyi function is $\beta = 4\,\frac{z^2\left(z^2+2\right)}{(z^2+1)^4}$.

7.12 Valencies $\langle 4,1,1,1,1|8\rangle$

$$\langle Tr(H^4)Tr^4(H)\rangle = 4\cdot 4!\left(\frac{1}{4}N\right).$$

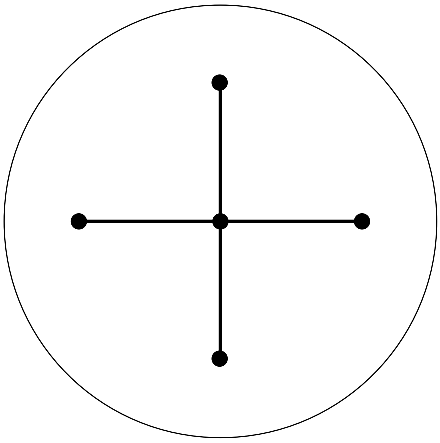


Figure 121: $S(41111|8)$. The automorphism group is Z_4. Dual dessin $S(8|41111)$, see Figure 28 on the page 25. Belyi function is $\beta = -4z^4(z-1)(z+1)(z^2+1)$.

7.13 Valencies $\langle 3,3,2|*\rangle$

$$\langle Tr^2(H^3)Tr(H^2)\rangle = 3^2\cdot 2\cdot 2!\left(2N^3+\frac{1}{2}N\right) =$$
$$= 3^2\cdot 2\cdot 2!\left(\left(1+2\cdot\frac{1}{2}\right)N^3+\frac{1}{2}N\right).$$

Valencies $\langle 3,3,2|6,1,1\rangle$

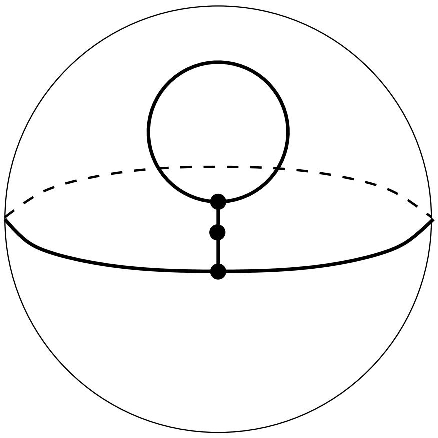


Figure 122: $S(332|611)$. The automorphism group is Z_2. Dual dessin $S(611|332)$, see Figure 75 on the page 49. Belyi function is $\beta = -1/4\,\frac{\left(z^2+4\right)^3 z^2}{2\,z^2+9}$.

Valencies $\langle 3,3,2|5,2,1\rangle$

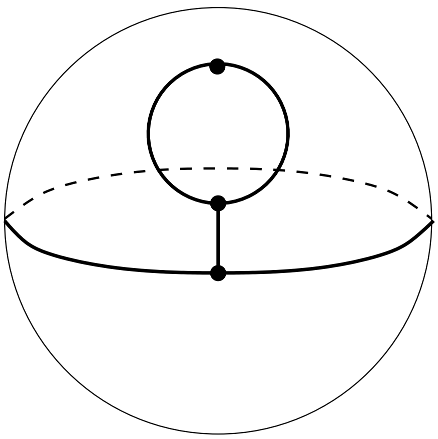


Figure 123: $S(332|521)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(521|332)$, see Figure 91 on the page 57. Belyi function is $\beta = \frac{4}{27}\,\frac{(z+4)^2\left(z^2-6\right)^3}{(2\,z+7)^2(4\,z-11)}$.

Valencies $\langle 3,3,2|3,3,2\rangle$

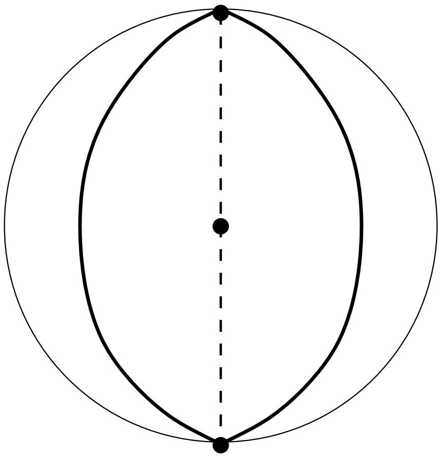


Figure 124: $S(332|332)$. Valencies $(3,3,2|3,3,2)$. The automorphism group is Z_2. This dessin is selfdual. Belyi function is $\beta = 64\,\frac{z^2\left(z^2+1\right)^3}{\left(8\,z^2-1\right)^3}$.

