# SPECTRA OF SOME DOMAINS IN COMPACT LIE GROUPS AND THEIR APPLICATIONS

Hajime Urakawa

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3
Federal Republic of Germany

and

Department of Mathematics
College of General Education
Tohoku University
Kawauchi, Sendai, 980 Japan



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# Spectra of some domains in compact Lie groups and their applications

Hajime URAKAWA

Abstract. In this paper, we determine explicitly the spectra of the Dirichlet problems of some domains in simply connected compact simple Lie groups. As their applications we can state, combining results of Hoffman [6], Mori [10], some stability conditions of these domains for the standard minimal isometric immersions into unit spheres.

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1. Introduction and results. Let $M$ be a simply connected compact simple Lie group and let $T$ be its maximal torus. We give a bi-invariant Riemannian metric $g$ on $M$ from the Killing form $B$ of the Lie algebra $\underline{m}$ of $M$ by

$$g_m(X_m, Y_m) = - B(X,Y), \quad X,Y \in \underline{m},\ m \in M,$$

where $X_m$, $Y_m$ are tangent vectors of $M$ at $m$ corresponding to $X$, $Y$. Let $d(x,y)$ be the distance of $(M,g)$ between two points $x,y$ in $M$. Then it is known (cf. Crittenden[4], Sakai [13]) that the cut locus $C$ of the identity $e$ in $M$ satisfies

$$C = \bigcup_{x \in M} x\, C(T)\, x^{-1},$$

where $C(T)$ is the cut locus of $e$ in the flat torus $T$ induced from the Riemannian metric $g$. For a positive number $\varepsilon$ with $0 < \varepsilon < d(e,C)$, consider a domain $\Omega(\varepsilon)$ containing the cut locus $C$ in $M$ defined by

$$\Omega(\varepsilon) = \bigcup_{x \in M} x\, \Omega(\varepsilon,T)\, x^{-1},$$

$$\Omega(\varepsilon,T) = \{ t \in T \,;\, d(t,C(T)) < \varepsilon \}.$$

These domains $\Omega(\varepsilon)$, which are invariant under all the inner automorphisms of $M$, shrink to the cut locus $C$ as $\varepsilon \to 0$ and converge to the whole $M$ as $\varepsilon \to d(e,C)$.

Now let $\Delta$ be the Laplace-Beltrami operator of $(M,g)$ acting on the space $C^\infty(M)$ of smooth functions on $M$, and for every $\varepsilon$ with $o < \varepsilon < d(e,C)$, let us consider the following Dirichlet problem for the above domains :

$$(\#)_\varepsilon \begin{cases} \Delta u + \lambda u = o & \text{on } M \setminus \overline{\Omega(\varepsilon)}, \text{ and} \\ u = o & \text{on } \Omega(\varepsilon). \end{cases}$$

For a solusion $u$ of the Dirichlet problem $(\#)_\varepsilon$, define a function $u^o$ on $M$ by

$$u^o(x) = \int_M u(yxy^{-1})\, dy,$$

where $dy$ is the Haar measure on $M$ normalized by $\int_M dy = 1$. Then ,if $u^o$ does not vanish identically, $u^o$ is also a solution of $(\#)_\varepsilon$, which is a zonal spherical function of $M$, i.e., invariant under all the inner automorphisms of $M$.

In this paper, we determine the spectra of the Dirichlet problem $(\#)_\varepsilon$ which have zonal spherical eigen-functions as follows :

