Quantum solvable algebras. Ideals and representations at roots of 1

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Abstract. There proved that every prime stable with respect to quantum adjoint action ideal I is completely prime and $\mathrm{Fract}(R/I)$ is isomorphic to the skew field of fractions of an algebra of twisted polynomials. We study correspondence between symplectic leaves and irreducible representations. The Conjecture of De Concini-Kac-Procesi on dimension of irreducible representations is proved for sufficiently great l.

1 Introduction.

A quantum solvable algebra is an iterated q-skew extension, constructed by derivations and diagonal automorphisms. Among examples are Quantum Matrices, Quantum Weyl algebra, Quantum Heisenberg algebra, $U_q(n)$ and also some their subalgebras and factor algebras. Representations of these algebras were treated separately ([DC-K-P],[DC-P],[JZ]). The main approach is correspondence between irreducible representations of a quantum algebra and symplectic leaves of Poisson manifold of the center.

In the paper we study quantum solvable algebras in general setting, obeying some natural and easily checkable conditions (see Conditions 2.2-2.4). We consider the specializations of these algebras at primitive l-root of unity ε.

We make some assumptions on l. First, we claim that ε is rather great: ε is a point of good reduction of stratification of prime spectra for intederminate q (see Definition 2.8, Theorem 2.9). Second, we claim that ε is relatively prime with all minors of integer matrix $\mathbb{S}$ (Definition 2.1). We say that ε is admissible, if it obeys the above conditions. There are infinitely many admissible roots of unity.

We study prime ideals stable with respect to quantum adjoint action. We prove that these ideals have prime intersections with components of filtration of quantum solvable algebras (Theorem 1.13). All prime and stable with respect to quantum adjoint action ideals are completely prime (Theorem 2.20). We get stratification of these ideals. This helps us to study irreducible representations and symplectic leaves (Theorem 3.3). In particular, we prove that dimension of any irreducible representation equals to l raising to the power of one-half of dimension of corresponding symplectic leaf (Conjecture of De Concini-Kac-Procesi).

2 On quantum adjoint action and stable ideals

In this section we study q-skew extensions $E = R[x;\tau,\delta]$. The main statement of this section is Theorem 1.13 on reduction of prime ideals.

Throughout the paper we use the following notations. Let K be a field of zero characteristic and q is an indeterminate. We denote by C a localization of $K[q,q^{-1}]$ over some finitely generated denominator set and $F = \mathrm{Fract}(C)$. Let Γ be the cyclic subgroup $\{q^n\}_{n\in\mathbb{Z}}$ in the group of invertible elements C^* of C. Let ε be a primitive l-root of unity. Denote by $K(\varepsilon)$ the field extension of K (if $\varepsilon \in K$, we put $K(\varepsilon) = K$). Suppose further that C admits specialization via $q \mapsto \varepsilon$ in the field $K(\varepsilon)$.

Throughout this section R is a C-algebra and a free C-module. We consider specialization $\rho_\varepsilon : R \mapsto R_\varepsilon = R\mathrm{mod}(q-\varepsilon)$.

Further we differ the elements in R with their images in R_ε, using the following notations. For $x \in R$ we denote $x_\varepsilon = x\mathrm{mod}(q-\varepsilon)R$ or for $y \in R_\varepsilon$ we denote by $\underline{y}$ an element in its preimage in R.

Let u be an element in R such that $u_\varepsilon = u\mathrm{mod}(q-\varepsilon)$ lies in the center Z_ε of R_ε. For any $\underline{a} \in R$ the element $\mathrm{ad}_u(\underline{a}) = u\underline{a} - \underline{a}u$ lies in $(q-\varepsilon)R$. Whence $\frac{\mathrm{ad}_u(\underline{a})}{(q-\varepsilon)} \in R$. We denote

$$\mathcal{D}_u(a) = \frac{\mathrm{ad}_u(\underline{a})}{(q-\varepsilon)}\mathrm{mod}(q-\varepsilon) \tag{1.1}$$

for $a = \underline{a}\mathrm{mod}(q-\varepsilon) \in R_\varepsilon$.

Definition 1.1. The map $\mathcal{D}_u : R_\varepsilon \mapsto R_\varepsilon$ is called the quantum adjoin action of $u \in R$. We recall some properties of the quantum adjoin action [DC-K-P].

Property 1.2. $\mathcal{D}_u$ is a derivation of R_ε. That is $\mathcal{D}_u(ab) = \mathcal{D}_u(a)b + a\mathcal{D}_u(b)$ for all $a, b \in R_\varepsilon$.

Property 1.3. If $u\mathrm{mod}(q-\varepsilon) = u_1\mathrm{mod}(q-\varepsilon)$, then $u_1 - u = (q-\varepsilon)r, r \in R$ and $\mathcal{D}_{u_1} = \mathcal{D}_u + \mathrm{ad}_r$. This implies that an ideal $\mathcal{P}$ is $\mathcal{D}_{u_1}$-stable whenever it is $\mathcal{D}_u$-stable.

Property 1.4. If $u_\varepsilon, v_\varepsilon \in Z_\varepsilon$ for $u, v \in R$, then

$$\mathcal{D}_{uv}(a) = \mathcal{D}_u(a)v_\varepsilon + u_\varepsilon\mathcal{D}_v(a)$$

Property 1.5. The bracket

$$\{u_\varepsilon, v_\varepsilon\} = \mathcal{D}_u(v_\varepsilon) = -\mathcal{D}_v(u_\varepsilon)$$

is a Poisson bracket on Z_ε.

Consider a skew polynomial extension $E = R[\underline{x};\tau,\delta]$ where τ is an automorphism and δ is τ-derivation of R. That is $\delta(ab) = \delta(a)b + \tau(a)\delta(b)$. We assume that $\tau|_C = \mathrm{id}$, $\delta|_C = 0$ and suppose that E is q^s-skew extension of R. That is $\tau\delta = q^s\delta\tau$ for some $s \in \mathbb{Z}$. One can extend the automorphism τ upto an automorphism of E, putting $\tau(\underline{x}) = q^{-s}\underline{x}$. We save the notation τ for this extension.

All prime ideals in q-skew iterated extension of C are completely prime ([GL1], Theorem 2.3). It follows that the intersection with R of a prime ideal of E is a prime (indeed, completely prime) ideal in R. But in the case of roots of unity the intersection with R_ε of a prime ideal of E_ε may be not prime (even not semiprime) (see Example 2.6). The main goal of this section is to prove that intersection remains to be prime in the case of roots of unity if we claim that a given ideal in stable with respect to the quantum adjoint action. The following assumptions are valid till the end of this section.

Assumptions 1.6 1) As above ε is a primitive l-root of unity. We suppose that $GCD(s,l)=1$ if $s\neq 0$.

2) Suppose that x^l, $x=\underline{x}\mathrm{mod}(q-\varepsilon)$ lie in the center Z_ε of $E_\varepsilon = E \bmod (q-\varepsilon)$. We shall denote by $\mathcal{D}_x$ the derivation $\mathcal{D}_{\underline{x}^l}$ defined by (1.1).

3) We assume that τ Γ-diagonalizable automorphism of R. That is, there exists a C-basis $\{a_\alpha\}$ of R such that $\tau(a_\alpha)=q^{m_\alpha}a_\alpha$ for some $m_\alpha\in\mathbb{Z}$. It follows that the similar basis also exists in E.

One can extend τ and δ upto automorphism and τ-derivation of R_ε. We save the same notations for this extensions. Further we use usial notations

1) $(n)=(n)_{q^s}=\frac{q^{sn}-1}{q-1}$;

2) $(n)!=(n)_{q^s}=(1)(2)\cdots(n)$;

3) $\binom{n}{k}=\binom{n}{k}_{q^s}=\frac{(n)!}{(k)!(n-k)!}$.

Lemma 1.7. Under the above assumptions $\delta^l=0\mathrm{mod}(q-\varepsilon)$, $\tau^l=\mathrm{id} \bmod (q-\varepsilon)$.

Proof. One can prove ([G],6.2;[GL],2.5)

$$\underline{x}^l a=\sum_{i=0}^{n}\binom{l}{i}\tau^{l-i}\delta^i(a)\underline{x}^{l-i}=\tau^l(a)\underline{x}^l+\sum_{i=1}^{n-1}\binom{l}{i}\tau^{l-i}\delta^i(a)\underline{x}^{l-i}+\delta^l(a) \tag{1.2}$$

Since the element $x^l=\underline{x}^l\mathrm{mod}(q-\varepsilon)$ lie in the center Z_ε, then $\tau^l(a)=a\mathrm{mod}(q-\varepsilon)$ and $\delta^l(a)=0\mathrm{mod}(q-\varepsilon)$.□

We shall use the following notations

$$\Delta=\frac{\delta^l}{q-\varepsilon}:R_\varepsilon\mapsto R_\varepsilon$$

$$\theta=\frac{\tau^l-\mathrm{id}}{q-\varepsilon}:E_\varepsilon\mapsto E_\varepsilon.$$

Remark that Δ is a linear operator on R_ε and θ is a derivation of E_ε. For any integer m we denote

$$\underline{m}=\left(\frac{q^{ml}-1}{q-\varepsilon}\right)_{q-\varepsilon}=ml\varepsilon^{-1}.$$

Notice that, if $\tau(a)=q^m a$ for some $a\in E$ and $m\in\mathbb{Z}$, then $\theta(a_\varepsilon)=\underline{m}a_\varepsilon$.

Consider reduction of C-basis a_α modulo $q-\varepsilon$. We get a $\mathbb{C}$-basis $a_{\alpha\varepsilon}=a_\alpha\mathrm{mod}(q-\varepsilon)$. The derivation θ is diagonalizable $\theta(a_{\alpha\varepsilon})=\underline{m}_\alpha a_{\alpha\varepsilon}$.

Lemma 1.8 Let R be a free C-algebra. Let τ be a diagolizable automorphism of R, obeying Assumption 1.6(3). Consider an ideal I in R_ε. By I_θ, we denote the largest θ-stable ideal in I. We assert that the ideal I is prime whenever I_θ is prime.

