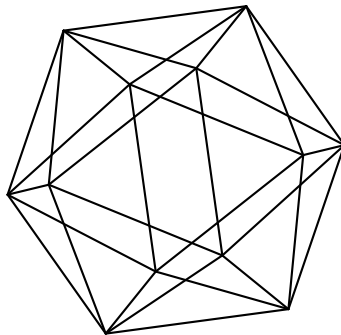# Max-Planck-Institut für Mathematik Bonn

Hidden symmetry of genus 0 modular operad and its stacky versions

by

Noémie C. Combe
Yuri I. Manin

Hidden symmetry of genus 0 modular
operad and its stacky versions

Noémie C. Combe
Yuri I. Manin

HIDDEN SYMMETRY OF GENUS 0 MODULAR OPERAD AND ITS STACKY VERSIONS

NOÉMIE C. COMBE, YURI I. MANIN

Dedicated to Werner Ballmann

CONTENTS

1. Introduction

1.1. **Belyi's theorem and the Grothendieck–Teichmüller group.** The problem of understanding the Galois group $G_{\mathbb{Q}} := Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ was approached by Grothendieck in his *"Esquisse d'un programme"* via a bold conjecture that any algebraic curve X over $\overline{\mathbb{Q}}$ admits a map $b : X \to \mathbf{P}^1$ ramified only over three points, say $\{0, 1, \infty\}$. This conjecture was proved in 1980 by G. Belyi.

This result represented $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ as fragment of a complicated groupoid including maps b as its main objects, and various isomorphisms between them induced by actions of geometric fundamental groups and the Galois group of algebraic numbers.

The *dessin d'enfants* formalism sketched by Grothendieck (cf.[20]) was a visually transparent encoding of the geometry of this groupoid via finite graphs including (in principle) also encoding of the Galois action upon these graphs, which led to the embedding of $G_{\mathbb{Q}}$ into the automorphism group of the profinite completion of the geometric fundamental group of $\mathbf{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$:

$$G_{\mathbb{Q}} \to \mathrm{Aut}\, \hat{\pi}_1(\mathbf{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}; (0, 1)).$$

The fundamental group involved here is simply a free group with two generators F_2. Using the de Rham formalism, V. Drinfeld later defined a highly nontrivial subgroup $\widehat{GT}$ of $\mathrm{Aut}\, \hat{\pi}_1$, *the Grothendieck–Teichmüller group*, such that the image of $G_{\mathbb{Q}}$ is contained in $\widehat{GT}$.

For a contemporary survey and many references, see [1].

In this paper, we focus upon the operad of moduli spaces of stable curves of genus zero with marked points, whose n–ary component is $\overline{M}_{n+3}$, and upon what we call "the hidden symmetry" of this operad. This symmetry can be seen in the simple structure "rigidifying" the action of $G_{\mathbb{Q}}$ upon $\hat{\pi}_1$ above. This structure lies also in the base of the definition of $\widehat{GT}$. In order to briefly describe it we adopt the approach and (temporarily) notations of Y. Ihara in [8].

Following Ihara, denote by $\overline{\mathbb{Q}}\{\{t\}\}$ the algebraically closed field of formal Puiseux series in fractional powers of a variable t. More precisely, any element h of this field is a formal Laurent series with coefficients in $\overline{\mathbb{Q}}$, in some formal root $t^{1/N}$,

$N \geq 1$ being an integer that can depend on h. Furthermore, denote by M the maximal Galois extension of $\overline{\mathbb{Q}}(t)$ in $\overline{\mathbb{Q}}\{\{t\}\}$ unramified outside $t = \{0, 1, \infty\}$. In the constructions below, we interpret $t^{1/N}$, resp. $(1-t)^{1/N}$, as functions whose restrictions to $(0, 1)$ take positive values.

The topological fundamental group $\pi_1(\mathbf{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}; (0, 1))$ is freely generated by elements x (resp. y) which are classes of loops starting at any point of the real interval $(0, 1)$ and passing once around 0 (resp. around 1) anticlockwise. This gives the identification of this group with F_2 mentioned above. By analytic continuation, this group acts upon M, leaving the subfield $\overline{\mathbb{Q}}(t) \subset M$ invariant.

At this point, the "hidden symmetry" $t \mapsto 1-t$ we had in mind enters the game. Denote by M' the maximal Galois extension of $\overline{\mathbb{Q}}(t)$ in $\overline{\mathbb{Q}}\{\{1-t\}\}$ unramified outside $\{0, 1, \infty\}$. There is a unique isomorphism $p : M \to M'$ over $\overline{\mathbb{Q}}(t)$ which is obtained by simply passing from the Puiseux expansion in t of a function near 0 to its expansion in $1-t$ near 1. The group $G_{\mathbb{Q}}$ acts upon both expansions by acting upon their coefficients, so that elements of $\mathbb{Q}(t)$ remain invariant. Denote these actions by σ_M, resp. $\sigma_{M'}$.

Now for each $\sigma \in G_{\mathbb{Q}}$ we can define the element $f_\sigma \in Gal(M/\overline{\mathbb{Q}}(t))$, whose action upon any $h \in M$ is defined as the composition of the following maps: apply σ_M^{-1} to h, then p to the result, then $\sigma_{M'}$, and finally p^{-1}.

Let us use σ_M in order to embed $G_{\mathbb{Q}}$ into $Gal(M/\mathbb{Q}(t))$ by action on coefficients of a rational function of t. This subgroup then acts upon whole $Gal(M/\mathbb{Q}(t))$ by conjugation $g \mapsto \sigma(g) := \sigma g \sigma^{-1}$. In particular, it acts upon the topological generators x, y, and its action can be described explicitly in the following way. Each σ defines an element $f_\sigma \in \hat{F}_2$ such that

$$\sigma(x) = x^{\chi(\sigma)}, \quad \sigma(y) = f_\sigma^{-1} y^{\chi(\sigma)} f_\sigma,$$

where $\chi : G_{\mathbb{Q}} \to \mathbb{Z}^\times$ is the cyclotomic character.

Drinfeld, Ihara et al. proved that the element f_σ for each fixed σ satisfies the three relations, called n–cycle relations, for $n = 2, 3$, and 5: see [8], sec. 1.7. These relations determine the Grothendieck–Teichmüller group $\widehat{GT}$.

1.2. Genus zero modular operad and its hidden symmetry.

The components and composition morphisms of the genus zero modular operad are smooth projective manifolds. A naive way to describe its n–ary component $\overline{M}_{n+3}$ is this. First, consider the moduli space M_{n+3} of configurations of pairwise distinct points $(x_1, \ldots, x_n)$ in $\mathbf{P}^1 \setminus \{0, 1, \infty\}$: it can be naturally identified with an obvious open subset in $(\mathbf{P}^1)^n$. Second, construct a compactification of this subset by adding as fibres "stable" curves of genus zero with marked points that can be described as degenerations of the generic stable curve.

Since we mentioned points $\{0, 1, \infty\}$, we have implicitly introduced in this description the coordinate t on $\mathbf{P}^1$ already used above, and thus we can extend the involution $t \mapsto 1 - t$ to induce it upon M_{n+3}, and then, with some efforts, to $\overline{M}_{n+3}$.

Actually, in this context there is a better way to define the rigidification involving t. To make explicit the geometry behind it, imagine first $\mathbf{P}^1(\mathbb{C})$ as a topological sphere S^2 endowed with one complex structure and three "equators" $S^1_j \subset S^2$ in general position. For each j, we can introduce a complex coordinate t_j upon S^2 identifying S^1_j with a naturally oriented $\mathbf{P}^1(\mathbb{R})$. Then the whole symmetry group behind this rigidification will be generated by $t_j \mapsto 1 - t_j$, and later it can be extended to the whole group of symmetries of the modular operad of genus zero. This is what we can call its hidden symmetry. The same group is used in [I] and elsewhere in order to treat the cycle relations in the Grothendieck–Teichmüller group.

However, in this draft of the article, we restrict ourselves to the study of only one hidden involution.

1.3. **Brief summary.** The Sections 2 and 3 introduce an appropriate categorical context and fix notations, that will be used afterwards. The Section 3 starts with description on the basic notions of operadic formalism in the form convenient for studying modular operad(s).

Finally, in the remaining part of the Section 4 and in the Sections 5, 6 we introduce the main new operad NY and study its first properties. We compare to the already known operads (such as the Gravity operad). A geometric perspective is taken under account and in particular, we discuss a Riemannian and Kähler framework, in the last section.

2. MONOIDAL CATEGORIES

2.1. **Monoidal categories: general setting.** A monoidal category $\mathbf{C} = (\mathbf{C_0}, \otimes, \mathbf{I}, \mathbf{a}, \mathbf{l}, \mathbf{r})$ consists of a category $\mathbf{C_0}$, a functor $\otimes : \mathbf{C_0} \times \mathbf{C_0} \to \mathbf{C_0}$, an object I of $\mathbf{C_0}$ and natural isomorphisms : $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $l_X : I \otimes X \to X$, $r_X : X \otimes I \to X$, subject to two *coherence axioms* expressing commutativity of the following diagrams:

$$
\begin{array}{ccccc}
((W {\otimes} X) {\otimes} Y) {\otimes} Z & \xrightarrow{a_{W \otimes X, Y, Z}} & (W {\otimes} X) {\otimes} (Y {\otimes} Z) & \xrightarrow{a_{W, X, Y \otimes Z}} & W {\otimes} (X {\otimes} (Y {\otimes} Z)) \\
\downarrow{\scriptstyle a \otimes 1} & & & & \downarrow{\scriptstyle 1 \otimes a} \\
(W {\otimes} (X {\otimes} Y)) {\otimes} Z & & \xrightarrow{\qquad a_{W, X \otimes Y, Z} \qquad} & & W {\otimes} ((X {\otimes} Y) {\otimes} Z)
\end{array}
$$

$$(X \otimes I) \otimes Y \xrightarrow{a_{X,I,Y}} X \otimes (I \otimes Y)$$

with arrows $r \otimes 1$ and $1 \otimes r$ to $X \otimes Y$.

We say that a monoidal category is strict if the *associativity* morphism is the identity. More precisely, a strict monoidal category is a category equipped with a functor $\otimes : \mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$.

(1) $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$
(2) $(f \otimes g) \otimes h = f \otimes (g \otimes h)$
(3) $X \otimes 1 = 1 \otimes X$
(4) $f \otimes id_1 = f = id \otimes f,$

for all objects X, Y, Z and all morphisms f, g, h in $\mathbf{C}$.

