ON THE SUPERSINGULAR LOCUS IN HILBERT-BLUMENTHAL 4-FOLDS

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ABSTRACT. We study the supersingular locus of Hilbert-Blumenthal four-folds modulo p when p is inert in the totally real field. The dimension, local moduli spaces, number of the irreducible components, and a description of intersections of these components are given. We also show that each irreducible component is a smooth algebraic stack which is a quotient of a ruled surface over $\mathbf{P}^1$ by a finite group.

CONTENTS

1. INTRODUCTION

We attempt to extend the work of K.-Z. Li and F. Oort [17] to Hilbert-Blumenthal modular varieties, and would like to answer the following questions:

1. What is the dimension of the supersingular locus?
2. How many irreducible components does the supersingular locus have?
3. Does the generic point of any irreducible component have a-number one?

We limit ourselves to the case where the moduli space has good reduction. This happens exactly when the rational prime p is unramified in the totally real field F [6, Thm 2.2]. In particular, we assume that the prime p is inert in F. This assumption does not lose the generality for answering the question 1 and 3 due to the theorem of Serre-Tate. For four-folds we give a fairly complete description of the supersingular locus. In particular, we obtain the following results:

1. The dimension of the supersingular locus is one half of the total space.
2. The number of the irreducible components is related to a class number.
3. There are some components whose generic points have a-number two.

The answer 3. gives rise to a natural question: what is the "generic" a-number for Hilbert-Blumenthal varieties analogue to "$a = 1$" in Siegel modular varieties? We give a simple combinatorial description for generic a-numbers (called *spaced* in [9]), and give some evidence, see Appendix.

The main tools are the Cartier-Dieudonné theory [38], deformations via displays [22, 23, 39], the method of Cayley-Hamilton from Oort's paper [29], and canonical isogenies (see Section 7, cf. [17, 16]).

Let F be a totally real field and let O_F be its ring of integers. Let $\mathcal{A}_g^{O_F}$ be the moduli space of principally polarized O_F-abelian varieties of dimension $g = [F : \mathbb{Q}]$. A polarized O_F-abelian variety, (A, λ, ι), consists of an abelian variety A, a polarization $\lambda : A \to A^t$ and an embedding $\iota : O_F \to \mathrm{End}(A)$ such that $\lambda\, \iota(a) = \iota(a)^t \lambda$, $\forall\, a \in O_F$. Such a polarization is called O_F-linear. Let p be a rational prime inert in F and k an algebraically closed field of characteristic p. For $n \geq 3$, $(n, p) = 1$, let $\mathcal{A}_{g,n}^{O_F}$ denote the moduli space of principally polarized O_F-abelian varieties of dimension g with symplectic n-level structure (with a choice of an n-th root of unity). It is a fine moduli scheme over $\mathrm{Spec}\, \mathbb{Z}[\zeta_n, \frac{1}{n}]$ and each fibre is geometrically connected. Denote by $\mathcal{S}_n$ the supersingular locus of the fine moduli scheme $\mathcal{A}_{g,n}^{O_F} \otimes_{\mathbb{Z}[\zeta_n, \frac{1}{n}]} k$. It is a Zariski closed subset of $\mathcal{A}_{g,n}^{O_F} \otimes_{\mathbb{Z}[\zeta_n, \frac{1}{n}]} k$, therefore we can regard it as a reduced closed subscheme of $\mathcal{A}_{g,n}^{O_F} \otimes k$. The Galois group H_n of $\mathcal{A}_{g,n}^{O_F} \otimes k \to \mathcal{A}_g^{O_F} \otimes k$ preserves the close subscheme $\mathcal{S}_n$. Let $\mathcal{S}$ be the algebraic stack $[\mathcal{S}_n / H_n]$, the supersingular locus of $\mathcal{A}_g^{O_F} \otimes k$. This definition of $\mathcal{S}$ does not depend on the choice of n.

Assume that $g = 4$. Let (A, λ, ι) be a superspecial polarized O_F-abelian variety over k such that the polarization λ has the degree p^4 the type $(p, 1, p, 1)$ (see the definition in Section 2). An abelian variety is *superspecial* if it is isomorphic to a product of supersingular elliptic curves. The automorphism group $\mathrm{Aut}_{O_F}(A, \lambda)$, consisting of automorphisms which preserve the O_F-action and the polarization, is finite. We regard $G = \mathrm{Aut}_{O_F}(A, \lambda)$ as an algebraic group over $\mathrm{Spec}\, \mathbb{Z}$. For any commutative ring R, the group of its R-points is

$$G(R) = \{\phi \in (\mathrm{End}_{O_F}(A) \otimes R)^\times | \phi' \phi = 1\},$$

where the map $\phi \to \phi'$ is the Rosati involution induced by λ. We summarize our main results in the theorem below; see Proposition 6.3, Theorem 9.7, 11.1, 11.2, and Proposition 8.2 and 8.10:

Main Theorem.

(1) *The supersingular locus* $\mathcal{S}$ *is equi-dimensional and* $\dim \mathcal{S} = 2$.
(2) *The supersingular locus* $\mathcal{S}$ *has* $6h$ *irreducible components, where* h *is the class number of* G.
(3) *The formal completion* $\hat{\mathcal{S}}_x$ *at a superspecial point* x *is isomorphic to* $\mathrm{Spf}\, k[[t_0, t_1, t_2, t_3]]/I$, *where* $I = \cap_{i=0}^3 (t_i, t_{i+1} + t_{i-1}^{p^2}) \cap (t_0, t_2) \cap (t_1, t_3)$.
(4) *Every irreducible component of* $\mathcal{S}$ *is smooth as an algebraic stack.*
(5) *Every irreducible component is isomorphic to a finite quotient of* X, *where* X *is either* $\mathbf{P}^1 \times \mathbf{P}^1$ *or a* $\mathbf{P}^1$-*fibration over* $\mathbf{P}^1$.

The following is an outline of the contents of this paper. In Section 2 we list the notations used in our work. In Section 3 we introduce the stratification induced by a-numbers and describe the change of the strata under the specialization and generalization. The analysis of the a-number stratification on the supersingular locus, $\mathcal{S}$, is the key step in studying $\mathcal{S}$. A classification of Dieudonné modules in question is given in Section 4. In Section 5, we introduce the method of "Cayley-Hamilton" due to F. Oort, and use it to compute the local moduli at the point of usual a-number one. In Section 6, we combine the slope stratification and the a-number stratification, and obtain the dimension of each slope stratum. An result states that the generic point of some component of $\mathcal{S}$ has usual a-number two, and not just one. In order to describe the geometry of $\mathcal{S}$ globally, we attach to each component of $\mathcal{S}$ with some isogeny level structure, which we call *canonical isogenies*. This idea of that may be traced back to the Manin's thesis [18]. An explicit form was constructed first in the work of Oda and Oort [25]. We follow Li and Oort [17] closely to construct such a model in Section 7–9. Some improvement of the Cayley-Hamilton method in the Hilbert-Blumenthal case is made in Section 10. This improvement, together with the knowledge of the a-number stratification, helps us to compute all the local moduli spaces of the supersingular locus. We give an explicit recipe of computing these local moduli spaces in Section 11. The intersections of the irreducible components near a point can be obtained directly from their local moduli spaces. We use the result of Section 9 to describe the intersections globally. Local and global descriptions of the intersections are discussed in Section 12. Some computation of local moduli is placed in Section 13 for interested readers. In Appendix, we show that the points of generic a-numbers form an open dense subset in each slope stratum.

The setting of this paper is for good reduction of Hilbert-Blumenthal modular varieties, although explicit information is given only for $g = 4$. We will extend our results to higher dimensions in our sequent paper [37]. The paper grows out of the author's thesis at University of Pennsylvania, where the main results were done in 1998. This work contains some results which were done in the work by E. Goren and F. Oort [9]. We don't try to eliminate our original proofs of the results they have obtained first. Rather we cite their results in an appropriate place.

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2. Notations and definitions

In the paper, we will fix a prime p and assume that p is inert in F, where F is a fixed totally field. Let f be the residue degree of p, hence $f = g = \dim A$. Let k be a perfect field containing $\mathbb{F}_q$ ($q = p^f$), $W := W(k)$ be the ring of Witt vectors, and σ be the absolute Frobenius morphism on W. We label the embeddings $\mathrm{Hom}(O_F, W) = \{\sigma_0, \sigma_1, \ldots, \sigma_{f-1}\}$ in a way that $\sigma_{i+1} = \sigma\sigma_i$ for $i \in \mathbb{Z}/f\mathbb{Z}$. For a polarized O_F-abelian variety (A, λ, ι) over k, the covariant Dieudonné module $M := M(A)$ of A has the following properties:

- M admits a $\mathbb{Z}/f\mathbb{Z}$-grading: $M = M^0 \oplus M^1 \oplus \cdots \oplus M^{f-1}$, where $M^i := \{\, x \in M \mid a \cdot x = \sigma_i(a)x \,\}$ is a W-module of rank 2. We make the convention that $M^f := M^0$, and $M^{-1} := M^{f-1}$.

- the polarization induces a non-degenerate alternating pairing on M

$$\langle\ ,\ \rangle : M \times M \to W$$

which is non-degenerate when restricted to M^i and $\langle M^i, M^j\rangle = 0$ if $i \neq j$.
- The operators F and V shift by degree 1 and degree -1, respectively.

For each $0 \leq i < f$, there exists a basis $X_i, Y_i \in M^i$ such that $\langle X_i, Y_i\rangle = p^{n_i}$. The number n_i does not depend on the choice of bases. The *polarization type* of (A, λ, ι) is defined to be $(p^{n_0}, p^{n_1}, \ldots, p^{n_{f-1}})$.

If L is a k-module or W-module equipped linear O_F-action, then the i-th component of L we denote by

$$L^i := \{\, x \in L \mid a \cdot x = \sigma_i(a)x \,\}.$$

We define the *a-number* of an O_F-abelian variety A to be

$$a(A) = a(M) := (a_0(M), a_1(M), \ldots, a_{f-1}(M)),\ \ a_i(M) := \dim_k(M/(F,V)M)^i,$$

where M is the covariant Dieudonné module of A.

If the Lie algebra condition holds for (A, λ, ι), i.e. $\mathrm{Lie}(A)$ is a free $O_F \otimes k$-module, then each $a_i = 0$, or 1 since $a_i \leq \dim \mathrm{Lie}(A)^i = 1$. In this case we define its *a-index* to be

$$\tau_A = \tau_M := \{0 \leq i < f \mid a_i(M) = 1\,\}.$$

When there is no confusion, write $a := |a(M)| = |\tau_M|$. This is the usual a-number defined in [20].

An a-number $(a_0, \ldots, a_{f-1})$ is called *generic* if $a_i a_{i+1} = 0$, for all $i \in \mathbb{Z}/f\mathbb{Z}$, and we define similarly for τ if there are no consecutive integers in τ. This is called *spaced* in [9]. We say τ_1 is higher than τ_2 if $\tau_1 \supset \tau_2$, and similarly for a-numbers. We will see that the a-indices induce a stratification of our moduli space, and the type of the stratum jumps up under the specialization.

In Section 3 we give the definition of the substack $\mathcal{M}_\tau$ of $\mathcal{M} := \mathcal{A}_g^{O_F} \otimes \mathbb{F}_p$ and we often identify the notations $\mathcal{M}_{\underline{a}}$ and $\mathcal{M}_\tau$, e.g. $\mathcal{M}_{1010} = \mathcal{M}_{\{0,2\}} = \mathcal{M}_{0,2}$. Although we are working on algebraic stacks, sometimes it is convenient to adapt the notation of moduli functors: $\mathcal{M}_\tau(k)$ means the objects in $\mathcal{M}_\tau$ over k.

Let $\mathcal{S}_\tau$ denote the intersection of $\mathcal{S}$ and $\mathcal{M}_\tau$, and let $\bar{\mathcal{S}}_\tau$ denote the Zariski closure of $\mathcal{S}_\tau$ in $\mathcal{S}$.

Let $(A_1, \lambda_1, \iota_1)$ and $(A_2, \lambda_2, \iota_2)$ be polarized O_F-abelian varieties. An isogeny $\phi : A_1 \to A_2$ between O_F-abelian varieties is *O_F-linear* if $\iota_2(a)\phi = \phi\iota_1(a)$, $\forall\, a \in O_F$. Call such an isogeny O_F-isogeny. By definition, the polarization of an polarized O_F-abelian variety is O_F-linear. An isogeny $\phi : A_1 \to A_2$ is *polarized O_F-linear* if it is O_F-linear and $\phi^*\lambda_2 = \lambda_1$. Let R be a commutative ring. An O_F-isogeny $\phi : A_1 \to A_2$ is *R-polarized* if $\phi^*\lambda_2 = a\,\lambda_1$ for some $a \in R^\times$.

In Section 7 we also use the classical contravariant Dieudonné modules, denoted by $M^*(A)$. We apologize for this inconsistence and any confusion it may cause.

We often write $x^{(n)}$ for x^{σ^n} for short.

3. a-NUMBER STRATIFICATION

In this section we study a stratification induced by the a-numbers of abelian varieties, which is the same as the one induced by their p-torsion subgroups, see [30, Thm 15.6]. This stratification is also studied in a joint paper of Goren and Oort [9], now called the *Ekedahl-Oort* stratification. The Ekedahl-Oort stratification is finer than a-number stratification for other modular varieties in general.

Let $\mathcal{M} := \mathcal{A}_g^{O_F} \otimes \mathbb{F}_p$. For any *non-empty* subset τ of $\{0, 1, \dots, f-1\}$. Define a substack $\overline{\mathcal{M}}_\tau$ over $\mathbb{F}_q$ whose objects are $(A/S, \lambda, \iota) \in \mathrm{Ob}(\mathcal{M}_S)$ such that the morphism $F : \mathrm{Lie}(A^t)^{(p),i} \to \mathrm{Lie}(A^t)^i$ is zero for all $i \in \tau$. It is clear that we have the following properties

(1) $\overline{\mathcal{M}}_\tau$ is representable by a closed subscheme of $\mathcal{M}$.
(2) $\overline{\mathcal{M}}_{\tau_1}$ is closed in $\overline{\mathcal{M}}_{\tau_2}$ if $\tau_1 \supset \tau_2$.
(3) The scheme-theoretic intersection of $\overline{\mathcal{M}}_{\tau_1}$ and $\overline{\mathcal{M}}_{\tau_2}$ is $\overline{\mathcal{M}}_{\tau_1 \cup \tau_2}$
(4) $\overline{\mathcal{M}}_\tau$ is proper over $\mathbb{F}_q$ since there is no intersection with the boundary of $\mathcal{M}$.

Theorem 3.1. *The inclusion $\overline{\mathcal{M}}_\tau \to \mathcal{M}$ is a local complete intersection of codimension $|\tau|$ and $\overline{\mathcal{M}}_\tau/\mathbb{F}_q$ is smooth.*

Proof. This result was first obtained by Goren and Oort in a slightly different form, op. cit.. First, we will show that locally $\overline{\mathcal{M}}_\tau$ is a closed subscheme of $\mathcal{M}$ defined by $|\tau|$ equations. It amounts to show that:

For any principally polarized O_F-abelian scheme $(A/S, \lambda, \iota)$, any $s \in S$ there exists an affine neighborhood U of s and sections $\{f_i\}_{i \in \tau}$ of $\underline{\mathrm{Hom}}_{O_F}(H^1(A, \mathcal{O}_A)^{(p)}, H^1(A, \mathcal{O}_A))$ over U such that U factors through $\overline{\mathcal{M}}_\tau$ if and only if these sections vanish.

However, it is obvious because once one chooses an affine chart trivializing the coherent sheaves $H^1(\mathcal{O}_A)^{(p)}$ and $H^1(\mathcal{O}_A)$, the Frobenius F is simply a multiplication by functions (f_i).

Now we will show that the tangent space of $\overline{\mathcal{M}}_\tau$ at any point has dimension $f - |\tau|$. This ensures that $\dim_x \overline{\mathcal{M}}_\tau = \dim \overline{\mathcal{M}}_\tau$ everywhere. Let $(A, \lambda, \iota) \in \overline{\mathcal{M}}_\tau(k)$. Set $M := H^1_{cris}(A/k), M^{(p)} := H^1_{cris}(A^{(p)}/k) = M \otimes_{k, Fr} k$. We have Hodge filtrations:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Fil^{(p)} & \longrightarrow & M^{(p)} & \longrightarrow & Q^{(p)} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle F} & & & & \\ 0 & \longrightarrow & Fil & \longrightarrow & M & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

Let $\{x_i, y_i\}$ be a basis of M such that $y_i \in Fil^i \subset M^i$ and $x_i \in M^i$. One can choose them such that $F(x_i \otimes 1) = -y_i$ for all $i \in \tau$. The differential forms x_i, y_i have unique liftings in $\tilde{M} := H^1_{cris}(A/R)$ with respect to the Gauss-Manin connection, where R is any PD-extension of k. We will denote the liftings by x_i, y_i again.

Let $R := k[[\underline{t}]]/(\underline{t})^2$ be the first order universal deformation ring of (A, λ, ι). The first order universal deformation $(\tilde{A}, \tilde{\lambda}, \tilde{\iota})$ gives rise to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{Fil}^{(p)} & \longrightarrow & \tilde{M}^{(p)} & \longrightarrow & \tilde{Q}^{(p)} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle F} & & & & \\ 0 & \longrightarrow & \widetilde{Fil} & \longrightarrow & \tilde{M} & \longrightarrow & \tilde{Q} & \longrightarrow & 0 \end{array}$$

where $\widetilde{Fil} = span\langle y_i + t_i x_i \rangle$, $\widetilde{Fil}^{(p)} = span\langle (y_i + t_i x_i) \otimes 1 \rangle = span\langle y_i \otimes 1 \rangle$, $\tilde{Q} = span\langle [x_i] \rangle$, $\tilde{Q}^{(p)} = span\langle [x_i \otimes 1] \rangle$.

For each $i \in \tau$, $F([x_i \otimes 1]) = [-y_i] = t_i[x_i]$. It follows that t_i must vanish in order to lift (A, λ, ι) inside $\overline{\mathcal{M}}_\tau$. Therefore $R/(t_i)_{i \in \tau}$ is the first order deformation ring in $\overline{\mathcal{M}}_\tau$, and the dimension of $\overline{\mathcal{M}}_\tau$ is $f - |\tau|$. $\square$

Second proof. Beginning with crystalline theory, we check the smoothness directly by using the deformation theory. Let $I \to R' \to R \to k$ be artinian k-extensions such that $I\mathfrak{m}_{R'} = 0$. We need to show that $\overline{\mathcal{M}}_\tau(R') \to \overline{\mathcal{M}}_\tau(R)$ is surjective. Let $(A/R, \lambda, \iota) \in \overline{\mathcal{M}}_\tau(R)$ and set $M := H^1_{cris}(A/R), Fil := H^0(A, \Omega_A)$. One can choose a basis x_i, y_i such that $y_i \in Fil^i$, $x_i \in M^i$ and $F(x_i \otimes 1) = -y_i\ \forall i \in \tau$. Set $M' := H^1_{cris}(A/R')$ for R' a PD-extension of R with the natural PD-structure. We lift the basis x_i, y_i over R' by the Gauss-Manin connection and denote them by x_i, y_i again, but $F(x_i \otimes 1) = (-1 + a_i)y_i + b_i x_i$ for some $a_i, b_i \in I$ for each $i \in \tau$.

