Scalar curvature of a metric with unit volume

by

Osamu Kobayashi

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3
Federal Republic of Germany

and

Department of Mathematics
Keio University
Hiyoshi
Yokohama 223
Japan

MPI 86-44

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The problem of finding Riemannian metrics on a closed manifold with prescribed scalar curvature function is now fairly well understood from the works of Kazdan and Warner in 1970's ([7] and references cited in it). In this paper we shall consider the same problem under a constraint on the volume. For this purpose it is useful to introduce an invariant $\mu(M)$ of a smooth closed manifold M defined as the supremum of $\mu(M,C)$ of all conformal classes C of Riemannian metrics of M,

$$\mu(M) = \sup_{C} \mu(M,C),$$

where $\mu(M,C)$ is defined as

$$\mu(M,C) = \inf_{g \in C} \frac{\int_M R_g dv_g}{(\int_M dv_g)^{(n-2)/n}},$$

where R_g is the scalar curvature of g, and n = dim M. Obviously $\mu(M) = 4\pi\chi(M)$ by Gauss-Bonnet, if n = 2. In general, $\mu(M) > 0$ iff M carries a metric of positive scalar curvature.

The method of Kazdan and Warner is modified to prove the following.

Theorem 1. (a) *Suppose* dim M = 2. *Then* $f \in C^\infty(M)$ *is the scalar curvature* (= *twice the Gaussian curvature*) *of some metric* g *of* M *with* Area(M,g) = 1 *iff either* $\min f < \mu(M) < \max f$ *or* f =

const. = $\mu(M)$.

(b) Suppose dim $M \geqq 3$, $\mu(M) \leqq 0$ and $\mu(M) = \mu(M,C)$ for some conformal class C. Then $f \in C^\infty(M)$ is the scalar curvature of some metric g with $\mathrm{Vol}(M,g) = 1$ iff either min $f < \mu(M)$ or f = const. = $\mu(M)$. Moreover, a metric g with $R_g = \mu(M)$ and $\mathrm{Vol}(M,g) = 1$ is an Einstein metric.

(c) Suppose dim $M \geqq 3$, $\mu(M) \leqq 0$ and $\mu(M) > \mu(M,C)$ for any C. Then $f \in C^\infty(M)$ is the scalar curvature of some metric g with $\mathrm{Vol}(M,g) = 1$ iff min $f < \mu(M)$.

It is known that $\mu(T^n) = 0$ and $\mu(T^n \# T^n) \leqq 0$ (cf. [6]). T^n, $n \geqq 3$, are examples of the case (b). $T^n \# T^n$, $n \geqq 3$, are examples of (c), because $T^n \# T^n$ carries no Ricci flat metrics, and because $\mu(T^n \# T^n) = 0$ for $n \geqq 3$, which we can see from the following.

Theorem 2. If dim M_1 = dim $M_2 \geqq 3$, $\mu(M_1) \geqq 0$ and $\mu(M_2) \geqq 0$, then $\mu(M_1 \# M_2) \geqq 0$.

There is a similar result that $\mu(M_1 \# M_2) > 0$ if $\mu(M_1) > 0$ and $\mu(M_2) > 0$ ([13]), but ous requies a more careful control on scalar curvature and volume, and basically the same idea yields the following, which shows a clear contrast to the case when $\mu(M) \geqq 0$.

Theorem 3. Suppose dim $M \geqq 3$ and $\mu(M) > 0$. Then any function $f \in C^\infty(M)$ which is either nonconstant or constant less than $\mu(M)$ is the scalar curvature of some metric g with $\mathrm{Vol}(M,g) = 1$.

The author hopes that this result will be improved and that in this case any function can be the scalar curvature of some metric with unit volume, which is shown to be true for many examples.

When we restric ourselves to a fixed conformal class, the problem becomes more involved.

Theorem 4. *Suppose* $\dim M \geqq 3$ *and* C *is a conformal class of Riemannian metrics of* M *with* $\mu(M,C) > 0$. *Then for any* $\varepsilon > 0$ *and any integer* $k \geqq 0$, *there is a metric* $g \in C$ *with* $\mathrm{Vol}(M,g) = 1$ *and* $|R_g - (\mu(M,C)^{n/2} + k\,\mu(S^n)^{n/2})^{2/n}| < \varepsilon$, *where* $\mu(S^n) = n(n-1)\mathrm{Vol}(S^n(1))^{2/n}$.

The metrics for $k \geqq 1$ and small $\varepsilon > 0$ cannot be obtained through the variational method of Aubin (cf.[1]), where the obtained scalar curvature necessarily satisfies the condition that $(\min R_g)\mathrm{Vol}(M,g)^{2/n} \leqq \mu(S^n)$. In general, we cannot take ε to be 0, for large k. For example, this is impossible for $k \geqq 1$, if C contains an Einstein metric (cf.[11]).

This paper is partly based on the last part of [10], and was completed during the author's stay at Max-Planck-Institut für Mathematik.

§1. Preliminaries.

Throughout this paper M denotes a smooth closed connected manifold of dimension n. We start with basic known facts on the Yamabe problem ([12] and references in [2]).

Fact 1.1. Any conformal class C contains a metric g such that $R_g = \text{const.} = \mathrm{Vol}(M,g)^{-2/n}\mu(M,C)$.

Fact 1.2. For any (M,C), $\mu(M,C) \leq n(n-1)\mathrm{Vol}(S^n(1))^{2/n}$, *and equality occurs iff* (M,C) *is conformal to the standard n-sphere.*

Fact 1.3. If $n \geq 3$, *there is a conformal class C such that* $\mu(M,C) = r$ *for any* $r < \mu(M)$.

