Pullbacks of Eisenstein series, Hecke-Jacobi theory and automorphic L-functions

By Bernhard E. Heim

Abstract

In this paper a new Rankin-Selberg integral is presented, which is a Petersson scalar product of a diagonal-restricted Eisenstein series against an elliptic modular form and two Siegel modular forms of degree n. This generalizes Paul Garrett's work on the Rankin triple product L-function. Surprisingly the case $n = 2$ leads, as $n = 1$, to an Euler product. We prove, by choosing one of the Siegel forms as a Saito-Kurokawa lift, that our approach gives a new integral representation of a degree eight L-function for $GSp_2 \times GL_2$, the twisted spinor L-function. [1]

This is a extended version of a talk given by the author at the NSF-CBMS conference: *Euler Products and Eisenstein Series* held at the Texas Christian University, Fort Worth, Texas, May 20-24, 1996 in honor of Professor Goro Shimura.

Introduction

The purpose of this paper is to present new insights into the connection between pullbacks of Eisenstein series and automorphic L-functions. Surprisingly, special properties of a certain Hecke-Jacobi algebra will play a decisive role. The paper has its origin in a question asked by S. Böcherer to the author.

Let us consider the following situation. Let $(n_i)_{i=1}^l$ be a partition of a natural number n and let $F_i \in S_{n_i}^k$ be Siegel cusp forms of degree n_i and weight k. Let further $E_n^k(z;s)$ be the analytic Eisenstein series of Siegel type, here z is in the Siegel half space $I\!H_n$ of degree n and $s \in \mathbb{C}$ with $2\,\mathrm{Re}(s) + k > n+1$. Let us further denote by $\langle\, ;\, \rangle$ the Petersson scalar product and let us assume that all the F_i's are Hecke eigenforms. Then P. Garrett [Ga84], S. Böcherer [Boe85b] and I. Piatetski-Shapiro and S. Rallis [P-R84], [G-P-R87] have shown, that the iterated scalar product

$$\Phi^n(F_1, \ldots, F_l; s) := \left\langle \ldots \left\langle E_n^k \left(\begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_l \end{pmatrix} ; s \right) ; F_1(z_1) \right\rangle \ldots F_l(z_l) \right\rangle \qquad (1)$$

for $l = 2$ is essentially the standard L-function $D_{F_1}^{n_1}$ of F_1 multiplied with $\langle F_1, F_2 \rangle$, if $n_1 = n_2$ and zero otherwise. By considering the case $l = 3$ with $n_1 = n_2 = n_3 = 1$,

[1] 1991 *Mathematics Subject Classification.* Primary 11F66, 11F67; Secondary 11F46, 11F60.

Key words and phrases. Pullbacks of Eisenstein series, Hecke-Jacobi theory, automorphic L-functions, Rankin triple-product L-function, special values of L-functions.

P. Garrett [Ga87] succeeded in refining his previous work to get an integral representation of the triple product L-function for the involved elliptic modular forms. This was generalized by I. Piatetski-Shapiro and S. Rallis [P-R87] to an integral representation of the "internal tensor product L-function" of an automorphic form on a semi-simple cubic abelian algebra over a number field (see also [Ga85]). (Of course, to get these integral representations a profound study of the pullbacks of the Eisenstein series via the embeddings $\mathbb{H}_{n_1} \times \ldots \times \mathbb{H}_{n_l} \hookrightarrow \mathbb{H}_n$ is needed.) These results were the starting point and a basis for a lot of important mathematical work, e.g. [Or87],[Sa87],[H-K91],[G-H93],[G-K95],[B-S95],[Sh95]. A good overview over the results concerning the triple product L-function can be found in the introduction to the paper [G-H93] by P. Garrett and M. Harris. Since the approach in (1) has been so successful it is certainly natural to ask:

> *Are there any other partitions of a natural number n, which lead to integral representations of an automorphic L-functions in the just described way?*

Already in 1985, at the conference of modular functions in Oberwolfach, Garrett mentioned at the end of his talk, that it seems to be unlikely. Nevertheless, this paper gives a positive answer. We produce a family of new pullbacks of Eisenstein series attached to partitions $1 + n + n$ of $2n + 1$, calculate $\Phi(F_1, F_2, F_3; s)$ explicitly as a Dirichlet series and show that in a special case we have established a new integral representation of an automorphic L-series.

To state our main results we have to introduce the following notation. Let Γ_n be the Siegel modular group. Writing an element of Γ_n in the form $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, with matrices a, b, c, d of size n, we can define the action of Γ_n on $\mathbb{H}_n$ by $\gamma < z >= (az + b)(cz + d)^{-1}$. Let $P_{n,0}$ the subgroup of Γ_n consisting of elements γ for which $c = 0$. Then we can introduce the Eisenstein series

$$E_n^k(z; s) = \sum_{\gamma \in P_{n,0} \backslash \Gamma_n} \det(cz + d)^{-k} \det(\operatorname{Im} \gamma < z >)^s.$$

Here $z \in \mathbb{H}_n$, $s \in \mathbb{C}$ and $k \in 2\mathbb{Z}$. Let $\mathbb{H}_1 \times \mathbb{H}_n \times \mathbb{H}_n$ be diagonally embedded in $\mathbb{H}_{2n+1}$. Let $\langle\, ; \rangle, \langle\, ; \rangle_J$ be the Petersson scalar products of Siegel modular and Jacobi forms. In the following, we assume for convenience the condition *totally real* for the Fourier coefficients of Siegel cusp forms, which can be easily dropped ([Sh76],[Boe85a]).

Theorem 1 *Let $F, G \in S_n^k$ and $h \in S_1^k$ be Hecke eigenforms with totally real Fourier coefficients and k even. Then for $s \in \mathbb{C}$ with $2\,Re(s) + k > 2n + 2$*

$$\Phi^{2n+1}(h, G, F; s) = a(n, s, k)\, \mathcal{L}_{h,G,F}^n(s + 2k - n - 1).$$

Here $a(n, s, k)$ is essentially a product of gamma functions and

$$\begin{aligned} \mathcal{L}_{h,G,F}^n(s + 2k - n - 1) \;\; &= \;\; \{\zeta(2s + k) \prod_{j=1}^n \zeta(4s + 2k - 2j)\}^{-1} \\ & \qquad D_F^n(2s + k - n)\, D_{h,G;F}(s), \end{aligned}$$

where D_F^n is the standard zeta function of F and $D_{h,G;F}(s)$ denotes a certain Dirichlet series (see section 2.4) involving the Fourier coefficients of h and the Fourier-Jacobi coefficients of F and G amalgamated together by the scalar product $\langle\, ; \,\rangle_J$.

Has this anything to do with Euler products? If $n = 2$ (the case $n = 1$ corresponds to the triple product L-function [Ga87]) we can prove the following **Main Theorem** *Let $F, G \in S_2^k$ and $h \in S_1^k$ be Hecke eigenforms with totally real Fourier coefficients. Let k be even and h normalized. Further we assume that F is in the Maass Spezialschar with corresponding $f \in S_1^{2k-2}$. Then for $s \in \mathbb{C}$ with $2\,Re\,(s) + k > 6$*

$$\begin{aligned} \Phi^5(h, G, F; s) &= a(2, s, k)\langle \Phi_1^F ; \Phi_1^G \rangle_J \\ &\quad \times \zeta(2s + k)^{-1} \zeta(4s + 2k - 2)^{-1} \zeta(4s + 2k - 4)^{-1} \\ &\quad \times L_f(2s + 2k - 4)\, Z_{G\otimes h}(s + 2k - 3). \end{aligned}$$

Here L_f denotes the Mellin transform of f and $Z_{G\otimes h}$ the convolution of the spinor zeta functions attached to G and h, in other words the L-functions to $GSp_2 \times GSp_1$. And with Φ_N^H we denote the N-th Fourier-Jacobi coefficient of a cusp form H.

If we assume that the first Fourier-Jacobi coefficient of G is not identically zero, we can conclude that $Z_{G\otimes h}(s)$ has a meromorphic continuation to the whole s-plane and get a functional equation (see Corollary 5.2). Taking G with totally real algebraic Fourier coefficients and of weight $k > 6$, and $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ we get

$$\left(\pi^{8-5k} \frac{Z_{G\otimes h}(2k - 3)}{\langle G; G\rangle \langle h; h\rangle}\right)^{\sigma} = \pi^{8-5k} \frac{Z_{G^\sigma \otimes h^\sigma}(2k - 3)}{\langle G^\sigma; G^\sigma \rangle \langle h^\sigma; h^\sigma \rangle} \in \overline{\mathbb{Q}}.$$

Related work: In [K-S89], W. Kohnen and N.-P Skoruppa studied analytic and arithmetic properties of $D_{F,G}(s) = \sum_{N=1}^{\infty} \langle \Phi_N^F ; \Phi_N^G \rangle_J\, N^{-s}$ with $F, G \in S_2^k$. They proved that $D_{F,G}(s)$ has a meromorphic continuation to the whole s-plane. This was generalized in [Ya90] with respect to $F, G \in S_n^k$. Let F be in the Maass Spezialschar and G a Hecke eigenform of degree 2, then

$$D_{F,G}(s) = \langle \Phi_1^F ; \Phi_1^G \rangle_J\, Z_G^2(s). \tag{2}$$

Here, $Z_G^2(s)$ denotes the spinor zeta function of G. If Φ_1^G is identically zero, then one replaces F by a certain paramodular cusp form such that one can still use the approach of Kohnen and Skoruppa to study $Z_G^2(s)$ for an arbitrary eigenform G. This refinement is due to V. Gritsenko [Gr93].

A different approach to study the L-function $Z_{G\otimes h}(s)$ was given in 1993 by M. Furusawa in [Fu93]. He showed that the L-function is essentially a scalar product of a Hermitian Eisenstein series attached to h and G, if G satisfies certain conditions. His results are compatible with our results concerning the meromorphic continuation, functional equation and the special value property of $Z_{G\otimes h}(s)$. For certain reasons it was impossible for him with his approach to state the Galois behaviour of the special value.

Brief outline of the Article: After reviewing some known results on Siegel modular forms, analytic Eisenstein series and a Hecke theory for the symplectic group, we state the Theorem of Garrett and Böcherer, which gives a first insight in the correspondence of pullbacks of Eisenstein series of Siegel and Klingen type. This leads us to the problem of understanding $P_{n+1,n}(\mathbb{Z})\Big\backslash Sp_{n+1}(\mathbb{Z})$ with respect to the action of $Sp_1(\mathbb{Z}) \times Sp_n(\mathbb{Z}) \hookrightarrow Sp_{n+1}(\mathbb{Z})$ from the right, here $P_{n+1,n}(\mathbb{Q}) = \{\left(\begin{smallmatrix} a & * \\ 0 & * \end{smallmatrix}\right) \in Sp_{n+1}(\mathbb{Q}) | a \in \mathbb{Q}^{\times}\}$.

In section 2, we present briefly a viewpoint, which reduces this problem to the calculation of the double coset space

$$P_{n+1,n}(\mathbb{Q})\Big\backslash Sp_{n+1}(\mathbb{Q})\Big/ Sp_1(\mathbb{Q}) \times Sp_n(\mathbb{Q})$$

and indicate how one can do that in a systematic manner. Behind all that stands a new method, which allows us to understand orbit structures of much more general shape. This will be the main subject of a forthcoming paper. After stating a pullback formula of the relevant Eisenstein series of Klingen type, we prove Theorem 1 and give some first applications.

In the next two sections we show, that $\mathcal{L}^n_{h,G,F}(s)$ has an Euler product for $n = 2$ (the case $n = 1$ is due to Garrett [Ga87]) and that Fourier-Jacobi coefficients of eigenfunctions of Siegel modular forms have certain multiplicative properties. For that, we have chosen the following strategy:

In section 3, we construct a modified Hecke-Jacobi algebra $\tilde{\mathcal{H}}^J$ in which one can embed the usual symplectic Hecke algebra $\mathcal{H}^2$ of degree 2. Highlights of our considerations are factorizations of the Hecke and Rankin polynomials of degree 2 (cf. [Gr82],[Gr93] and [An87]). We interpret Jacobi forms as automorphic forms (P-forms) on a higher dimensional space to maximal parabolic subgroups of the symplectic group (Klingen [Kl89]), which explains many properties of Jacobi forms in a transparent way.
With the results developed in section 2 and 3 we prove in section 4, that $\mathcal{L}^2_{h,G,F}(s)$ has an Euler product and calculate the local factors explicitly.

In section 5, we summarize our investigations and deduce important properties of the L-function $Z_{G\otimes h}(s)$, assuming that the first Fourier-Jacobi coefficient of G is not identically zero. The reader should not be surprised about that condition, because already in the case $n = 1$ ([Ga87]), the Fourier(-Jacobi) coefficients play a normalizing role. This paper is rounded out by a unifying presentation of some properties of the triple product L-function and the L-function $Z_{G\otimes h}(s)$. This makes it possible to translate and apply techniques, that have been developed to analyse the triple product L-function, to $Z_{G\otimes h}(s)$ and get a deeper understanding of it in the future.

$\mathcal{T}$his paper contains the main parts of my thesis ($\mathcal{M}$annheim 97) which was written

under the supervision of $\mathcal{P}$rof. $\mathcal{D}$r. $\mathcal{S}$. $\mathcal{B}$öcherer. $\mathcal{I}$ would like to thank him for introducing me into the subject and for his advice. $\mathcal{I}$ am grateful to the $\mathcal{M}$ax-$\mathcal{P}$lanck-$\mathcal{I}$nstitut für $\mathcal{M}$athematik in $\mathcal{B}$onn for its hospitality and support. $\mathcal{F}$inally $\mathcal{I}$ would like to take the opportunity to thank the organizers of the conference, $\mathcal{R}$obert $\mathcal{S}$. $\mathcal{D}$oran, $\mathcal{Z}$e-$\mathcal{L}$i $\mathcal{D}$ou and $\mathcal{G}$eorge $\mathcal{T}$. $\mathcal{G}$ilbert for their excellent job and their warm hospitality.

Notation: For an associative ring R with identity element, we denote by $R^\times$ the group of all invertible elements of R and by $M(m,n;R)$ the module of all $m \times n$ matrices with entries in R. If M is a matrix, M^t, $\det(M)$, and $\mathrm{tr}(M)$ stand for its transpose, determinant, and trace. We put $M_n(R) = M(n,n;R)$, $Gl_n(R) = M_n(R)^\times$, $Sl_n(R) = \{M \in Gl_n(R) | \det(M) = 1\}$ (when R is commutative). The identity and zero elements of $M_n(R)$ are denoted by 1_n and 0_n respectively (when n needs to be stressed). Let $J_n = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ then $Sp_n(R) = \{M \in Gl_{2n}(R) | M^t J_n M = J_n\}$.

$$
\begin{aligned}
P_{n,r}(R) &= \left\{ \begin{pmatrix} * & * \\ 0_{n+r,n-r} & * \end{pmatrix} \in Sp_n(R) \right\}, \\
C_{n,r}(R) &= \left\{ \begin{pmatrix} * & * \\ 0_{n-r,n+r} & * \end{pmatrix} \in Sp_n(R) \right\}, \\
C^+_{n,r}(R) &= \left\{ \begin{pmatrix} * & * \\ 0_{n-r,n+r} & 1_{n-r} \end{pmatrix} \in Sp_n(R) \right\},
\end{aligned}
$$

For convenience let $\Gamma_n = Sp_n(\mathbb{Z})$, $P_{n,r} = P_{n,r}(\mathbb{Z})$, $\Gamma = \Gamma_1$ and $\Gamma_\infty = P_{1,0}$. For a subring R of $\mathbb{R}$ we put $R^+ = \{r \in R \,|\, r > 0\}$ and define

$$G^+Sp_n(R) = \{M \in Gl_{2n}(R) | M^t J_n M = \mu(M) J_n \text{ with } \mu(M) \in R^+\}.$$

For real symmetric matrices A and B, we write $A > B$ ($A \geq B$) to indicate that $A - B$ is positive definite (positive semi-definite) and $A[B] = B^t A B$ if A, B are suitable. If $A_1, A_2, \ldots, A_n$ are square matrices, $[A_1, A_2, \ldots, A_n]$ denotes the matrix with $A_1, A_2, \ldots, A_n$ in the diagonal blocks and 0 in all other blocks. Further we write $[A_1, \ldots, A_m; A_{m+1} \ldots, A_{2m}]$ for numbers A_j, if $n = 2m$. The disjoint union of sets is denoted by $\bigsqcup$. Let $T, Z \in M_n(\mathbb{C})$, then we put $e\{TZ\} = e^{2\pi i\,(\mathrm{tr}\, TZ)}$ and $\mathrm{Re}(Z), \mathrm{Im}(Z)$ for the real and imaginary part of Z and for $M \in M(m,n;\mathbb{C})$ we put occasionally $x_M = \mathrm{Re}(M)$ and $y_M = \mathrm{Im}(M)$. Further $\delta(Z) = \det(\mathrm{Im}\,(Z))$. By $\overline{\mathbb{Q}}$ we denote the algebraic closure of $\mathbb{Q}$, which we assume to be embedded into $\mathbb{C}$ and $\overline{a}$ denotes the complex conjugation of an element $a \in \mathbb{C}$. Let us always assume, that p is a prime number.

1 Preliminaries

1.1 Siegel modular forms

The group of positive symplectic similitudes $G^+Sp_n(\mathbb{R})$ acts on Siegel's half space $I\!H_n = \{Z = Z^t \in M_n(\mathbb{C}) |\ \mathrm{Im}(Z) > 0\}$ of degree n as a group of biholomorphic automorphisms by

$$(M, Z) \mapsto M < Z >= (AZ + B)(CZ + D)^{-1},$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with $A, B, C, D \in M_n(\mathbb{R})$. We denote the factor of automorphy by $j(M, Z) = \det(CZ + D)$. For $F : I\!H_n \to \mathbb{C}$ and $k \in \mathbb{Z}$ we define the Petersson operator

$$(F|_k M)(Z) := \mu(M)^{nk - \frac{n(n+1)}{2}} j(M, Z)^{-k} F(M < Z >). \tag{3}$$

The real positive number $\mu(M)$ is called factor of similitude. Let M_n^k be the space of Siegel modular forms and S_n^k the subspace of cusp forms of degree n and weight k for the Siegel modular group Γ_n. Every $F \in M_n^k$ satisfies the transformation law

$$F|_k M = F, \text{ for all } M \in \Gamma_n = Sp_n(\mathbb{Z}) \tag{4}$$

and has a Fourier expansion of the form

$$F(Z) = \sum_{T \in A_n} a(T) e\{TZ\},$$

with $a(T) \in \mathbb{C}$ and A_n denotes the set of semi-integral positive semidefinite matrices of size n. For cusp forms we have only to sum over $A_n^+ = \{T \in A_n | T > 0\}$. It is well known ([Sh75]) that, if σ is an automorphism of $\mathbb{C}$ and $F \in S_n^k$, then we have

$$F^\sigma(Z) = \sum_{T \in A_n^+} a(T)^\sigma e\{TZ\} \in S_n^k. \tag{5}$$

For arbitrary complex valued functions F and G on $I\!H_n$, which satisfy (4), one can define the Petersson integral, convergence assumed,

$$\langle F, G \rangle = \int_{\Gamma_n \backslash I\!H_n} F(Z)\, \overline{G(Z)} \det(\mathrm{Im}\, Z)^k d^*Z. \tag{6}$$

Here $d^*Z = \det(Y)^{-(n+1)} dX dY$ denotes the symplectic volume element. For more details the reader is referred to Andrianov [An87] or Klingen [Kl90].