2.2. Monoidal Functor.

Definition 1 (Monoidal functor)**.** *A monoidal functor* $\Phi = (F_1, F_2, F_0) : C \rightarrow C'$ *between monoidal categories* C *and* C' *consists of the following items:*

(1) *An ordinary functor* $F_1 : C \rightarrow C'$ *between categories;*
(2) *For objects* a, b *in* C *morphisms:*

$$(1) \qquad\qquad F_2(a, b) : F(a) \otimes F(b) \rightarrow F(a \otimes b)$$

in C' *which are natural in* a *and* b
(3) *For the units* e *and* e', *a morphism in* C'

$$(2) \qquad\qquad F_0 : e' \rightarrow Fe$$

Together these must make all the following three diagrams involving the structural maps α, λ *and* ρ *commute in* C'.

$$
\begin{array}{ccc}
F(a) \otimes (F(b) \otimes F(c)) & \xrightarrow{\alpha} & (F(a) \otimes F(b)) \otimes F(c) \\
\downarrow{\scriptstyle 1 \otimes F_2} & & \downarrow{\scriptstyle F_2 \otimes 1} \\
F(a) \otimes (F(b \otimes c)) & & F(a \otimes b) \otimes F(c)) \\
\downarrow{\scriptstyle F_2} & & \downarrow{\scriptstyle F_2} \\
F(a \otimes (b \otimes c)) & \xrightarrow{F_1(\alpha)} & F(a \otimes b) \otimes c)
\end{array}
$$

$$
\begin{array}{ccc}
F(b)\otimes e' & \xrightarrow{\ \ \rho\ \ } & F(b) \\
\downarrow{\scriptstyle 1\otimes F_0} & & \uparrow{\scriptstyle F(\rho)} \\
F(b)\otimes F(e) & \xrightarrow{\ \ F_2\ \ } & F(b\otimes e)
\end{array}
$$

$$
\begin{array}{ccc}
e'\otimes F(b) & \xrightarrow{\ \ \lambda\ \ } & F(b) \\
\downarrow{\scriptstyle F_0\otimes 1} & & \uparrow{\scriptstyle F(\lambda)} \\
F(e)\otimes F(b) & \xrightarrow{\ \ F_2\ \ } & F(e\otimes b)
\end{array}
$$

Proposition 1. *Lax monoidal functors send monoids to monoids: if $F : (C, \otimes) \to (C', \otimes)$ is a lax monoidal functor and*

$$(3) \qquad\qquad A \in C, \mu_A : A \otimes A \to A, i_A : I \to A$$

is a monoid object in C, the object $F(A)$ is naturally equipped with the structure of a monoid in C' by setting

$$(4) \qquad\qquad i_{F(A)} : I_{C'} \to F(I_C) \underrightarrow{F(i_A)} F(A)$$

and

$$(5) \qquad\qquad \mu_{F(A)} : F(A) \otimes F(A) \to F(A \otimes A) \underrightarrow{F(\mu_A)} F(A).$$

This construction defines functor,

$$Mon(f) : Mon(C) \to Mon(C').$$

2.3. **Symmetric monoidal categories.** A symmetric (i.e. commutative) monoidal category is a monoidal category with a commutativity constraint i.e. a family of natural isomorphisms τ such that $\tau_{X,Y} : X \otimes Y \to Y \otimes X$, for all couples X, Y verifying:

$$\tau_{X,Y}^{-1} = \tau_{Y,X}, \quad \tau_{Y,X} \circ \tau_{X,Y} = Id.$$

The map τ is said to be a natural map if it verifies the following commutative diagram:

$$
\begin{array}{ccc}
X\otimes Y & \xrightarrow{\ \ \tau_{X,Y}\ \ } & Y\otimes X \\
\downarrow{\scriptstyle \phi\otimes\psi} & & \downarrow{\scriptstyle \psi\otimes\phi} \\
X'\otimes Y' & \xrightarrow{\ \ \tau_{X',Y'}\ \ } & Y'\otimes X'
\end{array}
$$

where ϕ and ψ are morphisms.

Note, that if we forget the condition $\tau \circ \tau = Id$, then we have only a braided monoidal category.

Let $\mathbf{C}$ be a strict monoidal category. A *braiding* is a commutativity constraint in $\mathbf{C}$ satisfying two relations:

(1) $c_{X \otimes Y, Z} = (c_{x,z} \otimes Id_Y)(id_X \otimes c_{Y,Z})$,
(2) $c_{X, Y \otimes Z} = (Id_Y \otimes c_{X,Z})(c_{X,Y} \otimes Id_Z)$

for all objects X, Y, Z.

Given a symmetric monoidal category $\mathbf{C}$, we have that for any $X \in Ob(\mathbf{C})$, where $Ob(\mathbf{C})$ are the objects of $\mathbf{C}$, there is a natural action of the symmetric group on $X^{\otimes n}$, given by :

$$(i, i+1) \mapsto Id_{X^{\otimes i-1}} \otimes \tau_{X,X} \otimes Id_{X^{\otimes n-i-2}}.$$

2.4. Monoid in a monoidal category.
More generally, a *monoid* in a monoidal category $\mathbf{C}$ is given by the following data:

(1) An object A of $\mathbf{C}_0$;
(2) Two morphisms $\mu : A \otimes A \to A$ (multiplication) and $\eta : I \to A$ (unit).

These morphisms verify the following axioms:

- Associativity Axiom

- Unit Axiom

A commutative monoid in a symmetric monoidal category $\mathbf{C}$, is a monoid A such that

$$\mu_A \circ \tau_{A,A} = \mu_A.$$

3. Groupoids

A prestable curve over a scheme T is a flat proper morphism $\pi : C \to T$ whose geometric fibers are reduced one-dimensional schemes with at most ordinary double singular points. Its genus is a locally constant function on T: $g(t) := dim H^1(C_t, \mathcal{O}_t)$. An S-pointed (S is a finite set) prestable curve over T is a family where $\pi : C \to T$ is a prestable curve and x_i are sections such that for any geometric point t of T, we have $x_i(t) \neq x_j(t)$.

Such a curve is stable if it is connected and the normalization of each irreducible component which has genus zero carries at least three special points. Let $(C, \pi, x_i | i \in S)$ is an S-pointed prestable curve. It is stable iff automorphism groups of its geometric fibers are finite and there are no infinitesimal automorphisms.

3.1. Groupoids: general setting.
Let $\mathcal{F}$ and $\mathcal{S}$ be two categories and $b : \mathcal{F} \to \mathcal{S}$ a functor. If $F \in Ob(\mathcal{F})$ and $b(F) = T$, we will call F a family with the base T inducing identitiy on T, or a T-family.

3.1.1. *Condition for groupoids.*
In order to form a groupoid, the data must satisfy the following:

VERSION 1: First, for any base $T \in \mathcal{S}$, any morphism of families over T inducing identity on T must be an isomorphism. There must be given the base change.

VERSION 2: For any arrow $\phi : T_1 \to T_2$, between the basis and any family F_2 over the target T_2, there must exist a T_1-family F_1 and a morphism $F_1 \to F_2$ lifting ϕ.

3.1.2. *1-morphisms of abstract groupoids.*
We will be considering only morphisms between groupoids over the same category of bases $\mathcal{S}$. Such a morphism $\{b_1 : \mathcal{F}_1 \to \mathcal{S}\} \to \{b_2 : \mathcal{F}_2 \to \mathcal{S}\}$ is a functor $\Phi : \mathcal{F}_1 \to \mathcal{F}_2$ such that $b_2 \circ \Phi = b_1$.

$$\mathcal{F}_1 \xrightarrow{\quad \Phi \quad} \mathcal{F}_2$$
$$b_1 \searrow \quad \swarrow b_2$$
$$\mathcal{S}$$

3.2. Groupoids of S-labeled stable curves.
Here $\mathcal{S}$ is the category of schemes, objects of $\mathcal{F}$ are stable S-labeled curves over $T \in \mathcal{S}$, and a morphism: $(C_1/T_1, x_{1i} | i \in S) \to (C_2/T_2, x_{2i} | i \in S)$ is a pair of compatible morphism $\phi : T_1 \to T_2; \psi : C_1 \to C_2$ such that ψ induces an isomorphism of labeled curves $C_1 \to \phi^*(C_2)$. Equivalently, the diagram

$$
\begin{array}{ccc}
C_1 & \xrightarrow{\psi} & C_2 \\
\downarrow & & \downarrow \\
T_1 & \xrightarrow{\phi} & T_2
\end{array}
$$

is cartesian, and induces the bijection of the two families of S-labeled sections.

3.3. Groupoids of universal curves.

The Objects are stable curves $(C/T, x_i | i \in S)$ endowed with an additional section $\tilde{\Delta} : T \to C$ not constrained by any restrictions.

The Morphisms must be compatible with this additional data.

3.4. Stacks.

Definition 1 (5.1in [13], 4.1 [3]). *A stack of groupoids is a quadruple*

$$(\mathcal{F}, \mathcal{S}, b : \mathcal{F} \to \mathcal{S}, \text{Grothendieck topology } \mathcal{T} \text{ on } \mathcal{S})$$

satisfying the following conditions:

(1) *$b : \mathcal{F} \to \mathcal{S}$ is groupoid (such a in 4.2). Each contravariant representable functor $S^{op} \to Sets$ is a sheaf on $\mathcal{T}$.*
(2) *Any isomorphism beween families over a given base is uniquely defined by its restrictions to the elements of any covering of the base. Given X_1, X_2 over T, the functor $T' \mapsto Iso_{T'}(X_1 \to X_2)$*
(3) *Any family over a given base is uniquely defined by its local restrictions.*

The stacks over T are the objects of a 2-category (stacks/T): 1-morphisms are functors from one stack to another, and 2-morphisms are morphisms of functors. In this 2-category every 2-morphism is an isomorphism.

4. OPERADS

4.1. Operad in a category.

Let $(\mathbf{C}, \otimes, 1_C, a, l, r, \tau)$ be a symmetric monoidal category (see section 2.3). Let **Fin** be the category of finite sets with bijections. Given any subset $X \subset Y$, we use the notation $Y/X := Y \setminus X \sqcup \{*\}$.

