Max-Planck-Institut für Mathematik
Bonn

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Max-Planck-Institut für Mathematik
Preprint Series 2010 (100)

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MPIM 10-100

The Szegö Kernel on a Sewn Riemann Surface

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November 9, 2010

Abstract

We describe the Szegö kernel on a higher genus Riemann surface in terms of Szegö kernel data coming from lower genus surfaces via two explicit sewing procedures where either two Riemann surfaces are sewn together or a handle is sewn to a Riemann surface. We consider in detail the examples of the Szegö kernel on a genus two Riemann surface formed by either sewing together two punctured tori or by sewing a twice-punctured torus to itself. We also consider the modular properties of the Szegö kernel in these cases.

1 Introduction

The purpose of this paper is to provide an explicit description of the Szegö kernel [Sz, HS, Sc, F1] on a higher genus Riemann surface in terms of Szegö kernel data coming from lower genus surfaces. We exploit two explicit sewing procedures where either two lower genus Riemann surfaces are sewn together or else a handle is sewn to a lower genus Riemann surface. We also consider in some detail the construction and modular properties of the Szegö kernel on a genus two Riemann surface formed either by sewing two tori together or by sewing a handle on to a torus. Our main motivation is

*Supported by a Science Foundation Ireland under the Research Frontiers Programme

to lay the foundations for the explicit construction of the partition and n-point correlation functions for a fermionic vertex operator super algebra on higher genus Riemann surfaces [TZ1, TZ2]. As such, this paper is a further development of the theory of partition and n-point functions on Riemann surfaces for vertex operator algebras (e.g. [FLM, Ka]) as described in [T, MT1, MT1, MT2, MT3, MT4, MTZ]. However, this present paper may also be of interest to readers outside the vertex operator algebra community.

We begin in Section 2 with a review of some basic aspects of the theory of Riemann surfaces [FK, Sp, F1, F2, Mu]. We then define and discuss properties of the Szegö kernel, which is a meromorphic $(\frac{1}{2}, \frac{1}{2})$ differential with a simple pole structure and prescribed multiplicities on the cycles of the Riemann surface [Sz, HS, Sc, F1].

In Section 3 we describe the Szegö kernel on a genus $g_1 + g_2$ Riemann surface $\Sigma^{(g_1+g_2)}$ obtained by sewing two lower genus Riemann surfaces $\Sigma^{(g_1)}$ and $\Sigma^{(g_2)}$. This is similar to the approach of refs. [Y] and [MT1], for computing the period matrix and other related structures on $\Sigma^{(g_1+g_2)}$ in terms of lower genus data. Following [MT1], we refer to this sewing scheme as the *ϵ-formalism* where ϵ is a complex sewing parameter which forms part of the data according to which the sewing is performed (see Figure 1 below). In particular, we introduce an infinite block matrix

$$Q = \begin{pmatrix} 0 & \xi F_1 \\ -\xi F_2 & 0 \end{pmatrix}, \tag{1}$$

where F_1 and F_2 are infinite matrices whose entries are certain weighted moments of the Szegö kernels on $\Sigma^{(g_1)}$ and $\Sigma^{(g_2)}$, respectively, and $\xi \in \{\pm\sqrt{-1}\}$. The matrix $I - Q$, where I is the infinite identity matrix, plays a crucial role here (and in the sequel [TZ1]). In particular, we show that $I - Q$ is invertible for small enough ϵ. $(I - Q)^{-1}$ then forms part of the expression of the genus $g_1 + g_2$ Szegö kernel in terms of the lower genus Szegö kernel data as proved in Theorem 3.6. In Theorem 3.9 we further show that the determinant $\det(I - Q)$ is well-defined and is a non-vanishing holomorphic function for small enough ϵ. Finally, we describe the example of the Szegö kernel on a genus two Riemann surface formed by sewing two tori and verify its modular transformation properties under the modular group which preserves the sewing scheme. This example is extensively exploited in [TZ1].