Any deformation (A', λ', ι') of $(A/R, \lambda, \iota)$ over R' is given by $Fil' = span\langle y_i + t_i x_i\rangle$ for some $t_i \in I$ and all deformations arise in this way. Take $t_i = -b_i\ \forall i \in \tau$. Then for each $i \in \tau$,

$$\begin{aligned} F([x_i \otimes 1]) &= [(-1 + a_i)y_i + b_i x_i] \\ &= [(-1 + a_i)y_i + (-1 + a_i)t_i x_i] + (t_i + b_i)[x_i] = 0 \end{aligned}$$

where $[- \otimes 1], [-]$ denote the class in $M'^{(p)}/Fil'^{(p)}, M'/Fil'$ respectively. Thus we get a lift in $\overline{\mathcal{M}}_\tau(R')$. $\square$

Let $\mathcal{M}_\tau$ be the substack of $\overline{\mathcal{M}}_\tau$ whose k-points have a-index τ, with the reduced structure. Since $\overline{\mathcal{M}}_\tau$ is reduced, $\mathcal{M}_\tau$ is an open substack of $\overline{\mathcal{M}}_\tau$ and its Zariski closure is $\overline{\mathcal{M}}_\tau$. This justifies the notation $\overline{\mathcal{M}}_\tau$. Using Cartier-Dieudonné theory, it is easy to deform any point in $\mathcal{M}_{\tau_1}$ to a point in $\mathcal{M}_{\tau_2}$ as long as τ_1 is higher than τ_2. Conversely, it is true that we can specialize any point of lower a-index to a higher one. Indeed, we know that $\overline{\mathcal{M}}_\tau$ is complete and that every stratum $\mathcal{M}_\tau$ is quasi-affine, op. cit..

4. Normal forms

From now on and in the rest of this paper, the base field k is algebraically closed of characteristic p. In this section, we will give a normal form (Proposition 4.1) of the Dieudonné modules in question and its variant (Proposition 4.2). Let (A, λ, ι) be a principally polarized O_F-abelian variety over k, and let M be its covariant Dieudonné module with a-index $\tau = \tau_M = \{n_1, n_2, \ldots, n_a\}$.

Proposition 4.1. *There exists a symplectic basis $\{X_i, Y_i\}_{0 \le i < f}$ for M such that*

(i) $X_i, Y_i \in M^i$ *and* $Y_i \in (VM)^i$,

(ii) *the pairings are zero except* $\langle X_i, Y_i\rangle = 1$ *for all* i, *and*

(iii) $$F\,X_i = \begin{cases} X_{i+1} & \text{if } i+1 \notin \tau \\ -Y_{i+1} + a_{i+1} p X_{i+1} & \text{if } i+1 \in \tau\ (*) \end{cases}$$

for some $a_{i+1} \in W(k)$,

$$F\,Y_i = \begin{cases} pY_{i+1} & \text{if } i+1 \notin \tau \\ pX_{i+1} & \text{if } i+1 \in \tau \end{cases}$$

Proof. We prove the following inductive statement (I_n): There exists a W- basis $\{X_i, Y_i\}_{0 \le i < f}$ for M such that

(i) $X_i, Y_i \in M^i$ and $Y_i \in (VM)^i$.

(ii) the pairings are zero mod p^n except $\langle X_i, Y_i\rangle \equiv 1 \mod p^n$ for all i.

(iii) $F\,X_i = X_{i+1}$ if $i+1 \notin \tau$.

$$F\,Y_i = \begin{cases} pY_{i+1} & \text{if } i+1 \notin \tau. \\ pX_{i+1} & \text{if } i+1 \in \tau. \end{cases}$$

Note that this would automatically imply $FX_i = -Y_{i+1} + a_{i+1}pX_{i+1}$ for some $a_{i+1} \in W(k)$ when $i+1 \in \tau$, namely, $(*)$ of (iii).

Let $\{X_{n_j} \in M^{n_j}\}$ be a set of generators of M as a Dieudonné module. There is a W-basis $\{X_i, Y_i\}$ of M such that (1) $F X_i = X_{i+1}$ if $i+1 \notin \tau$, (2) $V Y_i = Y_{i-1}$ if $i \notin \tau$, and (3) $V X_i = Y_{i-1}$ if $i \in \tau$. For example, when $f = 6$ and $\tau = (0, 3, 5)$, the basis is

$$\begin{pmatrix} X_0 & X_1 & X_2 & X_3 & X_4 & X_5 \\ Y_0 & Y_1 & Y_2 & Y_3 & Y_4 & Y_5 \end{pmatrix} = \begin{pmatrix} X_0 & FX_0 & F^2X_0 & X_3 & FX_3 & X_5 \\ V^3X_3 & V^2X_3 & VX_3 & V^2X_5 & VX_5 & VX_0 \end{pmatrix}$$

Begin with a set of generators X_{n_j} of M, extend a W-basis as above. Suppose $\langle X_{n_j}, Y_{n_j}\rangle \equiv \alpha_j \mod p$. Let $X'_{n_j} := \lambda_j X_{n_j}$, then conditions

$$\langle X'_{n_j}, Y'_{n_j}\rangle \equiv 1 \mod p \qquad \forall\, j = 1, \dots, a$$

give the equations

$$\bar{\lambda}_j \bar{\lambda}_{j+1}^{p^{n_j - n_{j+1}}} \alpha_j = 1 \qquad \forall\, j = 1, \dots, a$$

where $\lambda_{a+1} = \lambda_1, n_{a+1} = n_1 + f$ and $\bar{\lambda} = \lambda \mod p$. Raising to some power of p, we get equations

$$\lambda_j^{p^{n_{j+1} - n_j}} \lambda_{j+1} \beta_j = 1 \qquad \forall\, j = 1, \dots, a$$

where β_j is a some power of α_j. Since k is algebraically closed these equations have solutions. Adjust X_{n_j} by X'_{n_j} and extend the basis again as above, hence (I_1) is done.

Now assume (I_k). Suppose $\langle X_{n_j}, Y_{n_j}\rangle \equiv 1 + \alpha_j p^k \mod p^{k+1}$. Let $X'_{n_j} = (1 + \lambda_j p^k) X_{n_j}$. We may regard $\lambda_j \in k$, for simplicity. Then

$$\langle X'_{n_j}, Y'_{n_j}\rangle \equiv 1 + (\lambda_j + \lambda_{j+1}^{p^{n_j - n_{j+1}}} + \alpha_j) p^k \mod p^{k+1} \qquad \forall\, j = 1, \dots, a.$$

A similar procedure for replacing X_{n_j} by X'_{n_j} will yield the statement (I_{k+1}).

□

It might be simpler not to deal with signs, hence we provide the following variant.

Proposition 4.2. *Let M, τ be as in Proposition 4.1. Then there exists a basis $\{X_i, Y_i\}$, $i = 0, \dots, f-1$, of M such that*

(i) $X_i, Y_i \in M^i$, $Y_i \in (VM)^i$, $\forall\, i = 0 \dots f-1$,

(ii) $\langle X_i, Y_i\rangle = u_i$, *a unit, and all other pairings are zero, and*

(iii) $F X_i = \begin{cases} X_{i+1} & \text{if } i+1 \notin \tau \\ Y_{i+1} + a_{i+1} p X_{i+1} & \text{if } i+1 \in \tau \end{cases}$

for some $a_{i+1} \in W(k)$

$F Y_i = \begin{cases} pY_{i+1} & \text{if } i+1 \notin \tau \\ pX_{i+1} & \text{if } i+1 \in \tau \end{cases}$

Proof. Let $\{X_i, Y_i\}$ be a symplectic basis satisfying (i)(ii)(iii) of Proposition 4.1.

Let $X'_{n_i} = \lambda_i X_{n_i}$, $\lambda_i \in W^\times$, for $i = 1, \dots, a$. Then

$$F^{n_{i+1} - n_i} X'_{n_i} = \lambda_i^{\sigma^{n_{i+1} - n_i}} (-Y_{n_{i+1}} + p a_{n_{i+1}} X_{n_{i+1}})$$

$$Y'_{n_{i+1}} = V^{n_{i+2} - n_{i+1}} X'_{n_{i+2}} = \lambda_{i+2}^{\sigma^{-(n_{i+2} - n_{i+1})}} Y_{n_{i+1}}$$

where $n_{a+1} = n_1 + g$. Let $\lambda_i \in W^\times, i = 1, \cdots, a$ such that

$$-\lambda_i^{\sigma^{(n_{i+1}-n_i)}} = \lambda_{i+2}^{\sigma^{-(n_{i+2}-n_{i+1})}}.$$

Then

$$F^{n_{i+1}-n_i} X'_{n_i} = Y'_{n_{i+1}} + p a'_{n_{i+1}} X'_{n_{i+1}}$$

where

$$a'_{n_{i+1}} = a_{n_{i+1}} \lambda_i^{\sigma^{(n_{i+1}-n_i)}} \lambda_{i+1}^{-1}$$

. ☐

Lemma 4.3. *If M is superspecial, i.e. $|a(M)| = f$. Then*
(1) There exists a basis $\{X_i, Y_i\}_{0 \le i < f}$ of M such that

- $X_i, Y_i \in M^i,\ Y_i \in (VM)^i$,
- $FX_i = -Y_{i+1},\ FY_i = pX_{i+1}$,
- $\langle X_i, Y_i \rangle = 1$,

for all $i = 0, \ldots, f-1$.
(2) There exists a basis $\{X_i, Y_i\}$ of M such that

- $X_i, Y_i \in M^i,\ Y_i \in (VM)^i$,
- $FX_i = Y_{i+1},\ FY_i = pX_{i+1}$,
- $\langle X_i, Y_i \rangle \in W(\mathbb{F}_{p^r})^\times$, *where $r = lcm(2, f)$,*

for all $i = 0, \ldots, f-1$.

Proof. (1) Suppose f is even. Let $f = 2d$, $M^\circ := \{x \in M \mid F^d x = (-1)^d V x\}$. Since M is supersingular, we have $F^2 M^\circ = pM^\circ$ and $M^\circ \otimes_W (\mathbb{F}_{p^f}) W(k) \simeq M$. Choose $X_0, Y_0 \in (M^\circ)^0$ such that $\langle X_0, Y_0 \rangle = 1$. Define recursively

$$X_{i+1} = p^{-1} F Y_i,\ Y_{i+1} = -F X_i. \tag{4.1}$$

Then $X_{i+2} = \frac{-1}{p} F^2 X_i,\ Y_{i+2} = \frac{-1}{p} F^2 Y_i$. One verifies that

$$X_f = (-1)^d p^{-d} F^{2d} X_0 = X_0,$$

$$Y_f = (-1)^d p^{-d} F^{2d} Y_0 = Y_0.$$

and $\langle X_i, Y_i \rangle = 1$ for all $i = 0, \ldots, f-1$. The basis $\{X_i, Y_i\}$ is a desired one.

Suppose F is odd. Let $f = 2d+1$ and $M^\circ = \{x \in M \mid F^{2f} + p^f x = 0\}$. Choose $X_0 \in (M^\circ)^0 \setminus (VM)^0$ and let $Y_0 = (-1)^{d+1} p^{-d} F^f X_0$. Define X_i, Y_i as (4.1). One verifies that

$$X_f = \frac{1}{p} F Y_{f-1} = (-1)^d p^{-d-1} F^f Y_0 = X_0,$$

and $Y_f = Y_0$. Let $\mu = \langle X_0, Y_0 \rangle$. From $\langle F^f X_0, F^f Y_0 \rangle = \langle (-1)^{d+1} p^d Y_0, (-1)^d p^{d+1} X_0 \rangle$ we have $\mu^{(f)} = \mu$. Replace X_0 by λX_0, then μ will change to $\lambda \lambda^{(f)} \mu$. Since $B(\mathbb{F}_{p^{2f}})/B(\mathbb{F}_{p^f})$ is a quadratic unramified extension, the norm map is surjective onto local units. Therefore, we can choose X_0 such that $\langle X_0, Y_0 \rangle = 1$. The basis defined above is a desired one.

(2) We modify the proof of (1) as follows:

When f is even, one replaces $M\circ$ by $\{x \in M \mid F^f x = V^f x\}$ and (4.1) by

$$X_{i+1} = p^{-1} F Y_i,\ Y_{i+1} = F X_i. \tag{4.2}$$

When f is odd, one replaces definition of Y_0 by $p^{-d} F^f X_0$ and (4.1) by (4.2).

☐

5. Cayley-Hamilton for $a = 1$

A σ-linear version of "Cayley-Hamilton" was introduced by F. Oort. It is an effective method to compute the slopes of a Dieudonné module. Hence it provides a way to compute the local moduli spaces of a slope stratum. In [29] F. Oort used this method to (re)solve several problems of Manin, Grothendieck, and Koblitz. This section is completely inspired by his work.

The key point of this method is the following observation:

Proposition 5.1. *Let M be a Dieudonné module of height h over a perfect field k. Suppose there exists an element $X \in M$ such that*

(i) *the W-submodule $W[F]X$ is of full rank, and*
(ii) *X satisfies an equation*

$$F^h X + p_1 F^{h-1} X + \cdots + p_{h-1} FX + p_h X = 0, \ \ p_i \in W. \tag{5.1}$$

Then the Newton polygon of M is the same as that of the polynomial

$$x^h + p_1 x^{h-1} + \cdots + p_{h-1} x + p_h.$$

Proof. See a proof in op. cit.. □

The equation (5.1) is called a Cayley-Hamilton equation. A Cayley-Hamilton equation does dependent on the choice X and may not give the correct slopes if (i) is not satisfied. Most of the elements in M will satisfy this property. However, in order to obtain the coefficients p_i as simple as possible, the element X has to be carefully chosen. It will boil down to an explicit local calculation of Dieudonné modules.

The next step is to describe how to compute the closed subscheme of a given slope inside a deformation space, once one has obtained a "family" of Cayley-Hamilton equations.

Let (A_0, λ_0) be a polarized abelian variety over k such that its p-divisible group G is local-local. Let $(M_0, \langle\ ,\ \rangle)$ be the covariant quasi-polarized Dieudonné module of (A_0, λ_0). In the rest of this section R will denote a complete Noetherian local ring with residue field k.

Suppose that we have a deformation (G, λ) of (G_0, λ_0) over R, as formal groups together with the biextension [20]. Then the n^{th}-order deformations (G_n, λ_n) over $\operatorname{Spec} R/\mathfrak{m}^{n+1}$ are formal p-divisible groups. Hence (G, λ) induces a compatible system of deformations of p-divisible groups, thus a p-divisible group over $\operatorname{Spf} R$. On the other hand, by Serre-Tate, it also gives a compatible system of polarized abelian schemes (A_n, λ_n), thus a polarized formal abelian scheme $(\bar{A}, \bar{\lambda})$ over $\operatorname{Spf} R$. By Grothendieck's existence theorem, there is a polarized abelian variety (A, λ) over $\operatorname{Spec} R$ whose formal $\mathfrak{m}$-adic completion is isomorphic to $(\bar{A}, \bar{\lambda})$.

For any prime ideal $\mathfrak{p} \in \operatorname{Spec} R$, let $K(\mathfrak{p})$ be the perfect closure of the residue field $k(\mathfrak{p})$. If $A_{k(\mathfrak{p})}$ has p-rank 0, then $A_{K(\mathfrak{p})}[p^\infty]$ is a formal p-divisible group over a perfect field $K(\mathfrak{p})$ and its covariant Dieudonné module

$$\mathbf{D}_*(A_{K(\mathfrak{p})}) \simeq \mathbf{D}_*(G) \otimes_{Cart_p(R)} Cart_p(K(\mathfrak{p}))$$

where $\mathbf{D}_*$ is the covariant functor in Cartier-Dieudonné theory. This is a finite $W(K(\mathfrak{p}))$-module with F and V action. With the notion of display [22][23], everything can be explicitly written down: First, we display the Dieudonné module $(M_0, \langle\ ,\ \rangle)$ by choosing a basis with the relations

$$FX_j = \sum_i a_{ij} X_i, \quad j = 1, \dots, g,$$
$$X_j = V \sum_i a_{ij} X_i, \quad j = g+1, \dots, 2g.$$

Suppose that M is a deformation of M_0 over R such that the biextension induced by $\langle\ ,\ \rangle$ extends. By the theorem [22, Thm 1], M can be realized as a $Cart_p(R)$-module with generators $\{X_i\}$ and relations

$$FX_j = \sum_i \tilde{a}_{ij} X_i, \quad j = 1, \dots, g,$$
$$X_j = V \sum_i \tilde{a}_{ij} X_i, \quad j = g+1, \dots, 2g.$$

where $\tilde{a}_{ij}$ is an element in $W(R) \subset Cart_p(R)$ whose image in $Cart_p(k)$ is a_{ij}.

For $\mathfrak{p} \in \operatorname{Spec} R$, the Cartier module $M \otimes_{Cart_p(R)} Cart_p(K(\mathfrak{p}))$ is a Dieudonné module over $K(\mathfrak{p})$ with the generators and relations:

$$FX_j = \sum_i \tilde{a}_{ij}(\mathfrak{p}) X_i, \quad j = 1, \dots, g,$$
$$X_j = V \sum_i \tilde{a}_{ij}(\mathfrak{p}) X_i, \quad j = g+1, \dots, 2g.$$

where $\tilde{a}_{ij}(\mathfrak{p})$ is the image of $\tilde{a}_{ij}$ in $Cart_p(K(\mathfrak{p}))$. Denote this module by $M_{\mathfrak{p}}$. This gives a way to compute (the local moduli of) a slope stratum with reduced structure.

Proposition 5.2. *Let $(M_0, \langle\ ,\ \rangle), (M, \langle\ ,\ \rangle)$ as before, γ the Newton polygon of M_0. Suppose there exists an element $X \in \{X_i\}$ such that*

(i) *There exists an (Cayley-Hamilton) equation $F^{2g}X + p_1 F^{2g-1}X + \cdots + p_{2g-1}FX + p_{2g} = 0$, where $p_i = (p_{i0}, p_{i1}, \dots) \in W(R)$.*
(ii) *For all $\mathfrak{p} \in \operatorname{Spec} R$, the $W(K(\mathfrak{p}))$-submodule $W(K(\mathfrak{p}))[F]X$ of $M_{\mathfrak{p}}$ is of full rank $2g$.*

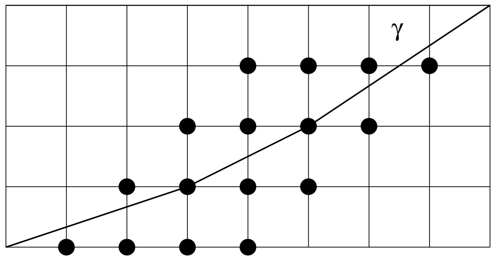
Then the closed subscheme with Newton polygon γ or higher is $\operatorname{Spec} R/\sqrt{I}$, where I is the ideal generated by $p_{ij} \in R$ for all (i,j) strictly below γ in the Newton polygon diagram.