Besides the above we add some other properties of $\mu(M)$, most of which are more or less well-known.

Lemma 1.4. (i) *If* $n = 2$, *then* $\mu(M) = \mu(M,C) = 4\pi\chi(M)$ *for any C.* (ii) $\mu(M) > 0$ *iff M carries a metric of positive scalar curvature.*

Proof. (i) is the Gauss-Bonnet formula. The only if part of (ii) is from Fact 1.1. So, suppose g is a metric of M with positive scalr curvature. Then it follows form the following lemma that $\mu(M,C) > 0$ for C to which g belongs, hence $\mu(M) > 0$.

Lemma 1.5. Suppose $n = 2$ *or* $\mu(M,C) \leq 0$. *Then the scalar curvature* R_g *of any metric* $g \in C$ *satisfies* $(\min R_g)\mathrm{Vol}(M,g)^{2/n} \leq \mu(M,C) \leq (\max R_g)\mathrm{Vol}(M,g)^{2/n}$, *and each of two equalities implies* R_g *is constant.*

Proof. The case when $n = 2$ is again from Gauss-Bonnet. If $n \geq 3$, any metric in C is written of the form $f^{4/(n-2)}g$ for some positive $f \in C^\infty(M)$ and an arbitrarily fixed $g \in C$. Then, $\mu(M,C)$ is rewritten as

$$(1.6) \qquad \mu(M,C) = \inf_{f>0} \frac{4\frac{n-1}{n-2}\int_M |df|^2 dv_g + \int_M R_g f^2 dv_g}{(\int_M f^{2n/(n-2)} dv_g)^{(n-2)/n}} .$$

From this expression and Sobolev's inequality, we get $\min R_g \leqq 0$ if $\mu(M,C) \leqq 0$. Then, applying Hölder's inequality to (1.6), we have $(\min R_g)\mathrm{Vol}(M,g)^{2/n} \leqq \mu(M,C)$ with equality only when R_g is constant. The latter inequality is obvious from (1.6).

<u>Corollary 1.7</u>. <u>If</u> $n = 2$ <u>or</u> $\mu(M,C) \leqq 0$, <u>then</u>

$$\sup_{g\in C} (\min R_g)\mathrm{Vol}(M,g)^{2/n} = \mu(M,C).$$

<u>In particular</u>, <u>if</u> $n = 2$ <u>or</u> $\mu(M) \leqq 0$, <u>then</u>

$$\sup_{g} (\min R_g)\mathrm{Vol}(M,g)^{2/n} = \mu(M).$$

In contrast to this, we can see from Theorem 4 that $\sup_{g\in C} (\min R_g)\mathrm{Vol}(M,g)^{2/n} = \infty$ if $n \geqq 3$ and $\mu(M,C) > 0$.

<u>Corollary 1.8</u>. <u>If</u> $n = 2$ <u>or</u> $\mu(M) \leqq 0$, <u>then</u>

$$\sup_{g} (\min_{|X|=1} \mathrm{Ric}_g(X,X))\mathrm{Vol}(M,g)^{2/n} \leqq \frac{1}{n}\mu(M).$$

In [5], Gromov has shown that the left side of the above inequality is strictly negative for M which carries a metric with negative curvature. In view of this fact, I want to pose the following question: Does $\mu(M) < 0$ hold for M which has a negative sectional curvature metric?

In Corollary 1.8, the equality occurs if $\mu(M) = \mu(M,C)$ for some conformal class C, and then the supremum is attained by some Einstein metric, which is shown from the following (cf. [3],[8]).

Lemma 1.9. *If* $\mu(M) \leq 0$ *and* $\mu(M) = \mu(M,C)$ *for some* C, *then* C *contains an Einstein metric* g, *and with this metric* $\mu(M) = \mu(M,C) = R_g \mathrm{Vol}(M,g)^{2/n}$ *holds.*

Proof. Let

$$\lambda(g) = \inf_{f>0} \frac{4\frac{n-1}{n-2}\int_M |df|^2 dv_g + \int_M R_g f^2 dv_g}{\int_M f^2 dv_g} .$$

Then in a way similar to the proof of Lemma 1.5, we have $\mu(M,C) \geq \lambda(g)\mathrm{Vol}(M,g)^{2/n}$ for all $g \in C$, if $\mu(M,C) \leq 0$ (the opposite inequality holds if $\mu(M,C) \geq 0$). Now, let C be such that $\mu(M) = \mu(M,C)$, and take $g \in C$ such that R_g is constant equal to $\mathrm{Vol}(M,g)^{-2/n}\mu(M,C)$ (cf. Fact 1.1). Let $g_t = g - t\,\mathring{Ric}_g$, where $\mathring{Ric}_g$ is the traceless part of the Ricci tensor of g, and t is sufficiently small. Then, $\lambda(g_t)$ is differentiable in t, and we have

$$\frac{d}{dt}\, \lambda(g_t)\mathrm{Vol}(M,g_t)^{2/n}\Big|_{t=0} = \mathrm{Vol}(M,g)^{(2-n)/n} \int_M |\mathring{Ric}_g|^2 dv_g$$

(cf.[8]). Therefore if $\mathring{Ric}_g \not\equiv 0$, then $\mu(M,C_t) \geq \lambda(g_t)\mathrm{Vol}(M,g_t)^{2/n} > \lambda(g)\mathrm{Vol}(M,g)^{2/n} = \mu(M,C) = \mu(M)$ for a small positive t, where $C_t = \{e^u g_t;\ u \in C^\infty(M)\}$, which contradicts to the definition of $\mu(M)$.