Section 4 is devoted to development of the corresponding formalism in the case that $\Sigma^{(g+1)}$ of genus $g + 1$ is obtained by self-sewing a handle to a genus

g Riemann surface $\Sigma^{(g)}$ with complex sewing parameter ρ. We refer to this as the *ρ-formalism*. This case is more technical due to the extra multiplicities on the two new cycles associated with the sewing handle. This leads us to introduce an analogue of (1), namely, an infinite matrix T whose entries are determined by weighted moments of certain genus g objects related to the Szegö kernel on $\Sigma^{(g)}$ and the new multiplicities. We show that $I - T$ is invertible for suitably small ρ and in Theorem 4.6 express the Szegö kernel on $\Sigma^{(g+1)}$ in terms of $(I-T)^{-1}$ and other genus g Szegö kernel data. In Theorem 4.7 we show that the determinant $\det(I-T)$ is well-defined and holomorphic for suitably small ρ. We conclude with two examples of sewing a handle to a Riemann sphere to obtain a torus and sewing a handle to a torus to obtain genus two Riemann surface. The modular transformation properties of the genus two Szegö kernel are also verified under the modular group preserving this ρ-sewing scheme. This example will be extensively exploited in [TZ2].

2 The Szegö Kernel on a Riemann Surface

Consider a compact Riemann surface Σ of genus g with canonical homology cycle basis $a_1, \ldots a_g, b_1, \ldots b_g$. In general there exists g holomorphic 1-forms ν_i, $i = 1, \ldots, g$ which we may normalize by (e.g. [FK, Sp])

$$\oint_{a_i} \nu_j = 2\pi i \delta_{ij}. \tag{2}$$

The genus g period matrix Ω is defined by

$$\Omega_{ij} = \frac{1}{2\pi i} \oint_{b_i} \nu_j, \tag{3}$$

for $i, j = 1, \ldots g$. Ω is symmetric with positive imaginary part i.e. $\Omega \in \mathbb{H}_g$, the Siegel upper half plane. The canonical intersection form on cycles is preserved under the action of the symplectic group $Sp(2g, \mathbb{Z})$ where

$$\begin{pmatrix} b \\ a \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{b} \\ \tilde{a} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z}). \tag{4}$$

This induces the modular action on $\mathbb{H}_g$

$$\Omega \rightarrow \tilde{\Omega} = (A\Omega + B)(C\Omega + D)^{-1}. \tag{5}$$

It is useful to introduce the *normalized differential of the second kind* defined by [Sp, Mu, F1]:

$$\omega(x,y) \sim \frac{dxdy}{(x-y)^2} \quad \text{for } x \sim y, \tag{6}$$

for local coordinates x, y, with normalization $\int_{a_i} \omega(x,\cdot) = 0$ for $i = 1, \dots g$. Using the Riemann bilinear relations, one finds that $\nu_i(x) = \oint_{b_i} \omega(x,\cdot)$.

We also introduce the *normalized differential of the third kind*

$$\omega_{p_2-p_1}(x) = \int_{p_1}^{p_2} \omega(x,\cdot), \tag{7}$$

for which $\oint_{a_i} \omega_{p_2-p_1} = 0$ and $\omega_{p_2-p_1}(x) \sim \frac{(-1)^a}{x-p_a}dx$ for $x \sim p_a$ and $a = 1, 2$.

We recall the definition of the theta function with real characteristics e.g. [Mu, F1, FK]

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\Omega) = \sum_{m \in \mathbb{Z}^g} \exp\left(i\pi(m+\alpha).\Omega.(m+\alpha) + (m+\alpha).(z+2\pi i\beta)\right), \tag{8}$$

for $\alpha = (\alpha_i), \beta = (\beta_i) \in \mathbb{R}^g$, $z = (z_i) \in \mathbb{C}^g$ and $i = 1, \dots g$ with

$$\begin{aligned} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z + 2\pi i(\Omega.r + s)|\Omega) &= e^{2\pi i\alpha.s} e^{-2\pi i\beta.r} e^{-i\pi r.\Omega.r - r.z} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\Omega), \\ \vartheta \begin{bmatrix} \alpha + r \\ \beta + s \end{bmatrix} (z|\Omega) &= e^{2\pi i\alpha.s} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\Omega), \end{aligned} \tag{9}$$

for $r, s \in \mathbb{Z}^g$.