Theorem 1. Let $M$ be a simply connected compact simple Lie group and let $\Delta$ be the Laplace-Beltrami operator of the Riemannian metric $g$ of $M$ induced from negative of the Killing form $B$ of the Lie algebra $\mathfrak{m}$ of $M$. Then for every $\varepsilon$ with $o < \varepsilon < d(e,C)$, the eigenvalues of the Dirichlet problem $(\#)_\varepsilon$ which have zonal spherical eigenfunctions are given by

$$(1) \qquad \left\{ \frac{d(e,C)}{d(e,C) - \varepsilon} \right\}^2 |\Lambda + \delta|^2 - |\delta|^2 \quad , \Lambda \in \mathbb{D},$$

and the corresponding zonal spherical eigenfunctions $u_{\Lambda,\varepsilon}$ are described explicitly by

$$(2) \quad u_{\Lambda,\varepsilon}(\exp H) = \begin{cases} \xi_{\Lambda+\delta}\left(\exp\left(\frac{d(e,C)}{d(e,C) - \varepsilon} H\right)\right) / \xi_\delta(\exp H), & \exp H \in T \setminus \Omega(\varepsilon, T), \\ o, & \exp H \in \Omega(\varepsilon, T). \end{cases}$$

Here $\mathbb{D}$ is the set of all dominant integral forms on $\underline{t}$ of $T$ the Lie algebra, $\delta$ is half the sum of all positive roots, $|\cdot|$ is the inner product of the dual space $\underline{t}^*$ of $\underline{t}$ induced from negative of the Killing form, and $\xi_\lambda$, $\lambda \in \mathbb{D}$, are the alternating characters of $T$ (cf. § 2 ).

Theorem 1 implies immediately

Corollary 1. Under the assumptions of Theorem 1, the first eigenvalue $\lambda_1(\varepsilon)$ of the Dirichlet problem $(\#)_\varepsilon$, $o < \varepsilon < d(e,C)$, is given by

$$\left\{\frac{d(e,C)}{d(e,C)-\varepsilon}\right\}^2 |\delta|^2 - |\delta|^2 = \left\{\frac{d(e,C)}{d(e,C)-\varepsilon}\right\}^2 \frac{d}{24} - \frac{d}{24},$$

where $d = \dim M$ ( cf. [p.291, 15] ). The corresponding eigenfunction with the eigenvalue $\lambda_1(\varepsilon)$ is $u_{o,\varepsilon}$.

Remark. In case of $S^3 = SU(2)$, the same formula as Theorem 1 was obtained in [p.201, 3] . Chavel and Feldman [3] investigated also the behavior of the eigenvalues $\lambda_1(\varepsilon)$ of the Dirichlet problems of the domains $X \setminus \overline{\Omega(\varepsilon)}$, where $\Omega(\varepsilon) = \{x \in X ; d(x,Y) < \varepsilon\}$ for every compact Riemannian manifold $X$ and a closed submanifold $Y$ of $X$ with codim $\geq 2$. A more precise behavior of the first eigenvalue has obtained in Ozawa [12], Matsuzawa and Tanno [9].

As a geometric application of Corollary 1, we can state some stability conditions of those domains $M \setminus \Omega(\varepsilon)$ in $M$ for the standard minimal isometric immersions $x_k$ of $M$ into the unit sphere as follows.

Let $\{o = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots\}$ be the set of all mutually distinct eigenvalues of negative of the Laplace-Beltrami operator $\Delta$ acting on $C^\infty(M)$. Let $V^k$, $k = 1,2,\ldots,$ be the eigenspace with the eigenvalue $\lambda_k$, and put $m(k) + 1 = \dim V^k$. We choose an orthonormal basis $\{f_j\}_{j=0}^{m(k)}$ of $V^k$ consisting of real valued functions with respect to the inner product $(\varphi, \psi) = \int_M \varphi(x)\psi(x)\, d\mu(x)$, where $d\mu(x)$ is the Haar measure of $M$ normalized by $\int_M d\mu(x) = m(k) + 1$. Consider the following mapping $x_k$ of $M$ into the Euclidean space $\mathbb{R}^{m(k)+1}$ defined by

$$x_k(p) = (f_0(p), f_1(p), \ldots, f_{m(k)}(p)), \quad p \in M.$$

Then it turns out that the image of $x_k$ is contained in the unit sphere $S^{m(k)}$, moreover the mapping $x_k$ is a minimal isometric immersion of $(M, \frac{\lambda_k}{d} g)$, $d = \dim M$, into the unit sphere $S^{m(k)}$ with the standard Riemannian metric of constant curvature 1 ( cf. [8] ) since $M$ is a simple Lie group.