§2. Proof of Theorem 1.

The only if part of each of the cases (a), (b) and (c), and the additional statement in the case (b) have already been proved in Lemmas 1.5 and 1.9. The remainning parts are from Facts 1.1, 1.3 and the following generalization of a result of

Kazdan and Warner [9].

<u>Proposition 2.1</u>. *Let* g_0 *be a smooth Riemannian metric of* M *with* $\mathrm{Vol}(M,g_0) = 1$, *and* f *a smooth function satisfying* $\min f < \min R_{g_0}$ *and* $\max R_{g_0} < \max f$. *Then there exists another smooth metric* g *such that* $R_g = f$ *and* $\mathrm{Vol}(M,g) = 1$.

Then, for example, the proof of the case (c) is as follows: We have only to find a metric g with $\mathrm{Vol}(M,g) = 1$ and $R_g = f$ for f satisfying $\min f < \mu(M)$. The case when f is constant is immediate from Facts 1.1 and 1.3. If f is not constant, we can choose $c \in \mathbb{R}$ so that $\min f < c < \min\{\mu(M), \max f\}$. Then from Facts 1.1 and 1.3, we have a metric g_0 such that $R_{g_0} = c$ and $\mathrm{Vol}(M,g_0) = 1$. Therefore applying the above proposition we get the desired metric.

The proofs of the cases (a) and (b) are completely similar. The same argument works also to some extent when $\mu(M) > 0$ and $\dim M \geqq 3$:

<u>Corollary 2.2</u>. *If* $\dim M \geqq 3$, *then any* $f \in C^\infty(M)$ *such that* $\min f < \mu(M)$ *is the scalar curvature of some metric* g *with* $\mathrm{Vol}(M,g) = 1$.

<u>Proof of Proposition 2.1.</u> Since the proof is similar to that given in [9], we shall only sketch it. Let $\mathcal{S}_p(M)$ denote the Sobolev space of $H_{2,p}$ symmetric covariant 2-tensor fields, where $H_{2,p}$ means that derivatives up to second order are L_p integrable. We always assume $p > n = \dim M$. Put $\mathcal{M}_p(M) =$

$\{h \in \mathcal{S}_p;$ h is everywhere positive definite$\}$ and $\mathring{\mathcal{S}}_p(M) =$ $\{h \in \mathcal{S}_p;\ \int_M (\mathrm{tr}_{g_0} h)\, dv_{g_0} = 0\}$.

Since $\mathrm{Vol}:\mathcal{M}_p(M) \longrightarrow \mathbb{R};\ g \mapsto \mathrm{Vol}(M,g)$ is a C^1 mapping whose differential is not zero, it follows from the implicit function theorem that there are a neighborhood U of 0 in $\mathring{\mathcal{S}}_p(M)$ and a C^1 function $s:U \to \mathbb{R}$ such that $S(h) := g_0 + h + s(h)g_0$ for $h \in U$ has the following properties; (i) $S(h) \in \mathcal{M}_p(M)$, (ii) $\mathrm{Vol}(S(h)) = 1$, (iii) $S(0) = g_0$, (iv) DS at 0 is the inclusion map $\mathring{\mathcal{S}}_p(M) \hookrightarrow \mathcal{S}_p(M)$. Note that S(h) is a C^∞ metric iff h is of class C^∞.

The scalar curvature $R:\mathcal{M}_p(M) \to L_p(M;\mathbb{R})$ is defined as a C^1 mapping. So, we get a C^1 mapping $R\circ S:U \to L_p(M;\mathbb{R})$, $U \subset \mathring{\mathcal{S}}_p(M)$, whose differential $A:\mathring{\mathcal{S}}_p(M) \to L_p(M;\mathbb{R})$ at 0 is computed as $A(h) = -\Delta h^i{}_i + h^{ij}{}_{;ij} - h^{ij}R_{ij}$, where the covariant differentiation, the Ricci curvature, etc. are relative to g_0. The formal L_2 adjoint A^* of A is given by $A^*(u) = \bar{A}^*(u) + a(u)g_0$, where $\bar{A}^*(u) = -(\Delta u)g_0 + \nabla^2 u - u\,\mathrm{Ric}_{g_0}$ and $a(u) = (\int_M u R_{g_0} dv_{g_0})/(n\,\mathrm{Vol}(M,g_0))$. $A^*:H_{4,p}(M;\mathbb{R}) \to \mathring{\mathcal{S}}_p(M)$ is a continuous linear map. Now, we remark that we may assume with no loss of generality that the scalar curvature $R(g_0)$ of g_0 is <u>not</u> constant. Then, we can show that $A\circ\bar{A}^*:H_{4,p}(M;\mathbb{R}) \to L_p(M;\mathbb{R})$ is a linear homeomorphism, and $A\circ A^*:H_{4,p}(M;\mathbb{R}) \to L_p(M;\mathbb{R})$ is injective (cf. [3],[4],[9]). Then, since $A\circ(A^* - \bar{A}^*)$ is a compact operator, we conclude that $A\circ A^*$ is invertible.