1.2 Classical Hecke theory for the symplectic group

We start by recalling the basic facts on abstract Hecke rings ([An87], §3.1; [Sh71] Chapter 3). Let (R,S) be a Hecke pair, this means R is a subgroup of the group S and for $s \in S$ the coset space $R\backslash RsR$ is finite. Let A be a principal ideal domain, then R acts from the right on the $A-$module

$$L_A(R,S) = \{X = \sum_{i=1}^{n} a_i\, Rs_i \mid a_i \in A,\ s_i \in S \text{ and } n \in I\!N\}.$$

The subset of elements $H_A(R,S)$ invariant under this action forms a ring with

$$(\sum_i a_i Rg_i) \circ (\sum_j b_j Rh_j) = \sum_{i,j} a_i b_j Rg_i h_j.$$

It is called Hecke ring or Hecke algebra to stress either the ring or algebra properties. Identifying double cosets $RsR = \bigsqcup_{i=1}^{n} Rs_i$ with $\sum_{i=1}^{n} Rs_i \in H_A(R,S)$ makes it possible to see them as a basis of the A-module $H_A(R,S)$.

Lemma 1.1 *Let (Γ_1, S_1) and (Γ_2, S_2) be two Hecke pairs included in a bigger group. If we assume that*

$$\Gamma_2 \subset \Gamma_1,\ S_1 \subset \Gamma_1 S_2,\ \text{and}\ \Gamma_1 \cap S_2 S_2^{-1} \subset \Gamma_2,$$

then we one can define an embedding i of $H_A(\Gamma_1, S_1)$ into $H_A(\Gamma_1, S_1)$ in the following way: write $X \in L_A(\Gamma_1, S_1)$ in the form $\sum_i a_i \Gamma_1 s_{i,2}$ with $s_{i,2} \in S_2$ and $a_i \in A$, then $i(X) := \sum_i a_i \Gamma_2 s_{i,2}$.

Lemma 1.2 *If (R,S) is a Hecke pair, then (R,S) is also a Hecke pair and*

$$j : \begin{cases} H_A(R,S) & \longrightarrow & H_A(R,S^{-1}) \\ RsR & \mapsto & Rs^{-1}R \end{cases} \tag{7}$$

is an anti-isomorphism which commutes with the embedding i defined in the last lemma.

1.2.1 Symplectic Hecke algebra $\mathcal{H}^n$

By applying the elementary divisor theorem, it is not difficult to see that the symplectic Hecke algebra $\mathcal{H}^n := H_{\mathbb{Q}}(\Gamma_n, G^+Sp_n(\mathbb{Q}))$ decomposes into an infinite tensor product

$$\mathcal{H}^n = \otimes_p \mathcal{H}^n_p.$$

The local Hecke algebra $\mathcal{H}^n_p = H_{\mathbb{Q}}(\Gamma_n, GSp_n^+(\mathbb{Z}[p^{-1}]))$ is isomorphic by the Satake- isomorphism Φ^n_p to $\mathbb{Q}[x_0^{\pm 1}, \dots, x_n^{\pm 1}]^{W_n}$. Here W_n denotes the Weyl group of Sp_n (for more

details see Freitag [Fr83] Kapitel 4).
The Hecke algebra $\mathcal{H}_p^n$ is generated by

$$\begin{aligned} T^{(n)}(p) &= \Gamma_n[1,\dots,1;p,\dots,p]\Gamma_n, \quad T_n^{(n)}(p)^{-1} \\ T_i^{(n)}(p^2) &= \Gamma_n[\underbrace{1,\dots,1}_{n-i},\underbrace{p,\dots,p}_{i};p^2,\dots,p^2,p,\dots,p]\Gamma_n \end{aligned}$$

for $i=0,\dots,n$. Furthermore we define $T^{(n)}(a)=\sum_{M\in SD_n(a)}\Gamma_n M\Gamma_n$ with

$$SD_n(a)=\Big\{[e_1,\dots,e_n;f_1,\dots,f_n]\,\Big|\,e_i,f_j\in\mathbb{N};\ e_i|e_{i+1};e_n|f_n \text{ and } e_if_i=a\Big\}.$$

Definition 1.3 *The Hecke polynomial $Q_p^n(X)$ and Rankin polynomial $D_p^n(X)$ of degree n are defined by*

$$\begin{aligned} Q_p^n(X) &:= (1-x_{0,p}X)\prod_{r=1}^{n}\prod_{1\le i_1<\dots<i_r\le n}(1-x_{0,p}x_{i_1,p}\cdot\ldots\cdot x_{i_r,p}X) \\ D_p^n(X) &:= (1-X)\prod_{i=1}^{n}(1-x_{i,p}X)(1-x_{i,p}^{-1}X). \end{aligned}$$

Taking the pre-images of the coefficients of these polynomials under Φ_p^n, we get polynomials in $\mathcal{H}_p^n[X]$ which we denote by $q_p^n(X)$ and $d_p^n(X)$. We denote the formal spinor- and standard zeta function of degree n by $Z^n(s)=\prod_p\left(Q_p^n(p^{-s})\right)^{-1}$ and $D^n(s)=\prod_p\left(D_p^n(p^{-s})\right)^{-1}$ for $s\in\mathbb{C}$.

By modifying the map j slightly we get an (anti-)automorphism

$$*:\ \mathcal{H}^n\longrightarrow\mathcal{H}^n,\ \Gamma_nM\Gamma_n\mapsto(\Gamma_nM\Gamma_n)^*=\Gamma_n\mu(M)\,M^{-1}\Gamma_n. \tag{8}$$

To simplify our notation in the case $n=1$, we put $T(a,d)=\Gamma\left(\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right)\Gamma$ and $T(n)=T^{(1)}(n)$. Then

$$T(p)=\sum_{a\,mod\,p}\Gamma\begin{pmatrix}1&a\\0&p\end{pmatrix}+\Gamma\begin{pmatrix}p&0\\0&1\end{pmatrix}=\Gamma\begin{pmatrix}1&0\\0&p\end{pmatrix}U_{[1,p]},$$

where $U_{[a,b]}$ denotes a representative system of $\left(\begin{pmatrix}a&0\\0&b\end{pmatrix}^{-1}\Gamma\begin{pmatrix}a&0\\0&b\end{pmatrix}\cap\Gamma\right)\setminus\Gamma$.
In $\mathcal{H}_p^n[X]$, we have the following formal identity

$$\sum_{n=0}^{\infty}T(p^n)X^n=(1-T(p)X+pT(p,p)X^2)^{-1}=q_p^1(X)^{-1}. \tag{9}$$

It is not difficult to see that

$$pT_1^{(2)}(p^2)+p(p^2+1)\Delta_p^{(2)}=T^{(2)}(p)^2-T^{(2)}(p^2)-p^2\Delta_p^{(2)} \tag{10}$$

holds in $\mathcal{H}_p^2$. Important for our purpose is the explicit form of $q_p^2(X)$ and $d_p^2(X)$.

Lemma 1.4 *In $\mathcal{H}_p^2[X]$, we have*

$$q_p^2(X) = 1 - A_1 X + A_2 X^2 - p^3 \Delta_p A_1 + p^6 (\Delta_p)^2 X^4 \tag{11}$$
$$d_p^2(X) = (1-X)(1 - B_1 X + B_2 X^2 - B_1 X^3 + X^4), \tag{12}$$

with $A_1 = T^{(2)}(p)$, $A_2 = p(T_1^{(2)}(p^2) + (p^2+1)\Delta_p)$ and

$$B_1 = \Delta_p^{-1} p^{-3} \Big(T^{(2)}(p)^2 - T^{(2)}(p^2) - p^2 \Delta_p - 2\Delta_p p^3 \Big),$$
$$B_2 = \Delta_p^{-1} p^{-3} \Big(2\, T^{(2)}(p^2) - T^{(2)}(p)^2 + 2p^3 \Delta_p + 2p^2 \Delta_p \Big).$$

The space S_n^k can be seen as an $\mathcal{H}^n$-module. Let $X = \sum a_i \Gamma_n g_i \in \mathcal{H}^n$, then

$$F \mapsto F|_k \mathcal{D}(X) := \sum_i a_i F|_k g_i. \tag{13}$$

It is well known that there exists a basis of S_n^k consisting of eigenforms. An eigenform F induces for every local algebra $\mathcal{H}_p^n$ an algebra homomorphism to $\mathbb{C}$, which can be characterized by $(\alpha_{0,p}, \alpha_{1,p}, \dots, \alpha_{n,p}) \in (\mathbb{C}^\times)^{n+1}$, the Satake-parameters of F. The eigenvalue $p^{nk-n(n+1)}$ of Δ_p^n gives $\alpha_{0,p}^2 \alpha_{1,p} \dots \alpha_{n,p} = p^{nk - \frac{n(n+1)}{2}}$. (The parameters are unique up to the action of the Weyl group). To such a F we can define $Q_{p,F}^n(X)$, $D_{p,F}^n(X)$, $Z_F^n(s)$, $D_F^n(s)$. It is known that the last two functions are absolutely convergent for Re(s) large.
In the case $n = 1$, it is convenient to use a different notation. One defines $\alpha_p = \alpha_{0,p}$ and $\beta_p = \alpha_{0,p}\alpha_{1,p}$ for Satake-parameters $\alpha_{0,p}, \alpha_{1,p}$ of a normalized ($a_1^f = 1$) Hecke eigenform $f \in S_1^k$ where a_m^f denotes the m^{th} Fourier coefficient of f. We usually want to assume that an eigenform in S_1^k is normalized. Then one has

$$\alpha_p \beta_p = p^{k-1} \text{ and } \alpha_p + \beta_p = a_p^f.$$

We write $L_f(s)$ for $Z_f^1(s)$ and $L_{p,f}(X)$ for $Q_{p,f}^1(X)$. With this notation we introduce the L-function $Z_{G\otimes h}$ corresponding to two eigenforms $G \in S_2^k$ and $h \in S_1^k$ with parameters $\alpha_{0,p}, \alpha_{1,p}, \alpha_{2,p}$ and α_p, β_p for every prime number p:

$$Z_{G\otimes h}(s) = \prod_p \left\{ Q_{p,G}^2(\alpha_p p^{-s})\, Q_{p,G}^2(\beta_p p^{-s}) \right\}^{-1}.$$

The triple product L-function of eigenforms $f_1, f_2, f_3 \in S_1^k$ is

$$L_{f_1\otimes f_2 \otimes f_3}(s) = \prod_p \det \left(1_8 - \begin{pmatrix} \alpha_p^{f_1} & 0 \\ 0 & \beta_p^{f_1} \end{pmatrix} \otimes \begin{pmatrix} \alpha_p^{f_2} & 0 \\ 0 & \beta_p^{f_2} \end{pmatrix} \otimes \begin{pmatrix} \alpha_p^{f_3} & 0 \\ 0 & \beta_p^{f_3} \end{pmatrix} p^{-s} \right)^{-1}.$$

Remark 1.5 *For each subring $A \subset \mathbb{C}$, let $S_n^k(A)$ be the $A-$module of all $F \in S_n^k$ for which all the Fourier coefficients are in A. Let F be an eigenform, especially we have $F|_k T^{(n)}(m) = \lambda_m^F F$ for all $m \in \mathbb{N}$ and define a subfield $\mathbb{Q}(F)$ of $\overline{\mathbb{Q}}$ generated by $\mathbb{Q}$ and the set of all λ_m^F. Then $\mathbb{Q}(F)$ is a totally real finite extension of $\mathbb{Q}$ and for $k \geq 2n+2$ there exists an orthogonal eigenbasis $\{F_j\}_{j=1}^d$ of S_n^k such that $F_j \in S_n^k(\mathbb{Q}(F_j))$ and the F_j are permuted by the action of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$), the absolute Galois group of $\mathbb{Q}$. Such a basis is called a distinguished basis. Every eigenform $F \in S_n^k(\mathbb{Q}(F))$ can be extended to such a basis ([Mi91], Appendix A).*

1.3 Eisenstein series

In this section, we recall some basic facts on Eisenstein series of Klingen type and study some abstract properties of their pullbacks.

Definition 1.6 *Let r, k be rational integers and n a natural number. Further let k be even and $0 \leq r \leq n$. If $s \in \mathbb{C}$ with $k + 2\,Re(s) > n + r + 1$ and $F \in S_r^k$, then we denote by*

$$E_{n,r}^k(Z, F; s) = \sum_{\gamma \in P_{n,r} \backslash \Gamma_n} F(\gamma < Z >_*)\, j(\gamma, Z)^{-k} \left(\frac{\text{det Im}\gamma < Z >}{\text{det Im}\gamma < Z >_*} \right)^s,$$

the Eisenstein series of Klingen type on $\mathbb{H}_n$.

The series is absolutely convergent and admits a meromorphic continuation to the whole $s-$plane ([La76]). With regard to k and Γ_n, it satisfies the transformation law (4) and it is orthogonal to S_n^k for $r < n$. When $s = 0$ and $k > n+r+1$, we get the holomorphic Klingen Eisenstein series ([Kl90], chapter 5). Siegel ([Si39]) has already proved that the Fourier coefficients of $E_{n,0}^k(Z, 1; 0)$ are rational numbers. For convenience we put $E_n^k(Z; s) = E_{n,0}^k(Z, 1; s)$, $\Gamma_m(s) = \prod_{j=1}^m \Gamma(s - \frac{j-1}{2})$ and $\xi(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$.

Lemma 1.7 *If we put $\mathbb{E}_n^k(Z; s) := \epsilon(n, s)\, E_n^k(Z; s - \frac{k}{2})$ with $\epsilon(n, s) = \xi(2s) \frac{\Gamma_n(s+\frac{k}{2})}{\Gamma_n(s)} \prod_{j=1}^{[\frac{n}{2}]} \xi(4s - 2j)$. Then we have*

$$\mathbb{E}_n^k(Z; s) = \mathbb{E}_n^k(Z; \frac{n+1}{2} - s).$$

Now we associate to a given partition $(n_i)_{i=1}^l$ of n (i.e. $\sum_{i=1}^l n_i = n$ with $n_i \in \mathbb{N}$) the embeddings $\mathbb{H}_{n_1} \times \ldots \times \mathbb{H}_{n_l} \hookrightarrow \mathbb{H}_n$ and $\Gamma_{n_1} \times \ldots \times \Gamma_{n_l} \hookrightarrow \Gamma_n$ given by

$$(z_1, \ldots, z_l) \mapsto [z_1, \ldots, z_l]$$

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \times \ldots \times \begin{pmatrix} a_l & b_l \\ c_l & d_l \end{pmatrix} \mapsto \begin{pmatrix} a_1 & & 0 & b_1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & a_l & 0 & & b_l \\ c_1 & & 0 & d_1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & c_l & 0 & & d_l \end{pmatrix}.$$

It is convenient to use the same notation for the images and pre-images.

Definition 1.8 *Let n, r, k be as before and $(n_i)_{i=1}^l$ a partition of n. Assume further that $F \in S_r^k$ and $s \in \mathbb{C}$ with $Re(s)$ big enough. Then we call*

$$\phi_F(z_1, \ldots, z_l; s) = E_{n,r}^k \left(\begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_l \end{pmatrix}, F; s \right)$$

a diagonal-restricted Eisenstein series attached to F of type $((n_i)_{i=1}^l; r)$.

It is easy to see that such functions satisfy (4) in each variable. If we can describe $\phi_F(z_1, \ldots, z_l; s)$ explicitly as a function in the variables $z_1, \ldots, z_l$, then we will call such an expression a **pullback formula** ([Ga84], Main formula). Let us put $(n_1, \ldots, n_l) = (n_1, \ldots, n_l; 0)$ and $\phi := \phi_1$. The following theorem has served as the main motivation for us to study pullback formulas.

Theorem 1.9 *Let k be an even rational integer and $s \in \mathbb{C}$ with $k + 2\,Re(s) > n + 1$. Further let $(n_i)_{i=1}^l$ be a partition of n and $F_i \in S_{n_i}^k$. Then*

$$\Phi(F_1, \ldots, F_l; s) := \langle \ldots \langle \phi(z_1, \ldots, z_l; s); F_1(z_1) \rangle \ldots F(z_l) \rangle$$

satisfies

(a) $\Phi(F_1, \ldots, F_l; s)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$ and inherits a functional equation from Lemma 1.7.

(b) If we assume that the F_i's are Hecke eigenforms with $F_i \in S_{n_i}^k(\mathbb{Q}(F_i))$ and $k >$ max$\{n + 1,$ max$\{n_i\} + 1\}$, then

$$\frac{\Phi(F_1^\sigma, \ldots, F_l^\sigma; 0)}{\prod_{i=1}^l \langle F_i^\sigma; F_i^\sigma \rangle} = \left(\frac{\Phi(F_1, \ldots, F_l; 0)}{\prod_{i=1}^l \langle F_i; F_i \rangle} \right)^\sigma \in \overline{\mathbb{Q}} \text{ with } \sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q}).$$

Proof:
Part (a) is clear. To prove (b), we assume for convenience that $l = 2$, for arbitrary l one proceeds in the same way. We can complete $f_1 = F_1$ and $g_1 = F_2$ to a *distinguished* basis

of $S^k_{n_1}$ and $S^k_{n_2}$ respectively and then to an orthogonal basis of $M^k_{n_1}$ and $M^k_{n_2}$. This can be done in such a way that the orthogonal complements of $S^k_{n_1}$ and $S^k_{n_2}$ are invariant under all automorphisms of $\mathbb{C}$ (cf. [Boe85a], section 6). From $\phi(z_1, z_2; 0) \in M^k_{n_1} \otimes M^k_{n_2}$ we can deduce

$$\phi(z_1, z_2; 0) = \sum_{i,j} c_{i,j} f_i(z_1) g_j(z_2) \qquad (c_{ij} \in \mathbb{C}).$$

The rationality of the Fourier coefficients of $\phi(z_1, z_2; 0)$ leads to

$$\left(\frac{\Phi(F_1, F_2; 0)}{\langle F_1; F_1 \rangle \langle F_2; F_2 \rangle} \right)^\sigma \;=\; c^\sigma_{11} \;=\; \frac{\langle \langle \phi(z_1, z_2; 0); F^\sigma_1 F^\sigma_2 \rangle \rangle}{\langle F^\sigma_1; F^\sigma_1 \rangle \langle F^\sigma_2; F^\sigma_2 \rangle},$$

for all automorphisms σ of $\mathbb{C}$. The assumptions in (b) imply that c_{11} is algebraic.

□

Remark 1.10 *If we denote by $\mathbb{Q}(F_1, \ldots, F_l)$ the totally real number field which is generated by $\mathbb{Q}(F_1), \ldots, \mathbb{Q}(F_l)$, then*

$$\frac{\Phi(F_1, \ldots, F_l; 0)}{\prod \langle F_i; F_i \rangle} \in \mathbb{Q}(F_1, \ldots, F_l).$$

It is of course possible to state the theorem under weaker assumptions. For example, the purely technical condition totally real *can easily be dropped (e.g. [Boe85a], Satz 14).*

A first insight into the connection between $\Phi(F_1, \ldots, F_l; s)$ and automorphic L-functions is given by the following

Theorem 1.11 *(Garrett, Böcherer)*
Let $m, n \in \mathbb{N}$, $k \in 2\mathbb{Z}$ and $s \in \mathbb{C}$ with $s \in \mathbb{C}$ with $k + 2Re(s) > m + n + 1$. If $F \in S^k_n$ is an eigenform then the following assertion holds

$$\langle E^k_{m+n} \left(\begin{pmatrix} z & 0 \\ 0 & * \end{pmatrix}; s \right); F \rangle = \begin{cases} \mu(n, k, s) \left\{ \zeta(2s + k) \prod_{j=1}^n \zeta(4s + 2k - 2j) \right\}^{-1} & \\ \quad \times D^n_F(2s + k - n) E^k_{m,n}(z, F^\#; s) & \text{if } m \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\#$ denotes complex-conjugation and

$$\mu(n, k, s) = 2^{\frac{n^2+3n}{2} - 2ns - nk + 1} (-1)^{\frac{nk}{2}} \pi^{\frac{n(n+1)}{2}} \frac{\Gamma_n(k + s - \frac{n+1}{2})}{\Gamma_n(k + s)}. \tag{14}$$

See [Ga84] and [Boe85b] for more details. In the special case $n = m$, $s = 0$ we have

Corollary 1.12 *Let $F \in S^k_n(\mathbb{Q}(F))$ be a Hecke eigenform and $k > 2n$ even. Then*

$$\left(\frac{\pi^{\frac{3n(n+1)}{2} - 2nk - k} D^n_F(k - n)}{\langle F; F \rangle} \right)^\sigma = \frac{\pi^{\frac{3n(n+1)}{2} - 2nk - k} D^n_{F^\sigma}(k - n)}{\langle F^\sigma; F^\sigma \rangle} \in \mathbb{Q}(F) \tag{15}$$

for all $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

2 A family of Dirichlet series

This section centers around the explicit calculation of the integral of an Eisenstein series of type $(1,n,n)$ against three cusp forms. To do this, we start with some LINEARE ZAHLENTHEORIE (cf. Siegel [Si68] 3. Abschnitt).

