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Algebraic curves $A^{\circ l}(x) - U(y) = 0$ and arithmetic of orbits of rational functions

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Abstract

We give a description of pairs of complex rational functions A and U of degree at least two such that for every $d \geq 1$ the algebraic curve $A^{\circ d}(x) - U(y) = 0$ has a factor of genus zero or one. In particular, we show that if A is not a "generalized Lattès map", then this condition is satisfied if and only if there exists a rational function V such that $U \circ V = A^{\circ l}$ for some $l \geq 1$. We also prove a version of the dynamical Mordell-Lang conjecture, concerning intersections of orbits of points from $\mathbb{P}^1(K)$ under iterates of A with the value set $U(\mathbb{P}^1(K))$, if A and U are defined over a number field K.

1 Introduction

In this paper we solve the following problem posed in [3].

Problem 1.1. *Describe pairs of complex rational functions A and U of degree at least two such that for every $d \geq 1$ the algebraic curve*

$$A^{\circ d}(x) - U(y) = 0 \tag{1}$$

has an irreducible factor of genus zero or one.

The motivation for this problem comes from the arithmetic dynamics. Specifically, in the paper [3] the following problem was investigated: what are rational functions A defined over a number field K that have a K-orbit containing infinitely many distinct mth powers of elements from K ? If such an orbit exists, then for every $d \geq 1$ the algebraic curve

$$A^{\circ d}(x) - y^m = 0 \tag{2}$$

has infinitely many K-points, implying by the Faltings theorem that it has a factor of genus zero or one. Thus, a geometric counterpart of the initial arithmetic problem is to describe rational functions A such that all curves (1) have a factor of genus zero or one. Considering now instead of intersections of orbits of A with the set of mth powers intersections with the value set $U(\mathbb{P}^1(K))$ of an arbitrary rational function U, we arrive to Problem 1.1.

The paper [3] is based on painstaking calculations of possible ramifications of rational functions A such that every curve (1) has a factor of genus zero or one, and provides a very explicit description of such functions. In contrast, our approach is more geometric and provides an answer in the general case in terms of semiconjugacies and Galois coverings. Notice that Problem 1.1 is somehow similar to the following problem considered in the paper [13]: what are rational functions U for which there exists a sequence of rational functions C_d, $d \geq 1$, such that $\deg C_d \to \infty$ and for every $d \geq 1$ the curve

$$C_d(x) - U(y) = 0$$

is irreducible and of genus zero. It was shown in [13] that U satisfies this condition if and only if the Galois closure of the field extension $\mathbb{C}(z)/\mathbb{C}(U)$ has genus zero or one. Thus this condition holds also for solutions of Problem 1.1 whenever curves (1.1) are irreducible. However, Problem 1.1 is distinct from the problem considered in [13] in two important aspects. First, curves (1.1) can be reducible. Second, Problem 1.1 asks for a description of all *pairs* A, U such that curves (1.1) have an irreducible factor of genus zero or one, and not for a description of functions U for which *some* A with this property exists.

Let A and B be rational functions of degree at least two. Recall that the function B is called semiconjugate to the function A if there exists a non-constant rational function X such that the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ \downarrow X & & \downarrow X \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \qquad (3)$$

commutes. Semiconjugate rational functions appear naturally in complex and arithmetic dynamics (see e. g. the recent papers [4], [8], [15]). They are also closely related to Problem 1.1. Really, since the commutativity of diagram (1) implies that

$$A^{\circ d} \circ X = X \circ B^{\circ d}, \qquad d \geq 1,$$

setting U equal X we see that for every $d \geq 1$ curve (1.1) has a component parametrized by rational functions, and hence the genus of this component is zero.

More generally, if A, B and X satisfy (1), then curves (1.1) have a factor of genus zero for any rational function U which is a "compositional left factor" of the function $A^{\circ l} \circ X$ for some $l \geq 0$, where by a compositional left factor of a holomorphic map $f : R_1 \to R_2$ between Riemann surfaces we mean any holomorphic map $g : R' \to R_2$ such that $f = g \circ h$ for some holomorphic map $h : R_1 \to R'$. Indeed, it follows from (1) and

$$A^{\circ l} \circ X = U \circ V$$

that

$$A^{\circ l+k} \circ X = U \circ V \circ B^{\circ k}$$

for every $k \geq 0$, implying as above that the pair A, U is a solution of Problem 1.1. In particular, setting $B = A$ and $X = z$ in (1), we see that curves (1.1) have a factor of genus zero whenever U is a compositional left factor of some iterate $A^{\circ l}$, $l \geq 1$.

Semiconjugate rational functions were studied at length in the recent series of papers [12], [14], [17], [18], using the theory of orbifolds on Riemann surfaces, and our approach to Problem 1.1 is based on ideas and methods of these papers. Roughly speaking, our main result states that, unless A belongs to a special family of functions, all corresponding solutions U of Problem 1.1 can be obtained in the way described above from some *fixed* semiconjugacy (1), where X is a *Galois covering* depending on A only. Moreover, for "most" rational functions A this Galois covering is equal to z, meaning that a rational function U is a solution of Problem 1.1 if and only if U is a compositional left factor of $A^{\circ l}$ for some $l \geq 1$. In order to formulate our results explicitly we need several definitions.

An *orbifold* $\mathcal{O}$ on $\mathbb{CP}^1$ is a ramification function $\nu : \mathbb{CP}^1 \to \mathbb{N}$ which takes the value $\nu(z) = 1$ except at a finite set of points. We assume that considered orbifolds are *good* meaning that we forbid $\mathcal{O}$ to have exactly one point with $\nu(z) \neq 1$ or two such points z_1, z_2 with $\nu(z_1) \neq \nu(z_2)$. Let f be a rational function and $\mathcal{O}_1$, $\mathcal{O}_2$ orbifolds with ramifications functions ν_1 and ν_2. We say that $f : \mathcal{O}_1 \to \mathcal{O}_2$ is *a covering map* between orbifolds if for any $z \in \mathbb{CP}^1$ the equality

$$\nu_2(f(z)) = \nu_1(z) \deg_z f$$

holds. In case if the weaker condition

$$\nu_2(f(z)) = \nu_1(z) \mathrm{GCD}(\deg_z f, \nu_2(f(z))$$

is satisfied, we say that $f : \mathcal{O}_1 \to \mathcal{O}_2$ is *a minimal holomorphic map* between orbifolds.

In the above terms, *a Lattès map* can be defined as a rational function A such that $A : \mathcal{O} \to \mathcal{O}$ is a covering self-map for some orbifold $\mathcal{O}$ (see [10]). Following [18], say that a rational function A is *a generalized Lattès map* if there exists an orbifold $\mathcal{O}$, distinct from the non-ramified sphere, such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map. Thus, A is a Lattès map if there exists an orbifold $\mathcal{O}$ such that

$$\nu(A(z)) = \nu(z) \deg_z A, \qquad z \in \mathbb{CP}^1,$$

and a generalized Lattès map if there exists an orbifold $\mathcal{O}$ such that

$$\nu(A(z)) = \nu(z) \mathrm{GCD}(\deg_z A, \nu(A(z))), \qquad z \in \mathbb{CP}^1. \tag{4}$$

Finally, say that a rational function is *special* if it is either a Lattès map, or is conjugate to $z^{\pm n}$ or $\pm T_n$, where T_n is the Chebyshev polynomial.

For rational functions A and U denote by $g_d = g_d(A, U)$, $d \geq 1$, the minimal number g such that curve (1.1) has a component of genus g. In this notation our main result concerning Problem 1.1 is following.

Theorem 1.2. *Let A be a non-special rational function of degree at least two. Then there exist a rational Galois covering X_A and a rational function B such that the diagram*

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ \downarrow{\scriptstyle X_A} & & \downarrow{\scriptstyle X_A} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \tag{5}$$

commutes, and for a rational function U of degree at least two the sequence g_d, $d \geq 1$, is bounded if and only if U is a compositional left factor of $A^{\circ l} \circ X_A$ for some $l \geq 0$. In particular, if A is not a generalized Lattès map, then g_d, $d \geq 1$, is bounded if and only if U is a compositional left factor of $A^{\circ l}$ for some $l \geq 1$.

Notice that our method provides an explicit description of the Galois covering X_A appearing in Theorem 1.2 via the "maximal" orbifold $\mathcal{O}$ for which (1) is satisfied. In particular, X_A is defined by A in a unique way up to the change

$$X_A \to X_A \circ \mu,$$

where μ is a Möbius transformation.

Theorem 1.2 can be illustrated as follows. A "random" rational function A is not a generalized Lattès map. Thus, a rational function U is a solution of Problem 1.1 if and only if U is a compositional left factor of $A^{\circ l}$ for some $l \geq 1$. A simple example of a generalized Lattès map is provided by any function of the form $A = z^r R^n(z)$, where R is a rational function, $n \geq 2$, $r \geq 1$, and $\mathrm{GCD}(r, n) = 1$. Indeed, one can easily check that (1) is satisfied for the orbifold $\mathcal{O}$ defined by the conditions

$$\nu(0) = n, \quad \nu(\infty) = n.$$

With a few exceptions, the rational function $A = z^r R^n(z)$ is not special, and diagram (1.2) from Theorem 1.2 has the form

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{z^r R(z^n)} & \mathbb{CP}^1 \\ \downarrow{\scriptstyle z^n} & & \downarrow{\scriptstyle z^n} \\ \mathbb{CP}^1 & \xrightarrow{z^r R^n(z)} & \mathbb{CP}^1 . \end{array}$$

Thus, a rational function U is a solution of Problem 1.1 if and only if there exists $l \geq 0$ such that U is a compositional left factor of the function

$$(z^r R^n(z))^{\circ l} \circ z^n = z^n \circ (z^r R(z^n))^{\circ l}.$$

Assume now that considered rational functions A and U are defined over a number field K. Then Theorem 1.2 combined with some arithmetical properties of solutions of (1) implies a statement, concerning intersections of orbits of points from $\mathbb{P}^1(K)$ under iterates of A with the value set $U(\mathbb{P}^1(K))$, which can be considered as a version of the dynamical Mordell-Lang conjecture.

Recall that the dynamical Mordell-Lang conjecture states that if f is an endomorphism of a quasiprojective variety V over $\mathbb{C}$, then for any point $z_0 \in V$ and any subvariety W of V, the set of indices n such that the n-th iterate of z_0 under f lies in W is a finite union of arithmetic progressions (see [2] and the bibliography therein). In particular, this implies that if the f-orbit of z_0 has an infinite intersection with a proper subvariety W, then its Zariski closure is contained in a finite union of proper subvarieties and therefore cannot coincide with whole V. Point out that singletons are considered as arithmetic progressions with the common difference equal zero, so any finite set is a union of arithmetic progressions.

Notice that the dynamical Mordell-Lang conjecture in a sense is complementary to the conjecture proposed in [8] (see also [1], [20]) which states that if f is a dominant endomorphism of a quasiprojective variety V defined over an algebraically closed field K of characteristic zero for which there exists no non-constant rational function g satisfying $g \circ f = g$, then there is a point $z_0 \in V(K)$ whose f-orbit is Zariski dense in V.