Proof. If I is prime, then I_θ is also prime ([G],6.4). Suppose that I_θ is prime and I is not prime. There exist the elements $a,b\in R_\varepsilon$ such that $aR_\varepsilon b\subset I$ and $a,b\notin I$. Decompose a and b into the sums of θ-eigenvectors with different eigenvalues: $a=a_1+\cdots+a_k$, $b=b_1+\cdots+b_p$, $\theta(a_i)=\underline{m}_i a_i$, $\theta(b_j)=\underline{n}_j b_j$, $m_1>\cdots>m_k$, $n_1>\cdots>n_p$. We consider that all eigenvectors $a_1,\ldots,a_k,b_1,\ldots,b_p$ don't lie in

I. Let c be θ-eigenvector: $\theta(c) = \underline{z}(c)$, $z \in \mathbb{Z}$. The element acb lies in I and have decomposition

$$acb = (a_1 + \cdots + a_k)c(b_1 + \cdots + b_p) = a_1cb_1 + \cdots + a_kcb_p$$

into a sum of θ-eigenvectors. The term a_1cb_1 have the greatest eigenvalue $m_1 + z + n_1$. Since $acb \in I$, then $a_1cb_1 \in I_\theta$. Whence $a_1R_\varepsilon b_1 \subset I_\theta$. As far as I_θ is prime, we obtain that $a_1 \in I$ or $b_1 \in I$. A contradiction. The ideal I is prime. □

Consider the localization R' of R over the set

$$\{f(q) \in C : f(\varepsilon) \neq 0\}.$$

Lemma 1.9. Let Assumption 1.6(1) holds. Let k be a positive integer. Suppose that $k = \xi l + \eta$ with positive integers ξ and $0 \leq \eta < l$. As above $\delta^l(R) \subset (q - \varepsilon)R$. Then $\frac{\delta^l}{(k)!}(R') \subset R'$

Proof. For $s = 0$ the statement is trivial. Let $s \neq 0$. For a in R we have

$$\delta^k(a) = \delta^{\xi l + \eta}(a) = \delta^\eta(\delta^l)^\xi(a) \subset (q - \varepsilon)^\xi \delta^\eta(R) \subset (q - \varepsilon)^\xi R$$

On the other hand, $(q - \varepsilon)$ divides $(i) = \frac{q^{is} - 1}{q - 1}$ if and only if l divides i. It follows $(k)! = (q - \varepsilon)^\xi c(q)$ where $c(\varepsilon) \neq 0$. This implies the assertion.□

For all $g \in R$ we denote

$$\Pi_n(g) = \tau^{n-1}(g) \cdots \tau(g)g$$

Lemma 1.10. Let τ, δ, be as above. Suppose that $s \neq 0$. Let $a \in R$ and let I_a be a minimal τ-ideal generated by a in R'. Let n, m be positive integers. We assert that

$$\frac{\delta^n(a^m)}{(n)!} = \begin{cases} \Pi_n(\delta(a))\mathrm{mod} I_a, & \text{for} \quad m = n, \\ 0 \ \mathrm{mod} I_a & \text{for} \quad n < m. \end{cases}$$

Proof. We shall prove be induction on n. The case $n = 1$ is easy. Suppose that statement is true for all positive integers less then n. We are going to prove for n. We need the formula ([G],6.2):

$$\delta^n(ab) = \sum_{i=0} \binom{n}{i} \tau^{n-i}\delta^i(a)\delta^{n-i}(b). \tag{1.3}$$

We get

$$\delta^n(a^m) = \delta(a^{m-1}a) = \sum_{i=0} \binom{n}{i} \tau^{n-i}\delta^i(a^{m-1})\delta^{n-i}(a).$$

We obtain

$$\frac{\delta^n(a^m)}{(n)!} = \sum_{i=0}^{n-2} \tau^{n-i} \frac{\delta^i(a^{m-1})}{(i)!} \frac{\delta^{n-i}(a)}{(n-i)!} + \frac{\tau\delta^{n-1}(a^{m-1})}{(n-1)!}\delta(a) + \frac{\delta^n(a^{m-1})}{(n)!}a \tag{1.4}$$

Let $n < m$. By assumption of induction, $\frac{\delta^i(a^{m-1})}{(i)!} \in I_a$ for $i \leq n - 1$. By Lemma 1.9, $\frac{\delta^{n-i}(a)}{(n-i)!} \in R'$. This proves that all terms in (1.4), besides the last one, lie in I_a. As about the last term, it obviously lie in I_a. This concludes case $n < m$.

Let $m = n$. Similarly to previous case, we get

$$\frac{\delta^n(a^n)}{(n)!} = \frac{\tau\delta^{n-1}(a^{n-1})}{(n-1)!}\delta(a)\mathrm{mod}I_a = \tau(\Pi_{n-1}(\delta(a)))\delta(a)\mathrm{mod}I_a = \Pi_n(\delta(a))\mathrm{mod}I_a$$

$\Box$

Let I be an ideal in E_ε. Denote $J = I \bigcap R_\varepsilon$.

Lemma 1.11. Suppose that I is θ-stable . Then
1) I is τ-stable;
2) The ideal J is (τ, δ)-stable. If in addition I is $\mathcal{D}_x$-stable, then J is Δ-stable ideal in R_ε.

Proof.
1) The automorphism $\tau : E \mapsto E$ is diagonalizable. This implies that $\tau, \theta : E_\varepsilon \mapsto E_\varepsilon$ are diagonalizable with the common system of basic eigenvectors $\{a_\alpha\}$. Recall that $\tau(a_\alpha) = \varepsilon^{m_\alpha}a_\alpha$ and $\delta(a_\alpha) = \underline{m}_\alpha a_\alpha$. Every θ-eigenvector lies in a span of basis elements $a_{\alpha\varepsilon}$ corresponding to a common θ-eigevalue $\underline{m}$. Then all these $a_{\alpha\varepsilon}$ have the common τ-eigenvalue ε^m. This proves that every θ-eigenvector is also τ-eigenvector (the converse is not true).

Let a be an element in I. Decompose a into a sum of θ-eigenvectors with different eigenvalues (call θ-components). Since I is θ-stable, all θ-components also lie in I. We see above that this θ-components are τ-eigenvectors. This proves that $\tau(a) \in I$. We conclude that I is τ-stable.
2) As far as I is (τ, θ)-stable, the ideal J is also (τ, θ)-stable. Since $xa = \tau(a)x + \delta(a)$ and $a \in J$, then $xa, \tau(a)x \in I$. Whence $\delta(a) \in J$. This proves that J is δ-stable.

Suppose that I is $\mathcal{D}_x$-stable. stableLet $a \in J$. By direct calculations,

$$\mathcal{D}_x(a) = \theta(a)x^l + \sum_{i=1}^{l-1} \frac{\binom{l}{i}_{q^s}}{q-\varepsilon}\tau^{l-i}(\delta^i(a))x^{l-i} + \Delta(a) \tag{1.5}$$

Since $\mathcal{D}_x(a) \in I$, $\theta(a) \in J$, we obtain $\Delta(a) \in J$. $\Box$

Lemma 1.12.
1) $\delta(\theta(a)) = (\theta + \underline{s})\delta(a)$ for all $a \in R_\varepsilon$;
2) $\mathcal{D}_x(\tau(a)) = \tau\mathcal{D}_x(a)$ and $\mathcal{D}_x(\theta(a)) = (\theta + \underline{sl})\mathcal{D}_x(a)$ for all $a \in E_\varepsilon$. Here $\underline{s} = sl\varepsilon^{-1}$ and $\underline{sl} = sl^2\varepsilon^{-1}$.

Proof. Direct calculations.
Recall that we denote $\mathcal{D}_x = \mathcal{D}_{\underline{x}^l}$.

Theorem 1.13. Let R be a Noetherian C-algebra and a free C-module. Given q^s-skew extension $E = R[\underline{x}; \tau, \delta]$, obeying Assumptions 1.6. Let I be a prime $\mathcal{D}_x$-stable ideal in E_ε. Then the ideal $J = I \bigcap R_\varepsilon$ is also prime.

Proof. The proof divides in several steps. In the points 1 and 2 we shall prove the statement in the case I is θ-stable ideal. In the last point 3 we shall consider the general case.
1) Suppose that the ideal I is θ-stable. In this point we are going to prove J is semiprime. In the case $s = 0$ this follows from ([G],6.5). Let further $s \neq 0$.

By Lemma 1.11, the ideal J is (τ, δ)-stable Then EJ is a two-sided ideal in E_ε. The ideal J is (τ, δ)-prime ideal in R_ε. ([GL2],2.1).

Let N be an ideal in R_ε such that $N/J = \mathrm{Rad}(R/J)$. Hence, $N^n \in J$ for some positive integer n. We are going to prove that $N = J$.

Radical N is stable under actions of automorphisms and derivations (see [MC-R],14.2.3). In particular N is (τ, θ)-stable.

Suppose that N is δ-stable. Then EN is a two-sided ideal in E_ε. We see that $(EN)^n \subset EN^n \subset EJ \subset I$. Since I is prime, we have $EN \subset I$, whence $N \subset J$. This proves that J is semiprime.

The above observation shows that it remains to prove that N is δ-stable. Consider the image $\mathcal{L} = \delta(N)\mathrm{mod}(N)$ in R_ε/N. Since

$$\delta(N) = \delta(R_\varepsilon N R_\varepsilon) = \delta(R_\varepsilon) N R_\varepsilon + \tau(R_\varepsilon)\delta(N)R_\varepsilon + \tau(R_\varepsilon)\tau(N)R_\varepsilon$$

we see that $\mathcal{L}$ is an ideal in $B = R/N$. Moreover, by Lemma 1.12(1), the ideal $\mathcal{L}$ is (τ, θ)-stable. The algebra B is semiprime. Its Goldie quotient algebra $A = \mathrm{Goldie}(B)$ is a direct sum of matrix algebras over division rings $A = \oplus_{i=1}^N A_i$. Automorphism τ acts on components A_i by permutations. The ideal $A\mathcal{L}A$ is the sum $A' = \oplus_{i=1}^s A_i$ which is τ-stable. There exist the elements $r_1, \ldots, r_k, p_1, \ldots, p_k \in A$ and $h_1, \ldots, h_k \in \mathcal{L}$ such that

$$1_{A'} = r_1 h_1 p_1 + \cdots r_k h_k p_k.$$

Multiplying this equality by suitible regular elements of B, we show that $\mathcal{L}$ contains a regular element of A', say h. This implies that $\Pi_n(h) \neq 0$ for all n.

Denote by a an element of R_ε such that $a \in N$, $\delta(a) = h$. Denote by $\underline{a} \in R$ a representative of $a \in R_\varepsilon$.