2.1 Representatives for $P_{n+1,n}\backslash\Gamma_{n+1}$

The fact, that for every $g \in Sp_m(\mathbb{Q})$ there exists $p_g \in \bigcap_{r=0}^{m-1} P_{m,r}(\mathbb{Q})$ with

$$p_g \cdot g \in \Gamma_m \tag{16}$$

allows us to find a suitable system of representatives for $RS^{\mathbb{Z}}$ of $P_{n+1,n}\backslash\Gamma_{n+1}$ in a rather simple way, that suits our purposes.

Proposition 2.1 *The left cosets space $P_{n+1,n}(\mathbb{Q})\backslash Sp_{n+1}(\mathbb{Q})$ has representatives*

$$RS^{\mathbb{Q}} = R_1^{\mathbb{Q}} \sqcup R_2^{\mathbb{Q}} \sqcup R_3^{\mathbb{Q}}.$$

$$\begin{aligned}
R_1^{\mathbb{Q}} &= P_{1,0}(\mathbb{Q})\backslash Sp_1(\mathbb{Q}) \times 1_{2n}, && (17)\\
R_2^{\mathbb{Q}} &= \pi_n \cdot (1_2 \times P_{n,n-1}(\mathbb{Q})\backslash Sp_n(\mathbb{Q})), && (18)\\
R_3^{\mathbb{Q}} &= \xi_n \cdot \left(P_{1,0}(\mathbb{Q})\backslash Sp_1(\mathbb{Q}) \times ((T \times 1_{2(n-1)}) \cdot P_{n,n-1}(\mathbb{Q})\backslash Sp_n(\mathbb{Q}))\right) && (19)
\end{aligned}$$

$$\textit{with } \pi_n = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 & & \\ 0 & 1_{n-1} & & 0 \\ & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \\ & 0 & 0 & 1_{n-1} \end{pmatrix}$$

$$\textit{and } \xi_n = \begin{pmatrix} \begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix} & 0 & & \\ 0 & 1_{n-1} & & 0 \\ & & \begin{matrix} 1 & -1 \\ 0 & 1 \end{matrix} & 0 \\ & 0 & 0 & 1_{n-1} \end{pmatrix}.$$

Furthermore $T = T(\mathbb{Q}) = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{Q}^\times\}$.

Proof:
By Witt's Theorem ([Bou59] §4, $n^0 3$; Corollaire 2), we can identify $P_{n+1,n}(\mathbb{Q})\backslash Sp_{n+1}(\mathbb{Q})$ with the set X of one-dimensional (isotropic) subspaces of $\mathbb{Q}^{2(n+1)}$. To prove the assertion of the proposition, we have to determine the orbits of the action of $Sp_1(\mathbb{Q}) \times Sp_n(\mathbb{Q}) \leq$

$Gl_{2(n+1)}(\mathbb{Q})$ on X. We decompose $\mathbb{Q}^{2(n+1)} = \mathbb{Q}^2 \oplus \mathbb{Q}^{2n}$ and denote by $p_1(U)$, $p_2(U)$ the projection of $U \in X$ on $\mathbb{Q}^2$ and $\mathbb{Q}^{2n}$ respectively. Then it is clear, that the orbits are characterized by the ranks $\big(\text{rank}(p_1(U)), \text{rank}(p_2(U))\big)$ of the projection: $(1,0)$, $(0,1)$ $(1,1)$. So we get three orbits. Then we fix an element in each orbit and determine its isotropy group. Translating these observations into the language of matrices proves the proposition.

$\square$

To get the desired representatives over $\mathbb{Z}$, we have to alter $RS^{\mathbb{Q}}$ by suitable elements of $P_{n+1,n}(\mathbb{Q})$ from the left.

Corollary 2.2 *The left cosets space $P_{n+1,n}\backslash\Gamma_{n+1}$ has representatives*

$$RS^{\mathbb{Z}} = R_1^{\mathbb{Z}} \sqcup R_2^{\mathbb{Z}} \sqcup R_3^{\mathbb{Z}}.$$

$$\begin{aligned} R_1^{\mathbb{Z}} &= P_{1,0}\backslash\Gamma_1 \times 1_{2n}, && (20)\\ R_2^{\mathbb{Z}} &= \pi_n \cdot (1_2 \times P_{n,n-1}\backslash\Gamma_n), && (21)\\ R_3^{\mathbb{Z}} &= \bigsqcup_{\substack{\varepsilon\in\mathbb{N},\nu\in\mathbb{Z}\backslash\{0\}\\ (\varepsilon,\nu)=1}} \xi_n^{\varepsilon,\nu} \cdot (P_{1,0}\backslash\Gamma \times P_{n,n-1}\backslash\Gamma_n) && (22)\end{aligned}$$

$$\textit{with } \xi_n^{\varepsilon,\nu} = \begin{pmatrix} * & * & 0 & 0\\ \varepsilon & \nu & 0 & 0\\ 0 & 0 & \nu & -\varepsilon\\ 0 & 0 & * & * \end{pmatrix} \times 1_{2(n-1)} \in \Gamma_{n+1}.$$

Proof:

The property (16) implies that there is an 1-1 correspondence between $P_{n,r}(\mathbb{Q})\backslash Sp_{n,r}(\mathbb{Q})$ and $P_{n,r}\backslash\Gamma_n$. So it remains only to translate $R_3^{\mathbb{Q}}$ back to $R_3^{\mathbb{Z}}$. That means we are looking for a $p_a \in P_{n,r}(\mathbb{Q})$ with

$$p_a \cdot \left(1_2 \times \begin{pmatrix} a & 0\\ 0 & a^{-1}\end{pmatrix} \times 1_{2(n-1)}\right) \in \Gamma_{n+1}.$$

If we parametrize $a \in \mathbb{Q}^\times$ with $\frac{\nu}{\varepsilon}$ ($\nu \in \mathbb{Z}\backslash\{0\}$, $\varepsilon \in \mathbb{N}$), then

$$p_{\varepsilon,\nu} = \begin{pmatrix} \nu^{-1} & \mu' & & \\ 0 & \varepsilon & & 0 \\ & & \nu & 0\\ & 0 & \mu & \varepsilon^{-1}\end{pmatrix} \times 1_{2(n-1)}$$

does the job, if we choose $\mu, \mu' \in \mathbb{Z}$ in such a way that $\mu\,\varepsilon \equiv 1 \bmod \nu$.

$\square$

The assertion of the Proposition 2.1 still holds for arbitrary fields. The corollary is of course not true for every integral domain.

2.2 Pullback formulas

This subsection is devoted to explain how one can use Corollary 2.2 to produce pullback formulas. We achieve this by giving two examples.

Theorem 2.3 *Let $k \in 2\mathbb{Z}$ and $s \in \mathbb{C}$ with $2\,Re(s) + k > 2n+2$. If $F \in S_n^k$ and $\tau \in I\!\!H$, $\tau' \in I\!\!H_n$, then we get*

$$E^k_{n+1,n}\left(\begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix}, F; s\right) = E^k_1(\tau, s) F(\tau') + \left\{ \begin{array}{ll} E^k_1(\tau'; s) F(\tau) & n = 1 \\ E^k_{n,n-1}(\tau', F_\tau; s) & n > 1 \end{array} \right.$$

$$+ \quad 2\,\delta(\tau)^s\,\delta(\tau')^s \sum_{\substack{\epsilon,\nu=1 \\ (\epsilon,\nu)=1}}^{\infty} \sum_{\substack{\gamma_1 \in \Gamma_\infty \backslash \Gamma \\ \gamma_2 \in P_{n,n-1}\backslash \Gamma_n}} \chi_j^{k,s}(\gamma_1, \tau)\, \chi_j^{k,s}(\gamma_2, \tau')$$

$$\times F\left(\begin{pmatrix} \varepsilon^2 \gamma_1 < \tau > & 0 \\ 0 & 0 \end{pmatrix} + (\gamma_2 < \tau' >)\left[\begin{pmatrix} \nu & 0 \\ 0 & 1_{n-1} \end{pmatrix}\right]\right)$$

$$\times \delta\left(\begin{pmatrix} \varepsilon^2 \gamma_1 < \tau > & 0 \\ 0 & 0 \end{pmatrix} + (\gamma_2 < \tau' >)\left[\begin{pmatrix} \nu & 0 \\ 0 & 1_{n-1} \end{pmatrix}\right]\right)^{-s}.$$

Here $F_\tau(z) = F\left(\begin{smallmatrix} \tau & 0 \\ 0 & z \end{smallmatrix}\right) \in S^k_{n-1}$ and $\chi_j^{k,s}(\gamma_j, \tau) = j(\gamma_j, \tau)^{-k} |j(\gamma_j, \tau)|^{-2s}$.

Proof:
It is convenient to rewrite the Eisenstein series in the following manner

$$E^k_{n+1,n}\left(\begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix}, F; s\right) \;=\; \delta(\tau)^s\,\delta(\tau')^s \sum_{g \in P_{n+1,n}\backslash \Gamma_{n+1}} \chi_j^{k,s}(g, \begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix})$$

$$\times F\left(g < \begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix} >_*\right) \;\; \delta\left(g < \begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix} >_*\right)^{-s},$$

because $\chi_j^{k,s}$ satisfies the same cocyle relation as the factor of automorphy j. Then it is easy to see that the subseries $E^k_1(\tau, s)\, F(\tau')$ corresponds to $R_1^{\mathbb{Z}}$. If $n = 1$, then $R_2^{\mathbb{Z}}$ leads to $E^k_1(\tau')F(\tau)$ and to $E^k_{n,n-1}(\tau', F_\tau; s)$ otherwise, because the $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ block of π_n interchanges τ and τ'. To interpret $R_3^{\mathbb{Z}}$ it suffices to look at

$$\left(\begin{pmatrix} \begin{matrix} * & * \\ \varepsilon & \nu \end{matrix} & 0 \\ 0 & 1_{n-1} \end{pmatrix} \begin{pmatrix} \tilde\tau & 0 \\ 0 & \begin{matrix} \tilde z_1 & \tilde z_2 \\ \tilde z_2^t & \tilde z_4 \end{matrix} \end{pmatrix} \begin{pmatrix} \begin{matrix} * & \varepsilon \\ * & \nu \end{matrix} & 0 \\ 0 & 1_{n-1} \end{pmatrix}\right)_* = \begin{pmatrix} \varepsilon^2\tau + \nu^2 \tilde z_1 & \nu \tilde z_2 \\ \nu \tilde z_2^t & \tilde z_4 \end{pmatrix}.$$

□

Remark 2.4

(1) *The case $n=1$ is essentially due to Garrett [Ga87].*

(2) *For $n>1$ a new phenomenon shows up, namely the occurrence of diagonal-restricted cusp forms. This will play no role in this paper.*

(3) *If we put $U_+ = \{ \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma \mid c, d > 0 \}$, then we can essentially describe $R_3^{\mathbb{Z}}$ by*

$$\bigsqcup_{u \in U_+} \left(\left(\begin{array}{cc} u & 0 \\ 0 & u^\tau \end{array} \right) \times 1_{2(n-1)} \right) \left(P_{1,0} \backslash \Gamma \times P_{n,n-1} \backslash \Gamma_n \right).$$

We are now in a good position. The pullback formula in Theorem 2.3 can easily be generalized to get pullback formulas of type $(n_1, \dots, n_l, 1; n)$. It is further possible to calculate $\Phi_F(F_1, \dots, F_{l+1}; s)$ in an explicit way (This vein will be developed in a forthcoming paper). To convince the reader, we present here the case $(1,1,1;2)$.

Corollary 2.5 *Let $k \in 2\mathbb{Z}$ and $F \in S_2^k$. Then for $s \in \mathbb{C}$ with Re$(s) > 6$ we have*

$$\begin{aligned}
&E_{3,2}^k \left([z_1, z_2, z_3], F; s \right) \\
&\quad = \sum_{(a,b,c) \in P} E_1^k(z_a; s) F \left(\begin{array}{cc} z_b & 0 \\ 0 & z_c \end{array} \right) \\
&\qquad + 2\, \delta(z_a)^s \delta(z_b)^s \sum_{u \in U_+} \sum_{\gamma_1, \gamma_2 \in \Gamma_\infty \backslash \Gamma} \chi_j^{k,s}(\gamma_1, z_a)\, \chi_j^{k,s}(\gamma_1, z_b) \\
&\qquad \times F_{z_c} \left(\left(\begin{array}{cc} \gamma_1 < z_a > & 0 \\ 0 & \gamma_2 < z_b > \end{array} \right) \left[u^t \right]_* \right) \delta \left(\left(\begin{array}{cc} \gamma_1 < z_a > & 0 \\ 0 & \gamma_2 < z_b > \end{array} \right) [u^t]_* \right)^{-s} \\
&\qquad + 2\, \delta(z_1)^s \delta(z_2)^s \delta(z_3)^s \sum_{u_1, u_2 \in U_+} \sum_{\gamma_1, \gamma_2, \gamma_3 \in \Gamma_\infty \backslash \Gamma} \prod_{i=1}^3 \chi_j^{k,s}(\gamma_i, z_i) \\
&\qquad \times F \left(\left(\begin{array}{ccc} \gamma_1 < z_1 > & 0 & 0 \\ 0 & \gamma_2 < z_2 > & 0 \\ 0 & 0 & \gamma_3 < z_3 > \end{array} \right) \left[\left(\begin{array}{cc} 1 & 0 \\ 0 & u_1^t \end{array} \right) \left(\begin{array}{cc} u_2^t & 0 \\ 0 & 1 \end{array} \right) \right]_* \right) \\
&\qquad \times \delta \left(\left(\begin{array}{ccc} \gamma_1 < z_1 > & 0 & 0 \\ 0 & \gamma_2 < z_2 > & 0 \\ 0 & 0 & \gamma_3 < z_3 > \end{array} \right) \left[\left(\begin{array}{cc} 1 & 0 \\ 0 & u_1^t \end{array} \right) \left(\begin{array}{cc} u_2^t & 0 \\ 0 & 1 \end{array} \right) \right]_* \right)^{-s},
\end{aligned}$$

with $P = \{(1,2,3), (3,1,2), (2,3,1)\}$.

2.3 The integral representation

We calculate the integral of a diagonal-restricted Eisenstein series of type $(1, n, n)$ against three cusp forms. To sharpen our result, we recall some fundamental principles on Jacobi forms (see [E-Z85],[Kl89] or section 3.4). Let $F \in S_n^k$, then one denotes by

$$F\begin{pmatrix} \tau & z \\ z^t & \tau' \end{pmatrix} = \sum_{N=1}^{\infty} \Phi_N^F(\tau, z)\, e\{N\tau'\} \quad (\tau' \in I\!H) \tag{23}$$

the Fourier-Jacobi expansion of F with respect to τ'. The Fourier-Jacobi coefficients $\Phi_N^F(\tau,z)$ are Jacobi forms $J_{n-1,1}^k(N)$ of degree $(n-1,1)$, weight k and index N. By $\langle\, ;\, \rangle_J = \langle\, ;\, \rangle_{J,N}$ we denote the Petersson scalar product for such spaces. Further we put

$$U_l : \begin{cases} J_{m,1}^k(N) & \longrightarrow \qquad J_{m,1}(Nl^2) \\ \Phi_N^F(\tau,z) & \mapsto \quad (\Phi_N^F|U_l)(\tau,z) = \Phi_N^F(\tau, lz). \end{cases} \tag{24}$$

In the case $n = 1$, we interpret the Fourier-Jacobi coefficients as the Fourier coefficients, the map U_l as the identity map and $\langle\, ;\, \rangle_J$ as the standard scalar product in $\mathbb{C}^2$.