For a piecewise smooth domain $D$ in $M$, we call $D$ is <u>stable</u> for the minimal immersion $x_k$ if for all normal variations $D_t$ which fix the boundary $\partial D$, the function $V(t) = \text{Volume } D_t$ satisfies $V''(0) > 0$. Combining Corollary 1 with results of Hoffman [6], Mori [10], we have

Corollary 2. Under the situations of Theorem 1, if a positive number $\varepsilon$ satisfies

$$d(e,C) > \varepsilon > d(e,C) - d(e,C)\left\{\frac{24\lambda_k}{d}(\|A\|^2 + d) + 1\right\}^{-1/2},$$

then for every $D \subset M \setminus \overline{\Omega(\varepsilon)}$, $D$ is stable for the minimal isometric immersion $x_k$. Here $\|A\|^2$ is the square of the length of the second fundamental form of the immersion $x_k$.

**Remark.** In case of $M = Sp(n)$ and $k = 1$, then it is known ( cf. Nagura [11], Kobayashi and Takeuchi [7] ) that $d = n(2n+1)$, $\|A\|^2 = n(n-1)(n+1)(2n+1)$ , and $\lambda_1 = \frac{2n+1}{4n+4}$ . Therefore for every $D \subset M \setminus \overline{\Omega(\varepsilon)}$ , $D$ is stable for the immersion $x_1$ if $d(e,C) > \varepsilon > d(e,C)\left\{1 - \sqrt{\frac{n+1}{7n+1}}\right\}$, in particular, if $d(e,C) > \varepsilon > 0.623\, d(e,C)$.

## 2. Preliminaries.

Since we will use the precise formula of the radial part ( cf. [2] ) of the Laplace-Beltrami operator and the structure of the cut loci $C$ and $C(T)$ (cf. [13]) in the proof of Theorem 1, we have to prepare some notations.

2.1. Let $M$ be a simply connected compact simple Lie group, and let $T$ be a maximal torus in $M$. Let $\underline{m}$ ( resp. $\underline{t}$ ) be the Lie algebra of $M$ ( resp. $T$ ). Since the Killing form $B$ is negative definite on $\underline{m}$, we define an Ad(M)-invariant positive definite inner product $( , )$ on $\underline{m}$ by $(X,Y) = -B(X,Y)$, $X,Y \in \underline{m}$, which induces a bi-invariant Riemannian metric $g$ on $M$ as in introduction. Let $\underline{\Sigma}$ be the root system of the complexification $\underline{m}^{\mathbb{C}}$ of $\underline{m}$ with respect to $\underline{t}$, i.e., the set of non-zero elements $\underline{\alpha}$ of the dual space $\underline{t}^*$ of $\underline{t}$ such that $\{E \in \underline{m}^{\mathbb{C}} ; [H,E] = \sqrt{-1}\,\underline{\alpha}(H)\, E$ for all $H \in \underline{t}\}$ is not zero. We give a lexicographic order $>$ on $\underline{\Sigma}$ and let $\underline{\Sigma}_+$ be the set of all positive roots. Let $\underline{\alpha}^o$ be the highest root of $\underline{\Sigma}_+$ with respect to the order $>$. Put $\overline{\underline{t}^+} = \{H \in \underline{t} \,;\, \underline{\alpha}(H) \geq 0$ for all $\underline{\alpha} \in \underline{\Sigma}_+\}$.