It was conjectured in the paper [3] that the conclusion of the dynamical Mordell-Lang conjecture remains true if V, f, and z_0 are defined over a number field K, while W is allowed to be the value set of a K-morphism instead of a subvariety. More precisely, it was conjectured that if A is a rational function of degree at least two and C is a curve defined over a number field K, then for any K-morphism $U : C \to \mathbb{CP}^1$ and $z_0 \in \mathbb{P}^1(K)$, the index set

$$I = \{n \geq 0 : \quad A^{\circ n}(z_0) \in U(C(K))\}$$

is a finite union of arithmetical progressions. In the paper we prove this conjecture[1] in the case where A is non-special and the morphism U is a rational function on $\mathbb{CP}^1$.

Theorem 1.3. *Let A and U be rational functions of degree at least two defined over a number field K, and z_0 a point from $\mathbb{P}^1(K)$. Assume that A is not special. Then the set of indices n such that $A^{\circ n}(z_0) \in U(\mathbb{P}^1(K))$ is a finite union of arithmetic progressions. Moreover, if A is not a generalized Lattès map, then either the above set is finite, or $A^{\circ n}(z_0)$ belongs to $U(\mathbb{P}^1(K))$ for all but finitely many n.*

The paper is organized as follows. In the second section we collect necessary definitions and results concerning orbifolds, fiber products, and generalized Lattès maps mostly proved in the papers [12], [18]. In particular, we explain the construction of the Galois covering X_A associated with a non-special rational function A.

In the third section, using lower bounds on the genus of an algebraic curve of the form

$$C(x) - U(y) = 0,$$

[1] A proof of this conjecture was announced by T. Hyde and M. Zieve on the "Workshop on Interactions between Model Theory and Arithmetic Dynamics" in July 2016 at the Fields Institut. However, a complete proof of their results is still not available to date.

where C and U are rational function, obtained in the paper [13], we prove Theorem 1.2. In fact, we consider a more general version of Problem 1.1 allowing U to be a holomorphic map

$$U : R \to \mathbb{CP}^1,$$

where R is a compact Riemann surface, and considering instead of curves (1.1) fiber products of coverings U and $A^{\circ d}$, $d \geq 1$. We also solve Problem 1.1 for special A. Namely, for A conjugate to $z^{\pm n}$ or $\pm T_n$ we list corresponding U explicitly, while for Lattès maps A we provide a description of U in terms of decompositions of certain meromorphic functions related with discrete subgroups of $Aut(\mathbb{C})$.

In the fourth section we prove Theorem 1.3. For this purpose we prove a result concerning fields of definiton of semiconjugate rational functions which is interesting on its own. Specifically, using the relation between semiconjugate rational functions and finite subgroups of $Aut(\mathbb{CP}^1)$ established in [12], we show that if functions A and X in (1) are defined over a number field K and the algebraic curve

$$A(x) - X(y) = 0$$

is irreducible, then some *iterate* of the function B is also defined over K. Combined with Theorem 1.2 this result permits to prove Theorem 1.3. Finally, we provide an example illustrating results of the paper.

2 Orbifolds and generalized Lattès maps

2.1 Riemann surface orbifolds

In this section we recall basic definitions concerning orbifolds on Riemann surfaces (see [9], Appendix E) and some results and constructions from the papers [12], [18]. We also prove some additional related results used later.

A Riemann surface orbifold is a pair $\mathcal{O} = (R, \nu)$ consisting of a Riemann surface R and a ramification function $\nu : R \to \mathbb{N}$ which takes the value $\nu(z) = 1$ except at isolated points. For an orbifold $\mathcal{O} = (R, \nu)$ the *Euler characteristic* of $\mathcal{O}$ is the number

$$\chi(\mathcal{O}) = \chi(R) + \sum_{z \in R} \left(\frac{1}{\nu(z)} - 1 \right),$$

the set of *singular points* of $\mathcal{O}$ is the set

$$c(\mathcal{O}) = \{z_1, z_2, \dots, z_s, \dots\} = \{z \in R \mid \nu(z) > 1\},$$

and the *signature* of $\mathcal{O}$ is the set

$$\nu(\mathcal{O}) = \{\nu(z_1), \nu(z_2), \dots, \nu(z_s), \dots\}.$$

For orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ write

$$\mathcal{O}_1 \preceq \mathcal{O}_2 \qquad (6)$$

if $R_1 = R_2$, and for any $z \in R_1$ the condition $\nu_1(z) \mid \nu_2(z)$ holds. Clearly, (2.1) implies that

$$\chi(\mathcal{O}_1) \geq \chi(\mathcal{O}_2).$$

Let $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ be orbifolds and $f : R_1 \to R_2$ a holomorphic branched covering map. Say that $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a *covering map between orbifolds* if for any $z \in R_1$ the equality

$$\nu_2(f(z)) = \nu_1(z)\deg_z f \tag{7}$$

holds, where $\deg_z f$ is the local degree of f at the point z. If for any $z \in R_1$ instead of (2.1) the weaker condition

$$\nu_2(f(z)) \mid \nu_1(z)\deg_z f \tag{8}$$

is satisfied, we say that $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a *holomorphic map between orbifolds.*

If $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds with compact R_1 and R_2, then the Riemann-Hurwitz formula implies that

$$\chi(\mathcal{O}_1) = d\chi(\mathcal{O}_2), \tag{9}$$

where $d = \deg f$. For holomorphic maps the following statement is true (see [12], Proposition 3.2).

Proposition 2.1. *Let $f : \mathcal{O}_1 \to \mathcal{O}_2$ be a holomorphic map between orbifolds with compact R_1 and R_2. Then*

$$\chi(\mathcal{O}_1) \leq \chi(\mathcal{O}_2)\deg f, \tag{10}$$

and the equality holds if and only if $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds. □

Let R_1, R_2 be Riemann surfaces and $f : R_1 \to R_2$ a holomorphic branched covering map. Assume that R_2 is provided with a ramification function ν_2. In order to define a ramification function ν_1 on R_1 so that f would be a holomorphic map between orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ we must satisfy condition (2.1), and it is easy to see that for any $z \in R_1$ a minimal possible value for $\nu_1(z)$ is defined by the equality

$$\nu_2(f(z)) = \nu_1(z)\mathrm{GCD}(\deg_z f, \nu_2(f(z))). \tag{11}$$

In case if (2.1) is satisfied for any $z \in R_1$, we say that $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a *minimal holomorphic map between orbifolds.* It follows from the definition that for any orbifold $\mathcal{O} = (R, \nu)$ and holomorphic branched covering map $f : R' \to R$ there exists a unique orbifold structure ν' on R' such that f becomes a minimal holomorphic map between orbifolds. We will denote the corresponding orbifold by $f^*\mathcal{O}$.

Below we will use the following property of the map $\mathcal{O} \to f^*\mathcal{O}$ (see [12], Corollary 4.2).

Proposition 2.2. *Let $f : R_1 \to R'$ and $g : R' \to R_2$ be holomorphic branched covering maps, and $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ orbifolds. Assume that $g \circ f : \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map (resp. a covering map). Then $f : \mathcal{O}_1 \to g^*\mathcal{O}_2$ and $g : g^*\mathcal{O}_2 \to \mathcal{O}_2$ are minimal holomorphic maps (resp. covering maps).* □

A universal covering of an orbifold $\mathcal{O}$ is a covering map between orbifolds $\theta_{\mathcal{O}} : \widetilde{\mathcal{O}} \to \mathcal{O}$ such that $\widetilde{R}$ is simply connected and $\widetilde{\mathcal{O}}$ is non-ramified, that is $\widetilde{\nu}(z) \equiv 1$. If $\theta_{\mathcal{O}}$ is such a map, then there exists a group $\Gamma_{\mathcal{O}}$ of conformal automorphisms of $\widetilde{R}$ such that the equality

$$\theta_{\mathcal{O}}(z_1) = \theta_{\mathcal{O}}(z_2)$$

holds for $z_1, z_2 \in \widetilde{R}$ if and only if $z_1 = \sigma(z_2)$ for some $\sigma \in \Gamma_{\mathcal{O}}$. A universal covering exists and is unique up to a conformal isomorphism of $\widetilde{R}$ whenever $\mathcal{O}$ is *good*, that is distinct from the Riemann sphere with one ramified point or with two ramified points z_1, z_2 such that $\nu(z_1) \neq \nu(z_2)$. Furthermore, $\widetilde{R} = \mathbb{D}$ if and only if $\chi(\mathcal{O}) < 0$, $\widetilde{R} = \mathbb{C}$ if and only if $\chi(\mathcal{O}) = 0$, and $\widetilde{R} = \mathbb{CP}^1$ if and only if $\chi(\mathcal{O}) > 0$ (see e.g. [5], Section IV.9.12). Below we always will assume that considered orbifolds are good. Abusing notation we will use the symbol $\widetilde{\mathcal{O}}$ both for the orbifold and for the Riemann surface $\widetilde{R}$.

Covering maps between orbifolds lift to isomorphisms between their universal coverings. More generally, the following proposition is true (see [12], Proposition 3.1).

Proposition 2.3. *Let $f : \mathcal{O}_1 \to \mathcal{O}_2$ be a holomorphic map between orbifolds. Then for any choice of $\theta_{\mathcal{O}_1}$ and $\theta_{\mathcal{O}_2}$ there exist a holomorphic map $F : \widetilde{\mathcal{O}_1} \to \widetilde{\mathcal{O}_2}$ and a homomorphism $\varphi : \Gamma_{\mathcal{O}_1} \to \Gamma_{\mathcal{O}_2}$ such that the diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{O}_1} & \xrightarrow{F} & \widetilde{\mathcal{O}_2} \\ \downarrow{\scriptstyle \theta_{\mathcal{O}_1}} & & \downarrow{\scriptstyle \theta_{\mathcal{O}_2}} \\ \mathcal{O}_1 & \xrightarrow{f} & \mathcal{O}_2 \end{array} \qquad (12)$$

is commutative and for any $\sigma \in \Gamma_{\mathcal{O}_1}$ the equality

$$F \circ \sigma = \varphi(\sigma) \circ F \qquad (13)$$

holds. The map F is defined by $\theta_{\mathcal{O}_1}$, $\theta_{\mathcal{O}_2}$, and f uniquely up to a transformation $F \to g \circ F$, where $g \in \Gamma_{\mathcal{O}_2}$. In the other direction, for any holomorphic map $F : \widetilde{\mathcal{O}_1} \to \widetilde{\mathcal{O}_2}$ which satisfies (2.3) *for some homomorphism $\varphi : \Gamma_{\mathcal{O}_1} \to \Gamma_{\mathcal{O}_2}$ there exists a uniquely defined holomorphic map between orbifolds $f : \mathcal{O}_1 \to \mathcal{O}_2$ such that diagram* (2.3) *is commutative. The holomorphic map F is an isomorphism if and only if f is a covering map between orbifolds.* □

With each holomorphic function $f : R_1 \to R_2$ between compact Riemann surfaces one can associate in a natural way two orbifolds $\mathcal{O}_1^f = (R_1, \nu_1^f)$ and

$\mathcal{O}_2^f = (R_2, \nu_2^f)$, setting $\nu_2^f(z)$ equal to the least common multiple of local degrees of f at the points of the preimage $f^{-1}\{z\}$, and

$$\nu_1^1(z) = \frac{\nu_2^f(f(z))}{\deg_z f}.$$

By construction,

$$f : \mathcal{O}_1^f \to \mathcal{O}_2^f$$

is a covering map between orbifolds. It is easy to see that this covering map is minimal in the following sense. For any covering map between orbifolds $f : \mathcal{O}_1 \to \mathcal{O}_2$ we have:

$$\mathcal{O}_1^f \preceq \mathcal{O}_1, \quad \mathcal{O}_2^f \preceq \mathcal{O}_2.$$

Notice that the orbifolds $\mathcal{O}_1^f$ and $\mathcal{O}_2^f$ are good (see [12], Lemma 4.2).