By Lemma 1.10, we have

$$\frac{\delta^n(\underline{a}^n)}{(n)!} = \Pi(h) + \underline{b}$$

with some $\underline{b} \in I_{\underline{a}}$. As in Lemma 1.9 decompose $n = \xi l + \eta$, $0 \leq \eta < l$ and ξ,η are positive integers. Denote

$$\underline{d} = \frac{\delta^n(\underline{a}^n)}{(q-\varepsilon)^\xi} = c(q)(\Pi(h) + \underline{b})$$

with $c(\varepsilon) \neq 0$. We have proved that the element $d = \underline{d}\mathrm{mod}(q-\varepsilon)$ equals to A'-regular element $c(\varepsilon)\Pi_n(h)$ modulo N.

On the other hand,

$$\underline{d} = \frac{\delta^n(\underline{a}^n)}{(q-\varepsilon)^\xi} = \left(\frac{\delta^l}{q-\varepsilon}\right)^\xi \delta^\eta(\underline{a}^n) = \Delta^\xi \delta^\eta(a^n)\mathrm{mod}(q-\varepsilon),$$

$$d = \Delta^\xi \delta^\eta(a^n).$$

Recall that $a^n \in N^n \in J$ and J is (δ, Δ)-stable. This follows $d \in J$. Whence $d = 0\mathrm{mod}N$. A contradiction. The ideal N is δ-stable and J is semiprime.

2) In this point we shall prove that J is prime in the case I is θ-stable.

Since J is semiprime, one can present it as an intersection of prime ideals $J = Q_1 \bigcap \cdots \bigcap Q_N$. If a derivation save J, then it save all Q_i ([MC-R], 14.2.3). This implies that all ideals Q_i are θ and δ stable. Whence Q_i is τ-stable. Automorphism τ-acts on the system of central minimal idempotents $e_1, \dots, e_N$ by permutations. Decompose $A = \oplus_{i=1}^N A_i$, $A_i = Ae_i$. Since $\tau(A_i) \subset A_i$, then $\tau(e_i) = e_i$ and

$$\delta(e_i) = \delta(e_i^2) = \delta(e_i)e_i + \tau(e_i)\delta(e_i) = 2e_i\delta(e_i).$$

This proves $\delta(e_i) = 0$. Whence $\delta(Q_i) \subset Q_i$ for all i. Since J is (τ, δ)-prime, $J = Q_i$. Finally, J is prime.

3) In this point we conclude the proof. Let I be a prime $\mathcal{D}_x$-stable ideal in E_ε. By Lemma 1.8, I_θ is also prime. Let us prove that I_θ is $\mathcal{D}_x$-stable. Note

$$I_\theta = \{a \in E_\varepsilon : \theta^k(a) \in I \text{ for all } k = 0, 1, 2 \dots\}.$$

Given $a \in I_\theta$. By Lemma 1.12, we have

$$\theta^k(\mathcal{D}_x(a)) = \mathcal{D}_x(\theta - \underline{sl})^k(a) \in \mathcal{D}(I) \subset I$$

This proves that $\mathcal{D}_x(a) \in I_\theta$ and I_θ is also $\mathcal{D}_x$-stable and prime.

By points 1-2 of the proof, the ideal $I_\theta \bigcap R_\varepsilon$ is prime. The last ideal coinsides with J_θ. By Lemma 1.8, J is prime. □

3 Stratification of $\mathcal{D}$-stable ideals

In this section we are going to get stratification of prime $\mathcal{D}$-stable ideals at roots of unity. This provides a reduction to the case of algebra of twisted Laurent polynomials. We prove that all prime $\mathcal{D}$-stable ideals are completely prime.

Recall K is a field of zero characteristic, q is an indeterminate and C is a localization of $K[q, q^{-1}]$ over some finitely generated denominator set. Let $\mathbb{S} = (s_{ij})$ be a $n \times n$ integer skew-symmetric matrix. Denote $q_{ij} = q^{s_{ij}}$ and form the matrix $\mathbb{Q} = (q_{ij})$.

Definition 2.1([P1],[P2]). We say that a ring R is quantum solvable over C, if R is generated by the elements $x_1, x_2, \dots, x_n, k_1^{\pm 1}, \dots, k_m^{\pm 1}$ such that the monomials $x_1^{t_1} \cdots x_n^{t_n} k_1^{s_1} \cdots k_m^{s_m}$ with $t_i \in \mathbb{N}$,$s_i \in \mathbb{Z}$ form a free C-basis with the relations

1) the elements $k_1, \dots, k_m$ q-commute with all generators $x_1, x_2, \dots, x_n, k_1^{\pm 1}, \dots, k_m^{\pm 1}$;

2) for $i < j$ the holds

$$x_i x_j = q_{ij} x_j x_i + r_{ij}, \qquad (2.1)$$

where r_{ij} is the element of subalgebra R_{i+1} generated by $x_{i+1}, \dots, x_n, k_1^{\pm 1}, \dots, k_m^{\pm 1}$.

There exists the chain of subalgebras $R = R_1 \supset R_2 \supset \cdots \supset R_n \supset R_{n+1}$. The last algebra R_{n+1} is an algebra of twisted Laurent polynomials generated by $k_1^{\pm 1}, \dots, k_m^{\pm 1}$. One can prove that a quantum solvable algebra is an iterated skew extension of C (see [MC-R],[GL1]). This means that for all $i \in [1, n]$ the map $\tau_i : x_j \mapsto q_{ij} x_j, i > j$ is extended to automorphism of R_{i+1} and the map $\delta_i : x_j \mapsto r_{ij}$ is extended to τ_i-derivation of R_{i+1}. Each algebra R_i is the skew extension $R_{i+1}[x_i; \tau_i.\delta_i]$. All automorphisms τ_i are identical on C and all τ_i-derivations δ_i are equal to zero on C. A quantum solvable algebra is a Noetherian domain ([MC-R],1.2.9). We put some more conditions on R.

This conditions are comparable with more general conditions Q1-Q4 of ([P2])
Condition 2.2. We claim that R is an iterated q-skew extension in terms of ([GL1,2]). This means that $\delta_i\tau_i = q_i\tau_i\delta_i$ for some $q_i = q^{s_i}$, $s_i \in \mathbb{Z}$.
Condition 2.3. All automorphisms τ_i are extended to diagonal automorphisms of R (i.e the monomials $x_1^{t_1}\cdots x_n^{t_n}k_1^{s_1}\cdots k_m^{s_m}$, $t_i \in \mathbb{N}$,$s_i \in \mathbb{Z}$ are eigenvectors) with eigenvalues in Γ. Denote by H the group of diagonal automorphisms of R generated by $\tau_1,\dots,\tau_n$.
Condition 2.4. The exists an infinite set of positive integers with the property. For any element l of this set the elements $x_1^l,\dots,x_n^l$ modulo $q-\varepsilon$ lie in the center of R_ε (ε is a primitive l-root of unity).

The last Condition claims that R_ε is finite over its center for almost all prime ideals of C. Such algebras R are called pure C-algebras in the paper [P1]. The Conditions 2.2-2.4 imply the Conditions Q1-Q4 of the paper [P2] (in the case C is a localization of $K[q,q^{-1}]$ over some finitely generated denominator set). See [P2] for examples. One get stratification of prime ideals with zero intersection with C.

Theorem 2.5([P2], Theorem 3.4). Let R be a quantum solvable algebra obeying Conditions 2.2-2.4.
1) There exists the finite set $\mathbb{M} = \{\mathcal{P}_\mu\}$, $\mu = (i_1,\dots,i_{k+1})$, $i_s \geq 0$, $i_s \in \mathbb{Z}$, $k + i_1 + \cdots + i_{k+1} = n$ of semiprime ideals with zero intersections with C. There exists the denominator subset $S_\mu \subset R/\mathcal{P}_\mu$ generated by k q-commuting elements. The localization $(R/\mathcal{P}_\mu)S_\mu^{-1}$ is isomorphic to a factor algebra of an algebra of twisted Laurent polynomials.
2) Any prime ideal $\mathcal{I}$ of R with zero intesection with C (recall that indeed all such ideals are completely prime) contains a unique ideal $\mathcal{P}_\mu$ such that $S_\mu \bigcap (\mathcal{I} \mathrm{mod} \mathcal{P}_\mu) = \emptyset$.

The stratification of Spec(R) takes place. We shall recall the stratification of prime ideals of R, refering to ([P2], Theorem 3.4) for complete proof.

Consider the subset $L_1 = \{x_n, x_{n-1},\dots,x_{n-i_1+1}\}$ of last i_1 generators of R. Let $\mathcal{J}_1$ be minimal semiprime ideal containing L_1. Denote by X_1 the set of minimal prime ideals over $\mathcal{J}$. We decompose $X_1 = X_1'\bigcup X_1''\bigcup X_1'''$ where

$$X_1' = \{\mathcal{Q} \in X_1 : \mathcal{Q}\bigcap C = 0, x_{n-i_1} \notin \mathcal{Q},\},$$

$$X_1'' = \{\mathcal{Q} \in X_1 : \mathcal{Q}\bigcap C \neq 0, x_{n-i_1} \notin \mathcal{Q}\},$$

$$X_1''' = \{\mathcal{Q} \in X_1 : x_{n-i_1} \in \mathcal{Q}\}.$$

We denote

$$\mathcal{P}_{i_1} = \bigcap_{Q\in X_1'} Q. \tag{2.2}$$

The ideal $\mathcal{P}_{i_1}$ contains L_1, does not contain x_{n-i_1} and $\mathcal{P}_{i_1}\bigcap C = 0$. Denote by S_{1*} the denominator subset $\{x_{n-i_1}^m\}_{m\in\mathbb{N}}$. Consider the localizaton $R^{(1)} = (R/\mathcal{P}_{i_1})S_{1*}^{-1}$. The algebra $R^{(1)}$ is generated by $x_1,\dots,x_{n-i_1-1},x_{n-i_1}^{\pm 1}$ and admits the filtration

$$R^{(1)} = R_1^{(1)} \supset \cdots \supset R_{n-i_1}^{(1)} \supset R_{n-i_1+1}^{(1)} = \cdots = R_n^{(1)} = 0$$

with $R_j^{(1)} = (R/\mathcal{P}_{i_1}\bigcap R_j)S_{1*}^{-1}$. The subalgebra $R_j^{(1)}$ is generated by x_j and $R_{j+1}^{(1)}$.

There exists a localization of C over some finitely generated denominator subset $N_1 \subset C$ such that the algebra $R^{(1)}N_1^{-1}$ obtains the new system of generators

$x'_1, \cdots, x'_{n-i_1-1}, x^{\pm 1}_{n-i_1}$ related $x'_j x_{n-i_1} = q_{j,n-i_1} x_{n-i_1} x'_j$ ([P2]. Lemma 3.3). The algebra $R^{(1)}_j N_1^{-1}$ is generated by x'_j and $R^{(1)}_{j+1} N_1^{-1}$. Notice that the above $q_{j,n-i_1}$ is the entries of matrix $\mathbb{Q}$ (see Definition 2.1). Denote by S_1 the denominator set generated by N_1 and S_{1*} and save the notaton $R^{(1)}$ for localization $R^{(1)} N_1^{-1}$.