Definition 2 (Operad, Definition 4.1). [14] *An operad in $\mathbf{C}$ is a presheaf $\mathcal{P} : Fin^{op} \to \mathbf{C}$ endowed with partial operadic composition $\circ_{X \subset Y}$:*

$$\mathcal{P}(Y/X) \otimes \mathcal{P}(X) \to \mathcal{P}(Y),$$

for any $X \subset Y$ and a unit $\eta : 1_{\mathbf{C}} \to \mathcal{P}(\{*\})$ such that the following diagrams commute.

$$\mathcal{P}(Z\backslash Y)\otimes\mathcal{P}(Y\backslash X))\otimes\mathcal{P}(X) \xrightarrow{\alpha} \mathcal{P}(Z\backslash Y)\otimes(\mathcal{P}(Y\backslash X)\otimes\mathcal{P}(X)) \xrightarrow{id\otimes\circ_{X\subset Y}} \mathcal{P}(Z\backslash Y)\otimes\mathcal{P}(Y))$$

$$\downarrow{\cong} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\circ_{Y\subset Z}}$$

$$\mathcal{P}((Z\backslash Y)\backslash(Y\backslash X)\otimes\mathcal{P}(Y\backslash X))\otimes\mathcal{P}(X)) \xrightarrow{\circ_{Y\backslash X\subset Z\backslash X}\otimes id} \mathcal{P}(Z\backslash X)\otimes\mathcal{P}(X)) \xrightarrow{\circ_{Y\subset Z}} \mathcal{P}(Z)$$

$$\mathcal{P}(Z\backslash X)\backslash Y)\otimes(\mathcal{P}(X)\otimes\mathcal{P}(Y)) \xrightarrow{id\otimes\tau} \mathcal{P}((Z\backslash X)\backslash Y))\otimes(\mathcal{P}(Y)\otimes\mathcal{P}(X)) \xrightarrow{\alpha^{-1}} (\mathcal{P}((Z\backslash X)\backslash Y)\otimes\mathcal{P}(Y))\otimes\mathcal{P}(X)$$

$$\downarrow{\alpha^{-1}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\circ_{Y\subset Z\backslash X}\otimes id}$$

$$\mathcal{P}((Z\backslash X)\backslash Y)\otimes\mathcal{P}(X)\otimes\mathcal{P}(Y) \qquad\qquad\qquad\qquad\qquad \mathcal{P}(Z\backslash X)\otimes\mathcal{P}(X)$$

$$\downarrow{\cong} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\circ_{X\subset Y}}$$

$$(\mathcal{P}((Z\backslash Y)\backslash X)\otimes\mathcal{P}(X))\otimes\mathcal{P}(Y) \xrightarrow{\circ_{X\subset Z\backslash Y}\otimes id} \mathcal{P}(Z\backslash Y)\otimes\mathcal{P}(Y) \xrightarrow{\circ_{Y\subset Z}} \mathcal{P}(Z)$$

From [14] we know that the skeletal category of **Fin** is the groupoid $\mathbb{S}$ whose objects are $\{1, 2, ..., n\}$ for any $n \in \mathbb{N}$ and whose morphisms are the elements of the symmetric groups $\mathbb{S}_n$. A presheaf on **Fin** is thus equivalent to a collection $\{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ of right $\mathbb{S}_n$-modules. In these terms the above structure of operad is equivalent to partial composition products $\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n + m - 1)$, for $1 \le i \le n$, and a unit map $\eta : 1_{\mathbf{C}} \to \mathcal{P}(1)$ satisfying the analogous axioms.

4.2. **Operad in groupoids (small categories).** An operad in the category of small categories consists of a *sequence* of small categories $\mathcal{P}(r), r \in \mathbb{N}$, each of which are equipped with a symmetric group action; together with a unit morphism $\eta : pt \to \mathcal{P}(1)$, and composition products $\mu : \mathcal{P}(r) \times \mathcal{P}(n_1) \times \cdots \times \mathcal{P}(n_r) \to \mathcal{P}(n_1 + \cdots + n_r)$, all formed in the category of categories, so that the identities expressed by the diagrams hold;

Since the category of groupoids forms a *symmetric monoidal subcategory* of the category of small categories, an operad in groupoids can be defined as an operad in categories $\mathcal{P}$ of which components $\mathcal{P}(r)$ are groupoids.

The composition structure of an operad in categories (resp. groupoids) can be defined by giving a collection of functors $\circ_k : \mathcal{P}(m) \times \mathcal{P}(n) \to \mathcal{P}(m + n - 1)$, $k = 1, ..., m$ satisfying the equivariance, unit and associativity relations.

4.2.1. *Morphisms.* A morphism of operads in categories $f : \mathcal{P} \to \mathcal{Q}$ is a sequence of functors $f : \mathcal{P}(r) \to \mathcal{Q}(r)$ preserving the internal structures attached to operads. The category of operads in groupoids forms a full subcategory of the category

of operads in categories. For operads in categories, we will naturally consider the operad morphisms of which all underlying functors $f : \mathcal{P}(r) \to \mathcal{Q}(r)$ are equivalences of categories.

4.3. **Operads stable of S-labeled curves.** For a finite set I denote by $\overline{\mathcal{M}}_{g,I}$ the Deligne-Mumford stack classifying stable curves of genus g and with marked points $(x_i)_{i \in I}$ labeled by I. For any injective map $\phi : I \to J$ of finite sets and any g such that $\overline{\mathcal{M}}_{g,I} \neq \emptyset$, there is a natural morphism of stacks $\overline{\mathcal{M}}_{g,J} \to \overline{\mathcal{M}}_{g,I}$ called the stable forgetting.

We consider the case $I = [n] = \{0, 1, ..., n\}$. The point x_0 on a curve from $\overline{\mathcal{M}}_{g,[n]}$ will be called the root point. The group $\mathbb{S}_n$ acts upon $\overline{\mathcal{M}}_{g,[n]}$ by renumbering all points except for x_0. We have the morphisms:

$$(6) \qquad \overline{\mathcal{M}}_{g,[l]} \times \overline{\mathcal{M}}_{0,[m_1]} \times \cdots \times \overline{\mathcal{M}}_{0,[m_l]} \to \overline{\mathcal{M}}_{g,[m_1+\cdots+m_l]}$$

glueing the root point of the universal curve parametrized by $\overline{\mathcal{M}}_{0,m_i}$ to the i-th labeled point of the universal curve parametrized by $\overline{\mathcal{M}}_{g,m_{[l]}}$, $i = 1, ..., l$.

If $g = 0$, we get an operad $\overline{\mathcal{M}}$ in the monoidal category of smooth projective manifolds with $\overline{\mathcal{M}}(n) = \overline{\mathcal{M}}_{0,n+1}$, $n \geq 2$ and $\overline{\mathcal{M}}(1) = \{pt\}$, $\overline{\mathcal{M}}(0) = \emptyset$.

The composition not involving $\overline{\mathcal{M}}(1)$ are given by equation (6) while the unique element of $\overline{\mathcal{M}}(1)$ is the unit.

4.4. **Hidden symmetry of the DM-Stack.**

4.4.1. *Oriented frame on the S-labeled stable curve.* Consider the DM-stack $\overline{\mathcal{M}}_{0,[n]}$. Chose a stable S-labeled curve $(C/T, \pi, (x_i)|i \in S, |S| = n + 1)$, where T is a base scheme in the category $\mathcal{S}$ and C in $\mathcal{F}$. Recall, that the marked points are sections over $t \in T$, where t is a geometric point. A section of C is a morphism of T-schemes $s : T \to C$, verifying $\pi \circ s = Id_T$.

We consider the contravariant functor $\overline{\mathcal{M}}_{0,[n]}$ sending a scheme T to a collection of $n + 1$-pointed curves of genus 0 over S modulo isomorphisms. Knudsen [12] shows that $\overline{\mathcal{M}}_{0,[n]}$ is represented by a smooth complete variety T_n together with a universal curve $(C, (x_i)_{\in S}) \to T_n$, where $(x_i)_{i \in S}$ are universal sections. In addition to representing $\overline{\mathcal{M}}_{0,[n]}$, T_n gives a compatification of the space of n distinct points on $\mathbb{P}^1$ modulo $PSL_2(\mathbb{C})$ which his isomorphic to: $\mathbb{P}^1 \setminus \{0, 1, \infty\} \times ... \times \mathbb{P}^1 \setminus \{0, 1, \infty\} \setminus \Delta$, where Δ is the discriminant variety. This space is contained in T_n as the open subset over which π is smooth, hence the open set parametrizing $n + 1$-pointed curves over $Spec(\mathbb{C})$, for which the curve is $\mathbb{P}^1$.

Consider the $S \cup \{*\}$-pointed stable curve, with sections $(x_i)_{i \in S \cup \{*\}}$ and having an additional section x_*. Knudsen shows that there exists (up to unique isomorphism) a unique $[n + 1]$-pointed curve $\pi_* : C^* \to T_{n+1}$ with sections $(x_i')_{i \in [n+1]}$ such that C is the contraction of C^* along $x_{[n+1]}'$ and such that x_{n+1}' is send to the section x_*. C^* is obtained from C by an explicitly described blow-up.

The important result is as follows. If $\overline{\mathcal{M}}_{0,[n]}$ is the stable S-labeled curve $C \to S$, with sections $(x_i)_{i \in S}$, then:

$\overline{\mathcal{M}}_{0,[n+1]}$ is $\pi : (C, (x_i)_{i \in [n+1]}) \to T_{n+1}$, with $T_{n+1} = (C, (x_i)_{i \in [n]})$ and $(C, (x_i)_{i \in [n+1]})$ is a blow up of $T_{n+1} \times_{T_n} T_{n+1}$.

The fixed points under $PSL_2(\mathbb{C})$ are $0, 1, \infty$. This induces on the 2-sphere an oriented frame, given by the equators on which those fixed points lie. In particular, these equators, cut the 2-sphere into 4 quadrants. Hence, using the induction argument from Knudsen above, this oriented frame is defined on the stable S-labeled curve C.

4.4.2. *Symmetry of the DM-Stack.* Consider an affine orientable symmetry group

$$G = \langle \rho | \rho^2 = Id \rangle,$$

where for any $x \in \mathbb{C}$,

$$\rho : x \mapsto 1 - x.$$

We consider the representation of this group as follows. Let C/T be a stable curve with n marked points:

$$G \to \mathrm{Aut}(C/T, (x_i)).$$

This group action has the property that the marked points $x_i(t)$ on C (which are sections at a geometric point t of T), lying in a given quadrant of the sphere, are mapped to their diagonally opposite quadrant, where quadrants here, intersect at the point $\frac{1}{2}$.