Valencies $\langle 3,3,2|8\rangle$

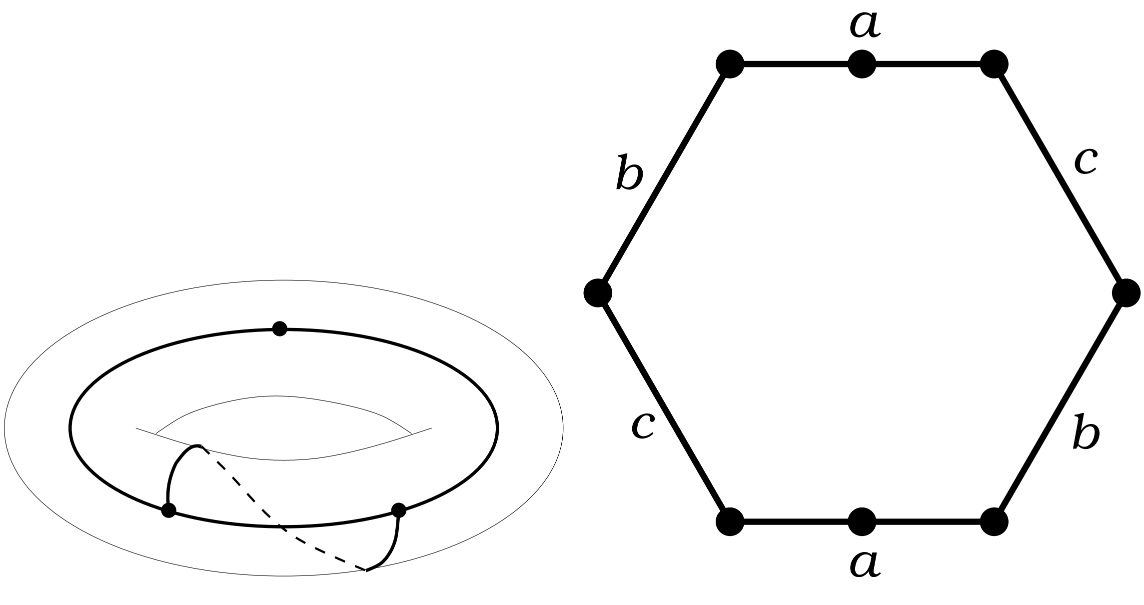


Figure 125: $T(332|8)$. The automorphism group is Z_2. Dual dessin $T(8|332)$, see Figure 41 on the page 32. Belyi function is $(X : y^2 = (x-2)\,(4\,x^2+4\,x+3)\,, \beta = -\frac{16}{27}\,(x-2)\,x^3)$.

7.14 Valencies $\langle 3,3,1,1|*\rangle$

$$\langle Tr^2(H^3)Tr^2(H)\rangle = 3^2\cdot 2!\cdot 2!\left(2N^2\right) =$$
$$= 3^2\cdot 2!\cdot 2!\left(\left(1+2\cdot\frac{1}{2}\right)N^2\right).$$

Valencies $\langle 3,3,1,1|7,1\rangle$

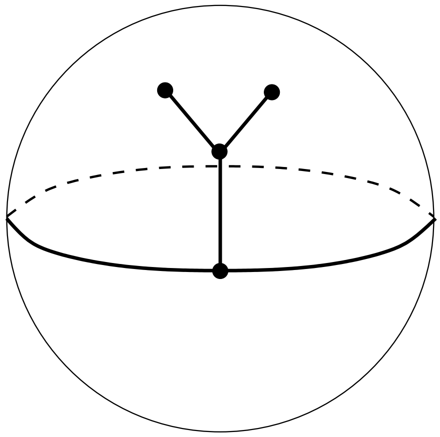


Figure 126: $S(3311|71)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(71|3311)$, see Figure 49 on the page 36. Belyi function is $\beta = -\frac{1}{1728}\frac{\left(1+z^2-5\,z\right)^3\left(49\,z^2-13\,z+1\right)}{z^7}$