We continue the stratification process. Consider the subset $L_2 = \{x'_{n-i_1-1}, \ldots, x'_{n-i_1-i_2}\}$ of i_2 generators of $R^{(1)}$. Let $\mathcal{J}_2$ be the minimal semiprime ideal containing L_2. We denote by X_2 the set of minimal prime ideals over $\mathcal{J}_2$. As above we decompose $X_2 = X'_2 \bigcup X''_2 \bigcup X'''_2$ where

$$X'_2 = \{\mathcal{Q}' \in X_2 : \mathcal{Q}' \bigcap C_1 = 0, x'_{n-i_1-i_2-1} \notin \mathcal{Q}, \},$$

$$X''_2 = \{\mathcal{Q}' \in X_2 : \mathcal{Q}' \bigcap C_1 \neq 0, x'_{n-i_1-i_2-1} \notin \mathcal{Q}\},$$
$$X'''_2 = \{\mathcal{Q}' \in X_2 : x'_{n-i_1-i_2-1} \in \mathcal{Q}\}.$$

We denote

$$\mathcal{P}'_{i_1 i_2} = \bigcap_{Q' \in X'_2} Q'. \tag{2.3}$$

There exists a semiprime ideal $\mathcal{P}_{i_1 i_2}$ is R such that

$$(\mathcal{P}_{i_1 i_2}/\mathcal{P}_{i_1}) S_1^{-1} = \mathcal{P}'_{i_1 i_2}.$$

Consider the denominator set S'_{2*} in $R^{(1)}$ generated by $x'_{n-i_1-i_2-1}$. Let $S_{2*} = S_1 \bigcap (R/\mathcal{P}_{i_1})$. Denote $R^{(2)} = (R/\mathcal{P}_{i_1 i_2}) S_{2*}^{-1}$. The algebra $R^{(2)}$ is generated by

$$x'_1, \cdots, x'_{n-i_1-i_2-2}, (x'_{n-i_1-i_2-1})^{\pm 1}, x^{\pm 1}_{n-i_1}$$

and admites the filtration

$$R^{(2)} = R^{(2)}_1 \supset \cdots \supset R^{(2)}_{n-i_1-i_2} = \cdots = R^{(2)}_{n-i_1} \supset R^{(2)}_{n-i_1+1} = \cdots = R^{(2)}_n = 0.$$

There exists localization of C over some finitely generated denominator subset $N_2 \subset C$ such that the algebra $R^{(2)} N_1^{-1}$ obtains the new system of generators

$$x''_1, \cdots, x''_{n-i_1-i_2-2}, (x'_{n-i_1-i_2-1})^{\pm 1}, x^{\pm 1}_{n-i_1}$$

where x''_j q-commutes with $x'_{n-i_1-i_2-1}$ and x_{n-i_1}. As above we denote by S_2 denominator set generated by S_{2*} and N_2. We save the notation $R^{(2)}$ for localization $R^{(2)} N_2^{-1}$.

We continue the stratification process. Finally, we get a semiprime ideal $\mathcal{P}_\mu = \mathcal{P}_{i_1 i_2 \ldots i_{k+1}}$ and a denominator subset S_μ in $R/\mathcal{P}_\mu$. Denote $B_\mu = (R/\mathcal{P}_\mu) S_\mu^{-1}$. The algebra B_μ is generated by q-commuting elements $u_1^{\pm 1}, \ldots, u_k^{\pm 1}$ with

$$u_1 = x_{n-i_1}, u_2 = x'_{n-i_1-i_2-1}, \cdots, u_k = x^{(k-1)}_{n-i_1-\cdots-i_k-(k-1)}.$$

For any $s = 1, 2, \cdots, k$ we denote $\psi(s) = n - i_1 - \cdots - i_s - (s-1)$ and

$$u_s = x^{(s-1)}_{\psi(s)}.$$

The generators $u_1, \cdots, u_k$ are related

$$u_i u_j = q'_{ij} u_j u_i \tag{2.4}$$

and the matrix

$$\mathbb{Q}_\mu = (q'_{ij})_{i,j=1}^k \tag{2.5}$$

is a submatrix of $\mathbb{Q}$. We denote by $\mathbb{S}_\mu$ the corresponding submatrix of $\mathbb{S}$. The algebra B_μ admits the filtration

$$B_\mu = B_{\mu,1} \supset B_{\mu,2} \supset \cdots \supset B_{\mu,n} \tag{2.6}$$

with

$$B_{\mu,j} = (R/(\mathcal{P}_\mu \bigcap R_i))S_\mu^{-1}.$$

Notice that some subalgebras $B_{\mu,j}$ and $B_{\mu,j+1}$ can coinside. If $B_{\mu,j} \neq B_{\mu,j+1}$, then $j = \psi(s)$ for some $s \in \{1, 2, \ldots, k\}$ and $B_{\mu,j}$ is generated by u_s and $B_{\mu,j+1}$. Recall that R_j is a skew extension of R_{j+1} and x_j (see Definition 2.1). We save the notation x_j for image of x_j in $B_{\mu,j}$. We get that $B_{\mu,j}$ is generated by x_j and $B_{\mu,j+1}$. The elements u_s and x_j are related as follows

$$u_s = x_j + r_{j+1} \tag{2.7}$$

with $r_{j+1} \in B_{\mu,j+1}$.

Our next goal is to study stratification of prime ideals of specializations of R. For any $\lambda \in K$ we consider specislization $\rho_\lambda : R \mapsto R_\lambda = R\mathrm{mod}(q - \lambda)$. For any ideal I in R_λ we denote $\mathcal{I} = \rho_\lambda^{-1}(I)$. The above ideal $\mathcal{I}$ is prime in R and $\mathcal{I} \bigcap C = C(q - \lambda)$. Denote as usual $\mathcal{P}_{\mu\lambda} = \mathcal{P}_\mu \mathrm{mod}(q - \lambda) = \mathcal{P}_\mu + R(q - \lambda)/R(q - \lambda)$ and $S_{\mu\lambda}$ the image of S_μ in $R_\lambda/\mathcal{P}_{\mu\lambda} = (R/\mathcal{P}_\mu)\mathrm{mod}(q - \lambda)$. Since $\mathcal{I} \bigcap C \neq 0$, we can't apply Theorem 2.5 directly. The next Example shows that above stratification is not valid for all prime ideals.

Example 2.6. Let $R_q = \mathbb{A}_{q,1}$ be Quantum Weyl algebra generated by x, y related $xy - qyx = 1$. Any prime ideal of R_q with zero intersection with $K[q, q^{-1}]$ admits localization over y. The localization of R_q over y is generated by $y^{\pm 1}$ and $u = (q - 1)yx + 1$ related $yu = quy$. Consider specialization $q \mapsto \varepsilon = -1$. The algebra R_{-1} is generated by x, y related $xy + yx = 1$. Let I be the kernel of irreducible representation

$$x \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad y \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The ideal I is prime, but does not admit localization over y. $\Box$

Definition 2.7. Let $\lambda \in K$ be not a root of unity. We say that λ is is a point of a good reduction of stratification $\mathbb{M} = \{\mathcal{P}_\mu\}$, if the following properties hold.

Property 1. Every ideal $\mathcal{P}_{\mu\lambda}$ is a semiprime ideal in R_λ, $S_{\mu\lambda}$ is a denominator set of $R_\lambda/\mathcal{P}_{\mu\lambda}$.

Property 2. For any prime ideal I in R_λ there exists a standart ideal $\mathcal{P}_{\mu\lambda}$ of R_λ such that $I \supset \mathcal{P}_{\mu\lambda}$ and $S_{\mu\lambda} \bigcap (I\mathrm{mod}\mathcal{P}_{\mu\lambda}) = \emptyset$. (i.e $\mathcal{I} \supset P_\mu$ and $S_\mu \bigcap (\mathcal{I}\mathrm{mod}\mathcal{P}_\mu) = \emptyset$).

For a point of good reduction we denote

$$B_{\mu\lambda} = (R_\lambda/\mathcal{P}_{\mu\lambda})S_{\mu\lambda}^{-1} = B_\mu \mathrm{mod}(q - \lambda)$$

and we can study any prime ideal I in R_λ as an ideal of $B_{\mu\lambda}$. That is we can reduce the study of I to the study of an ideal in the algebra of twisted Laurent polynomials.

Let R be a quantum solvable algebra obeying Conditions 2.2-2.4. As above ε if a primitive l-root of unity. Suppose that Condition 2.4 is true for choosen l. Denote by $\mathcal{D}_0$ the set of derivations $\mathcal{D}_{x_1^l}, \ldots, \mathcal{D}_{x_n^l}$ defined in Secton 1. We say that an ideal I of R_ε is $\mathcal{D}_0$-stable if I is stable with respect to all derivations of set $\mathcal{D}_0$.

Definition 2.8. Let $\lambda = \varepsilon$ be a primitive l-root of unity. We say that ε is is a point of a good reduction of stratification $\mathbb{M} = \{\mathcal{P}_\mu\}$, if Property 1 of above Definition 2.7 holds and Property 2 holds for any prime $\mathcal{D}_0$-stable ideal I.

We say that $\lambda \in K$ is a point of bad reduction is the statements of Definitions 2.7 and 2.8 is not true for λ. We say further that a property $\mathcal{E}(\lambda)$ is true for almost all $\lambda \in K$ if there exists a finite set F such that this property is true for all elements $\lambda \in K - F$.

Theorem 2.9 The number of points of bad reduction is finite.

Proof.

Step 1. Suppose that λ is not a root of unity. First consider the semiprime H-stable (see Conditon 2.3) ideal $\mathcal{P}_{i_1}$ defined in (2.2) as an intersection of prime ideals $\mathcal{Q} \in X_1'$. These prime ideals are H-stable ([P2], Prop.2.1). For almost all λ estableach ideal $\mathcal{Q}_\lambda$ is semiprime ([P2], Prop.2.7). Recall that $x_{n-i_1} \notin \mathcal{Q}$ for $\mathcal{Q} \in X_1'$. For almost all λ the element $x_{n-i_1,\lambda}$ is regular in $R_\lambda/\mathcal{Q}_\lambda$ ([P2], Prop.2.7). This proves $\mathcal{P}_{i_1,\lambda}$ is semiprime and the element $x_{n-i_1,\lambda}$ is regular in $R_\lambda/\mathcal{P}_{i_1,\lambda}$. The set $S_{1*} = x_{n-i_1}^m m \in \mathbb{N}$ obeys Ore condition in $R/\mathcal{P}_{i_1}$ ([P2], Theorem 3.4). Then $S_{1*,\lambda}$ obeys Ore condition in $R_\lambda/\mathcal{P}_{i_1,\lambda}$. We conclude that $S_{1*,\lambda}$ is a denominator set and consider localization $R^(1)_\lambda$ of $R_\lambda/\mathcal{P}_{i_1,\lambda}$ over $S_{1*,\lambda}$.