Proof.

$$\begin{aligned}
&\mathfrak{p} \in \operatorname{Spec} R \text{ such that } M_{\mathfrak{p}} \text{ has the Newton polygon } \gamma \text{, or higher.}\\
\Longleftrightarrow\ &\text{The Cayley-Hamilton equation, after base change to } W(K(\mathfrak{p})),\\
&\text{has the Newton polygon } \gamma \text{, or higher.}\\
\Longleftrightarrow\ &p_{ij} = 0 \text{ in } K(\mathfrak{p}) \quad \forall\,(i,j) \text{ below } \gamma.\\
\Longleftrightarrow\ &p_{ij} \in \mathfrak{p} \quad \forall\,(i,j) \text{ strictly below } \gamma.\\
\Longleftrightarrow\ &I \subset \mathfrak{p}, \quad i.e. \quad \mathfrak{p} \in \operatorname{Spec} R/\sqrt{I}.
\end{aligned}$$

□

Example. Let γ is a Newton polygon as follows.



As the Newton polygons of abelian varieties are symmetry, we may look at just the points (i,j) with $i \leq g$. In this case, the points strictly below γ are $(1,0),(2,0),(3,0),(4,0),(4,1)$. Therefore $I = (p_{10}, p_{20}, p_{30}, p_{40}, p_{41})$.

Theorem 5.3 (Oort)**.** *Let $(M_0, \langle\ ,\ \rangle)$ as before. If the polarization is separable and $a(M_0)$=1, then there is an X for the universal deformation $(M, \langle\ ,\ \rangle)$ and an explicit Cayley-Hamilton equation such that the conditions in the previous proposition hold.*

We apply this method, in the Hilbert case, to compute each slope stratum in the open subset $a = 1$. First note that the Dieudonné module of a point on $\mathcal{M}$ has slopes a/f and $1 - a/f$ with multiplicity f. This can be easily shown by using Katz's sharp slope estimate. Let us denote $\mathcal{M}^{(m)}$ (for $m < (f/2)+1$) the substack of $\mathcal{M}$, with the reduced structure, whose points have the lowest slope m/f.

Let $\tau = 10\ldots 0$ and $M = M(A)$ with $(A, \lambda, \iota) \in \mathcal{M}_\tau(k)$. We can choose a basis $\{X_0 = X, X_i, Y_0 = Y, Y_i\}$ of M such that

$$\begin{array}{ll} FX_i = X_{i+1}\ (i \neq f-1), & FX_{f-1} = pa_0 X - Y \\ FY_i = pY_{i+1}\ (i \neq f-1), & FY_{f-1} = pX. \end{array}$$

Iterating F f times, we have

$$F^f X = pa_0 X - Y, \quad F^f Y = p^f X.$$

It is easy to see that $F^{2f}X - pa_0^{(f)} F^f X + p^f X = 0$. This shows that the lowest slope of M is $\min(\frac{1}{f}(\mathrm{ord}(a_0)+1), 1/2)$.

Proposition 5.4. *Let $\tau = 10\ldots 0$ and M the Dieudonné module of a point $x = (A, \lambda, \iota) \in \mathcal{M}^{(m)}_\tau(k)$. Then the local moduli space $\widehat{\mathcal{M}}^{(m)}_x = \widehat{\mathcal{M}}^{(m)}_{\tau,x}$ is isomorphic to* $\mathrm{Spf}\, k[[t_m, \ldots, t_{f-1}]]$.

Proof. This is Theorem 5.3.3 in [9]. Choose a basis of M as above. The covariant Dieudonné module of the universal deformation of M inside $\mathcal{M}_\tau$ is

$$\begin{array}{ll} FX_i = X_{i+1}\ (i \neq f-1), & FX_{f-1} = pa_0 X - Y \\ FY_i = pY_{i+1} + pT_{i+1}\ X_{i+1}\ (i \neq f-1), & FY_{f-1} = pX. \end{array}$$

where T_i is the Teichmüller lift of t_i. Iterating F f-times, we obtain

$$F^f X = pa_0 X - Y, \quad F^f Y = (p^f + pa_0 u)X - uY$$

where $u = p^{f-1} T^{(1)}_{f-1} + p^{f-2} T^{(2)}_{f-2} + \cdots + pT^{(f-1)}_1$. It is easy to check that

$$F^{2f} X + (u - pa_0^{(f)}) F^f X + p^f X = 0$$

The coefficient $u - pa_0^{(f)} = (0, t_1^{p^f}, \ldots, t_{m-1}^{p^f}, \ldots)$, hence $\sqrt{I} = (t_1, \ldots, t_{m-1})$ and $\widehat{\mathcal{M}}^{(m)}_x \simeq \mathrm{Spf}\, k[[t_1, \ldots, t_{f-1}]]/(t_1, \ldots, t_{m-1}) \simeq \mathrm{Spf}\, k[[t_m, \ldots, t_{f-1}]]$. $\square$

6. Preliminary results for g=4

In this section $g = f$ is 4.

Lemma 6.1. *If $(A, \lambda, \iota) \in \mathcal{M}_{1010}(k)$ then A is supersingular .*

Proof. Let M be the Dieudonné module of (A, λ, ι). M can be written as in Proposition 4.1

$$\begin{array}{ll} FX_0 = X_1, & FY_0 = pY_1 \\ FX_1 = -Y_2 + pa_2X_2, & FY_1 = pX_2 \\ FX_2 = X_3, & FY_2 = pY_3 \\ FX_3 = -Y_0 + pa_0X_0, & FY_3 = pX_0, \end{array}$$

for some $a_0, a_2 \in W$. We iterate the Frobenius map four times

$$\begin{aligned} F^4 &: X_0 \to -p^2X_0 + a_2^{(2)}p(-Y_0 + a_0pX_0) \\ F^4 &: Y_0 \to p^2(-Y_0 + a_0pX_0), \end{aligned}$$

or

$$\begin{aligned} p^{-2}F^4 &: pX_0 \to -(pX_0) + a_2^{(2)}(-Y_0 + a_0(pX_0)) \\ p^{-2}F^4 &: Y_0 \to (-Y_0 + a_0(pX_0)). \end{aligned}$$

Therefore $p^{-2}F^4$ sends the submodule N generated by pX_0, Y_0 to itself. Since N is a submodule of the same rank as M, the slopes of M and those of N are the same. So the slopes of M are 1/2. □

Lemma 6.2. *Every point $(A, \lambda, \iota) \in \mathcal{S}_{1100}(k)$ can be deformed to a point of $\mathcal{S}_{0100}$ and also to a point of $\mathcal{S}_{1000}$.*

Proof. Let $x = (A, \lambda, \iota) \in \mathcal{S}_{1100}(k)$ and M be its Dieudonné module. Let $N \subset M \otimes FracW(k)$ such that $VN = M$. Then there is $(B, \lambda', \iota') \in \mathcal{S}_{0110}(k)$ and O_F-morphism $\varphi : (A, \lambda, \iota) \to (B, \lambda', \iota')$ such that $\varphi^*\lambda' = p\lambda$ and $N = M(B)$. Choose a basis $\{X_i, Y_i\}$ of N as in Proposition 4.2, i.e.

$$FX_i = \begin{cases} X_{i+1} & \text{if } i \notin \{0, 1\}, \\ Y_{i+1} + pa_{i+1}X_{i+1} & \text{if } i \in \{0, 1\}. \end{cases}$$

$$FY_i = \begin{cases} pY_{i+1} & \text{if } i \notin \{0, 1\}, \\ pX_{i+1} & \text{if } i \in \{0, 1\}. \end{cases}$$

Let $M_t = span\langle \tilde{Y}_i, \tilde{X}_i \rangle$, where $\tilde{Y}_0 = Y_0 + TX_0, \tilde{Y}_i = Y_i (i > 0), \tilde{X}_i = pX_i (\forall i)$ and $T = [t]$ is the Teichmüller lift of $t \in k$. It is easy to check that M_t is a Dieudonné module inside N and $\langle\ ,\ \rangle_N = p\langle\ ,\ \rangle_{M_t}$. This construction gives a one-parameter family in $\mathcal{S}$. Finally it is straightforward to check $a_1(M_t) = 0$ when $t \neq 0$, therefore we deform to a point of $\mathcal{S}_{1000}$. Similarly, we can deform the point to a point of $\mathcal{S}_{0100}$. □

Proposition 6.3.

(1) *The generic point of $\mathcal{M}$ is ordinary.*
(2) *For $|\tau| = 1$, the generic point of each irreducible component of $\mathcal{M}_\tau$ has slope 1/4 and the supersingular locus $\mathcal{S}_\tau$ is a one-dimensional smooth divisor of $\mathcal{M}_\tau$.*
(3) *For $|\tau| > 2$, every point on $\overline{\mathcal{M}}_\tau$ is supersingular .*
(4) *Every point on $\mathcal{M}_{1010}$ or $\mathcal{M}_{0101}$ is supersingular .*
(5) $\mathcal{S}_{1100} \subset \bar{\mathcal{S}}_{1000} \cap \bar{\mathcal{S}}_{0100}$.

(6) *The generic point of each irreducible component of* $\mathcal{M}_{1100}$ *has slope* $1/4$.
(7) *Every component of the supersingular locus has dimension* 2. *There are 6 types of them, namely* $\mathcal{S}_\tau$ *with generic a-index* τ.
(8) *The slope stratification behaves well:* $\overline{\mathcal{M}^{(2)}} \subset \overline{\mathcal{M}^{(1)}} \subset \overline{\mathcal{M}^{(0)}}$ *and each irreducible component of* $\overline{\mathcal{M}^{(m)}}$ *has codimension* m *in* $\mathcal{M}$. *Here* $\overline{\mathcal{M}^{(m)}}$ *is the Zariski closure of* $\mathcal{M}^{(m)}$ *in* $\mathcal{M}$.

Proof. (1) follows from Theorem 3.1. (2) is a consequence of Cayley-Hamilton for $a = 1$. (3) follows from (4). (4) is Lemma 6.1. (5) is Lemma 6.2. (6) follows from (5). (7) follows from (5). (8) follows from (7) and (2). □

7. Canonical isogenies

Let k be a algebraically closed field of characteristic $p > 0$. Let O_F be the ring of integers of a totally real field F of degree f, and assume that p is inert in O_F. Let H be an α-group over k with an (left) O_F-action. By definition, an *α-group* over S (of characteristic p) is a commutative finite flat group scheme $G \to S$ such that the Frobenius $F_{G/S} : G \to G^{(p)}$ and Verschiebung $V_{G/S} : G^{(p)} \to G$ are zero. Denote by $M^*(H)$ the classical *contravariant* Dieudonné module of H, which is simply a k-vector space with a k-linear action by O_F. The action gives a decomposition of $M^*(H)$ into eigenspaces. The only invariant other than its dimension is the multiplicity of each one-dimensional representation σ_i. We say an α-group H is *of type* $(n_0, \dots, n_{f-1})$ if n_i is the multiplicity of σ_i on $M^*(H)$. Denote by $\alpha_p^{(n_0,\dots,n_{f-1})}$ the α-group of type $(n_0, \dots, n_{f-1})$. When H is an α-group equipped with an O_F-action over a *connected* noetherian scheme S, we define $n_i(H)$ to be the rank of the i-th component of the *α-sheaf* of H defined in [17], Section 2. Locally in Zariski topology H is isomorphic to $\alpha_p^{(n_0,\dots,n_{f-1})} \times S$.

The Cartier dual H^t is again an α-group with a natural right O_F-action. Since O_F is commutative, we can regard it as a left action. According to the definition, we have the canonical pairing

$$(\ , \ ) : H^t \times H \to G_m \tag{7.1}$$

and

$$(a\,f, x) = (f, a\,x) \quad \forall a \in O_F,\ f \in H^t,\ x \in H. \tag{7.2}$$

The non-degeneracy of the canonical pairing forces the same multiplicity of each eigenspace in $M^*(H)$ and $M^*(H^t)$. Therefore H and H^t have the same representation type. The relation between the O_F-action and duality as (7.1) and (7.2) still holds for any finite flat groups with an O_F-action.

Let (A, ι) be an O_F-abelian variety of dimension g over k, M its covariant Dieudonné module. M is canonically isomorphic to $M^*(A)^t$, the (twisted) W-linear dual of $M^*(A)$. Let (A^t, ι^t) be its dual O_F-abelian variety. The Dieudonné module $M(A^t)$ is canonically isomorphic to the dual Dieudonné module M^t. The α-group $\alpha(A) := \ker F \cap \ker V^{(p^{-1})}$, which is the smallest α-subgroup of A containing all α_p in A, is of type $a(A^t)$. However, the invariant $a(A^t)$ is not equal to $a(A)$ in general. They are related by the following lemma.

Lemma 7.1. *Let* $(A, \iota), (A^t, \iota^t), M, M^t$ *as before. Assume that the Lie algebra* $\mathrm{Lie}(A) = M/VM$ *has the type* $(n_0, n_1, \dots, n_{f-1})$, $\sum n_i = g$ *and the a-number of A is* $(a_0, a_1, \dots, a_{f-1})$,

$a_i \le n_i$. Let $(n_0^t, n_1^t, \dots, n_{f-1}^t)$ be the type of $\mathrm{Lie}(A^t)$, *and $(a_0^t, a_1^t, \dots, a_{f-1}^t)$ be the a-number of A^t. Then $n_i^t = 2r - n_i$, $a_i^t = a_i + n_{i-1} - n_i$ for all $i \in \mathbb{Z}/f\mathbb{Z}$, where $r = g/f$.*

Proof. Let $\bar{M} := M/pM, \bar{M}^t := M^t/pM^t$. The exact sequence

$$0 \to V\bar{M} \to \bar{M} \xrightarrow{F} F\bar{M} \to 0$$

yields the exact sequence

$$0 \to V\bar{M}^{i+1} \to \bar{M}^i \xrightarrow{F} F\bar{M}^i \to 0.$$

The Lie algebra representation gives $\dim F\bar{M}^i = n_i$, $\dim V\bar{M}^{i+1} = 2r - n_i$. In the k-vector space $\bar{M}^i$, there are 2 subspaces $F\bar{M}^{i-1}$, $V\bar{M}^{i+1}$ of dimension n_{i-1}, $2r - n_i$ respectively. Let $W^i := F\bar{M}^{i-1} \cap V\bar{M}^{i+1}$. From

$$\dim W^i + \dim(F\bar{M}^{i-1} + V\bar{M}^{i+1}) = \dim F\bar{M}^{i-1} + \dim V\bar{M}^{i+1}$$

we get $\dim W^i + (2r - a_i) = n_{i-1} + 2r - n_i$, or $\dim W^i = a_i + n_{i-1} - n_i$.

The pairing $\bar{M}^t \times \bar{M} \to k$ induces a non-degenerate pairing $\bar{M}^{t,i} \times \bar{M}^i \to k$. The subspace $(F\bar{M}^{i-1})^\perp$ consists of elements killed by V, hence it is $F\bar{M}^{t,i-1}$ because of the short exact sequence

$$0 \to F\bar{M}^{t,i-1} \to \bar{M}^{t,i} \xrightarrow{V} V\bar{M}^{t,i} \to 0.$$

Similarly we can show that $(V\bar{M}^{i+1})^\perp = V\bar{M}^{t,i+1}$. Thus $(W^i)^\perp = F\bar{M}^{t,i-1} + V\bar{M}^{t,i+1}$. By definition, $a_i^t = 2r - \dim(F\bar{M}^{t,i-1} + V\bar{M}^{t,i+1}) = \dim W^i = a_i + n_{i-1} - n_i$. □

For any Dieudonné submodule N of M of index p^h, there is a unique isogeny $\varphi : B \to A$ of degree p^h up to isomorphism with $M(B) = N$. The submodule N is stable under the O_F-action if and only if there is an O_F-action on B such that φ is O_F-linear. And we can take the pull-back polarization for B so that φ is compatible with polarizations. The quotient M/N is canonically isomorphic to $M^*((\ker\varphi)^t)$ because

$$1 \to \ker\varphi \to B \to A \to 1$$

$$1 \to (\ker\varphi)^t \to A^t \to B^t \to 1$$

$$0 \to M^*(B^t) \to M^*(A^t) \to M^*((\ker\varphi)^t) \to 0.$$

For any Dieudonné over-module N of M of index p^h, there is a unique isogeny $\varphi : A \to B$ of degree p^h up to isomorphism with $M(B) = N$. The Dieudonné module N is stable under the O_F-action if and only if there is an O_F-action on B such that φ is O_F-linear. The polarization λ can descend to B if and only if $\langle N, N\rangle \subset W$; and $p^n\lambda$ does if n is large enough. In any case λ descends to a *fractional* polarization λ_B on B, i.e. $n\,\lambda_B$ is a polarization for some $n \in \mathbb{Z}$. The quotient N/M is canonically isomorphic to $M^*((\ker\varphi)^t)$. This is sort of a covariant version of the finite group $\ker\varphi$ although it does not make sense to talk about this.

To each M, the Dieudonné module of a polarized O_F-abelian variety, one can associate a unique Dieudonné lattice chain

$$\cdots \subset M_2 \subset M_1 \subset M_0 \subset M_{-1} \subset M_{-2} \cdots$$

such that $M_0 = M$ and $M_{n+1} = (F, V)M_n$ for all integer n. This Dieudonné lattice chain induces a chain of polarized O_F-isogenies

$$\varphi_n : (A_{n+1}, \lambda_{n+1}, \iota_{n+1}) \to (A_n, \lambda_n, \iota_n) \;\forall\; n$$

where $(A_0, \lambda_0, \iota_0) = (A, \lambda, \iota)$, λ_n is a fractional polarization such that $\varphi_n^* \lambda_n = \lambda_{n+1}$. We call this chain of isogenies the *canonical isogenies* attached to (A, λ, ι).

Remark. (1) The polarized O_F-abelian varieties $(A_n, \lambda_n, \iota_n)$ may not satisfy the freeness condition for their Lie algebras.

(2) The canonical isogenies can be also defined for polarized O_B-abelian varieties where O_B is an order of a more general semi-simple algebra B over $\mathbb{Q}$ with a positive involution.

8. Construction of models

In this section we compute the canonical isogenies of the "generic" principally polarized O_F-abelian varieties in question ($g = f = 4$). Using the information derived from these canonical isogenies, we construct a model attached to each component of the supersingular locus $\mathcal{S}$ as in [17]. This gives a global description of each component. In particular, it gives another proof of the Main theorem (1).

