ON THE CHARACTERISTIC FUNCTION OF A STRICTLY CONVEX DOMAIN AND THE FUBINI-PICK INVARIANT

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Introduction.

This paper is a continuation of the author's note [6] on the characteristic function and the associated differential equation defined on a strictly convex bounded domain in the euclidean space $\mathbf{R}^n$. We denote such a domain by Ω and assume that the boundary $\partial\Omega$ is smooth. Let Ω^* be the dual domain of Ω defined as the set $\mathrm{int}\{\xi \in \mathbf{R}^n ; 1 + \langle x,\xi\rangle \geq 0 , x \in \Omega\}$, $\langle,\rangle$ being the inner product. It is also a strictly convex domain and projecitvely equivalent to a bounded domain. Then the characteristic function χ_Ω is defined by

$$\chi_\Omega(x) = \int_{\Omega^*} n!(1+\langle x,\xi\rangle)^{-n-1}\, d\xi \tag{0.1}$$

It tends to infinity at the boundary. By the associated differential equation we mean an equation of Monge-Ampère type defined by

$$\begin{cases} I(u) := (-u)^{n+2} \det(u_{ij}) = 1 & \text{on } \Omega \\ u|_{\partial\Omega} = 0 . \end{cases} \tag{0.2}$$

The unique existence of a convex solution u, i.e. $u < 0$ and $(u_{ij}) > 0$, is known by S.Y. Cheng and S.T. Yau [2].

The purpose of this paper is to give a relation between these functions χ_Ω and u . The result is

Theorem. *There exists a smooth function* F *on* $\bar{\Omega}$ *such that*

$$\chi_\Omega = cu^{-n-1}\left(1 + \frac{5}{24(n-1)} Fu^2 + (\text{higher order of } u)\right) ,$$

where c *is a constant depending on* n . *The boundary value of* F *is the Fubini-Pick cubic invariant of the boundary.*

From this theorem follows

Corollary 2 [1]. *Assume the projective automorphism group is noncompact. Then it is an ellipsoid.*

We will define in §1 an approximate solution of (0.2). The process is very similar to that shown in [3]. In §2 we will give an expansion of χ with respect to u , where coefficients are computable by use of local geometric data of the boundary. In §3 we will explicitly compute the first non-trivial coefficient and prove Theorem.

Let us remark that the theorem is a real analogoue of the deep result due to C. Fefferman [4] on the Bergman kernel function on a strongly pseudoconvex domain. As was shown in [6], the equation (0.2) for the domain Ω is a restriction in simple way of a complex Monge-Ampère equation defined on the tube domain $V + i\mathbb{R}^{n+1}$, where V is the non-degenerate convex cone over the domain Ω . Note that a tube domain is not generally strongly pseudoconvex. It is easy to see that the expansion (2.4) given in §2 implies the expansion of the Bergman kernal function of this tube domain outside its Šilov boundary $V + i\{0\}$ with respect to the solution of this complex equation. However to give a geometric interpretation of this expansion is an open problem.

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§1. Approximate solutions.

We first modify the equation (0.2) introducing an operator

$$(1.1) \qquad K(v) = I(-\sqrt{-2v}) = -\begin{vmatrix} v_{ij} & v_i \\ v_j & 2v \end{vmatrix}$$

for a negative function v . Then the equation (0.2) is equivalent to the equation

$$(\#) \qquad \begin{cases} K(v) = 1 & \text{on } \Omega , \\ v\big|_{\partial\Omega} = 0 . & \end{cases}$$

We next fix a smooth defining function ϕ of the domain $\Omega : \Omega = \{\phi < 0\}$ and $(\phi_{ij}) > 0$, $d\phi \neq 0$ on the boundary. By (1.1) we have

$$K(\phi) = (\sum \phi^{ij}\phi_i\phi_j)\ \det(\phi_{ij}) + O(\phi) ,$$

where (ϕ^{ij}) is the inverse matrix of (ϕ_{ij}) . This shows $K(\phi)$ is positive at the boundary. Since ϕ satisfies the Dirichlet condition of (#) , the solution v may be supposed to be a slight modification of ϕ . We put

$$(1.2) \qquad w_1 = f\phi$$

for an undetermined function f . Then

$$(1.3) \qquad K(w_1) = -\begin{vmatrix} f\phi_{ij} + f_i\phi_j + f_j\phi_i + f_{ij}\phi & f\phi_i + f_i\phi \\ f\phi_j + f_j\phi & 2f\phi \end{vmatrix}$$

$$= - \begin{vmatrix} f\phi_{ij} + f_{ij}\phi & f\phi_i - f_i\phi \\ f\phi_j - f_j\phi & 2f\phi \end{vmatrix}$$

$$= f^{n+1}K(\phi) + O(\phi) .$$

If we define f by

(1.4) $f = K(\phi)^{-1/n+1}$,

then w_1 satisfies $K(w_1) = 1 + O(\phi)$. Assume here we have already obtained w_s with the property