Theorem 2.4. *Let $f : R_1 \to R_2$ be a holomorphic map between compact Riemann surfaces and $\mathcal{O}_2 = (R_2, \nu_2)$ an orbifold. Then f is a compositional left factor of $\theta_{\mathcal{O}_2}$ if and only if $\mathcal{O}_2^f \preceq \mathcal{O}_2$. Furthermore, for any decomposition $\theta_{\mathcal{O}_2} = f \circ \psi$, where $\psi : \widetilde{\mathcal{O}_2} \to R_1$ is a holomorphic map, the equality $\psi = \theta_{f^*\mathcal{O}_2}$ holds, and the map $f : f^*\mathcal{O}_2 \to \mathcal{O}_2$ is a covering map between orbifolds. In particular, $\theta_{\mathcal{O}_2^f} = f \circ \theta_{\mathcal{O}_1^f}$.*

Proof. The "only if" part follows from the definitions and the chain rule. In the other direction, let ψ be the analytic continuation of $f^{-1} \circ \theta_{\mathcal{O}_2}$, where f^{-1} is a germ of the function inverse to f. It follows easily from the definitions and the condition $\mathcal{O}_2^f \preceq \mathcal{O}_2$ that ψ has no ramification. Therefore, since $\widetilde{\mathcal{O}_2}$ is simply connected, ψ is single-valued, and $\theta_{\mathcal{O}_2} = f \circ \psi$.

Finally, it follows from the equality $\theta_{\mathcal{O}_2} = f \circ \psi$ by Proposition 2.2 that

$$f : f^*\mathcal{O}_2 \to \mathcal{O}_2, \qquad \psi : \widetilde{\mathcal{O}_2} \to f^*\mathcal{O}_2$$

are covering maps between orbifolds, implying that $\psi = \theta_{f^*\mathcal{O}_2}$, since $\widetilde{\mathcal{O}_2}$ is a non-ramified simply-connected Riemann surface. In particular, if $\mathcal{O}_2 = \mathcal{O}_2^f$, then $f^*\mathcal{O}_2^f = \mathcal{O}_1^f$. □

Corollary 2.5. *Let $f : R \to \mathbb{CP}^1$ be a holomorphic map between compact Riemann surfaces. Then $\chi(\mathcal{O}_2^f) > 0$ implies that $g(R) = 0$. On the other hand, $\chi(\mathcal{O}_2^f) = 0$ implies that $g(R) \leq 1$.*

Proof. If $\chi(\mathcal{O}_2^f) > 0$, then $\widetilde{\mathcal{O}_2^f} = \mathbb{CP}^1$. Thus, by Theorem 2.4, $\theta_{\mathcal{O}_1^f} : \mathbb{CP}^1 \to R$ is a holomorphic map, implying that $g(R) = 0$. Similarly, if $\chi(\mathcal{O}_2^f) = 0$, then $\theta_{\mathcal{O}_1^f} : \mathbb{C} \to R$ is a holomorphic map, implying that $g(R) \leq 1$, since otherwise lifting $\theta_{\mathcal{O}_1^f}$ to a map between universal coverings (in the usual sense) we would obtain a contradiction with the Liouville theorem. □

Finally, we will need the following simple statement.

Lemma 2.6. *Let $f : R \to \mathbb{CP}^1$ be a holomorphic map between compact Riemann surfaces. Assume that $\mathcal{O}_2^f$ is defined by the conditions*

$$\nu_2^f(0) = n, \qquad \nu_2^f(\infty) = n. \tag{14}$$

Then $g(R) = 0$, and $A = z^n \circ \mu$ for some Möbius transformation μ. On the other hand, if $\mathcal{O}_2^f$ is defined by the conditions

$$\nu_2^f(-1) = 2, \qquad \nu_2^f(1) = 2, \qquad \nu_2^f(\infty) = n, \tag{15}$$

then $g(R) = 0$, and either

$$f = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) \circ \mu,$$

or $f = \pm T_n \circ \mu$ for some Möbius transformation μ.

Proof. Since by Theorem 2.4 the map f is a compositional left factor of $\theta_{\mathcal{O}_2^f}$, and the universal coverings for orbifolds given by (2.6) and (2.6) are rational functions

$$C_n = z^n, \qquad D_n = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) \tag{16}$$

correspondingly, the statement follows from the well know fact that any compositional left factor of C_n has the form $C_d \circ \mu$ for some Möbius transformation μ and $d|n$, while any compositional left factor of D_n has the form $\pm T_d \circ \mu$ or $D_d \circ \mu$ for some Möbius transformation μ and $d|n$ (see e.g. [16], Subsection 4.1 and 4.2). □

2.2 Functional equations, fiber products, and orbifolds

Orbifolds $\mathcal{O}_1^f$ and $\mathcal{O}_2^f$ defined above are useful for the study of the functional equation

$$f \circ p = g \circ q, \tag{17}$$

where

$$p : R \to C_1, \quad f : C_1 \to \mathbb{CP}^1, \quad q : R \to C_2, \quad g : C_2 \to \mathbb{CP}^1$$

are holomorphic maps between compact Riemann surfaces. Recall that solutions of this equation for fixed f and g can be described in terms of the *fiber product* of f and g. For basic properties of fiber products and their relations with functional equation (2.2) we refer the reader to [11], Sections 2 and 3. In practical terms, such a product is a collection

$$(C_1, f) \times (C_2, g) = \bigcup_{j=1}^{n(f,g)} \{R_j, p_j, q_j\}, \tag{18}$$

where R_j are compact Riemann surfaces provided with holomorphic maps

$$p_j : R_j \to C_1, \quad q_j : R_j \to C_2$$

satisfying (2.2), such that any solution of (2.2) factors through some of these solutions. More precisely, the following statement holds.

Theorem 2.7. *For any holomorphic maps $f : C_1 \to \mathbb{CP}^1$ and $g : C_2 \to \mathbb{CP}^1$ there exist holomorphic maps $p_j : R_j \to C_1$, $q_j : R_j \to C_2$, $1 \le j \le n(f,g)$, such that the following conditions are satisfied:*

(i) $f \circ p_j = g \circ q_j$,

(ii) $\sum_j \deg p_j = \deg g$ *and* $\sum_j \deg q_j = \deg f$,

(iii) for any solution p, q of (2.2) *there exist an index j and a holomorphic function $w : R \to R_j$ such that $p = p_j \circ w$, $q = q_j \circ w$.* □

Notice that although the fiber product is symmetric with respect to f and g as a geometric object, notation (2.2) assumes that condition (i) holds, that is if we exchange f and g in the left side of (2.2) we must exchange p_j and q_j in the right side. Properties listed in Theorem 2.7 define R_j, p_j, q_j, $1 \le j \le n(f,g)$, in a unique way up to natural isomorphisms, and we will call the surfaces R_j, $1 \le j \le n(f,g)$, *irreducible components* of the fiber product of f and g. In case if f and g are rational functions, these irreducible components are simply normalizations of irreducible components of the algebraic curve

$$f(x) - g(y) = 0$$

(see e.g. [11], Proposition 2.4). Since the degree of every function

$$h_j = f \circ p_j = g \circ q_j$$

is divisible by $\mathrm{LCM}(\deg f, \deg g)$ and

$$\sum_{j=1}^{n(f,g)} h_j = \deg f \deg g$$

by (ii), for the number of irreducible components $n(f,g)$ the inequality

$$n(f,g) \le \mathrm{GCD}(\deg f, \deg g) \tag{19}$$

holds.

Say that holomorphic maps $p : R \to C_1$ and $q : R \to C_2$ *have no non-trivial common compositional right factor*, if the equalities

$$p = \widetilde{p} \circ w, \quad q = \widetilde{q} \circ w,$$

where $w: R \to \widetilde{R}$, $\widetilde{p}: \widetilde{R} \to C_1$, $\widetilde{q}: \widetilde{R} \to C_2$ are holomorphic maps between compact Riemann surfaces, imply that $\deg w = 1$. By Theorem 2.7, if such p and q satisfy (2.2), then

$$p = p_j \circ w, \qquad q = q_j \circ w$$

for some j, $1 \le j \le n(f,g)$, and an *isomorphism* $w: R_j \to R_j$.

A solution f, p, g, q of (2.2) is called *good* if $n(f,g) = 1$, and p and q have no non-trivial common compositional right factor. In this notation the following statement holds (see [12], Theorem 4.2).

Theorem 2.8. *Let f, p, g, q be a good solution of* (2.2)*. Then the commutative diagram*

$$\begin{array}{ccc} \mathcal{O}_1^q & \xrightarrow{\;p\;} & \mathcal{O}_1^f \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle f} \\ \mathcal{O}_2^q & \xrightarrow{\;g\;} & \mathcal{O}_2^f \end{array}$$

consists of minimal holomorphic maps between orbifolds. □

Of course, vertical arrows in the above diagram are holomorphic maps simply by definition. The meaning of the theorem is that the branching of f and q to a certain extent defines the branching of g and p and vice versa.

Below we will use the following criterion (see [12], Lemma 2.1).

Lemma 2.9. *A solution f, p, g, q of* (2.2) *is good whenever any two of the following three conditions are satisfied:*

- *the fiber product of f and g has a unique component,*
- *p and q have no non-trivial common compositional right factor,*
- $\deg f = \deg q, \quad \deg g = \deg p.$ □

Finally, we will need the following result concerning fiber products of compositions.

Theorem 2.10. *Let $f: C_1 \to \mathbb{CP}^1$, $g: C_2 \to \mathbb{CP}^1$, and $u: C_3 \to C_2$ be holomorphic maps between compact Riemann surfaces. Assume that*

$$(C_1, f) \times (C_2, g) = \bigcup_{j=1}^{n(f,g)} \{R_j, p_j, q_j\}$$

and

$$(R_j, q_j) \times (C_3, u) = \bigcup_{i=1}^{n(u,q_j)} \{R_{ij}, p_{ij}, q_{ij}\}, \qquad 1 \le j \le n(f,g).$$

Then

$$(C_1, f) \times (C_3, g \circ u) = \bigcup_{j=1}^{n(f,g)} \bigcup_{i=1}^{n(u,q_j)} \{R_{ij}, p_j \circ p_{ij}, q_{ij}\}.$$

Proof. Considering for j, $1 \le j \le n(f,g)$, and i, $1 \le i \le n(u,q_j)$, the commutative diagram

$$\begin{array}{ccccc} R_{ij} & \xrightarrow{p_{ij}} & R_j & \xrightarrow{p_j} & C_1 \\ \Big\downarrow{\scriptstyle q_{ij}} & & \Big\downarrow{\scriptstyle q_j} & & \Big\downarrow{\scriptstyle f} \\ C_3 & \xrightarrow{u} & C_2 & \xrightarrow{g} & \mathbb{CP}^1\,, \end{array}$$

we see that

$$(g \circ u) \circ q_{ij} = f \circ (p_j \circ p_{ij}),$$

$$\sum_{j=1}^{n(f,g)} \sum_{i=1}^{n(u,q_j)} \deg q_{ij} = \sum_{j=1}^{n(f,g)} \deg q_j = \deg f,$$

and

$$\sum_{j=1}^{n(f,g)} \sum_{j=1}^{n(u,p_j)} \deg\,(p_j \circ p_{ij}) = \sum_{j=1}^{n(f,g)} \deg p_j \deg u = \deg g \deg u.$$

Thus, conditions (i) and (ii) of Theorem 2.7 are satisfied.