Let S_1 be the set of isomorphism classes of polarized O_F-abelian varieties (A, λ, ι) over $k = \bar{k}$ of dimension 4 with $a(A) = 0202$ of polarization type $(p, 1, p, 1)$. Let S_2 be the set of isomorphism classes of polarized O_F-abelian varieties (A, λ, ι) over k of dimension 4 with $a(A) = 0202$ and the polarization type (p^2, p, p^2, p). Let Λ_i be a complete set of representatives of S_i for $i = 1, 2$. These sets are finite since there are finitely many ways to impose polarizations of fixed degree on a superspecial abelian variety and many ways to impose O_F-action on a fixed polarized abelian variety. There is a natural bijection between S_1 and S_2, sending λ to $(1/p)\lambda$. These sets will correspond to the irreducible components and its cardinality can be realized as class numbers.

As the points in $\mathcal{S}_\tau$ with generic a-number τ form an open dense subset in $\mathcal{S}$ (Proposition 6.3), we only need to determine the information from the canonical isogenies of the these points. Let $(A, \lambda, \iota) \in \mathcal{M}_{1010}(k)$, and let M be its Dieudonné module. Choose a presentation for M as in Proposition 4.1. A straightforward computation gives

$$\begin{aligned}
M_0 = M &= span \left\{ \begin{array}{cccc} X_0 & X_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \end{array} \right\} \\
M_1 = (F, V)M_0 &= span \left\{ \begin{array}{cccc} pX_0 & X_1 & pX_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \end{array} \right\} \\
M_2 = (F, V)M_1 &= span \left\{ \begin{array}{cccc} pX_0 & pX_1 & pX_2 & pX_3 \\ Y_0 & pY_1 & Y_2 & pY_3 \end{array} \right\}.
\end{aligned}$$

We see that there is a polarized O_F-abelian variety (B, λ_B, ι_B) and a polarized O_F-isogeny $\varphi : B \to A$ of degree p^2 such that $\varphi_*(M(B)) = M_1$ and $\ker \varphi \simeq \alpha_p^{1010}$. Clearly, $(B, \lambda_B, \iota_B) \in S_1$. Let $(A, \lambda, \iota) \in \mathcal{S}_{1000}(k)$, and let M be its Dieudonné module. Choose a presentation for M as in Proposition 4.1. Since A is supersingular, $a_0 \equiv 0 \mod p$. A straightforward computation gives

$$M_0 = M = span \left\{ \begin{matrix} X_0 & X_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \end{matrix} \right\}$$

$$VM_0 = span \left\{ \begin{matrix} pX_0 & pX_1 & pX_2 & pX_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \end{matrix} \right\}$$

$$M_1 = (F,V)M_0 = span \left\{ \begin{matrix} pX_0 & X_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \end{matrix} \right\}$$

$$VM_1 = span \left\{ \begin{matrix} pX_0 & pX_1 & pX_2 & pX_3 \\ Y_0 & Y_1 & Y_2 & pY_3 \end{matrix} \right\}$$

$$M_2 = (F,V)M_1 = span \left\{ \begin{matrix} pX_0 & pX_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & pY_3 \end{matrix} \right\}$$

$$VM_2 = span \left\{ \begin{matrix} p^2X_0 & pX_1 & pX_2 & pX_3 \\ Y_0 & Y_1 & pY_2 & pY_3 \end{matrix} \right\}$$

$$M_3 = (F,V)M_2 = span \left\{ \begin{matrix} p^2X_0 & pX_1 & pX_2 & X_3 \\ Y_0 & Y_1 & pY_2 & pY_3 \end{matrix} \right\}$$

$$(\because FX_3 = -Y_0 + pa_0X_0 \in span\{Y_0, p^2X_0\})$$

$$VM_3 = span \left\{ \begin{matrix} p^2X_0 & p^2X_1 & pX_2 & pX_3 \\ Y_0 & pY_1 & pY_2 & p^2Y_3 \end{matrix} \right\}$$

$$M_4 = (F,V)M_2 = span \left\{ \begin{matrix} p^2X_0 & p^2X_1 & pX_2 & pX_3 \\ Y_0 & Y_1 & pY_2 & p^2Y_3 \end{matrix} \right\}.$$

From the result above and using Lemma 7.1, we obtain

$$(8.1)\quad \begin{array}{llll} a(M_0) = 1000, & \mathrm{Lie}(A_0) = k \cdot 1111, & a(M_0^t) = 1000, & \text{pol.type} = (1,1,1,1) \\ a(M_1) = 0101, & \mathrm{Lie}(A_1) = k \cdot 0112, & a(M_1^t) = 2000, & \text{pol.type} = (p,1,1,1) \\ a(M_2) = 1020, & \mathrm{Lie}(A_2) = k \cdot 1021, & a(M_2^t) = 1101, & \text{pol.type} = (p,p,1,p) \\ a(M_3) = 0202, & \mathrm{Lie}(A_3) = k \cdot 0202, & a(M_3^t) = 2020, & \text{pol.type} = (p^2,p,p^2,p). \end{array}$$

The symbol $k \cdot 0112$ is an abbreviation for $0 \oplus k \oplus k \oplus k^2$, the decomposition with respect to the O_F-action.

Next step is to use this information to construct a model and to describe each irreducible component. We only need to deal with the cases $\tau = 1010$ and $\tau = 1000$, due to the symmetry.

Definition. (1) Let $\tau = 1010$. An *abelian scheme over S of τ-level structure* consists of data $(\varphi : (A_1, \lambda_1, \iota_1) \to (A_0, \lambda_0, \iota_0))$ where

(i) Each $(A_i, \lambda_i, \iota_i)$ is a polarized O_F-abelian scheme over S and φ is a polarized O_F-isogeny with degree p^2.

(ii) $(A_1, \lambda_1, \iota_1) \simeq \eta \times S$ for some $\eta \in S_1$.

(iii) The subgroup $\ker \varphi$ is an α-group of type 1010.

Two abelian schemes of τ-level structure $(\varphi : (A_1, \lambda_1, \iota_1) \to (A_0, \lambda_0, \iota_0))$ and $(\varphi' : (A_1', \lambda_1', \iota_1') \to (A_0', \lambda_0', \iota_0'))$ are isomorphic if there exist isomorphisms $\theta_i : A_i' \to A_i$ for $i = 0, 1$, as polarized O_F-abelian varieties.

(2) Let $\tau = 1000$. An *abelian scheme over S of τ-level structure* consists of data $(\varphi_n : (A_{n+1}, \lambda_{n+1}, \iota_{n+1}) \to (A_n, \lambda_n, \iota_n); n = 0, 1, 2)$ where

(i) Each $(A_i, \lambda_i, \iota_i)$ is a polarized O_F-abelian scheme over S and each φ_i is a polarized O_F-isogeny.
(ii) $(A_3, \lambda_3, \iota_3) \simeq \eta \times S$ for some $\eta \in S_2$.
(iii) The subgroups $\ker \varphi_2$, $\ker \varphi_1$, and $\ker \varphi_0$ are α-groups of type 1020, 0101 and 1000, respectively.

We define similarly for two abelian schemes of τ-level structure being isomorphic.

Case $\tau = 1010$. Let $\eta \in \Lambda_1$. We define

$$\mathcal{N}_{\tau,\eta}(S) = \left\{ \begin{array}{l} \text{isomorphism classes of abelian schemes over } S \\ \text{of } \tau\text{-level structure s.t. } (A_1, \lambda_1, \iota_1) = \eta \times S \end{array} \right\},$$

the moduli space parameterizing the abelian varieties of τ-level structure with the source η. Two objects $(\varphi : (A_1, \lambda_1, \iota_1) \to (A_0, \lambda_0, \iota_0))$ and $(\varphi' : (A'_1, \lambda'_1, \iota'_1) \to (A'_0, \lambda'_0, \iota'_0))$ are isomorphic if there exists an isomorphism $\theta_i : A_i \to A'_i$ of polarized O_F-abelian varieties for each $i = 0, 1$ such that

$$\begin{tikzcd} A_1 \arrow[r, "\varphi_0"] \arrow[d, equal, "\theta_1"] & A_0 \arrow[d, "\theta_0"] \\ A'_1 \arrow[r, "\varphi'_0"] & A'_0 \end{tikzcd}$$

Note that we rigidify the source, hence there is no non-trivial automorphism in $\mathcal{N}_{\tau,\eta}$.

Lemma 8.1. *$\mathcal{N}_{\tau,\eta}$ is representable by $\mathbf{P}^1 \times \mathbf{P}^1$.*

Proof. Every abelian scheme of τ-level structure is uniquely determined by $\ker \varphi$, which is an α-subgroup of type 1010 in $\alpha(\eta) = \alpha_p^{2020} \times S$. Therefore there is one-one correspondence between $\mathcal{N}_{\tau,\eta}(S)$ and α-subgroups of type 1010 in $\alpha_p^{2020} \times S$. The latter are isomorphism classes of rank two quotient bundles of $\mathcal{O}_S^2 \oplus 0 \oplus \mathcal{O}_S^2 \oplus 0$ (O_F-decomposition) of type 1010, which is representable by $\mathbf{P}^1 \times \mathbf{P}^1$. $\square$

Since $\mathcal{N}_{\tau,\eta}$ is irreducible, the last projection $\mathrm{pr} : \mathcal{N}_{\tau,\eta} \to \mathcal{M}$ induces a morphism

$$\coprod_{\eta \in \Lambda_1} \mathcal{N}_{\tau,\eta} \to \bar{\mathbb{S}}_\tau,$$

which is proper and dominant. For each abelian variety x in $\bar{\mathbb{S}}_\tau(k)$ there is a unique abelian variety y of τ-level structure, up to isomorphism, such that $\mathrm{pr}(y) = x$. Therefore we have a morphism

$$\coprod_{\eta \in \Lambda_1} \mathcal{N}_{\tau,\eta}/G_\eta \to \bar{\mathbb{S}}_\tau,$$

which restricts to each component $\mathcal{N}_{\tau,\eta}/G_\eta$ is an isomorphism, where G_η is the automorphism group of η. We have thus shown

Proposition 8.2. *The algebraic stack $\overline{\mathcal{M}}_{1010} = \bar{\mathbb{S}}_{1010}$ has $|S_1|$ irreducible components and each component is isomorphism to a quotient of $\mathbf{P}^1 \times \mathbf{P}^1$ by a finite group.*

We will describe the preimage of higher a-number strata via pr. We first need a lemma.

Lemma 8.3. *Let $(B, \lambda_B, \iota_B) \in S_1$ and N be its Dieudonné module. Then there is a basis $\{X_i, Y_i\}_{0 \le i < 4}$ for N such that*

- $X_i, Y_i \in N^i$.
- $\langle X_i, Y_i\rangle = p\langle X_{i+1}, Y_{i+1}\rangle = p$ *for* $i = 0, 2$.
- $\begin{array}{llll} FX_0 = -pY_1, & FX_1 = -Y_2, & FX_2 = -pY_3, & FX_3 = -Y_0 \\ FY_0 = pX_1, & FY_1 = X_2, & FY_2 = pX_3, & FY_3 = X_0. \end{array}$

Proof. By Proposition 8.2 there is a superspecial principally polarized O_F-abelian variety (A, λ, ι) and an O_F-isogeny $B \to A$ such that $\ker\varphi \simeq \alpha_p^{1010}$. By Lemma 4.3 we can choose a basis $\{X_i', Y_i'\}_{0\le i<4}$ for the Dieudonné module M of A such that $FX_i' = -Y_{i+1}'$, $FY_i' = pX_{i+1}'$ and all the pairings are zero except $\langle X_i, Y_i\rangle = 1$. Since $M/N = k\oplus 0\oplus k\oplus 0$, N is generated by $X_i := pX_i'$ $(i = 0, 2)$, $X_i = X_i'$ $(i = 1, 3)$ and $Y_i = Y_i'$ $(\forall\, 0 \le i < 4)$. This basis $\{X_i, Y_i\}$ is the desired one. ☐

Let Z be an irreducible component of $\overline{\mathcal{S}}_{1010}$. Let $\eta = (B, \lambda_B, \iota_B) \in \Lambda_1$ such that pr : $\mathbf{P}^1\times\mathbf{P}^1 = \mathcal{N}_{\tau,\eta} \to Z$. We can choose a presentation for N as in Lemma 8.3. Then

$$N_{-1} = (F, V)^{-1}N = span\left\{\begin{array}{cccc} \frac{1}{p}X_0 & X_1 & \frac{1}{p}X_2 & X_3 \\ \frac{1}{p}Y_0 & Y_1 & \frac{1}{p}Y_2 & Y_3 \end{array}\right\}.$$

For any point $p = ((x_0, y_0), (x_2, y_2)) \in \mathbf{P}^1 \times \mathbf{P}^1(k)$, the Dieudonné module of the image pr(p) in Z is

$$M_p = \langle N, \frac{\tilde{x}_0}{p}X_0 + \frac{\tilde{y}_0}{p}Y_0, \frac{\tilde{x}_2}{p}X_2 + \frac{\tilde{y}_2}{p}Y_2\rangle$$

where $\tilde{x}_i$ and $\tilde{y}_i$ are any lifts of x_i and y_i. Note that M_p is independent of the choice of the liftings. For simplicity, we write the affine coordinates $x = x_0/y_0,\ y = x_2/y_2$:

$$M_{(x,y)} = span\left\{\begin{array}{cccc} X_0 & X_1 & X_2 & X_3 \\ V_0 & Y_1 & V_2 & Y_3 \end{array}\right\} \tag{8.2}$$

where $V_0 := \frac{\tilde{x}}{p}X_0 + \frac{1}{p}Y_0,\ V_2 := \frac{\tilde{y}}{p}X_2 + \frac{1}{p}Y_2$. A simple computation shows that

$$a_0(M_{(x,y)}) = a_2(M_{(x,y)}) = 1$$

$$((F, V)\bar{M}_{(x,y)})^1 = span\{-x^pY_1 + X_1, y^{p^{-1}}Y_1 - X_1\}$$

$$((F, V)\bar{M}_{(x,y)})^3 = span\{-y^pY_3 + X_3, x^{p^{-1}}Y_3 - X_3\}.$$

Therefore $a_1(M_{(x,y)}) = 1 \iff x^{p^2} = y$ and $a_3(M_{(x,y)}) = 1 \iff x = y^{p^2}$. We deduce the points of $\mathbf{P}^1\times\mathbf{P}^1$ whose images in $\mathcal{S}$ have higher a-numbers:

Proposition 8.4. *Let $pr : \mathcal{N}_{\tau,\eta} = \mathbf{P}^1\times\mathbf{P}^1 \to Z$ be as above.*

(1) The point in $\mathbf{P}^1\times\mathbf{P}^1$ whose image lies in $(Z\cap\overline{\mathcal{M}}_{1110})$ is exactly in $\{(x, x^{p^2})\} \subset \mathbf{P}^1\times\mathbf{P}^1$; the point in $\mathbf{P}^1\times\mathbf{P}^1$ whose image lies in $(Z\cap\overline{\mathcal{M}}_{1011})$ is exactly in $\{(y^{p^2}, y)\} \subset \mathbf{P}^1\times\mathbf{P}^1$. In particular, $Z\cap\overline{\mathcal{M}}_{1110}$ (or $Z\cap\overline{\mathcal{M}}_{1011}$) is a finite quotient of $\mathbf{P}^1$, whose geometric quotient is isomorphic to $\mathbf{P}^1$.

(2) The stratum $\overline{\mathcal{M}}_{1110}$ (or $\overline{\mathcal{M}}_{1011}$) has $|S_1|$ connected components and the each connected component is a finite quotient of $\mathbf{P}^1$.

(3) The set of superspecial points on $\mathbf{P}^1\times\mathbf{P}^1$ is $\{(x, x^{p^2})\,|x \in \mathbb{F}_{p^4}\} \simeq \mathbf{P}^1(\mathbb{F}_{p^4})$.

Next we describe the action of G_η on $\mathbf{P}^1\times\mathbf{P}^1$. Since $G_\eta\subset \mathrm{Aut}_{O_F}(B)\subset \mathrm{Aut}_{O_F}(N)$, we first compute $\mathrm{End}_{O_F}(N)$. The module N is generated by X_1,Y_1,X_3,Y_3. Write $g\in\mathrm{End}_{O_F}(N)$ by

$$g\cdot\begin{pmatrix}X_1\\ Y_1\end{pmatrix}=\begin{pmatrix}a_1&b_1\\ c_1&d_1\end{pmatrix}\begin{pmatrix}X_1\\ Y_1\end{pmatrix},\quad g\cdot\begin{pmatrix}X_3\\ Y_3\end{pmatrix}=\begin{pmatrix}a_3&b_3\\ c_3&d_3\end{pmatrix}\begin{pmatrix}X_3\\ Y_3\end{pmatrix}$$

From the relation $F^2X_1+pX_3=0$ and $F^2Y_1+pY_3=0$, we obtain

$$\begin{pmatrix}a_3&b_3\\ c_3&d_3\end{pmatrix}=\begin{pmatrix}a_1&b_1\\ c_1&d_1\end{pmatrix}^{(2)}.$$

Let $g\in G_\eta$ and $\rho=\begin{pmatrix}a&b\\ c&d\end{pmatrix}\in\mathrm{Aut}_{O_F}(N/pN,\langle\,,\rangle)=\mathrm{SL}_2(\mathbb{F}_{p^4})$ be the image of g. Then

$$g\cdot(x,y)=(\rho x,\rho^{(2)}y),\quad\forall\,(x,y)\in\mathbf{P}^1\times\mathbf{P}^1.$$

Lemma 8.5. *The G_η-orbits of the superspecial points $\mathbf{P}^1(\mathbb{F}_{p^4})$ one-one correspond the superspecial points on Z.*

Proof. To assert that we need to prove for any two morphisms $\alpha_1:(B,\lambda_B,\iota_B)\to(A_1,\lambda_1,\iota_1)$ and $\alpha_2:(B,\lambda_B,\iota_B)\to(A_2,\lambda_2,\iota_2)$ such that there exits an isomorphism $\beta:A_1\to A_2$ of polarized O_F-abelian varieties, then the composition $\alpha_2^{-1}\beta\alpha_1$ should be an isomorphism.

$$\begin{array}{ccc}B&\xrightarrow{\alpha_1}&A_1\\ \big\downarrow{\scriptstyle ?}&&\big\downarrow{\scriptstyle\beta}\\ B&\xrightarrow{\alpha_2}&A_2\end{array}$$

In our situation the degree of α_2 is a power of p, hence it suffices to verify whether the quasi-isogeny map $\alpha_2^{-1}\beta\alpha_1$ induces a morphism on their Dieudonné modules. Let

$$\begin{array}{ccc}N&\xrightarrow{\alpha_1}&M_1\\ \big\downarrow{\scriptstyle ?}&&\big\downarrow{\scriptstyle\beta}\\ N&\xrightarrow{\alpha_2}&M_2\end{array}$$

be the corresponding Dieudonné modules that M_1 and M_2 are submodules of N_{-1}. Since β is Dieudonné , it induces an isomorphism form $M_1/(F,V)M_1$ to $M_2/(F,V)M_2$. Both of them are isomorphic to $k\cdot 1111$. The image $\bar N$ of N in either vector space is $k\cdot 1010$. Since the morphism β is also compatible with the O_F-action, it preserves $\bar N$, hence N. $\square$

From the smoothness of $\bar{\mathcal{S}}_{1010}$, $\bar{\mathcal{S}}_{1010}$ is a disjoint union of irreducible components. Every superspecial point lies in exact one component of $\bar{\mathcal{S}}_{1010}$. The number is superspecial points on $\mathcal{S}$ is

$$\sum_{\eta\in\Lambda_1}\#(\mathbf{P}^1(\mathbb{F}_{p^4})/G_\eta).\tag{8.3}$$

Remark. A classification of possible G_η will help compute the number of superspecial points on $\mathcal{S}$. It seems to me a delicate problem, so we leave this aside in this work.