(1.5) $w_s = f_s\phi$ and $K(w_s) = 1 + O(\phi^s)$,

f_s being smooth and positive near the boundary. Let us put

$$w_{s+1} = w_s + g(w_s)^{s+1}$$

and compute $K(w_{s+1})$. Denote w_s by w . Then

$$- K(w_{s+1}) = \begin{vmatrix} \begin{matrix} Lw_{ij} + (s+1)w^s(g_iw_j+g_jw_i) \\ +g_{ij}w^{s+1} + s(s+1)gw_iw_jw^{s-1} \end{matrix} & Lw_i + g_iw^{s+1} \\ Lw_j + g_jw^{s+1} & 2(w+gw^{s+1}) \end{vmatrix}$$

where $L = 1 + (s+1)gw^s$, which is invertible near the boundary. Hence

$$- K(w_{s+1}) = L^{n+1} \begin{vmatrix} \begin{matrix} w_{ij} + L^{-1} \{(s+1)w^s(g_iw_j+g_jw_i) \\ + s(s+1)gw^{s-1}w_iw_j\} \end{matrix} & w_i \\ w_j & 2L^{-1}w \end{vmatrix} + O(w^{s+1})$$

$$= L^{n+1}\left\{1 - 2s(s+1)L^{-1}gw^s\right\}(-K(w)) + O(w^{s+1}) \ .$$

Hence, by the assumption (1.5) for $w = w_s$,

$$K(w_{s+1}) = K(w) + (s+1)(n+1-2s)gw^s + O(w^{s+1}) \ .$$

Now define g by

$$(1.6) \qquad g = \frac{1 - K(w)}{(s+1)(n+1-2s)w^s}$$

unless $n+1-2s \neq 0$. Then (1.5) also holds for w_{s+1} . This argument shows

Proposition 1.

a) *Assume* n *is even. Then, for any* $s \geq 1$, *there exists a function* w *with the property* (1.5).

b) *Assume* n *is odd. Then there exists a function* w *with the property* (1.5) *for* $s \leq (n+1)/2$.

We call this w an *approximate solution* of (#) . Let v be a unique convex solution of (#) . By the convexity

$$(1.7) \qquad v = g\phi \quad \text{near} \quad \partial\Omega$$

for some positive function g . Assume an approximate solution w satisfies

$$w = v + h\phi^k \quad \text{for some} \quad k \geq 1 \quad \text{and} \quad K(w) = 1 + O(\phi^s).$$

Then the similar computation as above shows

$$K(w) = \left\{1 + k(n+3-2k)hg^{-1}\phi^{k-1}\right\} K(v) + O(\phi^k) \ .$$

Since this is equal to $1+O(\phi^{s})$, $h=O(\phi^{s+1-k})$ when $k-1<s$ and $n+3-2k\neq 0$. Therefore

<u>Proposition</u> 2.

a) *Assume* n *is even. Then, for any* s, *there exists an approximate solution* w *satisgying* $w=v+O(\phi^{s})$,

b) *Assume* n *is odd. Then there exists an approximate solution* w *satisfying* $w=v+O(\phi^{(n+3)/2})$.

The process defining w is dependent on the choice of ϕ. But this proposition implies that it is determined uniquely up to the ambiguity of order $O(\phi^{s})$ or $O(\phi^{(n+3)/2})$.

<u>Problem</u> 1. *Is the solution* u *smooth of class* C^{∞} *(resp.* $C^{n+1/2}$*) if* n *is even (resp. odd) ?*

§2. <u>The characteristic function of a strictly convex domain</u>.

The characteristic function χ_{Ω} is defined in Introduction. We introduce another function which we call the kernel function by

$$(2.1)\qquad k_{\Omega}(x) = \int_{\Omega^*}(2n+1)!\,(1+\langle x,\xi\rangle)^{-2n-2}\,\chi_{\Omega^*}(\xi)^{-1}\,d\xi\ .$$

The important property of these functions is that they are invariant under a projective transformation $A:\Omega_1 \longrightarrow \Omega_2 = A\Omega_1$ in the sense that

$$(2.2)\qquad \begin{aligned} \chi_{\Omega_2}(Ax) &= (\mathrm{Jac}\ A)^{-1}\,\chi_{\Omega_1}(x) \\ k_{\Omega_2}(Ax) &= (\mathrm{Jac}\ A)^{-2}\,k_{\Omega_1}(x)\ , \end{aligned}$$

where Jac A is the jacobian determinant. The solution v of (#) is also invariant:

$$(2.3)\qquad v_{\Omega_2}(Ax) = (\mathrm{Jac}\ A)^{2/n+1} v_{\Omega_1}(x),$$

see [6]. In [6] it is shown that χ and k have the following expansions.