Differently speaking, we can look at ρ as an affine map acting on the set of n-sections as follows:

$$A_\rho : \mathbb{C}^n \to \mathbb{C}^{2n}$$
$$(x_1, ..., x_n) \mapsto (x_1, ..., x_n, 1 - x_1, ..., 1 - x_n),$$

defined in matricial notation as:

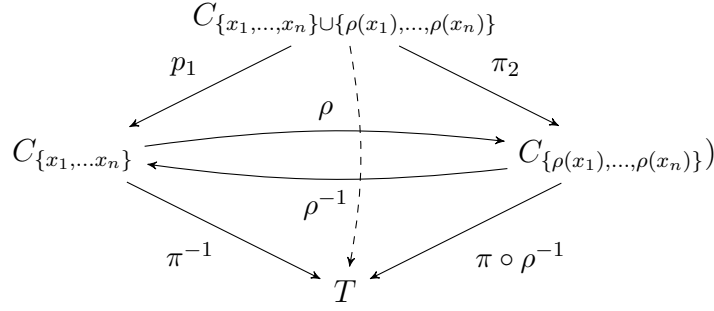$$\mathbf{z} \bigg([\mathbf{Id}]_{(\mathbf{n} \times \mathbf{n})}, [-\mathbf{Id}]_{(\mathbf{n} \times \mathbf{n})} \bigg)_{(\mathbf{n} \times \mathbf{2n})} + \bigg([\mathbf{0}]_{\mathbf{1} \times \mathbf{n}}, [\mathbf{1}]_{\mathbf{1} \times \mathbf{n}} \bigg)_{\mathbf{1} \times \mathbf{2n}}$$

where the matrix is:

$$\begin{bmatrix} 1 & 0 & \dots & 0 & | & -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & | & 0 & -1 & \dots & 0 \\ 0 & 0 & \dots & 1 & | & 0 & 0 & \dots & -1. \end{bmatrix}.$$

We can now consider the graph "function" of $\rho \in \mathrm{Aut}(C/T, (x_i))$. Indeed, given the set of S-labeled points on C/T, consider a binary relation ρ (endorelation) on $(C/T; (x_i)_{i \in S})$. This binary relation is given by the holomorphic involution $\rho : x \mapsto 1 - x$, which maps the n-tuple $(x_i)_{i \in S}$ of labeled marked points on C/T to $(\rho(x_i))_{i \in S}$. The graph of the binary relation ρ from C/T to itself is formed from the pairs $(\mathbf{x}, \rho(\mathbf{x}))$ and the relation is functional and entire.

We have for $(C/T, (x_i, \rho(x_i))_{i \in S})$:

$$C_{\{x_1,\dots,x_n\}\cup\{\rho(x_1),\dots,\rho(x_n)\}}$$

$$C_{\{x_1,\dots x_n\}} \qquad C_{\{\rho(x_1),\dots,\rho(x_n)\}})$$

$$T$$

with arrows labeled p_1, π_2, ρ, ρ^{-1}, π^{-1}, $\pi \circ \rho^{-1}$.

More simply,

$$(C,(x_i)_{i\in S}) \xrightarrow{\ \rho\ } (C,(\rho(x_i))_{i\in S})$$

$$\pi \qquad \pi\circ\rho^{-1}$$

$$T$$
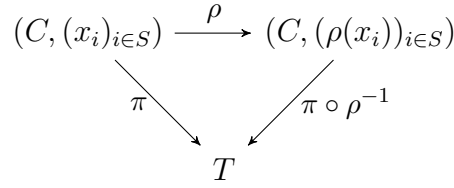
4.5. Monad structure.

Take the DM-stack of S-labeled stable curves. We have the functor $b : \mathcal{F} \to \mathcal{S}$, where $\mathcal{F}$ is the category of S-labeled stable genus 0 curves. We consider an involutive endofunctor $\rho : \mathcal{F} \to \mathcal{F}$ i.e. an endofunctor verifying the following condition:

$$\rho^2 = \rho \circ \rho = Id_{\mathcal{F}}.$$

This operation defines a monad in the category $\mathcal{F}$ since the transformations $\eta : I_{\mathcal{F}} \to^* \rho$ and $\mu : \rho^2 \to^* \rho$ are natural transformations and the following diagrams commute:

$$
\begin{array}{ccc}
\rho^3 = \rho & \xrightarrow{\ \rho_\mu\ } & Id_{\mathcal{F}} \\
{\scriptstyle \mu_\rho}\downarrow & & \downarrow{\scriptstyle \mu} \\
Id_{\mathcal{F}} & \xrightarrow[\ \mu\ ]{} & \rho
\end{array}
$$

$$
\begin{array}{ccccc}
\rho^3 = \rho & \xrightarrow{\ \eta\rho\ } & \rho^2 = Id_{\mathcal{F}} & \xrightarrow{\ \rho\eta\ } & \rho Id_{\mathcal{F}} \\
{\scriptstyle =}\downarrow & & & & \downarrow{\scriptstyle =} \\
\rho & \xrightarrow[\ =\ ]{} & \rho & \xrightarrow[\ =\ ]{} & \rho
\end{array}
$$

In particular, we chose the ρ-algebra for the monad ρ to be constructed on the model of the group action $G \times \mathcal{F}$,

with the structure map $H : G \times \mathcal{F} \to \mathcal{F}$ verifiying $h(g_1g_2, x) = h(g_1, h(g_2, x))$, $h(u, x) = x$.

Proposition 2. *Let $\mathcal{F}$ be the category of S-labeled stable curves of genus 0 and $\mathcal{F}^\rho$ the category of S-labeled stable curves of genus 0, obtained by the action of G on objects of $\mathcal{F}$. Then, $b : \mathcal{F} \times \mathcal{F}^\rho \to \mathcal{S})$ is a groupoid.*

Proof. Let $\mathcal{F}$ be the category of S-labeled stable curves. Construct the isomorphism of categories $\rho : \mathcal{F} \to \mathcal{F}^\rho$, where any object of $\mathcal{F}^\rho$ is the image of one object C/T by the map $\rho \in Aut(C/T)$; and any morphism of S-labeled stable curves $c, c' \in \mathcal{F}$ is mapped to a morphism of S-labeled stable curves in $\mathcal{F}^\rho$: $\rho f : \rho c \to \rho c'$. The definition of this morphism induces a bijection of the two families of S-labeled curves in $\mathcal{F}$ and respectively in $\mathcal{F}^{op}$, in other words the following diagram is commutative:

$$
\begin{array}{ccc}
c & \xrightarrow{\ \rho\ } & \rho c \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \rho f} \\
c' & \xrightarrow{\ \rho\ } & \rho c'
\end{array}
$$

Now, we show that $(b, b') : \mathcal{F} \times \mathcal{F}^\rho \to \mathcal{S}$ is a groupoid. Indeed, given a base $T \in \mathcal{S}$, any morphism of families over T inducing identity on T is an isomorphism for $\mathcal{F}$ and, by the construction above for $\mathcal{F}^\rho$. So, condition 1 of definition is verified and we have a groupoid. $\square$

5. Operad for the new stable-curve

In this section, we define the NY operad.

We know, from the previous section that the S-labeled stable curves, equipped with the bi-functor "$\otimes$" form a monoidal category. Therefore, the stable curves indexed by the graph $(\mathbf{x}, \rho(\mathbf{x}))$ inherit this monoidal structure. Let us call C' the monoidal category of those curves with symmetry.

To check if it is possible to construct an operad, on this new monoidal category, where objects are the stable curves $(C/T, (\mathbf{x}, \rho(\mathbf{x})))$, recall the definition of a Monoidal Functor.

So, we use this monoidal functor between the classical monoidal category C of S-labeled stable curves and the monoidal category C' of those curves with symmetry, in order to construct the operad.

Lemma 1. *Let C' be the monoidal category, where objects are S-labeled stable curves indexed by the graph of the binary relation ρ; morphisms are bijections. Then, there exists an operad structure on the monoidal category C'.*

Proof. Consider the presheaf: $\mathcal{P} : Fin^{op} \to \mathbf{C}$. It defines the operadic structure (see definition above). Consider the functor $Mon(f) : Mon(C) \to Mon(C')$ (equation 5), which maps the monoid in C to the monoid in C'. Then, by diagrams in section 2.2, definition 1,

$$
\Phi(\mathcal{P}) : Fin^{op} \to \mathbf{C}'
$$

is a presheaf. It follows, from the proposition 1 that the composition operation is conserved, in **C'**.

$$\Phi(\mathcal{P}(Y \setminus X) \otimes \mathcal{P}(X)) = \Phi(\mathcal{P}(Y \setminus X)) \otimes \Phi(\mathcal{P}(X)).$$

We have $\Phi(\mathcal{P}(Y \setminus X) \otimes \mathcal{P}(X)) \to \Phi(\mathcal{P}(Y))$. We obtain that:

$$\Phi(\mathcal{P}(Y \setminus X)) \otimes \Phi(\mathcal{P}(X)) \to \Phi(\mathcal{P}(Y)).$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Therefore, we may formulate the following definition of the topological NY operad.

Definition 3 (NY Operad). *Let* **C** *(resp.* **C'**) *be the monoidal category of S-labeled stable curves of genus 0 (resp. S-labeled stable curves indexed by the graph of the binary relation ρ) ; let $\mathcal{P}$ be the presheaf giving the operad structure on $\{\overline{\mathcal{M}}_{0,n+1}\}_{n\geq1}$ and let $\Phi : \mathbf{C} \to \mathbf{C'}$ be a functor.*
  The $\{NY(n+1)\}_{n\geq2}$ operad in the category **C'** *is the presheaf*
  $\mathcal{P'} = \Phi \circ \mathcal{P} =: Fin^{op} \to \mathbf{C'}$ *endowed with the partial operadic composition $\circ_{X \subset Y}$:*

$$\mathcal{P'}(Y/X) \otimes \mathcal{P'}(X) \to \mathcal{P}(Y),$$

for any $X \subset Y$ and a unit $\eta : 1_{\mathbf{C}} \to \mathcal{P'}(\{*\})$.

6. The DM-stack with hidden symmetry

Previously, we have shown that the monoidal category of DM-stacks $\overline{\mathcal{M}}_{0,[n]}$ enriched by a given symmetry, gives a collection of elements in $\mathbb{S} - Vect$ which form an operad, denoted $\{NY(n+1)\}_{n\geq1}$. We want to compare this new NY operad to the gravity operad $Grav$ and as well to other operads, which have been extensively studied in [6] [10],[11].