Let $V := (A^*)^{-1}(U)$, and defined a C^1 mapping $Q:V \to L_p(M;\mathbb{R})$ by $Q = R\circ S\circ A^*$. The differential of Q at 0 is $A\circ A^*$, hence it follows from the inverse function theorem that Q is locally invertible around 0. In particular, $Q(V)$ contains some ε-ball centered at $Q(0)$ in $L_p(M;\mathbb{R})$.

For the function f given in the proposition we can find a diffeomorphism φ of M such that $|Q(0) - f\circ\varphi|_{L_p} < \varepsilon$ ([9]). Therefore, we get $u \in V \subset H_{4,p}(M;\mathbb{R})$ with $Q(u) = f\circ\varphi$.

Although $Q(u)$ involves integrals of u, ∇u and $\nabla^2 u$, we can see the elliptic regularity argument is applicable, if we write down $Q(u)$ explicitly. Hence, the u for which $Q(u) = f\circ\varphi$ is of class C^∞. Thus, $\bar{g} = S\circ A^*(u)$ is a C^∞ metric with $\mathrm{Vol}(M,\bar{g}) = 1$ and $R_{\bar{g}} = f\circ\varphi$. Then the desired metric is given by $g = (\varphi^{-1})^* \bar{g}$.

§3. Approximation Lemmas.

Let (S^n, g_0) be the Euclidean unit n-sphere, and r be the intrinsic distance relative to g_0 from the north pole so that g_0 is written as $g_0 = dr^2 + \sin^2 r\, h_0$, where h_0 is the standard metric of the unit $(n-1)$-sphere. For an interval $I \subset [0,\pi]$, we denote by $A(I)$ the region $A(I) = \{x \in S^n;\ r(x) \in I\}$.

Lemma 3.1. *If $n \geq 3$, there exists, for any $\varepsilon_1 > 0$ and $0 < \varepsilon_2 < \pi$, a positive function $f = f(r)$ of S^n such that*

(i) $|R_{g'} - n(n-1)| < \varepsilon_1$, *where* $g' = f^{-2} g_0$;

(ii) $|\mathrm{Vol}(S^n, g') - 2\mathrm{Vol}(S^n, g_0)| < \varepsilon_1$;

(iii) $f(r) = 1$ *for* $r > \varepsilon_2$, *and* $(A([0,\varepsilon_2']), g')$ *is isometric to* $(A((\varepsilon_2,\pi]), g') = (A((\varepsilon_2,\pi]), g_0)$ *for some* $\varepsilon_2' < \varepsilon_2$;

(iv) $0 < f(r) \leq 1$ *and* $|\dot{f}(r)| \leq 2/\sin r$ *for all* r, *where* $\dot{}$ *means* d/dr.

Proof. The scalar curvature of g' is given as $R_{g'}/n(n-1) = \frac{2}{n} f(\ddot{f} + (n-1)\dot{f} \cot r) - \dot{f}^2 + f^2$. It is convenient to change the variables by

(3.2) $$\cos r = \tanh t, \quad 0 < r < \pi, \ \infty > t > -\infty,$$

and to put

(3.3) $$u(t) = f(r(t)) \cosh t.$$

Then $g' = u^{-2}(dt^2 + h_0)$ and

(3.4) $$\frac{R_{g'}}{n(n-1)} = \frac{2}{n} u u'' - (u')^2 + \frac{n-2}{n} u^2 ,$$

where $' = d/dt = -(\sin r) d/dr$. We put

(3.5) $$B(t) = u^{-n} ((u')^2 - u^2 + 1).$$

Then, from (3.4), we get

(3.6) $$B'(t) = (u^{-n})' \left(1 - \frac{R_{g'}}{n(n-1)}\right).$$

We fix $t_0 > \max\{0, \log \cot(\varepsilon_2/2)\}$, and put

(3.7) $$u(t) \equiv \cosh t \quad \text{for} \quad t \in (-\infty, t_0],$$

hence $B(t) \equiv 0$ for $t < t_0$. We shall consider the solution u of (3.5) with a suitably given B(t). First, we note that

(3.8) $$u(t) \leqq \cosh t \quad \text{for } t \geqq t_0, \text{ if } B(t) \leqq 0 \quad \text{for } t \geqq t_0 ,$$

which is easily seen by a simple comparison argument. Let

$$\begin{cases} B(t) = 0 & \text{for } t \leqq t_0; \\ B(t) \leqq 0 \text{ and } -2\delta \leqq B'(t) \leqq 0 & \text{for } t_0 \leqq t \leqq t_0 + 1; \\ B(t) = -\delta & \text{for } t_0 + 1 \leqq t \leqq t_1 . \end{cases}$$

If $\delta > 0$ is taken to be sufficiently small, then (3.5) with (3.7) is solvable for u in the interval $(-\infty, t_1)$ with arbitral $t_1 \geqq t_0 + 1$, and $u'(t) > 0$, $u''(t) > 0$ for $t_0 \leqq t \leqq t_0 + 1$. Therefore, taking $\delta > 0$ smaller, if necessary, we have from (3.6)

$$|R_{g'} - n(n-1)| \leqq 2(n-1)\frac{\cosh^{n+1}(t_0+1)}{\sinh t_0}\delta < \varepsilon_1 \quad \text{for } t \leqq t_0+1.$$

t_1 is then chosen so that $u'(t_1) = 0$ and $u'(t) > 0$ for $t_0 + 1 < t < t_1$, hence, in particular, $R_{g'} \equiv n(n-1)$ for $t_0+1 \leqq t \leqq t_1$. For $t \geqq t_1$, we put

$$B(t) = B(2t_1 - t).$$

Then

$$u(t) = u(2t_1 - t) \quad \text{for } t \geqq t_1 . \tag{3.9}$$

Thus, $|R_{g'} - n(n-1)| < \varepsilon_1$ for all t, and the assertion (iii) follows from (3.7) and (3.9) via (3.2) and (3.3). As for the volume, we have

$$\mathrm{Vol}(S^n, g') = \mathrm{Vol}(S^n, g_0) + 2\int_1^{u(t_1)} \frac{du}{u^n u'}$$

and we can see by a tedious but elementary calculation that the second term of the right side converges to $\mathrm{Vol}(S^n, g_0)$ as $\delta \to 0$. Therefore (ii) holds for a sufficiently small δ. From (3.8),