For almost λ the elements of denonimator set $N_1 \subset C$ (see stratification process) is nonsero modulo $q-\lambda$. We can consider localization over N_1 modulo $q-\lambda$. Following the stratification process, we see that all ideals $\mathcal{P}_{\mu\lambda}$ are semiprime and $S_{\mu\lambda}$ is a denominator set in factor algebra of R_λ over $\mathcal{P}_{\mu\lambda}$. This proves Property 1 of Definition 2.7.

Notice that, since λ is not a root of unity, then all prime ideals in R are completely prime ([GL],Theorem 2.3).

Consider the nonempty Zariski open set $\mathbb{O}_{st} \subset C$ defined as follows. A point λ lies in $K - \mathbb{O}_{st}$ if λ annihilate some of ideal in $X_m'' \bigcap C$ (see stratification process) or some element in $S \bigcap C$. The Property 2 of Definition 2.7 is true for all λ in $\mathbb{O}_{st}$ ([P2], Theorem 3.10).

Step 2. Suppose that $\lambda = \varepsilon$ is a primitive l-root of unity. The proof of Property 1) of Definition 2.8 is simillar to Step 1. We are going to prove Property 2) of the above definition. Let I be a prime $\mathcal{D}_0$-stable ideal in R_ε. As usual $\mathcal{I} = \rho_\varepsilon^{-1}(I)$ with $\mathcal{I} \bigcap C = (q-\varepsilon)C$.

Recall that Theorem 1.13 asserts that for all j the ideal $I \bigcap R_{\varepsilon,j}$ is prime in $R_{\varepsilon,j}$. It follows that $\mathcal{I}_j = \mathcal{I} \bigcap R_j$ is a prime ideal in R_j.

In particular, $\mathcal{I} \bigcap R_n$ is a prime ideal in $R_n = C[x_n]$. Hence, $\mathcal{I} \bigcap R_n$ is completely prime in R_n. If $x_n \notin \mathcal{I}$, then $\mathcal{I}$ has zero intersection with $\{x_n^m\}_{m \in \mathbb{N}}$. Therefore, the ideal $\mathcal{I}$ admits localization over x_n.

Otherwise, $x_n \in \mathcal{I}$. Suppose that some subset $L_1 = \{x_n, \ldots, x_{n-i_1-1}\} \subset \mathcal{I}$ and $x_{n-i_1} \notin \mathcal{I}$. The ideal $\mathcal{I}$ contains a minimal prime ideal, say $\mathcal{Q}$, in the set X_1 (see stratification process). Since $x_{n-i_1} \notin \mathcal{I}$, then $\mathcal{Q} \in X_1' \bigcup X_1''$. If $\mathcal{Q} \in X_1''$, then ε annihilates $Q \bigcap C$. This contradicts the claim $\varepsilon \in \mathbb{O}_{st}$. We get $\mathcal{Q} \in X_1'$ and $\mathcal{I} \supset \mathcal{P}_{i_1}$. We see above that $\mathcal{I}_{n-i_1} = \mathcal{I} \bigcap R_{n-i_1}$ is a prime ideal in R_{n-i_1}. Since $R_{n-i_1}/\mathcal{I}_{n-i_1}$ is a factor ring of $K[x_{n-i_1}]$, then the ideal $\mathcal{I}_{n-i_1}$ is completely prime. It follows that $\mathcal{I}_{n-i_1}$ (and $\mathcal{I}$) have empty intersection with denominator set $\{x^m_{n-i_1}\}_{m\in\mathbb{N}} \subset R/\mathcal{P}_{i_1}$. Ideal $\mathcal{I}$ admits localization over x_{n-i_1}. The ideal $\mathcal{I}^{(1)} = \mathcal{I}_0/\mathcal{P}_{i_1}$ admits localization over N_1 (see stratification process).

Suppose that $\mathcal{I}^{(1)}$ contains some subset $L_2 = \{x'_{n-i_1-1}, \ldots, x'_{n-i_1-i_2}\}$ and does not contain $x'_{n-i_1-i_2-1}$. As above the claim $\varepsilon \in \mathbb{O}_{st}$ implies $\mathcal{I}_0 \supset \mathcal{P}_{i_1i_2}$ (see also [P2], Theorem 3.10). Since $\mathcal{I}_{n-i_1-i_2-1}$ is prime ideal in $R_{n-i_1-i_2-1}$ and $R^{(1)}_{n-i_1}/\mathcal{I}^{(1)}_{n-i_1}$ is a factor ring of the algebra of twisted polynomials $K_q[x^{\pm1}_{n-i_1}, x'_{n-i_1-i_2-1}]$, then the ideal $\mathcal{I}^{(1)}_{n-i_1-i_2-1}$ (and $\mathcal{I}_{n-i_1-i_2-1}$) is completely prime. It follows that $\mathcal{I}^{(1)}_{n-i_1-i_2-1}$ have empty intersection with denominator set $\{(x'_{n-i_1-i_2-1})^m\}_{m\in\mathbb{N}}$. Ideal $\mathcal{I}^{(1)}$ admits localization over $x'_{n-i_1-i_2-1}$. The ideal $\mathcal{I}^{(2)} = \mathcal{I}/\mathcal{P}_{i_1}S^{-1}_{2*}$ admits localization over N_2. And so on.

Finally, we prove that ideal $\mathcal{I}$ contains a unique ideal $\mathcal{P}_\mu$ such that $S_\mu \bigcap (\mathcal{I}\mathrm{mod}\mathcal{P}_\mu) = \emptyset$.□

Notations 2.10.

1) Let R be a domain and a C algebra. Let I be an ideal in $R_\varepsilon = R\mathrm{mod}(q-\varepsilon)$. We denote by $I_\mathcal{D}$ the greatest $\mathcal{D}$-stable (i.e. stable with respect to quantum adjoint action) ideal in I.

2) Let R be quantum solvable algebra, obeying Condition 2.4. We denote by $I_0 = I(\mathcal{D}_0)$ the greatest $\mathcal{D}_0$ ideal in I.

Remark 2.11. If I is prime ideal in R_ε, then I_0 and $I_\mathcal{D}$ are also prime ideals in R_ε ([MC-R],14.2.3; [G],6.4).

Definition 2.12. We say, that two R_ε-ideals I_1, I_2 are $\mathcal{D}$-equivalent, if $I_{1\mathcal{D}} = I_{2\mathcal{D}}$. We shall refer the class of equivalent to I ideals, as an orbit of I with respect to quantum adjoint action.

We need further a few Lemmas.

Lemma 2.13. Let R be a domain and a C-algebra. Let $\mathcal{P}$ be a semiprime ideal in R with $\mathcal{P} \bigcap C = 0$ and $\mathcal{P}_\varepsilon = (\mathcal{P} + (q-\varepsilon)R)\mathrm{mod}(q-\varepsilon)$. We claim the following.

1) $\mathcal{P}_\varepsilon$ is $\mathcal{D}$-stable;

2) Let $u_\varepsilon = u\mathrm{mod}(q-\varepsilon) \in Z_\varepsilon$ and $\mathcal{D}_u$ the quantum adjoint action of u. We denote by $\bar{u}$ is the image of u in $B = R/\mathcal{P}$. As usual $B_\varepsilon = B\mathrm{mod}(q-\varepsilon)$. We denote by $\overline{\mathcal{D}}_{\bar{u}}$ the quantum adjoint action of $\bar{u}$ via reduction $B_\varepsilon = B\mathrm{mod}(q-\varepsilon)$. Then $\mathcal{D}_u : B_\varepsilon \mapsto B_\varepsilon$ coinsides with $\overline{\mathcal{D}}_{\bar{u}}$;

3) Denote by Z' the image of the center Z_ε of R_ε under the map $R_\varepsilon \mapsto R_\varepsilon/\mathcal{P}_\varepsilon = B_\varepsilon$. The subalgebra Z' lies in the center $Z(B_\varepsilon)$. We equip Z' with Poisson structure inherited by the Poisson structure in Z_ε. We consider the Poisson structure ob $Z(B_\varepsilon)$ defined via reduction $B_\varepsilon = B\mathrm{mod}(q-\varepsilon)$ (Section 1). We assert that Z' is a Poisson subalgebra of $Z(B_\varepsilon)$, i.e. the Poisson bracket of two elements in Z' coinsides with their Poisson bracket as elements $Z(B_\varepsilon)$.

Proof. 1) For any $a \in \mathcal{P}_\varepsilon = (\mathcal{P} + R(q-\varepsilon))\mathrm{mod}(q-\varepsilon)$ one can choose $\underline{a} \in \mathcal{P}$ such that $a = \underline{a}\mathrm{mod}(q-\varepsilon)$. Let $u \in R$ such that $u_\varepsilon = u\mathrm{mod}(q-\varepsilon) \in Z_\varepsilon$, then $b = u\underline{a} - \underline{a}u \in R(q-\varepsilon) \bigcap \mathcal{P} = (q-\varepsilon)\mathcal{P}$. Then $b \in (q-\varepsilon)\mathcal{P})$ and $\mathcal{D}_u(a) \in \mathcal{P}_\varepsilon$.

The claim 2) is obtained by reduction $ua - au \in (q-\varepsilon)R$ modulo $\mathcal{P}$. The claim 3) is a corollary of claim 2). $\square$

We shall remind some well knows statements on algeras of twisted polynomials. Let $\mathbb{Q}$ and $\mathbb{S}$ be as above matrices with $q_{ij} = q^{s_{ij}}$. Let $B = B_{\mathbb{Q}}$ be an algebra generated over C by the generators $u_1^{\pm 1}, \dots, u_k^{\pm 1}$ related $u_i u_j = q_{ij} u_j u_i$. That is B is a factor algebra of an algebra of twisted Laurent plynomials. One can decompose $B = A \otimes Z(B)$ where A is an algebra of twisted Laurent polynomials and $Z(B)$ is the center of B. Generators of A can be presented as $y_1^{\pm 1}, \dots, y_{2r}^{\pm 1}$ related $y_1 y_2 = q^{d_1} y_2 y_1, \dots, y_{2r-1} y_{2r} = q^{d_r} y_{2r} y_{2r-1}$. All other pair of generators commutes. Here $2r$ is the rank $\mathbb{S}$. We shall refer $2r$ as a rank of B. The center $Z(B)$ is generated by monomials. Any ideal of B is generated by its intersection with the center $Z(B)$. Any prime ideal of B is completely prime.