6.1. **Pointed stable curves and their graphs.** We recall some important properties of the stratification of stable S-labeled curves by stable trees (graphs). Consider an object $(C/T, (x_i)_{i \in S})$ in the category $\mathcal{F}$ discussed above. The scheme $T \in \mathcal{S}$ is decomposed into a disjoint union of strata, where each stratum $D(\tau)$ is indexed by a stable S-tree τ. The stratum $D(\tau)$ is a locally closed, reduced and irreducible subscheme of T, and parametrizing curves of the combinatorial type τ. Its codimension of the stratum $D(\tau)$ equals the cardinality of the set of edges of τ (i.e. the number of singular points of a curve of type τ). This subscheme depends only on the n-isomorphism class of τ.

The closure of the stratum $\overline{D}(\tau)$ is formed from the union of subscheme $D(\sigma)$, where $\tau > \sigma$ and where τ and σ have the same set of tails. In our case - where the genus of the curve is zero, the condition that $\tau > \sigma$ is uniquely specified by the *splitting* data, which can be resumed to be a certain type of Whitehead move. Let us recall it roughly. Chose a vertex v of τ and a partition of the set of flags incident to v: $F_\tau(v) = F'_\tau(v) \cup F''_\tau(v)$ such that both subsets are invariant under

the involution $j_\tau : F_\tau \to F_\tau$. To obtain σ, replace the vertex v by two vertices v' and v'' connected by an edge e, where the flags verify $F'_\tau(v') = F'_\tau(v) \cup \{e'\}$, $F'_\tau(v'') = F''_\tau(v) \cup \{e''\}$ where e', e'' are the two halves of the edge e. The remaining vertices, flags and incidence relations stay the same for τ and σ. For more details see [13] ch.III § 2.7, p.90.

Consider the stratification of the scheme T by graphs (as depicted in [13] ch.III §3). The symbol τ stands for a tree. If S is a finite set, then $\mathcal{T}((S))$ is the set of isomorphism classes of trees τ whose external edges are labeled by the elements of S . The set of trees is graded by the number of edges:

$$\mathcal{T}((S)) = \bigcup_{i=0}^{|S|-3} \mathcal{T}_i((S)).$$

$\mathcal{T}_i((S))$ is a tree with i edges. The tree $\mathcal{T}_0((S))$ is the tree with one vertex and the set of flags equals to S. We have the following theorem which makes a connection between the stratification to $H_*(\overline{\mathcal{M}}_{0,n})$.

Theorem 1. *Let $\top$ be a tree in $\mathcal{T}((n))$. The cycles $[\overline{\mathcal{M}}((\top))], \top \in \mathcal{T}((n))$ span $H_*(\overline{\mathcal{M}}_{0,n})$.*

We now proceed to a short description of the stratification of $\overline{\mathcal{M}}_{0,(x,\rho(x))}$ and compare it to the standard one for $\overline{\mathcal{M}}_{0,n}$.

Recall that the moduli space of genus 0 curves with n marked points has a singular set, which is defined as the zero locus of the set of analytic functions given by $\{x_i = x_j | i \neq j\}$. It is a $\mathcal{A}_{n-1}$ singularity. Blowing-up the singular locus, gives a divisor with normal crossings, which we call D.

Let D_τ be the divisor with normal crossings corresponding to the dual graph τ; it corresponds to a given stratum of $\overline{\mathcal{M}}_{0,n+3}$. This divisor $D = \sum_{n \in I} n_i D_i$, where the n_i are integers, is decomposed into a sum of irreducible components which are closed integral subschemes of codimension 1 on the blown-up algebraic variety. Here, we have $D_i \cong \mathbb{P}^1$. The normal crossings condition imply that each irreducible component is non singular and whenever r irreducible components $D_1, \ldots, D_r$ meet at a point P, then the local equations $f_1, \ldots, f_r$ of the D_i form part of a regular system of parameters at P.

If this divisor is considered as a Cartier divisor, then it can also be defined locally by $\{(U_i, f_i)\}$, where f_i are holomorphic functions and U_i are open subsets. To D one can make correspond the divisor: $D_\tau^\rho = \sum_{n \in I} n_i D_i^\rho$ defined by $\{(U_i, \rho(f_i))\}$. So, we have a pair of divisors (D_τ, D_τ^ρ), which are isomorphic. The intersection of (D_τ, D_τ^ρ) is non-empty if $f_i = \rho(f_i)$.

Proposition 3. *Consider a pair of divisors $D_\tau \cup D_\tau^\rho$, indexing a given stratum in $F \times \mathcal{F}^\rho$. D_τ is locally defined by $\{(U_i, f_i)\}$ and D_τ^ρ is locally defined by $\{(U_i, \rho(f_i))\}$. If there exists $f_i = \rho(f_i), D_\tau \cup D_\tau^\rho$ forms a connected set.*

Proof. One direction is easy: if D^ρ_τ are disjoint, then there are no points verifying $f_i = \rho(f_i)$. This occurs in the case where the sections are different from $\frac{1}{2}$, so lying in $\mathbb{P}^1 \setminus \{0, 1, \frac{1}{2}, \infty\}$. $\qquad\square$

Suppose that thee are k colliding points.

Lemma 2. *Given a stratification of $\overline{\mathcal{M}}_{0,n} \times Aut(\overline{\mathcal{M}}_{0,n})$, there exists a stratum of codimension greater than n.*

Proof. Indeed, if there exists a point in Δ, such that $x_i(t) \in Fix_\rho$ for all $i \in S$, then blowing up this point gives a divisor with $2n$ irreducible components. $\qquad\square$

6.2. **The stacky $\overline{\mathcal{M}}_{0,x,\rho(x)}$.** We prove that adding the symmetry structure onto the stack $\overline{\mathcal{M}}_{0,n+1}$ gives again a stack.

Proposition 4. *The DM-stack equipped with the symmetry group G is a stack.*

Proof. In order to show that $\overline{\mathcal{M}}_{0,(x,\rho(x))}$ is a stack, let us first equip the base space $\mathcal{S}$ with the étale topology $\mathcal{T}$. We need to verify the three conditions of definition 1.

(1) The first condition is to show that the contravariant functor from $\mathcal{S}^{op}$ to the category of sets Set is a sheaf.

We know that if $\mathcal{S}$ has a Grothendieck topology, then $\overline{\mathcal{M}}_{0,n}$ is a stack. The modification of this stack into $\overline{\mathcal{M}}_{0,(x,\rho(x))}$ implies a slight modification of the data. Indeed, $\mathcal{F} : \mathcal{S}^{op} \to Set$ is a sheaf. Properties of the category Set allow to consider the direct sum $Set \oplus Set$. So, we are dealing with the section $\mathcal{S}^{op} \to Set \oplus Set$, which turns out to be a direct sum of sheafs: $\mathcal{F} + \mathcal{F} : \mathcal{S}^{op} \to Set \oplus Set$, hence a sheaf. So, we have a sheaf $\mathcal{S}^{op} \to Set$.

(2) The second condition to check is that for any open T' in T, the functor $T' \mapsto Iso_{T'}(X_1 \oplus X_1^\rho, X_2 \oplus X_2^\rho)$ is a sheaf.

By hypothesis, we know that for any open T', the functor $T' \mapsto Iso_{T'}(X_1, X_2)$ is a sheaf. Clearly, the functor $T' \mapsto Iso_{T'}(X_1^\rho, X_2^\rho)$ is also a sheaf. So, the map from $T' \mapsto Iso_{T'}(X_1, X_2) \oplus Iso_{T'}(X_1^\rho, X_2^\rho)$ is a sheaf and by elementary properties of $\oplus$, we have $Iso_{T'}(X_1, X_2) \oplus Iso_{T'}(X_1^\rho, X_2^\rho) = Iso_{T'}(X_1 \oplus X_1^\rho, X_2 \oplus X_2^\rho)$.

(3) The last property is the so-called cocycle condition. Let $\{T_i \xrightarrow{\phi_i} T\}_i$ be an étale cover of T, where ϕ_i are étale maps and let F be a family over T. Then, applying to F the base change functors ϕ^*, we get localized families F_i over T_i, and similarly localized families F_{ij} over $T_{ij} := T_i \times_T T_j$, F_{ijk} over T_{ijk}, (etc).

They come along with the descent data, i.e isomorphisms $f_{ij} : pr_{ji,i}^* F_i \xrightarrow{\cong} pr_{ji,j}^* F_j$, which turn to satisfy the cocycle condition: $f_{ki} = f_{kj} \circ f_{ji}$ on T_{kji}. The family F is compatible with the direct sum operation.

Therefore, we have over T the family: $F \oplus F^\rho = (C/T, (x_i)) \oplus (C/T, (\rho(x_i)))$. The base change functors ϕ^* give localized families $F_i \oplus F_i^\rho$ over T_i (more generally, $F_{ij} \oplus F_{ij}^\rho$ over $T_{ij} := T_i \times_T T_j$, $F_{ijk} \oplus F_{ijk}^\rho$ over T_{ijk}, etc).

We have $(f_{ij}, f_{ij}^{\rho}) : (pr_{ji,i}^* F_i, pr_{ji,i}^* F_i^{\rho}) \xrightarrow{(\cong, \cong)} (pr_{ji,j}^* F_j, pr_{ji,j}^* F^{\rho} j)$, satisfying the cocycle condition: $(f_{ki}, f_{ki}^{\rho}) = (f_{kj} \circ f_{ji}, f_{kj}^{\rho} \circ f_{ji}^{\rho})$. The converse property comes from 2).

$\square$

6.3. **The NY Gravity operad.** In this subsection, we introduce the NY Gravity operad.

Definition 4. *Let* $Grav_{NY}((n))$ *be the sable cyclic* $\mathbb{S}$-*module defined as follows:*

$$Grav_{NY}((n)) = \begin{cases} sH_*((\mathcal{M}_{0,n} \times \mathrm{Aut}(\mathcal{M}_{0,n})) \setminus Fix_{\rho}), & n \geq 3 \\ 0, & n < 3 \end{cases}$$

Proposition 5. *Consider the sable cyclic* $\mathbb{S}$-*module* $\{Grav_{NY}((n))\}_{n \geq 3}$. *Then, there is a natural cyclic operad structure on* $Grav_{NY}$.