There exists a (nonsingular and odd) character $\left[\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix}\right]$ such that [Mu, F1]

$$\vartheta \begin{bmatrix} \gamma \\ \delta \end{bmatrix} (0|\Omega) = 0, \qquad \partial_{z_i} \vartheta \begin{bmatrix} \gamma \\ \delta \end{bmatrix} (0|\Omega) \neq 0. \tag{10}$$

Let

$$\zeta(x) = \sum_{i=1}^{g} \partial_{z_i} \vartheta \begin{bmatrix} \gamma \\ \delta \end{bmatrix} (0|\Omega) \nu_i(x), \tag{11}$$

a holomorphic 1-form, and let $\zeta(x)^{\frac{1}{2}}$ denote the form of weight $\frac{1}{2}$ on the double cover $\widetilde{\Sigma}$ of Σ. We also refer to $\zeta(x)^{\frac{1}{2}}$ as a (double-valued) $\frac{1}{2}$-form on

Σ. We define the prime form $E(x,y)$ by[1]

$$E(x,y) = \frac{\vartheta\left[\begin{smallmatrix}\gamma\\ \delta\end{smallmatrix}\right]\left(\int_y^x \nu|\Omega\right)}{\zeta(x)^{\frac{1}{2}}\zeta(y)^{\frac{1}{2}}} \sim (x-y)dx^{-\frac{1}{2}}dy^{-\frac{1}{2}} \quad \text{for } x \sim y, \tag{12}$$

where $\int_y^x \nu = (\int_y^x \nu_i) \in \mathbb{C}^g$. $E(x,y) = -E(y,x)$ is a holomorphic differential form of weight $(-\frac{1}{2}, -\frac{1}{2})$ on $\widetilde{\Sigma} \times \widetilde{\Sigma}$. $E(x,y)$ has multipliers along the a_i and b_j cycles in x given by 1 and $e^{-i\pi\Omega_{jj} - \int_y^x \nu_j}$ respectively [F1].

The normalized differentials of the second and third kind can be expressed in terms of the prime form [Mu]

$$\omega(x,y) = \partial_x \partial_y \log E(x,y) dx dy, \tag{13}$$

$$\omega_{p-q}(x) = \partial_x \log \frac{E(x,p)}{E(x,q)} dx. \tag{14}$$

Conversely, we can also express the prime form in terms of ω by [F2]

$$E(x,y) = \lim_{p\to x,\ q\to y} \left[\sqrt{(x-p)(q-y)} \exp\left(-\frac{1}{2}\int_y^x \omega_{p-q}\right)\right] dx^{-\frac{1}{2}} dy^{-\frac{1}{2}}. \tag{15}$$

We define the Szegö Kernel [Sc, HS, F1] for $\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0|\Omega) \neq 0$ as follows

$$S\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x,y|\Omega) = \frac{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\int_y^x \nu|\Omega\right)}{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0|\Omega)E(x,y)}, \tag{16}$$

where $\theta = (\theta_i),\ \phi = (\phi_i) \in U(1)^n$ for

$$\theta_j = -e^{-2\pi i\beta_j}, \quad \phi_j = -e^{2\pi i\alpha_j}, \quad j = 1, \ldots g. \tag{17}$$

It follows from (9) that (16) is a function of $e^{2\pi i\alpha_i}$ and $e^{2\pi i\beta_i}$. The further factors of -1 in (17) are included for later convenience. The Szegö kernel has multipliers along the a_i and b_j cycles in x given by $-\phi_i$ and $-\theta_j$ respectively and is a meromorphic $(\frac{1}{2}, \frac{1}{2})$-form on $\widetilde{\Sigma} \times \widetilde{\Sigma}$ satisfying:

$$S\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x,y) \sim \frac{1}{x-y} dx^{\frac{1}{2}} dy^{\frac{1}{2}} \quad \text{for } x \sim y, \tag{18}$$

$$S\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x,y) = -S\begin{bmatrix}\theta^{-1}\\ \phi^{-1}\end{bmatrix}(y,x), \tag{19}$$

[1] Note that our definition differs from that of refs. [Mu, F1] by a factor of -1.

where $\theta^{-1} = (\theta_i^{-1})$ and $\phi^{-1} = (\phi_i^{-1})$. Note that the skew-symmetry property (19) implies $S\begin{bmatrix}\theta\\\phi\end{bmatrix}(x,y)$ has multipliers along the a_i and b_j cycles in y given by $-\phi_i^{-1}$ and $-\theta_j^{-1}$ respectively.