$u(t) \leqq \cosh t$, and hence $|u'(t)| \leqq |\sinh t|$ from (3.5), because $B(t) \leqq 0$, which proves (iv).

In this lemma, when ε_1 tends to 0, the obtained metrics become closer to a singular metric isometric to $4dr^2 + \sin^2(2r)h_0$, $0 \leqq r \leqq \pi$, which has constant sectional curvature 1 in the non-singular part $r \neq \pi/2$, and whose volume is twice the volume of the unit n-sphere.

Lemma 3.10. *Let g be a fixed Riemannian metric of M, and g_0 another metric defined in a neighborhood of a point $o \in M$ such that $R_g(o) = R_{g_0}(o)$ and $j_o^1 g = j_o^1 g_0$. Then for any $\varepsilon > 0$ there is a metric $\bar{g}$ with the following properties;*

(i) *$\bar{g} = g$ outside the ε-ball centered at o;*

(ii) *$\bar{g} = g_0$ in a neighborhood of o;*

(iii) *$|R_{\bar{g}}(x) - R_g(x)| < \varepsilon$ for all $x \in M$;*

(iv) *$|\bar{g}(X,X) - g(X,X)| \leqq \varepsilon g(X,X)$ for all $X \in TM$;*

(v) *if g_0 is conformal to g, then so is $\bar{g}$.*

To prove this lemma, we need the following sublemmas.

Sublemma 3.11. *Let g and g' be two Riemannian metrics, $h = g' - g$ and $q(x) = \max\{g(X,X)/g'(X,X);\ X \in T_xM \setminus 0\}$. Then $R_{g'} - R_g = P_g(h) + Q_g(h)$, where*

$$P_g(h) = -h^i{}_{i;}{}^j{}_j + h^{ij}{}_{;ij} - h^{ij}R_{ij},$$

and

$$|Q_g(h)(x)| \leqq a_n(|\nabla h|^2 q^3 + |h||\nabla^2 h| q^2 + (|h||\nabla^2 h| + |Ric||h|^2 q)(x),$$

where the covariant derivation, the Ricci tensor, etc. are with respect to the metric g, and a_n is a constant depending only on the dimension n.

The proof is a straightforward computation of writting out the scalar curvature explicitly in terms of the metric and its derivatives, so it is omitted.

Sublemma 3.12. For any $\delta > 0$, there is a nonnegative function $w_\delta \in C^\infty(\mathbb{R})$ such that (i) $0 \leqq w_\delta \leqq 1$, $w_\delta(t) \equiv 1$ in a neighborhood of 0 and $w_\delta(t) \equiv 0$ for $t \geqq \delta$; (ii) $|t\dot{w}_\delta(t)| < \delta$ and $|t^2\ddot{w}_\delta(t)| < \delta$ for all t.

Proof. Take $\lambda > 0$ so that $\cosh\lambda \triangleq e^{1/2\delta}$, and define a piecewise smooth continuous function $u: \mathbb{R} \to \mathbb{R}$ as

$$\begin{cases} u(t) = 0 & \text{for } t \in (-\infty, \delta e^{-2\lambda}] \cup [\delta, \infty); \\ u(t) = e^{2\lambda} - \frac{\delta}{t} & \text{for } t \in [\delta e^{-2\lambda}, 2\delta/(e^{2\lambda}+1)]; \\ u(t) = \frac{\delta}{t} - 1 & \text{for } t \in [2\delta/(e^{2\lambda}+1), \delta]. \end{cases}$$

Then,

$$\int_0^\delta u(t)\,dt = 1, \quad 0 \leqq t\,u(t) \leqq \delta\tanh\lambda < \delta \quad \text{and}$$

$$t^2\dot{u}(t) = \begin{cases} 0 & \text{for } t \in (-\infty, \delta e^{-2\lambda}) \cup (\delta, \infty) \\ \delta & \text{for } t \in (\delta e^{-2\lambda}, 2\delta/(e^{2\lambda}+1)) \\ -\delta & \text{for } t \in (2\delta/(e^{2\lambda}+1), \delta). \end{cases}$$

Therefore, we can modify u to get a smooth function v with the following properties;

$$\int_0^{\delta} v(t)\,dt = 1, \quad 0 \leqq t\,v(t) < \delta, \quad |t^2\dot{v}(t)| < \delta \text{ and}$$

$$v(t) = 0 \quad \text{for } t \in (-\infty, \delta e^{-\delta}/4] \cup [\delta, \infty).$$

Thus, $w_{\delta}(t) = \int_{\delta}^{t} v(t)\,dt$ has the desired properties.