Proof. It is known that the collection of stable curves with labeled points forms a topological operad. As shown in equation (6), we have a well defined operation composition for this:

$$\overline{\mathcal{M}}_{0,[l]} \times \overline{\mathcal{M}}_{0,[m_1]} \times \cdots \times \overline{\mathcal{M}}_{0,[m_l]} \to \overline{\mathcal{M}}_{g,[m_1+\cdots+m_l]}.$$

Referring to this, we construct the composition operation for the NY Gravity operad. This can be modified into the following composition morphism :

$$\mathrm{Aut}(\overline{\mathcal{M}}_{0,[l]}) \times \mathrm{Aut}(\overline{\mathcal{M}}_{0,[m_1]}) \times \cdots \times \mathrm{Aut}(\overline{\mathcal{M}}_{0,[m_l]}) \to \mathrm{Aut}(\overline{\mathcal{M}}_{g,[m_1+\cdots+m_l]}),$$

where we use the automorphism $\rho \in \mathrm{Aut}(\overline{\mathcal{M}}_{0,[k]})$ such as defined above. Combining both morphisms together, we get the following:

$$\overline{\mathcal{M}}_{0,[l]} \times \mathrm{Aut}(\overline{\mathcal{M}}_{0,[l]}) \times \overline{\mathcal{M}}_{0,[m_1]} \times \mathrm{Aut} \times \cdots \times \overline{\mathcal{M}}_{0,[m_l]} \times \mathrm{Aut}(\overline{\mathcal{M}}_{0,[m_l]})$$
$$\to \overline{\mathcal{M}}_{0,[m_1+\cdots+m_l]} \times \mathrm{Aut}(\overline{\mathcal{M}}_{0,[m_1+\cdots+m_l]}).$$

For the gravity operad, this operation is not fully satisfactory, since only the smooth stratum (i.e. codimension 0 stratum) needs to be taken under consideration. Therefore, we omit fixed points of the automorphism group, and consider only the smooth stratum:

$$(\mathcal{M}_{0,[l]} \times \mathrm{Aut}(\mathcal{M}_{0,[l]})) \setminus Fix_{\rho} \times (\mathcal{M}_{0,[m_1]} \times \mathrm{Aut}(\mathcal{M}_{0,[m_1]})) \setminus Fix_{\rho} \times \cdots \times (\mathcal{M}_{0,[m_l]}$$
$$\times \mathrm{Aut}(\overline{\mathcal{M}}_{0,[m_l]})) \setminus Fix_{\rho} \to (\overline{\mathcal{M}}_{0,[m_1+\cdots+m_l]} \times \mathrm{Aut}(\overline{\mathcal{M}}_{0,[m_1+\cdots+m_l]})) \setminus Fix_{\rho}.$$

To define the product for the NY gravity operad, let us proceed as follows.

Consider the Poincaré residue map associated to the embedding [2] :

$$(\mathcal{M}_{0,[l]} \times \mathrm{Aut}(\mathcal{M}_{0,[l]})) \setminus Fix_{\rho} \times (\mathcal{M}_{0,[m]} \times \mathrm{Aut}(\mathcal{M}_{0,[m]})) \setminus Fix_{\rho}$$
$$\to \overline{\mathcal{M}}_{0,[m+l]} \times \mathrm{Aut}(\overline{\mathcal{M}}_{0,[m+l]}),$$

which is:

$$Res : H^*(\mathcal{M}_{0,[m+l]} \times \text{Aut}(\mathcal{M}_{0,[m+l]})) \to H^*(\mathcal{M}_{0,[m]} \times \text{Aut}(\mathcal{M}_{0,[m]}) \times \mathcal{M}_{0,[l]} \times \text{Aut}(\mathcal{M}_{0,[l]})).$$

Suitably, suspending the adjoint of this map, we obtain a product $\circ_i$ of $Grav_{NY}$. The $Grav_{NY}$ satisfies the equivariance and associativity axioms.

Indeed, the equivariance axiom is not modified. The main argument is that $t_i \in \mathbb{S}_i$ acts on the set of marked points $[i]$ and, thus on the set obtained under ρ. So, the following diagram remains commutative:

$$
\begin{array}{ccc}
\mathcal{P}(k) \otimes \mathcal{P}(r_1) \otimes \cdots \otimes \mathcal{P}(r_k) & \xrightarrow{id \otimes (t_1 \otimes \ldots \otimes t_k)} & \mathcal{P}(k) \otimes \mathcal{P}(r_1) \otimes \cdots \otimes \mathcal{P}(r_k) \\
\mu \downarrow & & \downarrow \mu \\
\mathcal{P}(r_1 + \ldots + r_k) & \xrightarrow{t_1 \oplus \ldots \oplus t_k} & \mathcal{P}(r_1 + \ldots + r_k)
\end{array}
$$

The associativity axioms of a cyclic operad also holds. Therefore, on $Grav_{NY}$ there is natural a cyclic operad structure. $\square$

6.4. Comparison of the NY gravity operad with gravity operad and $Hycom$ operad.

Theorem 1 (Comparison theorem). *Let $Grav((n)) = sH_*(\mathcal{M}_{0,n})$ be the gravity operad of $\mathcal{M}_{0,n}$ and $Grav_{NY}((n))$ be the NY Gravity operad. Then, for $n \geq 3$ we have:*

$$
\begin{aligned}
Grav_{NY}((n)) &= sH_*(\mathcal{M}_{0,n} \times \text{Aut}(\mathcal{M}_{0,n}) \setminus Fix_\rho) \\
&= s(H_*(\mathcal{M}_{0,n} \setminus Fix_\rho) \otimes H_*(\text{Aut}(\mathcal{M}_{0,n}) \setminus Fix_\rho)) \\
&= sH_*(\mathcal{M}_{0,n} \setminus Fix_\rho) \otimes H_*(\text{Aut}(\mathcal{M}_{0,n}) \setminus Fix_\rho)
\end{aligned}
$$

where Fix_ρ is the set of fixed points of the automorphism ρ.

Proof. By definition, the parametrizing space of $\mathcal{M}_{0,n}$ is given by $\{x_1, x_2, ..., x_{n-3} \in (\mathbb{P}^1 \setminus (0,1,\infty))^{n-3} | x_i \neq x_j\}$. We can map $\mathcal{M}_{0,n}$ to $\text{Aut}(\mathcal{M}_{0,n})$, where we have chose the automorphism ρ, in the following way:

$$(\mathbb{P}^1 \setminus (0,1,\infty))^{n-3} \setminus \Delta_{n-3} \to (\mathbb{P}^1 \setminus (0,1,\infty))^{n-3} \setminus \Delta_{n-3} \oplus (\mathbb{P}^1 \setminus (0,1,\infty))^{n-3} \setminus \Delta_{n-3}$$

$$(x_1, x_2, ..., x_{n-3}) \mapsto (x_1, x_2, ..., x_{n-3}) \oplus (1 - x_1, 1 - x_2, ..., 1 - x_{n-3}).$$

In order to compare the gravity operad and the NY Gravity operad, it is necessary to remove all fixed points under the automorphism, namely points that are equal to $\frac{1}{2}$. Indeed, removing those points in $\bar{\mathcal{M}}_{0,n}^\rho$ leaves us to consider the codimension 0 stratum of this space i.e. $\mathcal{M}_{0,n}^\rho$, whereas, it implies a modification regarding the classical $Grav((n))$ operad. So, by the Künneth formulae we have:

$$
\begin{aligned}
Grav_{NY}((n)) &= sH_*(\mathcal{M}_{0,n} \times \text{Aut}(\mathcal{M}) \setminus Fix_\rho) \\
&= sH_*(\mathcal{M}_{0,n} \setminus Fix_\rho) \otimes H_*(\text{Aut}(\mathcal{M}_{0,n}) \setminus Fix_\rho).
\end{aligned}
$$

$\square$

A great deal of information is already known for $sH_{n+1}(\mathcal{M}_{0,n})$ from [6] and is useful for the comparison between the standard Gravity operad and the NY gravity operad. Recall that the relations and the presentation for the Gravity operad are as follows: the degree 1 subspace of $Grav((k))$ is one dimensional for each $k \geq 2$, and is spanned by the operation

$$\{a_1, ..., a_n\} = \sum_{1 \geq i < j \geq n} (-1)^{\epsilon(i,j)} \{a_i, a_j\} a_1 .... \hat{a}_i .... \hat{a}_j .... a_n,$$

$\epsilon(i,j) = (|a_1| + ... + |a_{i-1}|)|a_i| + (|a_1| + ... + |a_{j-1}|)|a_j| + |a_i||a_j|$. The following theorem gives a detailed presentation (see [6]):

Theorem 2. *The operations* $\{a_1, ..., a_k\}$ *generate the gravity operad* $Grav((k))$, *and all relations among them follow from the generalized Jacobi identity :*

$$(7) \quad \sum_{1 \leq i < j \leq k} (-1)^{\epsilon(i,j)} \{\{a_i, a_j\}, a_1, ..., \hat{a}_i, ..., \hat{a}_j, \ldots, a_k, b_1, ..., b_l\}$$

$$= \{\{a_1, ..., a_k\}, b_1, ..., b_l\},$$

where the right hand side is 0 if $l = 0$.

There exists a strong relation from the gravity operad to the so-called hypercommutative operad, which we defne below:

$$Hycom = \begin{cases} H_*(\overline{\mathcal{M}}_{0,n}), & n \geq 3; \\ 0, & n \geq 3; \end{cases}$$

By a theorem of Getzler [6], we have the a result which guarantees the Koszul duality relation between the Gravity operad and the hypercommutative operad (i.e. $Hycom$ operad).

Theorem 3. *Let $V \subset Hycom$ be the cyclic $\mathbb{S}$-submodule spanned by the fundamental classes,*

$$[\overline{\mathcal{M}}_{0,n}] \in H_{2(n-3)}(\overline{\mathcal{M}}_{0,n}) \subset Hycom((n)).$$

The operad $Hycom$ is Koszul, with generators V, and $Hycom^! \cong Grav$.

$Hycom$ is quadratic with generators $\mathcal{V}$ and relations R, where $\mathcal{V}((n))$ is spanned by an element of degree $2(n-3)$ and weight $2(3-n)$. We have that $\mathcal{V}((n))$ is identified with $H_{2(n-3)}(\overline{\mathcal{M}}_{0,n})$.

Relations R, where $R((n))$ has dimension $\binom{n-1}{2}$-1, are given by the following generalized associativity equation.