$$(2.4)\qquad \begin{aligned} \chi(x) &= c_1 K(\phi)^{1/2}(-\phi)^{-(n+1)/2} + \textstyle\sum_{1\le j\le [n/2]} \delta_j (-\phi)^{j-(n+1)/2} + O(A(\phi)) \\ k(x) &= c_2 K(\phi)(-\phi)^{-n-1} + \textstyle\sum_{1\le j\le n} \varepsilon_j (-\phi)^{j-n-1} + O(\log|\phi|) \end{aligned}$$

where $A(\phi) = 1$ or $\log|\phi|$ according as n is even or odd respectively and c_i are constants depending on the dimension; $c_1 = (2\pi)^{(n-1)/2}\Gamma((n+1)/2)$, $c_2 = (n+1)!/2$. The boundary value of coefficients can be computed using derivatives of ϕ at the boundary. Since we may take an approximate solution w in §1 as a defining function, (2.4) holds for $\phi = w$. Then referring to Proposition 2, we have

$$(2.5)\qquad \begin{aligned} \chi(x) &= c_1(-v)^{-(n+1)/2} + \textstyle\sum_j P_j(-v)^{j-(n+1)/2} + O(B(v)) \\ k(x) &= c_2(-v)^{-n-1} + \textstyle\sum_j Q_j(-v)^{j-n-1} + O(C(v)) , \end{aligned}$$

where $B(v) = 1$ or $\log|v|$ and $C(v) = \log|v|$ or $v^{-(n+1)/2}$ according as n is even or odd respectively. The boundary values of P_j and Q_j are expressible by use of the derivatives of a defining function, i.e. by use of local geometric data of the boundary.

Problem 2. *In view of the projective invariance of* χ, k *and* v *the above coefficients are certain polynomials of the projective invariants of the boundary. Give a precise statement of this fact.*

In the next section we will compute P_1 explicitly and show that this is a fundamental projective invariant of the boundary.

§3. Explicit calculation of a coefficient.

Let us recall first a result in [6]. Fix a point p in the boundary and choose local coordinates $(x^1, \ldots, x^n)$ at p so that $\{x^n = 0\}$ is the tangent plane at p. We use the notation $y = x^n$ and $x = (x^1, \ldots, x^{n-1})$. Suppose the boundary is written as

$$(3.1)\qquad \begin{aligned} y &= -h(x) \\ h(x) &= \tfrac{1}{2}\delta_{ij}x^i x^j + \tfrac{1}{6}a_{ijk}x^i x^j x^k + \tfrac{1}{24}a_{ijk\ell}x^i x^j x^k x^\ell + O(|x|^5) \end{aligned}$$

Here $1 \leq i,j,\ldots \leq n-1$ and the summation convention is used. Performing a projective change of coordinates at p, we may assume

$$(3.2)\qquad \textstyle\sum_i a_{iij} = \sum_{i,j} a_{iijj} = 0 .$$

We define a scalar invariant F at p by

$$(3.3)\qquad F = \textstyle\sum_{i,j,k} a_{ijk}a_{ijk},$$

This is called the *Fubini-Pick cubic invariant* of the hypersurface $\partial\Omega$. It is covariant under a proj. change of of coordinates in the following sense. Take another coordinates satisfying the above conditions. It corresponds to a matrix of the form

$$(3.4)\qquad A = \begin{pmatrix} \lambda & & \\ b & a & \\ \mu & c & \lambda^{-1} \end{pmatrix}$$

where $\lambda, \mu \in \mathbf{R}$, $a \in O(n-1)$ and $b, c \in \mathbf{R}^{n-1}$ satisfying the relations ${}^t c = \lambda^{-1}ab$ and $\mu = \frac{1}{2}\lambda c\,{}^t c$. The transformation is given by $(x,y) \longrightarrow (a_i{}^j x^i + c^j y/\lambda + b_i x^i + \mu y\ ,\ \lambda^{-1}y/\lambda + b_i x^i + \mu y)$. The jacobian determinant at the origin is λ^{-n-1}. And we know that the Fubini-Pick invariant $\bar{F}$ in new coordinates is given by

$$(3.5)\qquad \bar{F} = \lambda^2 F .$$

See [7] for these facts. Now the result we need is

(3.6) $$\chi\big|_{x=0} = c_1(-y)^{-(n+1/2}(1 + \frac{F}{12(n-1)}y) + O(D(y)) ,$$

where $D(y) = y^{-(n-3)/2}$ for $n \geq 4$, $\log|y|$ for $n = 3$ and 1 for $n = 2$ (Theorem 6 in [6]) . Note that $F = 0$ when $n = 2$.