Then the cut locus $C$ of the identity $e$ in $(M,g)$ is given ( cf. Sakai [13]) by

$$(2.1) \qquad C = \bigcup_{x \in M} x\, C(T)\, x^{-1} .$$

Here $C(T)$ is the cut locus of the flat torus $T$ induced from the Riemannian metric $g$ which is given ( cf. Takeuchi [14], Sakai [13]) by

$$(2.2) \qquad C(T) = \exp \widetilde{C}(\underline{t}),$$

$$(2.3) \qquad \widetilde{C}(\underline{t}) = \bigcup_{s \in W} s\left\{H \in \overline{\underline{t}^+} \,;\, \underline{\alpha}^o(H) = 2\pi\right\},$$

Put

$$(2.4)\quad \tilde{D}^{+}(\underline{t}) = \{ H \in \overline{\underline{t}^{+}} \;;\; \alpha^{o}(H) \leq 2\pi \}, \quad \tilde{D}(\underline{t}) = \bigcup_{s \in W} s\, \tilde{D}^{+}(\underline{t}),$$

and $\tilde{D} = \bigcup_{x \in M} \mathrm{Ad}(x)\, \tilde{D}(\underline{t})$. Then $\tilde{C}(\underline{t})$ is the boundary $\partial \tilde{D}(\underline{t})$ of $\tilde{D}(\underline{t})$, both the exponential mappings $\exp : \tilde{D}(\underline{t}) \longrightarrow T$, $\exp : \tilde{D} \longrightarrow M$ are onto mappings, and the restriction to the interior of $\tilde{D}$ is a diffeomorphism. Moreover the distance $d(e,C)$ between the identity $e$ and the cut locus $C$ is given by

$$(2.5)\quad d(e,C) = 2\pi / |\alpha^{o}| .$$

Here $|\cdot|$ is the norm of the inner product $(\ ,\ )$ on $\underline{t}^{*}$ induced from the inner product $(\ ,\ )$ on $\underline{t}$ by $(\lambda,\mu) = (H_{\lambda},H_{\mu})$, $\lambda, \mu \in \underline{t}^{*}$, where $H_{\lambda} \in \underline{t}$, $\lambda \in \underline{t}^{*}$, is the unique element in $\underline{t}$ satisfying $(H_{\lambda},H) = \lambda(H)$ for every $H \in \underline{t}$. Note that the distance $d(x,y)$, $x,y \in T$, coincides with the one with respect to the Riemannian metric on $T$ induced from the metric $g$ on $M$ ( see Remark in [p.80, 5]). In fact, since $T$ is totally geodesic in $M$, we have only to show the existence of a distance minimizing geodesic in $T$ joining $e$ and every $x$ in $T$. But it follows immediately from Theorem 7.9 (ii) and Lemma 7.10 in [5].

Then we have :

Lemma 2.1. For every $\varepsilon$ with $0 < \varepsilon < d(e,C) = 2\pi/|\alpha^{o}|$,

(i) the set $\Omega(\varepsilon,T) = \{ t \in T \;;\; d(t,C(T)) < \varepsilon \}$ is given by

$$\Omega(\varepsilon,T) = \exp \tilde{\Omega}(\varepsilon,\underline{t}), \quad \tilde{\Omega}(\varepsilon,\underline{t}) = \bigcup_{s \in W} s \{ H \in \overline{\underline{t}^{+}} ;\; 2\pi(1-\varepsilon\tfrac{|\alpha^{o}|}{2\pi}) < \alpha^{o}(H) \leq 2\pi \}.$$

(ii) The set $M \setminus \overline{\Omega(\varepsilon)}$ is given as follows :

$$M \setminus \overline{\Omega(\varepsilon)} = \bigcup_{x \in M} x \exp \widetilde{D}^+(\varepsilon)\, x^{-1} ,$$

where $\widetilde{D}^+(\varepsilon) = \left\{ H \in \overline{t^+} \;;\; \alpha^0(H) < 2\pi\left(1 - \varepsilon \frac{|\alpha^0|}{2\pi}\right)\right\}$.

(iii) $\dfrac{d(e,C)}{d(e,C) - \varepsilon} \cdot \widetilde{D}^+(\varepsilon) = \left\{ H \in \overline{t^+} \;;\; \alpha^0(H) < 2\pi \right\}$.

Here, for every $r > 0$, $r \cdot \widetilde{D}^+(\varepsilon)$ means the set $\{ r\, H \;;\; H \in \widetilde{D}^+(\varepsilon)\}$.