Finally, we describe the modular invariance of the Szegö kernel under the symplectic group $Sp(2g,\mathbb{Z})$ where we find [F1]

$$S\begin{bmatrix}\tilde{\theta}\\\tilde{\phi}\end{bmatrix}(x,y|\tilde{\Omega}) = S\begin{bmatrix}\theta\\\phi\end{bmatrix}(x,y|\Omega), \tag{20}$$

with $\tilde{\Omega}$ of (5) and where $\tilde{\theta}_j = -e^{-2\pi i\tilde{\beta}_j}$, $\tilde{\phi}_j = -e^{2\pi i\tilde{\alpha}_j}$ for

$$\begin{pmatrix}-\tilde{\beta}\\\tilde{\alpha}\end{pmatrix} = \begin{pmatrix}A & B\\C & D\end{pmatrix}\begin{pmatrix}-\beta\\\alpha\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-\mathrm{diag}(AB^T)\\\mathrm{diag}(CD^T)\end{pmatrix}, \tag{21}$$

where $\mathrm{diag}(M)$ denotes the diagonal elements of a matrix M.

For a Riemann surface of genus one described by an oriented torus $\mathbb{C}/\Lambda$ for lattice $\Lambda = 2\pi i(\mathbb{Z}\tau\oplus\mathbb{Z})$ for $\tau\in\mathbb{H}_1$, the genus one prime form is $E^{(1)}(x,y) = K(x-y,\tau)dx^{-\frac{1}{2}}dy^{-\frac{1}{2}}$ where

$$K(z,\tau) = \frac{\vartheta_1(z,\tau)}{\partial_z\vartheta_1(0,\tau)}, \tag{22}$$

for $z\in\mathbb{C}$ and $\tau\in\mathbb{H}_1$ and where $\vartheta_1(z,\tau) = \vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(z,\tau)$.

For $(\theta,\phi)\neq(1,1)$ with $\theta = -e^{-2\pi i\beta}$ and $\phi = -e^{2\pi i\alpha}$ the genus one Szegö kernel is

$$S^{(1)}\begin{bmatrix}\theta\\\phi\end{bmatrix}(x,y|\tau) = P_1\begin{bmatrix}\theta\\\phi\end{bmatrix}(x-y,\tau)dx^{\frac{1}{2}}dy^{\frac{1}{2}}, \tag{23}$$

where

$$\begin{aligned}P_1\begin{bmatrix}\theta\\\phi\end{bmatrix}(z,\tau) &= \frac{\vartheta\begin{bmatrix}\alpha\\\beta\end{bmatrix}(z,\tau)}{\vartheta\begin{bmatrix}\alpha\\\beta\end{bmatrix}(0,\tau)}\frac{1}{K(z,\tau)},\\ &= -\sum_{k\in\mathbb{Z}}\frac{q_z^{k+\lambda}}{1-\theta^{-1}q^{k+\lambda}},\end{aligned} \tag{24}$$

is a 'twisted' Weierstrass function [MTZ] for $q_z = e^z$ and with $\phi = \exp(2\pi i\lambda)$ for $0\leq\lambda<1$. The genus one modular group $SL(2,\mathbb{Z})$ acts in this case with