We next compute an approximate solution w at p . Define a defining function ϕ near p by

$$\phi = y + h + \frac{1}{2}(y + h)^2 .$$

Then $K(\phi) = (1+y+h)^{n-1}\det h_{ij}$. By the definitions (1.2) and (1.4)

(3.7) $$w_1 = f\phi , \quad f = (1+y+h)^{-n+1}(\det h_{ij})^{-1/n+1} .$$

To find w_2 we compute the $O(\phi)$-term of $K(w_1)$. Put

$$1 - K(w_1) = Q\phi + O(\phi^2) .$$

Then by (1.3) we see that

$$Q = -2f^{n+2} \det\phi_{\alpha\beta} + \sum_{1\leq\gamma\leq n} Q_\gamma + Q_{n+1} ,$$

where Q_γ is the determinant of the matrix $\begin{pmatrix} f\phi_{\alpha\beta} & f\phi_\alpha \\ f\phi_\beta & 0 \end{pmatrix}$ whose γ-th row is replaced by the vector $(f_{\gamma 1}, \ldots, f_{\gamma n}, -f_\gamma)$ and Q_{n+1} is that of the same matrix whose last row is replaced by $(-f_1, \ldots, -f_n, 2f)$. Here $1 \leq \alpha, \beta, \gamma \leq n$. We will now evaluate Q at p . Recalling definitions we see

$$h = h_i = 0 , \quad h_{ij} = \delta_{ij} , f = 1 , \quad f_i = 0, \quad f_n = \frac{n-1}{n+1}.$$

$$f_{ij} = \frac{n-1}{n+1}\delta_{ij} - \frac{1}{n+1}b_{ij}, \quad f_{in} = 0$$

$$\phi_i = \phi_{in} = 0, \quad \phi_n = \phi_{nn} = 1, \quad \phi_{ij} = \delta_{ij}$$

at the origin, where b_{ij} denotes a coefficient of $\det h_{ij}$: $\det h_{ij} = 1 + \frac{1}{2}b_{ij}x^i x^j + O(|x|^3)$. In view of (3.2) it is given by

(3.8) $$b_{ij} = \sum_k a_{ijkk} - 2\sum_{k,\ell} a_{ik\ell}\, a_{jk\ell} \,.$$

From these identities we see

$$\det \phi_{\alpha\beta}(p) = 1,\ Q_i(p) = -f_{ii},\ Q_n(p) = -\frac{n-1}{n+1},\ Q_{n+1}(p) = 2 - \frac{n-1}{n+1}$$

Hence, by use of (3.8) and (3.2),

$$Q(p) = \frac{(n-1)(n-3)}{n+1} - \frac{2}{n+1}F$$

Now by definition (1.6) we can easily see

$$w_2\big|_{x=0} = y - \frac{1}{n^2-1}Fy^2 + O(y^3) \ .$$

This leads to, by Proposition 2,

(3.9) $$v\big|_{x=0} = y - \frac{1}{n^2-1}Fy^2 + O(y^3) \ .$$

Then, combining this with (3.6), we have

(3.10) $$\chi = c_1(-v)^{-(n+1)/2}\left(1 - \frac{5F}{12(n-1)}v\right) + O(D(v))$$

on the line $x = 0$. Here note that the projective invariance (2.2) and (2.3) of χ and v implies that they change by multiplication of λ^{n+1} and λ^{-2} respectively under the coordinate change by (3.4). Together with the property (3.5) this shows that (3.10) is independent of the choice of coordinates and that, referring to the transformation (1.1), we have completed the proof of

<u>Theorem</u>. *Let* Ω *be a strictly convex bounded domain with smooth boundary. Then the characteristic function* χ_Ω *of the domain* Ω *is expanded by use of the convex solution* u *of the equation* (0.2) *and the Fubini-Pick invariant* F *of the boundary as follows.*

$$\chi_\Omega = c(-u)^{-n-1}\left(1 + \frac{5F}{24(n-1)}u^2\right) + O(E(u)) \ ,$$

where $c = 2^n\pi^{(n-1)/2}\Gamma((n+1)/2)$ *and* $E(u) = u^{-n+3}$ *for* $n \geq 4$, $\log|u|$ *for* $n = 3$ *and* 1 *for* $n = 2$.

Since the vanishing of the Fubini-Pick invariant characterizes locally an ellipsoid (L. Berwald, see [5]) for $n \geq 3$, we have

Corollary 1. *In addition to the assumptions of Theorem, assume that* χu^{n+1} *becomes constant near some open set* U *in the boundary and that* $n \geq 3$. *Then each connected component of* U *is a part of an ellipsoid.*

Especially we have

Corollary 2 ([1]). *In addition to the assumption of Theorem, assume that the projective automorphism group is noncompact and that* $n \geq 3$. *Then its boundary is an ellipsoid.*

Proof. By the projective invariance (2.2) and (2.3) the function χu^{n+1} must be constant everywhere.

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