One can consider specialization B_ε of B at primitive l-root of unity ε. We obtain $B_\varepsilon = A_\varepsilon \otimes Z(B)_\varepsilon$. The elements $u_1^l, \dots, u_k^l$ lie in the center of $Z(B_\varepsilon)$ of B_ε. Further we denote by $\overline{\mathcal{D}}_u$ the quantum adjoint action of an element $u \in B$ with $u_\varepsilon \in Z(B_\varepsilon)$. For u_i^l we calculate

$$\overline{\mathcal{D}}_{u_i^l}(u_j) = s_{ij} l \varepsilon^{-1} u_j u_i^l.$$

Any $\overline{\mathcal{D}}$-ideal in generated by its intersection with $Z(B_\varepsilon)$.

Lemma 2.14. Suppose that l is relatively prime with elementary divisors of matrix $\mathbb{S}$. Then 1) the center $Z(B_\varepsilon)$ is generated as K-algebra by $u_1^{\pm l}, \dots, u_k^{\pm l}$ and the reduction $Z(B)_\varepsilon$ of $Z(B)$ modulo $q - \varepsilon$;

2) if the field K is algebraically closed, then the dimension of every irreducible representation of B_ε equals to l^r with $r = \frac{1}{2}\mathrm{rank}(\mathbb{S})$.

Proof. One can choose another system $y_1^{\pm 1}, \dots, y_{2r}^{\pm 1}, z_1^{\pm 1}, \dots, z_t^{\pm 1}$, $2r + t = k$ of generators of B related $y_1 y_2 = q^{d_1} y_2 y_1, \dots, y_{2r-1} y_{2r} = q^{d_r} y_{2r} y_{2r-1}$. All other pair of generators commutes. Since $(l, d_i) = 1$ for all i, then the l-powes of generators and $z_1, \dots, z_t$ generate the center of B_ε.$\square$

Let $B = B_{\mathbb{Q}}$ be as above. Consider the filtration of B by subalgebras B_j generated by $u_j^{\pm 1}, \dots, u_k^{\pm 1}$. We put $B_{k+1} = C$. The center $Z(B)$ have a filtration

$$Z(B) = Z(B)_1 \supset \dots \supset Z(B)_j \supset \dots \supset Z(B)_k \supset Z(B_{k+1}) = C$$

where $Z(B) \bigcap B_j = Z(B)_j$. Similarly, $B_\varepsilon = B\mathrm{mod}(q - \varepsilon)$ have the filtration $B_{j\varepsilon}$ with $Z(B_\varepsilon) \bigcap B_{j\varepsilon} = Z(B_\varepsilon)_j$.

Lemma 2.15. Let $\mathbb{S}_j$ be a S-submatrix of $(s_{i\beta}), i = \overline{1,k}, \beta = \overline{j,k}$. Suppose that l is relatively prime with elementary divisors of $\mathbb{S}_j$. We assert that if a $a = u_j^{n_1} \cdots u_k^{n_k}, n_i \in \mathbb{Z}$ lies in the center $Z(B_{\mathbb{Q},\varepsilon})$ then there exist the $m_\beta \in \mathbb{Z}$, $\beta = \overline{j,k}$, $m_\beta = n_\beta \mathrm{mod}(l)$ such that $a = u_j^{m_1} \cdots u_k^{m_k}$ lies in the center $Z(B)$.

Proof. A monomial a lies in $Z(B_\varepsilon)$ if $(n_j, \dots, n_k)$ are a solution of system of equations

$$\sum_{\beta=1}^{k} s_{i\beta} n_\beta = 0\mathrm{mod}(l), \qquad i = \overline{1,k}.$$

Under the requirements on l any solution modulo l is a reduction modulo l of a solution over $\mathbb{Z}$.$\square$

Corollary 2.16. Suppose that $B_j = B_{j+1}[u_j^{\pm 1}; \tau_j]$ with $\tau_j = \mathrm{Ad}_{u_j}$. Suppose that

$Z(B)_j = Z(B)_{j+1}$. If a polynomial $u_j^m b$ with $b \in B_{j+1}$, $0 \leq m < l$ lies the center $Z(B_\varepsilon)_j$, then $m = 0$ and $b \in Z(B_\varepsilon)_{j+1}$. The algebra $Z(B_\varepsilon)_j$ is generated by u_j^l and $Z(B_\varepsilon)_{j+1}$. $\Box$

Now we are returning to ideals. Let R be a quantum solvable algebra obeying Conditions 2.2-2.4.

Definition 2.17 Let $\mathbb{Q}$ and $\mathbb{S}$ are matrices of Definition 2.1. We say that primitive l-root of inity ε is admissible if ε is a point of good reduction and l is relatively prime with all nonzero minors of $\mathbb{S}$. By Proposition 2.9, the set of admissible roots of unity is infinite.

Suppose that ε is admissible. Consider the reduction of ideals $\mathcal{P}_\mu$ and denominator sets $S_\mu \subset R/\mathcal{P}_\mu$ modulo $q-\varepsilon$. We denote $B = B_\mu = (R/\mathcal{P}_\mu)S_\mu^{-1}$ and $B_\varepsilon = B\mathrm{mod}(q-\varepsilon)$. We get $\mathcal{D}$-stable semiprime ideal $\mathcal{P}_{\mu\varepsilon}$ in R_ε. Recall that B_ε is generated by q-commuting elements $u_1^{\pm 1}, \ldots, u_k^{\pm 1}$ (see (2.4),(2.5)). Since ε is admissible, l is relatively prime with elementary divisors of matrix $\mathbb{S}_\mu$.

One can extend $\mathcal{D}$-action from R_ε to B_ε. The algebra B_ε admites also the quantum adjoint action $\overline{\mathcal{D}}$ of its center $Z(B_\varepsilon)$. Our next goal is to compair this two actions. We consider $Z(B_\varepsilon)\overline{\mathcal{D}}$ the $Z(B_\varepsilon)$-module spanned by $z_\varepsilon\overline{\mathcal{D}}_b$ with $b \in B$, $b_\varepsilon = b\mathrm{mod}(q-\varepsilon) \in Z(B_\varepsilon)$ and $z_\varepsilon \in Z(B_\varepsilon)$. We consider the $Z(B_\varepsilon)$-submodule $\mathrm{ad}_{B_\varepsilon}$ of inner derivations in B_ε. We denote by $Z(B_\varepsilon)\mathcal{D}_0$ the $Z(B_\varepsilon)$-submodule

$$Z(B_\varepsilon)\mathcal{D}_{x_1^l} + \cdots + Z(B_\varepsilon)\mathcal{D}_{x_n^l}$$

of derivations of B_ε.

Proposition 2.18. As above R is quantum solvable algebra obeying Conditions 2.2-2.4 and ε is admissible. Then $Z(B_\varepsilon)\mathcal{D}_0 = Z(B_\varepsilon)\overline{\mathcal{D}}\mathrm{mod}(\mathrm{ad}_{B_\varepsilon})$.

Proof. First $Z(B_\varepsilon)\mathcal{D}_0 \subset Z(B_\varepsilon)\overline{\mathcal{D}}$ (the derivation $\mathcal{D}_{x_i^l} : B_\varepsilon \mapsto B_\varepsilon$ coinsides with $\overline{\mathcal{D}}$ derivation of the image of x_i^l in B). To conclude the proof we are going to prove $Z(B_\varepsilon)\mathcal{D}_0$ containes $\overline{\mathcal{D}}$ modulo inner derivations.

Consider the filtration of $B = B_\mu$ (see (2.4)). Recall that by $Z(B_\varepsilon)_j$ we denote the intersection of $Z(B_\varepsilon)$ with $B_{j\varepsilon}$. We consider

$$\overline{\mathcal{D}}_j = \overline{\mathcal{D}}_{Z(B_\varepsilon)_j} = \{\overline{\mathcal{D}}_u : u \in B, u_\varepsilon = u\mathrm{mod}(q-\varepsilon) \in Z(B_\varepsilon)_j\}.$$

Considering $\overline{\mathcal{D}}_u$ upto inner derivations, we may choose above $u \in B_j$. We denote $Z(B_\varepsilon)_j\mathcal{D}_{0j} = Z(B_\varepsilon)_j\mathcal{D}_{x_j^l} + \cdots + Z(B_\varepsilon)_j\mathcal{D}_{x_n^l}$. We shall prove by induction on j that $\overline{\mathcal{D}}_j$ equals to $Z(B_\varepsilon)_j\mathcal{D}_{0j}$ modulo inner derivations.

The statement is obviously true for $j = k+1$. Assume that the statement is true for $j+1$. Our goal is to prove it for j.

Let $B_j \neq B_{j+1}$. The algebra B_j is generated by B_{j+1}, x_j and also by B_{j+1}, u_s. The elements u_s and x_j are related as (2.7). To simplify the notations we put $u_s = u$. We save the notations for images of x_j and u in B_ε. Recall that $u^l \in Z(B_\varepsilon)$ and $x_j^l \in Z(R_\varepsilon)$.

1) Suppose that there exists a monomial $z = u^m b \in Z(B)_j$ with $m \neq 0$ and $b \in B_{j+1}$ (remark that here z may be zero). We denote by m_0 the least positive integer with the following property. The element $v = u^{m_0}b_0$ modulo $q-\varepsilon$ lies in $Z(B_\varepsilon)_j$ for some $b \in B_{j+1}$. Here one may consider b_0 to be a monomial.

The algebra $Z(B_\varepsilon)_j$ is generated by $v_\varepsilon = v\mathrm{mod}(q-\varepsilon)$ and $Z(B_\varepsilon)_{j+1}$ Then m_0 divides m. Put $m = pm_0$. We obtain $z = v^p a$ with a monomial $a \in B_{j+1}$. Notice that $a\mathrm{mod}(q-\varepsilon)$ lies in $Z(B_\varepsilon)_{j+1}$.