Case $\tau=1000$. Let $\eta\in\Lambda_2$. We define

$$\mathcal{N}_{\tau,\eta}(S)=\left\{\begin{array}{l}\text{isomorphism classes of abelian schemes over } S\\ \text{of }\tau\text{-level structure s.t. }(A_3,\lambda_3,\iota_3)=\eta\times S\end{array}\right\},$$

the moduli space parameterizing the abelian varieties of τ-level structure with the source η. Two objects $(A_3 \overset{\varphi_2}{\to} A_2 \overset{\varphi_1}{\to} A_1 \overset{\varphi_0}{\to} A_0)$ and $(A'_3 \overset{\varphi'_2}{\to} A'_2 \overset{\varphi'_1}{\to} A'_1 \overset{\varphi'_0}{\to} A'_0)$ are isomorphic if there exists an isomorphism $\theta_i : A_i \to A'_i$ of polarized O_F-abelian varieties for each $i = 0, 1, 2, 3$ such that

$$\begin{tikzcd} A_3 \arrow[r, "\varphi_2"] \arrow[d, equal, "\theta_3"] & A_2 \arrow[r, "\varphi_1"] \arrow[d, "\theta_2"] & A_1 \arrow[r, "\varphi_0"] \arrow[d, "\theta_1"] & A_0 \arrow[d, "\theta_0"] \\ A'_3 \arrow[r, "\varphi'_2"] & A'_2 \arrow[r, "\varphi'_1"] & A'_1 \arrow[r, "\varphi'_0"] & A'_0. \end{tikzcd}$$

Note that we rigidify the source, hence there is no non-trivial automorphism in $\mathcal{N}_{\tau,\eta}$.

Lemma 8.6. *$\mathcal{N}_{\tau,\eta}$ is representable by a projective scheme over k.*

Proof. Consider the truncated τ-level structure: for $m = 0, 1, 2$, denote by $\mathcal{N}^{(m)}_{\tau,\eta}$ the moduli space parameterizing a chain of isogenies $(A_3 \to A_2 \to \cdots \to A_m)$ of a τ-level structure $(A_3 \to A_2 \to \cdots \to A_0)$. By the same proof in Lemma 8.1, $\mathcal{N}^{(2)}_{\tau,\eta}$ is representable by $\mathbf{P}^1$. Suppose $\mathcal{N}^{(m)}_{\tau,\eta}$ is representable by a projective scheme S_m and let $(\varphi_i : \mathbf{A}_{i+1} \to \mathbf{A}_i)_{i \geq m}$ be the universal family. The fibre of the truncated morphism $t : \mathcal{N}^{(m-1)}_{\tau,\eta} \to \mathcal{N}^{(m)}_{\tau,\eta}$ is given by the α-subgroups of $\mathbf{A}_m[F] \times_{S_m} T$ of a prescribed type, hence representable by a relatively projective scheme, cf [17], Section 3.5. □

The last projection pr gives a proper morphism $\mathcal{N}_{\tau,\eta} \to \mathcal{S}$. If $\mathcal{N}_{\tau,\eta}$ is irreducible, then the last projection pr induces a morphism

$$\coprod_{\eta \in \Lambda_2} \mathcal{N}_{\tau,\eta} \to \bar{\mathcal{S}}_\tau,$$

which is proper and dominant.

We will prove the irreducibility of $\mathcal{N}_{\tau,\eta}$ by representing it as a $\mathbf{P}^1$-fibration over $\mathbf{P}^1$, see Corollary 8.9 below.

Lemma 8.7. *Let $(A_3 \overset{\varphi_2}{\to} A_2 \overset{\varphi_1}{\to} A_1 \overset{\varphi_0}{\to} A_0)$ be an abelian variety of τ-level structure over k. Then $a(A_1)$=01x1, $a(A_1^t)$=20x0, $a(A_2) = 1020$ and $a(A_2^t) = 1101$, where $x = 0$ or 1.*

Proof. Let M_i be the corresponding Dieudonné modules, and let $M_i = M_i^0 \oplus \cdots \oplus M_i^3$ be the decomposition with respect to O_F-action. Since $\ker \varphi_2 = \alpha_p^{1020}$, $M_2/M_3 = k \oplus 0 \oplus k^2 \oplus 0$. So $M_2^0/M_3^0 = k$, $M_2^1 = M_3^1$, $M_2^2 = (1/p)M_3^2$ and $M_2^3 = M_3^3$. Since $a(M_3) = 0202$ and $((F,V)M_2)^0 = FM_3^3 + VM_3^1 = M_3^0$, hence $a(A_2) = 1020$. And $VM_2 = M_3^0 \oplus M_3^1 \oplus M_3^2 \oplus VM_2^0$, hence $\mathrm{Lie}(A_2) = 1021$, so $a(A_2^t) = 1101$.

Since the polarization type of M_1 is $(p, 1, 1, 1)$, we have $a(A_1)$ =01x1. We also check that $\mathrm{Lie}(M_1) = 0112$, thus $a(A_1^t)$ =20x0. □

Corollary 8.8. *The stratum $\mathcal{S}_{1100}$ is empty.*

Proof. We will show that any $(A_0, \lambda_0, \iota_0) \in \mathcal{S}_{1100}(k)$ does not admit a τ-level structure. Let (A_i) be an abelian variety of τ-level structure with $a(A_0) = 1100$ and M_i be the corresponding Dieudonné modules. From $M_0/M_1 = k \cdot 1000$ and $M_1/M_2 = k \cdot 0101$, we compute that

$$M_0 = span \left\{ \begin{array}{cccc} X_0 & X_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \end{array} \right\}$$

$$M_1 = span\left\{ \begin{array}{cccc} pX_0 & X_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \end{array} \right\}$$

$$M_2 = span\left\{ \begin{array}{cccc} pX_0 & pX_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & pY_3 \end{array} \right\}$$

$$(F,V)M_2 = span\left\{ \begin{array}{cccc} p^2X_0 & pX_1 & pX_2 & X_3 \\ pY_0 & Y_1 & pY_2 & pY_3 \end{array} \right\}.$$

and $a(M_2) = 2020$. This contradicts with Lemma 8.7. By Lemma 6.2, this completes the corollary. ☐

Corollary 8.9.

(1) *For any $(A, \lambda, \iota) \in \bar{\mathcal{S}}_\tau(k)$ there is a unique abelian variety $(\varphi_n, (A_n, \lambda_n, \iota))$ of τ-level structure, up to isomorphism, such that $(A_0, \lambda_0, \iota_0) \simeq (A, \lambda, \iota)$.*

(2) *$\mathcal{N}_{\tau,\eta}$ is a $\mathbf{P}^1$-fibration over $\mathbf{P}^1$.*

Proof. For (1), it is immediate just to compare the a-numbers of $\ker \varphi_n$ and A_n.

(2) First, $\mathcal{N}^{(2)}_{\tau,\eta} = \mathbf{P}^1$. By Lemma 8.7, the map $\mathcal{N}^{(1)}_{\tau,\eta}(k) \to \mathcal{N}^{(2)}_{\tau,\eta}(k)$ is bijective and for each point $\operatorname{Spec} k \to \mathcal{N}^{(1)}_{\tau,\eta}$ the fibre $\mathcal{N}^{(0)}_{\tau,\eta} \times_{\mathcal{N}^{(1)}_{\tau,\eta}} \operatorname{Spec} k$ is isomorphic to $\mathbf{P}^1$. So the morphism $\mathcal{N}^{(0)}_{\tau,\eta} \to \mathcal{N}^{(2)}_{\tau,\eta}$ gives the desired fibration structure. ☐

Proposition 8.10. *The algebraic stack $\bar{\mathcal{S}}_\tau$ has $|S_2|$ irreducible components. Each irreducible component is isomorphic to $\mathcal{N}_{\tau,\eta}/G_\eta$ for a unique η.*

Proof. We proved (Corollary 8.9) that the projection pr induces a bijection between $\mathcal{N}_{\tau,\eta}(k)/G_\eta$ and $Z_{\tau,\eta}(k)$, where $Z_{\tau,\eta}$ is the irreducible component of $\bar{\mathcal{S}}_\tau$ corresponding to η. It is easy to check that pr is generically étale by a tangent space calculation. Therefore $\mathrm{pr} : \mathcal{N}_{\tau,\eta}/G_\eta \to Z_{\tau,\eta}$ is an isomorphism. ☐

We do the tangent space calculation in the following lemma.

Lemma 8.11. *The projection* pr *above is generically étale.*

Proof. It is enough to check the point where $a(M_0) = 1000$. Use the bases for M_i appeared in the canonical isogenies. Let $\widetilde{Fil}_i \subset \widetilde{M}_i$ be the first order universal deformations. We have

$$\widetilde{Fil}_0 = \langle Y_i + t_{i0}X_i \rangle$$

$$\widetilde{Fil}_1 = \langle pX_0, Y_0, Y_1 + t_{11}X_1, Y_2 + t_{21}X_2 \rangle$$

$$\widetilde{Fil}_2 = \langle Y_0 + t_{02}pX_0, pX_1, Y_1, pY_3 + t_{32}X_3 \rangle$$

$$\widetilde{Fil}_3 = \langle p^2X_0, Y_0, pX_2, pY_2 \rangle.$$

From $\widetilde{Fil}_{i+1} \subset \widetilde{Fil}_i$ we get the relations $t_{00} = 0, t_{10} = t_{11}, t_{20} = t_{21}, t_{11} = 0, t_{32} = 0$, and $t_{02} = 0$. The projection map induces an isomorphism on the tangent spaces. ☐

S. Shatz asked which ruled surface the model $\mathcal{N}_{\tau,\eta}$ is in the classification of algebraic surfaces. The answer turns out to be F_2, a Hirzebruch surface [1, Chapter IV]. We thank him for this interesting question.

Proposition 8.12. *The model $\mathcal{N}_{\tau,\eta}$ is isomorphic to the Hirzebruch surface F_2.*

Proof. Let M_3 be the Dieudonné module of η. Write $\mathcal{N}^{(2)}_{\tau,\eta} = \mathbf{P}^1 = \{[x_0,x_1]\}$. Let $U_0 = \{[x_0,x_1]; x_0 \neq 0\}$ and $x = x_1/x_0$ be its coordinate. Let $U_1 = \{[x_0,x_1]; x_1 \neq 0\}$ and $x' = x_0/x_1$ be its coordinate. Choose a presentation of M_3 as Lemma 8.3. For $[x_0,x_1] \in \mathbf{P}^1(k)$, the Dieudonné modules corresponding to this point are

$$M_3 = span\left\{ \begin{matrix} X_0 & X_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \end{matrix} \right\}$$

$$M_2 = span\left\{ \begin{matrix} X_0, Y_0 & X_1 & \frac{1}{p}X_2 & X_3 \\ V_0 & Y_1 & \frac{1}{p}Y_2 & Y_3 \end{matrix} \right\},$$

where $V_0 = \tilde{x}_0(\frac{1}{p}X_0) + \tilde{x}_1(\frac{1}{p}Y_0)$, $\tilde{x}_0$ and $\tilde{x}_1$ are any liftings of x_0 and x_1 in W, for example, the Teichmüller ones. Then

$$M_1 = span\left\{ \begin{matrix} X_0, Y_0 & X_1, Y_1 & \frac{1}{p}X_2 & X_3, \frac{1}{p}VV_0 \\ V_0 & \frac{1}{p}FV_0 & \frac{1}{p}Y_2 & Y_3 \end{matrix} \right\}.$$

Let $\pi : \mathcal{N}_{\tau,\eta} \to \mathbf{P}^1$ be the projection map. Any point in the fibre over $[x_0,x_1]$ corresponds to a subspace M_0^0

$$M_1^0 \subset M_0^0 \subset ((F,V)^{-1}M_1)^0,$$

and $((F,V)^{-1}M_1)^0 = span\{\frac{1}{p}X_0, \frac{1}{p}Y_0, \frac{1}{p}V_0\}$. Set $X := X_0$ and $Y := Y_0$. then we have $M_1^0 = span\{X, Y, \tilde{x}_0 X + \tilde{x}_1 Y\}$.

Over U_1: $M_1^0 = span\{X, Y'\}$, where $Y' = \tilde{x}'\frac{1}{p}X + \frac{1}{p}Y$. We have an isomorphism $\alpha_1 : U_1 \times \mathbf{P}^1 \xrightarrow{\sim} \mathcal{N}_{\tau,\eta}$ by sending $(x', [y_0', y_1'])$ to

$$M_0^0 = span\{X, Y', \tilde{y}_0'(\frac{1}{p}X) + \tilde{y}_1'(\frac{1}{p}Y')\}.$$

Over U_0: $M_1^0 = span\{X', Y\}$, where $X' = \frac{1}{p}X + \tilde{x}\frac{1}{p}Y$. We have an isomorphism $\alpha_0 : U_0 \times \mathbf{P}^1 \xrightarrow{\sim} \mathcal{N}_{\tau,\eta}$ by sending $(x, [y_0, y_1])$ to

$$M_0^0 = span\{X', Y, \tilde{y}_0(\frac{1}{p}X') + \tilde{y}_1(\frac{1}{p}Y)\}.$$

Over $U_0 \cap U_1$, we have

$$M_1^0 = span\{X, Y'\} = span\{X', Y\}$$

with relation $Y' = \tilde{x}'X'$ and $X = pX' - \tilde{x}Y$. Using this relation, the image of $(x', [y_0', y_1']) \in U_1$ in $\mathcal{N}_{\tau,\eta}$ is

$$span\{X, Y', \tilde{y}_0'(\frac{1}{p}X) + \tilde{y}_1'(\frac{1}{p}Y')\} = span\{X', Y, \tilde{x}'\tilde{y}_0(\frac{1}{p}X') - \tilde{x}\tilde{y}_1(\frac{1}{p}Y)\}.$$

Therefore $\mathcal{N}_{\tau,\eta} = U_0 \times \mathbf{P}^1 \coprod U_1 \times \mathbf{P}^1/\sim$, where $(x, [y_0, y_1]) \sim (x', [y_0', y_1'])$ if and only if $xx' = 1$ and $y_0y_0'x + y_1y_1'x' = 0$. Choose another trivialization of $\mathcal{N}_{\tau,\eta}|_{U_0}$ by precomposing the isomorphism $(x, [y_0, y_1]) \to (x, [y_0, -y_1])$. We have $\mathcal{N}_{\tau,\eta} = U_0 \times \mathbf{P}^1 \coprod U_1 \times \mathbf{P}^1/\sim$, where $(x, [y_0, y_1]) \sim (x', [y_0', y_1'])$ if and only if $xx' = 1$ and $y_0y_0'x - y_1y_1'x' = 0$. Set $y := y_1/y_0$ and $y' = y_1'/y_0'$. The equivalent relation above is $xx' = 1$ and $y' = x^2y$. Hence $\mathcal{N}_{\tau,\eta}$ is isomorphic to the Hirzebruch surface F_2.

$\square$

9. Superspecial points as class numbers

In this section we interpret the number of components, $|S_1|$, which appears in the previous section, as a class number. The main result is Theorem 9.6. The ground field k is an algebraically closed field of characteristic p.

Proposition 9.1. *Let $(A_1, \lambda_1, \iota_1)$ and $(A_2, \lambda_2, \iota_2)$ be two supersingular polarized O_F-abelian varieties. Then there exists an O_F-linear isogeny $\varphi : A_1 \to A_2$ such that $\varphi^*\lambda_2 = q\lambda_1$ for some $q \in \mathbb{Q}^\times$, i.e. φ is a $\mathbb{Q}$-polarized O_F-linear isogeny (see the definition in Section 2).*

Proof. Since they are supersingular, we can choose an isogeny $\varphi : A_1 \to A_2$. The action of O_F on A_2 induces an action ι_1' of O_F on "$A_1 \otimes \mathbb{Q}$", an abelian variety up to isogeny, by

$$\begin{array}{ccccc} A_1 & \xrightarrow{g} & A_1 & \xrightarrow{\varphi} & A_2 \\ \downarrow{\scriptstyle \iota_1(a)} & & \downarrow{\scriptstyle \iota_1'(a)} & & \downarrow{\scriptstyle \iota_2(a)} \\ A_1 & \xrightarrow{g} & A_1 & \xrightarrow{\varphi} & A_2 \end{array}$$

$\iota_1'(a) := \varphi^{-1}\iota_2(a)\varphi \in \mathrm{End}(A_1) \otimes \mathbb{Q}$ for $a \in O_F$. Therefore we have two morphisms $\iota_1, \iota_1' : F \to \mathrm{End}(A_1) \otimes \mathbb{Q}$. Since the center of $\mathrm{End}(A_1) \otimes \mathbb{Q}$ is $\mathbb{Q}$, by Noether-Skolem's theorem, there exists $g \in \mathrm{End}(A_1) \otimes \mathbb{Q}$ such that $\iota_1 = g^{-1}\iota_1' g$. The element g may be chosen in $\mathrm{End}(A_1)$. Replacing φ by φg, we obtain an O_F-linear isogeny between A_1 and A_2.

Next we can adjust φ to be compatible with the polarizations up to a rational number. Let $*_1$ be the Rosati involution induced by λ_1 and $*_1'$ the one induced by $\lambda_1' := \varphi^*\lambda_2$. The involutions $*_1$ and $*_1'$ are positive on $\mathrm{End}(A_1) \otimes \mathbb{Q}$. By Noether-Skolem's theorem, there is a positive element $c \in \mathrm{End}(A_1)$ such that $*_1 = c *_1' c^{-1}$.