Theorem 2.6 *Let k be an even number, $F, G \in S_n^k$ and $h \in S_1^k$ with totally real Fourier coefficients. Then for $s \in \mathbb{C}$ with $k + 2\,Re(s) > 2n + 2$*

$$\langle\langle E_{n+1,n}^k\left(\begin{pmatrix} W & 0 \\ 0 & Z \end{pmatrix}, F; s\right); G(Z)\rangle; h(W)\rangle \tag{25}$$

$$= 2\,(4\pi)^{-(2k+s-n-1)}\frac{\Gamma(s+2k-n-1)\Gamma(s+k-n)\Gamma(s+k-1)}{\Gamma(2k-n-1+2s)}$$

$$\times \sum_{\substack{N,\varepsilon,l=1 \\ (\varepsilon,l)=1}}^{\infty} \langle \Phi_N^F | U_l; \Phi_{Nl^2}^G\rangle_J\, a_{N\varepsilon^2}^h \quad \varepsilon^{-2(k+s-1)} l^{-2(k+s-n)} N^{-(2k+s-n-1)}.$$

Proof:
To calculate (25), we insert the pullback formula developed in Theorem 2.3. After interchanging the integration and summation, we get by "unwinding "

$$\langle\langle E_{n+1,n}^k\left(\begin{pmatrix} W & 0 \\ 0 & Z \end{pmatrix}, F; s\right); G(Z)\rangle; h(W)\rangle$$

$$= 2\int_{\Gamma_\infty\backslash I\!H} d^*W \int_{P_{n,n-1}\backslash I\!H_n} d^*Z \sum_{\substack{\varepsilon,l=1 \\ (\varepsilon,l)=1}}^{\infty} F\left(\begin{pmatrix} \varepsilon^2 W & 0 \\ 0 & 0 \end{pmatrix} + Z\left[\begin{pmatrix} l & 0 \\ 0 & 1_{n-1} \end{pmatrix}\right]\right) \tag{26}$$

$$\times\, \delta\left(\begin{pmatrix} \varepsilon^2 W & 0 \\ 0 & 0 \end{pmatrix} + Z\left[\begin{pmatrix} l & 0 \\ 0 & 1_{n-1} \end{pmatrix}\right]\right)^{-s} \delta(W)^{k+s}\delta(Z)^{k+s}\overline{G(Z)h(W)}.$$

To use the notation in [Kl89] we rewrite this as

$$2\int_{\Gamma_\infty\backslash I\!H} d^*W \int_{C_{n,n-1}\backslash I\!H_n} d^*Z \sum_{\substack{\varepsilon,l=1 \\ (\varepsilon,l)=1}}^{\infty} F\left(\begin{pmatrix} 0 & 0 \\ 0 & \varepsilon^2 W \end{pmatrix} + Z\left[\begin{pmatrix} 1_{n-1} & 0 \\ 0 & l \end{pmatrix}\right]\right)$$

$$\times\delta\left(\left(\begin{array}{cc}0 & 0\\ 0 & \varepsilon^2 W\end{array}\right)+Z\left[\left(\begin{array}{cc}1_{n-1} & 0\\ 0 & l\end{array}\right)\right]\right)^{-s}\delta(W)^{k+s}\delta(Z)^{k+s}\overline{G(Z)h(W)}.$$

If we denote by $\mathcal{B}_{n-1,1}$ a fundamental domain of the action of the group $C^+_{n,n-1}/Z^{n,n-1}$ on $I\!H_{n-1}\times\mathbb{C}^{n-1}$, here we denote by $Z^{n,n-1}$ the center of $C^+_{n,n-1}$, then

$$\mathcal{Q}_{n,1}=\{Z=\left(\begin{array}{cc}\tau & z\\ z^t & \tau'\end{array}\right)\in I\!H_n\mid(\tau,z)\in\mathcal{B}_{n-1,1}\text{ and }0\le x_{\tau'}<1\}\tag{27}$$

is a fundamental domain of $C^+_{n,n-1}\backslash I\!H_n$ ([Kl89]). This leads to

$$\langle\langle E^k_{n+1,n}\left(\left(\begin{array}{cc}W & 0\\ 0 & Z\end{array}\right),F;s\right);G(Z)\rangle;h(W)\rangle$$

$$\begin{aligned}
&= 2\sum_{\substack{\varepsilon,l=1\\(\varepsilon,l)=1}}^{\infty}\int_{\mathcal{B}_{n-1,1}}d\tau dz\int_{y_{\tau'}>y_\tau^{-1}[y_z]}dy_{\tau'}\int_0^1 dx_{\tau'}\int_0^1 dx_W\int_0^\infty dy_W\ \delta(Z)^{k+s-(n+1)}y_W^{k+s-2}\\
&\quad\times\sum_{N=1}^{\infty}\Phi^F_N(\tau,lz)e\{N(\varepsilon^2W+l^2\tau')\}\overline{\sum_{M=1}^{\infty}a^h_Me\{MW\}\sum_{L=1}^{\infty}\Phi^G_L(\tau,z)e\{L\tau'\}}\\
&\quad\times\delta\left(\left(\begin{array}{cc}0 & 0\\ 0 & \varepsilon^2W\end{array}\right)+Z[\left(\begin{array}{cc}1 & 0\\ 0 & l\end{array}\right)]\right)^{-s}\\
&= 2\int_{\mathcal{B}_{n-1,1}}d\tau dz\sum_{\substack{N,\varepsilon,l=1\\(\varepsilon,l)=1}}^{\infty}\Phi^F_N(\tau,lz)\overline{\Phi^G_{Nl^2}(\tau,z)a^h_{N\varepsilon^2}}\int_{y_{\tau'}>y_\tau^{-1}[y_z]}dy_{\tau'}\int_0^\infty dy_W\\
&\quad\delta\left(\left(\begin{array}{cc}0 & 0\\ 0 & \varepsilon^2W\end{array}\right)+Z[\left(\begin{array}{cc}1 & 0\\ 0 & l\end{array}\right)]\right)^{-s}\delta(Z)^{k+s-(n+1)}\,y_W^{k+s-2}\,e^{-4\pi N(\varepsilon^2y_W+l^2y_{\tau'})}.
\end{aligned}$$

In the last step, we have integrated over the real parts of τ' and W. It remains to calculate the integration over dy_W and $dy_{\tau'}$. We have

$$\begin{aligned}
&\int\limits_{y_{\tau'} > y_\tau^{-1}[y_z]} dy_{\tau'} \int\limits_0^\infty dy_W\, \delta\begin{pmatrix} y_\tau & ly_z \\ ly_z^t & \varepsilon^2 W + l^2 y_{\tau'} \end{pmatrix}^{-s} \delta\begin{pmatrix} y_\tau & y_z \\ y_z^t & y_{\tau'} \end{pmatrix}^{k+s-(n+1)} \\
&\quad y_W^{k+s-2}\, e^{-4\pi N(\varepsilon^2 y_W + l^2 y_{\tau'})} \\
\overset{(1)}{=}\; &\int\limits_0^\infty dt \int\limits_0^\infty dy_W\, \delta\begin{pmatrix} y_\tau & ly_z \\ ly_z^t & \varepsilon^2 y_W + l^2(t + y_\tau^{-1}[y_z]) \end{pmatrix}^{-s} \delta\begin{pmatrix} y_\tau & y_z \\ y_z^t & t + y_\tau^{-1}[y_z] \end{pmatrix}^{k+s-n-1} \\
&\quad y_W^{k+s-2}\, e^{-4\pi N \varepsilon^2 y_W} e^{-4\pi N l^2 t} e^{-4\pi N l^2 y_\tau^{-1}[y_z]} \\
=\; &\int\limits_0^\infty dt \int\limits_0^\infty dy_W\; \delta\begin{pmatrix} y_\tau & ly_z \\ ly_z^t & \varepsilon^2 y_W + l^2(t + y_\tau^{-1}[y_z]) \end{pmatrix}^{-s} \delta\begin{pmatrix} y_\tau & y_z \\ y_z^t & l^2 t + l^2 y_\tau^{-1}[y_z] \end{pmatrix}^{k+s-n-1} \\
&\quad l^{-2(k+s-(n+1))}\, y_W^{k+s-2}\, e^{-4\pi N \varepsilon^2 y_W} e^{-4\pi N l^2 t} e^{-4\pi N l^2 y_\tau^{-1}[y_z]} \\
\overset{(2)}{=}\; &\int\limits_0^\infty dt \int\limits_0^\infty dy_W\, t^{k+s-(n+1)} (\varepsilon^2 y_W + l^2 t)^{-s} \delta(y_\tau)^{k-(n+1)}\, y_W^{k+s-2} \\
&\quad e^{-4\pi N \varepsilon^2 y_W} e^{-4\pi N l^2 t} e^{-4\pi N l^2 y_\tau^{-1}[y_z]}.
\end{aligned}$$

In (1) we have substituted $y_{\tau'}$ by $t + y_\tau^{-1}[y_z]$ and in (2) we have used

$$\begin{aligned}
&\delta\begin{pmatrix} y_\tau & ly_z \\ ly_z^t & \varepsilon^2 y_W + l^2(t + y_\tau^{-1}[y_z]) \end{pmatrix}^{-s} \delta\begin{pmatrix} y_\tau & ly_z \\ ly_z^t & l^2 t + l^2 y_\tau^{-1}[y_z] \end{pmatrix}^{k+s-n-1} \\
=\;& (l^2 t)^{k+s-(n+1)} (\varepsilon^2 y_W + l^2 t)^{-s} \delta(y_\tau)^{k-(n+1)}
\end{aligned}$$

Recall that the invariant measure for the Petersson scalar product in $\mathcal{J}_{n-1,1}^k(m)$ is

$$\delta(y_\tau)^{k-((n-1)+2)} e^{-4\pi m y_\tau^{-1}[y_z]} d\tau dz. \tag{28}$$

So it remains only to calculate

$$\int\limits_0^\infty \frac{dt}{t} \int\limits_0^\infty \frac{dy_W}{y_W}\, t^{k+s-n} y_W^{k+s-1} (\varepsilon^2 y_W + l^2 t)^{-s} e^{-4\pi N \varepsilon^2 y_W} e^{-4\pi N l^2 t},$$

which is an Eulerian integral of first kind. Applying the formula

$$\int\limits_0^\infty \int\limits_0^\infty \frac{dx}{x}\frac{dy}{y} x^\alpha y^\beta \left(\frac{xy}{x+y}\right)^\gamma e^{-(x+y)} = \frac{\Gamma(\alpha+\beta+\gamma)\,\Gamma(\alpha+\gamma)\,\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+2\gamma)}$$

gives the desired result.

□

2.4 First results

The Dirichlet series we obtained in the last subsection we now denote by

$$D_{h,G;F}(s) = \sum_{\substack{N,\varepsilon,l=1 \\ (\varepsilon,l)=1}}^{\infty} \langle \Phi_N^F | U_l ; \Phi_{Nl^2}^G \rangle_J \, a_{N\varepsilon^2}^h \, \varepsilon^{-2(k+s-1)} l^{-2(k+s-n)} N^{-(2k+s-n-1)}. \tag{29}$$

Combining results of the Theorem 2.6 with the Theorem of Garrett and Böcherer and Theorem 1.9 gives the following first résumé.

Theorem 2.7 *Let $F, G \in S_n^k$, $h \in S_1^k$ Hecke eigenforms with totally real Fourier coefficients. Let $s \in \mathbb{C}$ with $2\,Re(s) + k > 2n + 2$ and k even. Then we have*

$$\Phi(h, G, F; s) = a(n, s, k)\, \mathcal{L}_{h,G,F}^n (s + 2k - n - 1) \quad \textit{with} \tag{30}$$

$$\begin{aligned} \mathcal{L}_{h,G,F}^n(s + 2k - n - 1) &= \{\zeta(2s + k) \prod_{j=1}^{n} \zeta(4s + 2k - 2j)\}^{-1} \\ &\quad \times D_F^n(2s + k - n)\, D_{h,G;F}(s) \end{aligned}$$

and $a(n, s, k) = a^*(n, s, k)\, \Gamma^*(n, s, k)$ with

$$\begin{aligned} a^*(n,s,k) &= 2^{-2s(n+1)-nk-4k+3+\frac{n^2+7n}{2}} (-1)^{\frac{nk}{2}} 2\pi^{\frac{n^2+3n}{2} - s - 2k + 1}, \\ \Gamma^*(n,s,k) &= \frac{\Gamma(s+2k-1-n)\Gamma(s+k-n)\Gamma(s+k-1)}{\Gamma(2k-n-1+2s)} \, \frac{\Gamma_n(k+s-\frac{n+1}{2})}{\Gamma_n(k+s)}. \end{aligned}$$

Corollary 2.8 *With the same assumptions as above, we have:*

(i) $\mathcal{L}_{h,G,F}^n$ has meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\mathcal{L}_{h,G,F}^{n,*}(s) = \mathcal{L}_{h,G,F}^{n,*}(3k - n - 1 - s), \tag{31}$$

where

$$\mathcal{L}_{h,G,F}^{n,*}(s) = \epsilon(2n + 1, s + n + 1 - 3/2\,k)\, a(n, n + 1 + s - 2k, k)\, \mathcal{L}_{h,G,F}^n(s).$$

(ii) Let $k > 2n$, $h \in S_1^k(\mathbb{Q}(h))$, $G \in S_2^k(\mathbb{Q}(G))$ and $F \in S_2^k(\mathbb{Q}(F))$. Then we have for all $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

$$\left(\frac{a(n,0,k)\mathcal{L}_{h,G,F}^n(2k-n-1)}{\langle h;h\rangle\langle G;G\rangle\langle F;F\rangle} \right)^{\sigma} = \left(\frac{a(n,0,k)\mathcal{L}_{h^\sigma,G^\sigma,F^\sigma}^{(n)}(2k-n-1)}{\langle h^\sigma;h^\sigma\rangle\langle G^\sigma;G^\sigma\rangle\langle F^\sigma;F^\sigma\rangle} \right) \in \overline{\mathbb{Q}}.$$

The restriction $k > 2n$ can be weakened. The assumption *totally real* is only of technical nature. It can easily be removed (see [Sh76], [Boe85a]). All of the results stated so far have only been known for $n = 1$. In that case, P. Garrett [Ga87] has shown that $\mathcal{L}^1_{h,G,F}$ is essentially the triple product L-function for three elliptic eigenforms $h, G, F \in S^k_1$. So there is some hope that $\mathcal{L}^n_{h,G,F}$ has something to do with Euler products in general. The rest of this paper is devoted to show that one really gets an Euler product in the case of $n = 2$, if one chooses the cusp forms in an appropriate way.

3 Hecke-Jacobi theory

In this section, a certain Hecke-Jacobi theory ([An77b],[An87],[Gr82],[Gr93]) is developed, which serves mainly as an aid for studying the arithmetic properties of the Dirichlet series $D_{h,G;F}(s)$ introduced in (29). We proceed in a self-contained way to make the underlying structure as transparent as possible and to fix a good notation. This enables us to prove new results which are essential in getting an Euler product for the Dirichlet series.

3.1 The Jacobi group $G^J_{\mathcal{R}}$

For $\mathcal{R} = \mathbb{Z}$, $\mathbb{Z}[\frac{1}{p}]$ or $\mathbb{Q}$, we define the Heisenberg group $H_{\mathcal{R}}$, the group $\Gamma_{\mathcal{R}}$ and the Jacobi group $G^J_{\mathcal{R}}$. Let

$$H_{\mathcal{R}} = \{(X;\kappa) |\, X \in \mathcal{R} \times \mathcal{R},\ \kappa \in \mathcal{R}\}$$

with group law $(X_1;\kappa_1) \circ (X_2;\kappa_2) = (X_1 + X_2; \kappa_1 + \kappa_2 + \det\binom{X_1}{X_2})$, further let $\Gamma_{\mathcal{R}}$ be the direct product of the groups $G^+Sp_1(\mathcal{R})$ and $\mathcal{R}^+$, and

$$G^J_{\mathcal{R}} = \left\{ M = \begin{pmatrix} a & 0 & b & \mu' \\ \lambda & \alpha & \mu & \kappa \\ c & 0 & d & -\lambda' \\ 0 & 0 & 0 & \beta \end{pmatrix} \,\middle|\, \begin{array}{l} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G^+Sp_1(\mathcal{R}), \beta > 0 \\ \alpha\beta = \mu(M) = \mu\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \text{ and} \\ (\lambda,\mu;\kappa) \in \mathcal{R}^3 \text{ with } \beta^{-1}(\lambda',\mu')\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (\lambda,\mu) \end{array} \right\}$$

a subgroup of $G^+Sp_2(\mathcal{R})$. The exact sequence

$$1 \longrightarrow H_{\mathcal{R}} \xrightarrow{\ i\ } G^J_{\mathcal{R}} \xrightarrow{\ p\ } \Gamma_{\mathcal{R}} \longrightarrow 1 \tag{32}$$

with $i(\lambda,\mu;\kappa) = \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $p \begin{pmatrix} a & 0 & b & \mu' \\ \lambda & \alpha & \mu & \kappa \\ c & 0 & d & -\lambda' \\ 0 & 0 & 0 & \beta \end{pmatrix} = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \beta\right)$

splits by

$$q : \left\{ \begin{array}{ccc} \Gamma_{\mathcal{R}} & \longrightarrow & G^J_{\mathcal{R}} \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \beta\right) & \mapsto & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} * & 0 \\ 0 & \beta \end{pmatrix} \end{array} \right. .$$

That is the same as to say that $G^J_{\mathcal{R}}$ can be seen as a semi-direct product of $H_{\mathcal{R}}$ and $\Gamma_{\mathcal{R}}$. If $\gamma = (M, \beta)$, then ρ_γ given by $\rho_\gamma(X;\kappa) = (\alpha X M^{-1}; \frac{\alpha}{\beta}\kappa)$ is an automorphism of $H_{\mathcal{R}}$. The semi-direct product $H_{\mathcal{R}} \rtimes \Gamma_{\mathcal{R}}$ is the set $H_{\mathcal{R}} \times \Gamma_{\mathcal{R}}$ with the group law

$$(h,\gamma)\,(\tilde{h},\tilde{\gamma}) := (h\rho_\gamma(\tilde{h}), \gamma\tilde{\gamma}). \tag{33}$$

Proposition 3.1 *The groups $H_{\mathcal{R}} \rtimes \Gamma_{\mathcal{R}}$ and $G^J_{\mathcal{R}}$ are isomorphic by*

$$\phi : \begin{cases} H_{\mathcal{R}} \rtimes \Gamma_{\mathcal{R}} & \longrightarrow \quad G^J_{\mathcal{R}} \\ \left((x_1,x_2;\kappa);\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \beta\right)\right) & \mapsto \begin{pmatrix} a & 0 & b & \beta x_2 \\ y_1 & \alpha & y_2 & \beta\kappa \\ c & 0 & d & -\beta x_1 \\ 0 & 0 & 0 & \beta \end{pmatrix}. \end{cases} \tag{34}$$

It is convenient not to distinguish between image and pre-image of the embeddings i, q and ϕ. Then the group law in $G^J_{\mathcal{R}}$ becomes

$$\begin{aligned} &\Big((X;\kappa);(M;\beta)\Big) \circ \Big((\tilde{X};\tilde{\kappa});(\tilde{M},\tilde{\beta})\Big) \\ = \; &\left(\left(X + \alpha\tilde{X}M^{-1}; \kappa + \frac{\alpha}{\beta}\tilde{\kappa} + \det\begin{pmatrix} X \\ \alpha\tilde{X}M^{-1}\end{pmatrix}\right); (M\tilde{M}, \beta\tilde{\beta})\right) \end{aligned}$$

and for $h = (X;\kappa) \in H_{\mathcal{R}}$ and $\gamma = (M,\beta) \in \Gamma_{\mathcal{R}}$ we have

$$(h;\gamma) = h\,\gamma = \gamma\,(\alpha^{-1}XM; \frac{\beta}{\alpha}\kappa).$$

3.2 Hecke-Jacobi algebra

In the seventies, Andrianov (e.g. [An77b]) recognized that it is possible to factorize the Hecke and Rankin polynomials in a convenient way, if one embeds the Hecke algebra $\mathcal{H}^n$ in a Hecke algebra belonging to the Siegel parabolic $C_{n,0}$ of Sp_n. This embedding has the advantage that one can see the action of $\mathcal{H}^n$ on M^k_n directly on the Fourier coefficients of the underlying modular form. So it is rather obvious that one should exchange, in the proceeding of Andrianov, the Siegel parabolic by another maximal parabolic, if one is interested in analyzing the arithmetic properties of the Fourier-Jacobi coefficients of a Siegel eigenform. This has been done by Gritsenko ([Gr82],[Gr88],[Gr95]) in some special cases. We want to recall some of his results by stressing the underlying general structures already developed by Andrianov and modify and generalize them for our purposes.

Let us now introduce a Hecke theory to the Hecke pair

$$\left(\Gamma^J := C^+_{2,1}(\mathbb{Z}),\, G^J := G^J_{\mathbb{Q}}\right).$$

That this is really a Hecke pair can be proven by a slight modification of the proof of Prop 1.1 in [S-Z89]. We call the corresponding Hecke algebra $\mathcal{H}^J$ the Hecke-Jacobi algebra and the local Hecke algebra $\mathcal{H}_p^J$ associated to the pair $(\Gamma^J, G_p^J = G^J_{\mathbb{Z}[p^{-1}]})$ the local Hecke-Jacobi algebra. It is an important question how a double coset decomposes into left cosets. The following result gives a first answer.

Lemma 3.2

a) Let r be a positive rational number and $g \in G^J$. Then

$$\begin{aligned}\Gamma^J([r,r;r,r])\,\Gamma^J &= \Gamma^J([r,r;r,r]) \ \textit{and}\\ \Gamma^J g\,\Gamma^J \circ \Gamma^J([r,r;r,r])\,\Gamma^J &= \Gamma^J g([r,r;r,r])\,\Gamma^J.\end{aligned}$$

b) Let a, d, α, β be natural numbers with $a \mid d$ and $ad = \alpha\beta$. Then

$$\bigsqcup_j \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \beta \right) \quad \bigsqcup_{\substack{\lambda \ mod\ a/(a,\alpha) \\ \mu \ mod\ d/(d,\alpha) \\ \kappa \ mod\ \beta/(\beta,\alpha)}} (\lambda, \mu; \kappa)\ (U_j \times 1_2) \tag{35}$$

is a representative system of $\Gamma^J \backslash \Gamma^J \left(\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right), \beta \right) \Gamma^J$. Here U_j runs through a representative system of $(\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)^{-1} \Gamma \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \bigcap \Gamma) \backslash \Gamma$.