Valencies $\langle 3,3,1,1|6,2\rangle$

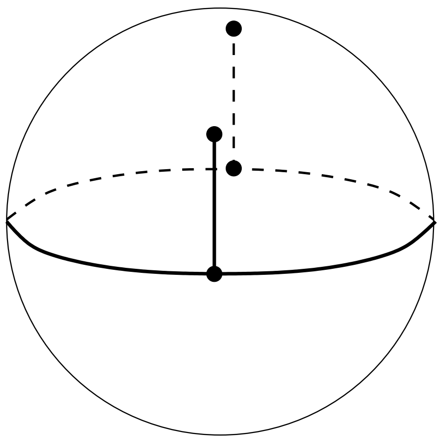


Figure 127: $S(3311|62)$. The automorphism group is Z_2. Dual dessin $S(62|3311)$, see Figure 62 on the page 43. Belyi function is $\beta = -\frac{1}{64}\frac{(z-3)(3+z)(z-1)^3(z+1)^3}{z^2}$

Valencies $\langle 3,3,1,1|4,4\rangle$

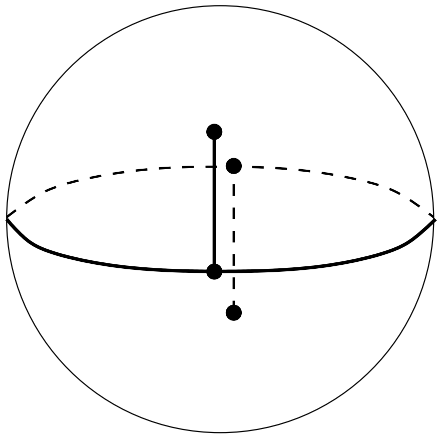


Figure 128: $S(3311|44)$. The automorphism group is Z_2. Dual dessin $S(44|3311)$, see Figure 98 on the page 60. Belyi function is $\beta = -\frac{1}{432}\frac{\left(z^2-10\,z-2\right)\left(z^2-2\,z-2\right)^3}{z^4}$.

7.15 Valencies $\langle 3,2,2,1|*\rangle$

$$\langle Tr(H^3)Tr^2(H^2)Tr(H)\rangle = 3\cdot 2^2\cdot 2!\left(3N^2\right).$$

Valencies $\langle 3,2,2,1|7,1\rangle$

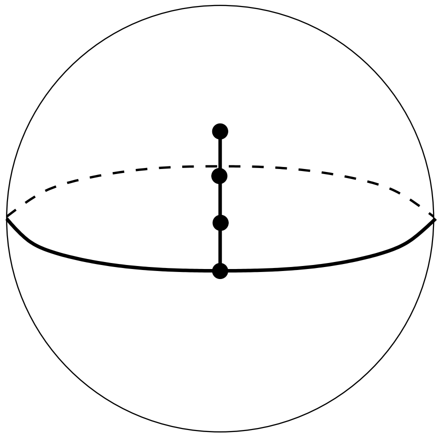


Figure 129: $S(3221|71)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(71|3221)$, see Figure 50 on the page 36. Belyi function is $\beta = 256\,\frac{z^3(z+1)\left(7+28\,z+24\,z^2\right)^2}{48\,z-1}$.

Valencies $\langle 3,2,2,1|6,2\rangle$

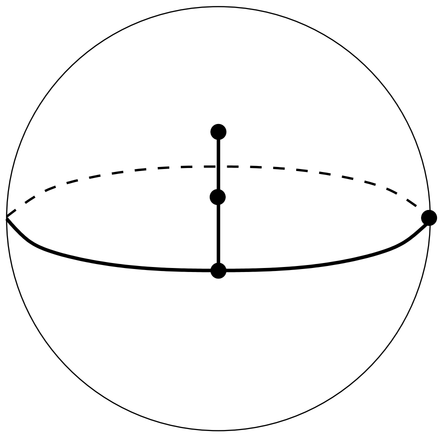


Figure 130: $S(3221|62)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(62|3221)$, see Figure 63 on the page 44. Belyi function is $\beta = -\frac{1}{4}\,\frac{z^3(4+z)\left(z^2+2\,z-2\right)^2}{(-1+2\,z)^2}$.

