Max-Planck-Institut für Mathematik Bonn

D-affinity and rational varieties

by

Dmitriy Rumynin

Max-Planck-Institut für Mathematik
Preprint Series 2018 (64)

D-affinity and rational varieties

Dmitriy Rumynin

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
University of Warwick
Conventry CV4 7AL
UK

Associated Member of
Laboratory of Algebraic Geometry
National Research University
Higher School of Economics
Russia

MPIM 18-64

D-AFFINITY AND RATIONAL VARIETIES

DMITRIY RUMYNIN

Abstract. We investigate geometry of D-affine varieties. Our main result is that a D-affine uniformly rational projective variety over an algebraically closed field of zero characteristic is a generalised flag variety of a reductive group. This is a partially converse statement for Beilinson-Bernstein Localisation Theorem.

Let us consider a connected smooth projective algebraic variety X over an algebraically closed field $\mathbb{K}$ of characteristic zero. By G/P we denote the generalised flag variety of a reductive algebraic group G.

Beilinson-Bernstein Localisation Theorem ([2] for G/B, [11, Th. 3.7] for G/P)**:** *If $X \cong G/P$, then X is D-affine.*

It is a long-standing problem whether the converse statement holds or there are other smooth projective D-affine varieties (weighted projective spaces are D-affine but singular [22]). The converse statement is known for toric varieties [21] and homogeneous varieties [8]. Our main result is the converse statement for the uniformly rational varieties: essentially classifying D-affine projective varieties:

Main Theorem: *If X is D-affine and uniformly rational, then $X \cong G/P$.*

In fact, we aim to cover the most general D-affine varieties with various intermediate statements. In particular, many results work in the positive characteristic as well. The reader should be aware that it is not known which of the partial flag varieties are D-affine in positive characteristic. Some of them are known to be D-affine: projective spaces [7], G/B in types A_2 [7] and B_2 [1, 19], quadrics [17]. On the other hand, the grassmannian $Gr(2,5)$ is not D-affine [16].

There are further notions of D-affinity in positive characteristic when instead of Grothendieck differential operators, either small differential

---

Date: November 25, 2018.

2010 *Mathematics Subject Classification.* 16S32, 14M20.

The work was done entirely during my stay at the Max Planck Institute for Mathematics in Bonn, whose hospitality and support are gratefully acknowledged. The research was partially supported by the Russian Academic Excellence Project '5–100'. The author would like to thank Michel Brion, Masaharu Kaneda and Viatcheslav Kharlamov for valuable discussions. The author is particularly indebted to Adrian Langer for finding a mistake in the first version of the paper.

operators [10, 13, 14, 20] or crystalline differential operators [3, 4] are studied. These are not covered by the present paper, although some of our methods may prove useful for these unusual differential operators.

Let us explain the context of the paper section-by-section. In Section 1 we define D-affine varieties and make general observations about them and quasicoherent sheaves on varieties.

In Section 2 we study divisors on D-affine varieties. The main result is Theorem 6, a positivity statement about effective Cartier divisors. We use it to study D-affine surfaces in this section. We also prove Theorem 8, a key technical result about complete intersections on a D-affine variety.

Section 3 is the heart of the paper, devoted to rational varieties. Here we prove Theorem 10, the key statement about the tangent sheaf of a D-affine uniformly rational variety. It would be interesting to prove it without rationality assumption. This would give the full converse of Beilinson-Bernstein Localisation Theorem.

The final section (Section 4) finishes the proof of the main theorem (Theorem 12) by extending Corollary 11 from complex numbers to an algebraically closed field of zero characteristic. Such extensions of results are known as Lefschetz Principle, but we need to fill the details.

1. Preliminaries

Pushing for greater generality of some of our results, we will work over three algebraically closed field: the complex numbers $\mathbb{C}$, a field $\mathbb{K}$ of characteristic zero and a field $\mathbb{F}$ of arbitrary characteristic.

Let X be an algebraic variety over $\mathbb{F}$, $\mathcal{O}_X-$Qcoh its category of quasicoherent sheaves. We say a quasicoherent sheaf $\mathcal{F}$ on X is *affine* if $\mathcal{F}$ is generated by global sections (i.e., the natural map $\mathcal{O}_X \boxtimes \Gamma(X, \mathcal{F}) \to \mathcal{F}$ is surjective) and all the higher cohomology vanish (i.e., $H^n(X, \mathcal{F}) = 0$ for all $n > 0$).