(20) and (21) following from

$$P_1\left(\gamma\begin{bmatrix}\theta\\ \phi\end{bmatrix}\right)(\gamma z|\gamma\tau) = (c\tau+d)P_1\begin{bmatrix}\theta\\ \phi\end{bmatrix}(z|\tau), \tag{25}$$

with

$$\gamma\tau = \frac{a\tau+b}{c\tau+d}, \quad \gamma z = \frac{z}{c\tau+d}, \tag{26}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ and

$$\gamma\begin{bmatrix}\theta\\ \phi\end{bmatrix} = \begin{bmatrix}\theta^a\phi^b\\ \theta^c\phi^d\end{bmatrix}. \tag{27}$$

We also have a Laurant expansion [MTZ]

$$P_1\begin{bmatrix}\theta\\ \phi\end{bmatrix}(z,\tau) = \frac{1}{z} - \sum_{n\geq 1} E_n\begin{bmatrix}\theta\\ \phi\end{bmatrix}(\tau)z^{n-1}, \tag{28}$$

for twisted Eisenstein series defined by

$$\begin{aligned} E_n\begin{bmatrix}\theta\\ \phi\end{bmatrix}(\tau) &= -\frac{B_n(\lambda)}{n!} + \frac{1}{(n-1)!}\sum_{r\geq 0}\frac{(r+\lambda)^{n-1}\theta^{-1}q^{r+\lambda}}{1-\theta^{-1}q^{r+\lambda}} \\ &\quad + \frac{(-1)^n}{(n-1)!}\sum_{r\geq 1}\frac{(r-\lambda)^{n-1}\theta q^{r-\lambda}}{1-\theta q^{r-\lambda}}, \end{aligned} \tag{29}$$

for $n \geq 1$ and where $B_n(\lambda)$ is the Bernoulli polynomial defined by

$$\frac{q_z^\lambda}{q_z - 1} = \frac{1}{z} + \sum_{n\geq 1}\frac{B_n(\lambda)}{n!}z^{n-1}.$$

For $(\theta,\phi) = (1,1)$ and $n \geq 2$ the twisted Eisenstein series reduce to the standard elliptic Eisenstein series with $E_n(\tau) = 0$ for n odd.

3 The Szegö Kernel on Two Sewn Riemann Surfaces

3.1 The ϵ-Formalism Sewing Scheme

We review the Yamada [Y] formalism for 'sewing' together two Riemann surfaces $\Sigma^{(g_a)}$ of genus g_a for $a = 1,2$ to form a surface of genus $g_1 + g_2$. Following [MT1], we refer to this sewing scheme as the ϵ-formalism.

Choose a local coordinate z_a on $\Sigma^{(g_a)}$ in the neighborhood of a point p_a, and consider the closed disk $|z_a| \leq r_a$ for $r_a > 0$, sufficiently small. Let ϵ be a complex sewing parameter with $|\epsilon| \leq r_1 r_2$ and excise the disk

$$\{z_a, |z_a| \leq |\epsilon|/r_{\bar{a}}\} \subset \Sigma^{(g_a)},$$

to form a punctured surface

$$\widehat{\Sigma}^{(g_a)} = \Sigma^{(g_a)} \backslash \{z_a, |z_a| \leq |\epsilon|/r_{\bar{a}}\}.$$

Here and below, we use the convention

$$\overline{1} = 2, \quad \overline{2} = 1.$$

Define the annulus $\mathcal{A}_a = \{z_a, |\epsilon|/r_{\bar{a}} \leq |z_a| \leq r_a\} \subset \widehat{\Sigma}^{(g_a)}$ and identify $\mathcal{A}_1$ and $\mathcal{A}_2$ as a single region $\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{A}_2$ via the sewing relation