Now, the proof of Lemma 3.10 proceeds as follows. Choose $r_0 > 0$ so that g_0 is defined in the r_0-ball at o and that $|g_0(X,X) - g(X,X)| \leq \min\{\varepsilon, 1/2\} g(X,X)$ for $X \in T_xM$ if x is in the ball, and define a metric g_{δ} of M as $g_{\delta} = g + w_{\delta}(r)(g_0 - g)$, where $\delta < \min\{r_0, \varepsilon\}$ and $r = r(x) = \mathrm{dist}(o,x)$. Then, $\bar{g} := g_{\delta}$ satisfies the conditions (i), (ii), (iv) and (v). From Sublemma 3.11, we have

$$R_{g_{\delta}} - R_g = P_g(w_{\delta}(r)(g - g_0)) + Q_g(w_{\delta}(r)(g - g_0))$$

$$- w_{\delta}(r)(P_g(g - g_0) + Q_g(g - g_0) - (R_{g_0} - R_g)).$$

Since $j_o^1 g = j_o^1 g_0$, we get from Sublemma 3.12

$$|P_g(w_{\delta}(r)(g - g_0)) - w_{\delta}(r) P_g(g - g_0)| < b_1 \delta,$$

$$|Q_g(w_{\delta}(r)(g - g_0))| + |w_{\delta}(r) Q_g(g - g_0)| < b_1 \delta^2,$$

for some constant b_1. And since $R_{g_0}(o) = R_g(o)$,

$$|w_{\delta}(r)(R_{g_0} - R_g)| < b_2 \delta$$

for some b_2. Therefore we have $|R_{g_{\delta}} - R_g| < \varepsilon$ for a sufficiently small δ, which completes the proof of Lemma 3.10.

Corollary 3.13. *If* $R_g(o) = n(n-1)$, *then for any* $\varepsilon > 0$ there is a metric $\bar{g}$ such that (i) $|R_g - R_{\bar{g}}| < \varepsilon$; (ii) $|Vol(g) - Vol(\bar{g})| < \varepsilon$; (iii) $\bar{g}$ *has constant sectional curvature* 1 *in a neighborhood of* o.

Proof. Put $g_0 = dr^2 + \sin^2 r\, h_0$ in the polar normal coordinates, h_0 being the standard metric of the $(n-1)$-sphere, and apply Lemma 3.10.

Here we make a digression to see that an argument like the above shows that *there is a metric of* $\mathbb{R}^n$, $n \geq 3$, *which is the ordinary Euclidean metric outside a compact set and whose scalar curvature is not positive, and negative somewhere*. The proof is as follows: From Facts 1.1 and 1.3 or a direct construction, which can be made in various ways, we can take a metric g_1 of S^n, $n \geq 3$, whose scalar curvature is negative everywhere and equal to $-n(n-1)$ at some point, say $o \in S^n$. Then by an argument quite similar to the above corollary, we may assume that g_1 is of the form $dr^2 + \sinh^2 r\, h_0$ in a neighborhood $V_1 = \{0 \leq r(x) = dist(o,x) < r_0\}$ of o. It is easy to show that there is a positive function $u \in C^\infty(\mathbb{R})$ such that $u(r) = \sinh r$ for $r \geq r_0$, $u(r) = r_0 - r$ for $r \leq 0$ and $2u\ddot{u} + (n-2)(\dot{u}^2 - 1) \geq 0$ for all r. Let $V_2 = (-\infty, r_0) \times S^{n-1}$, and g_2 be a metric of V_2 given by $g_2 = dr^2 + u^2 h_0$. Then we can glue $(S^n \setminus V_1, g_1)$ and (V_2, g_2) along their boundaries to obtain a smooth Riemannian manifold. Now it is easy to see that the resultant space is the desired one.

Corollary 3.14. *If* $R_g(o) = n(n-1)$, *then for any* $\varepsilon > 0$ *there is a metric* $\bar{g}$ *pointwise conformal to* g *such that* (i) $|R_{\bar{g}} - R_g| < \varepsilon$, (ii) $|\mathrm{Vol}(\bar{g}) - \mathrm{Vol}(g)| < \varepsilon$; (iii) $\mathrm{Ric}_{\bar{g}} = (n-1)\bar{g}$ *at the point* o.

Proof. Put $g_0 = f^{1/(n-2)}g$, $f(x) = \mathring{R}_{ij}x^i x^j + 1$, where x^i are normal coordinates around 0 and $\mathring{R}_{ij} = \mathrm{Ric}_g(\partial/\partial x^i, \partial/\partial x^j)$, and apply Lemma 3.10.

Corollary 3.14 is a geometric interpretation of necessity of Hölder continuity in the Schauder estimates, for the metric $\bar{g}$ is conformal to g and close to g in C^0 (cf. (iv) of Lemma 3.10) but not in C^2 in general because of the condition (iii) above, so then $R_{\bar{g}}$ cannot be C^α close to R_g for $0 < \alpha < 1$, since otherwise the Schauder estimates would imply $\bar{g}$ is $C^{2+\alpha}$ close to g, but actually $R_{\bar{g}}$ is C^0 close to R_g from the condition (i).

§4. Proof of Theorems 2 and 3.

We start with the following.

Lemma 4.1. *Let* (M_1, g_1) *and* (M_2, g_2) *be two Riemannian manifolds of same dimension* $n \geq 3$ *such that* $R_{g_i}(p_i) = n(n-1)$ *at some points* $p_i \in M_i$, $i = 1,2$. *Then for any* $\varepsilon > 0$, *there is a metric* g *of* $M_1 \# M_2$ *with the following properties;*

(i) $\left|\mathrm{Vol}(M_1 \# M_2, g) - \sum_{i=1}^{2} \mathrm{Vol}(M_i, g_i)\right| < \varepsilon$;

(ii) there are isometric imbeddings $\varphi_i : (M_i \setminus B_i, g_i) \longrightarrow (M_1 \# M_2, g)$, $i = 1, 2$, where B_i are small balls containning $p_i \in M_i$, and $|R_g(x) - n(n-1)| < \varepsilon$ for $x \in M_1 \# M_2 \setminus (\mathrm{Im}\, \varphi_1 \cup \mathrm{Im}\, \varphi_2)$.