Let $\mathcal{D}_X$ be the sheaf of Grothendieck differential operators, $D(X) = \Gamma(\mathcal{D}_X)$ its global sections. We consider the category of quasicoherent $\mathcal{D}_X$-modules $\mathcal{D}_X-$Qcoh (i.e., sheaves of $\mathcal{D}_X$-modules, quasicoherent as $\mathcal{O}_X$-modules). The variety X is called *D-affine* if

- $\Gamma : \mathcal{D}_X-\text{Qcoh} \to D(X)-\text{Mod}$ is exact,
- if $\mathcal{F} \in \mathcal{D}_X-$Qcoh and $\Gamma(\mathcal{F}) \cong 0$, then $\mathcal{F} \cong 0$.

Lemma 1. [12, 15] *The following statements (where statement (4) requires X to be quasiprojective with an ample line bundle $\mathcal{L}$) about an algebraic variety X over $\mathbb{F}$ are equivalent:*

(1) *X is D-affine.*
(2) $\Gamma : \mathcal{D}_X-\text{Qcoh} \to D(X)-\text{Mod}$ *is an equivalence.*

(3) *Each sheaf* $\mathcal{F} \in \mathcal{D}_X-\mathrm{Qcoh}$ *is affine.*
(4) *There exists* $N > 0$ *such that the following two statements hold for all* $n > N$*:*
  (a) $\mathcal{D}_X(-n) = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}^{-n\otimes}$ *is generated by global sections,*
  (b) *the map* $\Gamma(\mathcal{D}_X(-n)) \otimes_{\mathbb{K}} \Gamma(\mathcal{L}^{n\otimes}) \to \Gamma(\mathcal{D}_X)$ *is surjective.*

The following lemma is straightforward, so we skip a proof:

Lemma 2. *Let* $\mathcal{F}, \mathcal{F}' \in \mathcal{O}_X-\mathrm{Qcoh}$ *be generated by global sections.*

(1) *If* $\mathcal{G} \in \mathcal{O}_X-\mathrm{Qcoh}$ *is a quotient of* $\mathcal{F}$*, then* $\mathcal{G}$ *is generated by global sections.*
(2) *If* $Y \subseteq X$ *is a closed subscheme, then* $\mathcal{F}|_Y$ *is generated by global sections.*
(3) *If* $0 \to \mathcal{F}' \to \mathcal{G} \to \mathcal{F} \to 0$ *is an exact sequence in* $\mathcal{O}_X-\mathrm{Qcoh}$*, then* $\mathcal{G}$ *is generated by global sections.*

The third lemma is easy but contains not so well-known terminology, hence, we give a proof.

Lemma 3. *Let* $\mathcal{F} \in \mathcal{O}_X-\mathrm{Qcoh}$ *be normal in the sense of Barth. If for each* $p \in X$ *there exists an open neighbourhood* $p \in U \subseteq X$ *such that* $X \setminus U$ *is of codimension at least two and* $\mathcal{F}|_U$ *is generated by global sections, then* $\mathcal{F}$ *is generated by global sections.*

Proof. Let $F = \mathcal{F}_X(X)$. Consider an open $U \subseteq X$ with $X \setminus U$ is of codimension at least two. Normality means that the restriction map $F = \mathcal{F}(X) \to \mathcal{F}(U)$ is an isomorphism [9, p. 126].

Let Y be the support of the cokernel of the natural map $\gamma : \mathcal{O}_X \boxtimes F \to \mathcal{F}$. Generation of $\mathcal{F}|_U$ by global sections means that $U \cap Y = \emptyset$. Our condition means that no point p belongs to Y. Hence, $Y = \emptyset$ and γ is surjective. □

The following well-known observation is sufficient for our ends. We believe that it is true for singular varieties as well: it should follow from the description of $\mathcal{D}_X$ as the dual $\mathcal{O}_X|\mathbb{F}$-algebra of the algebroid of functions on the formal neighbourhood of the diagonal $X \to X \times X$ [18, 2.4].

Lemma 4. *If* X *and* Y *are smooth varieties over* $\mathbb{F}$*, then the natural map* $\varphi_{X,Y} : \mathcal{D}_X \boxtimes \mathcal{D}_Y \to \mathcal{D}_{X\times Y}$ *is an isomorphism of sheaves of* $\mathbb{F}$*-algebras on* $X \times Y$*.*

It is interesting whether the quasiprojectivty assumption is necessary in the next lemma.

Lemma 5. *If X and Y are D-affine quasiprojective varieties over $\mathbb{F}$ such the map $\varphi_{X,Y}$ from Lemma 4 is an isomorphism, then any open subset $U \subseteq X \times Y$ is a D-affine variety.*

Proof. Decompose the global sections as a composition of functors

$$\Gamma : \mathcal{D}_{X\times Y}\text{–Qcoh} \xrightarrow{(X\times Y\to X)_*} (D(X)\boxtimes\mathcal{D}_Y)\text{–Qcoh} \xrightarrow{\Gamma} D(X\times Y)\text{–Mod}.$$

The assumed tensor product decomposition together with D-affinity of X and Y imply that both functors are equivalences. Hence, $X \times Y$ is a D-affine variety

Finally, U is D-affine by criterion (4) in Lemma 1. ☐

2. Divisors

We go straight to the main result of this section.