Valencies $\langle 3,2,2,1|5,3\rangle$

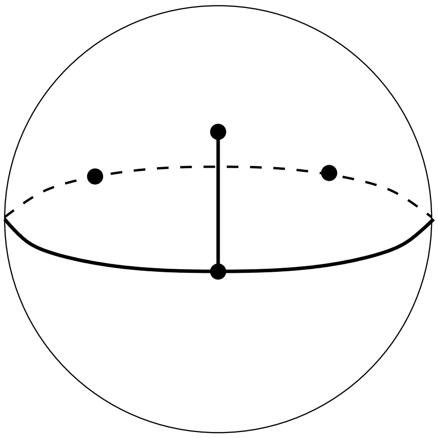


Figure 131: $S(3221|53)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(53|3221)$, see Figure 81 on the page 52. Belyi function is $\beta = \frac{1}{4}\,\frac{(z+7)(3+z)^3\left(z^2+1\right)^2}{(3+4\,z)^3}$.

7.16 Valencies $\langle 3, 2, 1, 1, 1|8\rangle$

$$\langle Tr(H^3)Tr(H^2)Tr^3(H)\rangle = 3 \cdot 2 \cdot 3!\,(N)\,.$$

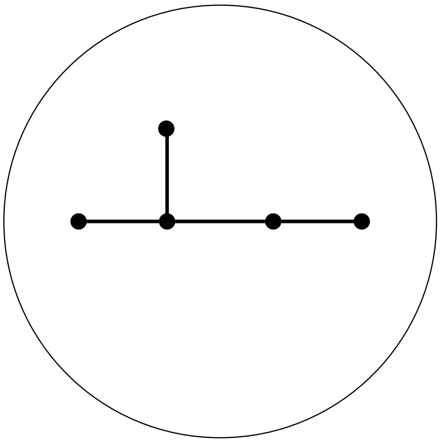


Figure 132: $S(32111|8)$. There are no nontrivial automorphisms of this dessin. Dual dessin $S(8|32111)$, see Figure 29 on the page 26. Belyi function is $\beta = -\frac{1024}{729}z^3(z-1)(16z^2+8z+3)(4z-3)^2$

7.17 Valencies $\langle 2,2,2,2|4,4\rangle$

$$\langle Tr^4(H^2)\rangle = 2^4 \cdot 4! \left(\frac{1}{8}N^2\right).$$

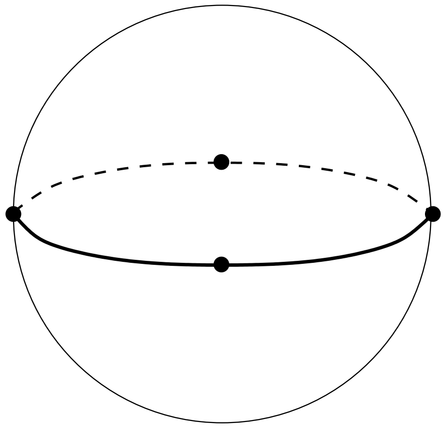


Figure 133: $S(2222|44)$. The automorphism group is $Z_4 \oplus Z_2$. Dual dessin $S(44|2222)$, see Figure 99 on the page 61. Belyi function is $\beta = \frac{(z^4+1)^2}{4z^4}$.

7.18 Valencies $\langle 2,2,2,1,1|8\rangle$

$$\langle Tr^3(H^2)Tr^2(H)\rangle = 2^3\cdot 3!\cdot 2!\left(\frac{1}{2}N\right).$$

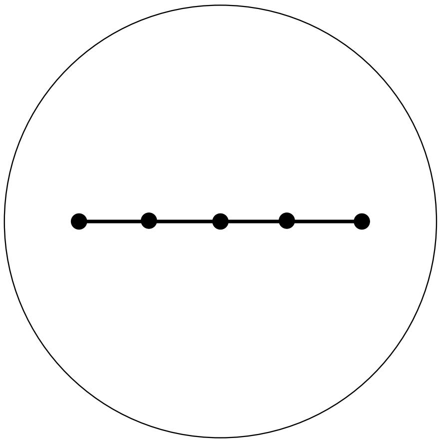


Figure 134: $S(22211|8)$. The automorphism group is Z_2. Dual dessin $S(8|22211)$, see Figure 30 on the page 26. Belyi function is $\beta = -4z^2(z^2-2)(z-1)^2(z+1)^2$

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