Lemma 9.2. *The element c can be chosen in $\mathrm{End}_{O_F}(A_1)$.*

Proof. We can restrict $*_1$ and $*_1'$ on $\mathrm{End}_{O_F}(A_1) \otimes \mathbb{Q}$ since both polarizations λ_1 and λ_1' are O_F-linear. Then there exists $c \in \mathrm{End}_{O_F}(A_1)$ such that $*_1 = c *_1' c^{-1}$ on $\mathrm{End}_{O_F}(A_1) \otimes \mathbb{Q}$. So they are the same, as involutions on $\mathrm{End}(A_1) \otimes \mathbb{Q}$, up to an inner conjugation by an integral element commuting with all elements in $\mathrm{End}_{O_F}(A_1)$, which is in O_F so as in $\mathrm{End}_{O_F}(A_1)$. Therefore c can be chosen in $\mathrm{End}_{O_F}(A_1)$. ☐

We recall some basic properties

- If φ is O_F-linear isogeny then $\varphi^*\lambda$ is O_F-linear $\iff$ λ is O_F-linear.
- If λ is an O_F-linear polarization then the Rosati involution preserves the subalgebra $\mathrm{End}_{O_F}(A) \otimes \mathbb{Q}$.
- If two polarizations λ_1 and λ_2 induce the same Rosati involution on $\mathrm{End}(A) \otimes \mathbb{Q}$, then $\lambda_2 = q\lambda_1$ for some $q \in \mathbb{Q}$.
- Let $\phi \in \mathrm{End}_{O_F}(A)$. Let $*_1$ and $'$ be the involutions induced by λ and $\phi^*\lambda$, respectively. Then $x' = (\phi^{*_1}\phi)^{-1}x^{*_1}(\phi^{*_1}\phi)$.

We check the last statement in the following:

$$\begin{aligned} x' &= (\phi^*\lambda)^{-1}x^t(\phi^*\lambda) = \phi^{-1}\lambda^{-1}(\phi^t)^{-1}x^t\phi^t\lambda\phi \\ &= \phi^{-1}(\phi x\phi^{-1})^{*_1}\phi = (\phi^{*_1}\phi)^{-1}x^{*_1}(\phi^{*_1}\phi). \end{aligned}$$

The question is reduced to asking: is the map $a \to a^{*_1}a$ surjective from $\mathrm{End}_{O_F}(A_1)$ to the positive elements of itself? In our case, the semi-simple algebra is $\mathrm{End}_{O_F}(A_1) \otimes \mathbb{Q} = F \otimes \mathrm{End}(E)$, where E is a supersingular elliptic curve. The positive element $c = c^{*_1}$ is a totally positive element in F. The assertion of the surjectivity of the norm map from $F \otimes \mathrm{End}(E)$ to F^+ is a well-known consequence of the Hasse principle. See a proof in the lemma below.

Let $g \in \mathrm{End}_{O_F}(A_1)$ such that $c = g^{*_1} g$, and replace φ by φg. Therefore we produce an O_F-linear isogeny φ from A_1 to A_2 such that $\varphi^*\lambda_2$ and λ_1 induce the same Rosati involution on $\mathrm{End}(A_1) \otimes \mathbb{Q}$, hence $\varphi^*\lambda_2 = q\lambda_1$ for some $q \in \mathbb{Q}$. This proves the proposition.

□

Lemma 9.3. *Let $D = \mathrm{End}(E) \otimes \mathbb{Q}$. The reduced norm map form $F \otimes D$ to F^+ is surjective.*

Proof. Let $c \in F^+$. Consider the short exact sequence of reductive algebraic groups

$$1 \to H \longrightarrow (F \otimes D)^\times \xrightarrow{N} F^\times \to 1$$

and we have the diagram of pointed sets

$$\begin{array}{ccccc} (F \otimes D)^\times & \xrightarrow{N} & F^\times & \xrightarrow{\delta} & H^1(\mathbb{Q}, H) \\ \downarrow & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r_\infty} \\ (\mathbb{H}^\times \times \cdots \times \mathbb{H}^\times) & \xrightarrow{\det} & (\mathbb{R}^\times \times \cdots \times \mathbb{R}^\times) & \xrightarrow{\delta_\infty} & H^1(\mathbb{R}, H) \end{array}$$

The algebraic group $H_{\bar{\mathbb{Q}}} \simeq \mathrm{SL}_2 \times \cdots \times \mathrm{SL}_2$ (f copies) is semi-simple and simply connected, hence $H^1(\mathbb{Q}_p, H) = 0$ for all primes p and r_∞ is bijective. Since $r(c)$ is totally positive, which is an image of the map det, therefore $r_\infty(\delta c) = \delta_\infty r(c) = 0 \Rightarrow \delta c = 0$ and c is an image of N. □

Corollary 9.4. *Let $(A_i, \lambda_i, \iota_i)$, for $i = 1, 2$, be two supersingular polarized O_F-abelian varieties. Then there is an element φ in $\mathrm{Hom}_{O_F}(A_1, A_2) \otimes \mathbb{Q}$ such that $\varphi^*\lambda_2 = \lambda_1$.*

Proof. By Proposition 9.1, there is an element φ_1 in $\mathrm{Hom}_{O_F}(A_1, A_2)$ such that $\varphi^*\lambda_2 = q\lambda_1$ for some $q \in \mathbb{Q}^\times$. Since λ_i are induced from ample line bundles, $q > 0$. By Lemma 9.3, there is $\alpha \in \mathrm{End}_{O_F}(A_1) \otimes \mathbb{Q}$ such that $\alpha^{*_1} \cdot \alpha = q^{-1}$, where the map $\alpha \to \alpha^{*_1}$ is the Rosati involution induced by λ_1. Then $(\varphi_1 \circ \alpha)^*\lambda_2 = \alpha^t q \lambda_1 \alpha = q\lambda_1 \alpha^{*_1}\alpha = \lambda_1$. □

Lemma 9.5. *Let $(A_1, \lambda_1, \iota_1)$ and $(A_2, \lambda_2, \iota_2)$ be polarized O_F-abelian varieties such that the degree of each λ_i is a power of p. Then their ℓ-adic Tate modules are isomorphic , as $O_F \otimes \mathbb{Z}_\ell$-modules with alternating pairings, for all $\ell \neq p$.*

Proof. The Tate module $M := T_\ell(A)$ is a rank 2 free $O_F \otimes \mathbb{Z}_\ell$-module together with a self-dual alternating $\mathbb{Z}_\ell$-valued pairing such that

$$\langle a\,x, y\rangle = \langle x, a\,y\rangle \quad \forall\, a \in O_F,\ x, y \in M.$$

We may assume $O_F \otimes \mathbb{Z}_\ell$ is a domain by decomposing M. Since the degree of the pairing is 1 and $O_F \otimes \mathbb{Z}_\ell$ is a PID, we are done. □

Recall that S_1 the set of isomorphism classes of polarized O_F-abelian varieties (A, λ, ι) over k such that $a(A) = 0202$ and $\deg \lambda = p^4$. We fix an object $(A_0, \lambda_0, \iota_0)$ as a reference. From Lemma 8.3, 9.5, the ℓ-divisible groups $A(\ell)$ and $A_0(\ell)$ (ℓ can be p) are isomorphic for each $(A, \lambda, \iota) \in S_1$. Let G the the automorphism group of $(A_0, \lambda_0, \iota_0)$, regarded as an algebraic group over $\mathrm{Spec}\, \mathbb{Z}$. That is, for any commutative ring R, the group of R-points of G is

$$G(R) = \{\phi \in (\mathrm{End}_{O_F}(A) \otimes R)^\times | \phi'\phi = 1\}$$

where the map $\phi \to \phi'$ is the Rosati involution induced by λ.

Theorem 9.6. *There is an one-one correspondence between the pointed set S_1 and the double coset $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\hat{\mathbb{Z}})$.*

Proof. For any $(A,\lambda,\iota) \in S_1$ there exists a $\mathbb{Q}$-polarized O_F-linear isogeny $\phi : (A_0,\lambda_0,\iota_0) \to (A,\lambda,\iota)$. Such ϕ gives rise to an element in $G(\mathbb{A}_f)$ up to a right translation of $G(\hat{\mathbb{Z}})$: for each prime ℓ (ℓ can be p) there exists $\phi_\ell \in G(\mathbb{Q}_\ell)$ such that $\phi\phi_\ell \in Isom(A_0(\ell),A(\ell))$ compatible with additional structure (the polarization and endomorphism). The element ϕ_ℓ lies in $G(\mathbb{Z}_\ell)$ for almost all ℓ, therefore $(\phi_\ell) \in G(\mathbb{A}_f)$. The element (ϕ_ℓ) is uniquely determined by ϕ up to a right translation of $G(\hat{\mathbb{Z}})$, since $\phi\phi_\ell, \phi\phi'_\ell \in Isom(A_0(\ell),A(\ell))$ implies $\phi'_\ell \in \phi_\ell G(\mathbb{Z}_\ell)$.

However, ϕ is not uniquely determined by (A,λ,ι). Suppose that ϕ_1 is another $\mathbb{Q}$-polarized O_F-linear isogeny from (A_0,λ_0,ι_0) to the same (A,λ,ι), then $\phi_1 \in \phi G(\mathbb{Q})$. That is, the elements $(\phi_\ell)G(\hat{\mathbb{Z}})$ determined by different ϕ's of a fixed (A,λ,ι) belong to an orbit by the left translation of $G(\mathbb{Q})$. Therefore we have a well-defined map from S_1 to $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\hat{\mathbb{Z}})$.

Proof of injectivity: Let $\alpha : (A_0,\lambda_0,\iota_0) \to (A,\lambda_A,\iota_A)$ and $\beta : (A_0,\lambda_0,\iota_0) \to (B,\lambda_B,\iota_B)$ be two $\mathbb{Q}$-polarized O_F-linear isogenies. Suppose that the images of α and β in the double coset are the same, then there exist $\phi \in G(\mathbb{Q})$ and $(\phi_\ell) \in G(\mathbb{A}_f)$ such that $\alpha\phi_\ell \in Isom(A_0(\ell),A(\ell))$ and $\beta\phi\phi_\ell \in Isom(A_0(\ell),B(\ell))$ for all ℓ. The polarized O_F-quasi-isogeny $\beta\phi\alpha^{-1} = \beta\phi\phi_\ell(\alpha\phi_\ell)^{-1} : A \to B$ is isomorphic at every place, hence an isomorphism.

Proof of surjectivity: Given $[(\phi_\ell)] \in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\hat{\mathbb{Z}})$, then there is a positive integer N such that $N\phi_\ell : A_0(\ell) \to A_0(\ell)$ is an *isogeny* for all ℓ. Let H be the product of all $\ker(N\phi_\ell)$. Since each $N\phi_\ell$ is O_F-linear, H is stable under the action of O_F. Take $A := A_0/H$, then there is a natural O_F-action, ι, on A such that the projection $\pi : A_0 \to A$ is an O_F-linear morphism. The degree of π is N^{2g}. The polarization $N\lambda_0$ descends to a fractional polarization λ on A. Since we have the induced isomorphism $A(\ell) \to A_0(\ell)$ w.r.t. the polarizations and endomorphisms, λ is ℓ-adic integral for all ℓ, hence a polarization on A. Comparing the degrees of $N\lambda_0$ and $\pi^*\lambda$, λ is principal. The polarized O_F-linear quasi-isogeny $N^{-1}\pi : (A_0,\lambda_0,\iota_0) \to (A,\lambda,\iota)$ gives the class $[(\phi_\ell)]$ by the construction. ☐

Remark. (1) In the proof of the theorem above, we use a special property of supersingular abelian varieties: $\mathrm{Hom}(A,B) \otimes \mathbb{Z}_p \simeq \mathrm{Hom}(M(A),M(B))$ and $\mathrm{Hom}(A,B) \otimes \mathbb{Z}_\ell \simeq \mathrm{Hom}(T_\ell(A),T_\ell(B))$ for all $\ell \neq p$.

(2) What we have proved for Theorem 9.6 is more general. Let (A_0,λ_0,ι_0) be a supersingular polarized O_F-abelian variety over k (p is not necessarily unramified in F, $[F:\mathbb{Q}]|g$, and the degree of the polarization can be arbitrary). Then the double coset for the automorphism group of (A_0,λ_0,ι_0) is in bijection corresponding the set of isomorphism classes of supersingular polarized O_F-abelian varieties (A,λ,ι) over k such that $A_0(\ell)$ and $A(\ell)$ are isomorphic as quasi-polarized ℓ-divisible groups with the $O_F \otimes \mathbb{Z}_\ell$-actions for all ℓ.

(3) We can consider the statement (2) for the "basic" points in the moduli space of polarized O_B-abelian varieties , e.g. see [35]. The statement (2) holds if the same statements for (1) and for Proposition 9.1 hold. In this way one can relate the irreducible components of the basic locus of Shimura varieties of PEL type and the class numbers of certain groups.

Theorem 9.7. *Let G be as in Theorem 9.6. The supersingular locus $\mathcal{S}$ has $6h$ irreducible components, where h is the class number of G, i.e. the cardinality of the double coset $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\hat{\mathbb{Z}})$.*

Proof. This is nothing but the collection of Proposition 8.2, 8.10 and Theorem 9.6. ☐

10. Cayley-Hamilton for higher a-numbers

The method of Cayley-Hamilton seems getting more complicated when the *usual* a-number is higher. Fortunately, for our case the situation is not too complicated even dealing with the higher a-number Dieudonné modules. Indeed, it turns out that we only have to deal with 2 or 3 cases separately.

Let M be the Dieudonné module of a point $(A, \lambda, \iota) \in \mathcal{M}(k)$. Let X, Y be any $W(k)$-basis of M^0. Suppose that

$$F^f X = aX + bY$$
$$F^f Y = cX + dY$$

where $a, b, c, d \in W$.

Case 1: If $b = c = 0$ then slopes of M are $\mathrm{ord}(a)/f$ and $\mathrm{ord}(d)/f$.

Case 2: If $b \neq 0$ then X generates a full-rank submodule and we obtain a Cayley-Hamilton equation

$$bF^{2f}X - (b^{(f)}d + ba^{(f)})F^f X + b^{(f)}(ad - bc)X = 0. \tag{10.1}$$

Therefore the lowest slope of M is $\min(\frac{1}{f}(\mathrm{ord}(b^{(f)}d + ba^{(f)}) - \mathrm{ord}(b)), \frac{1}{2})$.

Case 3: If $c \neq 0$ then we have the similar situation for Y. The lowest slope of M is $\min(\frac{1}{f}(\mathrm{ord}(c^{(f)}a + cd^{(f)}) - \mathrm{ord}(c)), \frac{1}{2})$.

We will write $p_0 F^{2f} X - p_1 F^f X + p_2 X = 0$ for the equation (10.1) in the rest of the paper.

11. Local moduli spaces I

In this section we compute the local moduli space $\widehat{\mathcal{S}}_x$ at any given supersingular point x. If $\mathcal{N}$ is a stratum of $\mathcal{M}$ and x is a point on $\mathcal{N}$, then we denote by $\mathcal{N}_x = \operatorname{Spec} R$ where $\widehat{\mathcal{N}}_x = \operatorname{Spf} R$, the local moduli space at x inside $\mathcal{N}$. We say $\mathcal{N}_x$ the ordinary scheme associated to the local moduli space $\widehat{\mathcal{N}}_x$. By Grothendieck's existence theorem, there is an abelian scheme over $\mathcal{N}_x$. As a topological space, $\mathcal{N}_x$ has more (non-closed) points than its formal completion. This enables us to test if a point $\mathfrak{p}$ of $\mathcal{N}_x$ lies in the supersingular locus by making the base change $\operatorname{Spec} k(\mathfrak{p}) \to \operatorname{Spec} R$.

For an a-index τ or the corresponding a-number, recall that $\mathcal{M}_\tau$ denote the locally closed substack of $\mathcal{M}$ whose points have the a-index τ; $\mathcal{S}_\tau = \mathcal{S} \cap \mathcal{M}_\tau$; and $\bar{\mathcal{S}}_\tau$ the Zariski closure of $\mathcal{S}_\tau$ in $\mathcal{S}$.

The idea to computing $\widehat{\mathcal{S}}_x$ is as follows:

First, the supersingular locus $\mathcal{S}$ decomposes into 6 pieces (Proposition 6.3):

$$\mathcal{S} = \bigcup_\tau \bar{\mathcal{S}}_\tau, \ \ \tau: \text{ generic } a\text{-indices}$$

Hence it suffices to compute the possible components $\bar{\mathcal{S}}_{i,x}$ and $\bar{\mathcal{S}}_{(i,i+2),x}$ in $\mathcal{S}_x$, and then take the union of them. We know when $\mathcal{S}_x$ contains a component $\bar{\mathcal{S}}_{(i,i+2),x}$ from the a-number of x. Also since $\bar{\mathcal{S}}_{i,i+2} = \overline{\mathcal{M}}_{i,i+2}$ is smooth, there is at most one irreducible component of such type passing through x, and this component is isomorphic to $\operatorname{Spec} \mathcal{O}/(t_i, t_{i+2})$, where $\mathcal{O} = k[[t_0, t_1, t_2, t_3]]$. Therefore, it remains to check if $\bar{\mathcal{S}}_{i,x} \subset \mathcal{S}_x$ and compute the defining equations.

Suppose that we want to compute $\bar{\mathcal{S}}_{\tau,x}$, where $\tau = 1000$, then it suffices to look at the supersingular locus, Z, in $\overline{\mathcal{M}}_{\tau,x}$. Compute the formal universal deformation over $\widehat{\overline{\mathcal{M}}}_{\tau,x} =$

Spf R_τ. In fact, using the explicit deformation theory of displays and the Cartier-Dieudonné theory, we directly write down the deformation over Spec R_τ. Compute the coefficients p_0, p_1 (defined at the end of the previous section) of the Cayley-Hamilton equation by iterating the Frobenius map. The next step, roughly speaking, is to "catch" each irreducible component of Z from the closed congruence condition "$\mathrm{ord}(p_1) - \mathrm{ord}(p_0) \geq 2$". The following is an explicit recipe.

Write $p_i = (p_{i0}, p_{i1}, \dots)$, the coordinates in the Witt ring $W(R_\tau)$. Denote by $p_i(\mathfrak{p})$ the image of p_i in $W((R_\tau/\mathfrak{p})^{pf})$. Note that $p_i(\mathfrak{p}) \equiv 0 \pmod{p^{n+1}}$ iff $\mathfrak{p} \in V(p_{i0}, p_{i1}, \dots, p_{i,n})$. Therefore, the reduced subscheme $V = V(p_{i0}, p_{i1}, \dots, p_{i,n})$ is characterized by the property that $\mathfrak{p} \in V$ iff $p_i(\mathfrak{p}) \equiv 0 \pmod{p^{n+1}}$. So we will say V is defined by the congruence condition $p_i \equiv 0 \pmod{p^{n+1}}$. The supersingular locus Z in $\overline{\mathcal{M}}_{\tau,x}$ is contained in the reduced subscheme, Z_1, of $\overline{\mathcal{M}}_{\tau,x}$ defined by $p_1 \equiv 0 \mod p^2$, or equivalently, by $p_{11} = 0$ (The condition $p_{10} = 0$ is satisfied since we are deforming inside $\overline{\mathcal{M}}_\tau$). Look at the irreducible components of Z_1, and then throw away the components which cause non-generic a-numbers locus since these are not whole components of Z. The remaining components are our candidates, and the supersingular locus Z is contained in the union of them. Let V_1 be one of these candidates. If V_1 is not contained in the reduced subscheme B_1 defined by $p_0 \equiv 0 \pmod p$, or equivalently by $p_{00} = 0$, then V_1 is a whole component of Z.