Theorem 6. *Let X be an irreducible D-affine algebraic variety over $\mathbb{F}$. The following statements hold for an effective Cartier divisor $Y \subset X$:*

1. *there exists a rational function $f \in \mathbb{K}(X)$ such that $Y = \mathrm{div}_{\mathrm{zer}}(f)$,*
2. *the normal sheaf $\mathcal{N}_Y$ is generated by global sections.*

Proof. Let $U \coloneqq X \setminus Y$ be the open complement and $j : U \hookrightarrow X$ its embedding. Observe that $j_*(\mathcal{O}_U) = \mathcal{O}_Y(*X)$ is a $\mathcal{D}_X$-submodule of the sheaf $\mathcal{M}_X$ of rational functions. On an open subset $V \subseteq X$ (from some cover of X) the divisor Y is defined by a single function h. Hence,

$$j_*(\mathcal{O}_U)(V) = \mathcal{O}_Y(*X)(V) = \{\frac{f}{h^n} \ \mid \ f \in \mathcal{O}_X(V)\} \ .$$

Let us verify that $\mathbf{d}(\mathcal{O}_Y(*X)(V)) \subseteq \mathcal{O}_Y(*X)(V)$ by induction on the order m of a differential operator $\mathbf{d} \in \mathcal{D}_X(V)$.

The statement is clear of $m = 0$. Suppose it is known for differential operators of order less than m. Since $[\mathbf{d}, h]$ is of order less than m,

$$[\mathbf{d}, h](\frac{f}{h^n}) = \frac{g}{h^k}$$

for some $g \in \mathcal{O}_Y(V)$ and an integer k. Opening the left side, we get a formula that permits an internal induction loop on n:

$$\mathbf{d}(\frac{f}{h^n}) = \frac{1}{h}\mathbf{d}(\frac{f}{h^{n-1}}) - \frac{g}{h^{k+1}}.$$

The sheaf of algebras $\mathcal{A} = j_*(\mathcal{O}_U)$ is D-module, so generated by global sections. It is filtered by $\mathcal{A}_n = \mathcal{O}_X(nY) = \{f \mid \mathrm{div}(f) + nY \geq 0\}$. The global sections $A_n = \Gamma(X, \mathcal{A}_n)$ define a filtration of the algebra $A = \Gamma(X, j_*(\mathcal{O}_U))$.

Pick the smallest n such that $A_n \neq A_0$ and some $h \in A_n \setminus A_0$. If $e \in \mathcal{O}_X(Y)(V)$ a non-constant local section, then $\mathrm{ord}_Y(e) = -1$.

Since A is generated by global sections, $e = g_0 \cdot 1 + g_1 h_1 + \ldots$ where $g_i \in \mathcal{O}_X(V)$, $h_i \in A_{k_i}$ with $k_i \geq n$. Since $\mathrm{ord}_Y(g_0 \cdot 1 + g_1 h_1 + \ldots) \leq -n$, it follows that $n = 1$.

(1) Let $f = h^{-1}$. Clearly, $Y = \mathrm{div}_{\mathrm{pol}}(h) = \mathrm{div}_{\mathrm{zer}}(f)$.

(2) As we have seen, $f \in A_1$. Moreover, the two functions f, $f+1$ generate the invertible sheaf $\mathcal{A}_1 = \mathcal{O}_X(Y)$ because at each point of U one of them is non-zero. Observe that we have the standard sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(Y) \to \mathcal{N}_{Y|X} \to 0$$

without any further restrictions on Y (such as smoothness). Indeed, pick an affine open U with $R = \mathcal{O}_X(U)$. Defining Y as zeroes of $g \in R$,

$$(\mathcal{O}_X(Y)/\mathcal{O}_X)|_U = Rg^{-1}/R, \ \mathcal{N}^*_{Y|X}|_U = Rg/Rg^2.$$

The multiplication gives a perfect R/Rg-module pairing

$$Rg^{-1}/R \times Rg/Rg^2 \to R/Rg, \quad (ag^{-1} + R, bg + Rg^2) \mapsto ab + Rg$$

that yields the isomorphism

$$(\mathcal{O}_X(Y)/\mathcal{O}_X)|_U = Rg^{-1}/R \xrightarrow{\cong} \hom_R(Rg/Rg^2, R/Rg) = \mathcal{N}_{Y|X}|_U.$$