Furthermore, the equality

$$(g \circ u) \circ q = f \circ p,$$

where p and q are holomorphic maps between compact Riemann surfaces, implies that

$$u \circ q = q_j \circ w, \qquad p = p_j \circ w$$

for some holomorphic map w and j, and the first from these equalities implies in turn that

$$q = q_{ij} \circ \widetilde{w}, \qquad w = p_{ij} \circ \widetilde{w}$$

for some holomorphic map $\widetilde{w}$ and i. Thus,

$$p = p_j \circ p_{ij} \circ \widetilde{w}, \qquad q = q_{ij} \circ \widetilde{w},$$

and hence condition (iii) is also satisfied. □

Corollary 2.11. *Let $f : C_1 \to \mathbb{CP}^1$, $g : C_2 \to \mathbb{CP}^1$, and $u : C_3 \to C_2$ be holomorphic maps between compact Riemann surfaces. Then $(C_1, f) \times (C_3, g \circ u)$ has a unique irreducible component $\{R, p, q\}$ if and only if $(C_1, f) \times (C_2, g)$ has a unique irreducible component $\{R_1, p_1, q_1\}$ and $(R_1, q_1) \times (C_3, u)$ has a unique irreducible component $\{R_2, p_2, q_2\}$. In case if this condition is satisfied, the equality $\{R, p, q\} = \{R_2, p_1 \circ p_2, q_2\}$ holds.* □

Corollary 2.12. *Let R be a compact Riemann surface, $U : R \to \mathbb{CP}^1$ a holomorphic map, and A a rational function. Then there exists $d_0 \ge 1$ such that*

$$n(A^{\circ d}, U) = n(A^{\circ d_0}, U) \tag{20}$$

for all $d \ge d_0$.

Proof. Clearly, Theorem 2.10 implies that for every $d \geq 1$ the inequality

$$n(A^{\circ(d+1)}, U) \geq n(A^{\circ d}, U)$$

holds. On the other hand, by (2.2), for every $d \geq 1$ we have:

$$n(A^{\circ d}, U) \leq \text{GCD}(A^{\circ d}, U) \leq \deg U.$$

Therefore, there exists $d_0 \geq 1$ such that (2.12) holds for all $d \geq d_0$. $\square$

2.3 Generalized Lattès maps

Most of orbifolds considered in this paper are defined on $\mathbb{CP}^1$. For such orbifolds we will omit the Riemann surface R in the definition of $\mathcal{O} = (R, \nu)$ meaning that $R = \mathbb{CP}^1$. Signatures of orbifolds on $\mathbb{CP}^1$ with non-negative Euler characteristic, and corresponding $\Gamma_{\mathcal{O}}$ and $\theta_{\mathcal{O}}$ can be described explicitly as follows. If $\mathcal{O}$ is an orbifold distinct from the non-ramified sphere, then $\chi(\mathcal{O}) = 0$ if and only if the signature of $\mathcal{O}$ belongs to the list

$$\{2,2,2,2\} \quad \{3,3,3\}, \quad \{2,4,4\}, \quad \{2,3,6\}, \tag{21}$$

and $\chi(\mathcal{O}) > 0$ if and only if the signature of $\mathcal{O}$ belongs to the list

$$\{n,n\}, \quad n \geq 2, \quad \{2,2,n\}, \quad n \geq 2, \quad \{2,3,3\}, \quad \{2,3,4\}, \quad \{2,3,5\}. \tag{22}$$

Groups $\Gamma_{\mathcal{O}} \subset Aut(\mathbb{C})$ corresponding to orbifolds $\mathcal{O}$ with signatures (2.3) are generated by translations of $\mathbb{C}$ by elements of some lattice $L \subset \mathbb{C}$ of rank two and the rotation $z \to \varepsilon z$, where ε is an nth root of unity with n equal to 2,3,4, or 6, such that $\varepsilon L = L$ (see [10], or [5], Section IV.9.5). Accordingly, the functions $\theta_{\mathcal{O}}$ may be written in terms of the corresponding Weierstrass functions as $\wp(z)$, $\wp'(z)$, $\wp^2(z)$, and $\wp'^2(z)$. Groups $\Gamma_{\mathcal{O}} \subset Aut(\mathbb{CP}^1)$ corresponding to orbifolds $\mathcal{O}$ with signatures (2.3) are the well-known finite subgroups C_n, D_{2n}, A_4, S_4, A_5 of $Aut(\mathbb{CP}^1)$, and the functions $\theta_{\mathcal{O}}$ are Galois coverings of $\mathbb{CP}^1$ by $\mathbb{CP}^1$ of degrees n, $2n$, 12, 24, 60, calculated for the first time by Klein in [7].

A *Lattès map* can be defined as a rational function A of degree at least two such that $A : \mathcal{O} \to \mathcal{O}$ is a covering self-map for some orbifold $\mathcal{O}$ (see [10]). Thus, A is a Lattès map if there exists an orbifold $\mathcal{O}$ such that for any $z \in \mathbb{CP}^1$ the equality

$$\nu(A(z)) = \nu(z)\deg_z A \tag{23}$$

holds. By formula (2.1), such $\mathcal{O}$ necessarily satisfies $\chi(\mathcal{O}) = 0$. Furthermore, for given A there might be at most one orbifold such that (2.3) holds (see [10] and [18], Corollary 4.5).

Following [18], say that a rational function A of degree at least two is a *generalized Lattès map* if there exists an orbifold $\mathcal{O}$, distinct from the non-ramified sphere, such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic self-map between orbifolds, that is for any $z \in \mathbb{CP}^1$ the equality

$$\nu(A(z)) = \nu(z)\text{GCD}(\deg_z A, \nu(A(z))) \tag{24}$$

holds. By inequality (2.1), such $\mathcal{O}$ satisfies $\chi(\mathcal{O}) \geq 0$. Naturally, since (2.3) implies (2.3), any ordinary Lattès map is a generalized Lattès map.

In general, for given A there might be several orbifolds $\mathcal{O}$ satisfying (2.3), and even infinitely many such orbifolds. For example, $z^{\pm d} : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map for any $\mathcal{O}$ defined by the conditions

$$\nu(0) = \nu(\infty) = n, \quad n \geq 2, \quad \mathrm{GCD}(d, n) = 1, \tag{25}$$

and $\pm T_d : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map for any $\mathcal{O}$ defined by the conditions

$$\nu(-1) = \nu(1) = 2, \quad \nu(\infty) = n, \quad n \geq 1, \quad \mathrm{GCD}(d, n) = 1. \tag{26}$$

For odd d, additionally, $\pm T_d : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map for $\mathcal{O}$ defined by

$$\nu(1) = 2, \quad \nu(\infty) = 2, \tag{27}$$

or

$$\nu(-1) = 2, \quad \nu(\infty) = 2. \tag{28}$$

Nevertheless, the following statement holds (see [18], Theorem 1.2).

Theorem 2.13. *Let A be a rational function of degree at least two not conjugate to $z^{\pm d}$ or $\pm T_d$. Then there exists an orbifold $\mathcal{O}_0^A$ such that $A : \mathcal{O}_0^A \to \mathcal{O}_0^A$ is a minimal holomorphic map between orbifolds, and for any orbifold $\mathcal{O}$ such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds the relation $\mathcal{O} \preceq \mathcal{O}_0^A$ holds. Furthermore, $\mathcal{O}_0^{A^{\circ l}} = \mathcal{O}_0^A$ for any $l \geq 1$.* □

Clearly, generalized Lattès maps are exactly rational functions for which the orbifold $\mathcal{O}_0^A$ is distinct from the non-ramified sphere, completed by the functions $z^{\pm d}$ or $\pm T_d$ for which the orbifold $\mathcal{O}_0^A$ is not defined. Furthermore, ordinary Lattès maps are exactly rational functions for which $\chi(\mathcal{O}_0^A) = 0$. Indeed, if $A : \mathcal{O} \to \mathcal{O}$ is a covering map, then it follows from $\mathcal{O} \preceq \mathcal{O}_0^A$ that $\chi(\mathcal{O}) \geq \chi(\mathcal{O}_0^A)$. Therefore, since $\chi(\mathcal{O}_0^A) \geq 0$ and $\chi(\mathcal{O}) = 0$, the equality $\chi(\mathcal{O}_0^A) = 0$ holds. Notice also that if A is a Lattès map, then $A : \mathcal{O}_0^A \to \mathcal{O}_0^A$ is a covering map by Proposition 2.1.

For exceptional functions $z^{\pm d}$ and $\pm T_d$ orbifolds for which (2.3) holds are described as follows (see [18], Theorem 5.2).

Theorem 2.14. *Let $\mathcal{O}$ be an orbifold distinct from the non-ramified sphere.*

1. *The map $z^{\pm d} : \mathcal{O} \to \mathcal{O}$, $d \geq 2$, is a minimal holomorphic map between orbifolds if and only if $\mathcal{O}$ is defined by conditions* (2.3).

2. *The map $\pm T_d : \mathcal{O} \to \mathcal{O}$, $d \geq 2$, is a minimal holomorphic map between orbifolds if and only if either $\mathcal{O}$ is defined by conditions* (2.3)*, or d is odd and $\mathcal{O}$ is defined by conditions* (2.3) *or* (2.3). □

If A is a generalized Lattès map, then $c(\mathcal{O}_0^A)$ is a subset of the set $c(\mathcal{O}_2^A)$ consisting of critical values of A, unless $\deg A \leq 4$. More generally, the following statement holds (see [18], Lemma 5.8).

Lemma 2.15. *Let A be a rational function of degree at least five, and $\mathcal{O}_1$, $\mathcal{O}_2$ orbifolds distinct from the non-ramified sphere such that $A : \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map between orbifolds. Assume that $\chi(\mathcal{O}_1) \geq 0$. Then $c(\mathcal{O}_2) \subseteq c(\mathcal{O}_2^A)$.* □

Say that a rational function is *special* if it is either a Lattès map, or is conjugate to $z^{\pm n}$ or $\pm T_n$, where T_n is the Chebyshev polynomial. If A is a generalized Lattès map which is not special, then $\chi(\mathcal{O}_0^A) > 0$ and diagram (2.3) takes the form

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{F} & \mathbb{CP}^1 \\ \Big\downarrow \theta_{\mathcal{O}_0^A} & & \Big\downarrow \theta_{\mathcal{O}_0^A} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \tag{29}$$

For such A the homomorphism (2.3) from Proposition 2.3 is an *automorphism.* More generally, for orbifolds $\mathcal{O}$ with $\chi(\mathcal{O}) > 0$ the following more precise version of Proposition 2.3 holds (see [12], Theorem 5.1).