Since $z \in Z(B)_j$, we have $\overline{\mathcal{D}}_z = 0$. We obtain

$$0 = \overline{\mathcal{D}}_z = pv^{p-1}a\overline{\mathcal{D}}_v + v^p\overline{\mathcal{D}}_a.$$

Then $\overline{\mathcal{D}}_v = -p^{-1}va^{-1}\overline{\mathcal{D}}_a$. By assumption of induction, the derivation $\overline{\mathcal{D}}_a$ lies in $Z(B_\varepsilon)_{j+1}\mathcal{D}_{0,j+1}$. This proves the statement.
2) Suppose that the condition $z = u^m b \in Z(B)_j$ implies $m = 0$. This proves that $Z(B)_j = Z(B)_{j+1}$. Consider the homomorphism Φ of the skew extension $B_{j+1}[u, \tau_j]$ onto B_j. Every ideal in $B_{j+1}[u, \tau_j]$ is generated by its intersection with the center. The kernel of Φ is zero (otherwise, there exists a central , in particular zero, element of the point 1). We identify $B_{j+1}[u, \tau_j]$ and B_j. By Lemma 2.16, $Z(B_\varepsilon)_j$ is generated by u^l and $Z(B_\varepsilon)_{j+1}$ By (2.7), $u = x_j + r_{j+1}$ with $r_{j+1} \in B_{j+1}$. Then $u^l = x_j^l + F(u)$. Where $F(u)$ a polynomial of u of degree $\leq l-1$ and $F(u)_\varepsilon$ lies in $Z(B_\varepsilon)_j$. The algebra $Z(B_\varepsilon)_j$ is generated by monomials. Applying Lemma 2.16, we prove that $F(u)\mathrm{mod}(q-\varepsilon) \in Z(B_\varepsilon)_{j+1}$. We obtain

$$\overline{\mathcal{D}}_{u^l} = \overline{\mathcal{D}}_{x_j^l} + \overline{\mathcal{D}}_{F(u)} \in \mathcal{D}_{x_j^l} + \overline{\mathcal{D}}_{Z(B_\varepsilon)_{j+1}}.$$

Using the assumption of induction, we prove the claim. □.
Corollary 2.19. Let L be either a $\mathcal{D}_0$-stable ideal in B_ε or $\mathcal{D}_0$-stable ideal in $Z(B_\varepsilon)$, then L is $\overline{\mathcal{D}}$-stable.
Theorem 2.20. As above R is a quantum solvable algebra obeying Conditions 2.2-2.4, and ε is admissible.
1) For any prime ideal I in R_ε the ideals $I_{\mathcal{D}_0}$ and $I_{\mathcal{D}}$ are completely prime. Any prime $\mathcal{D}_0$-stable (in particular, $\mathcal{D}$-stable) ideal P is completely prime.
2) For above P skew field of fractions $\mathrm{Fract}(R_\varepsilon/P)$ is isomorphic to a skew field of an algebra of twisted polynomials.
Proof. For any prime ideal I the ideals $I_{\mathcal{D}_0}$ and $I_{\mathcal{D}}$ are also prime ([MC-R],14.2.3). These ideals are correspond to prime $\overline{\mathcal{D}}$-ideals in $B_{\mu\varepsilon}$ which is a factor algebra of an algebra of twisted Laurent polynomials. Hence, these ideals are completely prime.□.

4 Representations at roots of unity

Let $K = \mathbb{C}$ be the field of complex numbers. Let R be a quantum solvable algebra obeying Conditions 2.2-2.4. Suppose that ε is admissible. As above, Z_ε is the center of R_ε. Let π be an irreducible complex representation of a quantum solvable algebra R_ε. Since R_ε is finite over its center, then π has finite dimentsion. Denote by $I(\pi)$ the kernal of representation π in R_ε. For $I(\pi)$ we consider $I(\pi)_{\mathcal{D}}$-the greatest $\mathcal{D}$-stable ideal in $I(\pi)$. We denote

$$R_\pi = R_\varepsilon/I(\pi)_{\mathcal{D}}.$$

The ideal $I(\pi)_{\mathcal{D}}$ is completely prime (Theorem 2.20) (i.e. R_π is a domain). There exists a unique standart semiprime $\mathcal{D}$-stable ideal $\mathcal{P}_{\mu\varepsilon}$ and a denominator set $S_{\mu\varepsilon}$ in

$R_\varepsilon/\mathcal{P}_{\mu\varepsilon}$ such that
1) $(R_\varepsilon/\mathcal{P}_{\mu\varepsilon})S_{\mu\varepsilon}^{-1}$ is a factor algebra of an algebra of an algebra twisted Laurent polynomials;
2) $I(\pi)_\mathcal{D} \supset \mathcal{P}_{\mu\varepsilon}$ and $S_{\mu\varepsilon} \bigcap I(\mathrm{mod}(\mathcal{P}_{\mu\varepsilon})) = \emptyset$. (see Theorem 2.9, Definitions 2.8, 2.17). The algebra R_π contains image of $S_{\mu\varepsilon}$ (we save the notaton $S_{\mu\varepsilon}$ for image of $S_{\mu\varepsilon}$ in R_π). The localization $B_\pi = R_\pi S_{\mu\varepsilon}^{-1}$ is the factor algebra of $B_{\mu\varepsilon} = (R_\varepsilon/\mathcal{P}_{\mu\varepsilon})S_{\mu\varepsilon}^{-1}$ over prime $\mathcal{D}$-stable ideal

$$\widehat{I}(\pi)_\mathcal{D} = (I(\pi)_\mathcal{D}/\mathcal{P}_{\mu\varepsilon})S_{\mu\varepsilon}^{-1}.$$

Recall that $B_\mu = A_\mu \otimes Z(B_\mu)$ where A_μ is generated by $y_1^{\pm 1}, \ldots, y_{2r}^{\pm 1}$ with $y_1y_2 = q^{d_1}y_2y_1, \ldots, y_{2r-1}y_{2r} = q^{d_r}y_{2r}u_{2r-1}$ and $Z(B_\mu)$ is the center of B_μ (see Section 2). Since ε is admissible, l is relatively prime with $d_1, \ldots, d_r$. We have $B_{\mu\varepsilon} = A_{\mu\varepsilon} \otimes Z(B_\mu)_\varepsilon$ (see Lemma 2.14). The center $Z(B_{\mu\varepsilon})$ of $B_{\mu\varepsilon}$ is generated by $y_1^{\pm l}, \ldots, y_{2r}^{\pm l}, Z(B_\mu)_\varepsilon$. Poisson bracket in $Z(B_{\mu\varepsilon})$ is determined by reduction $B_{\mu\varepsilon} = B_\mu \mathrm{mod}(q-\varepsilon)$. All symplectic leaves of $Z(B_{\mu\varepsilon})$ have dimension $2r$, algebraic and determined by characters of $Z(B_\mu)_\varepsilon$(see Section 2).

By Corollary 2.20, any $\mathcal{D}$-stable ideal (in particular $\widehat{I}(\pi)_\mathcal{D}$) is $\overline{\mathcal{D}}$-stable. Any $\overline{\mathcal{D}}$-stable ideal in $B_{\mu\varepsilon}$ is generated by its intersection with $Z(B_\mu)_\varepsilon$. We obtain $\widehat{I}(\pi)_\mathcal{D} = B_{\mu\varepsilon}(\widehat{I}(\pi)_\mathcal{D} \bigcap Z(B_\mu)_\varepsilon)$. We denote

$$E_\pi = \frac{Z(B_\mu)_\varepsilon}{\widehat{I}(\pi)_\mathcal{D} \bigcap Z(B_\mu)_\varepsilon}.$$

We have $B_\pi = A_{\mu\varepsilon} \otimes E_\pi$. Here E_π is a commutative domain (Later we shall prove that $E_\pi = \mathbb{C}$). The center if B_π is generated by images of $y_1^{\pm l}, \ldots, y_{2r}^{\pm l}$ and E_π. Quantum adjoint action $\overline{\mathcal{D}}$ provides the Poisson structure in $Z(B_\pi)$. All symplectic leaves of $Z(B_\pi)$ have dimension $2r$, algebraic and determined by characters of E.

Definition 3.1. We say that two irredicible representation π_1 and π_2 are $\mathcal{D}$-equivalent if $I(\pi_1)_\mathcal{D} = I(\pi_2)_\mathcal{D}$ (see Definition 2.12).

Denote by $\chi = \chi(\pi) : Z_\varepsilon \mapsto K$ the central character of π. The center Z_ε determines the algebraic Poisson manifold $\mathcal{M}$. Further we refer Z_ε as an algebra $\mathbb{C}[\mathcal{M}]$ of regular functions on $\mathcal{M}$ and χ is a point of $\mathcal{M}$. The intersection $I(\pi) \bigcap Z_\varepsilon$ is the maximal ideal $m(\chi)$ for $\chi \in Z_\varepsilon$. Let R_χ be $R_\varepsilon/m(\chi)$.

We denote by Ω_χ the symplectic leaf of χ with respect to Poisson bracket on Z_ε. We consider the greatest $\mathcal{D}$-stable ideal $m(\chi)_\mathcal{D}$ in $m(\chi)$. The ideal $m(\chi)_\mathcal{D}$ is prime. We denote by $\mathcal{M}_\chi$ the annihilator set $\mathrm{Ann}(m(\chi)_\mathcal{D})$. Remark that $\mathcal{M}_\chi$ is the Zariski closure symplectic leaf Ω_χ. We shall prove further (Theorem 3.3) that Ω_χ is open in its closure $\mathcal{M}_\chi$. The algebra of regular functions $Z_\chi = \mathbb{C}[\mathcal{M}_\chi]$ coinsides with $Z_\varepsilon/m(\chi)_\mathcal{D}$. The Poisson structure on Z_ε provides the Poisson structure on Z_χ.

Notice that an element a lies in $I(\pi)_\mathcal{D}$ iff for any system of elements $u_1, \ldots, u_s \in R$, obeying $u_{1\varepsilon}, \ldots, u_{s\varepsilon} \in Z_\varepsilon$, the element $\mathcal{D}_{u_1} \cdots \mathcal{D}_{u_s}(a)$ lies in $I(\pi)$. This proves that $I(\pi)_\mathcal{D} \bigcap Z_\varepsilon = m(\chi)_\mathcal{D}$ and $Z_\chi = Z_\varepsilon/m(\chi)_\mathcal{D} = Z_\varepsilon/(I(\pi)_\mathcal{D} \bigcap Z_\varepsilon)$. The algebra Z_χ lies in the center of $R_\pi = R_\varepsilon/I(\pi)_\mathcal{D}$ and R_π is finite over Z_χ.

According to Lemma 2.13, Z_χ is a Poisson subalgebra of $Z(R_\pi)$. Denote by Y_π the Poisson manifold of $Z(B_\pi)$.