Proposition 6. *Let $a, b, c, x_1, ..., x_n$ lie in Hycom, the hypercommutative operad. Then, all the relations in the operad Hycom are given by the following generalized associativity relation:*

$$(8) \qquad \sum_{S_1 \sqcup S_2} \pm((a, b, x_{S_1}), c, x_{S_2}) = \sum_{S_1 \sqcup S_2} \pm(a, (b, c, x_{S_1}), x_{S_2}),$$

where $S_1 \sqcup S_2 = \{1, ..., n\}$.

The symbol $\pm$ stands for the Quillen sign convention for $\mathbb{Z}_2$-graded vector spaces: it equals $+1$ if all the variables are of even degree.

In the scope of a more down to earth approach, consider the n-th component of the cohomology ring using Mayer-Vietoris exact sequence, where $X = \mathcal{M}_{0,n}$ and we chose the subsets A, B of X to be $A = \mathcal{M}_{0,n}^{\frac{1}{2}} \subset (\mathbb{P}^1 \setminus \{0, \frac{1}{2}, 1, \infty\})^n$ and B is an n-dimensional disc of radius ϵ, centered at $\frac{1}{2}$. The interiors of A and B cover $\mathcal{M}_{0,n}$ and $A \cap B$ is the n-dimensional disc, from which a point has been removed.

The Mayer-Vietoris (long) exact sequence is as follows:

$$\cdots \to H_{n+1}(\mathcal{M}_{0,n}) \to H_n(A \cap B) \to H_n(A) \oplus H_n(B)$$
$$\to H_n(\mathcal{M}_{0,n}) \to H_{n-1}(A \cap B) \to \ldots$$

This is used to give an estimate and calculate the n-th component of $sH_*(A)$. Applying the suspension onto the long exact sequence, we have the following:

$$\cdots \to sH_{n+1}(\mathcal{M}_{0,n}) \to sH_n(A \cap B) \to sH_n(A) \oplus sH_n(B)$$
$$\to sH_n(\mathcal{M}_{0,n}) \to sH_{n-1}(A \cap B) \to \ldots$$

and looks in the following way (if n is odd):

$$\cdots \to H_{n+1}(\mathcal{M}_{0,n}) \to 0 \to H_n(A) \to H_{n-1}(\mathcal{M}_{0,n}) \to \mathbb{Z} \to H_{n-1}(A)$$
$$\to H_{n-1}(\mathcal{M}_{0,n}) \to 0 \to \ldots$$

Going back to the cohomology ring of $\overline{\mathcal{M}}_{0,n}$, one may identify here Chow rings with cohomology. In particular $\overline{\mathcal{M}}_{0,n}$ has no odd homology and its Chow groups are finitely generated and free abelian.

It was Keel [11] who gave an explicit basis for $H^*(\overline{\mathcal{M}}_{0,n}(\mathbb{C}))$. The D_S are the divisors and generate the cohomology ring. The commutative ring is generated by elements of degree 2 D_S, one for each subset $S \subset \{1, 2, ..., n\}$ with $2 \leq |S| \leq n-1$ subject to the following relations:

- $D_S = D_{\{0,1,...,n\} \setminus S}$
- For distinct elements $i, j, k, l \in \{0, 1, ..., n\}$

$$\sum_{i,j \in S; k,l \notin S} D_S = \sum_{i,k \in S; j,l \notin S} D_S$$

- If $S \cap T \notin \{0, S, T\}$ and $S \cup T \neq \{0, 1, ..., n\}$ then $D_S D_T = 0$

Due to [5], we have a supplementary result which shows that $H^*(\overline{\mathcal{M}}_{0,n})$ is Koszul. They are useful to determine various homotopy invariants for the DM-compactification.

7. Kähler and Riemannian geometry approach

7.1. **Kähler aspect.** By passing to (co)homology, one should keep in mind that we may use Kähler logarithmic derivatives in place of holomorphic ones, as we can see in Arnold's result for the generators of $H^*(\mathcal{M}_{0,k+1})$:

Proposition 7. *For* $1 \leq j \neq k \leq n$, *let* $w_{jk} = \frac{dlog(x_j - x_k)}{2\pi\iota}$ *be the logarithmic differential form. The cohomology ring* $H^*(Conf_{0,n}, \mathbb{Z})$ *is the graded commutative ring with generators* $[w_{jk}]$, *and relations:*

- $w_{jk} = w_{kj}$
- $w_{ij}w_{jk} + w_{jk}w_{ki} + w_{ki}w_{ij} = 0$

The cohomology ring $H^*(\mathcal{M}_{0,n+1}, \mathbb{C})$ *may be identified with the kernel of the differential* ι *on* $H^*(Conf_{0,n}, \mathbb{C})$ *whose action on the generators is* $\iota w_{jk} = 1$.

The regular part of this moduli space (i.e. the union of codimension 0 strata) has an (incomplete) Riemannian metric, induced by the projective embedding. Indeed, recall that $\mathbb{P}^n$ is endowed with a natural Kähler metric ν, called Fubini-Study metric and defined by:

$$p^*\nu = \frac{i}{2\pi}\partial\overline{\partial}log||x_1|^2 + |x_2|^2 + \cdots + |x_n|^2|$$

$(x_1, \ldots, x_n) \in \mathbb{C}^{n+1}$ and $p : \mathbb{C}^{n+1} - 0 \to \mathbb{P}^n$ is the projection.

This gives a point-wise norm on smooth forms ω of type (p, q) on $\mathcal{M}_{0,n}$, and defines an L^2 norm $||\omega||_2$. Using this property, one defines a simplicial complex by setting $\mathcal{F}^{p,q}(\mathcal{M}_{0,n}) := \{\omega|||\omega||_2 < \infty, ||\bar{\partial}\omega||_2 < \infty\}$. From this definition, one can define a Dolbeault type of complex, given by $(\mathcal{F}^{p,*}(\mathcal{M}_{0,n}), \bar{\partial})$ for each $p \geq 0$. The existence of such a complex allows the definition of an L^2 q-th cohomology group, denoted by $H_2^{p,q}(\mathcal{M}_{0,n})$. See [18, 19] for more details.

Let us choose a ball B_r, centered at 0 and of radius r, such that $\sum_{i=1}^n |x_i|^2 \ll 1$. Consider the Euclidean distance between a pair of points x_i and x_j, where those points lie in the interior of the ball of radius r. Set

$$(9) \qquad\qquad\qquad F = -log(dist(x_i, x_j)),$$

and

$$(10) \qquad\qquad F_k := -log(dist(x_i, x_j)) - \frac{1}{k}log(-log\sum |x_i|^2)$$

where $x_i, x_j \in B_r$ and $k > 1$.

The latter strictly pluri-subharmonic function defines a Kahler metric $h_k := -i\partial\bar{\partial}F_k$ on the regular part of $\mathcal{M}_{0,n} \cap B_r$. The metric $h_k := -i\partial\bar{\partial}F_k$ on $\mathcal{M}_{0,n}$

is complete and decreases monotonically to $h := -i\partial\bar{\partial}F$. $\langle\partial F_k, \partial F_k\rangle$ is bounded, independently of k, where $\langle,\rangle_k^{\frac{1}{2}}$ denotes the pointwise norm on 1-forms with respect to h_k. We have the following lemmas:

Lemma 3 ([17]1.1, 2.1). *Let N be a complete Kahler manifold of dimension n, whose Kähler metric is given by the potential function $F : N \to \mathbb{R}$ such that $\langle\partial F, \partial F\rangle$ is bounded. Then the $L_2 - \bar{\partial}$-cohomology with respect to ω, $H_{(2)}^{p,q}(N,\omega) = 0$, $p + q \neq n$. In fact, if $\langle\partial F, \partial F\rangle^{\frac{1}{2}} \leq B$, and ϕ is a (p,q)-form on N with $\bar{\partial}\phi = 0$, $q > 0$ and $p + q \neq 0$, then there is a $(p, q-1)$-form ν such that $\bar{\partial}\nu = \phi$ and $||\nu|| \leq 4B||\phi||$*

Lemma 4 ([4], Theorem 4.1; [16], Proposition 4.1). *Let N be a complex manifold of dimension n with a decreasing sequence of complete hermitian metrics h_k, $k \geq 1$, which converges pointwise to a hermitian metric h. If $H_{(2)}^{n,q}(N, h_k)$ vanishes with an estimate that is independent of k, then $H_{(2)}^{n,q}(N, h)$ vanishes with an estimate. Here, $H_{(2)}^{n,q}(N, h_k)$ denotes $L_2 - \bar{\partial}$-cohomology with respect to the metric h_k.*

It is known that there exists a strong relation between h_k and a Riemannian metric in B_c, in the regular part of $\overline{\mathcal{M}}_{0,n}$.

7.2. Riemannian aspect.
In this subsection, we study the geometry around the divisor.

Define a conical structure on a smooth manifold M, to be

$$[0,1] \times M/\sim,$$

where $(0,x) \sim (0,x')$ and $x, x' \in M$. The terminology "conical" is employed in a purely geometric and topological context.

Proposition 8. *The (hidden) symmetry of $\overline{\mathcal{M}}_{0,n}$ induces a additional local conical structures on $\overline{\mathcal{M}}_{\mathbf{x}, \rho(\mathbf{x})}$.*

Proof. Indeed, $\mathcal{M}_{0,n} \times \mathrm{Aut}(\mathcal{M}_{0,n})$ is isomorphic to $\mathcal{M}_{0,n} \times \mathcal{M}_{0,n}/\sim$ where the relation $\sim$ is obtained by glueing together a pair of sections in $\mathcal{M}_{0,n}$ if one of them lies in the Fix_ρ. This defines a type of additional conical structure $\mathcal{M}_{0,n} \times \mathcal{M}_{0,n}$. Indeed, it is well known that around a small neighborhood of a singular point, there is a topological cone structure. Since pairs of points $(x_i, \rho(x_i))$ lying on Fix_ρ form singular points of $\overline{\mathcal{M}}_{\mathbf{x}, \rho(\mathbf{x})}$, the result follows. $\square$

This conical structure gives additional singular points to the scheme $\overline{\mathcal{M}}_{0,n} \times \mathrm{Aut}(\overline{\mathcal{M}}_{0,n})$. By blowing-up those points, irreducible components (isomorphic to $\mathbb{P}^1$) appear in the intersection $D \cap D^\rho$. This data modifies the geometry around the divisor of $\overline{\mathcal{M}}_{0,n} \times \mathrm{Aut}(\overline{\mathcal{M}}_{0,n})$, compared to $\overline{\mathcal{M}}_{0,n}$. Indeed, those points, lying in Fix_ρ, are conjugated to complex germs of the type $(z^{2k}, 0)$, where $1 \leq k \leq n$. Therefore, not only this modifies the stratification of the space, (comparing it to the one of $\overline{\mathcal{M}}_{0,n}$), but also implies differences in the metrics around the divisors.