If V_1 is contained in B_1, the another condition $p_1 \equiv 0 \mod p^3$ must be satisfied. The new equation $p_{12} = 0$ defines a reduced subscheme Z_2, and we throw away the bad components of Z_2 again. An irreducible component of Z contained in V_1 must be contained in $V_1 \cap V_2$ for some good component V_2 of Z_2. If $V_1 \cap V_2$ is not contained in the reduced subscheme B_2 defined by $p_0 \equiv 0 \mod p^2$, then it is a whole component. If it is in the other case, then another higher congruence condition of p_1 must be satisfied. Continuing this process until finding out one irreducible component of Z.

Now we are ready to compute $\bar{\mathbb{S}}_{\tau,x}$ for $|\tau| = 1$. Because of symmetry, we may assume $a(x) = 1111, 1110, 1100, 1010$ or 1000. The case $a(x) = 1000$ is done in Section 5. We see that $a(x) = 1100$ does not exist (Corollary 8.8). We will do the case $a(x) = 1111$ in this Section and put the remaining cases in Section 13.

Let $x \in \mathcal{M}_{1111}(k)$ be a superspecial point, M its Dieudonné module. We can choose a basis $\{X_i, Y_i\}$ of M such that $FX_i = -Y_{i+1}$, $FY_i = pX_{i+1}$ for $i = 0, \dots, 3$. The universal deformation of M over $\overline{\mathcal{M}}_{1000,x} = \mathrm{Spec}\,\mathcal{O}/(t_0)$ in $\overline{\mathcal{M}}_{1000}$ is given by the relation

$$\begin{aligned} FX_0 &= -Y_1 - T_1 X_1 \\ FX_1 &= -Y_2 - T_2 X_2 \\ FX_2 &= -Y_3 - T_3 X_3 \\ FX_3 &= -Y_0, \end{aligned}$$

where T_i is the Teichmüller lifting of t_i. Iterating the Frobenius map, we obtain

$$\begin{aligned} F^4 X_0 &= -pF^2 X_2 - T_1^{(3)} F^3 X_1 \\ F^3 X_1 &= pY_0 - T_2^{(2)} F^2 X_2 \\ F^2 X_2 &= -pX_0 + T_3^{(1)} Y_0 \\ F^4 Y_0 &= pF^3 X_1 \end{aligned}$$

We compute that

$$\begin{aligned}
a &= p(p - T_1^{(3)}T_2^{(2)}) \\
b &= T_1^{(3)}T_2^{(2)}T_3^{(1)} - p(T_3^{(1)} + T_1^{(3)}) \\
c &= p^2T_2^{(2)} \\
d &= p(p - T_2^{(2)}T_3^{(1)})
\end{aligned}$$

$$\begin{aligned}
p_0 &= b = T_1^{(3)}T_2^{(2)}T_3^{(1)} + \text{higher } p\text{-power terms} \\
p_1 &= (b^{(f)}\, d + b\, a^{(f)}) \\
&= -pT_2^{(2)}T_3^{(1)}T_1^{(7)}T_2^{(6)}(T_3^{(5)} + T_1^{(3)}) + \text{higher } p\text{-power terms.}
\end{aligned}$$

The condition $p_1 \equiv 0 \mod p^2$ defines a reduced subscheme $t_1t_2t_3(t_3^{p^2} + t_1) = 0$. The components $t_1 = 0$ and $t_3 = 0$ cause non-generic a-numbers, so they are out. The component $t_2 = 0$ is just the component corresponding to a-number 1010. The condition $p_0 \equiv 0 \mod p$ defines the reduced scheme B_1 by the equation $t_1t_2t_3 = 0$. Since the component $t_3^{p^2} + t_1 = 0$ is not contained in the bad locus B_1, it is a component of the supersingular locus. The generic fibre has the a-number 1000, hence $\bar{\mathcal{S}}_{1000,x} \simeq \operatorname{Spec} \mathcal{O}/(t_0, t_3^{p^2} + t_1)$. Similarly, we obtain that $\bar{\mathcal{S}}_{i,x} = \operatorname{Spec} \mathcal{O}/(t_i, t_{i-1}^{p^2} + t_{i+1})$, for $i \in \mathbb{Z}/4\mathbb{Z}$.

Theorem 11.1. *Suppose $x = (A, \lambda, \iota) \in \mathcal{M}_{1111}(k)$. Then the local moduli space $\widehat{\mathcal{S}}_x \simeq \operatorname{Spf} k[[t_0, t_1, t_2, t_3]]/I$, where I is the intersection of ideals $(t_0, t_2), (t_1, t_3)$ and $(t_i, t_{i-1}^{p^2} + t_{i+1})$ for $i = 0, \dots, 3$.*

Let $x \in \mathcal{S}_{1110}(k)$. From Section 13, one concludes that $\overline{\mathcal{M}}_{1000,x} \cap \mathcal{S} \subset \bar{\mathcal{S}}_{1010}$ and $\overline{\mathcal{M}}_{0010,x} \cap \mathcal{S} \subset \bar{\mathcal{S}}_{1010}$. Therefore there are two irreducible components of $\bar{\mathcal{S}}_x$:

$$\bar{\mathcal{S}}_{1010,x} \simeq \operatorname{Spec} \mathcal{O}/(t_0, t_1) \tag{11.1}$$

$$\bar{\mathcal{S}}_{0100,x} \simeq \operatorname{Spec} \mathcal{O}/(t_1, t_2 + t_0^{p^2} + \bar{a}_1^p t_2 t_0^{p^2}), \tag{11.2}$$

where $\bar{a}_1$ is the coefficient a_1 in (13.1) $\mod p$.

Let $x \in \mathcal{S}_{1010}(k)$. From Section 13, we have the local moduli spaces

$$\mathcal{S}_x \cap \overline{\mathcal{M}}_{1000,x} \simeq \operatorname{Spec} \mathcal{O}/(t_2, t_0(t_3 - \bar{a}_2^p)) \tag{11.3}$$

$$\mathcal{S}_x \cap \overline{\mathcal{M}}_{0010,x} \simeq \operatorname{Spec} \mathcal{O}/(t_0, t_2(t_1 - \bar{a}_0^p)), \tag{11.4}$$

where $\bar{a}_0$ and $\bar{a}_2$ the coefficients a_0 and a_2 in (13.3) $\mod p$. These local moduli spaces are irreducible or not depending on the divisibility of a_0 and a_2 by p. Nevertheless, each irreducible component is smooth at x.

From the local moduli spaces $\bar{\mathcal{S}}_{\tau,x}$ for all generic a-numbers τ and all points x, we get

Theorem 11.2. *Every irreducible component of $\mathcal{S}$ is smooth as an algebraic stack.*

12. Intersections

In this section, we will describe the intersection of two irreducible components. We content ourselves by describing the intersections in the supersingular locus *with an auxiliary* *level structure*. In this setting one avoids the issue whether the operations the quotient and intersection in our case commute. Therefore we change our notations a bit.

Let n be an integer such that $n \geq 3$ and $(n,p) = 1$. Let $\mathcal{M}$ be the moduli space of principally polarized O_F-abelian varieties in characteristic p with symplectic n-level structure, i.e. $\mathcal{M} := \mathcal{A}_{g,n}^{O_F} \otimes_{\mathbb{Z}[\zeta_n,1/n]} \bar{\mathbb{F}}_p$. Denote by $\mathcal{S}$ the supersingular locus of $\mathcal{M}$. For each a-index (or a-number) τ, we denote $\mathcal{S}_\tau$ the reduced subscheme of $\mathcal{S}$ whose points have the a-index (or a-number) τ, and $\bar{\mathcal{S}}_\tau$ the Zariski closure of $\mathcal{S}_\tau$ in $\mathcal{S}$. Finally, let S_i, Λ_i denote the sets as in Section 8 except the abelian varieties there with symplectic n-level structure.

From (8.3) and Theorem 9.7, one deduces that the supersingular locus has $6h_n$ irreducible components and $\#\mathbf{P}^1(\mathbb{F}_{p^4}) \cdot h_n$ superspecial points. Here the number h_n is

$$\#G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_n = [G(\hat{\mathbb{Z}}) : K_n] \left(\sum_c \frac{1}{|\Gamma_c|} \right), \tag{12.1}$$

where K_n is the kernel of the map $G(\hat{\mathbb{Z}}) \to G(\hat{\mathbb{Z}}/n\hat{\mathbb{Z}})$. The sum is taken over all the representatives of the double coset $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\hat{\mathbb{Z}})$, and $\Gamma_c = G(\mathbb{Q}) \cap cG(\hat{\mathbb{Z}})c^{-1}$ is the stabilizer of the coset $cG(\hat{\mathbb{Z}})$ in $G(\mathbb{Q})$. Each irreducible component of $\mathcal{S}$ is isomorphic to exact one model $\mathcal{N}_{\tau,\eta}$ (defined in Section 8) and has $\#\mathbf{P}^1(\mathbb{F}_{p^4})$ superspecial points.

As each irreducible component contains a superspecial point, one describes the intersections near a superspecial point first.

Lemma 12.1. *Let x be a superspecial point and $\mathcal{O} := k[[t_0,t_1,t_2,t_3]]$.*

(1) $\bar{\mathcal{S}}_{1010,x} \cap \bar{\mathcal{S}}_{0101,x} \simeq \operatorname{Spec} \mathcal{O}/(t_0,t_1,t_2,t_3)$.
(2) $\bar{\mathcal{S}}_{1010,x} \cap \bar{\mathcal{S}}_{1000,x} \simeq \operatorname{Spec} \mathcal{O}/(t_0,t_2,t_1 + t_3^{p^2})$.
(3) $\bar{\mathcal{S}}_{1010,x} \cap \bar{\mathcal{S}}_{0100,x} \simeq \operatorname{Spec} \mathcal{O}/(t_0,t_1,t_2)$.
(4) $\bar{\mathcal{S}}_{1000,x} \cap \bar{\mathcal{S}}_{0100,x} \simeq \operatorname{Spec} \mathcal{O}/(t_0,t_1,t_2,t_3^{p^2})$.
(5) $\bar{\mathcal{S}}_{1000,x} \cap \bar{\mathcal{S}}_{0010,x} \simeq \operatorname{Spec} \mathcal{O}/(t_0,t_1,t_2,t_3)$.

Proposition 12.2.

(1) *The intersection $\bar{\mathcal{S}}_{1010} \cap \bar{\mathcal{S}}_{0101}$ consists of all superspecial points with multiplicity one.*
(2) *The intersection $\bar{\mathcal{S}}_{1010} \cap \bar{\mathcal{S}}_{1000}$ is a smooth complete curve whose generic points have a-number* 1010.
(3) *The intersection $\bar{\mathcal{S}}_{1010} \cap \bar{\mathcal{S}}_{0100}$ is $\overline{\mathcal{M}}_{1110}$. Moreover, the intersection of any component of $\bar{\mathcal{S}}_{1010}$ and any component of $\bar{\mathcal{S}}_{0100}$, if not empty, is isomorphic to $\mathbf{P}^1$.*
(4) *The intersection $\bar{\mathcal{S}}_{1000} \cap \bar{\mathcal{S}}_{0100}$ consists of all superspecial points with multiplicity p^2.*
(5) *The intersection $\bar{\mathcal{S}}_{1000} \cap \bar{\mathcal{S}}_{0010}$ consists of all superspecial points with multiplicity one and some other points with multiplicity one.*

Proof. (1) is clear and (3) follows from Proposition 8.4.

(2) We have checked that the intersection is smooth at superspecial points. Let $x \in \mathcal{M}_{1010} \cap \bar{\mathcal{S}}_{1000}$. From the previous section, we know that $\bar{\mathcal{S}}_{1000,x} = \operatorname{Spec} \mathcal{O}/(t_0,t_3)$, hence the intersection of $\bar{\mathcal{S}}_{1010}$ and $\bar{\mathcal{S}}_{1000}$ at x has local equations t_0, t_2, t_3.

(4) Let $x \in \bar{\mathcal{S}}_{1000} \cap \bar{\mathcal{S}}_{0100}$. If $a(x) = 1110$, then $\bar{\mathcal{S}}_{1000,x} = \phi$ by (13.2). Similarly, $a(x) \neq 1101$. By Corollary 8.8, $a(x) = 1111$.

(5) Let $x \in \bar{\mathcal{S}}_{1000} \cap \bar{\mathcal{S}}_{0010}$, then $x \in \bar{\mathcal{S}}_{1010}$. By (13.2), $a(x) \neq 1110$ or 1011. If $a(x) = 1111$, we have Lemma 12.1 (5). If $a(x) = 1010$, it follows from (13.4) that $\bar{\mathcal{S}}_{1000,x} \cap \bar{\mathcal{S}}_{0010,x} = \operatorname{Spec} \mathcal{O}/(t_0,t_1,t_2,t_3)$. ∎

We will determine that the complete smooth curve in (2) is isomorphic to a disjoint union of $\mathbf{P}^1$'s. We need a lemma.

Lemma 12.3. *Let $M = M^0 \oplus \cdots \oplus M^3$ be a Dieudonné module given by the relation*

$$\begin{array}{ll} FX_0 = X_1, & FY_0 = pY_1 \\ FX_1 = -Y_2 + pa_2 X_2, & FY_1 = pX_2 \\ FX_2 = X_3, & FY_2 = pY_3 \\ FX_3 = -Y_0 + pa_0 X_0, & FY_3 = pX_0 \end{array}$$

for some $a_0, a_2 \in W(k)$. Then
(i) $a_2 \equiv 0 \mod p \iff F^4(M^0) \subset p^2 M^0$.
(ii) $a_0 \equiv 0 \mod p \iff F^4(M^2) \subset p^2 M^2$.

Proof. Set $\overline{M} := M/p^2 M$ and all elements are considered as their images in $\overline{M}$.

$$F\overline{M} = span\{X_1, Y_2, X_3, Y_0, pY_1, pX_2, pY_3, pX_0\}$$

$$\begin{aligned}(\because & span\{-Y_2 + pa_2 X_2, pX_2\} = span\{Y_2, pX_2\}, \\ & span\{-Y_0 + pa_0 X_0, pX_0\} = span\{Y_0, pX_0\})\end{aligned}$$

$$F^2\overline{M} = span\{-Y_2 + pa_2 X_2, pY_3, -Y_0 + pa_0 X_0, pY_1, pX_3, pX_1\}$$

$$F^3\overline{M} = span\{-pY_3 + pa_2^{(1)} X_3, -pY_1 + pa_0^{(1)} X_1, -pY_0, -pY_2\}$$

$$F^4\overline{M} = span\{pa_2^{(2)} Y_0, pa_0^{(2)} Y_2\}.$$

Therefore we are done. $\square$

Proposition 12.4. *The set of k-points of $\mathcal{N}_{1010,\eta} = \mathbf{P}^1 \times \mathbf{P}^1$ which belong to $\bar{\mathcal{S}}_{1000}$ is $\{(x,y) \in \mathbf{P}^1 \times \mathbf{P}^1(k) \mid x \in \mathbf{P}^1(\mathbb{F}_{p^4})\}$*

Proof. Use the affine coordinates as (8.2), for any point (x,y), the Dieudonné module corresponded is written by

$$M_{(x,y)} = span\left\{\begin{array}{cccc} X_0, & X_1, & X_2, & X_3 \\ V_0, & Y_1, & V_2, & Y_3 \end{array}\right\},$$

where $V_0 = \tilde{x}/pX_0 + 1/pY_0$ and $V_2 = \tilde{y}/pX_2 + 1/pY_2$. We have $FV_0 = -\tilde{x}^{(1)}/pY_1 + X_1$ and $FV_2 = -\tilde{y}^{(1)}/pY_3 + 1/pX_3$.

The point belongs to $\bar{\mathcal{S}}_{1000}$ iff the condition $F^4 M^0_{(x,y)} \subset p^2 M^0_{(x,y)}$ is satisfied. Iterating the Frobenius map F, we get

$$X_0 \xrightarrow{F} -pY_1 \xrightarrow{F} -pX_2 \xrightarrow{F} p^2 Y_3$$

$$\begin{aligned} V_0 &\xrightarrow{F} -\tilde{x}^{(1)} Y_1 + X_1 \\ &\xrightarrow{F} -\tilde{x}^{(2)} X_2 - Y_2 \\ &\xrightarrow{F} \tilde{x}^{(3)} pY_3 - pX_3 \\ &\xrightarrow{F} \tilde{x}^{(4)} pX_0 + pY_0 = (\tilde{x}^{(4)} - \tilde{x})pX_0 + p^2 V_0. \end{aligned}$$

The condition gives $\tilde{x}^{(4)} - \tilde{x} \equiv 0 \mod p$, i.e. $x^{p^4} - x = 0$. $\square$

Corollary 12.5. *Each connected component of the intersection $\bar{\mathcal{S}}_{1010} \cap \bar{\mathcal{S}}_{1000}$ is isomorphic to $\mathbf{P}^1$.*

Remark. The intersection of one component of $\bar{\mathcal{S}}_{1010}$ and one of component of $\bar{\mathcal{S}}_{1000}$ may not be connected. By Proposition 8.4 (3) and 12.4, every connected component of the intersection contains only one superspecial point. If the intersection is connected, then the number h_n of components of $\bar{\mathcal{S}}_{1000}$ must be at least $|\mathbf{P}^1(\mathbb{F}_{p^4})|$, the number of superspecial points on one component of $\bar{\mathcal{S}}_{1010}$. This is not true in general.

13. Local moduli spaces II

In this section we compute the local moduli space $\hat{\mathcal{S}}_x$ for $a(x) = 1110$ and $a(x) = 1010$, the remaining cases. We denote $x^{(n)} = x^{\sigma^n}$ for simplicity.

Let $x \in \mathcal{S}_{1110}(k)$ and M be its Dieudonné module. Take a basis of M such that

$$(13.1) \qquad \begin{array}{ll} FX_0 = -Y_1 + pa_1 X_1, & FY_0 = pX_1 \\ FX_1 = -Y_2 + pa_2 X_2, & FY_1 = pX_2 \\ FX_2 = X_3, & FY_2 = pY_3 \\ FX_3 = -Y_0 + pa_0 X_0, & FY_3 = pX_0. \end{array}$$

The universal deformation of M over $\mathcal{M}_x$ is given by

$$\begin{array}{ll} FX_0 = -Y_1 + (pa_1 - T_1)X_1, & FY_0 = pX_1 \\ FX_1 = -Y_2 + (pa_2 - T_2)X_2, & FY_1 = pX_2 \\ FX_2 = X_3, & FY_2 = pY_3 + pT_3X_3 \\ FX_3 = -Y_0 + (pa_0 - T_0)X_0, & FY_3 = pX_0. \end{array}$$

where T_i is the Teichmüller lift of t_i. Iterating the Frobenius map, one has

$$\begin{aligned} F^4X_2 &= -pFX_1 + (pa_0^{(2)} - T_0^{(2)})F^2X_0 \\ F^2X_0 &= -pX_2 + (pa_1^{(1)} - T_1^{(1)})FX_1 \\ FX_1 &= -Y_2 + (pa_2 - T_2)X_2 \\ F^4Y_2 &= p^2F^2X_0 + pT_3^{(3)}F^4X_2. \end{aligned}$$

Case 1: $t_0 = 0$, i.e. considering the components of $\mathcal{S}_x$ in $\overline{\mathcal{M}}_{1000,x}$. We compute that

$$p_0 = b = p(1 + a_0^{(2)}T_1^{(1)} - pa_0^{(2)}a_1^{(1)})$$

$$p_1 = p^2(1 + a_0^{(2)}T_1^{(1)})T_2^{(4)}(1 + a_0^{(6)}T_1^{(5)}) + p^3(\cdots)$$

Note that $\operatorname{ord}(p_0) = 1$, and the condition $p_1 \equiv 0 \mod p^3$ gives the equation $t_2 = 0$. The supersingular locus in $\overline{\mathcal{M}}_{1000,x}$ is given by equations $t_0 = t_2 = 0$, and it is $\bar{\mathcal{S}}_{1010,x}$.