The first condition is clear and the second can easily be proven (cf. [Gr82], Lemma 3.1).

By Lemma 1.1, it is easy to see that

$$i \,:\, \mathcal{H}^2 \hookrightarrow \mathcal{H}^J; \quad \sum_j a_j \Gamma_2\, g_j \mapsto \sum_j a_j \Gamma^J g_j$$

gives an embedding if one chooses $g_j \in \Gamma^J$. This map commutes with the anti-automorphism $*$ of Γ^J obtained by

$$\Gamma^J\,(h;(M,\beta))\,\Gamma^J \mapsto \left(\Gamma^J\,(h;(M,\beta))\,\Gamma^J\right)^* = \Gamma^J \mu(M)\,(h;(M,\beta))^{-1}\,\Gamma^J. \tag{36}$$

With (35) in Lemma 3.2, one sees that

$$j_- : \left\{ \begin{array}{ccc} \mathcal{H}^1 & \longrightarrow & \mathcal{H}^J \\ \Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma & \mapsto & \Gamma^J \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}; 1 \right) \Gamma^J \end{array} \right. \tag{37}$$

$$j_+ : \left\{ \begin{array}{ccc} \mathcal{H}^1 & \longrightarrow & \mathcal{H}^J \\ \Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma & \mapsto & \Gamma^J \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}; ad \right) \Gamma^J \end{array} \right. \tag{38}$$

lead to two algebra monomorphisms and that $j_+(X) = (j_-(X))^*$, if $X \in \mathcal{H}^1$. We denote the images of these maps by $\mathcal{H}^J_-$ and $\mathcal{H}^J_+$. One has $(\mathcal{H}^J_-)^* = \mathcal{H}^J_+$ and all elements of $\mathcal{H}^2$ are fixed by $*$. By these constructions we have distinguished certain subalgebras $\mathcal{H}^J_-, \mathcal{H}^J_+$ and $i(\mathcal{H}^2)$ of $\mathcal{H}^J$, which are commutative, have no zero divisors and decompose in their local components.

Next we want to introduce some important elements of $\mathcal{H}^J$. Let $r \in \mathbb{Q}$ and $n \in \mathbb{N}$:

$$\begin{aligned}
T^g_-(n) &:= j_-(T(n)), & T^g_+(n) &:= j_+(T(n)),\\
\Lambda^g_-(n) &:= j_-(T(n,n)), & \Lambda^g_+(n) &:= j_+(T(n,n)),\\
\nabla(r) &:= \Gamma^J\left((0;r);(1_2,1)\right)\Gamma^J, & \Delta_n &:= \Gamma^J[n,n;n,n]\Gamma^J,\\
\nabla^g_n &:= \Delta_n \sum_{b \bmod n} \nabla(\tfrac{b}{n}), & \Xi^g_n &:= \Delta_n \sum_{\lambda,\mu,\kappa \bmod n} \Gamma^J(\tfrac{\lambda}{n},\tfrac{\mu}{n};\tfrac{\kappa}{n}),\\
T^{J,g}(n) &:= \Gamma^J[1,n;n^2,n]\Gamma^J.
\end{aligned}$$

For $T(a,b) \in \mathcal{H}^1$ with $a, b \in \mathbb{N}$, we put $T^g_-(a,b) = \Gamma^J[a,ab;b,1]\Gamma^J$ and $T^g_+(a,b) = \Gamma^J[a,1;b,ab]\Gamma^J$. If $ab = n^2$ with $n \in \mathbb{N}$, we put $T^g_0(a,b) = \Gamma^J[a,n;b,n]\Gamma^J$. The double cosets $\nabla(a)$ and Δ_n decompose into only one left coset and Ξ^g_n, ∇^g_n, Δ_n and $T^{J,g}(n)$ are fixed by $*$. Furthermore, we have $(T^g_-(n))^* = T^g_+(n)$ and $(\Lambda^g_-(n))^* = \Lambda^g_+(n)$. Further we put $\nabla^r_n = (\Delta_n)^{-1}\nabla^g_n$, $\Xi^r_n = (\Delta_n)^{-1}\Xi^g_n$ and $T^{J,r}(n) = (\Delta_n)^{-1}T^{J,g}(n)$ (as a rule, we do this only when it is very convenient). By abuse of notation we put Δ_n for $\Gamma_2[n,n;n,n]\Gamma_2$ and $\Gamma^J[n,n;n,n]\Gamma^J$.

Remark 3.3 *(1) The Hecke-Jacobi algebra $\mathcal{H}^J$ is not commutative and has zero divisors, because $\Lambda^g_-(p)\cdot(\nabla^r_p - p) = 0$ and $\Lambda^g_-(p)T^g_+(p) \neq T^g_+(p)\Lambda^g_-(p)$.*

(2) It is not clear how one can decompose $\mathcal{H}^J$ into its local components. Let for example p, q be prime numbers, then

$$\nabla(\frac{1}{q})\nabla(\frac{1}{p}) = \nabla(\frac{p+q}{pq}),$$

and if we assume that $p^2 \not\equiv 1(q)$ we have $\Lambda^g_-(p)\,\nabla(\frac{1}{q}) \neq \nabla(\frac{1}{q})\,\Lambda^g_-(p)$.

The following proposition indicates how one can get a local-global result.

Proposition 3.4 *The elements T^g_-, T^g_+, $T^{J,g}$, ∇^g, Ξ^g, Λ^g_-, Λ^g_+ and Δ in $\mathcal{H}^J$ commute with each other when we only allow coprime arguments. For Λ^g_-, Λ^g_+ and Δ, the assumption on the coprimeness can be dropped.*

Proof:
Case by case. To indicate how one can do that, we prove that

$$T_-^g(n)T_+^g(m) = T_+^g(m)T_-^g(n) \qquad \text{for } (n,m)=1.$$

Since $\mathcal{H}^1 = \otimes_p \mathcal{H}_p^1$ and $\mathcal{H}_p^1$ is generated by $T(p), T(p,p)$ and $T(p,p)^{-1}$, it is enough to show that $T_-^g(p)T_+^g(q) = T_+^g(q)T_-^g(p)$ for two different prime numbers p and q. But this can be deduced directly from Lemma 3.2.

$\square$

Corollary 3.5 *If we denote by* $\tilde{\mathcal{H}}^J$ *the subalgebra of* $\mathcal{H}^J$ *generated by* T_-^g, T_+^g, T^{Jg}, ∇^g Ξ^g, Λ_-^g, Λ_+^g, Δ *and if we put* $\tilde{\mathcal{H}}_p^J = \mathcal{H}_p^J \bigcap \tilde{\mathcal{H}}^J$, *then we have*

$$\tilde{\mathcal{H}}^J = \otimes_p \tilde{\mathcal{H}}_p^J. \tag{39}$$

3.3 The Hecke algebra $\tilde{\mathcal{H}}_p^J$

First we want to show that $\mathcal{H}^2$ can be seen as a subalgebra of $\tilde{\mathcal{H}}^J$. To do that, it is enough to determine the image of the generators of $\mathcal{H}_p^2$ under the embedding i. A slight modification of Lemma 3.3.32 in [An87] gives us

$$\begin{aligned}
i(T^{(2)}(p)) &= T_-^g(p) + T_+^g(p) \\
i(T_1^{(2)}(p^2)) &= \Lambda_-^g(p) + \Lambda_+^g(p) + T^{J,g}(p) + \nabla_p^g - \Delta_p \\
i(T_0^{(2)}(p^2)) &= T_-^g(1,p^2) + T_+^g(1,p^2) + T^{J,g}(p)(\nabla_p^r - 1) + \Xi_p^g - \nabla_p^g \\
i(\Delta_p^{(2)}) &= \Delta_p,
\end{aligned}$$

which proves our claim. To simplify our notation we put $T_-^g, T_+^g, \Lambda_-^g, \Lambda_+^g, T^{J,g}$ for $T_-^g(p), T_+^g(p), \Lambda_-^g(p), \Lambda_+^g(p), T^{J,g}(p)$, if we are in a fixed local situation.
Gritsenko ([Gr95], Abschnitt 5.1) introduced the following very convenient invariants. Let $X = \Gamma^J(h;(M,\beta))\Gamma^J \in \tilde{\mathcal{H}}_p^J$, then we denote by $Sig(X) = \frac{\beta^2}{\mu(M)}$ the signature of X and for $r \in \mathbb{Q}^\times$ and $Y = \sum_i a_i X_i \in \tilde{\mathcal{H}}_p^J$ we put $Sig_r(Y) = \sum_{i, Sig_(X_i)=r} a_i X_i$. For $M_1, M_2 \in G^J$ we have

$$Sig\left(\Gamma^J M_1 \Gamma^J \circ \Gamma^J M_2 \Gamma^J\right) = Sig\left(\Gamma^J M_1 \Gamma^J\right) \cdot Sig\left(\Gamma^J M_2 \Gamma^J\right) \tag{40}$$

and for $X = \sum_u Sig_u(X)$ and $X = \sum_v Sig_v(X)$ we have

$$\begin{aligned}
Sig_r(XY) &= \sum_{uv=r} Sig_u(X) Sig_v(Y) &\quad (41)\\
&= \sum_{uv=r} Sig_v(Y) Sig_u(X). &\quad (42)
\end{aligned}$$

Applying this for example to the case $X = T^{(2)}(p)$ and $Y = T_1^{(2)}(p^2)$, leads us to

$$\begin{aligned}
T_+^g\Lambda_+^g &= \Lambda_+^g T_+^g \\
T_-^g\Lambda_+^g + T_+^g T^{J,g} + T_+^g(\nabla_p^g + \Delta_p) &= \Lambda_+^g T_-^g + T^{J,g}T_+^g + (\nabla_p^g + \Delta_p)T_+^g \\
T_+^g\Lambda_-^g + T_-^g T^{J,g} + T_-^g(\nabla_p^g + \Delta_p) &= \Lambda_-^g T_+^g + T^{J,g}T_-^g + (\nabla_p^g + \Delta_p)T_-^g \\
T_-^g\Lambda_-^g &= \Lambda_-^g T_-^g.
\end{aligned}$$

Such identities can be used to determine commutators in $\tilde{\mathcal{H}}_p^J$. Next, we want to simplify terms of the form $T_-^g\Lambda_+^g$, $\Lambda_-^g\Lambda_+^g$ and so on.

Lemma 3.6 *The following relations in $\tilde{\mathcal{H}}_p^J$ hold:*

$$\begin{aligned}
T_-^g\Lambda_+^g &= p^2\Delta_p T_+^g & (43)\\
\Lambda_-^g T_+^g &= p^2\Delta_p T_-^g & (44)\\
\Lambda_-^g\Lambda_+^g &= p^4(\Delta_p)^2 & (45)\\
T_-^g T_+^g &= pT^{J,g} + (p^2+p^3)\Delta_p. & (46)
\end{aligned}$$

Proof:
By Lemma 3.2, we have

$$\begin{aligned}
T_-^g &= \Gamma^J[1,p;p,1]\left(U_{[1,p]}\times 1_2\right)\\
\Lambda_-^g &= \Gamma^J[p,p^2;p,1]\\
T_+^g &= \Gamma^J[1,1;p,p]\sum_{\mu,\kappa\, mod\, p}(0,\mu;\kappa)\left(U_{[1,p]}\times 1_2\right)\\
\Lambda_+^g &= \Gamma^J[p,1;p,p^2]\sum_{\substack{\lambda,\mu\, mod\, p\\ \kappa\, mod\, p^2}}(\lambda,\mu;\kappa).
\end{aligned}$$

This leads to

$$\begin{aligned}
T_-^g\Lambda_+^g &= p^{-2}\Gamma^J[p,p;p^2,p^2]\left(U_{[1,p]}\times 1_2\right)\sum_{\lambda,\mu,\kappa\, mod\, p^2}(\lambda,\mu;\kappa) & (47)\\
&= p^{-2}\Delta_p\Gamma^J[1,1;p,p]\sum_{\lambda,\mu,\kappa\, mod\, p^2}(\lambda,\mu;\kappa)\left(U_{[1,p]}\times 1_2\right) & (48)
\end{aligned}$$

which proves (43). Here we used that $GL_2(\mathbb{Z})$ normalizes the Heisenberg group $H_{\mathbb{Z}/p^2\mathbb{Z}}$. If we apply the anti-automorphism $*$ to equation (43) we get (44).

It is easy to see that

$$\begin{aligned}
\Lambda^g_-\Lambda^g_+ &= \Gamma^J\,[p,p^2;p,1][p,1;p,p^2]\sum_{\substack{\lambda,\mu \, mod\, p\\ \kappa\, mod\, p^2}}(\lambda,\mu;\kappa)\\
&= \Gamma^J[p^2,p^2;p^2,p^2]\sum_{\substack{\lambda,\mu \, mod\, p\\ \kappa\, mod\, p^2}}(\lambda,\mu;\kappa)\\
&= p^4(\Delta_p)^2.
\end{aligned}$$

One can prove the last assumption of the lemma as follows:

$$\begin{aligned}
T^g_-T^g_+ &= \Big(\Gamma^J[1,p;p,1](U_{[1,p]}\times 1_2)\Big)\Big(\Gamma^J[1,1;p,p]\sum_{\mu,\kappa\, mod\, p}(0,\mu;\kappa)(U_{[1,p]}\times 1_2)\Big)\\
&= p^{-1}\Gamma^J\left((T^{(1)}(p)\cdot T^{(1)}(p))\times[p;p]\right)\sum_{\lambda,\mu,\kappa\, mod\, p}(0,\mu;\kappa)\\
&= p^{-1}\Gamma^J\left(\big(T^{(1)}(1,p^2)+T^{(1)}(p,p)+pT^{(1)}(p,p)\big)\times[p;p]\right)\sum_{\mu,\kappa\, mod\, p}(0,\mu;\kappa)\\
&= p^{-1}\Gamma^J\left(([1,p^2]U_{[1,p^2]})\times[p;p]\right)\left(\sum_{\mu,\kappa\, mod\, p}(0,\mu;\kappa)\right)\\
&\quad +(p^2+p^3)\Delta_p
\end{aligned}$$

which gives the desired result.

$\square$

We hope to have given the reader an impression of how one can prove such statements. The following list contains relations between the generators of $\tilde{\mathcal{H}}^J_p$. At the (i,j) position we have put the product of the corresponding row and column elements. With the subscript we want to indicate, which kind of relation is listed in the second chart. In the left column the position, in the middle one the relation and in the right eventually a reference is given. The number in the reference indicates another equation, which is the same as the given one. If the number comes with $*$, one has to apply $*$ to the corresponding equation. Finally, 15^m indicates a connection with equation 15.

$\circ$	T^g_-	Λ^g_-	T^g_+	Λ^g_+	$T^{J,g}$	∇^g_p	Ξ^g_p
T^g_-	11_F	12_F	$13_\surd$	$14_\surd$	$15_{[]}$	$16_\surd$	17_U
Λ^g_-	21_F	22_F	$23_\surd$	$24_\surd$	$25_\surd$	$26_\surd$	$27_\surd$
T^g_+	31_U	32_U	33_F	34_F	35_U	$36_?$	$37_?$
Λ^g_+	41_U	$42_\surd$	43_F	44_F	$45_?$	$46_?$	$47_?$
$T^{J,g}$	$51_{[]}$	$52_?$	$53_{[]}$	$54_\surd$	55_F	$56_?$	$57_?$
∇^g_p	$61_?$	$62_?$	$63_\surd$	$64_\surd$	$65_?$	66_F	$67_\surd$
Ξ^g_p	$71_?$	$72_?$	73_U	$74_\surd$	$75_?$	$76_\surd$	77_U

$\surd$	simplification
F	formula
U	easy transformation
$[\,]$	essentially the commutator (49)
$?$	no real simplification

$\boxed{11}$	$T^g_- T^g_-$	$=$	$T_-(p^2) + p\Lambda^g_-$	
$\boxed{12}$	$T^g_- \Lambda^g_-$	$=$	$T^g_-(p, p^2)$	
$\boxed{13}$	$T^g_- T^g_+$	$=$	$pT^{J,g} + (p^2 + p^3)\Delta_p$	
$\boxed{14}$	$T^g_- \Lambda^g_+$	$=$	$p^2 \Delta_p T^g_+$	
$\boxed{15}$	$T^g_- T^{J,g}$	$=$	$p(p-1)\Delta_p T^g_- + \nabla^g_p T^g_- - T^g_+ \Lambda^g_- + T^{J,g} T^g_-$	
$\boxed{16}$	$T^g_- \nabla^g_p$	$=$	$pT^g_- \Delta^g_p$	
$\boxed{17}$	$T^g_- \Xi^g_p$	$=$	$p^2(\Delta_p - p^{-1}\nabla^g_p)T^g_- + pT^g_+ \Lambda^g_-$	
$\boxed{21}$	$\Lambda^g_- T^g_-$	$=$	$T^g_-(p, p^2)$	12
$\boxed{22}$	$\Lambda^g_- \Lambda^g_-$	$=$	$\Lambda^g_-(p^2)$	
$\boxed{23}$	$\Lambda^g_- T^g_+$	$=$	$p^2 \Delta_p T^g_-$	14^*
$\boxed{24}$	$\Lambda^g_- \Lambda^g_+$	$=$	$p^4(\Delta_p)^2$	
$\boxed{25}$	$\Lambda^g_- T^{J,g}$	$=$	$p\Delta_p T^g_-(1, p^2)$	
$\boxed{26}$	$\Lambda^g_- \nabla^g_p$	$=$	$p\Delta_p \Lambda^g_-$	
$\boxed{27}$	$\Lambda^g_- \Xi^g_p$	$=$	$p^3 \Delta_p \Lambda^g_-$	
$\boxed{31}$	$T^g_+ T^g_-$	$=$	$\Delta_p^{-1} T^{J,g} \nabla^g_p + p\nabla^g + \Xi^g_p$	
$\boxed{32}$	$T^g_+ \Lambda^g_-$	$=$	$p^{-1} T^g_- \Xi^g_p - (p\Delta_p - \nabla^g_p)T^g_-$	
$\boxed{33}$	$T^g_+ T^g_+$	$=$	$T^g_+(p^2) + p\Lambda^g_+$	11^*
$\boxed{34}$	$T^g_+ \Lambda^g_+$	$=$	$T^g_+(p, p^2)$	12^*
$\boxed{35}$	$T^g_+ T^{J,g}$	$=$	$p(1-p)\Delta_p T^g_+ - T^g_+ \nabla^g_p + \Lambda^g_+ T^g_- + T^{J,g} T^g_+$	15^m
$\boxed{36}$	$T^g_+ \nabla^g_p$	$=$	$---$	
$\boxed{37}$	$T^g_+ \Xi^g_p$	$=$	$---$	
$\boxed{41}$	$\Lambda^g_+ T^g_-$	$=$	$p^{-1} \Xi^g_p T^g_+ - T^g_+(p\Delta_p - \nabla^g_p)$	32^*
$\boxed{42}$	$\Lambda^g_+ \Lambda^g_-$	$=$	$p^{-1} \Delta_p^{-1} \nabla^g_{p^2} \Xi^g_p$	
$\boxed{43}$	$\Lambda^g_+ T^g_+$	$=$	$T^g_+(p, p^2)$	34
$\boxed{44}$	$\Lambda^g_+ \Lambda^g_+$	$=$	$\Lambda^g_+(p^2)$	22^*
$\boxed{45}$	$\Lambda^g_+ T^{J,g}$	$=$	$---$	
$\boxed{46}$	$\Lambda^g_+ \nabla^g_p$	$=$	$---$	
$\boxed{47}$	$\Lambda^g_+ \Xi^g_p$	$=$	$---$	

