NONCOMMUTATIVE ALGEBRAIC GEOMETRY

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Introduction

In this paper I shall first, in §2-4, introduce a noncommutative deformation theory for modules on some k-algebra A, k a field. Then, in §5-6 I shall apply it to construct a noncommutative algebraic geometry. The final four paragraphs of the paper is devoted to invariant and moduli theory, and to some examples.

The basic idea of noncommutative deformation theory is very simple. Let $\underline{a}_r$ denote the category of r-pointed not necessarily commutative k-algebras R. The objects are the diagrams of k-algebras,

$$k^r \xrightarrow{\iota} R \xrightarrow{\rho} k^r$$

such that the composition of ι and ρ is the identity. Any such r-pointed k-algebra R is isomorphic to a k-algebra of $r \times r$-matrices $(R_{i,j})$. The radical of R is the bilateral ideal $Rad(R) := ker\rho$, such that $R/Rad(R) \simeq k^r$. The dual k-vectorspace of $Rad(R)/Rad(R)^2$ is called the tangent space of R.

For $r = 1$, there is an obvious inclusion of categories

$$\underline{l} \subseteq \underline{a}_1$$

where $\underline{l}$, as usual, denotes the category of commutative local artinian k-algebras with residue field k.

Fix a not necessarily commutative k-algebra A and consider a right A-module M. The ordinary deformation functor

$$Def_M : \underline{l} \to \underline{Sets}$$

is then defined. Assuming $Ext^i_A(M,M)$ has finite k-dimension for $i = 1,2$, it is well known, see [Sch], or [La 1], that Def_M has a noetherian prorepresenting hull H, *the formal moduli of* M. Moreover, the tangent space of H is isomorphic to $Ext^1_A(M,M)$, and H can be computed in terms of $Ext^i_A(M,M)$, $i = 1,2$ and their *matric* Massey products, see [La 1], [La 2].

In the general case, consider a finite family $\mathcal{V} = \{V_i\}_{i=1}^r$ of A-modules and define a deformation functor,

$$Def_{\mathcal{V}} : \underline{a}_r \to \underline{Sets}$$

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generalizing the functor Def_M above. Given an object $\rho : R = (R_{i,j}) \to k^r$ of $\underline{a}_r$, consider the k-vectorspace and R-left module $(R_{i,j} \otimes_k V_j)$. ρ defines a k-linear and left R-linear map,

$$\rho(R) : (R_{i,j} \otimes_k V_j) \to \oplus_{i=1}^r V_i,$$

inducing a homomorphism of R-endomorphism rings,

$$\tilde{\rho}(R) : (R_{i,j} \otimes_k Hom_k(V_i, V_j)) \to \oplus_{i=1}^r End_k(V_i).$$

The right A-module structure on the V_i's is defined by a homomorphism of k-algebras, $\eta_0 : A \to \oplus_{i=1}^r End_k(V_i)$. Let

$$Def_{\mathcal{V}}(R) \in \underline{Sets}$$

be the isoclasses of homomorphisms of k-algebras,

$$\eta' : A \to (R_{i,j} \otimes_k Hom_k(V_i, V_j))$$

such that,

$$\tilde{\rho}(R) \circ \eta' = \eta_0,$$

where the equivalence relation is defined by inner automorphisms in the k-algebra $(R_{i,j} \otimes_k Hom_k(V_i, V_j))$. One easily proves that $Def_{\mathcal{V}}$ has the same properties as the ordinary deformation functor:

Theorem 2.6. *The functor $Def_{\mathcal{V}}$ has a prorepresentable hull, i.e. an object H of $\underline{\hat{a}}_r$, together with a versal family,*

$$\tilde{V} = (H_{i,j} \otimes V_j) \in \varprojlim_{n \geq 1} Def_{\mathcal{V}}(H/\mathfrak{m}^n)$$

such that the corresponding morphism of functors on $\underline{a}_r$,

$$\rho : Mor(H, -) \to Def_{\mathcal{V}}$$

is smooth, and an isomorphism on the tangent level. Moreover, H is uniquely determined by a set of matric Massey products of the form

$$Ext^1(V_i, V_{j_1}) \otimes \cdots \otimes Ext^1(V_{j_{n-1}}, V_j) \cdots \to Ext^2(V_i, V_j).$$

The right action of A on $\tilde{V}$ defines a homomorphism of k-algebras,

$$\eta : A \longrightarrow O(\mathcal{V}) := End_H(\tilde{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j)),$$

and the k-algebra $O(\mathcal{V})$ acts on the family of A-modules $\mathcal{V} = \{V_i\}$, extending the action of A. If $dim_k V_i \leq \infty$, for all $i = 1, ..., r$, the operation of associating $(O(\mathcal{V}), \mathcal{V})$ to $(A, \mathcal{V})$ turns out to be a closure operation.

Moreover, if A is an object of $\underline{a}_r$, and if $\mathcal{V} = \{k_i\}_{i=1}^r$ is the correspondiong family of simple A-modules, then

$$\eta : A \longrightarrow H(\mathcal{V})$$

is an isomorphism.

In §3 we prove that there exists, in the noncommutative deformation theory, an obvious analogy to the notion of prorepresenting (modular) substratum H_0 of the formal moduli H. The tangent space of H_0 is determined by a family of subspaces

$$Ext^1_0(V_i,V_j) \subseteq Ext^1_A(V_i,V_j), \qquad i \neq j$$

the elements of which should be called the almost split extensions (sequences) relative to the family $\mathcal{V}$, and by a subspace,

$$T_0(\Delta) \subseteq \prod_i Ext^1_A(V_i,V_i)$$

which is the tangent space of the deformation functor of the full subcategory of the category of A-modules generated by the family $\mathcal{V} = \{V_i\}_{i=1}^r$, see [La 1]. If $\mathcal{V} = \{V_i\}_{i=1}^r$ is the set of all indecomposables of some artinian k-algebra A, we show that the above notion of *almost split sequence* coincides with that of Auslander.

In Remark (3.7) we associate to the family $\mathcal{V} = \{V_i\}$, and to any quotient R of H in $\underline{a}_r$ a quiver with vertices contained in the set $\{V_i\}$. The Auslander-Reiten quiver then turns out to correspond to $R = H_0/Rad(H_0)^2$.

Observe that, in general, the k-algebra H_0 and its corresponding modular family $\tilde{\mathcal{V}}_0$ contains much more information than what may be deduced from the tangent level.

In §4, we first prove,

Theorem (4.3) (A generalized Burnside theorem). *Let A be a finite dimensional k-algebra, k an algebraically closed field. Consider the family $\mathcal{V} = \{V_i\}_{i=1}^r$ of simple A-modules, then*

$$A \simeq O(\mathcal{V}) = (H_{i,j} \otimes Hom_k(V_i,V_j))$$

The remainder of the paragraph is concerned with the properties of the O-construction, the classification of extensions of extensions of the modules V_i, $i = 1.2...r$, and the classification of filtered modules with graded components in the family $\mathcal{V} = \{V_i\}$.

Based on this notion of noncommutative deformations, treated in §2.-§4. we shall in §5-6. propose a general definition of an affine noncommutative prescheme, and scheme, generalizing the classical notion of an affine algebraic scheme in the commutative case.

The construction is dependent upon the choice of a reasonable abelian category of objects $\underline{C}$, the universe. Since the process of generalizing will be clear, I shall assume that we are given an algebraically closed field k, a k-algebra A, and that we pick as our universe $\underline{C}$, the category $A-mod$ of right A-modules.

As a model we shall take the classical construction of the scheme $X := Spec(A)$ when A is a commutative finite type k-algebra. A point of X is a prime ideal $\mathfrak{p}$ of A, or rather the right (or left) A-module $A/\mathfrak{p}$. A closed point $x \in X$) is a simple module $k(x) = A/\mathfrak{m}_x$ corresponding to a maximal ideal $\mathfrak{m}_x \subset A$-module. Moreover, X is obviously the moduli space of its closed points, implying that the hull $H(k(x))$ of the deformation functor $Def_{k(x)} : \underline{a}_1 \to \underline{S}ets$, is the completion $\hat{A}_{\mathfrak{m}_x}$ of the local ring $O_{X,x}$ at the point $x \in X$. The regular functions f of X, i.e. the sections of the structure sheaf O_X, are analytically determined by the family of Taylor series

$\hat{f}_x \in \hat{A}_{\mathfrak{m}_x}$ at the different points $x \in X$, therefore by their (right multiplicative) *actions* on the hull of the different deformation functors $Def_{k(x)}$.

This completion process, and the corresponding identification of a regular function f on X as a (multiplicative) operator in the k-vectorspace A as well as in $\hat{A}_x$, for every $x \in X$, is going to replace the localization process of classical scheme theory.

Recall that to recover the affine ring from the scheme

$$(X, O_X) \tag{i}$$

we are dependent upon the Zariski topology on X and on the sheaf property of O_X, both stemming from the process of localization in commutative rings. We find,

$$A = H^0(X, O_X) \tag{ii}$$

Recall also that for noncommutative rings, the localization process functions only for Ore-sets, which are scarce. To obtain a good noncommutative theory we shall therefore have to change the notion of *space*, conserving the notion of points and incidences, but (seemingly) leaving out the *topology*. The notion of *structure sheaf* must therefore be changed.

Let $\underline{c}$ be any reasonnable *diagram* of objects and morphisms in $\underline{C}$. We shall not assume that the morphisms of $\underline{c}$ commute, but we shall assume, for the rest of this paper, that all objects V_i, V_j of $\underline{c}$ are such that,

$$dim_k Ext^1_A(V_i, V_j) < \infty.$$

Any such diagram will be called a *swarm*, see (5.1).

Let

$$\pi : \underline{C} \to k - mod.$$

be the obvious forgetful functor.

Let $|\underline{c}|$ denote the family of objects of $\underline{c}$, and assume first that $|\underline{c}| = \{V_i\}_{i=1}^r$ is finite. Let $H(|\underline{c}|) = (H_{i,j})$ be the hull of the noncommutative deformation functor of this family of A-modules, see [La 7]. To the swarm $\underline{c}$ we now associate, see §5., a subalgebra,

$$O^A(\underline{c}, \pi) \subseteq (H_{i,j}(|\underline{c}|) \otimes_k Hom_k(V_i, V_j))$$

of *preobservables*, together with a restriction of the canonical homomorphism η,

$$\eta(\underline{c}) : A \longrightarrow O^A(\underline{c}, \pi),$$

such that $O^A(\underline{c}, \pi)$ acts on $\underline{c}$, extending the action of A. Moreover, if for all i $dim_k V_i < \infty$, then the O-construction is a closure operator, i.e.

$$O^A(\underline{c}, \pi) \simeq O^O(\underline{c}, \pi). \tag{iii}$$

Notice that we shall, abusing the notations, write $\otimes$ where one should have written $\hat{\otimes}$, i.e. when $H(V_i) \in \hat{\underline{a}}_r$ but $H(V_i) \notin \underline{a}_r$ and where we therefore have to work with complete tensor products.

To extend this O-construction to infinite swarms, we have to "sheafify" the O-construction, obtaining for every finite swarm $\underline{c}$ a smaller k-algebra, $\mathcal{O}(\underline{c}, \pi)$ containing the image of $\eta(\underline{c})$. This new k-algebra of *observables* has good functorial properties and may be extended to the *permissible* infinite swarms, see (5.15). The final *noncommutative structure sheaf* $\mathcal{O}_\pi$, a certain quotient of this $\mathcal{O}(-, \pi)$, see §5, is a presheaf of k-algebras on the ordered set of sub-swarms of a given swarm $\underline{c}$. We then proclaim,

Definitions 5.8 and 5.17. *A swarm* $\underline{c}$ *of* $\underline{C}$ *will be called a prescheme for* A*, if*

$$\eta(\underline{c}) : A \longrightarrow \mathcal{O}(\underline{c}, \pi)$$

is an isomorphism. If this is the case, $(\underline{c}, A)$ *is called an affine prescheme and we shall refer to* A *as the affine ring of this prescheme. The swarm* $\underline{c}$ *will be called a scheme for* A*, if*

$$\eta(\underline{c}) : A \longrightarrow \mathcal{O}_\pi(\underline{c})$$

is an isomorphism. If this is the case, $(\underline{c}, A)$ *is called an affine scheme and we shall refer to* A *as the affine ring of this scheme.*

In particular, if $\underline{c}$ is a finite swarm of A-modules, the pair,

(iv) $$(\underline{c}, \mathcal{O}^A(\underline{c}, \pi))$$

is, by (5.7), an affine (noncommutative) prescheme. Thus, (ii) in the commutative scheme theory, is replaced by (iii) in the general case.

Notice that we are now talking about *a scheme* for A, not about *the scheme* for A. In fact it is easily seen that there may be several useful schemes for a given algebra A, depending upon what kind of properties of the algebra one would like to study.

The noncommutative algebraic geometry I propose, is concerned with these affine schemes, and their globalizations. The categorical properties of our universe $\underline{C}$ replace the topology, and the classical structure sheaf O_X is replaced by the $\mathcal{O}_\pi$-construction.

As an example, let us consider the 0-dimensional case. If A is a commutative k-algebra of dimension 0, then $A = \oplus_{i=1}^r O_{X,x_i}$ where $X = Spec(A) = \{x_1, \ldots, x_r\}$. In general, let A be a finite dimensional k-algebra, k algebraically closed, and $\mathcal{V} = \{V_i\}$ the (finite) family of all simple modules. We shall consider each module of this family as a point, and we shall consider the obvious forgetful functor $\pi : A-mod \to k-mod$. The local ring (or the infinitesimal neighbourhood) of a point V_i of $\mathcal{V}$, the analogue of the completion $\hat{O}_{X,x}$ of the local ring $O_{X,x}$ of a closed point $x \in X = Spec(A)$, is the algebra

(v) $$H(V_i) \otimes_k End_k(V_i)$$

where $H(V_i)$ is the hull of the deformation functor Def_{V_i}. The affine ring A is, however, no longer isomorphic to the sum of these local algebras.

Here is where the notion of noncommutative deformation enters. Let $H(\mathcal{V}) = (H_{i,j})$ be the hull of the noncommutative deformation functor of the family $\mathcal{V} = \{V_i\}$, see [La 7], then the *infinitesimal interactions* of the *points* of $\mathcal{V}$, translates into the components,

(vi) $$H_{i,j}(\mathcal{V}) \otimes_k Hom_k(V_i, V_j), i \neq j$$

of the ring of *observables* $O(\mathcal{V}) := O(\mathcal{V}, \pi)$. $H_{i,j}(\mathcal{V})$ is, as a $H_{i,i}(\mathcal{V}) - H_{j,j}(\mathcal{V})$ bi-module generated by a dual base of the *tangent space* $Ext^1_A(V_i, V_j)$ of the point V_i into the point V_j. There is a natural morphism of k-algebras,

$$\eta : A \longrightarrow O(\mathcal{V}) := (H_{i,j}(\mathcal{V}) \otimes_k Hom_k(V_i, V_j))$$

which, according to the *Generalized Burnside Theorem* is an isomorphism.

This is the *Serre theorem*, i.e. the analogue of (ii), in the 0-dimensional noncommutative algebraic geometry. The *discrete* swarm,

$$(\mathcal{V} := \{V_i\}_{i=1}^r, A)$$

is a scheme for A.

Notice that in the construction of $H(\mathcal{V})$ we only use the structure of the abelian category $\underline{C}$ (of A-modules) in which we consider our family of objects $\mathcal{V}$. $H(\mathcal{V})$ is therefore an invariant of the Morita equivalence class of A. To recover A, i.e. in the construction of the ring of observables, we must also know the dimensions of the different points V_i of the noncommutative scheme $\underline{c} := \mathcal{V}$, i.e. we must know the values of the forgetful functor π on $\mathcal{V}$. However, as one easily shows, $H(\mathcal{V})$ is Morita equivalent to A.

-Now, to call something a geometry, one should certainly have the possibility of defining some kind of hierarchy among the *geometrical subobjects*, something like a *quiver of incidences.* Given a geometrical subobject we should at least be able to decide which points *sit* on the subvariety. In our case, if $\underline{c}$ is an affine scheme for A, the morphisms of $\underline{c}$ correspond to incidences among the points. Moreover, as we have seen in the 0-dimensional case discussed above, there may also be some *infinitesimal* incidences between the points V_i and V_j, corresponding to a k-basis of $Ext^1_A(V_i, V_j)$. And these are essential in the (re)construction of the affine ring of observables.

-To qualify as a geometry, a model should include a *dynamical* element, i.e. either a topology and a differential structure, including vectorfields, or something taking its place. This is, in our case, provided by a differential calculus induced by the deformation theory, see (5.19) and (5.23), where the basic notions are introduced.

-To be taken seriously, a noncommutative algebraic geometry must certainly include the classical algebraic geometry as a special case. To see that our model satisfies this condition, let A be a commutative k-algebra of finite type. The points of the affine scheme Spec(A) may be identified with the members of the family of indecomposible modules $\mathcal{V} = \{A/\mathfrak{p}\}_{\mathfrak{p}\in Spec(A)}$. We shall consider this family of A-modules together with the obvious canonical morphisms, obtaining a (usually infinite) diagram (really an ordered set) $\underline{c} = Spec(A)$, of A-mod. Notice that $Spec(A)$ as a set, contains the set of closed points, $Simp(A)$, *the simple A-modules*, together with all the irreducible subvarieties of $Spec(A)$, considered both as points in their own right, and as defining subschemes, with the corresponding points as their *generic points.* This induces a notion of incidence among different points in the *geometry*, just as we have done above. Recall, however, that in classical scheme theory, a scheme is the moduli space of its closed points, but not necessarily of the non-closed points. There is, in fact, a dicotomy between the set of closed points, and the set of non-closed points, between the scheme and its *Hilbert schemes.*

In our noncommutative geometry, the general notion of scheme is an intermediate version, providing us with a set of *points* and incidences, such that all points are on equal footing, see section §5.

These considerations lead us to the swarm of A-modules, $Simp^*(A)$, consisting of A, the *projective generator*, and all the simple A-modules, together with the obvious incidences.

The embedding of the classical algebraic geometry (defined on an algebraically closed field k), into the proposed noncommutative algebraic geometry, is then taken care of by the following result,

Theorem 6.1. *Let A be any commutative k-algebra of finite type, k algebraically closed. Then the canonical morphism of k-algebras*

$$\eta(Simp^*(A),\pi) : A \to \mathcal{O}_\pi(Simp^*(A))$$

is an isomorphism.

We shall look at invariant theory, in this general setting and, in particular, we shall see that many problems of moduli in algebra, which cannot be treated in the classical framework of schemes, or of ringed spaces, have very satisfactory solutions expressible in the language of this generalized scheme theory, see section §10. We shall also, see section §5, discuss the relationship between this noncommutative *algebraic geometry*, and the noncommutative *geometry* and its *espaces quantiques* of Alain Connes, see [Connes]. It is clear that, although the starting point and the techniques used, are quite different, the basic ideas are closely related.

Now, a non-commutative version of differential geometry has been around for decades, spured by the needs of quantum mechanics and by the needs of invariant theory. In fact, there is a flora of proposed noncommutative geometries. The first ones were based on the notion of operator algebras. Von Neumanns work on quantum mechanics created a geometry where points, in some sense, were replaced by states or pure states in C* -algebras. Working on foliations, Connes has, in a most convincing way developed a theory of quotientspaces, or orbitspaces, related to the theory of moduli, which trancends the classical geometry. However, the basic notion of *space* as a set of points with a topology and a structure defined by a structure sheaf, disappeare in this model, see [Connes].

There are also purely algebraic attempts at the construction of a noncommutative geometry see e.g.[Manin], [Artin], and [Rosenberg]. The common aspect of these models have been that they do not include non-reduced schemes, and therefore cannot treat 0-dimensional *schemes*, and, subsequently contain no infinitesimal theory. However, noncommutative geometry is a field in progress, and there are hopes for a future convergence of views, and a common ground for noncommutative algebraic geometry.

To get a different view of what we are after, consider the following:

-Given an algebraic object, say a singularity, see [La-Pf]. We know via deformation theory what one should mean by its (infinitesimal) deformations. Consider a versal (flat) family. We are looking for an algebra of operators, called *observables*, with the property that the set of simple representations of this algebra is in one-to-one correspondence with the isomorphism classes of singularities in the given family. Moreover, this algebra of observables should contain all the information about the possible deformations, or changes of *states*, of the algebraic object, including the abrupt changes observed in families for which some discrete invariant jumps.

-This is analogous to the set-up of quantum theory. Replace the singularity with the (platonic?) idea of some reality *out there*, say a fundamental particle, then the state space of quantum theory is a module, or a representation, of a ring of generalized functions, the *observables.* Notice that if the simple modules of this ring had all been isomorphic to some field of *numbers*, the observables would have been provided with definite values characterizing the possible states of the fundamental particle, as in classical physics. When, however, the algebra of observables is noncommutative, all the simple representations may be of infinite dimensions on the base-field. The numerical trick does not function. The only invariant *values* of an observable is then the eigenvalues. We are therefore left with a new quantized description of the space of realities, in which measurement must be redefined, and time and dynamics rethought.

Earlier versions of this paper has appeared in the preprints [La 3-6].

1. Homological preparations.

Exts and Hochschild cohomology. Let k be a (usually algebraically closed) field, and let A be a k-algebra. Denote by A-mod the category of right A-modules and consider the exact forgetful functor

$$\pi : A - mod \longrightarrow k - mod$$

Given two A-modules M and N, we shall always use the identification

$$\sigma^i : Ext^i_A(M,N) \simeq HH^i(A, Hom_k(M,N)) \text{ for } i = 0,1,2,$$

where $Hom_k(M,N)$ is provided with the obvious left and right A-module structures. If L_* and F_* are A-free resolutions of M and N respectively, and if an element

$$\xi \in Ext^1_A(M,N)$$

is represented by the Yoneda cocycle,

$$\hat{\xi} = \{\xi_n\} \in \prod_n Hom_A(L_n, F_{n-1})$$

then $\sigma^1(\xi)$ is gotten as follows. Let σ be a k-linear section of the augmentation morphism

$$\rho : L_0 \longrightarrow M$$

and let for every $a \in A$ and $m \in M$, $\sigma(ma) - \sigma(m)a = d_0(x)$. Put,

$$\sigma^1(\hat{\xi})(a,m) = -\mu(\xi_1(x))$$

where

$$\mu : F_0 \longrightarrow N$$

is the augmentation morphism of F_*. Then,

$$\sigma^1(\hat{\xi}) \in Der_k(A, Hom_k(M,N))$$

and its class in $HH^1(A, Hom_k(M,N))$ equals $\sigma^1(\xi)$.

Recall the spectral sequence associated to a change of rings. If $\pi : A \longrightarrow B$ is a surjective homomorphism of commutative k-algebras, M a B-module and N an A-module, then $Ext^*_A(M,N)$ is the abuttment of the spectral sequence given by,

$$E_2^{p,q} = Ext^p_B(M, Ext^q_A(B,N)).$$

There is an exact sequence,

$$0 \longrightarrow E_2^{1,0} \longrightarrow Ext_A^1(M,N) \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0},$$

which, for a B-module N, considered as an A-module, implies the exactness of

$$\begin{aligned} 0 &\longrightarrow Ext_B^1(M,N) \longrightarrow Ext_A^1(M,N) \\ &\longrightarrow Hom_B(M, Hom_B(I/I^2, N)) \longrightarrow Ext_B^2(M,N) \end{aligned}$$

where I=ker π. The corresponding exact sequence,

$$\begin{aligned} 0 &\to HH^1(B, Hom_k(M,N)) \to HH^1(A, Hom_k(M,N)) \\ &\to Hom_{A\otimes A^{op}}(I, Hom_k(M,N)) \end{aligned}$$

in the noncommutative case is induced by the sequence

$$\begin{aligned} 0 &\to Der_k(B, Hom_k(M,N)) \to Der_k(A, Hom_k(M,N)) \\ &\to Hom_{A\otimes A^{op}}(I, Hom_k(M,N)). \end{aligned}$$

Notice that in general we do not know that the last morphism is surjective. This, however, is true if B=A/rad(A), where rad(A) is the radical of A, and A is a finite dimensional, i.e. an artinian, k-algebra. In this case, B is semisimple and the surjectivity above follows from the Wedderburn-Malcev theorem. Notice also that in the commutative case,

$$Hom_{A\otimes A^{op}}(I, Hom_k(M,N)) \simeq Hom_B(I/I^2, Hom_B(M,N))$$

as it must, since for $\phi \in Hom_{A\otimes A^{op}}(I, Hom_k(M,N))$, $a \in A$, and $i \in I$, $ai = ia$, and therefore

$$a\phi(i) = \phi(ai) = \phi(ia) = \phi(i)a, \text{ i.e. } \phi(i) \in Hom_B(M,N).$$

This implies that for $B = A/\mathfrak{p}$, $M = A/\mathfrak{p}$, $N = A/\mathfrak{q}$, where $\mathfrak{p} \subseteq \mathfrak{q}$ are (prime) ideals of A,

$$Ext_A^1(A/\mathfrak{p}, A/\mathfrak{q}) \simeq Hom_A(\mathfrak{p}/\mathfrak{p}^2, A/\mathfrak{q})$$

and, in particular

$$Ext_A^1(A/\mathfrak{q}, A/\mathfrak{q}) \simeq Hom_A(\mathfrak{q}/\mathfrak{q}^2, A/\mathfrak{q}) = N_{\mathfrak{q}},$$

the normal bundle of $V(\mathfrak{q})$ in Spec(A). If $\mathfrak{q} \subset \mathfrak{p}$ and $\mathfrak{q} \neq \mathfrak{p}$ we find,

$$Ext_A^1(A/\mathfrak{p}, A/\mathfrak{q}) \simeq Ext_{A/\mathfrak{q}}^1(A/\mathfrak{p}, A/\mathfrak{q}).$$

In [La 1], chapter 1., we considered the cohomology of a category $\underline{c}$ with values in a bifunctor, i.e. in a functor defined on the category $mor\underline{c}$ of morphismes of $\underline{c}$. Recall that a morphism between the objects ψ and ψ' is a commutative diagram,

$$\begin{tikzcd} c_1 \arrow[r, "\psi"] & c_2 \arrow[d] \\ c_1' \arrow[u] \arrow[r, "\psi'"] & c_2'. \end{tikzcd}$$

It is easy to see that this cohomology is an immediate generalization of the projective limit functor and its derivatives, or if one likes it better, the obvious generalization of the Hochschild cohomology of a ring. In fact, for every small category $\underline{c}$ and for every bifunctor,

$$G : \underline{c} \times \underline{c} \longrightarrow Ab$$

contravariant in the first variable, and covariant in the second, one obtains a covariant functor,

$$G : mor\underline{c} \longrightarrow Ab.$$

Consider now the complex,

$$D^*(\underline{c}, G)$$

where,

$$D^p(\underline{c}, G) = \prod_{c_0 \to c_1 \cdots \to c_p} G(c_0, c_p)$$

where the indices are strings of morphisms $\psi_i : c_i \to c_{i+1}$ in $\underline{c}$, and the differential,

$$d^p : D^p(\underline{c}, G) \longrightarrow D^{p+1}(\underline{c}, G)$$

is defined as usual,

$$(d^p\xi)(\psi_1, \ldots, \psi_i, \psi_{i+1}, \ldots, \psi_{p+1}) = \psi_1\xi(\psi_2, \ldots, \psi_{p+1})$$
$$+ \sum_{i=1}^{p}(-1)^i\xi(\psi_1, \ldots, \psi_i \circ \psi_{i+1}, \ldots, \psi_{p+1}) + (-1)^{p+1}\xi(\psi_1, \ldots, \psi_p)\psi_{p+1}.$$

As shown in [La 1], the cohomology of this complex is the higher derivatives of the projective limit functor $\varprojlim^{(*)}_{mor\underline{c}}$ applied to the covariant functor

$$G : mor\underline{c} \longrightarrow Ab.$$

This is the "Hochschild" cohomology of the category $\underline{c}$, denoted

$$H^*(\underline{c}, G) := H^*(D^*(\underline{c}, G)).$$

Example 1.1. Let $\underline{c}$ be a multiplicative subset of a ring R, considered as a category with one object, and let $\tilde{R} : \underline{c} \times \underline{c} \longrightarrow Ab$ be the functor, defined for $\psi, \psi' \in \underline{c}$, by $\tilde{R}(\psi, \psi') = \psi^*\psi'_*$, where ψ^* is left multiplication on R by ψ, and where ψ'_* is right multiplication on R by ψ', then

$$H^0(\underline{c}, \tilde{R}) = \{\phi \in R |\ \phi\psi = \psi\phi \text{ for all } \psi \in \underline{c}\},$$

i.e. the commutant of $\underline{c}$ in R.

Given a k-algebra A, and consider a subcategory $\underline{c}$ of the category of right A-modules. Let, as above $\pi : \underline{c} \to k - mod$ be the forgetful-functor, and consider the bifunctor,

$$Hom_\pi : \underline{c} \times \underline{c} \longrightarrow k - mod$$

defined by

$$Hom_\pi(V_i, V_j) = Hom_k(V_i, V_j).$$

Put,

$$O_0(\underline{c}, \pi) := H^0(\underline{c}, Hom_\pi).$$

It is clear that $O_0(\underline{c}, \pi)$ is a k-algebra, and that there is a canonical homomorphism of k-algebras,

$$\eta_0(\underline{c}, \pi) : A \longrightarrow O_0(\underline{c}, \pi),$$

see §5.

Example 1.2. Let A be a commutative k-algebra of finite type, k algebraically closed, and let $Spec(A)$ be the subcategory of A-mod consisting of the modules $A/\mathfrak{p}$, where $\mathfrak{p}$ runs through Spec(A), the morphisms being only the obvious ones. It is easy to see that the homomorphism

$$\eta_0(Spec(A), \pi) : A \longrightarrow O_0(Spec(A), \pi)$$

identifies $A/rad(A)$ with $O_0(Spec(A), \pi)$. If A is reduced we even find an isomorphism,

$$\eta_0(Simp^*(A), \pi) : A \simeq O_0(Simp^*(A), \pi).$$

Here $Simp^*(A)$ is the subcategory of A-mod where the objects are A and the simple A-modules, $A/\mathfrak{m}$, i.e. the closed points of $Spec(A)$, and where the morphisms are the obvious quotient morphisms $A \to A/\mathfrak{m}$. $\eta_0(Simp^*(A), \pi)$ is, however not, in general, an isomorphism. This is easily seen when A is a local k-algebra. To remedy this situation we shall in §5. introduce and study a generalization $O(\underline{c}, \pi)$ of $O_0(\underline{c}, \pi)$ defined in terms of the noncommutative deformation theory , see [La 7].

§2. Noncommutative deformations.

The category $\underline{a}_r$, test algebras and liftings of modules.

Let $\underline{a}_r$ be the category of "r-pointed" artinian k-algebras. Recall that an object R of $\underline{a}_r$ is a diagram of morphism of artinian k-algebras,

$$\begin{tikzcd} k^r \arrow[r, "\iota"] \arrow[dr, "="'] & R \arrow[d, "\rho"] \\ & k^r \end{tikzcd}$$

Put, $Rad(R) := ker\rho$, such that,

$$R/Rad(R) \simeq \prod_{j=1}^{r} k_j, \qquad k_j \simeq k.$$

A morphism $\phi : R \to S$ of $\underline{a}_r$ is a morphism of such diagrams inducing the identity on k^r, implying that the induced map,

$$k^r \simeq R/Rad(R) \to S/Rad(S) \simeq k^r$$

is the identity. Pick idempotents $e_i \in k^r \subseteq R$ such that

$$\sum_{i=1}^{r} e_i = 1, \qquad e_i e_j = 0 \text{ if } i \neq j.$$

For every (i, j), we shall consider the subspace $R_{ij} := e_i R e_j \subseteq R$, and the pairing

$$R_{ij} \otimes_k R_{jk} \to R_{ik}$$

given in terms of the multiplication in R.