By Lemma 2, $\mathcal{N}_{Y|X}$ is generated by global sections. Hence, so is its restriction $\mathcal{N}_Y = (\mathcal{N}_{Y|X})|_X$. □

We can derive some geometric consequences of D-affinity as soon as we can exhibit some interesting $\mathcal{D}_X$-modules. For example, $\mathcal{O}_X$ is a $\mathcal{D}_X$-module, thus, if X is complete, we know some of its Hodge numbers

$$h^{0,0}(X) = 1, \ h^{0,m}(X) = 0 \ \text{ for } \ m > 0.$$

For a smooth projective surface X this means that $p_a = p_g = 0$. Theorem 6 implies that the surface is minimal in a strong sense: $Y^2 \geq 0$ for any curve $Y \subseteq X$. Moreover,

$$c_2(X) = 2 + h^{2,2}(X), \ c_1^2(X) = 10 - h^{2,2}(X), \ 1 \leq h^{2,2}(X) \leq 10.$$

It would be interesting to classify minimal models with such numerical invariants that do not have any negative curves. Any example not covered by Corollary 7 is potentially D-affine, disproving the converse statement for Beilinson-Bernstein Theorem.

Notice that the next corollary does not follow from Theorem 12 because it covers any characteristic:

Corollary 7. *A rational smooth connected projective D-affine surface over $\mathbb{F}$ is isomorphic to either P^2 or $P^1 \times P^1$.*

Proof. A minimal smooth rational surface is either P^2 or the Hirzebruch surface H_n, $n \geq 0$. Since H_n contains an irreducible curve C with $C^2 = -n$, we conclude that $n = 0$. Finally, $H_0 \cong P^1 \times P^1$. □

Given an arbitrary closed subvariety $Z \subseteq X$, we can produce some $\mathcal{D}_X$-modules supported on Z, for instance, functions on the formal neighbourhood of Z or local cohomology sheaves $\mathcal{H}^n_Y(\mathcal{F})$ where $\mathcal{F}$ is an $\mathcal{D}_X$-module, e.g., $\mathcal{F} = \mathcal{O}_X$ or $\mathcal{F} = \mathcal{D}_X(m)$. It would be interesting to analyse how affinity of these sheaves affects geometry of X. The following observation is an example of such analysis:

Theorem 8. *If a smooth subvariety $Z \subset X$ is a scheme-theoretic complete intersection of effective Cartier divisors in an irreducible D-affine variety X over $\mathbb{F}$, then the normal sheaf $\mathcal{N}_Z$ is generated by global sections.*

Proof. Let Z be the scheme-theoretic intersection of effective Cartier divisors $Y_1, \ldots Y_n$. Define partial scheme-theoretic intersections $Z_m \coloneqq \cap_{k=1}^m Y_k$. Restrict all the sheaves of interest to Z:

$$\mathcal{F}_k \coloneqq \mathcal{N}_{Y_k \subset X}|_Z, \quad \mathcal{G}_k \coloneqq \mathcal{N}_{Z_k \subset X}|_Z, \quad \mathcal{H}_k \coloneqq \mathcal{N}_{Z_k \subset Z_{k-1}}|_Z.$$

The sheaves $\mathcal{F}_k$ are generated by global sections by a combination of Theorem 6 and Lemma 2.

The key observation is that $\mathcal{H}_k \cong \mathcal{F}_k$ under our assumptions. Both sheaves are subsheaves of $(\mathcal{N}^*_{Y_k \subset X}|_Z)^*$. It is a local question to compare them. Let $A = \mathcal{O}_{X,(p)}$ be the local ring for some $p \in Z$. The subschemes $Y_k \supseteq Z$ are locally defined by ideal $I \subseteq K \lhd A$ so that $\mathcal{O}_{Y_k,(p)} = A/I$ and $B \coloneqq \mathcal{O}_{Z,(p)} = A/K$ and

$$\mathcal{N}^*_{Y_k \subset X,(p)} = \frac{I}{I^2},\ \mathcal{N}^*_{Y_k \subset X}|_{Z,(p)} = \frac{I}{I^2} \otimes_A B \cong \frac{I/I^2}{(I/I^2)K} = \frac{I/I^2}{IK/I^2} \cong \frac{I}{IK}$$

where the former is an A/I-module and the latter is a B-module. Now

$$(\mathcal{N}^*_{Y_k \subset X}|_Z)^*_{(p)} \cong \hom_B(\frac{I}{IK}, B) = \hom_A(\frac{I}{IK}, B) \subseteq \hom_A(\frac{I}{I^2}, B),$$

but the last inclusion is actually equality: $f \in \hom_A(I/I^2, B)$ defines a map $\widetilde{f} : I \to B$ such that $\widetilde{f}(IK) \subseteq \widetilde{f}(I)K = 0$. On the other hand,

$$\mathcal{F}_{k,(p)} = (\frac{I}{I^2})^* \otimes_A B \cong \hom_A(\frac{I}{I^2}, A) \otimes_A \hom_A(A, B).$$