Case 2: $t_1 = 0$, i.e. considering the components of $\mathcal{S}_x$ in $\overline{\mathcal{M}}_{0100,x}$. We compute that

$$p_0 = b = p(1 + a_1^{(1)}T_0^{(2)} - pa_0^{(2)}a_1^{(1)})$$

$$p_1 = p^2(1 + a_1^{(1)}T_0^{(2)})(T_0^{(6)} + a_1^{(5)}T_0^{(6)}T_2^{(4)}) + p^3(\cdots)$$

Note that $\operatorname{ord}(p_0) = 1$, and the condition $p_1 \equiv 0 \mod p^3$ gives the equation $t_0^{p^2} + t_2 + \bar{a}_1^p t_0^{p^2} t_2 = 0$. The supersingular locus in $\overline{\mathcal{M}}_{1000,x}$ is given by equations $t_1 = t_0^{p^2} + t_2 + \bar{a}_1^p t_0^{p^2} t_2 = 0$, and it is $\bar{\mathcal{S}}_{0100,x}$.

Case 3: $t_2 = 0$, i.e. considering the components of $\mathcal{S}_x$ in $\overline{\mathcal{M}}_{0010,x}$. We compute that

$$p_0 = -T_0^{(2)}T_1^{(1)} + p(1 + a_1^{(1)}T_0^{(2)} + a_0^{(2)}T_1^{(1)} - pa_0^{(2)}a_1^{(1)})$$

$$p_1 = pT_0^{(2)}T_1^{(1)}T_0^{(6)}(T_1^{(5)}T_3^{(3)} - T_1^{(5)}T_2^{(4)} - 1) + p^2(\cdots).$$

The congruence condition $p_1 \equiv 0 \mod p^2$ gives the equation $t_0t_1 =$. The component $t_1 = 0$ will cause a non-generic a-number, so we throw it away. The only possible component $t_0 = 0$ has the a-number 1010, which is supersingular . Hence we get one component whose defining equations are $t_0 = t_2 = 0$.

We conclude that $\overline{\mathcal{M}}_{1000,x} \cap \mathcal{S} \subset \bar{\mathcal{S}}_{1010}$ and $\overline{\mathcal{M}}_{0010,x} \cap \mathcal{S} \subset \bar{\mathcal{S}}_{1010}$. Therefore there are two irreducible components of $\bar{\mathcal{S}}_x$:

$$\begin{aligned} \bar{\mathcal{S}}_{1010,x} &\simeq \operatorname{Spec} \mathcal{O}/(t_0, t_2) \\ \bar{\mathcal{S}}_{0100,x} &\simeq \operatorname{Spec} \mathcal{O}/(t_1, t_2 + t_0^{p^2} + \bar{a}_1^p t_2 t_0^{p^2}), \end{aligned} \tag{13.2}$$

where $\mathcal{O} := k[[t_0, t_1, t_2, t_3]]$.

Let $x \in \mathcal{S}_{1010}(k)$, and let M be its Dieudonné module. Take a basis of M such that

$$\begin{array}{ll} FX_0 = X_1, & FY_0 = pY_1 \\ FX_1 = -Y_2 + pa_2X_2, & FY_1 = pX_2 \\ FX_2 = X_3, & FY_2 = pY_3 \\ FX_3 = -Y_0 + pa_0X_0, & FY_3 = pX_0. \end{array} \tag{13.3}$$

The universal deformation of M over $\overline{\mathcal{M}}_{0010,x}$ is

$$\begin{array}{ll} FX_0 = X_1, & FY_0 = pY_1 + pT_1X_1 \\ FX_1 = -Y_2 + pa_2X_2, & FY_1 = pX_2 \\ FX_2 = X_3, & FY_2 = pY_3 + pT_3X_3 \\ FX_3 = -Y_0 + (pa_0 - T_0)X_0, & FY_3 = pX_0. \end{array}$$

where T_i is the Teichmüller lift of t_i. Iterating the Frobenius map, one has

$$F^4X_0 = -F^2Y_2 + pa_2^{(2)}FX_3$$

$$F^2Y_2 = p^2X_0 + pT_3^{(1)}FX_3$$

$$FX_3 = -Y_0 + (pa_0 - T_0)X_0$$

$$F^4Y_0 = p^2FX_3 + pT_1^{(3)}F^4X_0$$

We compute that

$$a = -pT_0^{(0)}(-T_3^{(1)} + a_2^{(2)})$$

$$d = -p^2(1 - T_1^{(3)}T_3^{(1)} + T_1^{(3)}a_2^{(2)}).$$

Since $p^2|d$ and $p_1 = b^{(f)}d + ba^{(f)}$, the slope will go up to $1/2$ if and only if $p^2|a$, which gives the equation $t_0(t_3 - \bar{a}_2^p) = 0$. Hence the supersingular locus in $\overline{\mathcal{M}}_{0010,x}$ is defined by the equations $t_2 = t_0(t_3 - \bar{a}_2^p) = 0$.

We conclude that

$$
\begin{aligned}
\mathcal{S}_x \cap \overline{\mathcal{M}}_{0010,x} &\simeq \operatorname{Spec} \mathcal{O}/(t_2, t_0(t_3 - \bar{a}_2^p)), \\
\mathcal{S}_x \cap \overline{\mathcal{M}}_{1000,x} &\simeq \operatorname{Spec} \mathcal{O}/(t_0, t_2(t_1 - \bar{a}_0^p)). \\
\bar{\mathcal{S}}_{0010,x} \neq \phi \iff \bar{a}_2 = 0 \implies \bar{\mathcal{S}}_{0010,x} &\simeq \operatorname{Spec} \mathcal{O}/(t_2, t_3) \\
\bar{\mathcal{S}}_{1000,x} \neq \phi \iff \bar{a}_0 = 0 \implies \bar{\mathcal{S}}_{1000,x} &\simeq \operatorname{Spec} \mathcal{O}/(t_0, t_1).
\end{aligned}
\tag{13.4}
$$

14. Appendix: Deformation via the Hecke correspondence

Let $\mathcal{N}$ be the moduli stack parameterizing the triple $((A_1, \lambda_1, \iota_1), (A_2, \lambda_2, \iota_2), \varphi)$ where $(A_1, \lambda_1, \iota_1)$ and $(A_2, \lambda_2, \iota_2)$ are in $\mathcal{M}$ and $\varphi : A_1 \to A_2$ is an O_F-linear isogeny morphism such that $\varphi^* \lambda_2 = p\lambda_1$. The algebraic stack $\mathcal{N}$ is a Hecke correspondence of $\mathcal{M}$, or called the p-parahoric level structure of $\mathcal{M}$. Let p_1, p_2 be the two projection morphisms to $\mathcal{M}$. The idea to deform a higher a-number point to a lower one along a slope stratum is the follows: for a point (A, λ, ι) of higher a-number, find a $B \in \mathcal{N}$ such that $p_1(B) = A$. Then we compute the fibre $p_2^{-1}(p_2(B))$ of $p_2(B)$ and determine the a-numbers of the points in the fibre.

Given M, the covariant Dieudonné module of a point $(A, \lambda, \iota) \in \mathcal{M}(k)$.
Let N be the Dieudonné module containing M such that $VN = M$. We have $pN \subset M \subset N$ and $\tau_N = \{n_1 + 1, \ldots, n_a + 1\}$ if $\tau_M = \{n_1, \ldots, n_a\}$.
Let $\{X_i, Y_i\}$ be a basis of N as in Proposition 4.2. We have the relation:

$$
FX_i = \begin{cases} X_{i+1} & \text{if } i \notin \tau_M, \\ Y_{i+1} + pa_{i+1}X_{i+1} & \text{if } i \in \tau_M. \end{cases} \qquad VX_i = \begin{cases} pX_{i-1} & \text{if } i \notin \tau_N, \\ Y_{i-1} & \text{if } i \in \tau_N. \end{cases}
$$

$$
FY_i = \begin{cases} pY_{i+1} & \text{if } i \notin \tau_M, \\ pX_{i+1} & \text{if } i \in \tau_M. \end{cases} \qquad VY_i = \begin{cases} Y_{i-1} & \text{if } i \notin \tau_N, \\ pX_{i-1} - pa_{i+1}^{\sigma^{-1}}pY_{i-1} & \text{if } i \in \tau_N. \end{cases}
$$

where a_i are elements of $W(k)$ for a suitable i.
For the Dieudonné module N, we have the following

$$
\begin{array}{llll}
i \notin \tau_N & FX_{i-1} = X_i & i+1 \notin \tau_N & VX_{i+1} = pX_i \\
i \in \tau_N & FX_{i-1} = Y_i + a_i pX_i & i+1 \in \tau_N & VX_{i+1} = Y_i \\
i \notin \tau_N & FY_{i-1} = pY_i & i+1 \notin \tau_N & VY_{i+1} = Y_i \\
i \in \tau_N & FY_{i-1} = pX_i & i+1 \in \tau_N & VY_{i+1} = pX_i - a_i^{\sigma^{-1}} pY_i
\end{array}
$$

For the Dieudonné module $M = span\langle Y_i, pX_i \rangle$, we have the following

$$
\begin{array}{llll}
i-1 \notin \tau_M & F(pX_{i-1}) = pX_i & i \notin \tau_M & V(pX_{i+1}) = p(pX_i) \\
i-1 \in \tau_M & F(pX_{i-1}) = pY_i + a_i p(pX_i) & i \in \tau_M & V(pX_{i+1}) = pY_i \\
i-1 \notin \tau_M & FY_{i-1} = pY_i & i \notin \tau_M & VY_{i+1} = Y_i \\
i-1 \in \tau_M & FY_{i-1} = (pX_i) & i \in \tau_M & VY_{i+1} = (pX_i) - a_i^{\sigma^{-1}} pY_i
\end{array}
$$

Consider a family of maximal isotropic subspaces of $\bar{N} = N/p$ stable under the action of O_F: $\{0 \subset \overline{M}_t = span\langle Y_i + t_i X_i \rangle \subset \overline{N}; t_i \in k\}$. This is an affine open neighborhood of M/pN in the (compact) Lagrangian subvariety compatible with the O_F. The preimage M_t is $span\langle Y_i + T_i X_i, pX_i \rangle$, where $T_i = [t_i]$ is the Teichmüller lifting of t_i. The W-module M_t is

Dieudonné if and only if the variables t_i satisfy the relations $F\bar{M}_t \subset \bar{M}_t$ and $V\bar{M}_t \subset \bar{M}_t$. But in fact both relations provide the same equations and we just apply F on $\bar{M}_t$.

$$i \notin \tau_M, \;\; F : Y_i + t_i X_i \to t_i^p X_{i+1} \Rightarrow t_i^p = 0$$

$$i \in \tau_M, \;\; F : Y_i + t_i X_i \to t_i^p Y_{i+1} = t_i^p (Y_{i+1} + t_{i+1} X_{i+1}) - t_i^p t_{i+1} X_{i+1}$$

$$\Rightarrow t_i^p t_{i+1} = 0$$

This is the local equations of the fibre mention above.

As we will "physically" deform M (t_i is taken the values in k), we may take the reduced structure of the deformation space.

Example.
1. When $a_M = (1,0,1,0,0)$, the defining equations are $t_0t_1, t_1, t_2t_3, t_3, t_4$, and the deformation space is $\operatorname{Spec} k[t_0, t_2]$.
2. When $a_M = (1,1,0,1,1,1,0)$, the defining equations are $t_0t_1, t_1t_2, t_2, t_3, t_3t_4, t_4t_5, t_5t_6, t_6$, and the deformation space is $\operatorname{Spec} k[t_0, t_1, t_3, t_4, t_5]/(t_0t_1, t_3t_4, t_4t_5)$
3. In general, the deformation space is $\operatorname{Spec} k[t_i]_{i\in\tau_M}/(t_it_{i+1})_{i,i+1\in\tau_M}$

We rewrite the deformed Dieudonné module M_t as $span\langle \tilde{X}_i, \tilde{Y}_i\rangle$, where $\tilde{X}_i = pX_i, \tilde{Y}_i = Y_i + T_iX_i$, where $T_i = 0$ if $i \notin \tau_M$. The relations in terms of this new basis are

$$\begin{aligned}
i-1 \notin \tau_M &: F\tilde{X}_{i-1} = \tilde{X}_i \\
i-1 \notin \tau_M &: F\tilde{Y}_{i-1} = p\tilde{Y}_i - T_i\tilde{X}_i \\
i-1 \in \tau_M &: F\tilde{X}_{i-1} = p\tilde{Y}_i + (a_ip - T_i)\tilde{X}_i \\
i-1 \in \tau_M &: F\tilde{Y}_{i-1} = T_{i-1}^{\sigma}\tilde{Y}_i + (1 + a_iT_{i-1}^{\sigma} - c_{i-1})\tilde{X}_i \\
i \notin \tau_M &: V\tilde{X}_{i+1} = p\tilde{X}_i \\
i \notin \tau_M &: V\tilde{Y}_{i+1} = \tilde{Y}_i + T_{i+1}^{\sigma^{-1}}\tilde{X}_i \\
i \in \tau_M &: V\tilde{X}_{i+1} = p\tilde{Y}_i - T_i\tilde{X}_i \\
i \in \tau_M &: V\tilde{Y}_{i+1} = (T_{i+1}^{\sigma^{-1}} - a_i^{\sigma^{-1}}p)\tilde{Y}_i + (1 + a_i^{\sigma^{-1}}T_i - c_i^{\sigma^{-1}})\tilde{X}_i
\end{aligned}$$

where $c_i = T_i^{\sigma}T_{i+1}/p$. In the following, we will deform by taking the values either $t_i = 0$ or $t_{i+1} = 0$, therefore we may assume $c_i = 0$.

We now look at the ith component $((F,V)\bar{M}_t)^i$ of $(F,V)M_t/pM_t$.

Case $i \notin \tau_M$:

(i) $i-1 \notin \tau_M$: $((F,V)\bar{M}_t)^i$ is generated by $\tilde{Y}_i + t_{i+1}^{p^{-1}}\tilde{X}_i, \tilde{X}_i$. Then $a_i(M_t) = 0$

(ii) $i-1 \in \tau_M$: $((F,V)\bar{M}_t)^i$ is generated by $\tilde{Y}_i + t_{i+1}^{p^{-1}}\tilde{X}_i$ and $t_{i-1}^p\tilde{Y}_i + (1 + a_it_{i-1}^p)\tilde{X}_i$. As the discriminant $\delta = 1 + a_it_{i-1}^p - t_{i+1}^{p^{-1}}t_{i-1}^p \neq 0$ for a general value of t_{i-1}, we have $a_i(M_t) = 0$.

Case $i \in \tau_M$: $((F,V)\bar{M}_t)^i$ contains $t_i\tilde{X}_i$ and $t_{i+1}^{p^{-1}}\tilde{Y}_i + (1 + a_i^{p^{-1}}t_i)\tilde{X}_i$, thus contains $t_i\tilde{X}_i$ and $t_{i+1}^{p^{-1}}\tilde{Y}_i + \tilde{X}_i$.

(i) $i-1 \notin \tau_M$: $((F,V)\bar{M}_t)^i$ is generated by $t_{i+1}^{p^{-1}}\tilde{Y}_i, \tilde{X}_i$.

(ii) $i-1 \in \tau_M$: $((F,V)\bar{M}_t)^i$ is generated by $t_{i+1}^{p^{-1}}\tilde{Y}_i + \tilde{X}_i$, $t_i\tilde{X}_i$, and $t_{i-1}^p\tilde{Y}_i + (1+a_i^p t_{i-1})\tilde{X}_i$. If $i+1 \notin \tau_M$, then $t_{i+1} = 0$ and it is generated by $\tilde{X}_i, t_{i-1}^p\tilde{Y}_i$.

In the case $i \in \tau_M$, we have the following
subcase 1. $i-1, i+1 \notin \tau_M$: $((F,V)\bar{M}_t)^i$ is generated by $\tilde{X}_i \Rightarrow a_i(M_t) = 1$.
subcase 2. $i-1 \notin \tau_M, i+1 \in \tau_M$: let $t_{i+1} \neq 0, t_i = 0 \Rightarrow a_i(M_t) = 0$.
subcase 3. $i-1 \in \tau_M, i+1 \notin \tau_M$: let $t_{i-1} \neq 0, t_i = 0 \Rightarrow a_i(M_t) = 0$.
subcase 4. $i-1 \in \tau_M, i+1 \in \tau_M$: take $t_i = 0, t_{i-1}^p \neq (1+a_i^p t_{i-1})t_{i+1}^{p^{-1}} \Rightarrow a_i(M_t) = 0$

Theorem 14.1. *Let $(A,\lambda,\iota) \in \mathcal{M}(k)$. Assume $a_i(A) = a_{i+1}(A) = 1$. Then there exist a deformation (A_t,λ_t,ι_t) keeping in the same slope such that either $a_i(A_t) = 0, a_{i+1}(A_t) = 1$ or $a_i(A_t) = 1, a_{i+1}(A_t) = 0$*

Corollary 14.2. *Let $\mathcal{M}^{(m)}$ be as in Section 5. Then $\overline{\mathcal{M}^{(m)}} = \cup_\tau \overline{\mathcal{M}_\tau^{(m)}}$. where τ runs over all generic a-numbers.*

Corollary 14.3. *If $\overline{\mathcal{M}_\tau^{(m+1)}} \subset \overline{\mathcal{M}_\tau^{(m)}}$ in $\mathcal{M}_\tau$ holds for all $m+1 \geq |\tau|$ and all generic a-numbers τ. Then $\overline{\mathcal{M}^{(m+1)}} \subset \overline{\mathcal{M}^{(m)}}$ for all $m < (f/2)$.*

Corollary 14.4. *The a-number of every generic point of $\mathcal{M}^{(m)}$ is generic.*

Conversely, we expect that the points of *every* generic a-numbers do form whole components of each slope stratum (if $\mathcal{M}_\tau^{(m)}$ is not empty). That is, we expect that if τ is generic each component of $\mathcal{M}_\tau^{(m)}$, if not empty, has dimension $g-m$. This is a refinement of Conjecture 5.2.5 in [9].

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