Let
$$R' = (R_{ij})$$
be the matrix algebra, the elements of which are matrices of the form
$$(\alpha_{ij})$$
with $\alpha_{ij} \in R_{ij},\ i,j = 1,\cdots,r$. There is an obvious isomorphism of k-algebras
$$\phi : R \to R'$$
defined by
$$\phi(\alpha) = (e_i \alpha e_j).$$
identifying the sub k-algebra k^r of R with the algebra of diagonal $r \times r$-matrices. Now, for any pair (i,j), $i,j = 1,\ldots,r$, consider the symbol ϵ_{ij}, and let's agree to put all products of such symbols equal to zero. Then we define the (i,j)-test algebra $R(i,j)$ as the matrix algebra

$$R(i,j) = k^r \oplus \begin{array}{cc} & j \\ i & \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & k \cdot \epsilon_{ij} & \cdots \\ 0 & \vdots & 0 \end{pmatrix} \end{array} \qquad \text{for } i \neq j$$

$$R(i,i) = \begin{array}{cc} & i \\ i & \begin{pmatrix} k & \vdots & 0 \\ \cdots & k[\epsilon_{ii}] & \cdots \\ 0 & \vdots & k \end{pmatrix} \end{array} \qquad \text{for } i = j$$

Denote by $HH^*(A,-)$ the Hochschild cohomology of the k-algebra A. If W is an A-bimodule denote by $Der_k(A,W)$ the k-vectorspace of derivations of A in W. Thus $\psi \in Der_k(A,W)$ is a linear map from A to W such that $\psi(a_1 \cdot a_2) = a_1\psi(a_2) + \psi(a_1)a_2$.

In particular, any element $w \in W$ determines a derivation $i(w) \in Der_k(A,W)$ defined by $i(w)(a) = aw - wa$. There is an exact sequence
$$0 \to HH^0(A,W) \to W \to Der_k(A,W) \to HH^1(A,W) \to 0$$
If V_i, V_j are right A-modules, then
$$Hom_k(V_i,V_j)$$
is an A-bimodule. In fact if $\phi \in Hom_k(V_i,V_j)$, then $a\phi$ is defined by $(a\phi)(v) = \phi(va)$, and ϕa is defined by $(\phi a)(v) = \phi(v)a$.

Moreover, we know that
$$HH^0(A, Hom_k(V_i,V_j)) = Hom_A(V_i,V_j)$$
$$HH^1(A, Hom_k(V_i,V_j)) = Ext^1_A(V_i,V_j).$$

Fix a finite family $\mathcal{V} = \{V_i\}_{i=1}^r$ of right A-modules, and consider for every

$$\psi \in Der_k(A, Hom_k(V_i, V_j))$$

the left $R(i,j)$-module and right A-module,

$$V_{ij}(\psi) = \begin{array}{cc} & \begin{array}{ccccc} & & & j & \end{array} \\ i & \begin{pmatrix} V_1 & & & \vdots & \\ \cdots & V_i & \cdots & \epsilon_{ij}V_j & \cdots \\ & & & V_j & \\ & & & \vdots & V_r \end{pmatrix} \end{array}$$

defined by

$$\begin{pmatrix} v_1 & & & \\ & v_i & \epsilon_{ij}v_j' & \\ & & v_j & \\ & & & v_r \end{pmatrix} \cdot a = \begin{pmatrix} v_1 a & & & \\ & v_i a & \epsilon_{ij}(\psi(a, v_i) + v_j' a) & \\ & & v_j a & \\ & & & v_r a \end{pmatrix}$$

and the obvious left $R(i,j)$-action. The $R(i,j)$- and the A-action commute, therefore we have got a $R(i,j) \otimes A$-module, such that

$$k^r \otimes_{R(i,j)} V_{ij}(\psi) \simeq \oplus_{i=1}^r V_i\,.$$

$V_{ij}(\psi)$ is called a lifting of $\mathcal{V}$ to $R(i,j)$. It is easy to see that if ψ maps to zero in $HH^1(A, Hom_k(V_i, V_j)) = Ext_A^1(V_i, V_j)$ then the lifting $V_{ij}(\psi)$ is trivial, i.e. isomorphic to the trivial one. Conversely, if $V_{ij}(\psi)$ is trivial, then ψ maps to zero in $Ext_A^1(V_i, V_j)$.

The noncommutative deformation functor.

We are now ready to start the study of noncommutative deformations of the family $\mathcal{V} = \{V_i\}_{i=1}^r$. We shall assume that$\mathcal{V}$ is a swarm, i.e. that for all $i, j = 1, 2, ..., r$,

$$dim_k Ext_A^1(V_i, V_j) < \infty.$$

Given an object $\rho : R = (R_{i,j}) \to k^r$ of $\underline{a}_r$, consider the left R-module $(R_{i,j} \otimes_k V_j)$. ρ defines a k-linear and left R-linear map,

$$\rho(R) : (R_{i,j} \otimes_k V_j) \to \oplus_{i=1}^r V_i,$$

inducing a homomorphism of R-endomorphism rings,

$$\tilde{\rho}(R) : (R_{i,j} \otimes_k Hom_k(V_i, V_j)) \to \oplus_{i=1}^r End_k(V_i).$$

The right A-module structure on the V_i's is defined by a homomorphism of k-algebras,

$$\eta_0 : A \to \oplus_{i=1}^r End_k(V_i).$$

Definition 2.1. *The deformation functor*

$$Def_{\mathcal{V}} : \underline{a}_r \to Sets$$

is defined for every $R \in \underline{a}_r$*, as the set,*

$$Def_{\mathcal{V}}(R) \in \underline{Sets}$$

of isoclasses of homomorphisms of k*-algebras,*

$$\{\eta' : A \to (R_{i,j} \otimes_k Hom_k(V_i, V_j))\}/\sim$$

such that,

$$\tilde{\rho}(R) \circ \eta' = \eta_0,$$

where the equivalence relation $\sim$ *is defined by inner automorphisms in the* k*-algebra*

$$End_R((R_{i,j} \otimes_k V_j)) = (R_{i,j} \otimes_k Hom_k(V_i, V_j)).$$

Any such isoclass $\tilde{\eta}'$ *will be called a deformation or a lifting of* V *to* R*, and usually denoted* V_R*.*

One easily proves that $Def_{\mathcal{V}}$ has the same properties as the ordinary deformation functor.

Let $\pi : R \to S$ be a morphism of $\underline{a}_r$, such that $Rad(R) \cdot \ker \pi = 0$. Morphisms like this will be called *small.* If $V_R \in Def_V(R)$ it is easy to see that $V_S := S \otimes_R V_R \in Def_V(S)$ and that $\overline{V} = \ker\{V_R \to S \otimes_R V_R\}$ is, as a left R-module, an $R/Rad(R) = k^r$-module. Put $\ker \pi = (K_{ij})$, then $\overline{V} = (\overline{V}_{ij})$ where $\overline{V}_{ij} = K_{ij} \otimes_k V_j$.

Consider now the k-vector spaces

$$E_{ij}^d = Ext_A^d(V_i, V_j)^*$$

i.e. the dual k-vectorspaces of $Ext_A^d(V_i, V_j)$, and consider the k-algebra of matrices,

$$T_2^d = \begin{pmatrix} k & & 0 \\ & \ddots & \\ 0 & & k \end{pmatrix} + (\epsilon_{ij} E_{ij}^d)$$

where as above, we assume all products of the ϵ_{ij}'s are equal to zero. Now let for every $i, j = 1, \ldots, r$, and $d = 1, 2$,

$$\left\{ t_{ij}^d(\ell) \right\}_{\ell=1}^{e_{ij}^d}$$

be a basis of E_{ij}^d, and let $\{\psi_{ij}^d(\ell)\}_{\ell=1}^{e_{ij}^d}$ be the dual basis. Thus $e_{ij}^k = dim_k E_{ij}^k$. Consider the k-algebra

$$T^d = \begin{pmatrix} k & & 0 \\ & \ddots & \\ 0 & & k \end{pmatrix} + (\widetilde{E}_{ij}^d)$$

freely generated as matrix algebra by the generators $\left\{t_{ij}^d(\ell)\right\}_{\ell=1}^{e_{ij}^d}$. An element of $\widetilde{E}_{ij}^d$ is then a matrix where the elements are linear combinations of elements of the form:

$$\begin{aligned}\tau_{ij} &= t_{ij_1}^d(l_1) \otimes t_{j_1 j_2}^d(l_2) \otimes \cdots \otimes t_{j_{m-1} j_m}^d(l_m) \\ j &= j_m,\ 1 \le l_s \le e_{j_{s-1} j_s}^d,\ 1 \le j_s \le r,\ m \ge 1 \\ &\text{of}\ \ E_{ij_1}^d \otimes E_{j_1 j_2}^d \otimes \cdots \otimes E_{j_{m-1} j}^d.\end{aligned}$$

Obviously

$$T_2^1 = T^1/Rad(T^1)^2.$$

where $Rad(T^1)$ is the two-sided ideal of T^1 generated by $(\widetilde{E}_{ij}^1)$.

Lemma 2.2. *Let R be an object of $\underline{a}_r$ and suppose that there exists a surjective homomorphism*

$$\phi_2 : T_2^1 \to R/Rad(R)^2,$$

then there exists a surjective homomorphism

$$\phi : T^1 \longrightarrow R$$

which lifts ϕ_2.

Definition 2.3. *For every object R of $\underline{a}_r$, put*

$$T_R = (Rad(R)/Rad(R)^2)^*$$

and call it the tangent space of R.

Lemma 2.4. *Let $\phi : R \to S$ be a morphism of $\underline{a}_r$. Assume ϕ induces a surjective homomorphism*

$$\phi^1 : T_R^* \to T_S^*$$

(or an injective homomorphism on the tangent space level). Then ϕ is surjective.

Notice that if we pick any finite dimensional k-vectorspaces F_{ij}, then there is a unique maximal pro-algebra $F = F(F_{ij})$ in $\underline{a}_r$ with tangent space

$$T_F \simeq (F_{ij}^*)$$

F is defined by the formula (2) above, with E replaced by F.

To prove the existence of a hull for the deformation functor Def_V the basic tool is the obstruction calculus, which in this case is easily established:

Proposition 2.5. *Suppose $R \overset{\phi}{\to} S$ is a surjective small morphism of $\underline{a}_r$, i.e. suppose $ker\phi \cdot Rad(R) = 0$. Put $ker\phi = (I_{ij})$. Consider any $V_S \in Def_V(S)$. Then there exists an obstruction*

$$o(\phi, V_S) \in (I_{ij} \otimes_k Ext_A^2(V_i, V_j))$$

which is zero if and only if there exists a lifting $V_R \in Def_V(R)$ of V_S. The set of isomorphism classes of such liftings is a torsor under

$$(I_{ij} \otimes_k Ext^1_A(V_i, V_j))\,.$$

Proof. As a k-vectorspace $V_R = (R_{ij} \otimes V_j)$ maps onto $V_S = (S_{ij} \otimes V_j)$. Since the right action of A commutes with the left S-action the action of an element $a \in A$ on V_S is uniquely given in terms of a family of k-linear maps,

$$a_{ij} : V_i \to S_{ij} \otimes V_j.$$

We may of course lift these to k-linear maps

$$\sigma(a)_{ij} : V_i \to R_{ij} \otimes V_j$$

inducing a lift of the action of each element of A on

$$\bigoplus_{j=1}^{r} S_{ij} \otimes V_j$$

to a k-linear action on

$$\bigoplus_{j=1}^{r} R_{ij} \otimes V_,.$$

The obstruction for this to be an A-module structure is, as usual, the Hochschild 2-cocycle

$$\psi^2(a,b) = \sigma(ab) - \sigma(a)\cdot\sigma(b) \in (I_{ij} \otimes_k Hom_k(V_i, V_j))\,.$$

The fact that this is a 2-cocycle follows from

(1) $\sigma(c)\cdot\psi^2(a,b) = c\cdot\psi^2(a,b)$
(2) $\psi^2(a,b)\cdot\sigma(c) = \psi^2(a,b)\cdot c$

and the obvious relation

$$\begin{aligned}
&d\psi^2(a,b,c) = a\psi^2(b,c) - \psi^2(ab,c) + \psi^2(a,bc) - \psi^2(a,b)\cdot c\\
&= \sigma(a)(\sigma(bc) - \sigma(a)\sigma(c)) - (\sigma(abc) - \sigma(ab)\sigma(c)) + (\sigma(abc) - \sigma(a)\sigma(bc))\\
&- (\sigma(ab) - \sigma(a)\sigma(b))\sigma(c) \equiv 0
\end{aligned}$$

Suppose the class of ψ^2 in $(I_{ij} \otimes_k Ext^2_A(V_i, V_j))$ is zero. This means that $\psi^2 = d\phi$, where $\phi \in Hom_k(A, (I_{ij} \otimes_k Hom_k(V_i, V_j))$, $\psi^2(a,b) = d\phi(a,b) = a\phi(b) - \phi(ab) + \phi(a)b$. Let $\sigma' = \sigma + \phi$ and consider,

$$\sigma'(ab) - \sigma'(a)\sigma'(b) = \sigma(ab) - \sigma(a)\sigma(b) + \phi(ab) - \sigma(a)\phi(b) - \phi(a)\sigma(b) - \phi(a)\phi(b)\,.$$

Since the matrix $\phi(a)\phi(b) = 0$ as $I_{ij}\cdot I_{jk} = 0$, $\forall i,j,k$ and since $\sigma(a)\phi(b) = a\phi(b)$, $\phi(a)\sigma(b) = \phi(a)b$ for the same reason, we find that $\sigma'(ab) - \sigma'(a)\sigma'(b) = 0$, i.e. there is a lifting of the A-module action to $V_R = (R_{ij} \otimes V_j)$.

If we have given one A-module action σ on V_R lifting the action on V_S, then for any other σ' we may consider the difference

$$\sigma' - \sigma : A \to (I_{ij} \otimes_k Hom_k(V_i, V_j))$$

Consider

$$d(\sigma' - \sigma)(a,b) = a(\sigma'(b) - \sigma(b)) - (\sigma'(ab) - \sigma(ab)) + (\sigma'(a) - \sigma(a))b$$

As above we may substitute $\sigma'(a)$ for a and $\sigma(b)$ for b, and the expression becomes zero. Thus $\sigma' - \sigma = \bar{\xi}$ defines a class

$$\xi \in (I_{ij} \otimes_k Ext^1_A(V_i, V_j)).$$

If $\xi = 0$, then $\bar{\xi} = d\phi$, $\phi \in (I_{ij} \otimes_k Hom_k(V_i, V_j))$ such that $\sigma'(a) - \sigma(a) = a\phi - \phi a$. Let $\phi = (\phi_{ij})$, then ϕ_{ij} defines an isomorphism

$$\bar{\phi} = id + \phi : \bigoplus_j R_{ij} \otimes V_j \to \bigoplus_j R_{ij} \otimes V_j$$

lifting the identity of $\bigoplus_j S_{ij} \otimes V_j$. Moreover

$$\begin{aligned} \sigma(a)(id + \phi)(v_i) &= \sigma(a)v_i + a\phi(v_i) \\ &= \sigma'(a)(v_i) + \phi(av_i) = (id + \phi)\sigma'(a)(v_i) \end{aligned}$$

since $\phi(\sigma'(a)v_i) = \phi(av_i)$.

Therefore the A-module structures on

$$V_R = (R_{ij} \otimes V_j)$$

defined by σ and σ' are isomorphic. The rest is clear. □

Theorem 2.6. *The functor $Def_{\mathcal{V}}$ has a prorepresentable hull, or a formal moduli of V, $H \in \hat{\underline{a}}_r$, together with a versal family*

$$\tilde{V} = (H_{i,j} \otimes V_j) \in \varprojlim_{n \geq 1} Def_{\mathcal{V}}(H/Rad(H)^n)$$

such that the corresponding morphism of functors on $\underline{a}_r$,

$$\rho : Mor(H, -) \to Def_{\mathcal{V}}$$

is smooth and an isomorphism on the tangent level. Moreover, H is uniquely determined by a set of matric Massey products defined on subspaces,

$$D_n \subset \bigoplus_{p=2}^{n} Ext^1(V_i, V_{j_1}) \otimes \cdots \otimes Ext^1(V_{j_{p-1}}, V_j),$$

with values in $Ext^2(V_i, V_j)$.

Proof. Notice first that ρ being an isomorphism at the tangent level means that ρ is an isomorphism for all objects R of $\underline{a}_r$ for which $Rad(R)^2 = 0$.

Word for word we may copy the proof (4.2) of [La 1]. In particular $H/Rad(H)^2 \simeq T^1_2$ and $Mor(H, R(i,j)) \simeq Hom_k(E^1_{ij}, k) \simeq Ext^1_A(V_i, V_j) \simeq Def_{\mathcal{V}}(R(i,j))$.

Notice that the universal lifting of V to T^1_2 is the $T^1_2 \otimes_k A$-module $\widetilde{V}_2$

$$\begin{pmatrix} V_1 & & 0 \\ & \ddots & \\ 0 & & V_2 \end{pmatrix} + (E^1_{ij} \otimes_k V_j)$$

with the obvious left T^1_2-action and the right A-action defined as,

$$((1 \otimes v_i) \cdot a)_{ii} = 1 \otimes v_i \cdot a + \sum_{\ell} t^1_{i,j}(\ell) \otimes (\psi^1_{i,j}(\ell)(a, v_i))$$

where $v_i \in V_i$, and where $\{t^1_{ij}(\ell)\}_{\ell=1}^{e_{ij}}$ is the chosen basis of E^1_{ij}. Recall that $\{\psi^1_{ij}(\ell)\}_{\ell=1}^{e_{ij}}$, the dual base, consists of elements $\psi^1_{ij}(\ell) \in Ext^1_A(V_i, V_j)$, which may be represented as elements of $Der_k(A, Hom_k(V_i, V_j))$.

To obtain H we kill obstructions for lifting $\widetilde{V}_2$ successively, to $T^1_3 := T^1/Rad(T^1)^3, T^1_4$ etc. just like in the commutative case. The proof of the existence of a prorepresentable hull for Def_V can, of course, also be modeled on the classical proof of M.Schlessinger [Sch]. This has been carried out by Runar Ile, see [Ile]. $\square$

A general structure theorem for artinian k-algebras.

For every deformation $V_R \in Def_{\mathcal{V}}(R)$ there exists, by definition an, up to inner automorphisms, unique homomorphism of k-algebras,

$$\eta_{V_R} : A \to End_R(V_R) = (R_{ij} \otimes Hom_k(V_i, V_j)).$$

Definition 2.7. *Let $\mathcal{V} = \{V_i\}_{i=1}^r$ be any finite swarm of A-modules, and let $H := H(\mathcal{V})$ be the formal moduli for $\mathcal{V}$, and $\tilde{V}$ the versal family. The k-algebra of observables of the family $\mathcal{V}$ is the k-algebra,*

$$O(\mathcal{V}) := End_(\tilde{V}) = (H_{ij} \otimes Hom_k(V_i, V_j))$$

We would like to describe the kernel and the image of the map,

$$\eta : A \to O(\mathcal{V})$$

To do this we need to consider the matric Massey products of the form,

$$D_n \to Ext^2_A(V_{i_1}, V_{i_n}),$$

the obvious generalizations of the matrix Massey products introduced in [La 2].

Here we shall describe these products using Hochschild cohomology. This is a more convenient way of describing the map η and maybe also an easier way of understanding the nature of the Massey products.

To simplify the notations, put

$$Ext^1_A(V) = (Ext^1_A(V_i, V_j))$$

For $l = 2$, the Massey product above is simply the cup product

$$Ext^1_A(V) \otimes Ext^1_A(V) \to Ext^2_A(V)$$

defined by: Let $(\psi^1_{ij}), (\psi^2_{ij}) \in Ext^1_A(V)$, and express ψ^k_{ij} as 1-Hochschild cocycles, i.e. $\bar{\psi}^1_{ij} \in Der_k(A, Hom_k(V_i, V_j)), \bar{\psi}^2_{ij} \in Der_k(A, Hom_k(V_i, V_j))$. The cup product $(\psi^1_{ij}) \cup (\psi^2_{ij}) \in \ Ext^2_A(V)$, now denoted

$$\langle(\psi^1_{ij}), (\psi^2_{ij})\rangle \in \ Ext^2_A(V)$$

is defined by the 2-cocycle in the Hochschild complex

$$\langle(\psi^1_{ij}), (\psi^2_{ij})\rangle_{ik}(a, b) = \sum_j \bar{\psi}^1_{ij}(a) \circ \bar{\psi}^2_{jk}(b) \in Hom_k(V_i, V_k)$$

Suppose $\langle(\psi^1_{ij}), (\psi^2_{ij})\rangle = 0$, this means that there exists, for each pair (i, k) a 1-cochain ϕ^{12}_{ik} in the Hochschild complex, i.e. a map

$$\phi^{12}_{ik} \in Hom_k(A, Hom_k(V_i, V_k))$$

such that $d\phi^{12}_{ik} = \langle(\psi^1_{ij}), (\psi^2_{ij})\rangle_{ik}$, i.e. such that for all $a, b \in A$,

$$a\phi^{12}_{ik}(b) - \phi^{12}_{ik}(ab) + \phi^{12}_{ik}(a)b = \sum_j \bar{\psi}^1_{ij}(a) \circ \bar{\psi}^2_{jk}(b)$$

Given classes $\psi^1 = (\psi^1_{ij}), \psi^2 = (\psi^2_{ij}), \psi^3 = (\psi^3_{ij}) \in Ext^1_A(V)$ such that $\langle\psi^1, \psi^2\rangle = \langle\psi^2, \psi^3\rangle = 0$ there exists $\phi^{12} = (\phi^{12}_{ik}), \phi^{23} = (\phi^{23}_{ik}) \in Hom_k(A, Hom_k(V_i, V_k))$ such that

$$d\phi^{12} = \langle\psi^1, \psi^2\rangle, \quad d\phi^{23} = \langle\psi^2, \psi^3\rangle\,.$$

Then there exists a matrix Massey product

$$\langle\psi^1, \psi^2, \psi^3\rangle \in \ Ext^2_A(V)$$

defined by the 2-cocycle

$$\langle\psi^1, \psi^2, \psi^3\rangle_{ik}(a, b) = \sum_j \phi^{12}_{ij}(a)\psi^3_{jk}(b) - \sum_j \psi^1_{ij}(a)\phi^{23}_{jk}(b)$$

in $Hom_k(A \otimes_k A, Hom_k(V_i, V_j))$.

As in [La 2] we may go on and obtain a sequence of *defining systems* $\{D_n\}_{n=2}^{\infty}$ and Massey products, computing the relations of $H(\mathcal{V})$.

Now if $a \in A$, denote by $\tilde{a}_i \in Hom_k(V_i, V_i)$ its action on V_i, $i = 1, \dots, d$. Let $End_0(V)$ be the diagonal matrix $(End_k(V_i, V_i))$, contained in the matrix $End_k(V) := (End_k(V_i, V_j))$. Put,

$$End(V)a = (\tilde{a}_1, \dots, \tilde{a}_d) \in End_0(V) \subseteq End(V)$$

If $a \in A$ is such that $End(V)a = 0$, this means that a acts trivially on each V_i. Let $\psi \in Ext^1_A(V)$ be represented by 1-cocycles $\psi_{ij} \in Der_k(A, End_k(V_i, V_j))$. If $End(V)a = End(V)b = 0$, we find,

$$\psi_{ij}(ab) = a\psi_{ij}(b) + \psi_{ij}(a)b = 0$$

This shows that $\psi \in Ext^1_A(V)$ defines a unique k-linear map,

$$\psi : \{a \in A \mid End(V)a = 0\} \to End_k(V)$$

vanishing on all squares.

Let $a \in A$, $End(V)a = 0$, and put

$$Ext^1_A(V)a = 0$$

when $\psi(a) = 0$, $\forall \psi \in Ext^1_A(V)$. Consider the sub k-vector space of A

$$K_1 = \{a \in A \mid End(V)a = Ext^1_A(V)a = 0\}$$

Let $\sum \alpha_{ij}\psi^i \otimes \psi^j \in Ext^1_A(V) \otimes Ext^1_A(V)$ such that its Massey (cup-)product is zero, i.e. such that:

$$\sum \alpha_{ij}\langle \psi^i, \psi^j \rangle = 0$$

Then there exists a 1-cochain $\phi \in Hom_k(A, (Hom_k(V_i, V_j)))$ such that

$$d\phi = \sum_{ij} \alpha_{ij}\langle \psi^i, \psi^j \rangle$$

Since $d\phi = 0$ implies that ϕ represents an element of $Ext^1_A(V)$ it is clear that ϕ defines a unique k-linear map

$$\phi : K_1 \to End_k(V)\,.$$

Let us denote by

$$ker\langle Ext^1_A(V), Ext^1_A(V)\rangle$$

the subset of $Ext^1_A(V) \otimes Ext^1_A(V)$ for which the Massey product (i.e. the cup product) is zero. Then we may put

$$ker\langle Ext^1_A(V), Ext^1_A(V)\rangle a = 0$$

if for every $d\phi \in ker\langle Ext^1_A(V), Ext^1_A(V)\rangle$, $\phi(a) = 0$.

Let

$$K_2 = \{a \in A \mid End(V)a = Ext^1_A(V)a = ker\langle Ext^1_A(V), Ext^1_A(V)\rangle a = 0\}$$

Continuing in this way we find a sequence of ideals $\{K_n\}_{n\geq 0}$, where $K_0 = ker\{A \to End(V)\}$ and, in general, $K_n = \{a \in A \mid D_n a = 0\}$.

Theorem 2.8. *Let A be any k-algebra and let $\mathcal{V} = \{V_i\}_{i=1}^r$ be a swarm of A-modules. Then the kernel of the canonical map*

$$\eta : A \to O(\mathcal{V})$$

is determined by the matrix Massey product structure of $Ext^i_A(V)$, $i = 1, 2$. In fact

$$\ker \eta = \bigcap_{n\geq 0} K_n$$

Proof. By definition, the homomorphism of k-algebras

$$\eta : A \to O\mathcal{V})$$

lifts the k-algebra homomorphism,

$$\eta_0 : A \to \prod_{i=1}^{d} End_A(V_i).$$

Modulo $Rad(H)^2$ η induces the homomorphism,

$$\eta_1 : A \to \prod_{i=1}^{d} End_k(V_i) \oplus (E^1_{ij} \otimes Hom_k(V_i, V_j))$$

with,

$$\eta_1(a)_{ij} = \delta_{ij} \otimes \eta_0(a)_i + \sum_l t_{ij}(l) \otimes \psi^1_{ij}(l)(a_i, -)\,, \qquad \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Now, by construction H is the quotient of the formally free k-algebra T^1 generated by the independent variables $\{t_{ij}(l),\, l = 1, \dots, l_{ij}\}$ as explained above. The relations of T^1 are generated by linear combinations of monomials in these variables of the form,

$$y_{ik} = \sum_{r=1}^{\infty} \sum_{\underline{j},\underline{l}} \alpha^{l_1,\dots,l_r}_{i,j_1,\dots,j_{r-1},k} t_{ij_1}(l_1) t_{j_1 j_2}(l_2) \cdots t_{j_{r-1},k}(l_r),$$

corresponding to elements,

$$y_{ik} \in Ext^2_A(V_i, V_k)^*.$$

The coefficients α are expressed in terms of partially, but inductively well defined, matric Massey products,

$$\langle\ \rangle_{\text{Ł}1401underliner} : \quad D_{\underline{r}} \quad \longrightarrow \quad Ext^2_A(V)$$

such that, if the Massey product $\langle \psi^1_{ij_1}(l_1), \dots, \psi^1_{j_{r-1},k}(l_r)\rangle$ is defined, then

$$y_{ik}(\langle \psi^1_{ij_1}(l_1), \dots, \psi^1_{j_{r-1},k}(l_r)\rangle) = \alpha^{l_1,\dots,l_r}_{ij_1,\dots,j_{r-1},k}.$$

We therefore obtain a basis for H, as k-vector space, by picking, in a coherent way, a k-basis for

$$coker\{Ext^2_A(V)^* \to D^*_{\underline{r}}\} = (\ker\langle\ \rangle_{\underline{r}})^*$$

Since $K_r = ker\langle\ \rangle_{\underline{r}}$, the conclusion of the Theorem follows. □

Remark 2.9. Let E_{ij} be an extension of V_i by V_j, then as a k-vector space $E_{ij} = V_j \oplus V_i$ and the right action by A is defined for $(v_j, v_i) \in E_{ij}$, $a \in A$ by,

$$(v_j, v_i)a = (v_j a + \psi^1_{ij}(a, v_i), v_i a),$$

where,

$$\psi^1_{ij} \in Der_k(A, Hom_k(V_i, V_j))$$

defines an element,

$$\bar{\psi}^1_{ij} \in Ext^1_A(V_i, V_j)$$

corresponding to E_{ij}. Suppose we consider an extension E_{ijk} of E_{ij} by V_k. Then as a k-vector space $E_{ijk} \simeq V_k \oplus E_{ij} = V_k \oplus V_j \oplus V_i$ and the action by A is defined by

$$(v_k, v_j, v_i)a = (v_k a + \phi(a, (v_j, v_i)), v_j a + \psi^1_{ij}(a, v_i), v_i a).$$

By additivity

$$\phi(a, (v_j, v_i)) = \phi(a, (v_j, 0)) + \phi(a, (0, v_i))\,.$$

Put

$$\psi^{1,0}_{ij}(a, v_i) = \psi^1_{ij}(a, v_i),\ \psi^{0,1}_{jk}(a, v_j) = \phi(a, (v_j, 0)), \quad \psi^{1,1}_{ik}(a, v_i) = \phi(a, (0, v_i)),$$

then the conditions on the action imply

$$\psi^{0,1}_{jk} \in Der_k(A, Hom_k(V_j, V_k))$$
$$\psi^{1,1}_{jk} \in Hom_k(A, Hom_k(V_i, V_k))$$

and

$$d\psi^{1,1}_{ik} = \psi^{0,1}_{jk} \circ \psi^{1,0}_{ij}.$$

This means that $\bar{\psi}^{0,1}_{jk} \in Ext^1_A(V_j, V_k)$ and that the cup product,

$$\bar{\psi}^{0,1} \cup \bar{\psi}^{1,0} \in Ext^2_A(V_i, V_k)$$

is zero.