Lemma 3.2. Let $f \in Z_\chi = \mathbb{C}[\mathcal{M}_\chi]$, $p \in \mathcal{M}_\chi$ and $f \neq 0$. There exists $p' \in \mathcal{M}_\chi$ such

that p' lie in the symplectic leaf of p, $f(p')\neq 0$ and R_p is isomorphic to $R_{p'}$.
Proof. Since f is nonzero element of $Z_\chi = Z_\varepsilon/m(\chi)_{\mathcal{D}}$, then there exist elements $u_1,\dots,u_s$ such that $u_{1\varepsilon},\dots,u_{n\varepsilon}\in Z_\varepsilon$ and the element $\mathcal{D}_{u_1}\cdots\mathcal{D}_{u_s}(f)(p)\neq 0$. Let $G_{u_1}(t_1),\dots,G_{u_s}(t_s)$ be local flows of the above derivations defined for small $|t_i|$. Put $p' = G_{u_s}(t_s)\cdots G_{u_1}(t_1)(p)$. We have $f(p')\neq 0$. The algebras R_p and $R_{p'}$ are isomorphic ([DC-P], 11.8). □
Theorem 3.3 As above R is quantum solvable algebra obeying Conditions 2.2-2.4 and ε is admissible. We assert that:
1) Dimension of any irreducible representation equals to $l^{\frac{1}{2}\dim(\Omega_\chi)}$;
2) Any symplectic leaf Ω_χ is Zariski-open in its Zariski closure;
3) If two points χ_1 and χ_2 belong to a common symplectic leaf, then the algebras R_{χ_1} and R_{χ_2} are isomorphic;
4) Let π,π' are irreducible representations of R_ε and χ,χ' their central characters. If π and π are quantun equivalent (see Definition 3.1), then χ and χ' lie in the common symplectic leaf.
Proof. Let χ be a central character of Z_ε. Let π be an irreducible representation over χ.

Recall that R_π is finite over $Z_\chi = \mathbb{C}[\mathcal{M}_\chi]$, $\mathcal{M}_\chi = \overline{\Omega_\chi}$. Let $v_1,\dots,v_k$ be q-commuting generators of S_μ. As above $v_{1\varepsilon}\dots v_{k\varepsilon}$ their images in R_π. Since R_π is finite over Z_χ, the $R_\pi Z_\chi^{-1}$ is a skew field. The element $v_{i\varepsilon}^{-1}$ can be presented in the form

$$v_{i\varepsilon}^{-1} = a_i w_i^{-1},$$

where $a_i\in R_\pi$, $w_i\in Z_\chi$. Let N denote the denominator subset generated by $w_1,\dots,w_k$. The localization $R_\pi N^{-1}$ contains $R_\pi S^{-1} = B_\pi$ and and $R_\pi N^{-1} = B_\pi N^{-1}$. Denote

$$f_0 = w_1\cdots w_k \in Z_\chi.$$

The algebra $B_\pi N^{-1}$ (as B_π) finite over its center $Z(B_\pi N^{-1})$. Denote by Y the Poisson manifold of $Z(B_\pi N^{-1})$. Notice that

$$Y = \{p\in Y_\pi : f_0(p)\neq 0\}.$$

As Y_π, symplectic leaves of Y are algebraic and have dimension $2r$. Any symplectic leaf $V\subset Y$ is $\mathrm{Ann}(J_V)$, where $J_V = Z(B_\pi N^{-1})(J_V\bigcap E_\pi)$ is a prime ideal generated by maximal ideal $J_V\bigcap E_\pi$ in E_π.

The algebra $Z_\chi N^{-1}$ is a commutative Poisson subalgebra of $Z(B_\pi N^{-1})$. Denote

$$\mathcal{O} = \{p\in\mathcal{M}_\chi : f_0(p)\neq 0\},$$

$$\mathbb{C}[\mathcal{O}] = Z_\chi N^{-1}.$$

Step 1. At this step we shall prove that $\mathcal{O}\subset\Omega_\chi$. Since Ω_χ is Zariski dense in $\mathcal{M}_\chi$ and $\mathcal{O}$ is Zariski open in $\mathcal{M}_\chi$, then the set $\mathcal{O}_\chi = \Omega_\chi\bigcap\mathcal{O}$ is Zariski dense in $\mathcal{O}$.

The inclusion of Poisson algebras $\mathbb{C}[\mathcal{O}] = Z_\chi N^{-1}\subset Z(B_\pi N^{-1}) = \mathbb{C}[Y]$ yields the Poisson map $\phi : Y\mapsto\mathcal{O}$ of Poisson manifolds. It follows that an image $\phi(V)$ of any symplectic leaf V is a Poisson subset (i.e a union of symplectic leaves).

Cosider the algebra $C^\infty(\mathcal{O})$ (respect. $C^\infty(Y)$) of C^∞-smooth functions on $\mathcal{O}$ (respect. Y). Let $J^\infty(\mathcal{O}_\chi)$ be an ideal in $C^\infty(\mathcal{O})$, cosisting of functions vanishing on $\mathcal{O}_\chi$. The ideal $J^\infty(\mathcal{O}_\chi)$ is $\mathcal{D}$-stable . We denote by J^∞_Y the ideal $C^\infty(Y)\phi^*(J^\infty(\mathcal{O}_\chi))$ of functions vanishing on $\pi^{-1}(\mathcal{O}_\chi)$. By Proposition 2.18, the ideal J^∞_Y is $\overline{\mathcal{D}}$-stable. This proves that $\phi^{-1}(\mathcal{O}_\chi)$ is a Poisson subset. Choose a symplectic leaf $V \subset \phi^{-1}(\mathcal{O}_\chi)$. We obtain

$$\phi(V) \subset \phi\phi^{-1}(\mathcal{O}_\chi) = \mathcal{O}_\chi.$$

Since $\phi(V)$ is Poisson subset, we obtain $\phi(V) = \mathcal{O}_\chi$. The symplectic leaf V is algebraic subset $V = \mathrm{Ann}(J_V)$. This implies that $\mathcal{O}_\chi$ is annihilated by $J_V \bigcap \mathbb{C}[\mathcal{O}]$. But $\mathcal{O}_\chi$ is Zariski dense in $\mathcal{O}$. We get $J_V \bigcap \mathbb{C}[\mathcal{O}] = 0$. The ideal J_V corresponds to an ideal in localization $F = \mathbb{C}[Y](\mathbb{C}[\mathcal{O}]^*)^{-1}$. Since $\mathbb{C}[Y]$ is finite over $\mathbb{C}[\mathcal{O}]$, the algebra F is a field. This proves that $J_V = 0$. Whence $Y = V$ and $\mathcal{O}_\chi = \phi(V) = \phi(Y) = \mathcal{O}$. This concludes Step 1. In addition we notice that $J_V = 0$ yields that $0 = J_V \bigcap E_\pi$ is a maximal ideal in E_π. We obtain $E_\pi = \mathbb{C}$.

Step 2. In this step we conclude the proof. Recall that $\chi \in \mathcal{M}$ and $\mathcal{M}_\chi$ is an annihilator of $m(\chi)_\mathcal{D}$.

1) Let π be an irreducible representation with Z_ε character χ. We apply Lemma 3.2 for $f = f_0$. There exists $p \in \Omega_\chi$ such that $f_0(p) \neq 0$ (i.e $p \in \mathcal{O}$) such that R_χ is isomorphic to R_p. Representations of R_p are passed through $R_\varepsilon \mapsto B_\pi N^{-1}$. The algebra B_π is an algebra of twisted Laurent polynomials. Statement 1) is well known for algebras of twisted Laurent polynomials. This proves that

$$\dim(\pi) = l^{\frac{1}{2}\dim(\mathcal{O})} = l^{\frac{1}{2}\dim(\Omega_\chi)}.$$

2) We have proved in Step 1 that Ω_χ contains a Zariski-open subset $\mathcal{O}$. For any element $p \in \overline{\Omega_\chi} - \Omega_\chi$ the dimension of corresponding symplectic leaf Ω_p is less than $r = \dim(\Omega_\chi)$. The set $\{p \in \mathcal{M} : \dim(\Omega_p) < r\}$ is Zariski-closed in $\mathcal{M}$. The set $\overline{\Omega_\chi} - \Omega_\chi$ is Zariski closed. This proves statement 2.

3) For any two points χ_1 and χ_2 of Ω_χ. Simillar to 1) there exist $p_1, p_2 \in \mathcal{O}$ such that R_{χ_1} is isomorphic to R_{p_1} and R_{χ_2} is isomorphic to R_{p_2}. The algebras R_{p_1} and R_{p_2} are isomorphic as fibers of an algebra of twisted Laurent polynomials. This proves 3.

4) Let π_1 and π_2 be two $\mathcal{D}$-equivalent representations of R_ε. Then $I(\pi_1)_\mathcal{D} = I(\pi_2)_\mathcal{D}$. Then $m(\chi_1)_\mathcal{D} = m(\chi_2)$. By 2), the points χ_1 and χ_2 lie in a common symplectic leaf.□

References

[DC-K-P] C.De Concini, V.G.Kac, C.Procesi, *Quantum adjoint action*, Journal of AMS, 5(1992),151-190.

[DC-P] C.De Concini, C.Procesi, *Quantum Groups*, Lecture Notes in Math., 1565, 1993, 31-140.

[G] K.R.Goodearl, *Prime ideals in Skew Poynomial Rings and Quantized Weyl Algebras*, J.Algebra, 150(1992), 324-377.

[GL1] K.R.Goodearl and E.S.Letzter, *Prime Ideals in Skew and q-Skew Polynomial Rings*, Memoirs AMS, N521 (1994).

[GL2] K.R.Goodearl and E.S.Letzter, *Prime factor algebras of the coordinate ring of quantum matrices*, Proc.Amer.Math.Soc., V.121, N4, P.1017-1025.

[MC-R] J.C.McConnel, J.C.Robson, *Noncommutative Noetherian Rings* (Wileys-Interscience,New York,1987).

[JZ] H.P.Jakobsen, H.Z.Zhang, *Quantized Heisenberg Space*, math.QA/9903028, Algebra and Representation Theory **3**, 2000, P.151-174.

[P1] A.N.Panov, *Skew fields of fractions of quantum solvable algebras*, math. QA/9906060, J.Algebra, **236**, 2001, P.110-121.

[P2] A.N.Panov, *Stratification of prime spectrum of quantum solvable algebras*, math.QA/0105131, Comm.Algebra **29**, 2001, N.9, 3801-3827.