$\boxed{51}$	$T^{J,g}T^g_-$	$=$	$T^g_+\Lambda^g_- - p(p-1)\Delta_p T^g_- - \nabla^g_p T^g_- + T^g_- T^{J,g}$	35^*
$\boxed{52}$	$T^{J,g}\Lambda^g_-$	$=$	$---$	45^*
$\boxed{53}$	$T^{J,g}T^g_+$	$=$	$p(p-1)\Delta_p T^g_+\nabla^g_p - \Lambda^g_+ T^G_- + T^g_+ T^{J,g}$	15^*
$\boxed{54}$	$T^{J,g}\Lambda^g_+$	$=$	$p\Delta_p T^g_+(1,p^2)$	25^*
$\boxed{55}$	$T^{J,g}T^{J,g}$	$=$	$T^{J,g}(p^2) + (p-1)\nabla^g_p T^{J,g}(p) + p\Delta_p(\Xi^g_p - \nabla^g_p + p^2 + p)$	
$\boxed{56}$	$T^{J,g}\nabla^g_p$	$=$	$---$	
$\boxed{57}$	$T^{J,g}\Xi^g_p$	$=$	$---$	
$\boxed{61}$	$\nabla^g_p T^g_-$	$=$	$---$	36^*
$\boxed{62}$	$\nabla^g_p\Lambda^g_-$	$=$	$---$	46^*
$\boxed{63}$	$\nabla^g_p T^g_+$	$=$	$p\Delta_p T^g_+$	16^*
$\boxed{64}$	$\nabla^g_p\Lambda^g_+$	$=$	$p\Delta_p\Lambda^g_+$	26^*
$\boxed{65}$	$\nabla^g_p T^{J,g}$	$=$	$---$	56
$\boxed{66}$	$\nabla^g_p\nabla^g_p$	$=$	$\nabla^g_{p^2}$	
$\boxed{67}$	$\nabla^g_p\Xi^g_p$	$=$	$p\Delta_p\Xi^g_p$	
$\boxed{71}$	$\Xi^g_p T^g_-$	$=$	$---$	37^*
$\boxed{72}$	$\Xi^g_p\Lambda^g_-$	$=$	$---$	47^*
$\boxed{73}$	$\Xi^g_p T^g_+$	$=$	$p^2 T^g_+(\Delta_p - p^{-1}\nabla^g_p) + p\Lambda^g_+ T^g_-$	17^*
$\boxed{74}$	$\Xi^g_p\Lambda^g_+$	$=$	$p^3\Delta_p\Lambda^g_+$	27^*
$\boxed{75}$	$\Xi^g_p T^{J,g}$	$=$	$---$	57^*
$\boxed{76}$	$\Xi^g_p\nabla^g_p$	$=$	$p\Delta_p\Xi^g_p$	67^*
$\boxed{77}$	$\Xi^g_p\Xi^g_p$	$=$	$p\Delta_p^{-1}\Xi^g_p\nabla^g_{p^2} + p^3\Delta_p\nabla^g_p - p^2\nabla^g_{p^2}$	

Now we are able to state and prove the main results of this section, the announced factorization of the Hecke and Rankin polynomials $q_p^2(X)$ and $d_p^2(X)$. It is tricky to construct such decompositions, in contrast it is easy to prove the correctness of them. A good overview over this topic is given in [An87], §3.5. For a proof of the next theorem the reader is referred to [Gr93].

Theorem 3.7 *(Gritsenko) The Hecke polynomial $q_p^2(X)$ has the following factorization in $\tilde{\mathcal{H}}_p^J[X]$:*

$$q_p^2(X) = j_-\left(q_p^1(X)\right)\left(1 + p\Delta_p(\nabla_p^r - p)X^2\right) j_+\left(q_p^1(X)\right)$$

with $q_p^1(X) = 1 - T^{(1)}(p)X + p\Delta_p^{(1)}X^2$.

Next, we want to present a new factorization of the Rankin polynomial. It is an unpleasant fact, that the suggested formula in [Gr88], Theorem 3, is not correct.

Theorem 3.8 *The Rankin polynomial $d_p^2(X)$ has the following factorization in $\tilde{\mathcal{H}}_p^J[X]$:*

$$d_p^2(X) = (1 - X)\,(1 - p^{-2}\Lambda_-^r X)\,S^{(2)}(X)\,(1 - p^{-2}\Lambda_+^r X) \tag{50}$$

with $S^{(2)}(X) = \sum_{j=0}^{3} (-1)^j S_j^{(2)} X^j$. *Here*

$$\begin{aligned} S_0^{(2)} &= 1 \\ S_1^{(2)} &= p^{-2}\left(T^{J,r} + \nabla_p^r - p^2\right) \\ S_2^{(2)} &= p^{-3}\left(T^{J,r}(\nabla_p^r - p) + \Xi^r - p\nabla_p^r + p^2\right) \\ S_3^{(2)} &= p^{-2}\left(\nabla_p^r - p\right). \end{aligned}$$

Proof:
Most of the work has already been done. We multiply the right side of the equation (50) out and get a polynomial of degree 5. With the aid of our charts, the stated formulas given in section 1.2 and at the beginning of section 3.3, we see that this polynomial is nothing else but the Rankin polynomial of degree 4.

□

3.4 Jacobi forms as P-forms

We begin with a very convenient definition.

Definition 3.9 *Let n, s and k be natural numbers. A P-Jacobi form of degree (n, s) and weight k is a holomorphic function F on $\mathbb{H}_{n+s}$, which satisfies*

(i) $F|_k\gamma = F$ *for all* $\gamma \in C^+_{n+s,n}(\mathbb{Z})$

(ii) There exists $t \in A_s$ *such that* F *has the Fourier expansion*

$$F(Z) = \sum_{T=\left(\begin{smallmatrix} * & * \\ * & t \end{smallmatrix}\right) \in A_{n+s}} a(T)\, e\{TZ\}. \tag{51}$$

We call such t the index of F. A P-Jacobi form is called P-Jacobi cusp form, (if the index is positive definite and) if the summation in (51) may be restricted to positive definite matrices. The space which is generated by all P-Jacobi forms (P-Jacobi cusp forms) of degree (n, s) and weight k is denoted by the space of P-forms $\mathcal{PJ}^k_{n,s}$ (space of P-cusp forms $\mathcal{PJ}^{k,cusp}_{n,s}$). Their elements are called P-forms and P-cusp forms respectively. Finally we denote by $\mathcal{PJ}^k_{n,s}(t)$ the subspace of P-Jacobi forms and by $\mathcal{PJ}^{k,cusp}_{n,s}(t)$ of P-Jacobi cusp forms for fixed index t. We use the convention $\mathcal{PJ}^{k,cusp}_{n,s}(t) = \{0\}$, if t is not half-integral.

To introduce the Petersson scalar product for P-forms, we put $\mathbb{C}^{n,s} = \mathrm{Mat}(n, s; \mathbb{C})$, $\mathbb{H}_{n,s} = \mathbb{H}_n \times \mathbb{C}^{n,s}$, $\Gamma^J_{n,s} = C^+_{n+s,n}(\mathbb{Z})$ and $\Gamma^{J,red}_{n,s} = \Gamma^J_{n,s}/Z^{n,s}$, where $Z^{n,s}$ denotes the center of

$\Gamma^J_{n,s}$. Let $\gamma \in \Gamma^J_{n,s}$, then

$$\gamma < \begin{pmatrix} z_1 & z_2 \\ * & * \end{pmatrix} >= \begin{pmatrix} z_1' & z_2' \\ * & * \end{pmatrix}$$

induces the operation $(z_1, z_2) \longmapsto \gamma < z_1, z_2 >= (z_1', z_2')$ of $\Gamma^J_{n,s}$ on $I\!H_{n,s}$, which is in fact an action of $\Gamma^{J,red}_{n,s}$. Then

$$\mathcal{Q}_{n,s} = \left\{ Z \in I\!H_{n+s} \,|\, (z_1, z_2) \in \mathcal{B}_{n,s} \text{ and } |\mathrm{Re}(z_4)_{ij}| \leq \tfrac{1}{2} \right\}$$

is a fundamental domain of the action of $\Gamma^J_{n,s}$ on $I\!H_{n+s}$, if $\mathcal{B}_{n,s}$ is a fundamental domain of the action of $\Gamma^{J,red}_{n,s}$ on $I\!H_{n,s}$. Every $F \in \mathcal{PJ}^k_{n,s}(t)$ has a decomposition of the form $F(Z) = \phi^F(z_1, z_2)\, e\{tz_4\}$.

Definition 3.10 *Let $F, G \in \mathcal{PJ}^k_{n,s}$. Then we denote by the value of*

$$\langle F, G\rangle_{\mathcal{A}} = \int_{Q_{n,s}} F(Z)\overline{G(Z)}\; (\det Y)^k \, d^*Z \tag{52}$$

the Petersson scalar product of F and G for $d^*Z = dX\,dY/(\det y)^{n+s+1}$, if it exists.

Remark 3.11

(i) The space of Jacobi forms of degree (n, s), weight k and index t is given by

$$\mathcal{J}^k_{n,s}(t) = \{\phi : I\!H_n \times \mathbb{C}^{n,s} \longrightarrow \mathbb{C} \,|\, \tilde{\phi}_t(Z) := \phi(z_1, z_2)\, e\{tz_4\} \in \mathcal{PJ}^k_{n,s}\}. \tag{53}$$

In the same way one defines the space of Jacobi cusp forms $\mathcal{J}^{k,cusp}_{n,s}(t)$ of index t, and more generally the spaces $\mathcal{J}^k_{n,s}$ and $\mathcal{J}^{k,cusp}_{n,s}$.

(ii) Let $F, G \in \mathcal{PJ}^{k,cusp}_{n,s}(t)$ with $k > n + s$ and $y_i =$Im(z_i). Then

$$\begin{aligned} \langle F, G\rangle_{\mathcal{A}} &= \beta_{s,k-n}\, (\det t)^{\frac{s+1}{2}+n-k} \\ &\quad \int_{\mathcal{B}_{n,s}} \phi^F(z_1, z_2)\, \overline{\phi^G(z_1, z_2)} e^{-4\pi \mathrm{tr}(ty_1^{-1}[y_2])} (\det y_1)^k \, d^*Z_{n,s}\,. \end{aligned}$$

Here $\beta_{s,k-n} = \pi^{\frac{s(s-1)}{4}} (4\pi)^{\frac{s(s+1)}{2} - s(k-n)} \prod_{\nu=1}^s \Gamma(k - n - \frac{s+\nu}{2})$ and $d^*Z_{n,s} = dz_1\, dz_2/(\det y_1)^{n+s+1}$ (cf. Klingen [Kl89] Satz 3, Dulinski [Du95]).

Let $\phi, \psi \in \mathcal{J}^{k,cusp}_{n,s}(t)$ and $k > n + s$, then we get the Petersson scalar product $\langle\phi, \psi\rangle_{J,t}$ for Jacobi forms by

$$\langle\tilde{\phi}; \tilde{\psi}\rangle_{\mathcal{A}} = \beta_{s,k-n}\, (\det t)^{\frac{s+1}{2}+n-k} \langle\phi, \psi\rangle_{J,t}\,.$$

Let us fix in the case $s = n = 1$ the following notation.

For $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in I\!H_2$, we put $\begin{pmatrix} \tau_x & z_x \\ z_x & \tau'_x \end{pmatrix} =$ Re(Z) and $\begin{pmatrix} \tau_y & z_y \\ z_y & \tau'_y \end{pmatrix} =$ Im(Z).

Let $F, G \in \mathcal{PJ}_{1,1}^{k,cusp}(t)$, $k > 2$ and $\beta_k = (4\pi)^{k-2}\Gamma(k-2)^{-1}$, then

$$\langle \phi^F, \phi^G \rangle_{J,t} = \beta_k t^{k-2} \langle F, G \rangle_{\mathcal{A}}. \tag{54}$$

If $F : I\!H_2 \longrightarrow \mathbb{C}$ is invariant under the $|_k$-action of Γ^J, then for $X = \sum a_j \Gamma^J g_j \in \tilde{\mathcal{H}}^J$

$$F|_k \mathcal{D}(X) := \sum_j a_j \, F|_k g_j \tag{55}$$

has the same property. Let $F(Z) = \sum_{n=1}^{\infty} f_n(\tau, z)\, e\{n\tau'\} \in S_2^k$ and $X \in \mathcal{H}^2$, then (55) is compatible with the action introduced in (13) and the embedding i. More generally, let $X \in \tilde{\mathcal{H}}^J$, then we put $f_m||_k X$ for the m-th Fourier-Jacobi coefficient of $F|_k\mathcal{D}(X)$ resp., τ'. Especially the action in (55) can be seen as an action of $\tilde{\mathcal{H}}^J$ on $\mathcal{PJ}_{1,1}^{k,cusp}$ ($\mathcal{J}_{1,1}^{k,cusp}$). To be more precise, if $X \in \tilde{\mathcal{H}}^J$ is a double coset with signature $r \in \mathbb{Q}^+$ and $F \in \mathcal{PJ}_{1,1}^{k,cusp}(t)$, then $F|_k\mathcal{D}(X) \in \mathcal{PJ}_{1,1}^{k,cusp}(tr)$. By Proposition 4.1.7, page 218 [An87], it is clear that

$$\langle F|_k\mathcal{D}(X); G \rangle_{\mathcal{A}} = \langle F; G|_k\mathcal{D}(X^*) \rangle_{\mathcal{A}}, \tag{56}$$

with $X \in \tilde{\mathcal{H}}^J$ and $F, G \in \mathcal{PJ}_{1,1}^{k,cusp}$. Further, if $\tilde{\phi} \in \mathcal{PJ}_{1,1}^{k,cusp}(t)$ and $(p,t) = 1$, then $\tilde{\phi}|_k\mathcal{D}(X) = 0$, if $X = \nabla_n^g, \Xi_n^g, T_+^g(n)$ or $\Lambda_+^g(n)$.

Finally, we are able to compute the adjoint operators of the classical Hecke-Jacobi operators (e.g. U_l, V_l, T_l) introduced in [E-Z85], §4 in a rather simple way. For T_p, we put T_p^{EZ}. Let A be a linear operator on $\mathcal{J}_{1,1}^{k,cusp}$, then we denote by $\tilde{A}$ the corresponding operator on $\mathcal{PJ}_{1,1}^{k,cusp}$, further by A^{ad} resp., $(\tilde{A})^{ad}$ the adjoint operators. Assuming that $\tilde{A}$ has signature r, we have $\widetilde{A^{ad}} = r^{k-2}(\tilde{A})^{ad}$. Then the dictionary

$$\begin{array}{rcl} \mathcal{D}(\Lambda_-^g(l)) & = & l^{3(k-2)}\tilde{U}_l \\ \mathcal{D}(T_-^g(l)) & = & l^{k-2}\tilde{V}_l \\ \mathcal{D}(T^{J,g}(p)) & = & p^{3-k}\widetilde{T_p^{EZ}} \end{array} \quad \text{leads to} \quad \begin{array}{rcl} \widetilde{U_l^{ad}} & = & l^{2-k}\mathcal{D}(\Lambda_+^g(l)) \\ \widetilde{V_l^{ad}} & = & \mathcal{D}(T_+^g(l)) \\ \widetilde{(T_p^{EZ})^{ad}} & = & \widetilde{T_p^{EZ}} \end{array} . \tag{57}$$

The following notation and observation will turn out to be convenient later on. If we put

$$S^{(2)}(X)^{factor} = 1 - p^{-2}T^{J,r}(p)X + p^{-1}X^2 \tag{58}$$

$$\begin{aligned} S^{(2)}(X)^{prim} &= 1 - (p^{-2}T^{J,r}(p) - 1)X + (p^{-1} - p^{-2})T^{J,r}(p)X^2 + p^{-1}X^3 \\ &= S^{(2)}(X)^{factor}\,(1 + X) \end{aligned} \tag{59}$$

$$S^{(2)}(X)^{n.prim} = 1 - (p^{-2}T^{J,r}(p) + p^{-1} - 1)X + p^{-3}\Xi_p^r X^2, \tag{60}$$

then $\mathcal{D}(S^{(2)}(X)) = \mathcal{D}(S^{(2)}(X)^{prim})$ on $\mathcal{PJ}_{1,1}^{k,cusp}(t)$, if $(p,t) = 1$ and $\mathcal{D}(S^{(2)}(X)) = \mathcal{D}(S^{(2)}(X)^{n.prim})$ otherwise. Further we have for all $F \in \mathcal{PJ}_{1,1}^{k,cusp}(p)$:

$$F|_k\mathcal{D}(S^{(2)}(X)T_+^g(p)) = F|_k\mathcal{D}(T_+^g(p)\,(1 - p^{-2}T^{J,r}(p)X + p^{-1}X^2)). \tag{61}$$

3.5 Weak rationality theorems

Let $F \in S_2^k$ be a Hecke eigenform and $(f_n)_{n\in\mathbb{N}}$ the Fourier-Jacobi coefficients of F. Further let $Q_{p,F}^2$ and $D_{p,F}^2$ the corresponding Hecke and Rankin polynomials. Then the following identities hold in the ring of formal power series.

Proposition 3.12 *(Gritsenko) Let $F \in S_2^k$ be a Hecke eigenform and $m \in \mathbb{N}$, then*

$$\begin{aligned} & Q_{p,F}^2(X)\sum_{\delta=0}^{\infty} f_{mp^\delta}|_k\mathcal{D}\left(T_+^g(p^\delta)X^\delta\right) \\ = \quad & \Big(f_m - f_{\frac{m}{p}}|_k\mathcal{D}\big(T_-^g(p)X\big) + pf_{\frac{m}{p^2}}|_k\mathcal{D}\big(\Lambda_-^g(p)X^2\big)\Big)|_k\mathcal{D}\big(1 + p(\nabla_p^r - p)\Delta_p X^2\big) \end{aligned}$$

with $f_m|_k\mathcal{D}\big(1 + p(\nabla_p^r - 1)\Delta_p X^2\big) = \begin{cases} f_m & if\, m\equiv 0(p) \\ (1-p^{2k-4}X^2)f_m & otherwise \end{cases}$.

Proposition 3.13 *Let $F \in S_2^k$ be a Hecke eigenform and $m \in \mathbb{N}$, then*

$$\begin{aligned} & D_{p,F}^2(X)\sum_{\delta=0}^{\infty} f_{mp^{2\delta}}|_k\mathcal{D}\left(\Lambda_+^r(p^\delta)\,(p^{-2}X)^\delta\right) \\ = \quad & (1-X)\,\Big(f_m|_k\mathcal{D}\big(S^{(2)}(X)\big) - f_{\frac{m}{p^2}}|_k\mathcal{D}\big(\Lambda_-^r(p)\,S^{(2)}(X)p^{-2}X\big)\Big). \end{aligned}$$

Theorem 3.7, Theorem 3.8 and results from our charts give the desired results. Also a proof of the Proposition 3.12 is given in [Gr93], the proof of the second one proceeds in the same way. It may seem that the two propositions are of the same type, but that is not the case. Let for example $(m,p) = 1$: then one can describe $f_{mp^\delta}|_k\mathcal{D}(T_+^g(p^\delta))$ in terms of eigenvalues and the function f_m, but this is not the case for $f_{mp^{2\delta}}|_k\mathcal{D}(\Lambda_+^g(p^\delta))$, where one needs to know more about the nature of $f_m|_k\mathcal{D}(T^{J,g}(p))$.

3.6 The Maass Spezialschar

There is a correspondence between certain elliptic modular forms, Jacobi forms and Siegel modular forms of degree 2, which can be described in the language of L-functions. To do that, we recall the definition of a certain subspace of S_2^k.