Now, consider an extension E_{ijkl} of E_{ijk} by V_l. As before the action of A on E_{ijkl} is given by

$$(v_l, v_k, v_j, v_i) \cdot a$$
$$= (v_l \cdot a + \phi(a, v_k, v_j, v_i), v_k \cdot a + \psi^2_{ik}(a, v_i) + \psi^{0,1}_{jk}(a, v_j), v_j \cdot a + \psi^1_{ij}(a, v_i), v_i \cdot a).$$

Put, as above,

$$\psi^{1,0,0} = \psi^{1,0}$$
$$\psi^{0,1,0} = \psi^{0,1}$$
$$\psi^{0,0,1}_{kl}(a, v_k) = \phi(a, v_k, 0, 0)$$
$$\psi^{0,1,1}_{jl}(a, v_j) = \phi(a, 0, v_j, 0)$$
$$\psi^{1,1,1}_{ik}(a, v_i) = \phi(a, 0, 0, v_i)$$

The conditions on ϕ are expressed by:

$$d\psi^{0,0,1}_{kl} = 0$$
$$d\psi^{0,1,1}_{jl} = \psi^{0,0,1}_{kl} \circ \psi^{0,1,0}_{jk}$$
$$d\psi^{1,1,1}_{il} = \psi^{0,1,1}_{jl} \circ \psi^{1,0,0}_{ij} + \psi^{0,0,1}_{kl} \circ \psi^{1,1,0}_{ik}$$

This means that $\bar{\psi}^{0,0,1}_{kl} \in Ext^1_A(V_k, V_l)$, that the cup product $\bar{\psi}^{0,0,1}_{kl} \cup \bar{\psi}^{0,1,0}_{jk} \in Ext^2_A(V_j, V_l)$ is zero, and that the Massey product

$$\langle \bar{\psi}^{0,0,1}_{kl}, \bar{\psi}^{0,1,0}_{jk}, \bar{\psi}^{1,0,0}_{ij} \rangle \in Ext^2_A(V_i, V_l)$$

is zero.

It is clear how to continue.

Corollary 2.10. *Suppose the k-algebra A is of finite dimension, and let the finite swarm $\mathcal{V} = \{V_i\}_{i=1}^r$ contain all simple representations, then*

$$\eta : A \to O(\mathcal{V})$$

is injective.

Proof. Let $a \in A$, and suppose $\eta(a) = 0$. Since A as a right A-module is an extension of the V'_is we may assume there are exact sequences of right A-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q_1 & \longrightarrow & A & \longrightarrow & \bigoplus_{i\in I_1} V_i & \longrightarrow & 0 \\
\\
0 & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & \bigoplus_{i\in I_2} V_i & \longrightarrow & 0 \\
\\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\\
0 & \longrightarrow & Q_N & \longrightarrow & Q_{N-1} & \longrightarrow & \bigoplus_{i\in I_N} V_i & \longrightarrow & 0
\end{array}$$

with $Q_N = \bigoplus_{i\in I_{N+1}} V_i$, $Q_{N+1} = 0$. Since $End(V)a = 0$ it follows from the first exact sequence above that $1 \cdot a = a \in Q_1$. Consider the exact sequence

$$0 \longrightarrow \bigoplus_{i\in I_2} V_i \longrightarrow A/Q_2 \longrightarrow \bigoplus_{i\in I_1} V_i \longrightarrow 0$$

Since $Ext^1_A(V)a = 0$ it follows that $1 \cdot a = a \in Q_2$. In fact, multiplication by a is zero on V_i, $i = 1, \dots, r$ and on A/Q_2 it is therefore given by the elements in $Ext^1_A(V)$. Continuing in this way, we consider the extensions of extensions,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i\in I_3} V_i & \longrightarrow & A/Q_3 & \longrightarrow & A/Q_2 & \longrightarrow & 0 \\
\\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\\
0 & \longrightarrow & \bigoplus_{i\in I_{N+1}} V_i & \longrightarrow & A & \longrightarrow & A/Q_N & \longrightarrow & 0.
\end{array}$$

Refering to (2.9), we know that the multiplication by $a \in A$ on the right in the middle term is given inductively, by a family of cochains $\psi_{ij}^{\underline{\epsilon}} \in Hom_k(A, Hom_k(V_i, V_j))$, with $\underline{\epsilon} \in \{0,1\}^n$, for $2 \leq n$, such that

$$d\psi_{ik}^{\underline{\epsilon}} = \sum_{\substack{\underline{\epsilon}_1 + \underline{\epsilon}_2 = \underline{\epsilon} \\ j}} \psi_{ij}^{\underline{\epsilon}_1} \circ \psi_{jk}^{\underline{\epsilon}_2}\,.$$

Now, this means that all these extensions are defined in terms of a series of well defined Massey products each one containing 0. By the proof of Theorem (2.8), we find that for all i, j and all $\underline{\epsilon}$, $\psi_{ij}^{\underline{\epsilon}}(a, -) = 0$.

This means that the action of $a \in A$ must be 0, so $a = 0$. $\square$

The same proof works for the following,

Corollary 2.10 bis. *Suppose the k-algebra A is an extension of extensions of the objects in the finite swarm $\mathcal{V} = \{V_i\}_{i=1}^r$. Then*

$$\eta : A \to O(\mathcal{V})$$

is injective.

Corollary 2.11. *Suppose A is an object of $\underline{a}_r$, and let $\mathcal{V} = \{V_i\}_{i=1}^r$ be the family of simple representations, with $V_i \simeq k_i$. Then*

$$A \simeq H$$

Proof. Obviously A is a left A-and a right A-module, flat over A, therefore $A \in Def_V(A)$. Let $R \in \underline{a}_r$ and pick an element $V_R \in Def_{\mathcal{V}}(R)$. Since

$$End(V) = \begin{pmatrix} k & \cdots & k \\ \vdots & & \vdots \\ k & \cdots & k \end{pmatrix},$$

this amounts to a homomorphism of k-algebras $A \to End_R(V_R) = R$, implying that A is versal. But then the unicity of the hull of Def_V gives us an isomorphism:

$$\phi : H \to A$$

□

Remark 2.12 Reconstructing an ordered set Λ and $k[\Lambda]$, from the swarm of simple modules.

Let Λ be an ordered set, see §1, and let $A = k[\Lambda]$, $V = \{k_\lambda\}_{\lambda \in \Lambda}$. Then the Corollary above implies that $H \simeq k[\Lambda]$.

1. By the general theory we know that $A = k[\Lambda]$ is the matrix algebra generated freely by the immediate relations $\lambda_1 \gg \lambda_2$, i.e. those for which $\{\lambda' \in \Lambda | \lambda_1 > \lambda' > \lambda_2\} = \emptyset$, modulo relations of the form

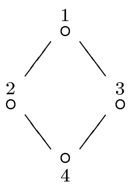
$$\begin{aligned} &(\lambda' > \lambda_2^1)(\lambda_2^1 > \lambda_3^1) \cdots (\lambda_{n_1}^1 > \lambda) \\ &= (\lambda' > \lambda_2^2)(\lambda_2^2 > \lambda_3^2) \cdots (\lambda_{n_2}^2 > \lambda) \end{aligned}$$

They correspond to the first obstructions, given by the n_i term well defined Massey products

$$\begin{aligned} Ext_A^1(k_{\lambda'}, k_{\lambda_2^1}) \otimes \cdots \otimes Ext_A^1(k_{\lambda_{n_1}^1}, k_\lambda) \to Ext_A^2(k_{\lambda'}, k_\lambda) \\ Ext_A^1(k_{\lambda'}, k_{\lambda_2^2}) \otimes \cdots \otimes Ext_A^1(k_{\lambda_{n_2}^2}, k_\lambda) \to Ext_A^2(k_{\lambda'}, k_\lambda) \end{aligned}$$

There are as many relations as there are base elements of $Ext_A^2(k_{\lambda'}, k_\lambda)$.

2. Let us check this for the dimond, i.e. for Λ:



One easily computes the Ext's,

$$Ext^1_A(k_{\lambda_i}, k_{\lambda_j}) = \begin{cases} 0 & i = j \\ k & \text{for } \; i = 1, \; j = 2,3 \\ k & \text{for } \; i = 2,3, \; j = 4 \end{cases}$$

$$Ext^2_A(k_{\lambda_i}, k_{\lambda_j}) = \begin{cases} 0 & \text{for } \; (i,j) \neq (1,4) \\ k & \text{for } \; i = 1, \; j = 4 \end{cases}$$

The two cup-products

$$Ext^1_A(k_{\lambda_1}, k_{\lambda_j}) \otimes Ext^1_A(k_{\lambda_j}, k_{\lambda_4}) \to Ext^2_A(k_{\lambda_1}, k_{\lambda_4}) \quad \text{for } j = 2,3,$$

are non-trivial. At the tangent level we have:

$$H_2 = \begin{pmatrix} k & k & k & 0 \\ 0 & k & 0 & k \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k \end{pmatrix}$$

Therefore H must be a quotient of the matrix ring,

$$T^1 = \begin{pmatrix} k & t_{12} \cdot k & t_{13} \cdot k & (t_{12}t_{24} \cdot k + t_{13}t_{34} \cdot k) \\ 0 & k & 0 & t_{24} \cdot k \\ 0 & 0 & k & t_{34} \cdot k \\ 0 & 0 & 0 & k \end{pmatrix}$$

The kernel of $T^1 \to H$ is given in terms of the cup products above. In fact, since we have $t^*_{13} \cup t^*_{34} = t^*_{12} \cup t^*_{24} = y^*$ where y^* is the generator of $Ext^2_A(k_{\lambda_1}, k_{\lambda_4})$, the kernel of $T^1 \to H$ is simply $t_{13} \otimes t_{34} + t_{12} \otimes t_{24}$ such that

$$H = \begin{pmatrix} k & k & k & k \\ 0 & k & 0 & k \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k \end{pmatrix} \simeq k[\Lambda]$$

as it should.

In general, we may reconstruct Λ from the tangent space T_H and the Massey-products above.

The corresponding problem for finite groups, i.e. reconstructing G from $k[G]$ is called the isomorphism problem. Due to some nice examples of Dade, we know that this is hopeless. In fact there are two non isomorphic finite groups such that their group algebras are isomorphic for all fields.

§**3. Noncommutative modular deformations.** Let V be any right A-module such that $dim_k Ext^1_A(V,V) < \infty$. Consider the formal moduli $H^A =: H$, the formal versal family $\tilde{V} = H \otimes V$, and the corresponding morphism of functors,

$$\rho : Mor_{\underline{a}_r}(H, -) \to Def_V.$$

We know that ρ is not, in general, injective. However, V is also a right $A \otimes End_A(V)$-module. As such it has a formal moduli $H^{A,End}$, and by functoriality there is a natural k-algebra homomorphism, $H^A \to H^{A,End}$. Let H_0^A be the unique maximal common quotient of H^A and $H^{A,End}$. Using the same construction as in [La, Pf], §2, we prove that the composition,

$$\rho_0 : Mor_{\underline{a}_r}(H_0, -) \to Mor_{\underline{a}_r}(H, -) \to Def_V$$

is injective.

At the tangent level, the homomorphisms,

$$H^A \to H^{A,End} \leftarrow H^{End},$$

looks like the canonical homomorphisms,

$$Ext^1_A(V,V) \leftarrow Ext^1_{A\otimes_k End}(V,V) \to Ext^1_{End}(V,V).$$

Representing elements of the ext-groups as derivations, it is easy to see that the two images are contained in the subspace $Ext^1_A(V,V)^{End}$, respectively $Ext^1_{End}(V,V)^A$. Therefore the tangent space of H_0 must be contained in the subspace of invariants under $End_A(V)$ of the tangent space of H, $Ext^1_A(V,V)^{End}$.

The tangent space of the modular (prorepresenting) substratum, and almost split sequences.

Consider as above a swarm $\mathcal{V} = \{V_i\}_{i=1}^r$ of A-modules, and consider the k^r-algebra

$$End_A(V) = (Hom_A(V_i, V_j)).$$

Suppose from now on that the modules V_i are non-isomorphic, indecomposables, and that for each $i = 1, .., r$. $End_A(V_i)$ is a commutative local ring with maximal ideal $\underline{m}_i$.

Lemma 3.1. *Under the above assumptions, the radical of $End_A(V)$ has the form*

$$rad(V) = \begin{pmatrix} m_1 End_A(V_1) & & & \vdots & \\ \cdots & \cdots & \underline{m}_i End_A(V_i) & Hom_A(V_i,V_j) & \cdots \\ & & & \underline{m}_j End_A(V_j) & \\ & & & \vdots & \underline{m}_r End_A(V_r) \end{pmatrix}$$

Proof. We need only check that $rad(V)$ is an ideal, and this amounts to proving that if $\phi_{ij} \in Hom(V_i, V_j)$ $i \neq j$ and $\phi_{ji} \in Hom(V_j, V_i)$ then

$$\phi_{ji}\phi_{ij} \in \underline{m}_i \subseteq End_A(V_i).$$

Suppose $\phi_{ji}\phi_{ij}$ is not in $\underline{m}_i$, then $\phi_{ji}\phi_{ij}$ is an isomorphism, and we may as well assume that $\phi_{ji}\phi_{ij} = id_{V_i}$. But then $V_j \simeq V_i \oplus \ker \phi_{ji}$ which contradicts the indecomposability of V_j. $\square$

In particular this lemma proves that if A is artinian and all V_i are of finite type, then for some N,

$$rad(V)^N = 0$$

Obviously there is a left and a right action of $End_A(V)$ on

$$T_H = (Ext^1_A(V_i, V_j)).$$

The difference between these actions defines the action of the Lie algebra $End_A(V)$ on T_H. The invariants of T_H under the Lie algebra $rad(V)$, is equal to the invariants under $End_A(V)$, therefore equal to,

$$T_{H_0} := \{\xi \in T_H | \forall \phi \in End_A(V), \quad \phi\xi - \xi\phi = 0\},$$

containing the tangent space of the *modular*, or the *prorepresentable* substratum H_0 of H.

Lemma 3.2. *Let $\xi \in T_{H_0}$, with $\xi = (\xi_{i,j})$, then for all $\phi = (\phi_{k,l}) \in rad(V)$ we have for $i \neq j$, and all l,*

$$\phi_{l,i}\xi_{i,j} = 0$$
$$\xi_{i,j}\phi_{j,l} = 0.$$

Moreover, for all i, j

$$\phi_{i,j}\xi_{j,j} = \xi_{i,i}\phi_{i,j}$$

Proof. Just computation. □

Definition 3.3. *In the above situation, an extension $\xi \in Ext^1_A(V_i, V_j)$ is called a left almost split extension (resp. a right almost split extension), lase (resp. rase) for short, if for all $\phi_{ki} \in r(V)_{ki}$ (resp. $\phi_{jk} \in r(V)_{jk}$)*

$$\phi_{ki}\xi = 0 \quad (resp. \quad \xi\phi_{jk} = 0).$$

An extension ξ which is both a lase and a rase is called an ase, an almost split extension.

This, of course, is nothing but a trivial generalization of the notion of almost split sequence, due to Auslander.

Denote by $Ext^1_l(V_i, V_j)$ (resp. $Ext^1_r(V_i, V_j)$) the subspace of $Ext^1_A(V_i, V_j)$ formed by the lase's (resp. rase's), and put

$$T^l_H = (Ext^1_l(V_i, V_j)) \subseteq T_H$$
$$T^r_H = (Ext^1_r(V_i, V_j)) \subseteq T_H$$
$$T^a_H = T^l_H \cap T^r_H =: (Ext^1_a(V_i, V_j)) \subseteq T_H.$$

Observe that since the left and the right action of $End(V)$ on T_H commute, $End(V)$ acts at right on T^l_H and at left on T^r_H. Moreover, by the lemma above

$$T^a_H = T^l_H \cap T^r_H \subseteq T_{H_0}.$$

Observe also that if $End_A(V_i) = k \oplus \underline{m}_i$ the diagonal part of T_{H_0} is exactly the tangent space of the deformation functor of the full subcategory of mod_A generated by V, see [La 1].

The structure of the modular substratum, and the existence of almost split sequences for artinian k-algebras.

Assume that A is artinian, and that the V_i's are of finite type. Then T_H is a k-vectorspace of finite dimension, and the radical $rad(V)$ of $End(V)$ acts nilpotently on T_H.

Corollary 3.4. *Given $i \in \{1, \dots, r\}$, assume there exists one $j \in \{1, \dots, r\}$ such that $Ext^1_A(V_i, V_j) \neq 0$. Then there exists a $\tau(i) \in \{1, \dots, r\}$ such that,*

$$Ext^1_r(V_i, V_{\tau(i)}) \neq 0.$$

Proof. This is simply Engels theorem for the right action of $rad(V)$ on T_H. □

Theorem 3.5. *Suppose $\mathcal{V}$ is such that every extension $\xi \in Ext^1_A(V_i, V_j)$ is of the form $0 \to V_j \to E \to V_i \to 0$ with E a direct sum of V_k's. Then, for every $i = 1, \dots, r$, such that there exists a $j = 1, \dots, r$ for which $Ext^1_A(V_i, V_j) \neq 0$, there is a unique ase of the form*

$$0 \to V_{\tau(i)} \to E_i \to V_i \to 0$$

Moreover, if we agree to put $\tau(i) = i$ for those i's for which $Ext^1_A(V_i, V_k) = 0$ for all k, then

$$\tau : \{1, \dots, r\} \to \{1, \dots, r\}$$

is a permutation.

Proof. We already know that there exists a rase of the form

$$\xi_i : 0 \to V_{\tau(i)} \to E_i \to V_i = 0.$$

We shall prove that ξ_i is also a lase. Let $\phi_{ki} \in Hom_A(V_k, V_i)$ for $k \neq i$, or pick an element $\phi_{ii} \in \underline{m}_i \subseteq End(V_i)$, and consider the commutative diagram,

$$\begin{tikzcd}
 & 0 \arrow[d] & & & & \\
0 \arrow[r] & V_{\tau(i)} \arrow[r] \arrow[d] & E_i \arrow[r] & V_i \arrow[r] & 0, & \xi_i \\
 & E_k = V_k \underset{V_i}{\times} E_i \arrow[d] \arrow[ur, "\psi_{ki}"] & & & & \\
 & V_k \arrow[d] \arrow[uurr, "\phi_{ki}"'] & & & & \\
 & 0 & & & &
\end{tikzcd}$$

Suppose $V_{\tau(i)} \to E_k$ is not split, then

$$0 \to E_k \to E_k \overset{V_{\tau(i)}}{\oplus} E_i \to V_i \to 0$$

is split, since ξ_i is a rase. Let pr.: $E_k \overset{V_{\tau(i)}}{\oplus} E_i \to E_k$ be the splitting. But then the two following diagrams commute:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & V_{\tau(i)} \arrow[r] \arrow[d] & E_i \arrow[r] \arrow[dl, shift left, "\psi_{ik}"] & V_i \arrow[r] \arrow[ddll, shift left, "\phi_{ik}"] & 0 \\
 & E_k \arrow[d] \arrow[ur, shift left, "\psi_{ki}"] & & & \\
 & V_k \arrow[d] \arrow[uurr, shift left, "\phi_{ki}"] & & & \\
 & 0 & & &
\end{tikzcd}$$

Here ψ_{ik} is the composition of $E_i \to E_k \overset{V_{\tau(i)}}{\oplus} E_i$ and the projection $E_k \overset{V_{\tau(i)}}{\oplus} E_i \to E_k$ and ϕ_{ik} the induced map.

This means that $(\phi_{ki}\phi_{ik})\xi_i = \xi_i$ which is impossible since $(\phi_{ki}\phi_{ik})$ acts nilpotently on $Ext^1_A(V_i, V_{\tau(i)})$, and ξ is nonzero. Therefore $V_{\tau(i)} \to E_k$ splits and ξ_i is also a lase, therefore an ase.

The unicity and the permutation property follows immediately from the following: Assume there exist two ase's ξ_i and ξ_i' of the form:

$$\begin{tikzcd}
 & & & 0 & & \\
0 \arrow[r] & V_{\tau(i)} \arrow[r] \arrow[drr, shift left, "\phi_i"] & E_i \arrow[r, "\rho_i"] \arrow[dr, shift left, "\psi_i"] & V_i \arrow[u] \arrow[r] & 0 & \xi_i : \\
 & & & E_i' \arrow[u, "\rho_i'"] \arrow[ul, shift left, "\psi_i'"] & & \\
 & & & V_{\tau(i')} \arrow[u] \arrow[uull, shift left, "\phi_i'"] & & \\
 & & & 0 \arrow[u] & &
\end{tikzcd}$$

Then, since ρ_i' is not split, there exist liftings ψ_i, ψ_i' inducing morphisms ϕ_i, ϕ_i'. But then $(\phi_i\phi_i')\xi_i = \xi_i$ which means that ξ_i is zero. Therefore an ase is unique and in particular, $\tau(i) = \tau(i')$. Dually we prove that $\tau(i) = \tau(i')$ implies $i = i'$, so that τ is a permutation.

We see that T_V^a looks like:

$$\begin{pmatrix} Ext^1_a(V_i, V_j) \end{pmatrix}$$

where

$$Ext^1_a(V_i, V_j) = \begin{cases} 0 & if\ j \neq \tau(i) \\ k & if\ j = \tau(i) \text{ and some } Ext^1_a(V_i, V_j) \neq 0 \end{cases}$$

□

Corollary 3.6. *With the assumptions of the theorem above, we find that*

$$T_{H_0} = \{(\alpha_{ij}) \big| \begin{cases} \alpha_{ij} \in k,\ \alpha_{ij} = 0 \ \mathit{if}\ j \neq \tau(i) \\ \alpha_{ii} \in Ext^1_A(V_i, V_i)\ \forall \phi_{ji} \in End_A(V_i),\ \phi_{ij}\alpha_{ii} = \alpha_{jj}\phi_{ji},\ \mathit{if}\ i{=}j \end{cases} \}$$

Remark 3.7. Consider again a not necessarily finite swarm $\mathcal{V} = \{V_i\}_{i=1}^{\aleph}$ of noetherian A-modules. The k^r-algebra

$$End_A(V)_r := (Hom_A(V_i, V_j)),\ i, j \leq r$$

acts on

$$(Ext^1_A(V_i, V_j)),\ i, j \leq r,$$

in the way described above. Suppose that the modules V_i are non-isomorphic, indecomposables, and that for each i, $End_A(V_i)$ is a local ring with maximal ideal $\underline{m}_i$. Suppose moreover that any extension of extensions is a direct sum of such V_i's. This is obviously the case when $\mathcal{V} = \{V_i\}_{i=1}^{\aleph}$ is the family of all indecomposible A-modules, but holds in many other interesting cases, see [R].

Let $H := H(\mathcal{V})$ and $\tilde{V}$ be the prorepresentable hull and the formal versal family, as defined in §5. For every quotient R of H in $\underline{a}_{\aleph}$, such that

$$dim_k Rad(R)/Rad(R)^2 < \infty$$

we consider the image $\tilde{V}(R) \in Def_{\mathcal{V}}(R)$ of $\tilde{V}$. Denote by $L_i(R)$ the $i^{th\cdot}$-line of $\tilde{V}(R)$. $L_i(R)$ is an A-module and a finite extension of extensions of the V_j, therefore a finite sum of indecomposables $\{L_i(R,p)\}$, from our family. Obviously there is a canonical surjection,

$$L_i(R) \to V_i,$$

and a homomorphism of k-algebras,

$$\iota : H \to End_A(\tilde{V}(R)),$$

defined by left multiplication. Any element,

$$r_{i,j} \in R_{i,j} \subset R$$

defines a homomorphism of right A-modules,

$$r_{i,j_*} : L_j(R) \to L_i(R).$$

In particular, if $r_{i,j}$ is in the socle of R, this morphism induces a homomorphism of right A-modules,

$$r_{i,j_*} : V_j \to L_i(R).$$

Using this, we may consider different quiver-structures on the set of indecomposible modules, $\{V_i\}_{i=1}^{\aleph}$. The Auslander-Reiten quiver is obtained by picking $R = H_0/Rad(H_0)^2$, a basis $\{h_{i,j}\}$ of $Rad(H_0)/Rad(H_0)^2$, the dual tangent space of H_0 and letting the arrows arriving at an indecomposible V_i be the compositions,

$$L_i(R,p) \to L_i(R) \to V_i,$$

and the arrows leaving an indecomposible V_j be the compositions,

$$V_j \stackrel{h_{i,j}}{\to} L_i(R) \to L_i(R,p).$$

For an arbitrary quotient R of H, we may construct another quiver containing more information than the A-R quiver. Consider representatives

$$\{r_{i,j}\} \in Rad(R)$$

of a basis of the dual tangent space $Rad(R)/Rad(R)^2$ of R, and let the arrows of the quiver be the compositions

$$L_j(R,q) \to L_j(R) \stackrel{r_{i,j_*}}{\to} L_i(R) \to L_j(R,p).$$

There is a ring homomorphism,

$$R \to End_A(\tilde{V}) = (Hom_A(L_j, L_i)) = (Hom_A(V_q, V_p)^{n_{p,q}}).$$

If this homomorphism is surjective, or an isomorphism, we find that the arrows of the quiver generate, in an obvious way, all morphisms of the full sub-category of A-modules defined by the family of indecomposibles $\mathcal{V} = \{V_i\}$. In both case, it is easy to see that the relations in the quiver correspond to non-trivial cup and Massey products of $Ext_a^*(V_i, V_j)$. When A is artinian, and the family $\{V_i\}$ generates the category of A-modules, it turns out that $\tilde{V}$ is a projective generator. H defines a quiver, with vertices corresponding to the indecomposible projectives, and

$$H = End_A(\tilde{V})$$

is Morita equivalent to A. Moreover, H is determined by the quiver (with relations). In particular, if $\{V_i\}$ is the family of simple A-modules, we shall see in the next paragraph that,

$$A \to (H_{i,j} \otimes Hom_k(V_i, V_j))$$

is an isomorphism, and that

$$L_i = \oplus_{j=1,2,\ldots} H_{i,j} \otimes V_j$$

is a projective A-module, for $i = 1, 2, ..r$. Since therefore

$$H \simeq End_A(\tilde{V}) = (Hom_A(L_j, L_i))$$

is Morita-equivalent to A, the quiver of (projective) summands of $\tilde{V}$ determines the Morita-equivalence class of A.

For the story of almost split sequences in the sense of Auslander, and the Auslander-Reiten quiver, see [R].

We shall end this paragraph by proving the following easy result, see [Kadison] for the notions of Frobenius extension and Frobenius bi-module.

Proposition 3.8. *Suppose the following conditions hold:*

(i) The family $\mathcal{V} = \{V_i\}$ of right A-modules are either finite dimensional as k-vector spaces, or such that,

$$Ext_A^p(V_i, V_j) = Ext_A^p(V_j^*, V_i^*).$$

(ii) The hull of the noncommutative deformation functor, $H(\mathcal{V}) = (H_{i,j})$ is a finite dimensional k-vectorspace.

(iii) For each i, the projective cover of V_i has a (finite) filtration with graded components contained in the family $\mathcal{V}$.

Then

$$\eta : A \to O(\mathcal{V})$$

is a Frobenius extension.

Proof. The assumption 1. implies that the versal family $P = \tilde{V}$ as a left H and right A-module has the duality property, ${}^*P = P^*$. The assumption 2. implies that P as left H-module is finite projective, and the assumption 3. garantees that P, as right A-module, is finite projective, therefore a Frobenius bi-module. □

§4. The generalized Burnside theorem.

In §2 we proved the following result,

Corollary 2.10. *Suppose the k-algebra A is of finite dimension and assume the swarm* $\mathcal{V} = \{V_i\}_{i=1}^r$ *contains all simple A-modules, then the natural k-algebra homomorphism*
$$\eta : A \to O(\mathcal{V}) = (H_{ij} \otimes_k Hom_k(V_i, V_j))$$
is injective.

Recall also the classical Burnside-Wedderburn-Malcev theorems, see [Lang], and [Curtis and Reiner].

Theorem (Burnside). *Let V be a finite dimensional k-vectorspace. Assume k is algebraically closed and let A be a subalgebra of $End_k(V)$. If V is a simple A-module, then $A = End_k(V)$.*

Theorem (Wedderburn). *Let A be a ring, and let V be a simple faithfull A-module. Put $D = End_A(V)$ and assume V is a finite dimensional D-vector space. Then $A \simeq End_D(V)$.*

Theorem (Wedderburn-Malcev). *Let A be a finite dimensional k-algebra, k-any field. Let $\mathfrak{r}$ be the radical of A, and suppose the residue class algebra $A/\mathfrak{r}$ is separable. Then there exists a semi-simple subalgebra $\mathcal{S}$ of A such that A is the semidirect sum of S and $\mathfrak{r}$. If S_1 and S_2 are subalgebras such that $A = S_i \oplus \mathfrak{r}$, $i = 1, 2$, then there exists an element $n \in \mathfrak{r}$, such that $S_1 = (1-n) \cdot S_2 \cdot (1-n)^{-1}$.*

In this § we shall prove a generalization of the theorem of Burnside. In fact, assuming the field k is algebraically closed and that $\mathcal{V} = \{V_i\}_{i=1}^r$ is the family of all simple A-modules we shall prove that the homomorphism η of the above Corollary (2.10), is an isomorphism.

When A is semi-simple we know that $Ext^1_A(V_i, V_j) = 0$ for all $i, j = 1, \ldots, r$, therefore the formal moduli H of V is isomorphic to k^r. This implies that
$$End_H(\tilde{V}) = \bigoplus_{i=1}^r End_k(V_i),$$
which is the classical extension of Burnsides theorem.

We shall need the following elementary lemma

Lemma 4.1. *Let the k-algebra A be a direct sum of the right-A-modules V_i, $i = 1, \ldots, d$ of the family $\mathcal{V} = \{V\}_{i=1}^r$. Then left multiplication with an element $a \in A$ induces A-module homomorphisms*
$$a_{ij} \in Hom_A(V_i, V_j), \qquad i, j = 1, \ldots, d.$$
Moreover, any k-linear map $x : A \to A$ expressed as $x = (x_{ij}) \in End_k(V) := (Hom_k(V_i, V_j))$, commuting with all $\varphi = (\varphi_{ij}) \in End_A(V) := (Hom_A(V_i, V_j))$ is necessarily a right multiplication by some element $\tilde{x} \in A$.

Proof. Trivial, since x commuting with all $\varphi \in (Hom_A(V_i, V_j))$ commutes with all left-multiplications by $a \in A$, and therefore $x(a) = a \cdot x(1)$, and we may put $\tilde{x} = x(1)$. □

Corollary 4.2. *Assume that the family of right A-modules $\mathcal{V} = \{V_i\}_{i=1}^r$ is such that*

(i)
$$A \simeq \bigoplus_{i=1}^{m} V_i^{n_i}$$

(ii)
$$Hom_A(V_i, V_j) = 0 \text{ for } i \neq j$$

Then the canonical morphism of k-algebras

$$\eta : A \to \bigoplus_{i=1}^{n_i} End_k(V_i)$$

is injective. Moreover, η induces an isomorphism

$$A \simeq \bigoplus_{i=1}^{n_i} End_{D_i}(V_i)$$

where $D_i = End_A(V_i)$.

This, in particular, implies the Wedderburn theorem for semisimple k-algebras A.

Theorem 4.3 (A generalized Burnside theorem). *Let A be a finite dimensional k-algebra, k an algebraically closed field. Consider the family $\mathcal{V} = \{V_i\}_{i=1}^r$ of simple A-modules, then*

$$A \simeq O(\mathcal{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j))$$

Proof. We know that the canonical map

$$\eta : A \to O(\mathcal{V})$$

is injective. Since $Rad(A)^n = 0$ for some n, we know that $\hat{A} = A$. The theorem therefore follows from the following lemmas.

Lemma 4.4. *Let A and B be finite type k-algebras and let $\varphi : A \to B$ be a homomorphism of k-algebras such that the induced morphism*

$$\varphi_2 : A \to B/Rad(B)^2$$

is surjective, then

$$\widehat{\varphi} : \hat{A} \to \hat{B}$$

is surjective.