Proof . By virtue of Corollary 3.13, we can take a metric $\bar{g}_i$ of M_i which coincides with g_i outside a small ball B_i containning p_i and satisfies the following; $|R_{\bar{g}_i}(x) - n(n-1)| < \varepsilon$ for $x \in B_i$, $|\mathrm{Vol}(M_i, \bar{g}_i) - \mathrm{Vol}(M_i, g_i)| < \varepsilon/4$ and B_i includes a smaller ball B_i' such that $(B_i', \bar{g}_i)$ is isometric to a geodesic δ-ball in the unit n-sphere and $\mathrm{Vol}(B_i', \bar{g}_i) < \varepsilon/4$.

On the other hand, from Lemma 3.1 with ε_1 and ε_2 sufficiently small less than ε and δ respectively, we have a small piece $(A([\delta', \delta]), g')$ for some $\delta' < \delta$ such that $|R_{g'} - n(n-1)| < \varepsilon$, $\mathrm{Vol}(A([\delta', \delta]), g') < \varepsilon/4$ and a neighborhood of each of the boundary components is isometric to a neighborhood of the boundary of the δ-ball of the unit n-sphere.

Thus, the desired space $(M_1 \# M_2, g)$ is obtained by putting together three pieces $(M_1 \setminus B_1', \bar{g}_1)$, $(M_2 \setminus B_2', \bar{g}_2)$ and $(A([\delta', \delta]), g')$.

Proof of Theorem 2. It follows from Corollary 2.2 that for any $\delta > 0$, M_i, $i = 1, 2$, has a metric g_i such that $\mathrm{Vol}(M_i, g_i) = 1$, $R_{g_i} = n(n-1)$ at some point, and $\min R_{g_i} > -\delta$, since $\mu(M_i) \leq 0$. Therefore from the above lemma we get a metric g of $M_1 \# M_2$ such that $\mathrm{Vol}(M_1 \# M_2, g) < 3$, $\min R_g > -\delta$. Then $\mu(M_1 \# M_2) > -3^{2/n}\delta$ from Corollary 1.7. Hence, $\mu(M_1 \# M_2) \geq 0$ because $\delta > 0$ can be chosen arbitrarily.

Proof of Theorem 3. For the sake of simplicity, we denote by $\alpha(M,r)$ and $\beta(M,v)$ respectively the following propositions; $\alpha(M,r)$ = "for any $\varepsilon > 0$ there is a metric g of M such that $|R_g - r| < \varepsilon$ and $Vol(M,g) = 1$" and $\beta(M,v)$ ="for any $\varepsilon > 0$ there is a metric g of M such that $|R_g - n(n-1)| < \varepsilon$ and $|Vol(M,g) - v| < \varepsilon$". It is obvious that $\alpha(M,v)$ is equivalent to $\beta(M,n(n-1)v^{2/n})$ if $v > 0$. From Facts 1.1 and 1.3, we know that (a) if $\mu(M) > 0$ and $n \geqq 3$, then $\beta(M,v)$ holds for $0 < v \leqq (\mu(M)/n(n-1))^{n/2}$, hence in particular, (b) if $n \geqq 3$, then $\beta(S^n,v)$ holds for $0 < v \leqq Vol(S^n(1))$. Moreover it follows from Lemma 4.1 that if $n \geqq 3$, then $\beta(M_1,v_1)$ and $\beta(M_2,v_2)$ imply $\beta(M_1 \# M_2, v_1+v_2)$. Hence, replacing M_1 and M_2 here by M and S^n respectively, we see that if $n \geqq 3$, $\beta(M,v)$ implies $\beta(M,v+a)$ for any $0 \leqq a \leqq Vol(S^n(1))$, because of the above fact (b). Therefore, $\beta(M,v)$ implies $\beta(M,v')$ for all $v' \geqq v$, if $n \geqq 3$. Thus from (a) above, $\beta(M,v)$ holds for any $v > 0$, hence so does $\alpha(M,r)$ for any $r > 0$, provided that $\mu(M) > 0$ and $n \geqq 3$.

On the other hand, we can see from Proposition 2.1 that any function $f \in C^\infty(M)$ with $\min f < r < \max f$ is the scalar curvature of some metric of unit volume if $\alpha(M,r)$. Therefore the above argument yields that any nonconstant function whose maximum is positive can be realized as the scalar curvature of a metric of unit volume, if $\mu(M) > 0$ and $n \geqq 3$. This, together with Corollary 2.2, completes the proof.

§5. Proof of Theorem 4.

We put $\gamma(M,C,v)$ = "for any $\varepsilon>0$ there is a metric $g\in C$ such that $|R_g - n(n-1)| < \varepsilon$ and $|\mathrm{Vol}(M,g) - v| < \varepsilon$". So we can rephrase Theorem 4 as follows; $\gamma(M,C,(\mu(M,C)/n(n-1))^{n/2} + k\,\mathrm{Vol}(S^n(1)))$ holds for any integer $k \geqq 0$, if $n \geqq 3$ and $\mu(M,C) > 0$. The case when $k = 0$ follows immediately from Fact 1.1. Hence, we have only to show that $\gamma(M,C,v)$ implies $\gamma(M,C,v+\mathrm{Vol}(S^n(1)))$ if $n \geqq 3$.