We prove the following proposition for $\overline{\mathcal{M}}_{0,n}$, where $3 \leq n < 6$.

Proposition 9. *Let* $K_{D \tilde{\cup} D^\rho}$ *(resp.* K_D*) be the scalar curvature around the irreducible components of the divisor* $D \tilde{\cup} D^\rho$ *of* $\overline{\mathcal{M}}_{0,n} \times \mathrm{Aut}(\overline{\mathcal{M}}_{0,n})$ *(resp.* D *in* $\overline{\mathcal{M}}_{0,n}$*). Then, there exists a stratum for which the irreducible component of the divisor* $D \tilde{\cup} D^\rho$ *verifies*

$$K_{D \tilde{\cup} D^\rho} < K_D.$$

As was already mentioned that topologically, the neighborhood of a singular locus can be considered as a topological cone (whatever the dimension of the singular locus is).

Theorem 2 (Hsiang-Pati)**.** [7] *Let* $(X, 0)$ *be the germ of an isolated complex surface singularity. Then there exists a desingularization* $\rho : (\tilde{X}, D) \to (X, 0)$ *where* $D = \rho^{-1}(0)$ *is a divisor with normal crossings such that near each smooth point of* D *the metric induced on* $\tilde{X} - D$ *by* ρ *extends to a pseudo-metric on* $\tilde{X}$ *quasi-isometric to*

- $\rho^{-1}(0) = u^m$ *locally*
- $\rho^{-1}(0) = u^{m_1} v^{m_2}$ *locally*

Let U' be the neighborhood of $\rho^{-1}(0) \subset \tilde{X}$. Choose a function R defined on U' with range in $[0, \infty)$. This function satisfies four conditions:

(1) $R|_{\rho^{-1}(0)} = 0$
(2) $R|_{U' - \rho^{-1}(0)}$ is smooth and positive
(3) $U' - \rho^{-1}(0) = R^{-1}(0, 1]$
(4) Using the R and certain appropriate flow lines in $U' - \rho^{-1}(0)$, we define a product structure

$$R^{-1}(0, 1] = (0, 1] \times R^{-1}(1)$$

This function is necessary to partition U' in a suitable manner. Indeed, set: $\Theta(\epsilon) = R^{-1}(0, \epsilon]$ and $\dot{\Theta}(\epsilon) = R^{-1}(\epsilon)$, $\epsilon > 0$. Now $\dot{\Theta}(1)$ can be decomposed into finite parts non overlapping except along heir boundaries. Using the product structure one can decompose U' as follows: $\Theta(1) = \cup_i U'_i$. Since it remains difficult to work with the subsets U'_i, we will work on less complicated manifolds. These manifolds are defined up to quasi-isometry with the U'_i. These manifolds are classified into two classes. To fix the notation denote by $W(+)$ the manifold quasi-isometric to the neighborhood U'_i of the intersection points of the irreducible components on the divisor. Denote by $W(-)$ the manifold quasi-isometric to the neighborhood U'_j of the divisor which is the complementary of the previous neighborhood U'_i. These manifolds are Riemanian manifolds and carry a Riemannian metric.

Proposition 1 (Hsiang-Pati; Nagase [15])**.**

(1) *Let* Y *be a compact polygon in* $\mathbb{R}^2$ *with standard metric* $\tilde{g}$*.* $W(-) = (0, 1] \times [0, 1] \times Y \ni (r, \theta, y)$ *carries the Riemannian metric*

$$g(-) := dr^2 + r^2 d\theta^2 + r^{2c} \tilde{g}(y).$$

(2) $W(+) = (0,1] \times [0,1]^3$ carries the Riemannian metric

$$g(-) := dr^2 + r^2 d\theta^2 + r^{2c}(ds^2 + h^2(r,s)d\phi).$$

$h(r,s) = \frac{f(r)}{l(\frac{s}{f(r)})}$ where $f(r)$ is a smooth function on $[0,1]$ such that $f'(r) \geq 0, \forall r \geq 0$ and $l(x)$ is is a smooth function on $[0,\infty)$ such that $l'(x) \geq 0$ and $l''(x) \geq 0$ for any $x \geq 0$. We define these two functions as :

- $f(r) = r^b$, if r is small and $r > 0, b > 0$.
- $f(r) = \frac{1}{2}$ if r is large and $r \leq 1$
- $l(x) = 1$, if $0 \leq x \leq 1 - \epsilon$.
- $l(x) = x$ if $1 + \epsilon \leq x$

$W(-) = (0,1] \times [0,1] \times Y$ and $W(+) = (0,1] \times [0,1]^3$ are models of the subsets in U, obtained up to quasi-isometry, where quasi-isometry means that for Riemannian manifolds $(Y_i, g_i$ the diffeomorphism $f : (Y_1, g_1) \to (Y_2, g_2)$ satisfies for a positive constant $C > 0$ the inequality: $C^{-1}g_1 \leq f^* g_2 \leq C g_1$.

We may now prove the proposition, mentioned above.

Proposition 9. The coefficient c, in the Hsiang-Pati metric, depends on the multiplicty of the irreducible component, of the divisor. We show that adding the symmetry discussed above, on $\overline{\mathcal{M}}_{0,n}$, implies the existence of new strata in $\overline{\mathcal{M}}_{0,n} \times \text{Aut}(\overline{\mathcal{M}}_{0,n})$, which have additional irreducible components, when compared to the divisor D indexing the analogous stratum in $\overline{\mathcal{M}}_{0,n}$. Those additional irreducible components are obtained by blowing-up points in Fix_ρ. These singular points are conjugated to complex germs of type $(z^{2k}, 0)$, where $1 \leq k \leq n$ is an integer.

Choose the specific stratum in $\overline{\mathcal{M}}_{0,n}$ and its analog in $\overline{\mathcal{M}}_{0,n} \times \text{Aut}(\overline{\mathcal{M}}_{0,n})$, where k points collide to $\frac{1}{2}$ in $\mathcal{M}_{0,n}$. Suppose that the $n - k$ remaining points are either all distinct, or that, among those $n - k$ points, the number of colliding points is smaller than $2k - 1$. Then, for this stratum in $\overline{\mathcal{M}}_{0,n} \times \text{Aut}(\overline{\mathcal{M}}_{0,n})$ there exists an irreducible component in $D \tilde{\cup} D^\rho$, for which the constant $c = 2k$. This is greater than the multiplicity of any irreducible component obtained for the divisor D. We can easily calculate the scalar curvature for the Hsiang-Pati metric and find $K = -6c^2 r^{-2}$. So, the result follows. $\qquad\square$

References

[1] F. Brown, *From the Deligne-Ihara conjecture to multiple modular values.* arXiv:1904.00179 [math.AG], 24 pp.

[2] P. Deligne, *Théorie de Hodge II* Publ. Maths tome **40**, (1971), p. 5-58, Publications mathématiques de l'I.H.E.S,

[3] P. Deligne, D. Mumford *The irreducibility of the space of curves of given genus* tome **36**, (1969), p. 75-109, Publications mathématiques de l'I.H.E.S

[4] J-P. Demailly, *Estimations L^2 pour operateur $-\bar{\partial}$ d'un fibré vectoriel holomorphe semipositif au dessus d'une variété Kählerienne complète* Ann. Sci. ENS (4) **11** (1982), 457-511

[5] V. Dotsenko, *Homotopy invariants of $\overline{\mathcal{M}}_{0,n}$ via Koszul duality* Arxiv: 1902.06318v2 [math. AT]

[6] E. Getzler, *Operad and moduli spaces of genus 0 Riemann surfaces* Progr. Math. **129**, Birkhäuser, Boston, 1995, pp. 199-230.

[7] W. C. Hsiang, V. Pati. L^2-*cohomology of normal algebraic surfaces. I.* Inventiones Mathematicae **81**, (1985), 395-412.

[8] Y. Ihara. *On the embedding of* $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ *into* $\widehat{GT}$. In [21], pp. 304–305.

[9] M. Kapranov, Y. Manin *Modules and Morita theorem for operads* arXiv:math/9906063

[10] M. Kapranov, V. Ginzburg *Koszul duality for Operads* arXiv:0709.1228 [math.AG]

[11] S. Keel. *Intersection theory of moduli space of stable N-pointed curves of genus zero.* Trans. Amer. Math. Soc., 330(2):545574, 1992.

[12] F. Knudsen and David Mumford. *The projectivity of the moduli space of stable curves. I.* Preliminaries on det and Div. Math. Scand., 39(1):19-55, 1976.

[13] Y. Manin *Frobenius manifolds, quantum cohomology and moduli spaces* (1999) AMS colloquium publications, vol 47.

[14] Y. Manin, B. Valette *Monoidal structures on the category of quadratic data* arXiv:1902.03778

[15] M. Nagase, *Remarks on the L^2-cohomology*, J. Math. Soc. Japan **41**, (1989), 97-1016.

[16] T. Ohsawa, *Hodge spectral sequence on compact Kähler spaces* Publ. Res. Inst. Math. Sci. **23** (1987), 265-274

[17] H. Donnelly, C. Fefferman, L^2-*Cohomology and index theorem for the Bergman metric*, Ann. of Maths **118**, (1983), 593-618.

[18] W. L. Pardon, M. A. Stern, $L^2 - \overline{\partial}$-*Cohomology of an algebraic surface*, Topology. **28**, No.2 (1989), 171-195.

[19] W. L. Pardon, M. A. Stern, $L^2 - \overline{\partial}$-*Cohomology of Complex Projective Varieties*, Jour. of the A.M.S. **4**, No.3 (1991), 603-621.

[20] L. Schneps, *Dessins d'enfants on the Riemann sphere.* In [21], pp. 47–77.

[21] L. Schneps, ed. *The Grothedieck Theory of Dessins d'enfants.* London Math. Soc. Lecture Note Series 200. Cambridge UP, 1994.

NOÉMIE C. COMBE, MAX PLANCK INSTITUTE FOR MATHEMATICS, VIVATSGASSE 7, 53111 BONN

YURI I. MANIN, MAX PLANCK INSTITUTE FOR MATHEMATICS, VIVATSGASSE 7, 53111 BONN