Proof. Well-known. $\square$

Lemma 4.5. *Let A be a finite dimensional k-algebra, k an algebraically closed field. Let $\mathcal{V} = \{V_i\}_{i=1}^r$ be the family of simple A-modules. Then the homomorphism*

$$\eta : A \to = (\mathcal{V})$$

induces an isomorphism

$$Rad(A)/Rad(A)^2 \simeq (Ext_A^1(V_i, V_j)^* \otimes_k Hom_k(V_i, V_j)).$$

Proof. The classical Burnside theorem implies that the canonical homomorphism of k-algebras

$$A \twoheadrightarrow \bigoplus_{i=1}^{r} End_k(V_i)$$

induces an isomorphism,

$$A/Rad(A) \simeq \bigoplus_{i=1}^{r} End_k(V_i).$$

According to the Wedderburn-Malcev theorem we may assume that $A/Rad(A)^2$ is a semidirect sum,

$$A/Rad(A) \oplus Rad(A)/Rad(A)^2.$$

Since $Rad(A)/Rad(A)^2$ is both a left and a right $\bigoplus_{i=1}^r End_k(V_i)$-module

$$Rad(A)/Rad(A)^2 = (E_{ij})$$

each E_{ij} being an $End_k(V_i)^{op} \otimes_k End_k(V_j)$-module. This, however, means that

$$E_{ij} \simeq Hom_k(V_i, V_j) \otimes k^{r_{ij}}$$

as a right $End_k(V_i)^{op} \otimes_k End_k(V_j)$-module. Since we already know that η is an injection, we must have,

$$E_{ij} \simeq Hom_k(V_i, V_j) \otimes k^{r_{ij}} \subset Ext_A^1(V_i, V_j) \otimes Hom_k(V_i, V_j).$$

We must show that this inclusion is an equality. Applying Hochschild cohomology as in §1, we find:

$$Ext_A^1(V_i, V_j) = HH^1(A, Hom_k(V_i, V_j)) = Der_k(A, Hom_k(V_i, V_j))/im\, d^\circ$$

where d° is the differential

$$Hom_k(V_i, V_j) \to Der_k(A, Hom_k(V_i, V_j)).$$

Clearly any derivation

$$\xi \in Der_k(A, Hom_k(V_i, V_j))$$

which is zero on $Rad(A)$ induces a derivation

$$\xi_0 \in Der_k(A/Rad(A), Hom_k(V_i, V_j))$$

which, since $A/Rad(A)$ is semisimple, obviously is a coboundary, i.e. an element of $im\, d^\circ$.

Moreover, any derivation $\xi \in Der_k(A, Hom_k(V_i, V_j))$ induces the zero map on $Rad(A)^2$ since $\xi(r_1 \cdot r_2) = r_i\xi(r_2) + \xi(r_1)r_2 = 0$ for $r_1, r_2 \in Rad(A)$, and any coboundary $\nu \in im\, d^\circ$ must vanish on $Rad(A)$ since $\nu(r) = \varphi r - r\varphi$, for some $\varphi \in Hom_k(V_i, V_j)$. Now, every $A^{op} \otimes_k A$-linear map $Rad(A)/Rad(A)^2 \to End_k(V_i, V_j)$ extends to a derivation of $Der_k(A/Rad(A)^2, Hom_k(V_i, V_j))$. In fact, let φ be an $A^{op} \otimes_k A$-linear map

$$Rad(A)/Rad(A)^2 \to End_k(V_i, V_j)$$

and define the map

$$\psi : A/Rad(A)^2 = A/Rad(A) \bigoplus Rad(A)/Rad(A)^2 \to End_k(V_i, V_j)$$

by

$$\psi(s, r) = \varphi(r) + \varphi(\rho(s))$$

where ρ is the 1-Hochschild cochain on $A/Rad(A)$ with values in $Rad(A)/Rad(A)^2$ that, according to the Wedderburn-Malcev theorem,defines the semidirect sum refered to above. Then,

$$\begin{aligned}
&\psi((s_1, r_1) \cdot (s_2, r_2)) = \psi((s_1 \cdot s_2, s_1\rho(s_2) - \rho(s_1 \cdot s_2) + \rho(s_1)s_2 + s_1r_2 + r_1s_2))\\
&= \varphi(s_1r_2 + r_1s_2 + s_1\rho(s_2) - \rho(s_1 \cdot s_2) + \rho(s_1) \cdot s_2) + \varphi(\rho(s_1 \cdot s_2))\\
&= (s_1, r_1)\psi((s_2, r_2)) + \psi((s_1, r_1))(s_2, r_2)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&Ext^1_A(V_i, V_j) = Hom_{A^{op}\otimes A}(Rad(A)/Rad(A)^2, Hom_k(V_i, V_j))\\
&= \{\varphi : Rad(A)/Rad(A)^2 \to Hom_k(V_i, V_j) |\ \forall a \in A,\ r \in Rad(A),\ s.t.\\
&\varphi(a \cdot r) = a \cdot \varphi(r) \text{ and } \varphi(ra) = \varphi(r) \cdot a\}
\end{aligned}$$

Since $Rad(A)/Rad(A)^2 \simeq (E_{ij})$ with

$$E_{ij} \simeq (V_i^* \otimes V_j)^{r_{ij}}$$

it is clear that

$$\begin{aligned}
&Hom_{A^{op}\otimes A}(Rad(A)/Rad(A)^2, Hom_k(V_i, V_j))\\
&\simeq Hom_{End_k(V_i)^{op}\otimes_k End_k(V_j)}((V_i^* \otimes V_j)^{r_{ij}}, (V_i^* \otimes V_j))\\
&\simeq k^{r_{ij}}
\end{aligned}$$

which means that

$$E_{ij} \simeq Ext^1_A(V_i, V_j)^* \otimes_k Hom_k(V_i, V_j).$$

□

Now suppose, as above, that A is a finite dimensional k-algebra, and let $\mathcal{V}_A = \{V_i\}_{i=1}^r$ be any family of finite dimensional A-modules. Obviously

$$dim_k Ext_A^p(V_i, V_j) < \infty$$

for all $p = 0, 1, 2, \dots$ and therefore the endomorphism ring

$$O(\mathcal{V}_A) := End_H(\tilde{V})$$

is a k-algebra such that

$$O(\mathcal{V})/Rad(O) = \bigoplus_{i=1}^{r} End_k(V_i).$$

This implies that $\mathcal{V} = \{V_i\}_{i=1}^r$ is the family of all simple $\mathcal{O}(V)$-modules. The generalized Burnside theorem aplies also in this case, showing that the operation

$$(A, \mathcal{V}) \mapsto (O(\mathcal{V}), \mathcal{V})$$

is a closure operation. Moreover, we have the following,

Proposition 4.6. *Let $\tau : A \to B$ be any homomorphism of finite dimensional k-algebras. Consider a family $\mathcal{V}_B = \{V_i\}_{i=1}^r$ of finite dimensional B-modules and let $\mathcal{V}_A$ be the corresponding family of A-modules. Suppose moreover that $\mathcal{V}_B$ is the family of all simple B-modules. Then there exists an, up to isomorphisms, unique homomorphism of k-algebras*

$$O(\tau) : O(\mathcal{V}_A) \to O(\mathcal{V}_B) \simeq B$$

extending τ.

Proof. There is an obvious forgetful functor defining a morphism of functors on $\underline{a}_r$,

$$\tau^* : Def_{\mathcal{V}_B} \to Def_{\mathcal{V}_A}$$

which in its turn induces a k-algebra homomorphism

$$\eta : H(\mathcal{V}_B) \to H(\mathcal{V}_A)$$

unique up to isomorphisms, and therefore a k-algebra homomorphism

$$O(\mathcal{V}_A) := (H(\mathcal{V}_A)_{i,j} \otimes Hom_k(V_i, V_j)) \to (H(\mathcal{V}_B)_{i,j} \otimes Hom_k(V_i, V_j)) =: O(\mathcal{V}_B)$$

obviously extending τ. By the generalized Burnside theorem, $O(\mathcal{V}_B) \simeq B$, and the Proposition follows.

□

Remark 4.7. Up to now we have only considered finite families of A-modules such that

$$\dim_k Ext_A^p(V_i, V_j) < \infty, \qquad p = 1, 2.$$

Neither of these conditions are essential. Introducing natural topologies we may, as in [La 1], treat general families of finite type A-modules. Notice also that if $r_1 \leq r_2$, there is an obvious canonical morphism

$$\underline{a}_{r_1} \to \underline{a}_{r_2}$$

inducing a restriction morphism of functors

$$Def_{\mathcal{V}(2)} \to Def_{\mathcal{V}(1)}$$

where $\mathcal{V}(1) = \{V_i\}_{i=1}^{r_1}, \mathcal{V}(2) = \{V_i\}_{i=1}^{r_2}$. Therefore we obtain an up to isomorphisms unique k-algebra homomorphism

$$r_{2,1} : H_{A,V(2)} \to H_{A,V(1)}.$$

However, this *restriction* morphism is not, in general, unique. The resulting problems will be dealt with later.

Filtered modules and extensions of extensions. Let as above $\mathcal{V} = \{V_i\}_{i=1}^{r}$ be a family of right A-modules, and let $E_{i_1,\ldots,i_s} : E_s \subset E_{s-1} \subset \cdots \subset E_1 = E$ be a filtered module such that $E_k/E_{k+1} \simeq V_{i_k}$. We shall, as before, refer to any such filtered module as an extension of extensions of V. Notice that for every p there is an extension,

$$\xi_{p,p+1} \in Ext^1_A(V_{i_p}, V_{i_{p+1}})$$

given by the exact sequence.

$$0 \to E_{p+1}/E_{p+2} \to E_p/E_{p+2} \to E_p/E_{p+1} \to 0.$$

Corresponding to the extension of extensions $E_{i_1,\ldots,i_s}$ we shall associate two directed graphs, $\Gamma(\underline{i})$ and $\Gamma(E_{\underline{i}})$. The first is gotten as the graph with nodes in bijection with the modules of the family V, and with arrows $\epsilon(i_p, i_{p+1})$ connecting the node i_p with the node i_{p+1}. The second is obtained from the first identifying two arrows $\epsilon(i_p, i_{p+1})$ and $\epsilon(i_q, i_{q+1})$ if the corresponding extensions $\xi_{p,p+1}$ and $\xi_{q,q+1}$ coincide. The corresponding k-algebra is the object of $\underline{a}_r$ generated by the arrows $\epsilon(i_p, i_{p+1})$ with relations given by composable monomials of the form

$$\epsilon(j_1, j_2)\epsilon(j_3, j_4)\ldots\epsilon(j_w, j_{w+1})$$

where the sequence

$$\{j_1, j_2, j_3, \ldots, j_{w+1}\}$$

is not contained in the sequence

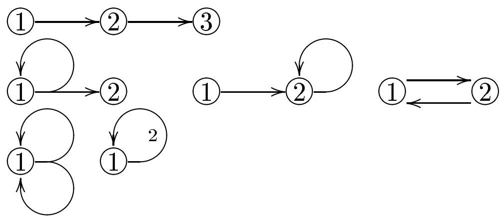
$$\{i_1, i_2, i_3, \ldots, i_s\}.$$

Example 4.8. Let us draw up all extension types for $r, s \leq 3$.

$r = 3, s = 3 :$

$r = 2, s = 3 :$

$r = 1, s = 3 :$



The last example is a $\Gamma(E_{\underline{i}})$ corresponding to the situation, $i_1 = i_2 = i_3 = 1$, and $\xi_{i_1,i_2} = \xi_{i_2,i_3}$ The associated k-algebras are, respectively, the matrix algebras,

$$\begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix} \begin{pmatrix} k[\epsilon] & k[\epsilon] \\ 0 & k \end{pmatrix} \begin{pmatrix} k & k[\epsilon] \\ 0 & k[\epsilon] \end{pmatrix} \begin{pmatrix} k[t_{1,2}t_{2,1}] & kt_{1,2} \\ kt_{2,1} & k \end{pmatrix}$$

with the obvious relations, and the k-algebras,

$$k\{t, u\}/(t^2, u^2, ut), \quad k[t]/(t^3)$$

Lemma 4.9. *Let H be any object of $\hat{\underline{a}}_r$, and let $R \in \underline{a}_r$. Then Mor(H,R) has a natural structure of an affine algebraic scheme $Mor(H, R) = Spec(A(H, R))$, and there is a universal morphism,*

$$\tilde{\phi} : H \longrightarrow A(H, R) \otimes_k R$$

Proof. Put $(E_{i,j}) = rad(H)/rad(H)^2$, and consider the affine space,

$$\mathbf{A}^N = \prod_{i,j} E^*_{i,j} \otimes_k R_{i,j}$$

with coordinates $z_{i,j}(l, m) = t_{i,j}(l) \otimes x_{i,j}(m)$, where $\{t_{i,j}(l)\}_{i,j}$ is a basis of $E_{i,j}$ and $\{r_{i,j}(m)\}_{i,j}$ is a basis, and $\{x_{i,j}(m)\}_{i,j}$ is a dual basis of $R_{i,j}$. An element

$$(\alpha_{i,j}(l, m)) \in \mathbf{A}^N$$

corresponds to a morphism $\phi \in Mor(H, R)$ if, and only if, the corresponding map

$$t_{i,j}(l) \longmapsto \sum_m \alpha_{i,j}(l, m) r_{i,j}(m) \in R_{i,j}$$

satisfies the relations of H. Let these, modulo a high ennough power of the radical, be polynomials in the generators $t_{i,j}(l)_{i,j}$ of the form

$$f_p(t_{i,j}(l)) = 0, \; p = 1, \ldots s \; ,$$

and let the relations of R be expressed in terms of,

$$r_{i,j}(m) r_{j,k}(n) = \sum_p \beta^p_{i,j,k}(m, n) r_{i,k}(p), \; i, j = 1, \ldots r.$$

Then we obtain equations for $Mor(H, R)$ given by (commutative) polynomial relations of the form,

$$F_p(z_{i,j}(l, m)) = 0, \; p = 1, \ldots t.$$

But then

$$t_{i,j}(l) \longmapsto \sum_m z_{i,j}(l, m) r_{i,j}(m) \in A(H, R) \otimes_k R_{i,j}$$

where the coordinates $z_{i,j}(l, m)$ are subject to the conditions above, defines the universal morphism $\tilde{\phi}$. $\square$

Proposition 4.10. *Let A be any k-algebra, $\mathcal{V} = \{V_i\}_{i=1}^r$ any family of A-modules such that*

$$\dim_k Ext^1_A(V_i, V_j) < \infty \quad \text{for all } i, j = 1, \ldots, r$$

(i): Consider an extension of extensions E of $\mathcal{V}$, with directed graph Γ. Then there exists a morphism of k-algebras

$$\phi : H(\mathcal{V}) \to k[\Gamma]$$

such that

$$E \simeq k[\Gamma] \otimes_\phi \tilde{V}$$

in the above sense.

(ii): The set of equivalence classes of extensions of extensions of $\mathcal{V}$ with extension type Γ, is a quotient of the affine algebraic scheme

$$\underline{A}[\Gamma] = Mor(H(\mathcal{V}), k[\Gamma])$$

(iii): There is a versal family $\tilde{V}[\Gamma]$ of A-modules defined on $A[\Gamma]$, containing as fibres all the isomorphism classes of extensions of extensions of the V_i's with extension type Γ.

Proof. Any morphism $\varphi : H \to k[\Gamma]$ in $\underline{a}_r$ correspond to an extension of extensions of the V_i's. This may be expressed in the following way. As vector spaces, we have an isomorphism,

$$k[\Gamma] \otimes_\varphi \tilde{V} \simeq V(\Gamma) \simeq V_{i_1} \times V_{i_2} \times \cdots \times V_{i_s}$$

Any A-module structure on this vectorspace is given by a homomorphism of k-algebras,

$$\psi : A \longrightarrow End_k(V_{i_1} \times V_{i_2} \times \cdots \times V_{i_s})$$

inducing a family of linear maps

$$\psi_{p,p+1,\ldots,p+q} : A \longrightarrow End_k(V_{i_p}, V_{i_{p+q}})$$

for $0 \leq p < p+q \leq s$.

Consider these maps as 1-cochains in the Hochschild complex

$$HC^*(A, Hom_k(V(\Gamma), V(\Gamma)))$$

The maps $\psi_{p,p+1}$ correspond to the extensions $\xi_{p,p+1}$ above, and must therefore be a 1-cocycle, or a derivation. To obtain an A-module structure, as an extension of extensions of V, the conditions on these cochains are: For all $a, b \in A$,

$$\psi_{p,p+1}(a)\psi_{p+1,p+2}(b) = d\psi_{p,p+1,p+2}(a,b)$$
$$\psi_{p,p+1}(a)\psi_{p+1,p+2,p+3}(b) + \psi_{p,p+1,p+2}(a)\psi_{p+2,p+3}(b) = d\psi_{p,p+1,p+2,p+3}(a,b)$$
$$\ldots$$
$$\sum_{m=2,\ldots,s-1} \psi_{1,2,\ldots,m}(a)\psi_{m,\ldots,s}(b) = d\psi_{1,2,\ldots,s}(a,b),$$

which means that all Massey products of the form

$$< \xi_{i_p,i_{p+1}}, \xi_{i_{p+1},i_{p+2}}, \ldots, \xi_{i_{p+q-1},i_{p+q}} >$$

are defined and contain zero.

Now (i) follows from the very definition of H, generated as it is by a basis of the dual Ext^1's, with relations exactly expressing the vanishing of the above Massey products. (ii) and (iii) then follows from deformation theory, together with the Lemma (4.9), above. □

Example 4.11. Consider the extension E_{ijk} of length 3 given as the composite extension of $\xi_{i,j} : 0 \leftarrow V_i \leftarrow E_{ij} \leftarrow V_j \leftarrow 0$ and $\xi_{i,j,k} : 0 \leftarrow E_{ij} \leftarrow E_{ijk} \leftarrow V_k \leftarrow 0$. Take the pullback $\xi_{i,k}$ of $\xi_{i,j,k}$ via $V_j \rightarrow E_{ij}$ and consider the diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 & V_i \arrow[l] \arrow[u] & E_{ij} \arrow[l] \arrow[u] & V_j \arrow[l] \arrow[u] & 0 \arrow[l] \\
 & E_{ik} \arrow[u] & E_{ijk} \arrow[l] \arrow[u] & E_{jk} \arrow[l] \arrow[u] & \\
 & V_k \arrow[u] \arrow[r, equal] & V_k \arrow[u] \arrow[r, equal] & V_k \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Let $\psi_{i,j} \in Der_k(A, Hom_k(V_i, V_j))$ be a Hochschild cocycle representing the class $\xi_{i,j}$. The multiplication with $a \in A$ on E_{ij}, identified with $V_j \times V_i$ as k-vectorspace, is given by

$$(v_j, v_i)a = (v_j a + \psi_{i,j}(a, v_i), v_i a)$$

and the multiplication with $a \in A$ on E_{ijk} identified with $V_k \times V_j \times V_i$ as k-vectorspace, must be such that,

$$(v_k, v_j, 0)a = (v_k a + \psi_{j,k}(a, v_j), v_j a, 0).$$

If there exists an action of a on E_{ijk} consistent with the above, then one proves the existence of a Hochschild cochain

$$\psi_{i,j,k} \in Hom_k(A, Hom_k(V_i, V_k))$$

such that

$$d\psi_{i,j,k}(a,b) = \psi_{i,j}(a)\psi_{j,k}(b).$$

From this follows that the cup-product $\xi_{ij} \cup \xi_{jk}$ is 0 in $Ext^2_A(V_i, V_k)$. This is also the criterion for the existence of $\xi_{i,j,k}$. Moreover if ξ and ξ' are two extensions

$$\xi : 0 \leftarrow E_{ij} \leftarrow E_{ijk} \leftarrow V_k \leftarrow 0$$
$$\xi' : 0 \leftarrow E_{ij} \leftarrow E'_{ijk} \leftarrow V_k \leftarrow 0$$

with the same pullback ξ_{jk}, then there is an extension

$$\xi_{ik} : 0 \leftarrow V_i \leftarrow E_{ik} \leftarrow V_k \leftarrow 0$$

such that its pullback, via $E_{ij} \rightarrow V_i$, is the difference $\xi - \xi'$.

Consider for the extension of extensions E_{ijk}, the extension diagram

$$\Gamma : \qquad \begin{tikzcd} i \arrow[r] & j \arrow[r] & k \end{tikzcd}$$

the first one of the example (4.8) above, then the corresponding k-algebra is given by,

$$k[\Gamma] = \begin{array}{cc} & \begin{array}{ccccc} & i & j & k & \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ j \\ \\ k \\ \\ \\ \end{array} & \begin{pmatrix} k & 0 & 0 & 0 & 0 \\ \ddots & \cdots & \cdots & \cdots & \vdots \\ & k & k & k & 0 \\ & \ddots & \cdots & \cdots & \vdots \\ & & k & k & 0 \\ & & \ddots & \cdots & \vdots \\ & & & k & 0 \\ & & & \ddots & \cdots \\ & & & & k \end{pmatrix} \end{array}$$

Notice that E_{ijk} then corresponds to a morphism

$$\phi : H \to k[\Gamma]$$

determined, modulo $Rad^2(k[\Gamma])$ (i.e. the radical squared), by

$$\xi_{i,j} = \phi|E_{i,j} \to k, \text{ and } \xi_{j,k} = \phi|E_{j,k} \to k$$

Here

$$Rad^2(k[\Gamma]) = \begin{array}{cc} & \begin{array}{ccccc} & i & j & k & \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ j \\ \\ k \\ \\ \\ \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \ddots & \cdots & \cdots & \cdots & \vdots \\ & 0 & 0 & k & 0 \\ & \ddots & \cdots & \cdots & \vdots \\ & & 0 & 0 & 0 \\ & & \ddots & \cdots & \vdots \\ & & & 0 & 0 \\ & & & \ddots & \cdots \\ & & & & 0 \end{pmatrix} \end{array}$$

and ϕ is, according to the analysis above, "calibrated" by the morphism $\phi_{i,k} : E_{i,k} \to k$, i.e. by $Ext^1_A(V_i, V_k)$, as it should.

Corollary 4.12. *(i): Given any finitely generated module M on a Noetherian ring A, there is a finite set of primes $E(M)$, containing the set of associated primes Ass(M), such that the module M is an extension of extensions of the corresponding modules $A/\mathfrak{p}$ for $\mathfrak{p} \in E(M)$. The (extension) type of this extension is determined by an ordered directed graph $\Gamma(M)$ the nodes of which is E(M).*

(ii): For any finite ordered directed graph Γ, with nodes corresponding to a set of primes $\mathcal{P} \subset Spec(A)$, there is an affine versal family of A-modules $\tilde{M}$ with extension type Γ, and $E(M) = \mathcal{P}$.

Proof. Obvious $\square$

Example 4.13. Let $A = k[x,y]/(xy,y^2)$. Consider A as an A-module. Then $Ass(A) = \{(y),(x,y)\}$. Put $V_1 = A/(y) = k[x]$, $V_2 = A/(x,y) = k$, and $\mathcal{V} = \{V_1, V_2\}$. We shall compute $H(\mathcal{V})$. One easily computes the following free resolutions,

$$0 \longleftarrow k[x] \longleftarrow A \xleftarrow{(\, y\,)} A \xleftarrow{(\, x\ y\,)} A^2 \xleftarrow{\begin{pmatrix} y & 0 & 0 \\ 0 & x & y \end{pmatrix}} A^3 \longleftarrow$$

$$0 \longleftarrow k \longleftarrow A \xleftarrow{(\, x\ y\,)} A^2 \xleftarrow{\begin{pmatrix} y & 0 & 0 \\ 0 & x & y \end{pmatrix}} A^3 \xleftarrow{\begin{pmatrix} x & y & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & x & y \end{pmatrix}} A^5 \longleftarrow$$

From these we obtain,

$$Ext^1_A(V_i, V_j) = \begin{cases} 0, \text{if } i = j = 1 \\ k, \text{if } i = 1,\ j = 2 \\ k, \text{if } i = 2,\ j = 1 \\ k^2, \text{if } i = j = 2 \end{cases}$$

and

$$Ext^2_A(V_i, V_j) = \begin{cases} k, \text{if } i = j = 1 \\ k^2, \text{if } i = 1,\ j = 2 \\ k, \text{if } i = 2,\ j = 1 \\ k^3, \text{if } i = j = 2 \end{cases}$$

Using the same notations for the bases of these vectorspaces, and their representatives, we obtain the following cup-products,

$$\begin{aligned}
&\psi_{1,2} \cup \psi_{2,1} = 1 \in k = Ext^2_A(V_1, V_1) \\
&\psi_{1,2} \cup \psi_{2,2}(1) = (0,1) \in k^2 = Ext^2_A(V_1, V_2) \\
&\psi_{1,2} \cup \psi_{2,2}(2) = (1,0) \in k^2 = Ext^2_A(V_1, V_2) \\
&\psi_{2,2}(1) \cup \psi_{2,1} = 1 \in k = Ext^2_A(V_2, V_1) \\
&\psi_{2,2}(2) \cup \psi_{2,1} = 0 \in k = Ext^2_A(V_2, V_1) \\
&\psi_{2,1} \cup \psi_{1,2} = (1,0,0) \in k^3 = Ext^2_A(V_2, V_2) \\
&\psi_{2,2}(1) \cup \psi_{2,2}(1) = (0,0,1) \in k^3 = Ext^2_A(V_2, V_2) \\
&\psi_{2,2}(1) \cup \psi_{2,2}(2) = (0,1,0) \in k^3 = Ext^2_A(V_2, V_2) \\
&\psi_{2,2}(2) \cup \psi_{2,2}(1) = (1,0,0) \in k^3 = Ext^2_A(V_2, V_2) \\
&\psi_{2,2}(2) \cup \psi_{2,2}(2) = (0,0,0) \in k^3 = Ext^2_A(V_2, V_2)
\end{aligned}$$

showing that the formal moduli $H(\mathcal{V})$ is given as the 2-pointed k-algebra,

$$H(\mathcal{V}) = \begin{pmatrix} k & < t_{1,2} > \\ k[t_{2,2}(2)]t_{2,1} & k\{t_{2,2}(1), t_{2,2}(2)\}/(t_{2,2}(1)^2, t_{2,2}(1)t_{2,2}(2)) \end{pmatrix}$$

where the relations not shown are

$$\begin{aligned}
&t_{1,2}t_{2,2}(l) = 0 \text{ for } l = 1,2. \\
&t_{2,2}(1)t_{2,1} = 0 \\
&t_{1,2}t_{2,1} = 0 \\
&t_{2,1}t_{1,2} = t_{2,2}(2)t_{2,2}(1)
\end{aligned}$$

Consider now the ordered directed graphs

$$\Gamma_1 : \qquad ① \longrightarrow ②$$

$$\Gamma_1' : \qquad ① \longleftarrow ②$$

and let,

$$\Gamma_2 : \qquad ① \underset{j}{\overset{i}{\rightleftarrows}} ②$$

for $i, j \in \{1,2\}$, be the first of the two obvious orderd graphs with the indicated arrows, and let Γ_2' be the other.The corresponding k-algebras are,

$$k[\Gamma_1] = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$$

$$k[\Gamma_1'] = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$$

and

$$k[\Gamma_2] = \begin{pmatrix} k[t_{1,2}t_{2,1}] & kt_{1,2} \\ kt_{2,1} & k \end{pmatrix}$$

$$k[\Gamma_2'] = \begin{pmatrix} k & kt_{1,2} \\ kt_{2,1} & k[t_{2,1}t_{1,2}] \end{pmatrix}$$

with all further products between the generators vanishing. Therefore the affine spaces

$$\underline{A}[\Gamma_1] = Mor(H(V), k[\Gamma_1]) = \mathbf{A}^1$$

with coordinate $t_{1,2}$,

$$\underline{A}[\Gamma_1'] = Mor(H(V), k[\Gamma`_1]) = \mathbf{A}^1$$

with coordinate $t_{2,1}$, and

$$\underline{A}[\Gamma_2] = Mor(H(V), k[\Gamma_2]) \subset \mathbf{A}^2$$

with coordinates $t_{1,2}, t_{2,1}$, and relation $t_{1,2}t_{2,1} = 0$, parametrizes all extensions of extensions with the extension types given. It is easy to see that the generic fiber of the induced family of modules on $\underline{A}[\Gamma_1] = \mathbf{A}^1$ is A, and on $\underline{A}[\Gamma_1'] = \mathbf{A}^1$ is V_1. The two components of the basis of the last family, correspond to the two different extension types of $A \oplus V_1$, as extension of extensions of V_1 and V_2. Now the base of the versal family of deformations of $M = V_1 \oplus V_2$ is given by

$$H(M) = k[t_{1,2}, t_{2,1}, t_{2,2}(1), t_{2,2}(2)]/\mathfrak{a}$$

with,

$$\mathfrak{a} = \{t_{2,2}(1)^2, t_{2,2}(1)t_{2,1}, t_{1,2}t_{2,2}(1), t_{1,2}t_{2,2}(1),$$
$$t_{2,1}t_{1,2} + t_{2,2}(2)t_{2,2}(1), t_{1,2}t_{2,1}, t_{2,2}(2)t_{2,2}(1)\}.$$

Using this, the last examples shows that the family $\tilde{V}[\Gamma_2']$ defined on $\underline{A}[\Gamma_2']$ is neither prorepresentable nor miniversal, as a family of modules. In particular we find that the canonical morphism

$$\rho : Mor(H(\mathcal{V}), k[\Gamma_2']) \to Def_V(k[\Gamma_2'])$$

is surjective but not injective.

Now, let us expand on Proposition (4.10), and apply the Corollary (4.12) to the situation of an affine family of modules, i.e. to the situation where we have given a k-algebra S, and an S-flat $S \otimes_k A$-module $\tilde{M}$, of finite type.

Lemma 4.14. *Suppose k is algebraically closed, then,*

(i) $$Ass_{S\otimes_k A}(\tilde{M}) \subset Ass(S) \times Spec(A),$$

There exists a finite locally closed stratification,

(ii) $$\{\underline{S}(\lambda)\}_{\lambda\in\Lambda}$$

of $\underline{S} := Spec(S)$, and for each $\lambda \in \Lambda$ a subset $L(\lambda) \subset Spec(A)$ and an ordered directed graph $\Gamma(\lambda)$ with $L(\lambda)$ as set of nodes, such that the restriction $\tilde{M}(\lambda)$ of $\tilde{M}$ to $\underline{S}(\lambda)$ is a family of extensions of extensions of the the set of modules $V(\lambda) = \{A/\mathfrak{p} | \mathfrak{p} \in L(\lambda)\}$, with extension graph $\Gamma(\lambda)$.