In these presentations the natural map $\gamma : \mathcal{F}_{k,(p)} \to (\mathcal{N}^*_{Y_k \subset X}|_Z)^*_{(p)}$ becomes the composition

$$\gamma : \hom_A(\frac{I}{I^2}, A) \otimes_A \hom_A(A, B) \longrightarrow \hom_A(\frac{I}{I^2}, B).$$

Observe that the point p is smooth not only in Z but also in Y_k. Indeed, if it were not smooth, then $\dim T_pY_k = \dim X > \dim Y_k$. Intersecting with one of Y_j, $j \neq k$ can bring down dimension by at most 1. This would mean that $\dim T_pZ > \dim Z$, contradicting smoothness of Z.

Smoothness of p in Y_k implies that I/I^2 is a projective A-module. This ensures that the map γ is an isomorphism.

Let $J \lhd A$ be the ideal that defines Z_{k-1} near p. Then $I + J$ defines Z_k so that $\mathcal{O}_{Z_{k-1},(p)} = A/J$ and $\mathcal{O}_{Z_k,(p)} = A/(I + J)$. Moreover,

$$\mathcal{N}^*_{Z_k\subset Z_{k-1},(p)} = \frac{(I+J)/J}{((I+J)/J)^2} = \frac{(I+J)/J}{(I^2+J)/J} \cong \frac{I+J}{I^2+J} \cong \frac{I}{I^2+I\cap J}.$$

The scheme-theoretic complete intersection property implies that the intersection of Z_{k-1} and Y_k is pure so that $I \cap J = IJ$. Thus,

$$\mathcal{N}^*_{Z_k\subset Z_{k-1},(p)} \cong \frac{I}{I^2+IJ} = \frac{I}{I(I+J)}.$$

Using this,

$$\mathcal{N}_{Z_k\subset Z_{k-1},(p)} \cong \hom_A(\frac{I}{I(I+J)}, \frac{A}{I+J}) \subseteq \hom_A(\frac{I}{I^2}, \frac{A}{I+J})$$

and the inclusion is an equality by the same argument as earlier in this proof. Hence,

$$\mathcal{H}_{k,(p)} \cong \hom_A(\frac{I}{I^2}, \frac{A}{I+J})\otimes_A B \cong \hom_A(\frac{I}{I^2}, \frac{A}{I+J})\otimes_A \hom_A(\frac{A}{I+J}, B).$$

The natural map $\delta : \mathcal{H}_{k,(p)} \to (\mathcal{N}^*_{Y_k\subset X}|_Z)^*_{(p)}$ becomes the composition

$$\delta : \hom_A(\frac{I}{I^2}, \frac{A}{I+J}) \otimes_A \hom_A(\frac{A}{I+J}, B) \longrightarrow \hom_A(\frac{I}{I^2}, B)$$

as earlier in the proof. The argument as before allows as to conclude that p is smooth in Z_k and δ is an isomorphism.

Finally, let us proceed by induction on k to prove that all $\mathcal{G}_k$ are generated by global sections. We would be done at the end of induction since $\mathcal{N}_X = \mathcal{G}_n$.

If $k = 1$, then $Z_1 = Y_1$ and $\mathcal{G}_1 = \mathcal{F}_1$ is generated by global sections.

Now suppose that $k > 1$ and we have proved that $\mathcal{G}_{k-1}$ is generated by global sections. The embedding $Z_k \subseteq Z_{k-1} \subset X$ leads to an exact sequence of $\mathcal{O}_Z$-modules:

$$0 \to \mathcal{H}_k \to \mathcal{G}_k \to \mathcal{G}_{k-1} \to 0.$$

Since $\mathcal{H}_k \cong \mathcal{F}_k$ is generated by global sections, Lemma 2 leads to the conclusion that $\mathcal{G}_k$ is generated by global sections. □

3. Rationality

Let X be an irreducible algebraic variety over $\mathbb{F}$, $p \in X$ a factorial point, which means that the local ring $\mathcal{O}_{X,(p)}$ is a unique factorisation domain. Let I_p be its maximal ideal. Since $\mathbb{F}(X)$ is the quotient field of $\mathcal{O}_{X,(p)}$, we can represent each non-zero rational function h uniquely (up to scalars) as $h = f/g$, with coprime $f, g \in \mathcal{O}_{X,(p)}$. This controls the interaction of p and h:

- $p \in \mathrm{div}_{\mathrm{zer}}(h) \iff f \in I_p$,
- $h(p) = 0 \iff g = 1, f \in I_p$,
- $p \in \mathrm{div}_{\mathrm{pol}}(h) \iff g \in I_p$,
- h has a pole at $p \iff f = 1, g \in I_p$,
- h is not determined at $p \iff f, g \in I_p$.