Theorem 2.16. *Let A and F be rational functions of degree at least two and $\mathcal{O}$ an orbifold with $\chi(\mathcal{O}) > 0$ such that $A : \mathcal{O} \to \mathcal{O}$ is a holomorphic map between orbifolds and the diagram*

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{F} & \mathbb{CP}^1 \\ \Big\downarrow \theta_{\mathcal{O}} & & \Big\downarrow \theta_{\mathcal{O}} \\ \mathcal{O} & \xrightarrow{A} & \mathcal{O} \end{array}$$

commutes. Then the following conditions are equivalent:

1. *The holomorphic map A is a minimal holomorphic map.*
2. *The homomorphism $\varphi : \Gamma_{\mathcal{O}} \to \Gamma_{\mathcal{O}}$ defined by the equality*

$$F \circ \sigma = \varphi(\sigma) \circ F, \quad \sigma \in \Gamma_{\mathcal{O}},$$

   is an automorphism of $\Gamma_{\mathcal{O}}$.
3. *The functions $\theta_{\mathcal{O}}$, F, A, $\theta_{\mathcal{O}}$ form a good solution of equation (2.2).*

3 Solution of Problem 1.1

In this section we solve Problem 1.1. Our approach is based on the following result proved in [13].

Theorem 3.1. *Let R be a compact Riemann surface and $W : R \to \mathbb{CP}^1$ a holomorphic map of degree n. Then for any rational function P of degree m such that the fiber product of P and W consists of a unique component E the inequality*

$$g(E) > \frac{m - 84n + 168}{84}$$

holds, unless $\chi(\mathcal{O}_2^W) \geq 0$.

Proof. For the case $R = \mathbb{CP}^1$ Theorem 3.1 was proved in [13], Section 3, and the proof holds verbatim for an arbitrary compact Riemann surface R. $\square$

We start with several definitions. Denote by $D = D\Big[R_i, A, W_i, h_i\Big]$ an infinite commutative diagram

$$\begin{array}{ccccccccc} \cdots \longrightarrow & R_3 & \xrightarrow{h_3} & R_2 & \xrightarrow{h_2} & R_1 & \xrightarrow{h_1} & R_0 \\ & \downarrow{W_3} & & \downarrow{W_2} & & \downarrow{W_1} & & \downarrow{W_0} \\ \cdots \longrightarrow & \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array}$$

consisting of holomorphic maps between compact Riemann surfaces. Say that D is *good* if for any $d_2 > d_1 \geq 0$ the maps

$$W_{d_1}, \quad h_{d_1+1} \circ h_{d_1+2} \circ \cdots \circ h_{d_2}, \quad A^{\circ(d_2-d_1)}, \quad W_{d_2}$$

form a good solution of equation (2.2). Notice that, by Lemma 2.9, if D is good, then

$$\deg W_d = \deg W_0, \qquad d \geq 1. \tag{30}$$

Say that D is *preperiodic* if there exist $k_0 \geq 0$ and $l \geq 1$ such that for every $d \geq k_0$ the Riemann surfaces R_d and R_{d+l} are isomorphic and

$$W_d = W_{d+l} \circ \alpha_d, \qquad h_d = h_{d+l} \circ \alpha_d, \tag{31}$$

for some isomorphism $\alpha_d : R_d \to R_{d+l}$.

Combined with general properties of fiber products and generalized Lattès maps, Theorem 3.1 implies the following statement.

Theorem 3.2. *Let $D = D\Big[R_i, A, W_i, h_i\Big]$ be a diagram consisting of holomorphic maps of degree at least two. Assume that D is good and the sequence $g(R_d)$, $d \geq 0$, is bounded. Then D is preperiodic, A is a generalized Lattès map, and $g(R_d) \leq 1$, $d \geq 0$. Furthermore, unless A is a Lattès map, $g(R_d) = 0$, $d \geq 0$.*

Proof. Show that the set of orbifolds $\mathcal{O}_2^{W_d}$, $d \geq 0$, contains only finitely many different orbifolds. Applying Theorem 3.1 for $W = W_d$ and $P = A^{\circ j}$ with j big enough, we see that $\chi(\mathcal{O}_2^{W_d}) \geq 0$, $d \geq 0$. Further, by Theorem 2.8, for any $d_2 > d_1 \geq 0$ the map

$$A^{\circ(d_2-d_1)} : \mathcal{O}_2^{W_{d_2}} \to \mathcal{O}_2^{W_{d_1}} \tag{32}$$

is a minimal holomorphic map between orbifolds. In particular, every map

$$A : \mathcal{O}_2^{W_d} \to \mathcal{O}_2^{W_{d-1}}, \qquad d \geq 1,$$

is a minimal holomorphic map. Therefore, if $\deg A > 4$, then by Lemma 2.15 all the sets $c(\mathcal{O}_2^{W_d})$, $d \geq 0$, are subsets of the set $c(\mathcal{O}_2^A)$, that is the set of possible singular values of the orbifolds $\mathcal{O}_2^{W_d}$, $d \geq 0$, is finite. Moreover, this set is finite also if $\deg A \leq 4$, since $\deg A^{\circ 3} > 4$ in view of $\deg A \geq 2$, and hence the sets $c(\mathcal{O}_2^{W_d})$, $d \geq 0$, are subsets of the set $c(A^{\circ 3})$, since

$$A^{\circ 3} : \mathcal{O}_2^{W_{d+3}} \to \mathcal{O}_2^{W_d}, \qquad d \geq 0,$$

are also minimal holomorphic maps. On the other hand, possible signatures of the orbifolds $\mathcal{O}_2^{W_d}$, $d \geq 0$, belong to lists (2.3), (2.3), and, although list (2.3) contains infinite series, Lemma 2.6 implies that if $\nu(\mathcal{O}_2^{W_d}) = \{n, n\}$, $n \geq 2$, or $\nu(\mathcal{O}_2^{W_d}) = \{2, 2, n\}$, $n \geq 2$, then either $n = \deg W_0$ or $n = \deg W_0/2$. Therefore, the set of possible signatures of the orbifolds $\mathcal{O}_2^{W_d}$, $d \geq 0$, is also finite and hence the set $\mathcal{O}_2^{W_d}$, $d \geq 0$, contains only finitely many different orbifolds.

The finiteness of the set $\mathcal{O}_2^{W_d}$, $d \geq 0$, implies that there exists an orbifold $\mathcal{O}$ with $\chi(\mathcal{O}) \geq 0$ and a sequence $d_k \to \infty$ such that $\mathcal{O}_2^{W_{d_k}} = \mathcal{O}$, $k \geq 0$. Moreover, by Theorem 2.4, the equalities

$$\theta_{\mathcal{O}} = W_{d_k} \circ \theta_{\mathcal{O}_1^{W_{d_k}}}, \quad k \geq 0, \tag{33}$$

hold. Since, by (3), all the groups $\Gamma_{\mathcal{O}_1^{W_{d_k}}}$, $k \geq 0$, are subgroups of the group $\Gamma_{\mathcal{O}}$ of the same finite index equal $\deg W_0$, and the group $\Gamma_{\mathcal{O}}$ is finitely generated, the set $\Gamma_{\mathcal{O}_1^{W_{d_k}}}$, $k \geq 0$, contains only finitely many different groups, implying that

$$\Gamma_{\mathcal{O}_1^{W_{d_{k_1}}}} = \Gamma_{\mathcal{O}_1^{W_{d_{k_0}}}}$$

for some $k_1 > k_0$. Since the equality

$$\theta_{\mathcal{O}_1^{W_{d_k}}}(x) = \theta_{\mathcal{O}_1^{W_{d_k}}}(y), \qquad x, y \in \widetilde{\mathcal{O}_1^{W_{d_k}}},$$

holds if and only if x and y are in the same orbit of the group $\Gamma_{\mathcal{O}_1^{W_{d_k}}}$, this yields that

$$\theta_{\mathcal{O}_1^{W_{d_{k_1}}}}(x) = \alpha \circ \theta_{\mathcal{O}_1^{W_{d_{k_0}}}}(y)$$

for some isomorphism $\alpha : R_{d_{k_0}} \to R_{d_{k_1}}$, implying by (3) the equality

$$W_{d_{k_0}} = W_{d_{k_1}} \circ \alpha.$$

Since R_d, W_d, h_d, $d \geq 1$, are defined by R_{d-1} and W_{d-1} up to natural isomorphisms, this implies that (3) holds for $l = d_{k_1} - d_{k_0}$.

Further, since $\chi(\mathcal{O}_2^{W_d}) \geq 0$, $d \geq 0$, it follows from Corollary 2.5 that $g(R_d) \leq 1$, $d \geq 0$. Finally, since maps (3) are minimal holomorpic maps between orbifolds, setting $d_2 = d + l$ and $d_1 = d$, where $d \geq k_0$, we see that $A : \mathcal{O}_2^d \to \mathcal{O}_2^d$ is a minimal holomorphic map for every $d \geq k_0$. Thus, A is a generalized Lattès map. Moreover, unless A is a Lattès map, $\chi(\mathcal{O}_2^{W_d}) > 0$, $d \geq k_0$, implying that $g(R_d) = 0$, $d \geq 0$. □

Four theorems below provide a solution of Problem 1.1. The first theorem imposes no restrictions on the function A and relates Problem 1.1 with semiconjugacies. The other three provide a more precise information for different classes of A. In particular, Theorem 3.5 implies Theorem 1.2 stated in the introduction. In fact, we consider a more general version of Problem 1.1, allowing U to be a holomorphic map $U : R \to \mathbb{CP}^1$, where R is a compact Riemann surface, and considering instead of curves (1.1) the fiber products of coverings U and $A^{\circ d}$, $d \geq 1$. For such U and A denote by $g_d = g_d(A, U)$, $d \geq 1$, the minimal number g such that the fiber product of U and $A^{\circ d}$ has a component of genus g.

Theorem 3.3. *Let R be a compact Riemann surface, $U : R \to \mathbb{CP}^1$ a holomorphic map of degree at least two, and A a rational function of degree at least two. Then the sequence g_d, $d \geq 1$, is bounded if and only if there exist a compact Riemann surface S of genus 0 or 1, and holomorphic maps $F : S \to S$ and $W : R \to \mathbb{CP}^1$ such that the diagram*

$$\begin{array}{ccc} S & \xrightarrow{F} & S \\ \big\downarrow{\scriptstyle W} & & \big\downarrow{\scriptstyle W} \\ \mathbb{CP}^1 & \xrightarrow{A^{\circ l_1}} & \mathbb{CP}^1 \end{array} \tag{34}$$

commutes for some $l_1 \geq 1$, the fiber product of W and $A^{\circ l_1}$ consists of a unique component, and U is a compositional left factor of $A^{\circ l_2} \circ W$ for some $l_2 \geq 0$. Furthermore, unless A is a Lattès map, $g(S) = 0$.