Definition 3.14 *The Maass Spezialschar $S_2^{k,M}$ of weight k is the space*

$$\left\{F(Z) = \sum_{l=1}^{\infty} l^{2-k}\left(\tilde{\phi}_1|_k\mathcal{D}(T_-^g(l)\right)(Z)\;\Big|\;\tilde{\phi}_1 \in \mathcal{PJ}_{1,1}^{k,cusp}(1)\right\}. \tag{62}$$

It is well known that $S_2^{k,M}$ is a subspace of S_2^k and that it is stable under the action of $\mathcal{H}^2$. For simplicity we have omitted reference to Eisenstein series. But much more is known ([E-Z85] §6):

Theorem 3.15 *(Saito-Kurokawa Conjecture) The space $S_2^{k,M}$ is spanned by Hecke eigenforms. These are in 1-1 correspondence with normalized Hecke eigenforms $f \in S_1^{2k-2}$, the correspondence being such that*

$$Z_F^2(s) = \zeta(s-k+1)\zeta(s-k+2)L_f(s). \tag{63}$$

The degeneration of the spinor zeta function leads to the degeneration of the standard zeta function.

Theorem 3.16 *Let us choose F and f as in Theorem 3.15. Then*

$$D_F^2(s) = \zeta(s)L_f(s+k-1)L_f(s+k-2). \tag{64}$$

Proof:
To prove the latter theorem one has to determine the Satake-parameters $\alpha_{0,p}, \alpha_{1,p}, \alpha_{2,p}$, that is to say, one has to determine the tuple $(\alpha_{1,p}, \alpha_{1,p}^{-1}, \alpha_{2,p}, \alpha_{2,p}^{-1})$, which is unique up to permutations of the entries. If α_p and β_p denote the p-parameters of f, then it turns out, that one can choose $\alpha_{1,p} = \alpha_p p^{1-k}$ and $\alpha_{2,p} = \beta_p p^{1-k}$, which leads to the desired result.

□

Remark 3.17 *There exists a distinguished basis $(F_j)_j$ of $S_2^{k,M}$ with corresponding $f_j \in S_1^{2k-2}(\mathbb{Q}(F_j))$. Let $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, then f_j^σ corresponds to F_j^σ.*

This remark, the explicit form of D_F^2 and Corollary 1.12 lead to

Proposition 3.18 *Let $F \in S_2^{k,M}(\mathbb{Q}(F))$ be a Hecke eigenform with corresponding $f \in S_1^{2k-2}$ and $k > 4$ even, then for all $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ we have*

$$\left(\frac{\pi^{7-4k}L_f(2k-3)\,L_f(2k-4)}{\langle F;F\rangle}\right)^\sigma = \frac{\pi^{7-4k}L_{f^\sigma}(2k-3)\,L_{f^\sigma}(2k-4)}{\langle F^\sigma;F^\sigma\rangle} \in \mathbb{Q}(F). \tag{65}$$

4 Euler product of a Dirichlet series of type (1,2,2)

From now on let $F, G \in S_2^k, h \in S_1^k$ Hecke eigenforms with h normalized and F in the Maass Spezialschar with corresponding $f \in S_1^{2k-2}$. We show that

$$D_{h,G;F}(s) = \sum_{\substack{n,\varepsilon,l=1\\(\varepsilon,l)=1}}^{\infty} \langle \Phi_n^F|_k U_l; \Phi_{nl^2}^G\rangle_J \, a_{n\varepsilon^2}^h \, \varepsilon^{-2(k+s-1)} l^{-2(k+s-2)} n^{-(2k+s-3)} \tag{66}$$

has an Euler product by using the language of P-forms and the Hecke-Jacobi theory developed in the last section.

4.1 Preparation of the Dirichlet series

First interpreting the Fourier-Jacobi coefficients of F and G as P-forms, then using our definition of the Maass Spezialschar and the translation rules in (57), leads to the formula

$$\begin{aligned} D_{h,G;F}(s) &= \beta_k \sum_{\substack{n,\varepsilon,l=1\\(\varepsilon,l)=1}}^{\infty} \langle \tilde{\Phi}_1^F|_k \mathcal{D}\big(T_-^g(n)\,\Lambda_-^g(l)\big);\tilde{\Phi}_{nl^2}^G\rangle_{\mathcal{A}}\, a_{n\varepsilon^2}^h \\ &\qquad \varepsilon^{-2(k+s-1)} l^{-3k-2s+6} n^{-(2k+s-3)}. \end{aligned} \tag{67}$$

Next we put $X = p^{-(s+k-1)}$, $N = p^{k-1}$ and define

$$\begin{aligned} D_{h,G;F}^{(p)}(s) &= \sum_{\substack{n,\varepsilon,l=0\\ min(\varepsilon,l)=0}}^{\infty} \langle \tilde{\Phi}_1^F|_k \mathcal{D}\left(T_-^g(p^n)\,\Lambda_-^g(p^l)\right);\tilde{\Phi}_{p^{n+2l}}^G\rangle_{\mathcal{A}}\, a_{p^{n+2\varepsilon}}^h \\ &\qquad X^{2\varepsilon}\,(X^2N^{-1}p^3)^l\,(pXN^{-1})^n, \end{aligned} \tag{68}$$

which will later lead to $D_{h,G;F}(s) = \beta_k \langle \tilde{\Phi}_1^F;\tilde{\Phi}_1^G\rangle_{\mathcal{A}} \prod_p (D_{h,G;F}^{(p)}(s)/\langle \tilde{\Phi}_1^F;\tilde{\Phi}_1^G\rangle_{\mathcal{A}})$. Our first goal is to calculate $D^{(p)}$ as explicitly as possible. To do that, we decompose $D^{(p)}$ by the condition $\min(\varepsilon,l)=0$ into three sums:

$$D_{h,G;F}^{(p)}(s) = D_{(\varepsilon)}(s) + D_{(l)}(s) - D_{(\varepsilon,l)}(s). \tag{69}$$

Let α and α' be the p-parameters of h determined by $\alpha+\alpha' = a_p^h$ and $\alpha\alpha' = N$. Then

$$\begin{aligned} D_{(\varepsilon)}(s)\,(\alpha-\alpha') &= \alpha \sum_{n,l=0}^{\infty} \langle \tilde{\Phi}_1^F;\tilde{\Phi}_{p^{n+2l}}^G|_k\mathcal{D}\big(T_+^g(p^n)\Lambda_+^g(p^l)\big)\rangle_{\mathcal{A}}(X^2N^{-1}p^3)^l\,(\alpha pXN^{-1})^n \\ &\quad -\alpha' \sum_{n,l=0}^{\infty} \langle \tilde{\Phi}_1^F;\tilde{\Phi}_{p^{n+2l}}^G|_k\mathcal{D}\big(T_+^g(p^n)\Lambda_+^g(p^l)\big)\rangle_{\mathcal{A}}(X^2N^{-1}p^3)^l\,(\alpha' pXN^{-1})^n \\ D_{(l)}(s)\,(\alpha-\alpha') &= \alpha \sum_{n,\varepsilon=0}^{\infty} \langle \tilde{\Phi}_1^F;\tilde{\Phi}_{p^n}^G|_k\mathcal{D}\big(T_+^g(p^n)\big)\rangle_{\mathcal{A}} \quad (\alpha X)^{2\varepsilon}\,(\alpha pXN^{-1})^n \\ &\quad -\alpha' \sum_{n,\varepsilon=0}^{\infty} \langle \tilde{\Phi}_1^F;\tilde{\Phi}_{p^n}^G|_k\mathcal{D}\big(T_+^g(p^n)\big)\rangle_{\mathcal{A}} \quad (\alpha' X)^{2\varepsilon}\,(\alpha' pXN^{-1})^n \\ D_{(\varepsilon,l)}(s)\,(\alpha-\alpha') &= \alpha \sum_{n=0}^{\infty} \langle \tilde{\Phi}_1^F;\tilde{\Phi}_{p^n}^G|_k\mathcal{D}\big(T_+^g(p^n)\big)\rangle_{\mathcal{A}}\,(\alpha pXN^{-1})^n \\ &\quad -\alpha' \sum_{n=0}^{\infty} \langle \tilde{\Phi}_1^F;\tilde{\Phi}_{p^n}^G|_k\mathcal{D}\big(T_+^g(p^n)\big)\rangle_{\mathcal{A}}\,(\alpha' pXN^{-1})^n. \end{aligned}$$

4.2 Local calculations

Using Proposition 3.12, we get the following two lemmas.

Lemma 4.1 *Let $X_1 = \alpha p X N^{-1}$ and $X_1' = \alpha' p X N^{-1}$, then*

$$D_{(l)} - D_{(\varepsilon,l)} = \frac{\langle \tilde{\Phi}_1^F ; \tilde{\Phi}_1^G \rangle_{\mathcal{A}}}{\alpha - \alpha'} \left(\frac{\alpha^3 X^2}{Q_{p,G}^2(X_1)} - \frac{\alpha'^3 X^2}{Q_{p,G}^2(X_1')} \right). \tag{70}$$

Lemma 4.2 *Let $Y = pNX^2$ and l a non-negative rational integer, then*

$$\sum_{n=0}^{\infty} \tilde{\Phi}^G_{p^{n+2l}} |_k \mathcal{D}\Big(T_+^g(p^n) \Lambda_+^g(p^l) \Big) X_1^n (X^2 N^{-1} p^3)^l \tag{71}$$

$$= Q_{p,G}^2(X_1)^{-1} \Big(\Big\{ \tilde{\Phi}^G_{p^{2l}} - \tilde{\Phi}^G_{p^{2l-1}} |_k \mathcal{D}(T_-^g(p) X_1) + p \tilde{\Phi}^G_{p^{2l-2}} |_k \mathcal{D}(\Lambda_-^g(p) X_1^2) \Big\}$$

$$|_k \mathcal{D}\Big((1 + p(\nabla_p^r - p) \Delta_p X_1^2) \Lambda_+^r(p^l) \, (p^{-2}Y)^l \Big). \tag{72}$$

Notice that the eigenvalue of $\Delta_p p^5 N^{-1} X^2$ is Y and that X, Y are independent of h, in contrast to X_1 and X_1'. For simplicity, let us use T_-, T_+, Λ_- and Λ_+ for $T_-(p), T_+(p)$, $\Lambda_-(p)$ and $\Lambda_+(p)$. To calculate $D_{(\varepsilon)}$, we have to analyze the infinite sums over l corresponding to $\tilde{\Phi}^G_{p^{2l}}, \tilde{\Phi}^G_{p^{2l-1}}, \tilde{\Phi}^G_{p^{2l-2}}$ as suggested in the last lemma.

Proposition 4.3 *We have*

$$\alpha \, Q_{p,G}^2(X_1)^{-1} \sum_{l=0}^{\infty} \langle \tilde{\Phi}_1^F ; \tilde{\Phi}^G_{p^{2l}} |_k \mathcal{D}\left((1 + p(\nabla_p^r - p) \Delta_p X_1^2) \Lambda_+^r(p^l) \right) \rangle_{\mathcal{A}} \, (p^{-2}Y)^l$$

$$- \alpha' \, Q_{p,G}^2(X_1')^{-1} \sum_{l=0}^{\infty} \langle \tilde{\Phi}_1^F ; \tilde{\Phi}^G_{p^{2l}} |_k \mathcal{D}\left((1 + p(\nabla_p^r - p) \Delta_p (X_1')^2) \Lambda_+^r(p^l) \right) \rangle_{\mathcal{A}} \, (p^{-2}Y)^l$$

$$= \left(\alpha Q_{p,G}^2(X_1)^{-1} - \alpha' Q_{p,G}^2(X_1')^{-1} \right) \frac{S_F^{(2)}(Y)^{prim} \langle \tilde{\Phi}_1^F ; \tilde{\Phi}_1^G \rangle_{\mathcal{A}}}{D_{p,G}^{2,*}(Y)}$$

$$+ \left(\frac{\alpha'^3 X^2}{Q_{p,G}^2(X_1')} - \frac{\alpha^3 X^2}{Q_{p,G}^2(X_1)} \right) \langle \tilde{\Phi}_1^F ; \tilde{\Phi}_1^G \rangle_{\mathcal{A}} \tag{73}$$

with $S_F^{(2)}(Y)^{prim} \tilde{\Phi}_1^F = \tilde{\Phi}_1^F |_k \mathcal{D}(S^{(2)}(Y)^{prim})$ and $D_{p,G}^{2,*}(Y) = \frac{D_{p,G}^2(Y)}{1-Y}$.

Proof:

Let X_* be X_1 or X_1', then

$$\sum\nolimits_1^G (X_*) := \sum_{l=0}^{\infty} \tilde{\Phi}^G_{p^{2l}} |_k \mathcal{D}\left((1 + p(\nabla_p^r - p) \Delta_p X_*^2) \Lambda_+^r(p^l) \right) \, (p^{-2}Y)^l$$

$$= \sum_{l=0}^{\infty} \tilde{\Phi}^G_{p^{2l}} |_k \mathcal{D}(\Lambda_+^r(p^l)) \, (p^{-2}Y)^l - p^2 \tilde{\Phi}_1^G |_k \mathcal{D}(\Delta_p X_*^2).$$

Here we used the peculiarity of $l = 0$.
Applying Proposition 3.13 leads to

$$\sum\nolimits_1^G (X_*) = (\tilde{\Phi}_1^G|_k \mathcal{D}(S^{(2)}(Y))\, D_{p,G}^{2,*}(Y)^{-1} - (\alpha_*)^2 X^2 \tilde{\Phi}_1^G,$$

where α_* is adapted to the meaning of X_*, that means $\alpha_* = \alpha$, if $X_* = X_1$ and α' otherwise. The property that $\tilde{\Phi}_1^F$ is an eigenform of $\tilde{\mathcal{H}}_0^J$ (the subalgebra of all elements of $\tilde{\mathcal{H}}^J$ with signature one) and that $S^{(2)}(Y)^{prim}$ is hermitian proves the proposition.

□

We want to point out the cancellations caused by Lemma 4.1 and Proposition 4.3. By using again Proposition 3.13 and the already calculated relations between the generators of $\tilde{\mathcal{H}}_p^J$, we get

Proposition 4.4 *We have*

$$\begin{aligned} &\alpha Q_{p,G}^2(X_1)^{-1} \sum_{l=0}^{\infty} p \langle \tilde{\Phi}_1^F ; \tilde{\Phi}_{p^{2l-2}}^G |_k \mathcal{D}\left((\Lambda_-^g X_1^2)(1 + p(\nabla_p^r - p)\Delta_p X_1^2)(\Lambda_+^r p^{-2} Y)^l \right) \rangle_{\mathcal{A}} \\ &- \alpha' Q_{p,G}^2(X_1')^{-1} \sum_{l=0}^{\infty} p \langle \tilde{\Phi}_1^F ; \tilde{\Phi}_{p^{2l-2}}^G |_k \mathcal{D}\left((\Lambda_-^g {X_1'}^2)(1 + p(\nabla_p^r - p)\Delta_p {X_1'}^2)(\Lambda_+^r p^{-2} Y)^l \right) \rangle_{\mathcal{A}} \\ &= \left(\frac{\alpha^3 N p^2 X^4}{Q_{p,G}^2(X_1)} - \frac{{\alpha'}^3 N p^2 X^4}{Q_{p,G}^2(X_1')} \right) \left(\frac{S_F^{(2)}(Y)^{prim} \langle \tilde{\Phi}_1^F ; \tilde{\Phi}_1^G \rangle_{\mathcal{A}}}{D_G^{2,*}(Y)} \right). \end{aligned}$$

To calculate the sum corresponding to $\tilde{\Phi}_{p^{2l-1}}^G$ is the main point here. Of course the specialness of F is essential.

Proposition 4.5 *Let $\lambda_p^{F,J}$ be defined by $\tilde{\Phi}_1^F |_k\ \mathcal{D}(T^{J,r}(p)) = \lambda_p^{F,J} \tilde{\Phi}_1^F$. Then*

$$\begin{aligned} &\alpha\, Q_{p,G}^2(X_1)^{-1} \sum_{l=0}^{\infty} \langle \tilde{\Phi}_1^F ; \tilde{\Phi}_{p^{2l-1}}^G |_k \mathcal{D}\left((T_-^g X_1)(1 + p(\nabla_p^r - p)\Delta_p X_1^2)\Lambda_+^r(p^l)\,(p^{-2}Y)^l \right) \rangle_{\mathcal{A}} \\ &- \alpha'\, Q_{p,G}^2(X_1')^{-1} \sum_{l=0}^{\infty} \langle \tilde{\Phi}_1^F ; \tilde{\Phi}_{p^{2l-1}}^G |_k \mathcal{D}\left((T_-^g X_1')(1 + p(\nabla_p^r - p)\Delta_p (X_1')^2)(\Lambda_+^r p^{-2} Y)^l \right) \rangle_{\mathcal{A}} \\ &= \ \left(\alpha^2 p^2 X^3 Q_{p,G}^2(X_1)^{-1} - (\alpha')^2 p^2 X^3 Q_{p,G}^2(X_1')^{-1} \right) \\ &\quad \lambda_p^G (1 - p^{-2} \lambda_p^{F,J} Y + p^{-1} Y^2) \langle \tilde{\Phi}_1^F , \tilde{\Phi}_1^G \rangle_{\mathcal{A}} \left(D_{p,G}^{2,*}(Y) \right)^{-1}. \end{aligned}$$

Proof:

First step

We start with the calculation of

$$\sum\nolimits_3^G (X_*) := \sum_{l=0}^{\infty} \tilde{\Phi}_{p^{2l-1}}^G |_k \mathcal{D}\left((T_-^g X_*)(1 + p(\nabla_p^r - p)\Delta_p X_*^2)\Lambda_+^r(p^l)\,(p^{-2}Y)^l \right). \tag{74}$$

By $T_-^g \Lambda_+^r = p^2 T_+^g$, we have

$$\begin{aligned} \sum\nolimits_3^G (X_*) &= \sum_{l=1}^{\infty} \tilde{\Phi}^G_{p^{2l-1}}|_k \mathcal{D}\left(\Lambda_+^r(p^{l-1})\,(p^{-2}Y)^{l-1} Y T_+ X_*\right) \\ &= \sum_{l=0}^{\infty} \tilde{\Phi}^G_{p^{2l+1}}|_k \mathcal{D}\left(\Lambda_+^r(p^{l})\,(p^{-2}Y)^{l} Y X_* T_+^g\right) \\ &= D^{2,*}_{p,G}(Y)^{-1} (\tilde{\Phi}^G_p|_k \mathcal{D}(S^{(2)}(Y) T_+^g)\, Y X_*. \end{aligned}$$

Here we used Proposition 3.13 and that the terms corresponding to Λ_-^r provide no contribution. With the identity given in (61), we have

$$\sum\nolimits_3^G (X_*) = \alpha_* p^2 X^3\, D^{2,*}_{p,G}(Y)^{-1}\, \tilde{\Phi}^G_p|_k \mathcal{D}\Big(T_+^g(p) S^{(2)}(Y)^{prim} (1+Y)^{-1}\Big). \tag{75}$$

<u>Second step</u>

The P-form $\tilde{\Phi}_1^F$ is an eigenfunction of $\tilde{\mathcal{H}}_0^J$ and $S^{(2)}(Y)^{prim}$ is self adjoint. This gives us

$$\begin{aligned} &\langle \tilde{\Phi}_1^F; \tilde{\Phi}_p^G|_k \mathcal{D}\left(T_+^g S^{(2)}(Y)^{prim}\right)\rangle_{\mathcal{A}}/(1+Y) \\ &= \langle \tilde{\Phi}_1^F|_k \mathcal{D}(1 - p^{-2} T^{J,r}(p) Y + p^{-1} Y^2); \tilde{\Phi}_p^G|_k \mathcal{D}(T_+^g)\rangle_{\mathcal{A}}\,(1+Y)^{-1} \\ &= (1 - p^{-2}\lambda_p^{F,J} Y + p^{-1} Y^2)\,\langle \tilde{\Phi}_1^F; \tilde{\Phi}_p^G|_k \mathcal{D}(T_+^g)\rangle_{\mathcal{A}}\,(1+Y)^{-1} \\ &= \lambda_p^G\,(1 - p^{-2}\lambda_p^{F,J} Y + p^{-1} Y^2)\,\langle \tilde{\Phi}_1^F; \tilde{\Phi}_1^G\rangle_{\mathcal{A}}. \end{aligned}$$

Here $\tilde{\Phi}_1^F|_k \mathcal{D}(T^{J,r}(p)) = \lambda_p^{F,J} \tilde{\Phi}_1^F$ and $\tilde{\Phi}_p^G|_k \mathcal{D}(T_+^g) = \lambda_p^G \tilde{\Phi}_1^G$. Thus the Proposition is proven.