Associated to h, we have two open loci in X:

$$X_h^{\mathrm{sm}} = \{p \in X_h^{\mathrm{det}} | h \text{ is smooth at } p\} \subseteq X_h^{\mathrm{det}} = \{p \in X | \neg(f, g \in I_p)\}.$$

The determinacy locus X_h^{det} contains all the points where $h(p)$ can be assigned a value in $\mathbb{F} \cup \{\infty\}$. Its complement, the indeterminacy locus $X_h^{\mathrm{ind}} \coloneqq X \setminus X_h^{\mathrm{det}}$ is a closed set of codimension at least 2. The singular locus $X_h^{\mathrm{sing}} \coloneqq X \setminus X_h^{\mathrm{sm}}$ is a closed set of codimension at least 1.

Recall that a uniformly rational variety is a connected variety where every point admits an open neighbourhood, isomorphic to an open subset of an affine space [5]. Note that this condition implies smoothness. It is a problem stated by Gromov [6] whether any smooth rational variety is uniformly rational. To the best of our knowledge, the problem is still open.

Proposition 9. *Let X be a uniformly rational quasiprojective variety, $p \in X$ its point, $\delta : X \to X^2$ the diagonal embedding. There exist non-empty open set $U \subseteq X$ such that $p \in U$, the complement $X \setminus U$ has a codimension at least 2 and $\delta : U \to U^2$ is a scheme-theoretic complete intersection.*

Proof. Uniform rationality implies that $\mathcal{O}_{X,(p)} = \mathbb{F}[f_1, \dots f_n]_{(f_1,\dots,f_n)}$, the localisation of the polynomials at the maximal ideal. Then $\mathbb{F}(X) = \mathbb{F}(f_1, \dots f_n)$. Define $U \coloneqq \cap_k X_{f_k}^{\mathrm{det}}$.

We claim that all functions f_i are smooth at any $q \in U$. Indeed, let I_q be be the maximal ideal of $\mathcal{O}_{X,(q)}$. If $f_i(q) \in \mathbb{F}$, let $h_i = f_i - f_i(q)$. If $f_i(q) = \infty$, let $h_i = 1/f_i$. Clearly, $p \in U$, $U = \cap_k X_{h_k}^{\mathrm{det}}$, and $h_1, \dots h_n \in I_q$. It follows that $\mathcal{O}_{X,(q)} \supseteq \mathbb{F}[h_1, \dots h_n]_{(h_1,\dots,h_n)}$. Moreover, these two local rings are isomorphic and have the same quotient field $\mathbb{F}(X)$. Hence,

$$\mathcal{O}_{X,(q)} = \mathbb{F}[h_1, \dots h_n]_{(h_1,\dots,h_n)},$$

which is clear because a further localisation will change the isomorphism class of this ring, e.g., it will change the dimensions of J^m/J^{m+1} where J is the maximal ideal. It follows that the differentials $\mathrm{d}_q h_1, \ldots, \mathrm{d}_q h_n$ form a basis of the cotangent space $T_q X = I_q/I_q^2$. Hence, so are the differentials $\mathrm{d}_q f_1, \ldots, \mathrm{d}_q f_n$ The claim is proved.

Observe that $\mathbb{F}(U^2) = \mathbb{F}(X^2)$ is the quotient field of $\mathbb{F}(X) \otimes_{\mathbb{F}} \mathbb{F}(X)$. Let $g_k = f_k \otimes 1 - 1 \otimes f_k \in \mathbb{F}(X^2)$. Since X is quasiprojective, any two points $q \neq s \in X$ lie on the same open affine subset, thus, $\mathcal{O}_{X,(q)} \neq \mathcal{O}_{X,(s)}$. It follows that the functions f_k separate points of U and U is a set-theoretic complete intersection of $Y_k = \mathrm{div}_{\mathrm{zer}}(g_k) \subseteq U^2$, $k = 1, \ldots, n$.

Moreover, U is a scheme-theoretic complete intersection of Y_k as follows from smoothness of U (X is assumed to be smooth) and smoothness of each Y_k at any diagonal point $(s,s) \in U^2$ (that follows from the differentials $\mathrm{d}_{(s,s)} g_k$ being surjective). □

We are ready for the main result of the section.