There is a, in general non-unique, morphism

(iii) $$\kappa(\lambda) : \underline{S}(\lambda) \to \underline{A}(\Gamma(\lambda))$$

consistent with the families $\tilde{M}(\lambda)$ and $M(\Gamma(\lambda))$

Proof. Clearly $Ass_{S\otimes_k A}(\tilde{M}) \subseteq Spec(S) \times Spec(A)$. Suppose

$$(\mathfrak{q},\mathfrak{p}) \in Ass_{S\otimes_k A}(\tilde{M}),\ \mathfrak{q} \in Spec(S),\ \mathfrak{p} \in Spec(A)$$

and let $\tilde{m} \in \tilde{M}$ be an element such that $(\mathfrak{q},\mathfrak{p}) = ann(\tilde{m})$. Since $\tilde{M}_\mathfrak{q} \simeq S_\mathfrak{q} \otimes_k M$, as S-modules, we may write

$$\tilde{m} = \sum_{i=1}^{n} s_i \otimes m_i,\ s_i \in S,\ m_i \in M$$

where the m_i's are linearly independent. Therefore $\mathfrak{q} = \bigcap_{i=1}^n ann_S(s_i)$, which implies that $\mathfrak{q} = ann_S(s_i)$ for at least one i. From this (i) follows. Now, let

$$E(\tilde{M}) \subseteq Spec(S) \times Spec(A)$$

such that there exists a filtration,

$$\tilde{M} = \tilde{M}_1 \supset \tilde{M}_2 \supset \cdots \supset \tilde{M}_t \supset 0$$

with

$$\tilde{M}_i/\tilde{M}_{i+1} \simeq \tilde{V}_i := S/\mathfrak{q}_i \otimes_k A/\mathfrak{p}_i$$

We know that $Ass(\tilde{M}) \subseteq E(\tilde{M})$. Let

$$\underline{S}_1' := \bigcup_{j=1}^{t} V(\mathfrak{q}_j),\ \mathfrak{q}_j \text{ not minimal}$$

and put

$$\underline{S}_0 := \underline{S} \setminus \underline{S}_1'.$$

Let $\{\underline{S}(\lambda)\}_{\lambda\in\Lambda_0}$ be the reduced components of $\underline{S}_0$, then by construction, the induced family $\tilde{M}(\lambda)$ on $\underline{S}(\lambda), \lambda \in \Lambda_0$ will be filtered as above,

(1) $$\tilde{M}(\lambda) = \tilde{M}_1(\lambda) \supset \tilde{M}_2(\lambda) \supset \cdots \supset \tilde{M}_s(\lambda) \supset 0$$

with
$$\tilde{M}_i(\lambda)/\tilde{M}_{i+1}(\lambda) \simeq \tilde{V}_i(\lambda) := O_{\underline{S}(\lambda)} \otimes_k A/\mathfrak{p}_i.$$
Put
$$V(\lambda) := \{V(\lambda)_i\}_{i=1}^s,\ V(\lambda)_i := A/\mathfrak{p}_i$$
and let $\Gamma(\lambda)$ be the corresponding ordered graph.

Now, repeat this procedure for $\tilde{M}$ restricted to $\underline{S}'$, and we get the stratification we want, i.e. (ii).

To prove (iii) we shall be prepared to change the above stratification slightly. In fact, let for every $\lambda \in \Lambda$, $K(\lambda)$ be the field of fractions of $O_{\underline{S}(\lambda)}$, and tensorize (1) with $K(\lambda)$ as module on $O_{\underline{S}(\lambda)}$. We obtain a $K(\lambda) \otimes_k A$-module $\bar{M}(\lambda)$, filtered with extension type $\Gamma(\lambda)$, and with family of components

$$\bar{V}(\lambda) := \{\bar{V}(\lambda)_i := K(\lambda) \otimes_k A/\mathfrak{p}_i\}.$$

An obvious computation shows that

$$H(\bar{V}(\lambda)) = K(\lambda) \otimes_k H(V)$$

and therefore $\bar{M}(\lambda)$ corresponds to a, in general non-unique, morphism

$$\bar{\phi}(\lambda) \in Mor(H(\bar{V}(\lambda)), K(\lambda)[\Gamma(\lambda)]) = Mor(H(V(\lambda)), K(\lambda)[\Gamma(\lambda)])$$

which induces a rational map,

$$\kappa(\lambda) : \underline{S}(\lambda) \to \underline{A}(\Gamma(\lambda)) := Mor(H(V(\lambda)), k[\Gamma(\lambda)])$$

with the property of (iii). Now, restrict $S(\lambda)$ such that $\kappa(\lambda)$ becomes defined, and the lemma is proved. $\square$

Remark 4.15. Recall that the formalism used above is unchanged if we instead of modules on a k-algebra A, consider modules on a scheme X. The obvious changes in the statements of the preceeding results are left to the readers.

Remark (4.16). Let A be the k-algebra $Sym_k(U^*)$ where U is some finite dimensional k-vectorspace of dimension n. Assume there is given a finite group G acting on U, and therefore on A. Let the action be given by the homomorphism,

$$\rho : G \longrightarrow Aut_k(U) \subset Aut_k(A).$$

Consider the skew group-algebra $A(G)$, in which for $a \in A,\ g \in G$, we have

$$ag = g\rho(g)(a).$$

The modules over $A(G)$ are the A-modules on which G acts in a manner consistent with the action on A. We shall first be interested in the finitely generated simple $A(G)$-modules. Let V be one such. We know that V must be a quotient of $A(G)$ by a right maximal ideal. Consider the resulting morphism of right A-modules,

$$A \longrightarrow V,$$

and let $x \in Supp(V) \subset Spec(A)$ be a closed point, $\mathfrak{m}_x$ and $k(x)$ be the corresponding maximal ideal and residue field considered as A-module. Let $I_x \subseteq G$ be the isotropy subgroup, i.e. the subgroup of G fixing the point x. Put $\mathfrak{m} = \bigcap_{g \in G} \mathfrak{m}_{g(x)}$, then

$$A/\mathfrak{m} = \prod_{g \in I_x \backslash G} k(g(x))$$

as k-algebra. Moreover

$$A/\mathfrak{m} \subseteq V,$$

and V is a quotient of $(A/\mathfrak{m})(G)$. Suppose that $I_x = G$ then V is a quotient of $k(x)(G)$, i.e. any irreducible representation of G. Consider the family $\mathcal{V} = \{V_i\}_{i=1,..r}$of such representations. They are all, in a unique way, simple modules of $A(G)$. Let us compute the noncommutative formal moduli for this family. Assume G is reductive. We first have to compute the different ext-functors, see §7,

$$Ext^1_{A(G)}(V_i, V_j) = (Ext^1_A(k(x), k(x)) \otimes_k V_i^* \otimes_k V_j)^G = Hom_G(V_i, U \otimes_k V_j)$$

since $Ext^1_A(k(x), k(x)) = U$. Moreover,

$$Ext^2_{A(G)}(V_i, V_j) = (Ext^2_A(k(x), k(x)) \otimes_k V_i^* \otimes_k V_j)^G = Hom_G(V_i, U \wedge U \otimes_k V_j).$$

The cup product is given in terms of the obvious composition,

$$Hom_G(V_i, U \otimes_k V_j) \otimes Hom_G(V_j, U \otimes_k V_k) \to Hom_G(V_i, U \wedge U \otimes_k V_k).$$

We obtain a formal moduli given in the form,

$$H(\mathcal{V}) = (H_{i,j}).$$

Now we shall want to classify the $A(G)$-module structures on $k(G)$, as the extension of extensions of, first the trivial representation V_0, then some V_{i_1},V_{i_2}, etc., fixing the "S-structure", i.e. fixing the graph Γ as above. Then the isomorphism classes of the $A(G)$-module structures fixing the graph Γ, are classified by the orbits of a Lie algebra action on the affine scheme $\underline{A}(\Gamma) = Mor(H, k(\Gamma))$.

One interesting problem is to identify the subscheme

$$\underline{E}(\Gamma) \subseteq \underline{A}(\Gamma) = Mor(H, k(\Gamma))$$

corresponding to the quotient modules of A. It is easy to see that this can be done by studying the following ordered graph,

$$\tilde{E} : \ \ldots \circ^{d_i} \xrightarrow{e_{i,j}} \circ^{d_j} \ldots$$

where $d_i = dim_k V_i$, and $e_{i,j} = dim_k Ext^1_{A(G)}(V_i, V_j) = dim_k Hom_G(V_i, V_j \otimes U) =: hom_G(V_i, V_j \otimes U)$, $i, j = 1 \ldots r$.

Another equally interesting problem is to find the deformations of the corresponding $A(G)$-modules. Fixing Γ and the sequence of vertices $\{V_i\}_{i=0,\ldots s}$, running through the vertices of Γ, it is clear that for any module M corresponding to a point of $\underline{A}(\Gamma)$, we have

$$\begin{aligned}\chi(M,M) :&= \sum_{i=1,\dots,n} (-1)^i ext^i_{A-G}(M,M)\\ &= \sum_{p=1,\dots,n,i,j=1,\dots,r} (-1)^p hom_G(V_i, V_j \otimes \wedge^p U).\end{aligned}$$

Now, go back to the Proposition (4.10), and consider the versal family $\tilde{V}[\Gamma]$ defined on the affine k-algebra $A[\Gamma]$. In the example (4.13) above, we have shown that the family $\tilde{V}[\Gamma]$ defined on $\underline{A}[\Gamma]$ is neither prorepresentable nor miniversal, as a family of modules. (We also noted that the canonical morphism

$$\rho : Mor(H(V), k[\Gamma_2']) \to Def_V(k[\Gamma_2'])$$

is surjective but not injective. This just tells us that the noncommutative formal moduli, in general, is not prorepresentable, which is not a surprise.) If the k-algebra A is commutative, we shall therefore be interested in the Kodaira-Spencer map for the family $\tilde{V}[\Gamma]$ defined on the (commutative) affine k-algebra $A[\Gamma]$. This is a morphism,

$$g: \ Der_k(A[\Gamma]) \longrightarrow Ext^1_{A[\Gamma]\otimes A}(\tilde{V}[\Gamma], \tilde{V}[\Gamma])$$

inducing, for every point $\alpha := (\alpha_{i,j}(l,m)) \in \underline{A}[\Gamma]$, a k-linear map,

$$g(\alpha): \ Der_k(A[\Gamma], k(\alpha)) \longrightarrow Ext^1_A(\tilde{V}[\Gamma](\alpha), \tilde{V}[\Gamma](\alpha))$$

which we shall now study in a special case. Let us consider the extension type with $r = s$, see Example (4.8).

$$\Gamma: \quad ① \longrightarrow ② \longrightarrow \cdots \qquad \longrightarrow (r)$$

If $R = k[\Gamma]$, then $R_{i,j} = k$ for all $i < j = 1, \dots, r$, and the affine scheme $\underline{A}[\Gamma]$ is cut out of the affine space $\mathbf{A}^N = \prod_{1\le i<j\le r} Ext^1_A(V_i, V_j)$ by, at best, quadratic equations. We may therefore identify the tanget space of any point $\alpha \in \underline{A}[\Gamma]$ with the k-vector space $\prod_{1\le i<j\le r} Ext^1_A(V_i, V_j)$. The point α corresponds to an A-module structure

$$\psi_\alpha : A \longrightarrow End_k(\prod_{i=1,\dots,r} V_i)$$

on the k-vector space $\prod_{i=1,\dots,r} V_i$, extending the given one, see the proof of Proposition (4.10). ψ_α defines an A-bi-module structure on $End_k(\prod_{i=1,\dots,r} V_i)$. Call this bi-module $End^\psi_k(\prod_{i=1,\dots,r} V_i)$. Corresponding to this bi-module structure, there is the canonical exact sequence,

$$0 \to End_k(\prod_{i=1,\dots,r} V_i) \to Der_k(A, End^\psi_k(\prod_{i=1,\dots,r} V_i) \to Ext^1_A(\tilde{V}[\Gamma](\alpha), \tilde{V}[\Gamma](\alpha)) \to 0.$$

Using the fact that α satisfies the relations given in terms of the obstructions above, there is a (non unique) map,

$$\tilde{g}(\alpha):\prod_{1\leq i<j\leq r}Der_k(A,End_k(V_i,V_j))\to Der_k(A,End_k^{\psi}(\prod_{i=1,\dots,r}V_i),$$

inducing the canonical $g(\alpha)$ above. The most interesting invariant of this is the dimension of the image of $g(\alpha)$. In good cases, this is the dimension of the modular stratum of the family $\tilde{V}[\Gamma]$ through the point $\alpha\in\underline{A}[\Gamma]$. See §8, for the consequences of this in the study of the McKay correspondence.

Example 4.17. Given any scheme $\underline{H}=Spec(H)$, say the 2-dimensional affine space given by $H=k[x_1,x_2]$. We shall be interested in the noncommutative moduli space parametrizing subschemes of length 2 of $\underline{H}$. We may do this by simply considering a point in the space $Spec(H)$ together with a tangent direction, i.e. the right H-module of the form,

$$V=k[x_1,x_2]/(x_1^2,x_2),$$

and compute the formal moduli of V.

Lemma 4.18. *The formal moduli, $H(V)$ of the H-module $H/(x_1^2,x_2)$, is given as the completion of the k-algebra,*

$$\Omega=k\{t_1,t_2,\omega_1,\omega_2\}/(y_1,y_2)$$

where

$$y_1=[t_1,t_2]-t_1[\omega_1,\omega_2]\quad y_2=[t_1,\omega_2]-[t_2,\omega_1]-\omega_1[\omega_1,\omega_2],$$

and where the family of left Ω-and right H-modules,

$$\Omega\otimes_k k^2$$

is defined by the actions of x_1 and x_2, given by,

$$x_1=\begin{pmatrix}0&t_1\\1&\omega_1\end{pmatrix},\ x_2=\begin{pmatrix}t_2&t_1\omega_2\\\omega_2&t_2+\omega_1\omega_2\end{pmatrix}$$

Proof. Consider the obvious free resolution of $V:=H/(x_1^2,x_2)$ as an H-module,

$$V\xleftarrow[\rho]{}H\xleftarrow[d_0]{}H^2\xleftarrow[d_1]{}H\xleftarrow[d_2]{}0$$

where we have,

$$d_0=(x_1^2,x_2),\ d_1=\begin{pmatrix}x_2\\-x_1^2\end{pmatrix}.$$

Consider the Yoneda complex, and pick a basis

$$\{\hat{t}_1,\hat{t}_2;\hat{\omega}_1,\hat{\omega}_2,\}$$

of $Ext^1_H(V,V)$ represented by the morphisms of the diagram,

$$\begin{tikzcd}
V & H \arrow[l, "\rho"'] & H^2 \arrow[l, "d_0"'] \arrow[dl, "\hat{\omega}_j"', "\hat{t}_i"] & H \arrow[l, "d_1"'] \arrow[dl, "\hat{\omega}_j^2"', "\hat{t}_i^2"] & & \\
V & H \arrow[l, "\rho"'] & H^2 \arrow[l, "d_0"'] \arrow[dl, "\hat{\omega}_j"', "\hat{t}_i"] & H \arrow[l, "d_1"'] \arrow[dl, "\hat{\omega}_j^2"', "\hat{t}_i^2"] & H \arrow[l, "d_2"'] & 0 \arrow[l] \\
V & H \arrow[l, "\rho"] & H^3 \arrow[l, "d_0"] & H^3 \arrow[l, "d_1"] & H \arrow[l, "d_2"] & 0 \arrow[l]
\end{tikzcd}$$

Here,

$$\hat{t}_1 = (1,0),\ \hat{t}_2 = (0,1,);$$
$$\hat{\omega}_1 = (x_1,0),\ \hat{\omega}_2 = (0,x_1)$$

and,

$$\hat{t}_1^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \hat{t}_2^2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

and finally,

$$\hat{\omega}_1^2 = \begin{pmatrix} 0 \\ x_1 \end{pmatrix},\ \hat{\omega}_2^2 = \begin{pmatrix} -x_1 \\ 0 \end{pmatrix}.$$

Using this it is easy to see that ,

$$\hat{t}_i \cup \hat{t}_i = 0,\ \hat{t}_1 \cup \hat{t}_2 = -\hat{t}_2 \cup \hat{t}_1 = \hat{y}_1,$$

and that

$$\hat{t}_1\hat{\omega}_2^2 = \hat{\omega}_1 t_2^2 = -\hat{y}_2,\ \hat{\omega}_i\hat{t}_i^2 = 0,\ \hat{\omega}_i\hat{\omega}_j^2 = 0,\ \hat{t}_2\hat{\omega}_1^2 = \hat{\omega}_2\hat{t}_1^2 = \hat{y}_2,$$

therefore

$$-y_2 = \hat{t}_1 \cup \hat{\omega}_2 = \hat{\omega}_1 \cup \hat{t}_2 = -\hat{t}_2 \cup \hat{\omega}_1 = -\hat{\omega}_2 \cup \hat{t}_1,\ \hat{\omega}_i \cup \hat{\omega}_j = \hat{t}_i \cup \hat{t}_j = 0.$$

Now, consider the dual basis $\{t_1, t_2; \omega_1, \omega_2\}$ generating the hull of the deformation functor $Def_{k[\epsilon]}$, we find after a simple computation of the 3. order Massey products the formulas we want.

Notice that we just have to compute the tangent situation and check that our formulas give us a lifting of the quadratic relations and of the corresponding H-action, to know that our result holds.

$\square$

By a simple computation one checks that the k-points of Ω form an open dense part of $Hilb^2\mathbf{A}^2$ containing V. $Hilb^2\mathbf{A}^2$ is the blow-up of $(\mathbf{A}^2 \times \mathbf{A}^2)/\mathbf{Z}_2$ along the diagonal. However, there are other simple representations of Ω. The homomrphism,

$$\Omega \to k|t_1, t_2, \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}]$$

maping ω_i to $\dfrac{\partial}{\partial t_i}$, shows that $k|t_1, t_2]$ is a simple representation of Ω.

§5. Noncommutative schemes.

Swarms, trivializations and observables. Let $\underline{C}$ be any abelian category with Massey products. The last proviso is satisfied if $\underline{C}$ has enough projectives, but there are other cases where Massey products exist even though projectives are scarce. See [La 2] and [Siq] for an exposition of the Massey product structure in the category of all O_X-modules for X a scheme defined on some field k.

Let $\underline{c} \subseteq \underline{C}$ be a diagram, i.e. a family of objects $\mathcal{V} = \{V_i\}_i$ and morphisms between them, but without any commutation relations. Put $|\underline{c}| := \mathcal{V}$. We shall assume that,

$$dim_k^1(V_i, V_j) < \infty,$$

for all $V_i, V_j \in \mathcal{V}$. Assume moreover that there exists an exact and faithful functor,

$$\pi : \underline{c} \longrightarrow k - mod.$$

Definition 5.1. Any such $\underline{c}$ will be called a swarm, and the functor π will be called a *trivialization* of $\underline{c}$.

Example 5.2. The obvious example of this set up is the following: Let A be any k-algebra, k a field, put $\underline{C} = A - mod$ and let

$$\pi : A - mod \longrightarrow k - mod$$

be the forgetful functor. Then π will be a trivialization for any diagram

$$\underline{c} \subseteq \underline{C} = A - mod$$

Unless we specifically mention another choice of trivialization, this is the one we shall use in the sequel.

Fix the trivialization π of $\underline{c} \subseteq \underline{C}$, and consider the functor,

$$Hom_\pi : mor\ \underline{c} \longrightarrow k - mod$$

defined for $\psi : c_1 \to c_2$ in $\underline{c}$, by

$$Hom_\pi(\psi) = Hom_k(\pi(c_1), \pi(c_2)).$$

Definition 5.3. $O_0 := O_0(\underline{c}, \pi) := H^0(\underline{c}, Hom_\pi)$ is the k-algebra of *immediate observables*

It is clear that O_0 acts on each object $\pi(c) \in k - mod,\ c \in ob\ \underline{c}$, in the sense that there is a canonical k-algebra homomorphism

$$O_0 \longrightarrow \mathrm{End}_k(\pi(c))$$

such that the image diagram

$$\pi(\underline{c}) \subseteq k - mod$$

becomes a diagram of O_0 -representations.

In the example above, we obtain for every diagram $\underline{c} \subseteq A - mod$, a k-algebra $O_0(\underline{c}, \pi)$ acting on every A-module in $\underline{c}$ such that $\underline{c}$ becomes a diagram of $O_0(\underline{c}, \pi)$-modules. Moreover there is a canonical homomorphism of k-algebras

$$\eta_0 : A \longrightarrow O_0(c, \pi)$$

which is, in an obvious sense, a universal "extension" of the algebra A, by an algebra acting on the diagram $\underline{c}$. Since we have,

$$\underline{c} \subseteq O_0 - mod$$

and since the trivialization π induces a trivialization,

$$\pi_0 : O_0 - mod \longrightarrow k - mod$$

we may repeat the construction of trivial observables. We obtain,

$$O_0(\underline{c}, \pi_0) = O_0(\underline{c}, \pi) = O_0$$

This implies that the operation of constructing trivial observables, is a closure operation.

Example 5.4. Consider any reduced commutative k-algebra A of finite type. Recall from (1.2) that if $\underline{c} = Spec(A)$, or if $\underline{c} = Simp^*(A)$ then

$$\eta_0 : A \longrightarrow O_0(\underline{c}, \pi)$$

is an isomorphism, provided k is algebraically closed. Denote by $Ind(A)$ the full subcategory of A-mod defined by the indecomposible modules and let $Prim(A)$ denote the subdiagram of $Ind(A)$ composed by the A-modules of the form $A/\mathfrak{q}$, where $\mathfrak{q}$ is a primary ideal, and where the morphisms are the obvious ones. It is easy to see that the canonical homomorphism

$$\eta_0 : A \longrightarrow O_0(Prim(A), \pi)$$

is an isomorphism when A is reduced. However, in general there is no such isomorphism, just look at the case $A = k[\epsilon]$, where

$$O_0(Prim(A), \pi) \simeq \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}.$$

Notice that there is a generalized Zariski topology on $Prim(A)$, due to Jacobson, defined as follows. Let $a \in A$ and consider the full subdiagram $D(a)$ of $Prim(A)$ defined by the objects V for which a is not a zerodivisor. Obviously $D(a) \cap D(b) = D(ab)$ and $D(a)$ is simply the localization of $Prim(A)$ at $\{a\}$. There are canonical isomorphisms

$$O_0(D(a), \pi) \simeq A_{(a)} = O_S(D(a)|_{Spec(A)})$$

where S is the affine scheme $Spec(A)$, and where O_S is the structure sheaf. This shows that there exists a ringed space $(Prim(A), O_P)$, and a continous map

$$S = Spec(A) \longrightarrow Prim(A) = P$$

inducing isomorphisms of the structure sheaves

$$O_S \simeq O_P.$$

The problem with $Prim(A)$ is that it is too big, that the topology is too coarse, and that it has some unsatisfactory functorial properties. On the other hand, $Spec(A)$ and $Simp^*(A)$ seem to be too small since the trivial observables for $Spec(A)$ kills the nilpotents of A, even for finite type k-algebras, and $\eta_0 : A \to O_0(Simp^*(A), \pi)$ is far from an isomorphism when A is local.

These problems stem from the trivial nature of the trivial observables. In the construction of O_0, we use only the trivial categorical structure of A-mod, restricted to $\underline{c}$. To get to the goal, we have to take into account the infinitesimal structure of the category A-mod, i.e. the abelian structure of A-mod, and, in particular, the family of multiple extensions of the objects of $\underline{c}$.

The goal is to construct, for every diagram $\underline{c}$, an extension of $O_0(\underline{c}, \pi)$, which we shall denote $\mathcal{O}_\pi(\underline{c})$, and canonical homomorphisms

$$A \xrightarrow{\eta} \mathcal{O}_\pi(\underline{c}) \xrightarrow{\rho} O_0(\underline{c}, \pi)$$

extending η_0. We shall show that $\mathcal{O}_\pi$, has good functorial properties, mimicking the notion of *structure sheaf* in commutative algebra, and providing us with a generalized, noncommutative, algebraic geometry. We shall be guided by the principles of the Introduction.

So consider a swarm $\underline{c}$ in $\underline{C} = A{-}mod$, together with the trivialization π. Assume first that $\underline{c}$ is finite. Let $|\underline{c}| = \{V_i\}_{i=1}^r$, be the family of objects, and construct the noncommutative formal moduli $H(|\underline{c}|) = (H_{i,j})$ as in §3. Let $\tilde{V} = (H_{i,j} \otimes V_j)$ be the versal family and consider the k-algebra

$$O(|\underline{c}|, \pi) := End_H(\tilde{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j))$$

and the k-algebra homomorphism,

$$\eta(|\underline{c}|) : A \longrightarrow O(|\underline{c}|, \pi)$$

defined by the action of A on $\tilde{V}$.

Recall that the noncommutative formal moduli is unique up to isomorphisms, and that having fixed a versal family, as a deformation, the action of A on $\tilde{V}$ is unique up to isomorphisms. This means that for any other homomorphism

$$\eta(|\underline{c}|)' : A \longrightarrow O(|\underline{c}|, \pi)$$

defining the same deformation, there exists an automorphism

$$\omega \in (H_{i,j} \otimes Hom_k(V_i, V_j))$$

such that

$$\eta(|\underline{c}|)' = \omega\ \eta(|\underline{c}|)\ \omega^{-1}.$$

Notice that ω, as an element of $O(|\underline{c}|, \pi)$, is a unit.

Recall also that, for an artinian algebra A, and for the family $Simp(A)$ of all the simple A-modules $\eta(Simp(A))$ is an isomorphism.

Notice that, by definition of the terms, there is a canonical morphism of k-algebras,

$$\rho_0 : O(|\underline{c}|, \pi) \longrightarrow O_0(|\underline{c}|, \pi)$$

which, together with η and η_0 form a commutative diagram. Therefore $|\underline{c}|$ is, in an obvious sense, a family of $O(|\underline{c}|, \pi)$-modules. Notice also that if $\underline{c}_1 \subseteq \underline{c}_2$ is an inclusion of swarms, there exist an, up to automorphisms, unique surjective homomorphism,

$$h(\underline{c}_1 \subseteq \underline{c}_2) : H(|\underline{c}_2|) \longrightarrow H(|\underline{c}_1|)$$

induced by the natural imbedding,

$$\underline{a}_{r_1} \to \underline{a}_{r_2}$$

where $r_i,\ i = 1, 2$, is the number of objects in $\underline{c}_i$. At the tangent level this morphism corresponds to the inclusion,

$$(Ext^1_A(V_i, V_j))_{i,j=1,..,r_1} \subseteq (Ext^1_A(V_i, V_j))_{i,j=1,..,r_2}.$$

Beware, this k-algebra homomorphism does not necessarily admit a section! The morphism $h(\underline{c}_1 \subseteq \underline{c}_2)$ induces a unique homomorphism of k-algebras,

$$o(\underline{c}_1 \subseteq \underline{c}_2) : O(|\underline{c}_2|, \pi) \to O(|\underline{c}_1|, \pi).$$

If $\underline{c}$ is infinite we shall later put,

$$O(|\underline{c}|, \pi) = \varprojlim_{\underline{c}_0 \subseteq \underline{c}} O(|\underline{c}_0|, \pi),$$

where $\underline{c}_0$ runs through all finite subdiagrams of $\underline{c}$. (This is possible, since $\underline{c}_0$ being fixed, we may choose bases $\{t_{i,j}(\ell)\}_\ell$ for each $Ext^1_A(V_i, V_j)$, for all $V_i, V_j \in |\underline{c}|$, and construct in one sweep all the cup and Massey products defining all $H(|\underline{c}_0|)$. However, there is a problem related to the A-action. See (5.14) and the definition (5.15).)

The k-algebra we are heading for is now a subquotient of $O(|\underline{c}|, \pi)$, singled out by the *incidences* of our geometry, i.e. by the morphisms,

$$\phi_{i,j} : V_i \to V_j,$$

of our diagram.

Let $\Gamma(\underline{c})$ be the quiver corresponding to the swarm $\underline{c}$, i.e. with set of nodes equal to the set of objects of $\underline{c}$, and with arrows corresponding to the morphisms $\phi_{i,j}$ of $\underline{c}$. Notice that $\underline{c}$ is a diagram of $\underline{C}$ not a subcategory, therefore we do not require that the identities of the objects be morphisms of $\underline{c}$. Assume first that $\underline{c}$ is finite, and assume, for simplicity, that $\Gamma(\underline{c})$ (or $\underline{c}$) is connected.

Corresponding to $\Gamma(\underline{c})$ there is the universal k-algebra $k[\Gamma(\underline{c})]$. Consider the obvious representation of $k[\Gamma(\underline{c})]$ on $V := \oplus_{i=1}^r V_i$ mapping $k[\Gamma(\underline{c})]$ to the k-algebra $end(\underline{c})$, generated by the morphisms of $\underline{c}$, in $End_A(V)$. Now V is a $A \otimes k[\Gamma(\underline{c})]$-module, and as such an A-module, as well as a $k[\Gamma(\underline{c})]$-module. We may consider the ordinary (noncommutative) deformation functors of this module, as $A\otimes k[\Gamma(\underline{c})$-module, as A-module, and as $k[\Gamma(\underline{c})]$-module. Let the formal moduli of these functors be, $H^{A,\Gamma}(V)$, H(V) and $H^{\Gamma}(V)$, respectively. There are natural (non unique) morphisms,

$$H^{\Gamma}(V) \to H^{A,\Gamma}(V) \leftarrow H(V).$$

Recall from §3. that the modular, or prorepresentable, substratum $H^{A,\Gamma}(V)_0$ of $H^{A,\Gamma}(V)$ is the unique maximal quotient of $H^{A,\Gamma}(V)$ such that the composition,

$$Mor(H^{A,\Gamma}(V)_0,-) \to Mor(H^{A,\Gamma}(V),-) \to Def_{A\otimes k[\Gamma]}V$$

is injective. There is a universal deformation of V to $H^{A,\Gamma}(V)_0$, i.e. an action of $A\otimes k[\Gamma]$ on $H^{A,\Gamma}(V)_0\otimes V$, uniquely inducing all other modular deformations. Finally, let $H(\underline{c})$ be the unique common quotient of $H^{A,\Gamma}(V)_0$ and $H(V)$ defined by the induced morphism,

$$H^{A,\Gamma}(V)_0 \leftarrow H(V).$$

Now, given any deformation ξ_S of V to some k-algebra $S\in\underline{\hat{a}}_1$, denote by $\underline{m}$ the maximal ideal of S. Let $\tilde{S}$ be the r-pointed matrix k-algebra $(\tilde{S}_{i,j})$ where $\tilde{S}_{i,i}=S$, on the diagonal, and $\tilde{S}_{i,j}=\underline{m}$ at the other places, i.e. for $i\neq j$. Clearly $\tilde{S}$ is in $\underline{\hat{a}}_r$, and the i-th. row of the matrix $(\tilde{S}_{i,j}\otimes V_j)$ is

$$(\underline{m}\otimes V_1)\oplus\cdots\oplus(S\otimes V_i)\oplus\cdots\oplus(\underline{m}\otimes V_r)\subseteq H(V)\otimes V.$$

Let $v_i\in V_i$, and $a\in A$. The component of $(1\otimes v_i)a$ in $S\otimes V_j$ for $i\neq j$ sits in $\underline{m}\otimes V_j$. This shows that A acts on each line of the matrix $(\tilde{S}_{i,j}\otimes V_j)$, commuting with the left action of $(\tilde{S}_{i,j})$, implying that $(\tilde{S}_{i,j}\otimes V_j)$ is, in a natural way, a noncommutative deformation of the family of right A-modules $|\underline{c}|$, to $\tilde{S}$. Therefore there is a morphism,

$$\iota_S : H(|\underline{c}|)\longrightarrow\tilde{S}$$

compatible with the specified deformations of right A-modules. This induces a morphism of k-algebras,

$$(H_{i,j}\otimes Hom_k(V_i,V_j))\longrightarrow(\tilde{S}_{i,j}\otimes Hom_k(V_i,V_j)).$$

Since the right hand side k-algebra is a subalgebra of

$$End_S(S\otimes(\oplus_{i=1}^r V_i))=(S\otimes Hom_k(V_i,V_j)),$$

we obtain a (non-unique) homomorphism of k-algebras,

$$\kappa_S : (H_{i,j}\otimes Hom_k(V_i,V_j))\longrightarrow(S\otimes Hom_k(V_i,V_j))$$

such that the action $\eta(|\underline{c}|)$ is mapped to the A-action on $S\otimes(\oplus_{i=1}^r V_i)$ defining the deformation ξ_S. In particular, for the versal deformation of V to $H(V)$, and for the versal A-action on $H(V)\otimes(\oplus_{i=1}^r V_i)$, there is a homomorphism of k-algebras,

$$\kappa_{H(V)} : (H_{i,j}\otimes Hom_k(V_i,V_j))\longrightarrow(H(V)\otimes Hom_k(V_i,V_j))$$

compatible with the actions. Notice that by construction of the terms involved, it is clear that $\kappa_{H(V)}$ is injective, and that $H(V)$ is generated by the images of the components $\iota_{i,j}:H_{i,j}\to H(V)$ of $\iota_{H(V)}$. Therefore we have the adjunction relation,

$$Mor_{\underline{a}_r}(H(|\underline{c}|),\tilde{S})\simeq Mor_{\underline{a}_1}(H(V),S).$$

Now, compose $\kappa_{H(V)}$ with the homomorphism induced by the quotient map $H(V)\to H(\underline{c})$ and get a k-algebra homomorphism,

$$\kappa_{H(\underline{c})} : (H_{i,j}\otimes Hom_k(V_i,V_j))\longrightarrow(H(\underline{c})\otimes Hom_k(V_i,V_j))$$

Definition 5.5. The k-algebra of *preobservables* $O(\underline{c}, \pi)$ of the finite swarm $\underline{c}$, is the subalgebra of

$$(H_{i,j}(|\underline{c}|) \otimes Hom_\pi(V_i, V_j))$$

commuting, via the morphism,

$$\kappa_S : (H_{i,j}(|\underline{c}|) \otimes Hom_\pi(V_i, V_j)) \to (S \otimes Hom_k(V_i, V_j)),$$

induced by any surjective k-algebra homomorphism

$$H(\underline{c}) \longrightarrow S,$$

with the corresponding representation of $k[\Gamma]$ in $(S \otimes Hom_k(V_i, V_j))$.