□

We have proven the last lemmas and propositions in such a way that one can easily extract the following crucial fact.

Corollary 4.6 *Let n', l', ε' be coprime to p. Then we have*

$$\sum_{\substack{n,\varepsilon,l=0 \\ min(\varepsilon,l)=0}}^{\infty} \langle \tilde{\Phi}_1^F|_k \mathcal{D}\left(T_-^g(n'p^n)\,\Lambda_-^g(l'p^l)\right); \tilde{\Phi}^G_{n'l'^2p^{n+2l}}\rangle_{\mathcal{A}}\, a^h_{n'(\varepsilon')^2 p^{n+2\varepsilon}} X^{2\varepsilon}\,(X^2N^{-1}p^3)^l (pXN^{-1})^n$$

$$= \langle \tilde{\Phi}_1^F|_k \mathcal{D}\,(T_-^g(n')\,\Lambda_-^g(l'))\,; \tilde{\Phi}^G_{n'l'^2}\rangle_{\mathcal{A}}\, a^h_{n'(\varepsilon')^2} \left(D^{(p)}_{h,G;F}(s) \langle \tilde{\Phi}_1^F; \tilde{\Phi}_1^G\rangle_{\mathcal{A}}^{-1}\right).$$

Thus we have proven the following

Theorem 4.7 *Let $F, G \in S_2^k$ and $h \in S_1^k$ Hecke eigenforms, with h normalized. Let further F be in the Maass Spezialschar. Then we have for k even and s big enough:*

$$D_{h,G;F}(s) = \beta_k \langle \tilde{\Phi}_1^F, \tilde{\Phi}_1^G \rangle_{\mathcal{A}} \prod_p (D_{h,G;F}^{(p)}(s) / \langle \tilde{\Phi}_1^F, \tilde{\Phi}_1^G \rangle_{\mathcal{A}}) \tag{76}$$

with

$$\begin{aligned} D_{h,G;F}^{(p)}(s) &= \frac{\langle \tilde{\Phi}_1^F ; \tilde{\Phi}_1^G \rangle_{\mathcal{A}} S_F^{(2)}(Y)^{prim}}{(\alpha - \alpha') D_{p,G}^{2,*}(Y)} \\ &\times \left(\frac{\alpha}{Q_{p,G}^2(X_1)} - \frac{\alpha'}{Q_{p,G}^2(X_1')} + \frac{\alpha^3 N p^2 X^4}{Q_{p,G}^2(X_1)} \right. \\ &\left. - \frac{\alpha'^3 N p^2 X^4}{Q_{p,G}^2(X_1')} + \frac{\alpha'^2 p^2 X^3 \lambda_p^G}{Q_{p,G}^2(X_1')(1+Y)} - \frac{\alpha^2 p^2 X^3 \lambda_p^G}{Q_{p,G}^2(X_1)(1+Y)} \right). \end{aligned} \tag{77}$$

It is easy to see, that the expression in the big brackets in (77), is equal to

$$\begin{aligned} &\left\{ \left((\alpha + \alpha^3 N p^2 X^4)(1+Y) - \alpha^2 p^2 X^3 \lambda_p^G \right) Q_{p,G}^2(X_1') \right. \\ &\quad \left. - \left((\alpha' + \alpha'^3 N p^2 X^4)(1+Y) - \alpha'^2 p^2 X^3 \lambda_p^G \right) Q_{p,G}^2(X_1) \right\} \\ &\quad Q_{p,G}^2(X_1)^{-1} Q_{p,G}^2(X_1')^{-1} (1+Y)^{-1}. \end{aligned}$$

Inserting $Y = pNX^2$, $N = p^{k-1}$ and $X_* = \alpha_* p N^{-1} X$ gives us

$$\begin{aligned} &\left\{ \left(\alpha + \alpha p N X^2 - \alpha^2 p^2 X^3 \lambda_p^G + \alpha^3 N p^2 X^4 + \alpha^3 N^2 p^3 X^6 \right) Q_{p,G}^2(\alpha' p N^{-1} X) \right. \\ &\quad \left. - \left(\alpha' + \alpha' p N X^2 - \alpha'^2 p^2 X^3 \lambda_p^G + \alpha'^3 N p^2 X^4 + \alpha'^3 N^2 p^3 X^6 \right) Q_{p,G}^2(\alpha p N^{-1} X) \right\} \\ &\quad \left\{ Q_{p,G}^2(X_1) Q_{p,G}^2(X_1')(1+Y) \right\}^{-1}. \end{aligned}$$

If we multiply out the numerator and rearrange the terms according to the powers of y, then we get

$$\begin{aligned} D_{h,G;F}^{(p)}(s) &= \frac{\langle \tilde{\Phi}_1^F ; \tilde{\Phi}_1^G \rangle_{\mathcal{A}} S_F^{(2)}(Y)^{prim}}{D_{p,G}^{2,*}(Y)} \left\{ Q_{p,G}^2(X_1) Q_{p,G}^2(X_1')(1+Y) \right\}^{-1} \\ &\times (1-y) \Big(1 + \left(2pN - p^2 N^{-1} A_2^G \right) Y + \left(2 - 2pN^{-2} A_2^G + pN^{-2} (\lambda_P^G)^2 \right) Y^2 \\ &\quad + \left(2pN - p^2 N^{-1} A_2^G \right) Y^3 + Y^4 \Big). \end{aligned} \tag{78}$$

Here $A_2^G = p \left(\lambda_{T_1(p^2)}^G + (p^2+1) p^{2k-6} \right)$, with $G|_k T_1^{(2)}(p^2) = \lambda_{T_1(p^2)}^G G$. The expression (78) in the last big bracket turns out to be just $D_{p,G}^2(pNX^2)$.

Lemma 4.8 *Let $F \in S_2^{k,M}$ be a Hecke eigenform with corresponding $f \in S_1^{2k-2}$. Then*

$$S_F^{(2)}(p^{-2s-k+2})^{factor} = L_{p,f}(p^{-(2s+2k-3)}).$$

Proof:
We have $f|_k T(p) = \lambda_f(p)\, f \iff \tilde{\Phi}_1^F|_k \mathcal{D}(\widetilde{T}_p^{EZ}) = \lambda_f(p)\tilde{\Phi}_1^F$ (see for example [K-S89]§2) and $T_p^{J,r} = p^{3-k}\tilde{T}_p^{EZ}$, which gives the desired result.

□

Thus we have

Theorem 4.9 *Let $F, G \in S_2^k$, $h \in S_1^k$ be Hecke eigenforms with totally real Fourier coefficients, h normalized and k even. Let F be in the Maass space with corresponding $f \in S_1^{2k-2}$. Then we have for $s \in \mathbb{C}$ with $k + 2Re(s) > 6$ and $X = p^{-(s+k-1)}$:*

$$D_{h,G;F}(s) = \langle \Phi_1^F; \Phi_1^G \rangle_J \times \prod_p \frac{S_F^{(2)}(pNX^2)^{factor}(1 - pNX^2)}{Q_{p,G}^2(\alpha p N^{-1}X) Q_{p,G}^2(\alpha' p N^{-1}X)} \tag{79}$$

with $S_F^{(2)}(pNX^2)^{factor} = L_{p,f}(pX^2)$.

5 The L-function $Z_{G\otimes h}(s)$

In this last section we present our final result, a new integral representation of $Z_{G\otimes h}(s)$ for $G \in S_2^k$ and $h \in S_1^k$. Assuming that the first Fourier-Jacobi coefficient of G is not identically zero enables us to read off properties of the L-function. For example, we show that there is a close connection between the triple product L-function and $Z_{G\otimes h}(s)$.

Theorem 5.1 *Let $F, G \in S_2^k$ and $h \in S_1^k$ be Hecke eigenforms with totally real algebraic Fourier coefficients. Let k be even and h normalized. Further we assume that F is in the Maass Spezialschar with corresponding $f \in S_1^{2k-2}$. Then for $s \in \mathbb{C}$ with $2\,Re(s) + k > 6$:*

$$\begin{aligned} &\langle\langle\langle E_5^k\,([z_1, z_2, z_3]; s)\,; h(z_1)G(z_2)F(z_3)\rangle\rangle\rangle \\ &\quad = a(2, s, k)\langle \Phi_1^F; \Phi_1^G\rangle_J \\ &\quad\quad \times\zeta(2s+k)^{-1}\zeta(4s+2k-2)^{-1}\zeta(4s+2k-4)^{-1} \\ &\quad\quad \times L_f(2s+2k-4)\, Z_{G\otimes h}(s+2k-3). \end{aligned}$$

Proof:
By Theorem 2.7, we know that $\Phi(h, G, F; s) = a(2, s, k)\mathcal{L}_{h,G,F}^2(2k + s - 3)$. Here $\mathcal{L}_{h,G,F}^2(2k + s - 3)$ is equal to

$$\{\zeta(2s+k)\prod_{j=1}^{2}\zeta(4s+2k-2j)\}^{-1}\, D_F^2(2s+k-2)\, D_{h,G;F}(s).$$

Since F is a Hecke eigenform in the Maass Spezialschar, we have

$$D_F^2(2s+k-2) = \zeta(2s+k-2)\, L_f(2s+\tilde{k}-1)\, L_f(2s+\tilde{k}-2),$$

with $\tilde{k} = 2k-2$ and $f \in S_1^{\tilde{k}}$. Finally, with the explicit form of $D_{h,G;F}(s)$ obtained in the last section the theorem becomes obvious.

□

Supplementarily we want to note that $a(2,s,k)$ is equal to

$$2 \cdot 2^{-6s-6k+12} \pi^{-s-2k+6} \frac{\Gamma_2(k+s-\frac{3}{2})\Gamma(s+2k-3)\Gamma(s+k-2)\Gamma(s+k-1)}{\Gamma_2(k+s)\Gamma(2k-3+2s)}. \tag{80}$$

From now on, we assume that $\tilde{\Phi}_1^G$ is not identically zero. It is obvious that one can choose a Hecke eigenform $F \in S_2^{k,M}$ in such a way (use Definition 3.14 and Theorem 3.15), that

$$\langle \tilde{\Phi}_1^F; \tilde{\Phi}_1^G \rangle \neq 0 \text{ and } F \in S_2^{k,M}(\mathbb{Q}(F)).$$

Corollary 5.2 *The L-function $Z_{G\otimes h}(s)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$. If we put*

$$Z^*_{G\otimes h}(s) := (2\pi)^{-4s}\Gamma(s-k+1)^2\Gamma(s-k+2)\Gamma(s)Z_{G\otimes h}(s), \tag{81}$$

then we have the functional equation

$$Z^*_{G\otimes h}(s) = Z^*_{G\otimes h}(3k-3-s). \tag{82}$$

This follows from elementary calculations of the already proven general functional equation stated in (31) and the well known functional equation of $L_f(s)$.

Remark 5.3 *It is pleasant that in our formula the help function F has no longer any influence on the functional equation of $Z_{G\otimes h}(s)$. Further, in contrast to the situation in the case of the triple product L-function, the value of $Z_{G\otimes h}(s)$ at the center of symmetry is not forced to be zero (the minus sign of the functional equation of $L_f(s)$ plays a decisive role).*

Corollary 5.4 *Let $k > 6$ be even, $G \in S_2^k(\mathbb{Q}(G))$ and $h \in S_1^k$ be Hecke eigenforms with h normalized. If the first Fourier-Jacobi coefficient of G is not zero, then*

$$Z'_{G\otimes h}(2k-3) := \pi^{8-5k} \frac{Z_{G\otimes h}(2k-3)}{\langle G; G\rangle\langle h; h\rangle} \in \overline{\mathbb{Q}}. \tag{83}$$

Corollary 5.5 *Let h and G be as in Corollary 5.4. Then*

$$\left(Z'_{G\otimes h}(2k-3)\right)^\sigma = Z'_{G^\sigma\otimes h^\sigma}(2k-3); \qquad \forall \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}). \tag{84}$$

Proof:
By Theorem 1.9 and Theorem 5.1, it is enough to show that

$$\begin{aligned}&\left(\frac{a(2,0,k)\langle\phi_1^F;\phi_1^G\rangle_J L_f(2k-4)}{\zeta(2s+k)\zeta(4s+2k-2)\zeta(4s+2k-4)\langle F;F\rangle}\right)^\sigma\\ &= \frac{a(2,0,k)\langle\Phi_1^{F^\sigma};\Phi_1^{G^\sigma}\rangle_J L_{f^\sigma}(2k-4)}{\zeta(2s+k)\zeta(4s+2k-2)\zeta(4s+2k-4)\langle F^\sigma;F^\sigma\rangle} \in \overline{\mathbb{Q}}.\end{aligned}$$

We put $a \sim b$ for $a, b \in \mathbb{C}$, $b \neq 0$, if $ab^{-1} \in \overline{\mathbb{Q}}$. Then $a(2,0,k) \sim \pi^{6-2k}$ and $(\zeta(2s+k)\zeta(4s+2k-2)\zeta(4s+2k-4))^{-1} \sim \pi^{-5k+6}$,
thus $\pi^{5k-8}a(2,0,k)\,(\zeta(2s+k)\zeta(4s+2k-2)\zeta(4s+2k-4))^{-1} \sim \pi^{4-2k}$.
By Proposition 3.18 and $L_f(2k-3)\,L_f(2k-4) \neq 0$, it is enough to prove:

$$\left(\frac{\pi^{-3+2k}\langle\Phi_1^F;\Phi_1^G\rangle_J}{L_f(2k-3)}\right)^\sigma = \frac{\pi^{-3+2k}\langle\Phi_1^{F^\sigma};\Phi_1^{G^\sigma}\rangle_J}{L_{f^\sigma}(2k-3)} \in \overline{\mathbb{Q}}.$$

But this is an easy consequence of the properties 1.12a and 1.12b and Theorem 1 stated in [Sh81]. Of course, we have used the well known translation of Jacobi forms of index one into modular forms of half-index weight, where the corresponding scalar products are equal up to a non-zero rational number [E-Z85], [Za80]).

□

The L-function $Z_{G\otimes h}(s)$ has already been investigated by M. Furusawa [Fu93]. He constructed an integral representation by integrating a Hermitian Klingen-Eisenstein series for h against a Siegel cusp form G. Under the assumption that certain non-vanishing properties of the Fourier coefficients of G hold, he was able to deduce the meromorphic continuation, the functional equation and the algebraicity of $Z'_{G\otimes h}(2k-3)$. Because of the not sufficiently known Galois behaviour of the Fourier coefficients of a Hermitian Klingen-Eisenstein series [Ha84], he was not able to study the Galois action at the special value.
We want to conclude our work with the following observation. For simplification let $F \in S_n^k(\mathbb{Q}(F)), G \in S_n^k(\mathbb{Q}(G))$ and $h \in S_1^k(\mathbb{Q}(h))$ be Hecke eigenforms (and every elliptic cusp form is assumed to be normalized). Let $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, then

$$\begin{aligned}\mathcal{L}_{h,G,F}^{n,*}(s) &= \mathcal{L}_{h,G,F}^{n,*}(3k-n-1-s)\\ \left(\frac{\pi^{\frac{n^2+3n-4k+2}{2}}\mathcal{L}_{h,G,F}^n(2k-n-1)}{\langle h;h\rangle\langle G;G\rangle\langle F;F\rangle}\right)^\sigma &= \frac{\pi^{\frac{n^2+3n-4k+2}{2}}\mathcal{L}_{h^\sigma,G^\sigma,F^\sigma}^n(2k-n-1)}{\langle h^\sigma;h^\sigma\rangle\langle G^\sigma;G^\sigma\rangle\langle F^\sigma;F^\sigma\rangle} \in \bar{\mathbb{Q}}.\end{aligned}$$

This enables us to give an unique approach to and description of the triple product L-function $L_{h,G,F}(s)$ for $h, G, F \in S_1^k$ (P. Garrett [Ga87]) and the L-function $Z_{G\otimes h}(s)$.
Let us assume that n is equal to 1 or 2 and that the first Fourier-Jacobi coefficient of G is

not identically zero. Then we have

$$\mathcal{L}^n_{h,G,F}(s+2k-n-1) = \begin{cases} \zeta(2s+k)^{-1}\zeta(4s+2k-2)^{-1}L_{h,G,F}(s+2k-2) & \text{for } n=1 \\ \zeta(2s+k)^{-1}\zeta(4s+2k-2)^{-1}\zeta(4s+2k-4)^{-1} & \\ \times\langle\Phi_1^F;\Phi_1^G\rangle_J L_f(2s+2k-4)Z_{G\otimes h}(s+2k-3) & \text{for } n=2, \end{cases}$$

and

$$\begin{array}{rcll} L^*_{h,G,F}(s) & = & (-1)L^*_{h,G,F}(3k-2-s) & \text{for } n=1, \\ Z^*_{G\otimes h}(s) & = & Z^*_{G\otimes h}(3k-3-s) & \text{for } n=2 \end{array}$$

with $L^*_{h,G,F}(s) = (2\pi)^{-4s}\Gamma(s-k+1)^3\Gamma(s)L_{h,G,F}(s)$ and $Z^*_{G\otimes h}(s) = (2\pi)^{-4s}\Gamma(s-k+1)^2$ $\Gamma(s-k+2)\Gamma(s)Z^*_{G\otimes h}(s)$. The central value of $\mathcal{L}^n_{h,G,F}(s)$ is equal to $\frac{3k-n-1}{2}$, which coincides for $k=n+1$ with our described special value $s=2k-n-1$. The Galois behaviour shows up as

$$\left(\frac{\pi^{-5k+5}L_{h,G,F}(2k-2)}{\langle h;h\rangle\langle G;G\rangle\langle F;F\rangle}\right)^\sigma = \frac{\pi^{-5k+5}L_{h^\sigma,G^\sigma,F^\sigma}(2k-2)}{\langle h^\sigma;h^\sigma\rangle\langle G^\sigma;G^\sigma\rangle\langle F^\sigma;F^\sigma\rangle} \in \bar{\mathbb{Q}} \quad \text{and}$$

$$\left(\frac{\pi^{-5k+8}Z_{G\otimes h}(2k-3)}{\langle h;h\rangle\langle G;G\rangle}\right)^\sigma = \frac{\pi^{-5k+8}Z_{G^\sigma\otimes h^\sigma}(2k-3)}{\langle h^\sigma;h^\sigma\rangle\langle G^\sigma;G^\sigma\rangle} \in \bar{\mathbb{Q}}.$$

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Max-Planck-Institut für Mathematik, Gottfried-Claren-Strasse 26, 53225 Bonn, Germany
E-mail address: heim@mpim-bonn.mpg.de, heim@euklid.math.uni-mannheim.de