Theorem 10. *If X is a D-affine uniformly rational quasiprojective variety over $\mathbb{F}$, then the tangent sheaf $\mathcal{T}_X$ is generated by global sections.*

Proof. Let $U \subseteq V$ be as in Proposition 9, in particular, U is a smooth scheme-theoretic complete intersection. By Lemma 5, V is D-affine. By Theorem 8, $\mathcal{N}_{U \subseteq V}$ is generated by global sections. Clearly, $\mathcal{N}_{U\subseteq V} = \mathcal{N}_{X \subseteq X^2}|_U$. Moreover, it is a locally free coherent sheaf, consequently, reflexive and normal in the sense of Barth. Thus, Lemma 3 implies that $\mathcal{T}_X \cong \mathcal{N}_{X \subseteq X^2}$ is generated by global sections. □

The final corollary immediately follows, thanks to [8, Th. 2]:

Corollary 11. *If X is a D-affine uniformly rational projective variety over $\mathbb{C}$, then X is a generalised flag variety.*

4. Lefschetz Principle

Lefschetz Principle is a metatheorem that if a statement about algebraic varieties over $\mathbb{C}$ should also hold for algebraic varieties over $\mathbb{K}$. It is true on the nose for first-order statements. It is not clear to us whether Corollary 11 is a first order statement, yet we can push it through to $\mathbb{K}$:

Theorem 12. *Suppose X is a uniformly rational projective D-affine variety over an algebraically closed field $\mathbb{K}$ of characteristic zero. Then X is isomorphic to a generalised flag variety.*

Proof. We will construct fields $\mathbb{A} \leq \mathbb{B} \leq \mathbb{E} \leq \mathbb{K}$, $\mathbb{E} \leq \mathbb{C}$ and a form $X_{\mathbb{A}}$ of X over $\mathbb{A}$ throughout the proof that will let us to push all the statements through. We can do it because the statements depend on finitely many field elements.

Embed X into a projective space. Let $\mathbb{A}$ be the field obtained by adding to $\mathbb{Q}$ all coefficients of the equations defining X and all coefficients of the rational functions $f_1, \dots f_n$ such that $\mathbb{K}(X) = \mathbb{K}(f_1, \dots f_n)$ as well as all coefficients that appear in isomorphisms between affine subsets of some finite affine cover of X and open subsets in affine spaces. This yields a form $X_{\mathbb{A}}$, which is a uniformly rational projective variety.

Choose a basis $\mathbf{e}_i$ of $\Gamma(\mathcal{T}_X)$ and a finite affine cover $X_{\mathbb{A}} = \cup_j U_j$. Express each $\mathbf{e}_i|_{(U_j)_{\mathbb{K}}}$ as a $\mathbb{K}$-linear combination of some sections from $\mathcal{T}_{X_{\mathbb{A}}}(U_j)$. Let $\mathbb{B}$ be the field obtained by adding all coefficients of these linear combinations to $\mathbb{A}$. Notice that $\mathcal{T}_X$ is generated by global sections by Theorem 10. Then $X_{\mathbb{B}} \coloneqq (X_{\mathbb{A}})_{\mathbb{B}}$ is a uniformly rational form of X such that $\mathcal{T}_{X_{\mathbb{B}}}$ is generated by global sections.

Consider the Lie algebra of global sections $\mathfrak{g}_{\mathbb{B}} \coloneqq \Gamma(\mathcal{T}_{X_{\mathbb{B}}})$. Choose a finite extension $\mathbb{E} \geq \mathbb{B}$ such that the semisimple part $\mathfrak{g}_{\mathbb{B}}/\mathrm{rad}(\mathfrak{g}_{\mathbb{B}})$ splits over $\mathbb{E}$. Then $X_{\mathbb{E}} \coloneqq (X_{\mathbb{B}})_{\mathbb{E}}$ is a form of X with all desired properties.

Since $\mathbb{E}$ is a finite extension of $\mathbb{Q}$, it admits an embedding $\mathbb{E} \hookrightarrow \mathbb{C}$. Fix such an embedding and consider the complex algebraic variety $X_{\mathbb{C}} \coloneqq (X_{\mathbb{E}})_{\mathbb{C}}$. It has the same Hodge number as X, in particular, $h^{0,m}(X_{\mathbb{C}}) = 0$ for $m > 0$. Its tangent sheaf is generated by global sections. Hence, the proof of [8, Th. 2] carries through. We conclude that $X_{\mathbb{C}}$ is a generalised flag variety G/P with G acting faithfully. Then the Lie $\mathfrak{g}_{\mathbb{C}} \coloneqq \Gamma(\mathcal{T}_{X_{\mathbb{C}}})$ is isomorphic to $\mathrm{Lie}(G)$ and, in particular, semisimple.