Proof. In order to prove the sufficiency observe that (3.3) and

$$A^{\circ l_2} \circ W = U \circ V \tag{35}$$

imply that

$$A^{\circ (l_2 + l_1 k)} \circ W = U \circ V \circ F^{\circ k}$$

for every $k \geq 0$. Therefore, for every $d \geq 1$ there exist holomorphic maps of the form

$$\varphi_d = A^{\circ s_d} \circ W, \qquad \psi_d = V \circ F^{\circ r_d},$$

where $s_d \geq 0$, $r_d \geq 0$, satisfying

$$A^{\circ d} \circ \varphi_d = U \circ \psi_d.$$

By the universality property, this implies that for every $d \geq 1$ the exist a component $\{C, p, q\}$ of the fiber product of U and $A^{\circ d}$ and a holomorphic map $w : S \to C$ such that

$$\varphi_d = p \circ w, \quad \psi_d = q \circ w.$$

Clearly, for such C we have:

$$g(C) \leq g(S) \leq 1.$$

Prove now the necessity. Let d_0 be a number satisfying condition (2.12). Set $s = n(A^{\circ d_0}, U)$, and let

$$(\mathbb{CP}^1, A^{\circ (d_0 + k)}) \times (R, U) = \bigcup_{j=1}^{s} \{R_{j,k}, W_{j,k}, H_{j,k}\}, \qquad k \geq 0. \tag{36}$$

It follows from equality (2.12) by the universality property that for every $k \geq 0$ and j, $1 \leq j \leq s$, there exists a uniquely defined j' such that

$$H_{j',k+1} = H_{j,k} \circ h$$

for some holomorphic map $h : R_{j',k+1} \to R_{j,k}$, and without loss of generality we may assume that the numeration in (3) is chosen in such a way that $j = j'$. Thus, we can assume that for every j, $1 \leq j \leq s$, there exist holomorphic maps $h_{j,k}$, $k \geq 1$, such that

$$H_{j,k} = h_{j,1} \circ h_{j,2} \circ \cdots \circ h_{j,k}$$

and the diagram

$$\begin{array}{ccccccc} & R_{j,3} & \xrightarrow{h_{j,3}} & R_{j,2} & \xrightarrow{h_{j,2}} & R_{j,1} & \xrightarrow{h_{j,1}} & R_{j,0} \\ \cdots & \downarrow{W_{j,3}} & & \downarrow{W_{j,2}} & & \downarrow{W_{j,1}} & & \downarrow{W_{j,0}} \\ & \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \qquad (37)$$

commutes. Moreover, this diagram is good by Lemma 2.9, since in view of (2.12) Corollary 2.11 implies that for any $k_2 > k_1 \geq 0$ the fiber product of W_{j,k_1} and $A^{\circ k_2 - k_1}$ consists of a unique component,

$$(R_{j,k_1}, W_{j,k_1}) \times (\mathbb{CP}^1, A^{\circ k_2 - k_1}) = \{R_{j,k_2},\ h_{j,k_1+1} \circ h_{j,k_1+2} \circ \cdots \circ h_{j,k_2},\ W_{j,k_2}\},$$

and

$$\deg W_{j,k_2} = \deg W_{j,k_1}, \qquad k \geq 0,$$

by Theorem 2.7, (ii). Finally, since obviously

$$g(R_{j,k+1}) \geq g(R_{j,k}), \qquad k \geq 0, \qquad (38)$$

it follows from the boundness of the sequence g_d, $d \geq 1$, that for at least one j, $1 \leq j \leq s$, the sequence $g(R_{j,k})$, $k \geq 0$, is bounded. In particular, for such j we can apply Theorem 3.2 to diagram (3), unless $\deg W_{j,0} = 1$.

Since each $R_{j,0}$, $1 \leq j \leq s$, is a component of the fiber product of U and $A^{\circ d_0}$, we can complete diagram (3) to the diagram

$$\begin{array}{ccccccccc} & R_{j,3} & \xrightarrow{h_{j,3}} & R_{j,2} & \xrightarrow{h_{j,2}} & R_{j,1} & \xrightarrow{h_{j,1}} & R_{j,0} & \xrightarrow{H_j} & R \\ \cdots & \downarrow{W_{j,3}} & & \downarrow{W_{j,2}} & & \downarrow{W_{j,1}} & & \downarrow{W_{j,0}} & & \downarrow{U} \\ & \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 & \xrightarrow{A^{\circ d_0}} & \mathbb{CP}^1 \end{array} \qquad (39)$$

where $H_j : R_{j,0} \to \mathbb{CP}^1$ is a holomorphic map. Fix now j such that the sequence $g(R_{j,k})$, $k \geq 0$, is bounded. If $\deg W_{j,0} = 1$, then $R = \mathbb{CP}^1$ and the equality

$$A^{\circ d_0} = U \circ H_j \circ W_{j,0}^{-1}$$

implies that the function U is a compositional left factor of $A^{\circ d_0}$. Therefore, in this case the theorem is true for

$$S = \mathbb{CP}^1, \quad W = z, \quad F = A, \quad l_1 = 1, \quad l_2 = d_0.$$

On the other hand, if $\deg W_{j,0} \geq 2$, then by Theorem 3.2 there exist $l \geq 1$, $k_0 \geq 0$, and an isomorphism $\alpha : R_{j,k_0} \to R_{j,k_0+l}$ such that

$$W_{j,k_0} = W_{j,k_0+l} \circ \alpha,$$

implying that the theorem is true for

$$S = R_{j,k_0}, \quad W = W_{j,k_0}, \quad F = h_{j,k_0+1} \circ h_{j,k_0+2} \circ \cdots \circ h_{j,k_0+l} \circ \alpha,$$

and $l_1 = l$, $l_2 = d_0 + k_0$, since

$$A^{\circ d_0+k} \circ W_{j,k_0} = U \circ H_j \circ h_{j,1} \circ h_{j,2} \circ \cdots \circ h_{j,k_0}. \qquad \square$$

Remark 3.4. Notice that in the proof of the sufficiency we did not use the assumption that the fiber product of W and $A^{\circ l_1}$ has one component. Thus, the theorem implies that if U satisfy (3.3) and (3) for some W, F, and V, then it satisfies (3.3) and (3) for W, F, and V such that the fiber product of W and $A^{\circ l_1}$ has one component (cf. [18], Section 3).

Theorem 3.5. *Let R be a compact Riemann surface, $U : R \to \mathbb{CP}^1$ a holomorphic map of degree at least two, and A a non-special rational function of degree at least two. Then the sequence g_d, $d \geq 1$, is bounded if and only if U is a compositional left factor of $A^{\circ l} \circ \theta_{\mathcal{O}_0^A}$ for some $l \geq 1$. In particular, if A is not a generalized Lattès map, then g_d, $d \geq 1$, is bounded if and only if U is a compositional left factor of $A^{\circ l}$ for some $l \geq 1$.*

Proof. Keeping the notation of Theorem 3.3, we see that, by Theorem 2.13, the orbifold $\mathcal{O}_0^A$ is well-defined and

$$\mathcal{O}_2^W \preceq \mathcal{O}_0^{A^{\circ l_1}} = \mathcal{O}_0^A.$$

Furthermore, by Theorem 2.4,

$$\theta_{\mathcal{O}_0^A} = W \circ \psi$$

for some rational function ψ, implying that any compositional left factor of $A^{\circ l_2} \circ W$ is a compositional left factor of $A^{\circ l_2} \circ \theta_{\mathcal{O}_0^A}$. This proves the necessity. On the other hand, since A is not a Lattès map, the inequality $\chi(\mathcal{O}_0^A) > 0$ holds and hence $\theta_{\mathcal{O}_0^A}$ is a rational function. Thus, A and $\theta_{\mathcal{O}_0^A}$ satisfy (2.3) for some rational function F, and the sufficiency can be proved as in Theorem 3.3. $\square$

Theorem 3.6. *Let R be a compact Riemann surface, $U : R \to \mathbb{CP}^1$ a holomorphic map of degree at least two, and A a Lattès map. Then the sequence g_d, $d \geq 1$, is bounded if and only if U is a compositional left factor of $\theta_{\mathcal{O}_0^A}$.*

Proof. Since in the proof of the necessity in the previous theorem we used only the condition that A is not conjugated to $z^{\pm n}$ or $\pm T_n$, in order to prove the necessity we only must show that if A is a Lattès map, then any compositional left factor of $A^{\circ d}\circ\theta_{\mathcal{O}_0^A}$, $d \geq 1$, is a compositional left factor of $\theta_{\mathcal{O}_0^A}$. Recall that for a Lattès map A the equality $\chi(\mathcal{O}_0^A) = 0$ holds and $A : \mathcal{O}_0^A \to \mathcal{O}_0^A$ is a covering map between orbifolds (see the remarks after Theorem 2.13). Therefore, by Proposition 2.3, the function F in diagram (2.3) is an isomorphism, implying that (2.3) takes the form

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F=az+b} & \mathbb{C} \\ \Big\downarrow{\scriptstyle \theta_{\mathcal{O}_0^A}} & & \Big\downarrow{\scriptstyle \theta_{\mathcal{O}_0^A}} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1\,, \end{array} \tag{40}$$

where $a, b \in \mathbb{C}$, $a \neq 0$. Thus, for every $d \geq 1$ the equality

$$\theta_{\mathcal{O}_0^A} = A^{\circ d} \circ \theta_{\mathcal{O}_0^A} \circ (F^{-1})^{\circ d}$$

holds, implying the necessary statement.

In distinction with the case $\chi(\mathcal{O}_0^A) > 0$, in order to prove the sufficiency we cannot now simply to refer to diagram (3), since the function $\theta_{\mathcal{O}_0^A}$ is transcendental. So, we modify the proof as follows. For every $d \geq 1$ consider the set $V_d \subset \mathbb{CP}^1 \times R$ consisting of points (x, y) such that $A^{\circ d}(x) = U(y)$. It is easy to see that V_d is a union of singular Riemann surfaces. Furthermore, Theorem 2.7 implies that irreducible components of the fiber product $(\mathbb{CP}^1, A^{\circ d}) \times (R, U)$ are normalizations of irreducible components of V_d. Namely, if

$$V_d = C_{1,d} \cup C_{2,d} \cup \cdots \cup C_{s_d,d}, \qquad d \geq 1,$$

then, under an appropriate numeration,

$$(\mathbb{CP}^1, A^{\circ d}) \times (R, U) = \bigcup_{j=1}^{s_d} \{R_{j,d}, W_{j,d}, H_{j,d}\}, \qquad d \geq 1,$$

where the map

$$\varphi_{j,d} : R_{j,d} \to C_{j,d} \subset R \times \mathbb{CP}^1 \tag{41}$$

given by

$$z \to (W_{j,d}, H_{j,d})$$

is the normalization map.

Assume now that $\theta_{\mathcal{O}_0^A} = U \circ \psi$, where $\psi : \mathbb{C} \to R$ is a holomorphic map. Since diagram (3) commutes, for every $d \geq 1$ the equality

$$A^{\circ d} \circ \theta_{\mathcal{O}_0^A} = U \circ (\psi \circ F^{\circ d})$$

holds, implying that the map $\psi_d : \mathbb{C} \to \mathbb{CP}^1 \times R$ given by

$$z \to (\theta_{\mathcal{O}_0^A}, \psi \circ F^{\circ d})$$

maps $\mathbb{C}$ to some $C_{j,d}$. Since maps (3) are normalization maps, this implies that we can lift ψ_d to a holomorphic map $\overline{\psi_d} : \mathbb{C} \to R_{j,d}$, implying, as in the proof of Lemma 2.5, that $g(R_{j,d}) \leq 1$. □

Theorem 3.7. *Let R be a compact Riemann surface, $U : R \to \mathbb{CP}^1$ a holomorphic map of degree at least two, and A a rational function of degree at least two.*

1. *If $A = z^m$, then the sequence g_d, $d \geq 1$, is bounded if and only if $U = z^s \circ \mu$, where μ is a Möbius transformation,*
2. *If $A = T_m$, then the sequence g_d, $d \geq 1$, is bounded if and only if either $U = \pm T_s \circ \mu$, or*
$$U = \frac{1}{2}\left(z^s + \frac{1}{z^s}\right) \circ \mu,$$
where μ is a Möbius transformation.