Proof. (i) By definition of $\widetilde{\Omega}(\varepsilon, t)$, (2.3), and the invariance of the distance $d$ under the inner automorphisms of $M$, we have

$$\widetilde{\Omega}(\varepsilon, t) = \bigcup_{s \in W} s \left\{ H \in \widetilde{D}^+(t) \;;\; d(\exp H, \exp \widetilde{C}(t)) < \varepsilon \right\}.$$

We denote $d_e(X,Y) = |X - Y|$, for $X, Y \in t$. Then, for each $H \in \widetilde{D}^+(t)$,

$$(2.6) \quad d(\exp H, \exp \widetilde{C}(t)) = d(\exp H, \exp(\widetilde{C}(t) \cap \overline{t^+}))$$
$$= d_e(H, \widetilde{C}(t) \cap \overline{t^+}).$$

In fact, putting $\Gamma = \{ X \in t \;;\; \exp X = e \}$, we have $d(\exp H, \exp \widetilde{C}(t)) = d_e(H, \widetilde{C}(t) + \Gamma)$ and $d(\exp H, \exp(\widetilde{C}(t) \cap \overline{t^+})) = d_e(H, (\widetilde{C}(t) \cap \overline{t^+}) + \Gamma)$. Since $(\widetilde{C}(t) + \Gamma) \cap \widetilde{D}(t) = \widetilde{C}(t)$ and $((\widetilde{C}(t) \cap \overline{t^+}) + \Gamma) \cap \widetilde{D}(t) = \widetilde{C}(t) \cap \overline{t^+}$, $d_e(H, \widetilde{C}(t) + \Gamma) = d_e(H, \widetilde{C}(t))$ and $d_e(H, (C(t) \cap \overline{t^+}) + \Gamma) = d_e(H, \widetilde{C}(t) \cap \overline{t^+})$. Since $H \in \widetilde{D}^+(t)$, we have $d_e(H, \widetilde{C}(t)) = d_e(H, \widetilde{C}(t) \cap \overline{t^+})$.
For the proof of the second equality, choose an element $X$ in $\widetilde{C}(t) \cap \overline{t^+}$ such that $d(\exp H, \exp(\widetilde{C}(t) \cap \overline{t^+})) = d(\exp H, \exp X)$. Then $d(\exp H, \exp X) = d(e, \exp(-H+X))$ and $-H+X \in \widetilde{D}(t)$, because of $0 \leq \alpha(H), \alpha(X) \leq 2\pi$ for $\alpha \in \Sigma_+$, and the definition (2.4) of $\widetilde{D}(t)$. Then $d(e, \exp(-H+X)) = |-H+X|$, which implies $d(\exp H, \exp(\widetilde{C}(t) \cap \overline{t^+})) \geq d_e(H, \widetilde{C}(t) \cap \overline{t^+})$. The converse inequality is clear.

By (2.3) and (2.6), we have

$$\{H\in\tilde{D}^+(\underline{t});\ d(\exp H,\exp\tilde{C}(\underline{t}))<\varepsilon\}=\{H\in\tilde{D}^+(\underline{t});\ d_e(H,\tilde{C}(\underline{t})\cap\overline{\underline{t}^+})<\varepsilon\}$$

$$=\left\{(1-r)\frac{2\pi H_{\alpha^0}}{(\alpha^0,\alpha^0)}+X\ ;\ |r|<\frac{\varepsilon|\alpha^0|}{2\pi},\ X\in\underline{t},\ \alpha^0(X)=0\right\}\cap\tilde{D}^+(\underline{t})$$

$$=\left\{H\in\overline{\underline{t}^+}\ ;\ 2\pi\left(1-\varepsilon\frac{|\alpha^0|}{2\pi}\right)<\alpha^0(H)\leq 2\pi\right\}.$$

For (ii), we have only to show $X = Y$ when $g_1 \exp X\, g_1^{-1} = g_2 \exp Y\, g_2^{-1}$, $X\in\tilde{D}^+(\underline{t})$, $Y\in\tilde{D}^+(\underline{t})$, $g_1, g_2\in M$. But in this case, we have $\exp X = \exp sY$ for some $s\in W$, because of Lemma 7.1o in [5]. Since $sY\in\tilde{D}(\underline{t})$ and $X\in\tilde{D}^+(\underline{t})$, $\exp X = \exp sY$ implies $X = sY$, and then $X = Y$. (iii) follows immediately from (ii).