Since $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g}_{\mathbb{E}} \otimes_{\mathbb{E}} \mathbb{C}$, the Lie algebra $\mathfrak{g}_{\mathbb{E}}$ is also semisimple. Since $\mathfrak{g}_{\mathbb{E}}$ is split, it is spanned by nilpotent elements. Exponentiation of nilpotent elements of $\mathfrak{g}_{\mathbb{E}}$ gives a family of automorphisms of $X_{\mathbb{E}}$. This family defines an action of the split semisimple group $G_{\mathbb{E}}$ because all relations (defining for $G_{\mathbb{E}}$) can be verified in the action on $X_{\mathbb{C}}$. It follows that $X_{\mathbb{E}}$ is a generalised flag variety, and so is X. □

References

[1] H. Andersen, M. Kaneda, On the D-affinity of the flag variety in type B_2, *Manuscripta Math.* **103** (2000), 393–399.

[2] A. Beilinson, J. Bernstein, Localisation de g-modules, *C. R. Acad. Sci. Paris Ser. I Math.* **292** (1981), 15–18.

[3] R. Bezrukavnikov, I. Mirković, D. Rumynin, Localization of modules for a semisimple Lie algebra in prime characteristic, *Ann. of Math. (2)* **167** (2008), 945–991.

[4] R. Bezrukavnikov, I. Mirković, D. Rumynin, Singular localization and intertwining functors for reductive Lie algebras in prime characteristic, *Nagoya Math. J.*, 184 (2006), 1–55.

[5] F. Bogomolov, C. Böhning, On uniformly rational varieties, *Topology, geometry, integrable systems, and mathematical physics*, 33–48, AMS Transl. Ser. 2, 234, Adv. Math. Sci., 67, AMS, Providence, 2014.

[6] M. Gromov, Oka's principle for holomorphic sections of elliptic bundles, *J. Amer. Math. Soc.*, **2** (1989), 851–897.

[7] B. Haastert, Über Differentialoperatoren und D-Moduln in positiver Charakteristik, *Manuscripta Math.* **58** (1987), 385–415.

[8] K. El Haloui, D. Rumynin, D-modules and projective stacks, *Journal of Algebra* **502** (2018), 515–537.

[9] R. Hartshorne, Stable reflexive sheaves, *Math. Ann.* **254** (1980), 121–176.

[10] Y. Hashimoto, M. Kaneda, D. Rumynin, On localization of $\overline{D}$-modules, *Representations of algebraic groups, quantum groups, and Lie algebras*, 43–62, Contemp. Math., 413, Amer. Math. Soc., Providence, 2006.

[11] M. Holland, P. Polo, K-theory of twisted differential operators on flag varieties, *Invent. Math.* **123** (1996), 377–414.

[12] R. Hotta, K. Takeuchi, T. Tanisaki, *D-Modules, Perverse Sheaves, and Representation Theory*, Birkhauser, Boston, 2008.

[13] M. Kaneda, J. Ye, Equivariant localization of $\overline{D}$-modules on the flag variety of the symplectic group of degree 4, *J. Algebra* **309** (2007), 236–281.

[14] M. Kaneda, On the Frobenius direct image of the structure sheaf of a homogeneous projective variety, *J. Algebra* **512** (2018), 160–188.

[15] M. Kashiwara, Representation theory and D-modules on flag varieties, *Orbites unipotentes et représentations, III*, 55–109, Astérisque **173–174** (1989).

[16] M. Kashiwara, N. Lauritzen, Local cohomology and D-affinity in positive characteristic, *C. R. Math. Acad. Sci. Paris* **335** (2002), 993–996.

[17] A. Langer, D-affinity and Frobenius morphism on quadrics, *Int. Math. Res. Not.* (2008), no. 1, Art. ID rnm 145.

[18] D. Rumynin, Duality for Hopf algebroids, *J. Algebra* **223** (2000), 237–255.

[19] A. Samokhin, On the D-affinity of flag varieties in positive characteristic, *J. Algebra* **324** (2010), 1435–1446.

[20] A. Samokhin, A vanishing theorem for differential operators in positive characteristic, *Transform. Groups* **15** (2010), 227–242.

[21] J. Thomsen, D-affinity and toric varieties, *Bull. London Math. Soc.*, **29** (1997), 317–321.

[22] M. Van den Bergh, *Differential operators on semi-invariants for tori and weighted projective spaces*, Topics in invariant theory (Paris, 1989/1990), 255–272, Lecture Notes in Math., 1478, Springer, Berlin, 1991.

Department of Mathematics, University of Warwick, Coventry, CV4 7AL, UK

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany

Associated member of Laboratory of Algebraic Geometry, National Research University Higher School of Economics, Russia

E-mail address: D.Rumynin@warwick.ac.uk