It is clear that $O(\underline{c}, \pi)$ and $\eta(\underline{c}, \pi)$ are uniquely defined, up to isomorphisms, and that $\eta(|\underline{c}|$ induces a homomorphism of k-algebras,

$$\eta(\underline{c} : A \longrightarrow O(\underline{c}, \pi)$$

Remark 5.6. Let A be a finite type k-algebra, and let $\{\phi\}$ be the diagram defined by the canonical homomorphism,

$$\phi : A \to k(x)$$

of A onto its closed point $k(x)$. The tangent space of $H := H(|\{\phi\}|)$ is,

$$(radH/rad^2H)^* = \begin{pmatrix} 0 & 0 \\ Ext^1_A(k, A) & Ext^1_A(k, k) \end{pmatrix}$$

and the tangent space of $H(\{\phi\})$ looks like

$$(RadH(\{\phi\})/Rad^2H(\{\phi\}))^* = \begin{pmatrix} 0 & 0 \\ Ext^1_A(k, A)^\phi & Ext^1_A(k, k) \end{pmatrix},$$

where $Ext^1_A(k, A)^\phi$ is the kernel of $\phi_* : Ext^1_A(k, A) \to Ext^1_A(k, k)$. The morphism

$$O(|\{\phi\}|, \pi) \to \begin{pmatrix} H(\{\phi\}) \otimes_k End_k(A) & H(\{\phi\}) \otimes_k Hom_k(A, k) \\ H(\{\phi\}) \otimes_k Hom_k(k, A) & H(\{\phi\}) \end{pmatrix}$$

maps an element

$$\alpha = \begin{pmatrix} \alpha_{1,1} & 0 \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \in O(|\{\phi\}|, \pi)$$

to an element of the same form,

$$\tilde{\alpha} = \begin{pmatrix} \alpha_{1,1} & 0 \\ \tilde{\alpha}_{2,1} & \alpha_{2,2} \end{pmatrix} \in \begin{pmatrix} H(\{\phi\}) \otimes_k End_k(A) & H(\{\phi\}) \otimes_k Hom_k(A, k) \\ H(\{\phi\}) \otimes_k Hom_k(k, A) & H(\{\phi\}) \end{pmatrix}.$$

Moreover a versal lifting of $\phi_{1,2}$ has the form,

$$\Phi = \begin{pmatrix} 0 & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{pmatrix} \in \begin{pmatrix} H(\{\phi\}) \otimes_k End_k(A) & H(\{\phi\}) \otimes_k Hom_k(A, k) \\ H(\{\phi\}) \otimes_k Hom_k(k, A) & H(\{\phi\}) \end{pmatrix}.$$

Suppose now that,

$$\Phi\tilde{\alpha} = \tilde{\alpha}\Phi$$

then, in particular,

$$\Phi_{1,2}\tilde{\alpha}_{2,1} = 0,$$

which implies that $\tilde{\alpha}_{2,1} = 0$, and then

$$\Phi_{1,2}\alpha_{2,2} = \alpha_{1,1}\Phi_{1,2}.$$

Since $\alpha_{2,2} =: \alpha_x \in H_{2,2} = \hat{A}_{\{x\}}$ is the obvious multiplication endomorphism, and since $\Phi_{1,2}$ reduces to the obvious completion map, $\rho_x : A \to H_{2,2} = \hat{A}_{\{x\}}$ we find that the conditions on α are,

$$\tilde{\alpha}_{2,1} = 0, \ \alpha_{1,1}\rho_x = \rho_x\alpha_x$$

for some $\alpha_x \in \hat{A}_{\{x\}}$, together with the condition

$$\Phi_{2,1}\alpha_{1,1} = \alpha_x\Phi_{2,1}.$$

O is a closure operation. The most important property of the O-construction is a kind of functoriality (up to isomorphisms) and the closure property, given by the following result:

Theorem 5.7. *Let $\psi : A \to B$ be a k-algebra homomorphism, and let $\underline{c}$ be a finite swarm of A-B-modules. Consider the O-constructions, $O^A(\underline{c},\pi)$, resp.$O^B(\underline{c},\pi)$.*

a. Assume the natural morphism,

$$\psi_* : H^{A,\Gamma}(V) \to H^{B,\Gamma}(V)_0$$

induces a surjective homomorphism

$$\psi_* : H^A(\underline{c}) \to H^B(\underline{c}).$$

Then there exists an, up to isomorphisms, unique extension of ψ, i.e. a commutative diagram,

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\ O^A(\underline{c},\pi) \arrow[r, "O(\psi)"'] & O^B(\underline{c},\pi) \end{tikzcd}$$

b. There is a natural isomorphism,

$$O(\eta_A) : O^A(\underline{c},\pi) \to O^{O^A}(\underline{c},\pi)$$

implying that the O-construction is a closure operation.

Proof. Let the noncommutative formal moduli of the family of B-modules $\{V_i\} = |\underline{c}|$, considered as A-and B-modules be H^A resp. H^B. Since the versal family of B-modules $(H^B_{i,j} \otimes V_j)$ is also a family of A-modules, there is a morphism, $H^A \to H^B$ inducing the morphism of families of A-modules, $(H^A_{i,j} \otimes V_j) \to (H^B_{i,j} \otimes V_j)$ consistent with the induced A-module structure on the latter. In the same way we find that there exists a morphism of the formal moduli

$$H^{A,\Gamma}(V) \longrightarrow H^{B,\Gamma}(V)$$

of $V = \oplus_{i=1}^{r} V_i$ as a $A \otimes k[\Gamma(\underline{c})]$-module, resp. a $B \otimes k[\Gamma(\underline{c})]$-module, consistent with the families. By assumption, the above homomorphism induces a surjection

$$\psi_p^* : H^A(\underline{c}) \longrightarrow H^B(\underline{c})$$

By definition of O, we have a commutative diagram,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\psi"'] & O^A(\underline{c},\pi) \arrow[r, hook] \arrow[d, dashed, "O(\psi)"'] & End_{H^A}(H^A_{i,j} \otimes V_j) \arrow[r] \arrow[d] & (H^A(\underline{c}) \otimes Hom_k(V_i,V_j)) \arrow[d] \\
B \arrow[r] & O^B(\underline{c},\pi) \arrow[r, hook] & End_{H^B}(H^B_{i,j} \otimes V_j) \arrow[r] & (H^B(\underline{c}) \otimes Hom_k(V_i,V_j)).
\end{tikzcd}$$

where, $O^A(\underline{c},\pi)$ and $O^B(\underline{c},\pi)$ are the commutants of the actions of $k[\Gamma(\underline{c})]$ in $(S \otimes Hom_k(V_i,V_j))$, for all quotients S of $H^A(\underline{c})$, respectively of $H^B(\underline{c})$. The surjectivity of ψ_*, together with the commutativity of the diagram defines the morphism $O(\psi)$, and proves (a).

To prove (b), we just have to observe that O^A acts on the H^A-family $(H^A_{i,j} \otimes V_j)$, consistent with the action of A via η_A, and that $O \otimes k[\Gamma(\underline{c})]$ acts on the $H^A(\underline{c})$-family $(H^A(\underline{c}) \otimes V)$ consistent with the A-action via the obvious composition,

$$\eta(V) : A \longrightarrow (H^A(\underline{c}) \otimes Hom_k(V_i,V_j)).$$

Therefore there must exist morphisms,

$$\left(H^{O^A}_{i,j}\right) \xrightarrow{\mu} (H^A_{i,j})$$
$$H^{O^A,\Gamma}(V) \xrightarrow{\mu(\underline{c})} H^{A,\Gamma}(V)$$

consistent with the obvious families. Moreover the composed morphisms

$$(H^A_{i,j}) \to \left(H^{O^A}_{i,j}\right) \xrightarrow{\mu} (H^A_{i,j})$$
$$H^{A,\Gamma}(V) \to H^{O^A,\Gamma}(V) \xrightarrow{\mu_p} H^{A,\Gamma}(V)$$

must be surjections inducing injections on the tangent spaces. Let us show that $\mu(\underline{c})$ induces a surjective homomorphism,

$$H^{O^A}(\underline{c}) \xrightarrow{\mu_0} H^A(\underline{c}).$$

Consider the diagram,

$$\begin{tikzcd}
Mor(H^A(\underline{c}),-) \arrow[r] \arrow[d] \arrow[dr, "\alpha"] & Mor(H^{O,\Gamma}(V),-) \arrow[r] \arrow[d] & Mor(H^{A,\Gamma}(V),-) \arrow[d] \\
Def_V^{A-\Gamma} \arrow[r, dashed] & Def_V^{O-\Gamma} \arrow[r, "\beta"'] & Def_V^{A-\Gamma}
\end{tikzcd}$$

Define α by the composition, i.e. by considering the canonical $O \otimes k[\Gamma]$-structure on $H^A(\underline{c}) \otimes V$, and tensorization. Since the composition of α and β is injective, α is injective. But this implies, by definition, and by the the unicity of the modular substratum, that the surjective homomorphism

$$\mu_0 : H^{O,\Gamma}(V) \longrightarrow H^A(\underline{c})$$

induces a surjective homomorphism,

$$\mu_0(\underline{c}) : H^{O,\Gamma}(V)_0 \longrightarrow H^A(\underline{c})$$

and therefore also a surjection,

$$\mu_0 : H^O(\underline{c}) \longrightarrow H^A(\underline{c}).$$

Consider now the commutative diagram,

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\eta"'] & (H^A_{i,j} \otimes Hom_k(V_i, V_j)) \arrow[r] & (H^A(\underline{c}) \otimes Hom_k(V_i, V_j)) \\ O(\underline{c}, \pi) \arrow[r] & (H^O_{i,j} \otimes Hom_k(V_i, V_j)) \arrow[r] \arrow[u, "\mu"] & (H^O(\underline{c}) \otimes Hom_k(V_i, V_j)). \arrow[u, "\mu_0"'] \end{tikzcd}$$

Since μ_0 is consistent with the actions of $k[\Gamma]$, we obtain a cosection,

$$\mu : O^O(\underline{c}, \pi) \longrightarrow O^A(\underline{c}, \pi)$$

of the morphism,

$$\eta_1 : O^A(\underline{c}, \pi) \longrightarrow O^O(\underline{c}, \pi).$$

Now use (a) for the case $\psi = \mu$. Let us put $A' = O^O(\underline{c}, \pi), B' = O^A(\underline{c}, \pi)$. Since the composition of $\mu \circ \nu_1$ is the identity, the homomorphism,

$$H^{A'}(\underline{c}) \to H^{B'}(\underline{c})$$

must be surjective. We therefore obtain a commutative diagram,

$$\begin{tikzcd} O^O(\underline{c}, \pi) \arrow[r, "\mu"] \arrow[d, "\eta_2"'] & O^A(\underline{c}, \pi) \arrow[d, "\eta_1"'] \\ O^{O^O}(\underline{c}, \pi) \arrow[r, "O(\mu)"'] & O^O(\underline{c}, \pi). \end{tikzcd}$$

Since, by construction, the composition $\eta_2 \circ O(\mu)$ is an isomorphism, μ must be injective, therefore an isomorphism, proving (b).

□

Noncommutative preschemes.

Definition 5.8. Let A be any k-algebra. A finite swarm $\underline{c}$ of A-modules is called a prescheme for A, if the morphism

$$\eta(\underline{c},\pi): A \longrightarrow O(\underline{c},\pi)$$

is an isomorphism. In this case we shall refer to A as the *affine k-algebra* of $\underline{c}$.

Corollary 5.9. *Any finite swarm, $\underline{c}$, of right A-modules is a prescheme for $O^A(\underline{c},\pi)$. In particular, if $\underline{c}$ is a diagram of finite dimensional k-vector spaces, then $\underline{c}$ is a swarm of right modules over $A = O^k(\underline{c},\pi) = O_0(\underline{c},\pi)$, and, as such, a prescheme for A.*

Proof. This follows from the isomorphism

$$O(\eta_A): O^A(\underline{c},\pi) \to O^{O^A}(\underline{c},\pi)$$

of (5.7).

□

Example 5.10. According to the Generalized Burnside Theorem, if A is any finite dimensional k-algebra, k algebraically closed, the family of simple A-modules V form a (0-dimensional) prescheme for A. In particular, if Λ is a finite partially ordered set, then the set of nodes of Λ, considered as the set of simple $k[\Lambda]$-modules, is a scheme for $k[\Lambda]$.

Example 5.11. Consider the following trivial example where A=k is a field, $V_1 = k^2$ and $V_2 = k$, and $\underline{c}$ is given by the diagram of right-modules,

$$V_1 \xrightarrow{\phi} V_2$$

where ϕ is the second projection. Obviously all $Ext's$ vanish, so that

$$H(|\underline{c}|) = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

$$(H_{i,j} \otimes V_j) = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$$

Moreover $H(\underline{c}) = k$, and the maximal ideal $\underline{m} \subset H(\underline{c})$ is zero. Therefore,

$$\bar{H}(\underline{c}) = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

and

$$(\bar{H}(\underline{c})_{i,j} \otimes V_j) = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$$

Therefore $O(\underline{c},\pi)$ is the commutant algebra of the subalgebra,

$$end(\underline{c}) \subseteq \begin{pmatrix} Hom_k(V_1,V_1) & Hom_k(V_1,V_2) \\ Hom_k(V_2,V_1) & Hom_k(V_2,V_2) \end{pmatrix}$$

generated by ϕ, in the sub k-algebra,

$$\begin{pmatrix} Hom_k(V_1,V_1) & 0 \\ 0 & Hom_k(V_2,V_2) \end{pmatrix}$$

which is easily seen to be equal to,

$$O_0(\underline{c},\pi) = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}.$$

Now it is equally easy to see that V_1 identifies with the second line of

$$O(\underline{c},\pi) = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$$

as a right $O(\underline{c},\pi)$-module, and is therefore projective, and that V_2 identifies with the second simple module of $O(\underline{c},\pi)$. Trivial calculation gives that

$$Ext^1_O(V_i,V_j) = 0,\ \forall i,j.$$

Put $O := O(\underline{c},\pi)$. If we consider $\underline{c}$ as a diagram of O-modules, and repeat the O-construction, we therefore obtain $O^O =: O^{(2)}(\underline{c},\pi) = O(\underline{c},\pi)$, implying that the diagram $\underline{c}$ is a prescheme for O, as it must be, see (5.9). Since the $O(\underline{c},\pi)$ we have found above is the algebra of the (A_1) diagram,

$$\circ^1 \longleftarrow \circ^2$$

we find that *as O-modules*, the discrete diagram $\mathcal{V} = \{k(1), k(2)\}$ consisting of the two simple O-modules is also a prescheme for O. Notice, however, that as family of k-vector spaces $\mathcal{V}$ is a prescheme for k^2.

Example 5.12. That the claim of the Corollary (5.9) is not obvious, is seen by classifying the simple schemes of the form,

$$\Gamma = \begin{tikzcd} \circ \arrow[loop above, "\phi"] \end{tikzcd}$$

Let $\underline{c} = \{\phi : V \to V\}$ be a diagram of k-vector spaces. Assume $dim_k V = 3$, then according to the Jordan decomposition of ϕ, we obtain the following k-algebras $A = O(\underline{c},\pi)$ with $\underline{c}$ as affine scheme,

(1) ϕ has 3 different eigenvalus, then $A = k^3$
(2) ϕ has only two different eigenvalus, then $A = k \times End_k(k^2)$ or $A = k \times k[\epsilon]$
(3) ϕ has all eigenvalues equal, then $A = End_k(k^3)$ or $A = k[x]/(x^3)$ or

$$A = \begin{pmatrix} k & k\cdot\epsilon_{1,2} \\ k\cdot\epsilon_{2,1} & k[\epsilon_{2,2}] \end{pmatrix}$$

where we have the following relations, $\epsilon_{1,2}\cdot\epsilon_{2,1} = 0,\ \epsilon_{1,2}\cdot\epsilon_{2,2} = 0,\ \epsilon_{2,2}\cdot\epsilon_{2,1} = 0,\ \epsilon_{2,1}\cdot\epsilon_{1,2} = \epsilon_{2,2}$

The last algebra is evidently artinian with two simple representations, both of dimension 1. Therefore it has an affine scheme consisting of the discrete diagram consisting of those two A-modules. As such, it has infinitesimal incidences given by the family $\{\epsilon_{i,j}\}$. By (5.9) A is also the affine algebra of the scheme $\underline{c}$, with quiver Γ.

Warning. NB! According to Example (5.4) if A is any reduced finite type commutative k-algebra, and if k is algebraically closed, $A \simeq O^k(Spec(A), \pi)$. However $Spec(A)$ is usually infinite. Therefore we cannot conclude from Corollary (2.9), that $Spec(A)$ is a prescheme for A. In fact, $Spec(A)$ is, in general, not a prescheme for A, since $O(Spec(A), \pi)$ may well be noncommutative.

Example 5.13. Consider the special case of Remark (5.6), where A is a local k-algebra, and $\phi : A \to k$ is the canonical homomorphism of A onto its residue field. Since there is a surjective homomorphism,

$$H(\{\phi\}) \longrightarrow H_{2,2} = \hat{A}.$$

and since the completion map ρ_x defines a lifting,

$$\Phi_{1,2} : \hat{A} \otimes A \to H_0(\{\phi\}) \otimes k = \hat{A}$$

of ϕ, we observe that $\alpha_{1,1}$ is the right multiplication by some element $a = \alpha_{1,1}(1) \in A$. Moreover $\alpha_x = \rho_x(a)$ and the condition

$$\Phi_{2,1}\alpha_{1,1} = \alpha_x \Phi_{2,1},$$

is automatically satisfied, since $\Phi_{1,2}$ is A-linear.Therefore,

$$O(\{\phi\}, \pi) = \{ \begin{pmatrix} \alpha & 0 \\ \alpha_{2,1} & \alpha_x \end{pmatrix} \mid \alpha \in A, \alpha_x = \rho_x(\alpha), \alpha_{2,1} \in H_{2,1} \otimes Hom_k(k, A)\}.$$

In particular, the natural morphism

$$\eta : A \to O(\{\phi\}, \pi)$$

is injective, and an isomorphism provided $Ext^1_A(k, A)^\phi = Ext^1_A(k, A)$.

Functoriality.

The k-algebra of preobservables of a diagram of modules is not, however, properly functorial with respect to inclusions between diagrams. The problem is the following: Let $\underline{c}_0 \subseteq \underline{c}$ be a finite subdiagram. There is a corresponding inclusion of quivers, $\Gamma(\underline{c}_0) \subseteq \Gamma(\underline{c})$, inducing a homomorphism of k-algebras,

$$k[\Gamma(\underline{c}_0)] \longrightarrow k[\Gamma(\underline{c})].$$

Consider the diagram,

$$\begin{tikzcd}
(H(|\underline{c}|)_{i,j} \otimes Hom_k(V_i, V_j)) \arrow[r, "o(\underline{c}_0 \subseteq \underline{c})"] \arrow[d] & (H(|\underline{c}_0|)_{i,j} \otimes Hom_k(V_i, V_j)) \arrow[d] \\
(H_0(\underline{c}) \otimes Hom_k(V_i, V_j)) & (H_0(\underline{c}_0) \otimes Hom_k(V_i, V_j)) \\
k[\Gamma] \arrow[u] & k[\Gamma_0]. \arrow[l] \arrow[u]
\end{tikzcd}$$

We would now like to conclude that this induces a natural morphism

$$O(\underline{c}_0 \subseteq \underline{c}) : O(\underline{c}, \pi) \longrightarrow O(\underline{c}_0, \pi)$$

since there are fewer conditions to be satisfied in the definition of the right hand algebra. However, the diagram above shows that this is not obvious. An element in $O(\underline{c}, \pi)$ certainly commutes with the action of $k[\Gamma_0]$ in $(H_0(\underline{c}) \otimes Hom_k(V_i, V_j))$ but not necessarily with the action of $k[\Gamma_0]$ in $(H_0(\underline{c}_0) \otimes Hom_k(V_i, V_j))$. But we are interested in the smallest k-algebra O extending A, and preserving both the diagram and the system of extensions of extensions of the objects of the diagram. Therefore we define the refined k-algebra of observables of the swarm $\underline{c}$, as,

Definition 5.14. The k-algebra of *observables* of the finite swarm $\underline{c}$, is the subalgebra

$$\mathcal{O}(\underline{c}, \pi) = \bigcap_{\underline{c}_0 \subseteq \underline{c}} o(\underline{c}_0 \subseteq \underline{c})^{-1}(O(\underline{c}_0, \pi)) \subseteq O(\underline{c}, \pi),$$

where $\underline{c}_0$ runs through all subdiagrams of $\underline{c}$.

Now there is a natural homomorphism,

$$\mathcal{O}(\underline{c}_0 \subseteq \underline{c}) : \mathcal{O}(\underline{c}, \pi) \to \mathcal{O}(\underline{c}_0, \pi),$$

and it is easy to see that the k-algebra of observables is a contravariant functor on the ordered set of subdiagrams of a given finite diagram.

We are now able to extend the definition of observables to certain infinite swarms.

Definition 5.15. A swarm is called permissible, if there exist a k-algebra homomorphism,

$$\eta(|\underline{c}|, \pi) : A \longrightarrow O(|\underline{c}|, \pi),$$

compatible with the morphisms $\eta(|\underline{c}_0|, \pi)$ and $o(\underline{c}_0 \subseteq \underline{c})$, where $\underline{c}_0$ runs through all subdiagrams of $\underline{c}$.

For any *permissible* swarm $\underline{c}$, we define,

$$\mathcal{O}(\underline{c}, \pi) = \varprojlim_{\underline{c}_0 \subseteq \underline{c}} \mathcal{O}(\underline{c}_0, \pi)$$

where $\underline{c}_0$ runs through all finite subdiagrams of $\underline{c}$.

Clearly

$$\mathcal{O}(\underline{c}, \pi) \subseteq O(|\underline{c}|, \pi)$$

and $\eta(|\underline{c}|, \pi)$ induces a natural homomorphism of k-algebras,

$$\eta(\underline{c}, \pi) : A \longrightarrow \mathcal{O}(\underline{c}, \pi)$$

the obvious limit of the family of morphisms $\eta(\underline{c}_0, \pi)$, where $\underline{c}_0$ runs through all finite subdiagrams of $\underline{c}$.

Definition 5.16. *The swarm $\underline{c}$ will be called a prescheme for A, if $\eta(\underline{c},\pi)$ is an isomorphism.*

Notice that if the finite swarm $\underline{c}$ is a prescheme in the sense of (2.8) then it is also a prescheme in the sense of (5.16). Therefore, any finite swarm $\underline{c}$ of A-modules is necessarily a prescheme for $O(\underline{c},\pi)$. This is, however, not the case for infinite swarms, see Warnig above.

Noncommutative schemes.

Finaly, we arrive at the notion of *structure sheaf* $\mathcal{O}_\pi$. For every finite subdiagram $\underline{c}_0$ of the swarm $\underline{c}$, consider the natural morphism,

$$\kappa(\underline{c}_0) : \mathcal{O}(\underline{c},\pi) \to (H(\underline{c}_0) \otimes Hom_k(V_i, V_j))$$

and consider the two-sided ideal $\mathfrak{n} \subset \mathcal{O}(\underline{c},\pi)$, defined by

$$\mathfrak{n} = \bigcap_{\underline{c}_0 \subseteq \underline{c}} ker\kappa(\underline{c}_0).$$

Here $\underline{c}_0$ runs through all finite subdiagrams of $\underline{c}$.

Definition 5.17. (i) In the above situation, put,

$$\mathcal{O}_\pi(\underline{c}) = \mathcal{O}(\underline{c},\pi)/\mathfrak{n}.$$

(ii) $\mathcal{O}_\pi$ is a presheaf on the ordered set of subdiagrams of a given permissible swarm $\underline{c}$.

Definition 5.18. (i) Let A be any k-algebra. A permissible swarm $\underline{c}$ of A-modules is called a scheme for A, if the canonical morphism

$$\eta(c,\pi) : A \longrightarrow \mathcal{O}_\pi(\underline{c})$$

is an isomorphism. In this case we shall call A the affine k-algebra of $\underline{c}$.

(ii) We shall consider the objects V_i of $\underline{c}$ as *points*, and the morphisms as *incidences* in our *geometry.*

(iii) Let for a point V_i of $\underline{c}$, $H_0(V_i)$ be the modular substratum of the local moduli $H(V_i)$ of V_i, as A-module. There exists a natural morphism,

$$\mathcal{O}_\pi(\underline{c}) \to H_0(V_i) \otimes End_k(V_i) = \mathcal{O}_\pi(V_i).$$

When $\underline{c}$ is a scheme we shall refer to $O_\pi(V_i)$ as the local ring of $\underline{c}$ at V_i.

This is in tune with the general setup, see the Introduction. In fact, according to the Introduction, a *space* $\underline{c}$ should be the moduli space of its points, subject only to the conditions imposed by the incidences, i.e. by the morphisms of $\underline{c}$, and this is exactely the property of a scheme $\underline{c}$. The k-algebra $A = \mathcal{O}_\pi(\underline{c})$ is the universal k-algebra parametrizing the (modular) deformations of its points, subject only to the conditions imposed by the incidences.

Note that if a diagram $\underline{c}$ of A-modules is a prescheme for A, then it is not necessarily a scheme for A. However, we have the following obvious,

Lemma 5.19. *(i) The diagram,*

$$\begin{tikzcd} \mathcal{O}(\underline{c},\pi) \arrow[r, two heads] & \mathcal{O}_\pi(\underline{c}) \arrow[d] \\ A \arrow[u] \arrow[r] & O^k(\underline{c},\pi) \end{tikzcd}$$

commutes.

(ii) If $\underline{c}$ is a prescheme for A and the homomorphism $A \to O_0(\underline{c},\pi) = O^k(\underline{c},\pi)$ is an isomorphism, then $\underline{c}$ is a scheme for A.

Now, by (5.17) (ii), there exist for every finite family of objects $\{V_l\}_l$ of $\underline{c}$, considered as a discrete subdiagram of $\underline{c}$, a natural morphism,

$$\mathcal{O}_\pi(\underline{c}) \to \mathcal{O}_\pi(\{V_l\}_l).$$

If all V_l's are simple A-modules, we find that $H_0(V_l) = H(V_l)$, and,

$$\mathcal{O}_\pi(\{V_l\}_l) = (H(\{V_l\}_l)_{i,j} \otimes Hom_k(V_i,V_j))$$

We shall refer to $\mathcal{O}_\pi(\{V_l\}_l)$ as the semilocal ring of $\underline{c}$ at the family $\{V_l\}_l$.

Generic points.

Consider for any finite type k-algebra A, the discrete diagram, $Simp(A)$, consisting of all the simple A-modules. If A is finite dimensional, we know that the morphism,

$$\eta(Simp(A)) : A \longrightarrow \mathcal{O}(Simp(A),\pi)$$

is an isomorphism provided k is algebraically closed. In general $\eta(Simp(A))$ is, however, far from an isomorphism. In fact, when A is commutative, it is easy to see that

$$\mathcal{O}(Simp(A),\pi) \simeq \prod_{\mathfrak{m}} \hat{A}_{\mathfrak{m}}$$

where $\mathfrak{m}$ runs through all maximal ideals of A. To obtain a result analogous to the General Burnside Theorem, see [La 7], we must add some *generic points* to $Simp(A)$, obtaining a diagram $Simp^*(A)$ such that $\mathcal{O}(Simp^*(A),\pi)$ becomes algebraic, and hopefully $\eta(Simp^*(A))$ an isomorphism. In fact, we should include modules like the components of A, as in (5.20), some *generating projectives*, or just A, see sections §3. and §4.

In the general case, given a diagram $\underline{c}$, we should be prepared to include in $\underline{c}$ a finite subset $\gamma(\underline{c})$ of *generic points*, and modify the construction of $\mathcal{O}_\pi(\underline{c})$, such that the result,

$$\mathcal{O}_\pi(\underline{c},\gamma(\underline{c}))$$

parametrizes the deformations of all points of $\underline{c}$, subject to the conditions imposed by the incidences and, moreover, *keeping the generic points fixed.*

This construction also turns out to give us the right tool to define a reasonable notion of noncommutative *subscheme.*

To obtain $\mathcal{O}_\pi(\underline{c},\gamma(\underline{c}))$ we copy the construction of $\mathcal{O}_\pi(\underline{c})$, replacing $O(|\underline{c}|,\pi)$ by the subring

$$O(|\underline{c}|,\gamma(\underline{c});\pi) = (H(|\underline{c}|,\gamma(\underline{c}))_{i,j} \otimes Hom_k(V_i,V_j))$$

where,

$$\begin{aligned}H(|\underline{c}|,\gamma(\underline{c}))_{i,j} =& H(|\underline{c}|)_{i,j} \text{ if } V_i \notin \gamma(\underline{c})\\ H(|\underline{c}|,\gamma(\underline{c}))_{i,j} =& 0 \text{ if } i \neq j,\ V_i \in \gamma(\underline{c})\\ H(|\underline{c}|,\gamma(\underline{c}))_{i,i} =& k \text{ if } V_i \in \gamma(\underline{c}),\end{aligned}$$

and, for every finite subdiagram $\underline{c}_0$ of $\underline{c}$, while insisting that $\underline{c}_0$ contain $\gamma(\underline{c})$, by replacing $H(\underline{c}_0)$ by $H(\underline{c}_0, \gamma(\underline{c}))$ given by the cocartesian diagram,

$$\begin{tikzcd}(Ext^1_A(U,V)^*) \arrow[r, hook] \arrow[d] & H(V) \arrow[r] & H(\underline{c}) \arrow[d] \\ k \arrow[rr] & & H(\underline{c}_0, \gamma(\underline{c})).\end{tikzcd}$$

Here,

$$U = \oplus_{V_i \in \gamma(\underline{c})} V_i \subseteq V = \oplus_{V_i \in |\underline{c}|} V_i$$

and $(Ext^1_A(U,V)^*)$ is the ideal of $H(V)$ generated by a dual basis of $Ext^1_A(U,V)$, considered as part of a dual basis of $Ext^1_A(V,V)$ generating $H(V)$.