First we prepare the following formula.

Lemma 5.1. *Let* $o \in M$, $r(x) = \mathrm{dist}(o,x)$ *and* $f = f(r)$ *be a smooth function of* M. *Then*

$$\Delta f = \ddot{f} + \left(\frac{n-1}{r} - \frac{r}{3}\,\mathrm{Ric}_g(\mathrm{grad}\ r,\ \mathrm{grad}\ r) + \rho\right)\dot{f}$$

in a neighborhood of o, *where* $\dot{}\ = d/dr$ *and* ρ *is a function such that* $|\rho(x)| \leqq a_1 r(x)^2$ *with a constant* a_1 *depending only on the metric*.

The proof is omitted as it is a direct calculation.

Let us assume $\gamma(M,C,v)$. Then we see from Corollary 3.14 that for any $\varepsilon > 0$ there is a metric $g \in C$ such that $|R_g - n(n-1)| < \varepsilon$, $|\mathrm{Vol}(M,g) - v| < \varepsilon$ and $\mathrm{Ric}_g = (n-1)g$ at some point $o \in M$.

Let $f = f(r)$ be the positive function of r in Lemma 3.1 with a small ε_2. Then naturally we can regard f as a smooth function of M through the identification $r = \mathrm{dist}(x,o)$. Let Δ_0 be the Laplacian of the metric $g_0 = dr^2 + \sin^2 r\, h_0$ of constant sectional curvature 1, defined on $0 \leqq r < \varepsilon_2$. Note that the Ricci curvatures of g and g_0 coincides at the point o. Hence from (iii), (iv) of Lemma 3.1 and the above lemma, we have

(5.2) $$\left|\Delta f - \Delta_0 f\right| \leqq a_2 \varepsilon_2^2 / \sin \varepsilon_2 ,$$

for some constant a_2. Putting $\bar{g} = f^{-2} g$, we have

$$R_{\bar{g}} = 2(n-1) f (\Delta f - \Delta_0 f) + (R_g - n(n-1)) f^2 + R_{g'} ,$$

where $R_{g'} = R_{g'}(r)$ is the scalar curvature of $f^{-2} g_0$. Thus, from the assumption, (i), (iv) of Lemma 3.1 and (5.2), we get

(5.3) $$\left|R_{\bar{g}} - n(n-1)\right| \leqq 2(n-1) a_2 \varepsilon_2^2 / \sin \varepsilon_2 + \varepsilon + \varepsilon_1 .$$

On the other hand, if we choose a constant a_3 so that $\left|dv_g(x) - dv_{g_0}(x)\right| < a_3 \varepsilon_2^2 dv_{g_0}(x)$ for $r \leqq \varepsilon_2$, it is easily seen that

(5.4) $$\left|\mathrm{Vol}(M,\bar{g}) - (v + \mathrm{Vol}(S^n(1)))\right| \leqq a_3 \varepsilon_2^2 (\mathrm{Vol}(S^n(1)) + \varepsilon_1) + \varepsilon_1$$

from Lemma 3.1.

Thus letting $\varepsilon, \varepsilon_1, \varepsilon_2 \to 0$ in (5.2) and (5.4), we get $\gamma(M, C, v + \mathrm{Vol}(S^n(1)))$.

References

[1] T. Aubin, Nonlinear analysis on manifolds, Monge-Ampère equations. Die Grundlehren der Math. Wissenschaften, vol 252, Springer-Verlag, 1982.

[2] L. Bérard Bergery, La coubure scalaire des variétés riemanniennes, Springer Lecture Notes 842 (1981), 225-245.

[3] J.P. Bourguignon, Une stratification de l'espace des structures riemanniennes, Composition Math., 30(1975), 1-41.

[4] A.E. Fischer and J.E. Marsden, Linearization stability of

nonlinear partial differential equations, Proc. Symp. Pure Math. 27(1975), 219-262.

[5] M. Gromov, Volume and bounded cohomology, Publ. Math. IHES 56(1983), 213-307.

[6] M. Gromov and H.B. Lawson, Jr., Positive scalar curvature and the Dirac operator on complete Riemannian manifolds, Publ. Math. IHES 58(1983), 83-196.

[7] J.L. Kazdan, Prescribing the curvature of a Riemannian manifold, Regional conference series in math. no. 57, Amer. Math. Soc. 1985.

[8] J.L. Kazdan and F.W. Warner, Prescribing curvatures, Proc. Symp. Pure Math. 27(1975), 309-319.

[9] ———, A direct approach to the determination of Gaussian and scalar curvature functions, Inv. Math. 28(1975), 227-230.

[10] O. Kobayashi, On total conformal curvature, Thesis, Tokyo Metropolitan University, 1985.

[11] M. Obata, The conjectures on conformal transformations of Riemannian manifolds, J.Diff.Geom. 6(1971), 247-258.

[12] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, J.Diff.Geom.20(1984), 479-496.

[13] R. Schoen and S.T. Yau, On the structure of manifolds with positive scalar curvature, Manuscripta Math. 28(1979), 159-183.

Max-Planck-Institut
für Mathematik
5300 Bonn 3 B.R.D.

Department of Mathematics
Keio University
Hiyoshi, Yokohama 223
Japan