Q.D.E.

2.2. For $\lambda\in\underline{t}^*$, $\lambda\neq 0$, put $H^*_\lambda = \frac{2}{(\lambda,\lambda)}H_\lambda$. Then since $M$ is simply connected, the lattice $\Gamma = \{H\in\underline{t}\ ;\ \exp H = e\}$ is given by $\Gamma = 2\pi\sum_{i=1}^{\ell}\mathbb{Z}\,H^*_{\alpha_i}$, where $\{\alpha_i\}_{i=1}^{\ell}$ is a fundamental system of $\Sigma$ with respect to the order $>$, and let $\ell = \dim T$. Put

$$I=\{\lambda\in\underline{t}^*;\ \lambda(H^*_{\alpha_i})\in\mathbb{Z},\ i=1,\dots,\ell\}$$

$$=\{\lambda\in\underline{t}^*;\ \lambda(\Gamma)\subset 2\pi\mathbb{Z}\},$$

$$\mathbb{D}=\{\lambda\in I\ ;\ (\lambda,\alpha)\geq 0\ \text{ for every }\alpha\in\Sigma^+\}.$$

An element of $\mathbb{D}$ is called a dominant integral form on $\underline{t}$. For $\lambda\in I$, define an function $\xi_\lambda$ on $T$, called the alternating character, by

$$\xi_\lambda(\exp H) = \sum_{s\in W}(-1)^s e^{s\lambda(H)},\quad H\in\underline{t}.$$

Put $\delta = \frac{1}{2}\sum_{\alpha\in\Sigma^+}\alpha$. Then $\delta$ belongs to $\mathbb{D}$. Moreover it is

known that $\xi_{\delta}(\exp H) = \prod_{\alpha \in \Sigma^+} ( e^{\frac{\sqrt{-1}}{2}\alpha(H)} - e^{-\frac{\sqrt{-1}}{2}\alpha(H)} )$,
every $\xi_{\lambda}$, $\lambda \in I$, can be divided by $\xi_{\delta}$, and $\xi_{\Lambda+\delta}/\xi_{\delta}$, $\Lambda \in D$, coincides with the restriction to $T$ of the character $\chi_{\Lambda}$ of the irreducible unitary representation of $M$ with highest weight $\Lambda$ ( cf. [14]). For every $C^{\infty}$ zonal spherical function $f$ on $M$, let $\bar{f}$ be its restriction to $T$. Then $\bar{f}(\exp sH) = \bar{f}(\exp H)$, $s \in W$, $H \in t$, and we have ( cf. Berezin[2], or [14])

$$(2.7) \qquad \xi_{\delta}\, \overline{(\Delta f)} = \{ \Delta_o + |\delta|^2 \}(\xi_{\delta} \bar{f}) \, ,$$

on $T$, where $\Delta_o$ is the standard Laplacian on $T$ induced from the Euclidean Laplacian of $t$ with respect to the inner product $( \, , \, )$.

## 3. Proof of Theorem 1.

For $0 < \varepsilon < d(e,C)$, assume that a zonal spherical function $u$ on $M$ satisfies

$$(3.1)\quad \begin{cases} \Delta u + \lambda u = 0 & \text{on } M \setminus \overline{\Omega(\varepsilon)}, \text{ and} \\ u = 0 & \text{on } \Omega(\varepsilon). \end{cases}$$

Then by (2.7), we have

$$\begin{cases} (\Delta_0 + |\delta|^2)(\xi_\delta \bar{u}) + \lambda \xi_\delta \bar{u} = 0 & \text{on } T \setminus \overline{\Omega(\varepsilon, T)}, \text{ and} \\ \bar{u} = 0 & \text{on } \Omega(\varepsilon, T). \end{cases}$$