So far we have defined a presheaf of observables on the family of subdiagrams of a permissible diagram $\underline{c}$ of A-modules. Starting with the *geometry*, i.e. the collection of points and incidences , we have thus defined, and studied, the algebra $\mathcal{O}_\pi(\underline{c})$ of *operators parametrizing the geometry.* When $\underline{c}$ is finite, the fact that $O(-,\pi)$ is a closure operator implies that the pair $(O := O(\underline{c},\pi), \underline{c})$ is a prescheme, such that $\eta : O \to O^O(\underline{c},\pi)$ is an isomorphism. If also $\eta_0 : O \to O^k(\underline{c},\pi)$ is an isomorphism it follows from (2.18) that $(O,\underline{c})$ is a scheme.

Now, if we start with a k-algebra A, how do we find its *scheme*? Obviously, there is no such thing, in this field, as a unique scheme associated to a given k-algebra. However we have the following examples:

Examples 5.20. (i) Let A be any k-algebra, and let $\underline{c} := Ind(A)$ be the diagram consisting of the essential (i.e. generating all others) morphisms between (all) the indecomposable A-modules. Suppose A is a sum of a finite number of indecomposibles. Using the fact that the only k-linear endomorphisms of A that are right A-linear, are the left multiplication by elements of A, we easily prove that $\eta_0 : A \to O_0(Ind(A),\pi)$ is an isomorphism. Since $Ind(A)$ is, essentially, a finite diagram, we may use the closure property of $\mathcal{O}$ and prove that $Ind(A)$ is a prescheme for A and therefore, by (5.18), also a scheme.

(ii) Consider a discrete diagram $\underline{c} = \mathcal{V}$ of A-modules. There is, by definition of $\mathcal{O}_\pi$, a homomorphism,

$$\mathcal{O}_\pi(\mathcal{V}) \longrightarrow (H_0(V) \otimes Hom_k(V_i, V_j)).$$

The tangent space of the image is

$$(E_{i,j} \otimes Hom_k(V_i, V_j)),$$

where the $E_{i,j}$ are defined by,

$$(E_{i,j}) = (Ext^1_A(V_i, V_j))^{End_A(V)}.$$

(iii) In particular, if $\mathcal{V}$ is the family of all simple right modules of an artinian k-algebra A, and k is algebraically closed, then $End_A(V)$ is trivial, and we obtain,

$$A \simeq \mathcal{O}_\pi(\mathcal{V})$$

so $\mathcal{V}$ is a scheme for A.

(iv) Going back to Example (5.13), where A was a local k-algebra with $\phi : A \to k$ as the residue map, we found that when $\phi_* : Ext^1_A(k, A) \to Ext^1_A(k, k)$ is zero, the natural morphism,

$$\eta : A \to \mathcal{O}(\{\phi\}, \pi)$$

is an isomorphism. It is now clear that, in all cases,

$$\eta : A \to \mathcal{O}_\pi(\{\phi\})$$

is an isomorphism, implying that $\{\phi\}$ is a scheme for A.

Example 5.21: The Hairy Line. Consider the k-algebra of matrices,

$$A = \begin{pmatrix} k[y] & k[y] \\ 0 & k[y] \end{pmatrix}.$$

The scheme $Ind(A)$ contains the two projectives,

$$V_{1,2} = \begin{pmatrix} k[y] & k[y] \\ 0 & 0 \end{pmatrix}, \; V_2 = \begin{pmatrix} 0 & 0 \\ 0 & k[y] \end{pmatrix}$$

such that $A = V_{1,2} \oplus V_2$. Therefore $Ind(A)$ is a scheme for A. Notice that there is an incidence, i.e. a morphism of (right) A-modules $V_2 \to V_{1,2}$, the cokernel of which is the indecomposible A-module,

$$V_1 = \begin{pmatrix} k[y] & 0 \\ 0 & 0 \end{pmatrix}.$$

Notice also that $A_i := V_i$, i=1,2 are quotient algebras of A both isomorphic to the polynomial k-algebra $k[y]$. The closed points of $Ind(A)$ are the different simple representations of A, that is, the different closed points of $Spec(A_i)$, for i=1,2. Thus we find that the closed points correspond to the points of two different affine lines, L_1 and L_2, both (canonically) isomorphic to $Spec(k[y])$. However, while there are no ordinary incidences between these points, there are infinitesimal incidences between pairs of *equal* points $(p, p) \in L_1 \times L_2$. In fact if $p \in L_1, \; q \in L_2$, then

$$\begin{aligned}
&Ext^1_A(k(p), k(p)) = k \\
&Ext^1_A(k(q), k(q)) = k \\
&Ext^1_A(k(p), k(q)) = k \text{ if } p = q \\
&Ext^1_A(k(p), k(q)) = 0, \;\text{ if }\; p \neq q \\
&Ext^1_A(k(q), k(p)) = 0, \text{ for all } \; p \in L_1, q \in L_2.
\end{aligned}$$

The picture of this is is an ordinary line L_2 with *hairs*, corresponding to the points of the other line L_1, stuck into the first line at the corresponding point.

Let as above, $Simp(A)$ be the discrete diagram of the simple A-modules, and let $Simp^*(A)$ be the diagram consisting of A and $Simp(A)$ together with the obvious morphisms. We claim that $Simp^*(A)$ is a scheme for A. In fact this follows from the following calculations: Let $p = q$, and let $\xi_{1,1} \in Ext^1_A(k(p), k(p))$, $\xi_{2,2} \in Ext^1_A(k(q), k(q))$ and $\xi_{1,2} \in Ext^1_A(k(p), k(q))$, be generators. Then the cup products $\xi_{1,1} \cup \xi_{1,2} \in Ext^2_A(k(p), k(q))$ and $\xi_{1,2} \cup \xi_{2,2} \in Ext^2_A(k(p), k(q))$ are equal and different from 0. Since $Hom_k(k, k) = k$ this implies,

$$\mathcal{O}_\pi(\{k(p), k(q)\}) = H(\{k(p), k(q)\}) = \begin{pmatrix} k[[y]] & k[[y]] \\ 0 & k[[y]] \end{pmatrix}.$$

and the natural morphism

$$\eta : A \to \mathcal{O}_\pi(\{k(p), k(q)\})$$

is the obvious completion. From this follows immediately,

$$\mathcal{O}_\pi(Simp^*(A)) = \begin{pmatrix} k[y] & k[y] \\ 0 & k[y] \end{pmatrix}.$$

Noncommutative subschemes.

Let $(\underline{c}_0, \gamma(\underline{c}_0))$ be a swarm of A-modules and its subset of generic points. Suppose first that $\gamma(\underline{c}_0) = \{V_0\}$, i.e. is reduced to one single object. Let $\rho : V_0 \to V_1$ be any homomorphism of A-modules. Consider now the morphisms $\phi : V_1 \to V_p$ of $\underline{C} = A - mod$, for which the composition $\rho\phi$ is a morphism of $\underline{c}_0$. The resulting diagram $\underline{c}_1$ with generic point $\gamma(\underline{c}_1)) = \{V_1\}$ should be considered as a *closed* subobject of $(\underline{c}_0, \gamma(\underline{c}_0))$ cut out by ρ. Notice that $(\underline{c}_1$ deprived of its generic point, is a subdiagram of $\underline{c}_0$. It is clear how to generalize this notion, but we shall leave that for later work. Consider the following example.

Example 5.22: The quantum plane. Let $A = k_q[x, y] := k\{x, y\}/(xy - qyx)$, k algebraically closed, with $q \neq 1$. The discrete diagram $Simp(A)$ may be identifyed with the union of the x-and the y-axes in the affine plane. Consider points $p_1, p_2 \in Simp(A)$, then we easily compute,

$$Ext^1_A(k(p_1), k(p_2)) = k,$$

if $p_1 = p_2 \neq 0$ or if p_1 and p_2 sit on the x-axis and $qp_2 = p_1$.

$$Ext^1_A(k(p_1), k(p_2)) = k$$

if p_1 and p_2 sit on the y-axis and $qp_1 = p_2$. In particular,

$$Ext^1_A(k(0), k(0)) = k^2$$
$$Ext^2_A(k(0), k(0)) = k.$$

All other $Ext's$ are 0, and there exists a basis $\{\xi_1, \xi_2\}$ of $Ext^1_A(k(0), k(0))$, such that

$$0 \neq \xi_1 \cup \xi_2 = q\xi_2 \cup \xi_1 \in Ext^2_A(k(0), k(0)).$$

Let $Simp^*(A)$ be the diagram $Simp(A)$ with A added. One easily proves that $\mathcal{O}_\pi(Simp^*(A))$ contains A, but is bigger. However, if we add $V_0 = A/(xy)$ to $Simp(A)$ as a generic point, we obtain a diagram $\underline{c}_0$, such that $A \simeq \mathcal{O}_\pi(\underline{c}_0, \{V_0\})$. Moreover, if $V_1 = A/(ax + by + c)$ is a line in the quantum plane, and if we consider the corresponding closed subobject $(\underline{c}_1, \{V_1\})$ of $(\underline{c}_1, \{V_1\})$, in the manner described above, we obtain, $\mathcal{O}_\pi(\underline{c}_1, \{V_1\}) \simeq A/(ax + by + c)$, as we would expect.

Espaces quantiques de Connes, trivial case. Let A be a commutative k-algebra, k algebraically closed. Consider an algebraic equivalence relation, $\underline{R} = Spec(R)$ on the affine scheme $X = Spec(A)$. It corresponds to the affine diagram,

$$A \overset{i_1}{\underset{i_2}{\rightrightarrows}} A \otimes A \xrightarrow{\rho} R \longrightarrow A$$

with the obvious relations. Consider the morphisms $\tau_i = i_i \circ \rho$, and let

$$Spec(A:R) = \{R/\tau_1(\mathfrak{p})R | \mathfrak{p} \in Spec(A)\}$$

be the diagram consisting of the objects $R/\mathfrak{p}R := R/\tau_1(\mathfrak{p})R$ considered as A-modules via τ_2, and with the obvious morphisms. Let $x \in X$ be a closed point, corresponding to the maximal ideal $\underline{m}_x$ of A, then the composition,

$$\kappa_x : A \xrightarrow{i_2 \circ \rho} R \xrightarrow{\pi_x} R/\underline{m}_x R$$

identifies $Spec(R/\underline{m}_x R)$ with the equivalence class $\bar{x}$ of $x \in X = Spec(A)$. In particular, if $x \sim_R y$ then

$$R/\underline{m}_x R = R/\underline{m}_y R$$

as objects of $Spec(A:R)$.

Now compute

$$O_0 := O_0(Spec(A:R), \pi).$$

It is clear that, $\alpha \in O_0$ is a family,

$$\alpha = \{\alpha_{\mathfrak{p}} \in End_k(R/\tau_1(\mathfrak{p})R) | \pi^{\mathfrak{p}}_{\mathfrak{q}} \circ \alpha_{\mathfrak{q}} = \alpha_{\mathfrak{p}} \circ \pi^{\mathfrak{p}}_{\mathfrak{q}},\ \forall\ \mathfrak{p}, \mathfrak{q} \in Spec(A)\}$$

where $\pi^{\mathfrak{p}}_{\mathfrak{q}} : R/\tau_1(\mathfrak{p})R \to R/\tau_1(\mathfrak{q})R$ is the obvious morphism corresponding to an inclusion $\mathfrak{p} \subseteq \mathfrak{q}$. Since each one of the $\pi^{\mathfrak{p}}_{\mathfrak{q}}$ is surjective, it is easy to see that α is determined by $\alpha_0 \in End_k(R)$ and the components $\alpha_x \in End_k(R/\underline{m}_x R)$. Moreover, since for $x \sim y$, $R/\underline{m}_x R = R/\underline{m}_y R$, as objects, it is clear that α_x depends only upon the equivalence class $\bar{x}$ of x. Each $\alpha_{\bar{x}}$ is a k-linear endomorphism. Suppose the equivalence classes of R are finite reduced, i.e. $R/\underline{m}_x R \simeq \oplus_{z \in \bar{x}} k(z)$, then $\alpha_{\bar{x}}$ is a matrix

$$\alpha_{\bar{x}} = (\alpha(z, z')),\ z, z' \in \bar{x},\ \alpha(z, z') \in k.$$

Moreover, in this case, given $\alpha, \alpha' \in O_0$ then $\alpha = \alpha'$ if and only if $\alpha_{\bar{x}} = \alpha'_{\bar{x}}$ for all closed points $x \in X$. This follows immediately from the defining relations, $\pi_x(\alpha(a)) = \alpha_{\bar{x}}(\pi_x(a))$. This means that α is determined as a *function* on $X \times X$ with $\alpha(z, z') = 0$ unless $z \sim z'$, and with addition defined as for functions, but with multiplication defined by the matrix nature of each $\alpha_{\bar{x}}$, i.e. $(\alpha\alpha')(x, y) = \sum_{z \in \bar{x} = \bar{y}} \alpha(x, z)\alpha'(z, y)$. But this is the way Connes defines *les espaces quantiques*, see [Connes].

Infinitesimal structures on schemes. Let $\underline{c}$ be a swarm of A-mod, and consider a point x=V_i . We would like to be able to talk about vectorfields, their values at points, if possible also about energy operators and time etc. as in quantum mechanics. We start with the following,

Definition 5.23. Given a point $x = V_i \in \underline{c}$, we put

$$
\begin{aligned}
T_x :=& Ext^1_A(V_i, V_i)^\Gamma : \\
=& \{\xi \in Ext^1_A(V_i, V_i) |\ \forall p \exists\ \xi_p \in Ext^1_A(V_p, V_p) \text{ such that } \forall \phi := \phi_{i,p} : V_i \to V_p,\ , \\
& \phi_*(\xi) = \phi^*(\xi_p) \text{ and } \forall \phi := \phi_{p,i} : V_p \to V_i,\ \phi_*(\xi_p) = \phi^*(\xi)\ \}
\end{aligned}
$$

and we shall call it the *tangent space* of $\underline{c}$ at x.

There is a canonical map

$$\kappa_x : Der_k(A, A) \longrightarrow T_x$$

the compositions of the natural maps,

$$Der_k(A, A) \longrightarrow Der_k(A, End_k(V))$$

the surjection

$$Der_k(A, End_k(V)) \longrightarrow Ext^1_A(V, V) = \oplus_{i,j} Ext^1_A(V_i, V_j)$$

and the projection onto $Ext^1_A(V_i, V_i)$. Recall that the tangent space T_0 of $H(\underline{c})$ is contained in,

$$T_0 = (Ext^1_A(V, V)^\Gamma)^{End_{k[\Gamma]\otimes A}(V)}$$

and see that there are natural homomorphisms,

$$Der_k(A, A) \to T_0.$$

For every $\delta \in Der_k(A, A)$, let $\delta(x) \in T_x$, be the image $\kappa_x(\delta)$ in T_x. Thus $Der_k(A, A)$ is a k-vectorspace of *vector fields* defined on $\underline{c}$. In particular if $E \in A$ is some element, then the k-linear map $adE : A \to A$ defined by $adE(a) = Ea - aE$, is a k-derivation of A. Obviously, $adE(x) = 0$ for every point x of $\underline{c}$. Thus, operators of this type (like time evolutions), i.e. $ad(E)$, although they act on the observables, leave the *points* fixed.

Definition 5.24. Given a point x=V_i of $\underline{c}$, we shall say that x is A-smooth if the map κ_x is surjective. If this is true for all points of $\underline{c}$, we shall say that $\underline{c}$ is A-smooth.

This generalizes the classical notion of smoothness in algebraic geometry. There is also the noncommutative version of Grothendiecks notion of smoothness, the Quillen, or Q-smoothness. The k-algebra A is Q-smooth if whenever $R \to S$ is a surjective small homomorphism of k-algebras, the map $Mor(A, R) \to Mor(A, S)$ is surjectiv. The following results are easily proved.

Proposition 5.25. *A is Q-smooth implies that for all finite dimensional simple A-module V, all cup and Massey products,*

$$Ext^1_A(V, V) \otimes Ext^1_A(V, V) \otimes \ldots Ext^1_A(V, V) \to Ext^2_A(V, V)$$

vanish.

From this we easily deduce the well known result,

Corollary 5.26. *If a finite type commutative k-algebra A is Q-smooth then either $A \simeq k^n$ for some n, or A is a smooth curve.*

§6. The commutative case.

The main Theorem. To show that the noncommutative algebraic geometry, introduced above, is a good extension of classical algebraic geometry, one would be tempted to prove that, for commutative k-algebras A,

$$\text{(S)} \qquad \eta(Spec(A),\pi) : A \longrightarrow \mathcal{O}_\pi(Spec(A))$$

is an isomorphism. This is, however, not reasonnable, see the Introduction. The problem is that $Spec(A)$ contains too many points. The closed points are special since $Spec(A)$ is the moduli space for such. The non closed points are not treated as bona fide points, but as *generic points* for subschemes. We would like to include as few as possible such generic points, and therefore we add only the projective generator of A. We have already proved the essentials of the following,

Theorem 6.1. *Let A be any commutative k-algebra, essentially of finite type, with k algebraically closed. Let, as above $Simp^*(A)$ be the diagram $Simp(A)$ augmented by the generic point A. Then the canonical morphisms of k-algebras,*

$$\eta(Simp^*(A),\pi) : A \to \mathcal{O}_\pi(Simp^*(A))$$

is an isomorphism.

Proof. For every closed point $x \in Simp(A)$ consider the corresponding homomorphisms of A-modules,

$$\phi_x : A \to k(x)$$

and use the Remark (5.6) together with see (5.20). Clearly the versal deformation of ϕ_x is the canonical morphism of k-algebras

$$\Phi_x : A \to H_{x,x} \otimes End_k(k(x)) = \hat{A}_x.$$

If $\alpha \in End_k(A)$ and $\alpha_x \in End_{H_{x,x}}(H_{x,x} \otimes End_k(k(x)))$ commute via the action of Φ_x, then the composition $(\alpha - R_{\alpha(1)}) \circ \Phi_x = 0$, where $R_{\alpha(1)}$ is the right multiplication by $\alpha(1)$. Since this is true for all $x \in Simp(A)$ we obtain that α is the right multiplication of some element $\alpha(1)$, proving our theorem. $\square$

We have also to show that the $\mathcal{O}_\pi$-construction applied to the obvious subdiagrams of the category $O_X - Mod$, where X is a k-scheme, gives us a globalization procedure.

According to paragraph (2) above, the globalization procedure is a consequence of the functorial properties of $\mathcal{O}_\pi$. To fix the ideas let us consider the following example.

Example 6.3 Blowing-ups. Let $A = k[x,y]$ and and consider the A-module $V = (x,y)$, i.e. the maximal ideal of A. The diagram of A-modules, $Simp^*(A - V)$ is, by definition, the diagram consisting of V together with all simple quotients of V. These correspond to all points of $\mathbf{A}^2 = Spec(A)$ different from (0,0), together with all tangent lines through (0,0) in $\mathbf{A}^2$. Now consider $x \in V$ and let $D(x)$ be the diagram obtained from $Simp^*(A - V)$ by localizing, i.e. removing the points where x becomes zero, and inverting the multiplicative action by x. In particular, in $D(x)$ we have all points of $\mathbf{A}^2$ minus the y-axis, but preserving all tangent lines through (0,0), except the x-axis. We find,

$$\mathcal{O}(Simp^*(A-V),\pi) = A,\ \mathcal{O}(D(x),\pi) = k[y,y/x]$$

as we should, proving that the scheme $(Simp^*(A-V),\mathcal{O}_\pi)$ is the blow up of the origin in $\mathbf{A}^2$

Notice that in this paragraph we have assumed that A is a commutative k-algebra essentially of finite type on an algebraically closed field. The extension of the theory to include schemes on general base rings, seems difficult.

§7. More homological preparations.

The category of A-G-modules. Let A be any k-algebra and let $g : A \to A$ be an automorphism. Given an A-module M_i, i=1,2 consider an automorphism of k-modules $\nabla_g^i : M_i \to M_i$, such that for $m_i \in M_i$ and $a \in A$ we have,

$$\nabla_g^i(m_i a) = \nabla_g^i(m_i)g(a) \text{ for i=1,2}$$

i.e. such that ∇_g^i is g-linear. Then there is an automorphism,

$$\theta_g^p := \theta_g^p(\nabla^1,\nabla^2) : Ext_A^p(M_1,M_2) \longrightarrow Ext_A^p(M_1,M_2)$$

induced via the isomorphism,

$$Ext_A^p(M_1,M_2) \simeq HH^p(A, Hom_k(M_1,M_2))$$

by the g^{-1}-linear automorphism of bi-modules,

$$\zeta_g : Hom_k(M_1,M_2) \longrightarrow Hom_k(M_1,M_2)$$

defined by,

$$\psi \longmapsto \nabla_g^1 \circ \psi \circ \nabla_{g^{-1}}^2.$$

Notice that we compose morphisms in the natural order. For $a \in A$ we compute,

$$\zeta_g(g(a)\psi) = \nabla_g^1 \circ g(a)\psi \circ \nabla_{g^{-1}}^2 = a(\nabla_g^1 \circ \psi \circ \nabla_{g^{-1}}^2) = a\zeta_g(\psi)$$

$$\zeta_g(\psi g(a)) = \nabla_g^1 \circ \psi g(a) \circ \nabla_{g^{-1}}^2 = (\nabla_g^1 \circ \psi \circ \nabla_{g^{-1}}^2)a = \zeta_g(\psi)a.$$

This implies that there is an automorphism of Hochschild cohomology,

$$\zeta_g^p : HH^p(A, Hom_k(M_1,M_2)) \longrightarrow HH^p(A, Hom_k(M_1,M_2))$$

defined on cochain form by,

$$\xi^p \longmapsto \{(a_1,a_2,\ldots,a_p) \mapsto \nabla_g^1 \circ \xi^p(g(a_1),\ldots,g(a_p)) \circ \nabla_{g^{-1}}^2\}.$$

In particular the automorphism,

$$\zeta_g^1 : Ext_A^1(M_1,M_2) \longrightarrow Ext_A^1(M_1,M_2)$$

is induced by the map

$$\zeta_g^1 : Der_k(A, Hom_k(M_1, M_2)) \longrightarrow Der_k(A, Hom_k(M_1, M_2))$$

defined by

$$\zeta_g^1(\delta)(a) = \nabla_g^1 \circ \delta(g(a)) \circ \nabla_{g^{-1}}^2 .$$

When $\mathfrak{p} \subseteq A$ is a g-invariant ideal of A contained in the annihilator of M_2, we know that the restriction of the derivations of $Der_k(A, Hom_k(M_1, M_2))$ to $\mathfrak{p}$ induces an isomorphism,

$$Hom_A(\mathfrak{p}/\mathfrak{p}^2, Hom_A(A/\mathfrak{p}, M_2)) \simeq Ext_A^1(A/\mathfrak{p}, M_2)$$

such that the automorphism ζ_g^1 takes the form,

$$\zeta_g^1(\psi)(x) = \nabla_{g^{-1}}^2(\psi(gx)) \text{ for } x \in \mathfrak{p}/\mathfrak{p}^2.$$

Suppose $\xi \in Ext_A^1(M_1, M_2)$ is represented by the exact sequence of A-modules,

$$(*) \qquad 0 \longrightarrow M_2 \longrightarrow E \longrightarrow M_1 \longrightarrow 0$$

Since the g-linear automorphisms $\nabla_g^i : M_i \to M_i$ correspond to an A-linear isomorphism,

$$\nabla_g^i : M_i \to M_i \otimes_{g^{-1}} A$$

we deduce from $(*)$ the exact sequence of A-modules,

$$(**) \qquad 0 \longrightarrow M_2 \otimes_{g^{-1}} A \longrightarrow E \otimes_{g^{-1}} A \longrightarrow M_1 \otimes_{g^{-1}} A \longrightarrow 0$$

which represents the element $\zeta_g^1(\xi) \in Ext_A^1(M_1, M_2)$. The ζ_g^1-invariant elements ξ of $Ext_A^1(M_1, M_2)$ therefore corresponds to the extensions $(*)$ for which there exists an isomorphism

$$(***) \qquad \nabla_g : E \longrightarrow E \otimes_{g^{-1}} A$$

compatible with the ∇_g^i, for i=1,2. Another way of viewing this is to look at $\zeta_g^1(\xi) - \xi$ as an obstruction for the existence of such an isomorphism $(**)$.

Given one $\nabla_g : E \longrightarrow E \otimes_{g^-1} A$ compatible with the ∇_g^i, another ∇_g‘ will differ from the first one by the composition Γ_g of the homomorphism $E \longrightarrow M_1$ and some A-linear map $\alpha : M_1 \to M_2 \otimes_{g^{-1}} A$, and any such Γ_g added to $(***)$, will again be compatible with the ∇_g^i, for i=1,2. In the category of (A-g)-modules, we therefore find,

$$Ext_{A-g}^1(M_1, M_2) \simeq Ext_A^1(M_1, M_2)^{\zeta_g} \oplus Hom_A(M_1, M_2 \otimes_{g^{-1}} A)/\sim$$

The equivalence $\sim$ identifies $(E‘, \nabla_g‘)$ and $(E“, \nabla_g“)$ if there exists an isomorphism of extensions $\zeta : E \simeq E“$ compatible with the ∇‘s. Since

$$\nabla_g^2 : Hom_A(M_1, M_2) \simeq Hom_A(M_1, M_2 \otimes_{g^{-1}} A)$$

the equivalence relation $\sim$ is trivial.

Now, suppose G is a group acting on the k-algebra A, i.e. suppose there exists a homomorphism of groups,

$$\rho : G \longrightarrow Aut_k(A).$$

Consider A-modules M_i, i=1,2, with G-actions compatible with ρ, i.e. homomorphisms

$$\nabla^i : G \longrightarrow Aut_k(M_i)$$

such that for $g \in G$, $m_i \in M_i$, and $a \in A$,

$$\nabla^i_g(m_i a) = \nabla^i_g(m_i)g(a) \text{ for i=1,2}$$

where we denote by g(a) the action of ρ(g) on $a \in A$.

Given an invariant $\xi \in Ext^1_A(M_1, M_2)$ under the action of the group G, as explained above, there exists for every $g \in G$ an isomorphism

$$\nabla_g : E \longrightarrow E \otimes_{g^{-1}} A$$

Since

$$(E \otimes_{g_1^{-1}} A) \otimes_{g_2^{-1}} A = E \otimes_{(g_1 g_2)^{-1}} A$$

we find an obstruction for the existence of a homomorphism of groups,

$$\nabla : G \longrightarrow Aut_k(E)$$

compatible with the $\nabla^{i'}$s, which is a 2-cocycle of G with values in the G-bimodule $Hom_A(M_1, M_2)$,

$$(g_1, g_2) \longmapsto (\nabla_{g_1} \circ \nabla_{g_2} - \nabla_{g_1 g_2}).$$

When the corresponding 2-class,

$$\sigma_\xi \in H^2(G, Hom_A(M_1, M_2))$$

vanishes, there exists a ∇ and the set of such will be a torsor under

$$H^1(G, Hom_A(M_1, M_2)).$$

Proposition 7.1. *Suppose $H^i(G, Hom_A(M_1, M_2)) = 0$ for i=1,2, then,*

$$Ext^1_{A-G}(M_1, M_2) \simeq Ext^1_A(M_1, M_2)^G.$$

Notice that a 1-coboundary of the form

$$g \longmapsto (g\alpha - \alpha)$$

corresponds to an automorphism $\theta_\alpha : E \longrightarrow E$ inducing an automorphism of (E, ∇_g).

The category of A- $\mathfrak{g}$-modules. Suppose

$$\rho : \mathfrak{g} \longrightarrow Der_k(A)$$

is a k-Lie homomorphism, e.g. a Lie-Cartan pair. We shall treat this as the tangent map of a Lie-group action ρ studied in the previous section. Let M_i, i=1,2 be A-modules with $\mathfrak{g}$-integrable connections

$$\nabla^i : \mathfrak{g} \longrightarrow End_k(M_i),$$

and consider for every $\delta \in \mathfrak{g}$ and every $\psi \in Hom_k(M_1, M_2)$ the map

$$\delta \longmapsto \nabla^1_\delta \psi - \psi \nabla^2_\delta.$$

This defines a Lie algebra homomorphism,

$$\rho : \mathfrak{g} \longrightarrow End_k(Hom_k(M_1, M_2))$$

such that, if ρ is a Lie-Cartan pair, $\rho(\delta a) = a\rho(\delta) - \rho(\delta)a$.

Let $D \in Der_k(A, Hom_k(M_1, M_2))$, then the map

$$a \longmapsto \nabla_\delta(D)(a) := D(\delta(a)) + \nabla^1_\delta D(a) - D(a)\nabla^2_\delta$$

is a derivation, and we obtain a connection

$$\nabla : \mathfrak{g} \longrightarrow End_k(Ext^1_A(M_1, M_2)).$$

As above, every $\xi \in Ext^1_A(M_1, M_2)^{\mathfrak{g}}$ is associated to an obstruction,

$$\sigma(\xi) \in H^2(\mathfrak{g}, Hom_k(M_1, M_2))$$

which vanishes if and only if there exists an integrable connection on the middle term E of the exact sequence representing ξ,

$$0 \longrightarrow M_2 \longrightarrow E \longrightarrow M_1 \longrightarrow 0$$

compatible with the connections ∇^i on M_i. The set of isomorphism classes of such (ξ, ∇) is then a torsor under

$$H^1(\mathfrak{g}, Hom_A(M_1, M_2)).$$

Proposition 7.2. *Suppose*

$$H^i(\mathfrak{g}, Hom_A(M_1, M_2)) = 0 \textit{ for i=1.2}$$

then,

$$Ext^1_{A-\mathfrak{g}}(M_1, M_2) = Ext^1_A(M_1, M_2)^{\mathfrak{g}}$$

§8. Invariant theory and moduli.

Consider a commutative finite type reduced k-algebra A, k algebraically closed. Suppose that there is a Lie-algebra $\mathfrak{g}$ of vectorfields (i.e. derivations), acting on A. Consider the category $\underline{C} = A - \mathfrak{g} - mod$, of A-modules with integrable $\mathfrak{g}$-covariant derivations. In this category we define the trivialization functor,

$$\pi_{\mathfrak{g}} : \underline{C} \longrightarrow k - mod$$

by

$$\pi_{\mathfrak{g}}(V) = H^0(\mathfrak{g}, V).$$

Recall that when $\mathfrak{g}$ is reductive $\pi := \pi_{\mathfrak{g}}$ is exact. Moreover, Ext, in this category, is then simply the $\mathfrak{g}$-invariants of the Ext in A-mod. The exactness of $\pi_{\mathfrak{g}}$, is, however, not necessary for the constructions we have in mind.

Given any permissible swarm $\underline{c}$ of $\underline{C}$, we may of course consider the ring of preobservables $O(\underline{c}, \pi_{\mathfrak{g}})$. In particular, let us consider the diagram, $Simp(A - \mathfrak{g})$, of simple $A - \mathfrak{g}$-modules, and add to it A, as a generic point, to obtain $Simp^*(A - \mathfrak{g})$. We shall want to construct the k-algebra of observables for this *space of orbits*.