Proof. Consider the case $A = T_m$. The proof for $A = z^m$ is similar. Prove the necessity. Keeping the notation of Theorem 3.3, observe first that if $\deg W = 1$, then U is a compositional left factor of $T_{m^{l_2}}$. Therefore, since any compositional factor of T_n has the form $T_d \circ \mu$ for some $d|n$ and Möbius transformation μ, in this case the statement is true.

Assume now that $\deg W > 1$. Applying Lemma 2.9 and Theorem 2.8 to diagram (3.3), we see that the map

$$T_{m^{l_1}} : \mathcal{O}_2^W \to \mathcal{O}_2^W$$

is a minimal holomorphic map between orbifolds, implying by Theorem 2.14 that $\mathcal{O}_2^W$ is defined by one of conditions (2.3), (2.3), (2.3). Further, observe that any compositional left factor of the map $A^{\circ l_2} \circ W$ is a compositional left factor of the map $A^{\circ l_2} \circ \theta_{\mathcal{O}_2^W}$, since

$$\theta_{\mathcal{O}_2^W} = W \circ \theta_{\mathcal{O}_1^W}$$

by Theorem 2.4. Since the universal coverings of the orbifolds given by (2.3), (2.3), (2.3) are the function D_n in (2.1) and the functions $-T_2$, T_2 correspondingly, this implies that U is a compositional left factor either of the function

$$T_{m^{l_2}} \circ D_n = D_{nm^{l_2}}$$

or of the function

$$T_{m^{l_2}} \circ \pm T_2 = \pm T_{2m^{l_2}}.$$

Since any compositional left factor of D_n has the form $\pm T_d \circ \mu$ or $D_d \circ \mu$ for some Möbius transformation μ and $d|n$, this proves the necessity. On the other hand, since $\pm T_s \circ \mu$ and $D_s \circ \mu$ are compositional left factors of D_s, the sufficiency follows from the equality $T_m \circ D_s = D_s \circ z^m$ as in Theorem 3.3. □

4 Proof of Theorem 1.3

In this section we prove Theorem 1.3. Show first that modifying the proof of Theorem 3.3 one can reduce the proof of Theorem 1.3 to a certain arithmetical property of functions F appearing in (3.3).

Denote by $O_A(z_0)$ the A-orbit of $z_0 \in \mathbb{P}^1(K)$ and suppose that the set $I \subseteq \mathbb{N}$ consisting of indices n such that $A^{\circ n}(z_0) \in U(\mathbb{P}^1(K))$ is infinite. Setting $z_n = A^{\circ n}(z_0)$, $n \geq 0$, we see that if z_{n_k}, $k \geq 1$, is any subsequence of z_n such that

$$z_{n_k} = U(y_k), \quad y_k \in K,$$

then $(z_{n_k - d}, y_k)$, $n_k \geq d$, is a sequence of points of curve (1.1). In particular, for every $d \geq 1$ algebraic curve (1.1) has infinitely many K-points (x, y) such that $x \in O_A(z_0)$. Let $d_0 \geq 1$ be a number such that (2.12) holds, and let (3) be the corresponding fiber products. Recall that for rational functions f and g irreducible components of the fiber product of f and g are normalizations of irreducible components of the curve

$$f(x) - g(y) = 0.$$

For the Riemann surfaces $R_{j,k}$ appearing in (3) we will denote the corresponding irreducible components of the curve

$$A^{\circ(d_0+k)}(x) - U(y) = 0$$

by $\mathring{R}_{j,k}$.

Let us define a subset $J \subset \{1, 2, \dots, s\}$ and natural numbers S, L_1, L_2 as follows. By definition, $j \in J$ if for infinitely many $k \geq 0$ the curve $\mathring{R}_{j,k}$ has infinitely many K-points (x, y) such that $x \in O_A(z_0)$. Notice that since for every $d \geq 1$ curve (1.1) has infinitely many K-points (x, y) such that $x \in O_A(z_0)$, the set J is non-empty. Since by the Faltings theorem every curve $\mathring{R}_{j,k}$ as above has genus zero or one, it follows from inequality (3) that for every $j \in J$ the sequence $g(R_{j,k})$, $k \geq 0$, is bounded, and the same arguments as in the proof of Theorem 3.3 show that diagram (3) is preperiodic, so that there exist $k_0 = k_0(j)$ and $l = l(j)$ such that for every $d \geq k_0$ the equalities

$$W_{j,d} = W_{j,d+l} \circ \alpha_{j,d}, \qquad h_{j,d} = h_{j,d+l} \circ \alpha_{j,d}$$

hold for some isomorphism $\alpha_{j,d} : R_{j,d} \to R_{j,d+l}$. Set

$$S = \max_{j \in J} \{k_0(j)\}, \qquad L_1 = \underset{j \in J}{\mathrm{LCM}} \{l(j)\}, \qquad L_2 = d_0 + S.$$

It is clear that as in the proof of Theorem 3.3, starting from diagram (3), we can construct for every $j \in J$ rational functions W_j, F_j, and V_j such that the equalities

$$A^{\circ L_2} \circ W_j = U \circ V_j$$

and

$$A^{\circ L_1} \circ W_j = W_j \circ F_j \tag{42}$$

hold, and the fiber product of $A^{\circ L_1}$ and W_j consists of a unique component. Furthermore, without loss of generality we may assume that if a curve $\mathring{R}_{j,k}$ has infinitely many K-points (x, y) such that $x \in O_A(z_0)$ but $j \notin J$, then $k < S$.

In order to prove Theorem 1.3 it is enough to show that if for given k, $0 \le k < L_1$, there exist infinitely many $n \in I$ such that $n \equiv k \bmod L_1$, then I contains an arithmetic progression $n_0 + mL_1$, $m \ge 0$, for some $n_0 \equiv k \bmod L_1$. Let k be such a number. Then for at least one $j \in J$ the curve $\mathring{R}_{j,S}$ has infinitely many K-points (x, y) such that

$$x = z_{n-L_2}, \qquad n \equiv k \bmod L_1. \tag{43}$$

Recall that an algebraic curve

$$C : \, f(x, y) = 0$$

of genus zero defined over K admits a parametrization by rational functions defined over K if and only if C has at least one simple K-point, and, if φ, ψ is such a parametrization, then any K-point (x, y) of C with finitely many exceptions has the form

$$x = \varphi(t), \qquad y = \psi(t)$$

for some $t \in K$ (see [6] and [19], Section 5). Therefore, since the curve $\mathring{R}_{j,S}$ has infinitely many K-points of form (4), it has a rational parametrization defined over K, and by the Lüroth theorem without loss of generality we may assume that the components of this parametrization have no common compositional right factor. Thus, these components coincide, up to an isomorphism of $\mathbb{CP}^1$, with the functions W_j and V_j, attached to the component $R_{j,S}$ of the fiber product of $A^{\circ L_2}$ and U, implying that there exists a Möbius transformation μ such that the rational functions

$$\widetilde{W}_j = W_j \circ \mu, \qquad \widetilde{V}_j = V_j \circ \mu,$$

are defined over K. Clearly,

$$A^{\circ L_2} \circ \widetilde{W}_j = U \circ \widetilde{V}_j \tag{44}$$

and

$$A^{\circ L_1} \circ \widetilde{W}_j = \widetilde{W}_j \circ \widetilde{F}_j, \tag{45}$$

where

$$\widetilde{F}_j = \mu^{-1} \circ F_j \circ \mu.$$

Furthermore, we can find $n_0 \equiv k \bmod L_1$ such that $z_{n_0} = U(y_0)$, $y_0 \in K$, and

$$z_{n_0 - L_2} = \widetilde{W}_j(t_0), \qquad y_0 = \widetilde{V}_j(t_0)$$

for some $t_0 \in K$.

Assume for a moment that $\widetilde{F}$ is defined over K. Then for every $m \geq 0$ we have:

$$z_{n_0+mL_1} = A^{\circ L_2+mL_1}(z_{n_0-L_2}) = (A^{\circ L_2+mL_1} \circ \widetilde{W}_j)(t_0) =$$
$$(A^{\circ L_2} \circ A^{\circ mL_1} \circ \widetilde{W}_j)(t_0) = (A^{\circ L_2} \circ \widetilde{W}_j \circ \widetilde{F}_j^{\circ m})(t_0) = (U \circ \widetilde{V}_j \circ \widetilde{F}_j^{\circ m})(t_0).$$

Therefore, since $\widetilde{V}_j$, $\widetilde{F}_j$ are defined over K and $t_0 \in K$, the set I contains an arithmetic progression $n_0 + mL_1$, $m \geq 0$.

However, actually, $\widetilde{F}_j$ is not necessary defined over K (see e.g. the example below). We overcome this difficulty by showing that there exists $r = r(j)$ such that the *iterate* $\widetilde{F}_j^{\circ r}$ of $\widetilde{F}_j$ is defined over K (Theorem 4.2 below). Then

$$A^{\circ L_2+mL_1 r}(z_{n_0-L_2}) = (U \circ \widetilde{V}_j \circ (\widetilde{F}_j^{\circ r})^{\circ m})(t_0),$$

implying that the progression $z_{n_0+mL_1 r}$, $m \geq 0$, is contained in $U(\mathbb{P}^1(K))$. Setting now

$$R = \operatorname*{LCM}_{j \in J}\{r(j)\}$$

and considering residue classes modulo $L_1 R$ instead of residue classes modulo L_1, we conclude as above that whenever there exist infinitely many $n \in I$ such that $n \equiv k \bmod L_1 R$, the set I contains an arithmetic progression $n_0 + mL_1 R$, $m \geq 0$, for some $n_0 \equiv k \bmod L_1 R$. Thus, Theorem 1.3 is still true.

Finally, if A is not a generalized Lattès map, then the degree of $\widetilde{W}_j$ in (4) equals one and equality (4) can be replaced by the equality

$$A^{\circ L_2} = U \circ \widetilde{V_j},$$

where $\widetilde{V_j}$ is defined over K. Thus,

$$z_{L_2+m} = A^{\circ L_2+m}(z_0) = U \circ \widetilde{V_j} \circ A^{\circ m}(z_0)$$

belongs to $U(\mathbb{P}^1(K))$ for every $m \geq 0$.