Now define a function $(\xi_\delta \bar{u})_\varepsilon$ on $T$ by

$$(\xi_\delta \bar{u})_\varepsilon(\exp H) = (\xi_\delta \bar{u})\left(\exp\left(\frac{d(e,C)-\varepsilon}{d(e,C)} H\right)\right), \quad H \in \widetilde{D}(\underline{t}).$$

It is well-defined on $T$ because of Lemma 3.1(iii), and $\bar{u} = 0$ on $\Omega(\varepsilon, T)$. Define also a function $\widetilde{(\xi_\delta \bar{u})_\varepsilon}$ on $\widetilde{D}(\underline{t})$ by

$$\widetilde{(\xi_\delta \bar{u})_\varepsilon}(H) = (\xi_\delta \bar{u})_\varepsilon(\exp H), \quad H \in \widetilde{D}(\underline{t}).$$

Then the function $\widetilde{(\xi_\delta \bar{u})_\varepsilon}$ satisfies

$$\begin{cases} \Delta_0 \widetilde{(\xi_\delta \bar{u})_\varepsilon} + \left\{\dfrac{d(e,C)-\varepsilon}{d(e,C)}\right\}^2 (|\delta|^2 + \lambda)\widetilde{(\xi_\delta \bar{u})_\varepsilon} = 0, \\ \qquad\qquad\qquad \text{on the interior of } \widetilde{D}(\underline{t}), \text{ and} \\ \widetilde{(\xi_\delta \bar{u})_\varepsilon} = 0 \quad \text{on } \partial \widetilde{D}(\underline{t}). \end{cases}$$

Moreover $\widetilde{(\xi_\delta \bar{u})_\varepsilon} = 0$ on $\widetilde{D}^+(\underline{t})$ since $\xi_\delta = 0$ on $\partial \widetilde{D}^+(\underline{t})$. Therefore $\widetilde{(\xi_\delta \bar{u})_\varepsilon}$ is the eigenfunction of the Dirichlet problem for the domain $\widetilde{D}^+(\underline{t})$. Since the domain $\widetilde{D}^+(\underline{t})$ is a fundamental domain of the affine Weyl group of the Lie group $M$ acting on $\underline{t}$, by a theorem of Bèrard [1], we have

$$\widetilde{(\xi_\delta \bar{u})_\varepsilon}(H) = \sum_{s \in W} (-1)^s e^{\sqrt{-1}\, s(\Lambda + \delta)(H)},$$

for some $\underline{\Lambda} \in \underset{\sim}{D}$, and $\left\{\frac{d(e,C)-\underline{\varepsilon}}{d(e,C)}\right\}^2 (|\underline{\delta}|^2 + \underline{\lambda}) = |\underline{\Lambda} + \underline{\delta}|^2$.

Therefore we obtain

$$(3.2) \quad \underline{\lambda} = \left\{\frac{d(e,C)}{d(e,C)-\underline{\varepsilon}}\right\}^2 |\underline{\Lambda} + \underline{\delta}|^2 - |\underline{\delta}|^2 \quad , \text{ and}$$

$$(3.3) \quad u(\exp H) = \begin{cases} \xi_{\underline{\Lambda}+\underline{\delta}}\left(\exp\left(\frac{d(e,C)}{d(e,C)-\underline{\varepsilon}} H\right)\right)/\xi_{\underline{\delta}}(\exp H), & H \in \Omega(\underline{\varepsilon},\underline{t}) \\ 0 , & H \notin \Omega(\underline{\varepsilon},\underline{t}). \end{cases}$$

Conversely, the function $u$ defined by (3.3) is a zonal spherical function on $M$ and satisfies (3.1) with the eigenvalue (3.2). We have Theorem 1.

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Department of Mathematics,
College of General Education,
Tohoku University,
Kawauchi, Sendai, 980, Japan

and

Max-Planck-Institut für Mathematik,
Gottfried-Claren-Straß 26,
5300, Bonn 3, Federal Republic of Germany