We propose, tentatively,

Definition 8.1. The quotient $Spec(A)/\mathfrak{g}$, is the presheaf of k-algebras of observables,

$$\mathcal{O} := \mathcal{O}(-, \pi_{\mathfrak{g}})$$

defined on the subdiagrams of the diagram $Simp^*(A - \mathfrak{g})$

Notice that we do not assert that $Simp^*(A - \mathfrak{g})$, as a diagram of $A - \mathfrak{g} - mod$, is a *scheme* for $\mathcal{O}$. This would be meaningless. In good cases we may, however, hope that $Simp^*(\mathcal{O})$ is a scheme for $\mathcal{O}$, and that $Simp(\mathcal{O})$ is, in a natural way, *isomorphic* to $Simp(A - \mathfrak{g})$.

In many cases the simple $A - \mathfrak{g}$-modules occur in a finite number of families, corresponding to quotients of some $U_l = A/\mathfrak{a}_l$. In this case it is reasonable to add the U_l's as generic points to $Simp(A - \mathfrak{g})$ forming a new diagram $Simp^*(A - \mathfrak{g})$. We would then consider $\mathcal{O}_{\pi_{\mathfrak{g}}}(Simp^*(A - \mathfrak{g}), \{U_l\})$ as a candidate for the affine ring of the *orbit space*.

That this invariant theory fits with the classical invariant theory, is shown by the following result,

Theorem 8.2. *Let A be any irreducible and reduced commutative k-algebra of finite type, k algebraically closed, and $\mathfrak{g}$ a reductive Lie-algebra of vectorfields (i.e. derivations), acting on A. Assume that the geometric quotient of $Spec(A)$ with $\mathfrak{g}$ exists and is affine. Then*

$$Spec(A)/\mathfrak{g} = A^{\mathfrak{g}}.$$

Proof. By assumption, the diagram $Simp^*(A^{\mathfrak{g}})$ induces the diagram $Simp^*(A - \mathfrak{g})$. Moreover the trivialization $\pi_{\mathfrak{g}}$ maps the diagram $Simp^*(A - \mathfrak{g})$ onto the diagram $Simp^*(A^{\mathfrak{g}})$, or rather, to the image of this diagram under the canonical trivialization π. But then the exactnes of $\pi_{\mathfrak{g}}$ and the smoothness of the morphism of the geometric quotient, proves that the formal moduli of the family $|Simp^*(A - \mathfrak{g})|$ in the category of $A - \mathfrak{g}$-mod is isomorphic to the corresponding formal moduli of the family $|Simp^*(A^{\mathfrak{g}})|$ in the category of $A^{\mathfrak{g}}$-mod. Since the trivializations coincide, the Theorem (6.1) shows that

$$O(Simp^*(A-\mathfrak{g}),\pi_{\mathfrak{g}}) \cong A^{\mathfrak{g}}$$

which is exactly what we wanted. $\square$

Now, if we are given a group action by the group G on a scheme X, there are different notions of quotients X/G. Assuming that $X = Spec(A)$ is affine, we may consider the category $A-G-mod$ of $A(G)$-modules, see §7. In this category we have the diagram

$$Simp^*(A-G)$$

of simple $A(G)$-modules that are quotients of A, together with as many *generic* objects $V_0^p = A,\ p = 1,\dots,n$ as there are families of simple modules. We also have the obvious forgetful functor,

$$\pi_G := H^0(G,-):\ (A-G)-mod \longrightarrow k-mod.$$

There is now a reasonable definition for the quotient,

Definition 8.3. Suppose the group G is reductive. The quotient of Spec(A) with G, denoted by $Spec(A)/G$, is the presheaf of k-algebras of observables,

$$\mathcal{O}(-,\pi_G).$$

defined on subdiagrams of $Simp^*(A-G)$.

We also have the possibility to restrict, or extend the kind of orbits that we want to include in our diagram. Instead of the simple $(A-G)$-modules, $Simp(A-G)$, one might consider the orbits of maximal dimension, the semistable or the stable orbits (for a given polarization), the orbits of length equal to the order of G when G is a finite group, etc. In the last case, denote by $Simp^{s.s}(A-G)$ the diagram of quotients of A that are isomorphic to $k(G)$ as G-modules. Let $\underline{o} \subseteq Spec(A)$ be an orbit. Suppose first that the isotropy subgroup $I_x \subseteq G$ for one of the points $x \in \underline{o}$ is trivial. Then the affine ring o of $\underline{o}$ is the k-algebra,

$$o = \prod_{g\in G} k(g(x))$$

where $k(x)$ is the residue field of x. It is easy to see that,

$$Ext^1_{A(G)}(o,o) = Ext^1_A(k(x),k(x)) = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*.$$

Moreover, the morphism of $A-G-mod$,

$$A \longrightarrow \hat{H}(o)\otimes_k o$$

is simply the homomorphism,

$$A \longrightarrow \prod_{g\in G} \hat{A}_{\mathfrak{m}_{g(x)}}$$

From this one easily proves that,

$$O(Simp(A-G),\pi_G) = A^G.$$

The situation is far more complicated when the finite group G is not reductive.

Example 8.4 The McKay correspondence. Let k be an algebraically closed field of characteristic 0, and let U be a finite dimensional k-vectorspace. Put $n = dim_k U$. Let G be any group acting on U, and let $\{V_i\}_{i=1,\ldots,r}$ be the family of simple G-modules, such that

$$k(G) = \prod_{i=1,\ldots,r} V_i^{n_i}$$

as $k(G)$-modules. Let $A = Sym_k(U^*)$, and consider now a quotient module o of A which is isomorphic to $k(G)$ as G-module, i.e. an object of the diagram $Simp^{s.s}(A-G)$. According to the Remark (4.16), there must exist a filtration of o with graded module consisting of the different V_i strung out in some order, defined by an ordered graph Γ of length r. If $x \in \underline{o}$ is a closed point, and I_x the corresponding isotropy subgroup of G, we know that o has an $A-G$-quotient of the form,

$$A/\mathfrak{m} = \prod_{g \in I_x/G} k(g(x))$$

which is an object of $Simp^*(A-G)$. In this way we find a map,

$$\kappa : \ Simp^{s.s}(A-G) \longrightarrow Simp^*(A-G)$$

which is *the McKay map.* There are interesting cases in which

$$\mathcal{O}(-, \pi_G)$$

is a non singular presheaf of commutative k-algebras on the (*open*) subdiagrams of $Simp^{s.s}(A-G)$, with a canonical birational morphism,

$$\kappa^* : \ \mathcal{O}(Simp^*(A-G), \pi_G) = A^G \to \mathcal{O}(-, \pi_G)$$

showing that κ is a desingularization of $Spec(A)/G$. Suppose G acts effectively, such that $Simp^{s.s}(A-G) \cap Simp^*(A-G)$ is an open dense set of $Simp^*(A-G) \subseteq Spec(A^G)$, in the Zariski topology. Then o is a non singular point of $Simp^{s.s}(A-G)$, if and only if,

$$\begin{aligned} \chi(o,o) :&= \sum_{i=1,\ldots,n} (-1)^i ext^i_{A-G}(o,o) \\ &= \sum_{p=1,\ldots,n, i,j=1,\ldots,r} (-1)^p hom_G(V_i, V_j \otimes \wedge^p U) = 0. \end{aligned}$$

Notice that for the scheme to be commutative we must have,

$$ext^2_{A-G}(o,o) \geq dim_k(Ext^1_{A-G}(o,o) \wedge Ext^1_{A-G}(o,o)) \geq 1/2n(n-1).$$

Moreover, we always have,

$$hom_{A-G}(o,o) = 1.$$

If o is a non singular point of $Simp^{s.s}(A-G)$, then the tangent space $Ext^1_{A-G}(o,o)$ of o must be of dimension $n = dim_k U$. This proves the only if condition above. The *if* part, is a little more involved.

Remark 8.5. If the Lie algebra $\mathfrak{g}$ (or the Lie group G) is not reductive,

$$Ext^1_{A-\mathfrak{g}}(V_i, V_j) = H^0(\mathfrak{g}, Ext^1_A(V_i, V_j)) \oplus H^1(\mathfrak{g}, Hom_A(V_i, V_j))$$

where the component $H^1(\mathfrak{g}, Hom_A(V_i, V_j))$ may be non-zero. This component of the tangent space of the deformation functor measures the number of different $\mathfrak{g}$-structures on the same deformation of the family of A-modules $\{V_i\}_i$. If we are not interested in the specific $\mathfrak{g}$-structures, but only in isomorphism classes as A-modules, then we may suppress this component, by taking an appropriate quotient of the resulting H, and of $O(Simp(A_{\mathfrak{g}}, \pi)$, and we obtain a k-algebra

$$O^{red}(Simp(A - \mathfrak{g}), \pi)$$

which is simpler, therefore contains less information, but for which the set of simple modules is in one-to-one correspondence with the orbits of $\mathfrak{g}$. See §10 for examples.

§**9. Tensor products and quantum groups.** Let $\underline{c}$ be a subcategory of $A-mod$, with a trivializing functor π. Suppose given a tensor product on the category $\underline{c}$, i.e. a bi-functor

$$\otimes : \underline{c} \times \underline{c} \longrightarrow \underline{c}$$

which is a faithful imbedding, consistent with π, with some extra structure. In particular there should exist natural isomorphisms,

$$\alpha_{-,-,-} : ((- \otimes -) \otimes -) \simeq (- \otimes (- \otimes -))$$

satisfying the Mac Lane pentagon condition,

$$\begin{aligned} &id_X \otimes \alpha_{Y,Z,W} \circ \alpha_{X,Y\otimes Z,W} \circ \alpha_{X,Y,Z} \otimes id_W \\ &= \alpha_{X,Y,Z\otimes W} \circ \alpha_{X\otimes Y,Z,W}. \end{aligned}$$

Consider the exact functor,

$$\Delta : \underline{c} \to \underline{c} \times \underline{c},$$

defined by $\Delta(V) = V \times V$. Then, since all of these functors are imbeddings, there exist homomorphisms of k-algebras,

$$\mathcal{O}(\underline{c}, \pi) \to \mathcal{O}(\underline{c} \otimes \underline{c}, \pi) \simeq \mathcal{O}(\underline{c} \times \underline{c}, \pi) \to \mathcal{O}(\underline{c}, \pi) \tag{1}$$

such that,

$$\mathcal{O}(\underline{c} \times \underline{c}, \pi) \simeq \mathcal{O}(\underline{c}, \pi) \otimes \mathcal{O}(\underline{c}, \pi),$$

and such that the last morphism of (1) is the multiplication morphism of the k-algebra $\mathcal{O}(\underline{c}, \pi)$.

The Mac Lane pentagon condition garanties that the first morphism of (1) becomes an associative co-algebra structure on $\mathcal{O}(\underline{c}, \pi)$. Clearly any extra functorial *symmetry* one may want to consider on $\underline{c}$, will show up in the corresponding k-algebra $\mathcal{O}(\underline{c}, \pi)$.

§10. Examples.

The noncommutative projective line. Let $A = k[x_0, x_1]$, and consider the usual k^*-action. We shall compute the space Spec(A)/k^*. The subcategory Simp(A-k^*) of A-k^*-modules consists of the origin V_3, the lines through the origin $V_2(l)$, and the generic point $V_1 = A$. The trivializing functor (see §8.),

$$\pi : A - k^* - mod \longrightarrow k - mod$$

has the values,

$$\pi(V_1) = k,\ \pi(V_2(l)) = k,\ \pi(V_3) = k.$$

The noncommutative orbit space is given by the hull of the deformation functor, i.e. by $(H_{i,j})$. Since $H^p(k^*, -) = 0$ for p≥1, we may use the Proposition (7.2), and we obtain,

$$Ext_{A-k^*}(V_i, V_j) = Ext_A(V_i, V_j)^{k^*}.$$

It is easy to compute the different ext-groups, we find:

$$\begin{aligned}
&Ext^1_A(V_i, V_j) = 0, \text{for i=1, j=1,2,3.}\\
&Ext^1_A(V_2(l), V_1) = V_2(l) = A/(\alpha x_0 + \beta x_1)\\
&Ext^1_A(V_2(l), V_2(l)) = V_2(l)\\
&Ext^1_A(V_2(l), V_2(l`)) = 0 \ \text{ if } l \neq l`\\
&Ext^1_A(V_2(l), V_3) = V_3 = k\\
&Ext^1_A(V_3, V_1) = 0\\
&Ext^1_A(V_3, V_2(l)) = V_3 = k\\
&Ext^1_A(V_3, V_3) = k^2.
\end{aligned}$$

Using the results of §7. we obtain for the invariants

$$\begin{aligned}
&Ext^1_A(V_i, V_j)^{k^*} = 0, \ \text{for i=1, j=1,2,3.}\\
&Ext^1_A(V_2(l), V_1)^{k^*} = k \text{ represented by } \xi = l\\
&Ext^1_A(V_2(l), V_2(l))^{k^*} = k \text{ represented by } \xi = l\\
&Ext^1_A(V_2(l), V_2(l`))^{k^*} = 0 \text{ if } l \neq l`\\
&Ext^1_A(V_2(l), V_3)^{k^*} = 0\\
&Ext^1_A(V_3, V_1)^{k^*} = 0\\
&Ext^1_A(V_3, V_2(l))^{k^*} = 0\\
&Ext^1_A(V_3, V_3)^{k^*} = 0.
\end{aligned}$$

The corresponding quotient becomes the infinite matrix algebra of the form,

$$Spec(A)/k^* := O(Simp(A - k^*), \pi) = \begin{pmatrix} k & 0 & 0 \\ k[t_2(l)]t_{2,1} & k[t_2(l)] & 0 \\ 0 & 0 & k \end{pmatrix}$$

where l runs through all the points in the ordinary projective line. We observe that the special point, corresponding to the isolated orbit, i.e. the origo, stays isolated, even infinitesimally. There are, however, adjacencies between the formal points corresponding to the lines through the origin, and the *generic* point corresponding to the generic point of the ordinary projective line.

Suppose that we localize, say in x_0, i.e. that we restrict to the

$$Spec(A_{\{x_0\}} - k^*) = \{V_1 = A_{\{x_0\}}, V_2(l) = A_{\{x_0\}}/(l)\}$$

then we find,

$$\pi(V_1) = k[x_1/x_0],\ \pi(V_2(l)) = k,$$

for all l. The exts in the new category looks like,

$$\begin{aligned}
&Ext^1_{A_{\{x_0\}}}(V_i, V_j)^{k^*} = 0, \text{ for i=1, j=1,2.} \\
&Ext^1_{A_{\{x_0\}}}(V_2(l), V_1)^{k^*} = k \text{ represented by } \xi = l \\
&Ext^1_{A_{\{x_0\}}}(V_2(l), V_2(l))^{k^*} = k \text{ represented by } \xi = l \\
&Ext^1_{A_{\{x_0\}}}(V_2(l), V_2(l'))^{k^*} = 0 \text{ if } l \neq l'
\end{aligned}$$

With this we find that $Spec(A_{\{x_0\}})/k^*$ is the subalgebra,

$$\mathcal{O}(Simp^*(A_{\{x_0\}} - k^*), \pi) = \left\{ \begin{pmatrix} f(x_1/x_0) & 0 \\ \psi(f(x_1/x_0))t_{2,1} & f(x_1/x_0)] \end{pmatrix} \right\},$$

of,

$$\begin{pmatrix} End_k(k[x_1/x_0]) & 0 \\ k[[x_1/x_0]]t_{2,1} & k[[t_2(l)]] \end{pmatrix}$$

where ψ is some derivation of $Der_k(k[x_1/x_0])$ and f runs through $k[x_1/x_0]$, as expected, see the Theorem (8.2).

It is therefore clear that the noncommutative version of the projective line *contains* the geometric projective line.

If we consider, instead of the action by the group k^*, the action of the Lie algebra $\mathfrak{g}$ generated by the Euler vectorfield $\delta_0 = x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1}$, we get a different picture steming from the fact that $\mathfrak{g}$ has cohomology. The subcategory Spec(A-$\mathfrak{g}$) of A-$\mathfrak{g}$-modules consists of the origin V_3, the lines through the origin $V_2(l)$, and the generic point V_1. The trivializing functor

$$\pi : A - \mathfrak{g} - mod \longrightarrow k - mod$$

has the values,

$$\pi(V_1) = k,\ \pi(V_2(l)) = k,\ \pi(V_3) = k$$

Since there are no π-incidences, the noncommutative orbit space $Spec(A)/\mathfrak{g}$ is given by the hull of the deformation functor, i.e. by $(H_{i,j})$, as above. However, here we cannot use the result (8.2), since for most $\mathfrak{g}$-modules V, $H^1(\mathfrak{g}, V) = V/\delta_0 V \neq 0$. In fact we get,

$$\begin{aligned}
&Ext^1_{A-\mathfrak{g}}(V_i, V_j) = Ext^1_A(V_i, V_j)^{\mathfrak{g}} \oplus H^1(\mathfrak{g}, Hom_A(V_i, V_j)) \\
&Ext^2_{A-\mathfrak{g}}(V_i, V_j) = H^1(\mathfrak{g}, Ext^1_A(V_i, V_j))
\end{aligned}$$

This implies that

$$
\begin{aligned}
&Ext^1_{A-\mathfrak{g}}(V_1,V_j) = H^1(\mathfrak{g}, Hom_A(V_1,V_j)) = k \text{ for j=1,2,3.}\\
&Ext^1_{A-\mathfrak{g}}(V_2(l),V_j) = Ext^1_A(V_2,V_j)^{\mathfrak{g}} \oplus H^1(\mathfrak{g}, Hom_A(V_2,V_j))\\
&= k\oplus 0 \text{ for j=1}\\
&= k\oplus k \text{ for } V_j = V_2(l)\\
&= 0\oplus 0 \text{ for } V_j = V_2(l')\ l \neq l'\\
&= 0\oplus k \text{ for j=3}\\
&Ext^1_{A-\mathfrak{g}}(V_3,V_j) = Ext^1_A(V_3,V_j)^{\mathfrak{g}} \oplus H^1(\mathfrak{g}, Hom_A(V_3,V_j))\\
&= 0\oplus 0 \text{ for j=1}\\
&= 0\oplus 0 \text{ for j=2}\\
&= 0\oplus k \text{ for j=3}\\
&Ext^2_{A-\mathfrak{g}}(V_i,V_j) = H^1(\mathfrak{g}, Ext^1_A(V_i,V_j))\\
&= 0 \text{ for i=1, j=1,2,3.}\\
&= k \text{ for i=2, j=1}\\
&= k \text{ for} V_i = V_2(l), V_j = V_2(l)\\
&= 0 \text{ for } V_i = V_2(l), V_j = V_2(l'),\ l \neq l'\\
&= k \text{ for i=3, j=3.}
\end{aligned}
$$

It follows that $Spec(A)/\mathfrak{g}$ is given by the rather complicated looking k-algebra, generated by,

$$
\begin{pmatrix} k[[t_1]] & t_{1,2}(l) & t_{1,3}(l)\\ u_{2,1}(l) & k[[t_2(l),u_2(l)]] & t_{2,3}(l)\\ 0 & 0 & k[[t_3]] \end{pmatrix}
$$

with some relations.

The moduli space of simple singularities, the A_2 case. We shall consider the Weierstrass family $F := F(t_0,t_1,x,y) = x^3 - y^2 + t_1x + t_0$, parametrized by the k-algebra $A := k[t_0,t_1]$, and the corresponding Kodaira-Spencer kernel $\mathfrak{g} \subseteq Der_k(A)$ see [La-Pf], generated by,

$$
\begin{aligned}
\delta_0 &= 3t_0\frac{\partial}{\partial t_0} + 2t_1\frac{\partial}{\partial t_1}\\
\delta_1 &= 2t_1^2\frac{\partial}{\partial t_0} - 9t_0\frac{\partial}{\partial t_1}
\end{aligned}
$$

We claim that the *moduli space* consisting of the three singularities in the family F, is given as the quotient space $Spec(A)/\mathfrak{g}$. We must therfore consider the diagram $Simp(A-\mathfrak{g})$, consisting of the 3 $A-\mathfrak{g}$-modules,$V_1 = k[t_0,t_1]$, $V_2 = k[t_0,t_1]/(\Delta)$, where $\Delta = 27t_0^2 + 4t_1^3$ is the discriminant of F, and finally $V_3 = k$ coresponding to the origin.

As above we find that

$$\pi = H^0(\mathfrak{g}, -) : A - \mathfrak{g} - mod \longrightarrow k - mod$$

defines three points,

$$\pi(V_1) = k, \ \pi(V_2) = k, \ \pi(V_3) = k$$

with no incidences. Since it is easy to see that

$$H^2(\mathfrak{g}, Hom_A(V_i, V_j)) = 0$$

we find,

$$Ext^1_{A-\mathfrak{g}}(V_i, V_j) = H^0(\mathfrak{g}, Ext^1_A(V_i, V_j)) \oplus H^1(\mathfrak{g}, Hom_A(V_i, V_j))$$

which implies,

$$\begin{aligned}
&Ext^1_{A-\mathfrak{g}}(V_1, V_j) = H^1(\mathfrak{g}, Hom_A(V_1, V_j)) = k \text{ for j=1,2,3.}\\
&Ext^1_{A-\mathfrak{g}}(V_2, V_j) = H^0(\mathfrak{g}, Ext^1_A(V_2, V_j)) \oplus H^1(\mathfrak{g}, Hom_A(V_2, V_j))\\
&= 0 \oplus 0 \text{ for j=1}\\
&= 0 \oplus k \text{ for } j = 2\\
&= 0 \oplus k \text{ for j=3}\\
&Ext^1_{A-\mathfrak{g}}(V_3, V_j) = Ext^1_A(V_3, V_j)^{\mathfrak{g}} \oplus H^1(\mathfrak{g}, Hom_A(V_3, V_j))\\
&= 0 \oplus 0 \text{ for j=1}\\
&= 0 \oplus 0 \text{ for j=2}\\
&= 0 \oplus k \text{ for j=3.}
\end{aligned}$$

Moreover,

$$\begin{aligned}
&Ext^2_{A-\mathfrak{g}}(V_i, V_j) \subseteq H^0(\mathfrak{g}, Ext^2_A(V_i, V_j)) \oplus H^1(\mathfrak{g}, Ext^1_A(V_i, V_j))\\
&= 0 \text{ for all i,j=1,2,3.}
\end{aligned}$$

The moduli space is therefore given by the k-algebra (freely generated by),

$$O(Simp(A - \mathfrak{g}), \pi) = \begin{pmatrix} k[[t_{1,1}]] & t_{1,2} & t_{1,3} \\ 0 & k[[t_{2,2}]] & t_{2,3} \\ 0 & 0 & k[[t_{3,3}]] \end{pmatrix}$$

which has a *reduced* quotient, given by the matrices of the form,

$$\begin{pmatrix} k & kt_{1,2} & kt_{1,3} \oplus kt_{1,2}t_{2,3} \\ 0 & k & kt_{2,3} \\ 0 & 0 & k \end{pmatrix}$$

which is the k-algebra of the non-commuting adjacency diagram corresponding to the Weierstrass family, see [La-Pf],

$$\begin{aligned}
&t_{2,3} : \text{cusp} \to \text{node}\\
&t_{1,2} : \text{node} \to \text{ellipt}\\
&t_{1,2}t_{2,3} : \text{cusp} \longrightarrow \text{ellipt}\\
&t_{1,3} : \text{cusp} \to \text{ellipt}
\end{aligned}$$

Notice that $\mathfrak{g}$ is a rank 2 A-module, such that we may expect to find exact sequences of $A-\mathfrak{g}$-modules,

$$\begin{aligned}
&0 \longrightarrow A \longrightarrow \mathfrak{g} \longrightarrow A \longrightarrow 0\\
&0 \longrightarrow A/(\Delta) \longrightarrow \mathfrak{g} \otimes_A A/(\Delta) \longrightarrow A/(\Delta) \longrightarrow 0\\
&0 \longrightarrow A/(t_0,t_1) \longrightarrow \mathfrak{g} \otimes_A k(o) \longrightarrow A/(t_0,t_1) \longrightarrow 0
\end{aligned}$$

explaining the diagonal tangent structure of the moduli space,

$$\begin{pmatrix} k[[t_{1,1}]] & t_{1,2} & t_{1,3} \\ 0 & k[[t_{2,2}]] & t_{2,3} \\ 0 & 0 & k[[t_{3,3}]] \end{pmatrix}.$$

The reduced moduli is, however, given by,

$$O^{red}(Simp(A-\mathfrak{g}),\pi) = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix},$$

i.e. the obvious 3-point space, see Remark (8.5).

The moduli of endomorphisms. The dimension 2 case. We shall compute the (noncommutative) space of invariants

$$End_k(k^n)/Gl_n(k)$$

for $k = \mathbf{C}$ and n=2. (For $n \geq 3$ see a forthcoming paper by Arvid Siqveland). Refering to §8. it suffices, to compute the noncommutative formal moduli for the longest chain of infinitesimal incidences. Let

$$A = O_{End_k(k^2)}(End_k(k^2))$$

then

$$A = k[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]$$

The group $G := Gl_2(k)$ acts on $End_k(k^2)$ by conjugation, and there are two Jordan formes of interest,

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

and

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

corresponding to orbits V_2 of dimension 0 and V_1 of dimension 2, respectively, in $End_k(k^2)$. According to §8., since

$$H^0(G,V_i) = k,\ i = 1,2$$

we should expect $End_k(k^2)/Gl_2(k)$ to be an algebraization of the formal moduli of the family of A-G-modules $\{V_i\}_{i=1.2}$, i.e.

$$H := H(\{V_i\}_{i=1.2})$$

The orbits, as A-modules, are given by:

$$
\begin{aligned}
V_1 &= k[x_{i.j}]/(s_1 - 2\lambda, s_2 - \lambda^2) \\
V_2 &= k[x_{i.j}]/(x_{1,1} - \lambda, x_{1,2}, x_{2,1}, x_{2,2} - \lambda).
\end{aligned}
$$

Now it is clear that we may assume $\lambda = 0$. We find A-free resolutions,

$$
\begin{tikzcd}
V_1 & A \arrow[l] & A^2 \arrow[l, "{(s_1,s_2)}"'] \arrow[dl, shift right, "\psi(1)_1"'] \arrow[dl, shift left, "\psi(2)_1"] & A \arrow[l, "\begin{pmatrix} -s_2 \\ s_1 \end{pmatrix}"'] \arrow[dl, shift right, "\psi(1)_2"'] \arrow[dl, shift left, "\psi(2)_2"] & 0 \arrow[l] \\
V_1 & A \arrow[l] & A^2 \arrow[l] \arrow[dl, shift right, "\psi_{1,2}(1)_1"'] \arrow[dl, shift left, "\psi_{1,2}(2)_1"] & A \arrow[l] \arrow[dl, shift right, "\psi_{1,2}(1)_2"'] \arrow[dl, shift left, "\psi_{1,2}(2)_2"] & 0 \arrow[l] \\
V_2 & A \arrow[l] & A^4 \arrow[l, "\rho"] \arrow[dl, "\psi_1"'] & A^6 \arrow[l, "d_1"] \arrow[dl, "\psi_2"'] & A^4 \arrow[l] \\
V_2 & A \arrow[l] & A^4 \arrow[l, "\rho"] & A^6 \arrow[l, "d_1"] & A^4. \arrow[l]
\end{tikzcd}
$$

Here,

$$
\begin{aligned}
\psi(1)_1 &= (1,0) \\
\psi(2)_1 &= (0,1) \\
\psi(1)_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\psi(2)_2 &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\
\psi_{(1,2)}(1)_1 &= (1,0) \\
\psi_{(1,2)}(2)_1 &= (0,1) \\
\psi_{(1,2)}(1)_2 &= \begin{pmatrix} x_{2,2} \\ -x_{2,1} \\ 0 \\ 0 \end{pmatrix} \\
\psi_{(1,2)}(2)_2 &= \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}
\end{aligned}
$$

and

$$\rho = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{2,1} & x_{2,2} \end{pmatrix}$$
$$d_1 = \begin{pmatrix} x_{1,2} & x_{2,1} & x_{2,2} & 0 & 0 & 0 \\ -x_{1,1} & 0 & 0 & x_{2,1} & x_{2,2} & 0 \\ 0 & -x_{1,1} & 0 & -x_{1,2} & 0 & x_{2,2} \\ 0 & 0 & -x_{1,1} & 0 & -x_{1,2} & -x_{2,1} \end{pmatrix}$$
$$\psi_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$
$$\psi_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

From this diagram we easily compute $Ext^1_A(V_i, V_j)$, and since G is reductive, also $Ext^1_{A-G}(V_i, V_j) = Ext^1_A(V_i, V_j)^G$. We obtain,

$$Ext^1_{A-G}(V_i, V_j) = \begin{cases} = k^2 \text{ for} i = 1 \\ = k \text{ for} i = j = 2 \\ = 0 \text{ for} i = 2, j = 1 \end{cases}$$

which means that the tangent space of H is given by,

$$\begin{pmatrix} k^2 & k^2 \\ 0 & k^1 \end{pmatrix}.$$

Now,

$$Ext^2_{A-G}(V_1, V_1) = k \cdot \eta_{1,1}$$
$$Ext^2_{A-G}(V_1, V_2) = k \cdot \eta_{1,2}$$

and we compute the cup products and the Massey products of the basis elements of the $Ext^1_{A-G}(V_i, V_j) = Ext^1_A(V_i, V_j)^G$,

$$s_1^* \cup s_2^* = -s_2^* \cup s_1^* = \eta_{1,1}, s^* \cup s^* = 0$$
$$t_1^* \cup s^* = 0, t_2^* \cup s^* = -2 \cdot \eta_{1,2},$$
$$< t_1^*, s^*, s^* >= 1/2 \cdot \eta_{1,2}$$
$$s_1^* \cup t_1^* = 0, s_1^* \cup t_2^* = \eta_{1,2}, s_2^* \cup t_1^* = -\eta_{1,2}, s_2^* \cup t_2^* = 0,$$

This proves that in H there are relations of the form,

(rel) $$s_1 s_2 = s_2 s_1, \; 1/2 \cdot t_1 s^2 - 2 \cdot t_2 s + s_1 t_2 - s_2 t_1 = 0$$

Dividing the generators s_1 and s_2 by 2, keeping the notations, it follows that,

$$H(\{V_i\}) = \begin{pmatrix} k[[s_1, s_2]] & < t_1, t_2 > \\ 0 & k[[s]] \end{pmatrix}$$

subject to the relation $t_1s^2 - 4\cdot t_2s + 4\cdot s_1t_2 - 4\cdot s_2t_1 = 0$. From this it is clear that,

$$End_k(k^n)/Gl_n(k) = \tilde{H}(\{V_i\}) = \begin{pmatrix} k[s_1,s_2] & <t_1,t_2> \\ 0 & k[s] \end{pmatrix}$$

subject to the relation $t_1s^2 - 4\cdot t_2s + 4\cdot s_1t_2 - 4\cdot s_2t_1 = 0$.

The k-points of $End_k(k^n)/Gl_n(k)$ are therefore stratified into two strata, a plane parametrized by (s_1, s_2), corresponding to the semi-simple (max.dim.) orbits, and

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

and a line parametrized by (s), corresponding to the orbits of dimension 0, i.e. to the Jordan forms,

$$\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}.$$

There are two tangents from a point

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