Theorem 4.1. *Let F_1, F_2, A, X be rational functions of degree at least two such that the diagrams*

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{F_i} & \mathbb{CP}^1 \\ \big\downarrow X & & \big\downarrow X \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \tag{46}$$

$i = 1, 2$, commute. Asssume that A is non-special and the fiber product of A and X consists of a unique component. Then

$$F_1^{\circ r} = F_2^{\circ r}, \tag{47}$$

where

$$r = |\Gamma_{\mathcal{O}_2^X}||Aut(\Gamma_{\mathcal{O}_2^X})|. \tag{48}$$

Proof. It follows from Lemma 2.9 and Theorem 2.8 that the maps

$$A : \mathcal{O}_2^X \to \mathcal{O}_2^X, \qquad F_i : \mathcal{O}_1^X \to \mathcal{O}_1^X, \quad i = 1, 2,$$

are minimal holomorphic maps between orbifolds, implying that $\chi(\mathcal{O}_X^2) \geq 0$ by (2.1). Moreover, $\chi(\mathcal{O}_X^2) > 0$, since A is not a Lattès map. Since $\chi(\mathcal{O}_X^1) > 0$ by (2.1), it follows from Proposition 2.3 that we can complete commutative diagrams (4.1) to the commutative diagrams

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{\widetilde{F}_i} & \mathbb{CP}^1 \\ \Big\downarrow \theta_{\mathcal{O}_1^X} & & \Big\downarrow \theta_{\mathcal{O}_1^X} \\ \mathbb{CP}^1 & \xrightarrow{F_i} & \mathbb{CP}^1 \\ \Big\downarrow X & & \Big\downarrow X \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1\,, \end{array}$$

where $\widetilde{F}_i : \mathbb{C} \to \mathbb{C}$, $i = 1, 2$, are some rational functions. On the other hand, since

$$\theta_{\mathcal{O}_2^X} = X \circ \theta_{\mathcal{O}_1^X}$$

by Theorem 2.4, Proposition 2.3 implies that there exist a homomorphism $\varphi : \Gamma_{\mathcal{O}_2^X} \to \Gamma_{\mathcal{O}_2^X}$ and $\nu \in \Gamma_{\mathcal{O}_2^X}$ such that

$$\widetilde{F}_1 \circ \sigma = \varphi(\sigma) \circ \widetilde{F}_1, \qquad \sigma \in \Gamma,$$

and

$$\widetilde{F}_2 = \nu \circ \widetilde{F}_1.$$

Finally, φ is an automorphism by Theorem 2.16, since $A : \mathcal{O}_2^X \to \mathcal{O}_2^X$ is a minimal holomorphic map.

In order to prove the theorem it is enough to prove the equality

$$\widetilde{F}_1^{\circ r} = \widetilde{F}_2^{\circ r}. \tag{49}$$

Indeed, if (4) holds, then it follows from the equalities

$$F_i^{\circ r} \circ \theta_{\mathcal{O}_1^X} = \theta_{\mathcal{O}_1^X} \circ \widetilde{F}_i^{\circ r}, \quad i = 1, 2,$$

that

$$F_1^{\circ r} \circ \theta_{\mathcal{O}_1^X} = F_2^{\circ r} \circ \theta_{\mathcal{O}_1^X},$$

implying (4.1).

We have:

$$\widetilde{F}_2^{\circ r} = (\nu \circ \widetilde{F}_1)^{\circ r} =$$

$$= \nu \circ \varphi(\nu) \circ \varphi^{\circ 2}(\nu) \circ \cdots \circ \varphi^{\circ \left(|\Gamma_{\mathcal{O}_2^X}| |Aut(\Gamma_{\mathcal{O}_2^X})| - 1 \right)}(\nu) \circ \widetilde{F}_1^{\circ r}.$$

On the other hand, since $\varphi^{\circ|Aut(\Gamma_{\mathcal{O}_2^X})|}$ is the identical automorphism,

$$\nu \circ \varphi(\nu) \circ \varphi^{\circ 2}(\nu) \circ \cdots \circ \varphi^{\circ\left(|\Gamma_{\mathcal{O}_2^X}||Aut(\Gamma_{\mathcal{O}_2^X})|-1\right)}(\nu) =$$

$$= \left(\nu \circ \varphi(\nu) \circ \varphi^{\circ 2}(\nu) \circ \cdots \circ \varphi^{\circ\left(|Aut(\Gamma_{\mathcal{O}_2^X})|-1\right)}(\nu)\right)^{\circ|\Gamma_{\mathcal{O}_2^X}|}. \tag{50}$$

Therefore, since

$$\nu \circ \varphi(\nu) \circ \varphi^{\circ 2}(\nu) \circ \cdots \circ \varphi^{\circ\left(|Aut(\Gamma_{\mathcal{O}_2^X})|-1\right)}(\nu)$$

is an element of $\Gamma_{\mathcal{O}_2^X}$, the Möbius transformation (4) is identical. This finishes the proof. □

Theorem 4.2. *Let F, A, X be rational functions of degree at least two such the diagram*

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{F} & \mathbb{CP}^1 \\ \downarrow X & & \downarrow X \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \tag{51}$$

commutes. Asssume that A is non-special, the fiber product of A and X consists of a unique component, and A and X are defined over a number field K. Then the function $F^{\circ r}$, where r is defined by (4.1)*, is also defined over K.*

Proof. It is clear that F is defined over $\bar{\mathbb{Q}}$, and that for any $\gamma \in \mathrm{Gal}\,(\bar{\mathbb{Q}}/K)$ the function ${}^{\gamma}F$ satisfies (4.2) along with F. In order to prove the corollary we only must show that for any $\gamma \in \mathrm{Gal}\,(\bar{\mathbb{Q}}/K)$ the equality

$${}^{\gamma}\left(F^{\circ r}\right) = F^{\circ r} \tag{52}$$

holds. On the other hand, since

$${}^{\gamma}\left(F^{\circ r}\right) = \left({}^{\gamma}F\right)^{\circ r},$$

equality (4) is a corollary of Theorem 4.1. □

Using Theorem 4.2 we can finish the proof of Theorem 1.3. Indeed, since the fiber product of the functions $A^{\circ L_1}$ and W_j in (4) consists of a unique component, the fiber product of the functions $A^{\circ L_1}$ and $\widetilde{W}_j$ in (4) also consists of a unique component. Therefore, since A and $\widetilde{W}$ are defined over K, it follows from Theorem 4.2 that there exists $r = r(j)$ such that $\widetilde{F}^{\circ r}$ is defined over K.

In conclusion, we illustrate some of the definitions and results of this paper with the following example:

$$A = 144\,\frac{z\,(z+3)}{(z-9)^2}, \qquad U = z^2, \qquad z_0 = 1, \qquad k = \mathbb{Q}.$$

The function A is obtained from a one-parameter series introduced in the paper [3] for the value of parameter equal one. It is shown in [3] that

$$I = \{0, 2\} \cup \{1 + 2m : m \geq 0\},$$

so, by Theorem 1.3, the function A should be a generalized Lattès map.

Specifically, $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorpic map for the orbifold $\mathcal{O}$ defined by the equalities

$$\nu(0) = 2, \qquad \nu(-3) = 2.$$

Indeed,

$$A^{-1}(0) = \{0, -3\},$$

and the multiplicity of A at the points 0 and -3 equals one so (2.3) holds at $z = 0$ and $z = 3$. Moreover,

$$A^{-1}(-3) = -9/7,$$

and the multiplicity of A at $-9/7$ equals two so (2.3) holds at $z = -9/7$. On the other hand, for any point z distinct from $0, -3$, and $-9/7$, equality (2.3) also holds since for such a point $\nu(z) = 1$ and $\nu(A(z)) = 1$.

Further,

$$\theta_{\mathcal{O}} = 27\,\frac{49\,z^2 + 14\,z + 1}{4183\,z^2 - 20254\,z + 21895},$$

and for the functions

$$V = 12\,\frac{119\,z^2 - 242\,z - 37}{1009\,z^2 - 5074\,z + 5473},$$

and

$$F = \frac{\left(97818816\,\sqrt{3} - 330844979\right) z^2 + \left(-209746752\,\sqrt{3} + 681958646\right) z - 3206016\,\sqrt{3} + 79866685}{-114015643\,z^2 + \left(75747648\,\sqrt{3} + 167828806\right) z - 190457664\,\sqrt{3} + 196457813}$$

the diagram

$$\begin{array}{ccccc}
\mathbb{CP}^1 & \xrightarrow{F} & \mathbb{CP}^1 & \xrightarrow{V} & \mathbb{CP}^1 \\
\downarrow \theta_{\mathcal{O}} & & \downarrow \theta_{\mathcal{O}} & & \downarrow U \\
\mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1
\end{array}$$

commutes, so U is a compositional left factor of $A \circ \theta_{\mathcal{O}}$. The group $\Gamma_{\mathcal{O}}$ is generated by the involution

$$\mu = \frac{121\,z + 37}{119\,z - 121},$$

and the automorphism φ from Theorem 2.16 is the identical automorphism, that is F commutes with μ.

Finally, the function $\bar{F}$ Galois conjugated to F satisfies $\bar{F} = \mu \circ F$, and the function

$$F^{\circ 2} = \bar{F}^{\circ 2} = \frac{374987315\,z^4 - 2202904300\,z^3 + 4416412818\,z^2 - 2863172908\,z - 389726893}{114015643\,z^4 - 638648204\,z^3 + 1081149954\,z^2 - 117500876\,z - 852810821}$$

has rational coefficients.

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References

[1] E. Amerik, F. Bogomolov, M. Rovinsky, *Remarks on endomorphisms and rational points*, Compos. Math. 147 (2011), no. 6, 1819-1842.

[2] J. Bell, D. Ghioca, T. Tucker, *The dynamical Mordell-Lang conjecture.* Mathematical Surveys and Monographs, 210. American Mathematical Society, Providence, RI, 2016.

[3] J. Cahn, R. Jones, J. Spear, *Powers in orbits of rational functions: cases of an arithmetic dynamical Mordell-Lang conjecture,* arXiv:1512.03085.

[4] A. Eremenko, *Invariant curves and semiconjugacies of rational functions*, Fundamenta Math., 219, 3 (2012), 263-270.

[5] H. Farkas, I. Kra, *Riemann surfaces*, Graduate Texts in Mathematics, 71. Springer-Verlag, New York, 1992.

[6] D. Hilbert, A. Hurwitz, *Über die diophantischen Gleichungen vom Geschlecht Null*, Acta Math. 14 (1890), no. 1, 217-224.

[7] F. Klein, *Lectures on the icosahedron and the solution of equations of the fifth degree*, New York: Dover Publications, (1956).

[8] A. Medvedev, T. Scanlon, *Invariant varieties for polynomial dynamical systems,* Annals of Mathematics, 179 (2014), no. 1, 81 - 177.

[9] J. Milnor, *Dynamics in one complex variable*, Princeton Annals in Mathematics 160. Princeton, NJ: Princeton University Press (2006).

[10] J. Milnor, *On Lattès maps,* Dynamics on the Riemann Sphere. Eds. P. Hjorth and C. L. Petersen. A Bodil Branner Festschrift, European Mathematical Society, 2006, pp. 9-43.

[11] F. Pakovich, *Prime and composite Laurent polynomials*, Bull. Sci. Math., 133 (2009), 693-732.

[12] F. Pakovich, *On semiconjugate rational functions,* Geom. Funct. Anal., 26 (2016), 1217-1243.

[13] F. Pakovich, *On algebraic curves A(x)-B(y)=0 of genus zero,* Math. Z. (2017), https://doi.org/10.1007/s00209-017-1889-9.

[14] F. Pakovich, *Recomposing rational functions,* IMRN (2017), https://doi.org/10.1093/imrn/rnx172.

[15] F. Pakovich, *Polynomial semiconjugacies, decompositions of iterations, and invariant curves*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), Vol. XVII (2017), 1417-1446.

[16] F. Pakovich, *On rational functions whose normalization has genus zero or one*, Acta Arith., 182 (2018) , 73-100.

[17] F. Pakovich, *Finiteness theorems for commuting and semiconjugate rational functions,* arxiv:1604:04771.

[18] F. Pakovich, *On generalized Lattès maps*, preprint, arxiv:1612.01315.

[19] J. Sendra, F. Winkler, S. Péréz-Díaz, *Rational algebraic curves. A computer algebra approach*, Algorithms and Computation in Mathematics, 22. Springer, Berlin, 2008.

[20] S. W. Zhang, Distributions in Algebraic Dynamics, *Surveys in differential geometry*, Vol. X, 381430, Int. Press, Somerville, MA, 2006