$$z_1 z_2 = \epsilon. \tag{30}$$

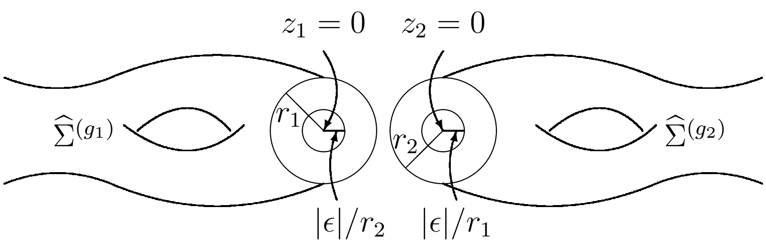

Fig. 1: Sewing Two Riemann Surfaces

In this way we obtain a compact Riemann surface $\Sigma^{(g_1+g_2)} = \{\widehat{\Sigma}^{g_1} \backslash \mathcal{A}_1\} \cup \{\widehat{\Sigma}^{(g_2)} \backslash \mathcal{A}_2\} \cup \mathcal{A}$ of genus $g_1 + g_2$. By construction, $\Sigma^{(g_1+g_2)}$ degenerates into $\Sigma^{(g_1)}$ and $\Sigma^{(g_2)}$ in the limit $\epsilon \to 0$.

The form $\omega^{(g_1+g_2)}$ on $\Sigma^{(g_1+g_2)}$ can be found in terms of data coming from $\omega^{(g_a)}$ on $\widehat{\Sigma}^{(g_a)}$ [Y]. $\Sigma^{(g_1+g_2)}$ inherits a homology cycle basis labeled $\{a_{s_1}, b_{s_1} | s_1 = 1, \ldots, g_1\}$ and $\{a_{s_2}, b_{s_2} | s_2 = g_1 + 1, \ldots, g_1 + g_2\}$ from $\Sigma^{(g_1)}$ and $\Sigma^{(g_2)}$ respectively. This allows us to compute the normalized 1-forms $\nu_i^{(g_1+g_2)}$ and the period matrix $\Omega_{ij}^{(g_1+g_2)}$. In particular, we find [Y, MT1]

Theorem 3.1 *$\omega^{(g_1+g_2)}$, $\nu_i^{(g_1+g_2)}$ and $\Omega_{ij}^{(g_1+g_2)}$ are holomorphic in ϵ for $|\epsilon| < r_1 r_2$ with*

$$\begin{aligned}
\omega^{(g_1+g_2)}(x,y) &= \delta_{ab}\omega^{(g_a)}(x,y) + O(\epsilon),\\
\nu_{s_b}^{(g_1+g_2)}(x) &= \delta_{ab}\nu_{s_a}^{(g_a)}(x) + O(\epsilon),\\
\Omega_{s_a t_b}^{(g_1+g_2)} &= \delta_{ab}\Omega_{s_a t_a}^{(g_a)} + O(\epsilon),
\end{aligned}$$

for $x \in \widehat{\Sigma}^{(g_a)}$, $y \in \widehat{\Sigma}^{(g_b)}$ and $a, b = 1, 2$ and where s_a, t_b label the inherited homology basis.

The explicit form of $\omega^{(g_1+g_2)}$, $\nu_i^{(g_1+g_2)}$ and $\Omega_{ij}^{(g_1+g_2)}$ is described in [Y, MT1].

3.2 The Szegö Kernel in the ϵ-Formalism

We now determine the Szegö kernel on the Riemann surface $\Sigma^{(g_1+g_2)}$ in terms of data coming from Szegö kernel

$$S^{(g_a)}(x,y) = S^{(g_a)}\begin{bmatrix}\theta^{(g_a)}\\ \phi^{(g_a)}\end{bmatrix}(x,y), \tag{31}$$

on the surface $\Sigma^{(g_a)}$ for $a = 1, 2$. We adopt the abbreviated notation of the left hand side of (31) when there is no ambiguity. Similarly, the Szegö kernel on $\Sigma^{(g_1+g_2)}$ is denoted by

$$S^{(g_1+g_2)}(x,y) = S^{(g_1+g_2)}\begin{bmatrix}\theta^{(g_1+g_2)}\\ \phi^{(g_1+g_2)}\end{bmatrix}(x,y), \tag{32}$$

with periodicities $(\theta_{s_a}^{(g_1+g_2)}, \phi_{s_a}^{(g_1+g_2)}) = (\theta_{s_a}^{(g_a)}, \phi_{s_a}^{(g_a)})$ on the inherited homology basis.[2]

We next describe $S^{(g_1+g_2)}(x,y)$ in terms of $S^{(g_a)}(x,y)$. We first show that

Theorem 3.2 *$S^{(g_1+g_2)}$ is holomorphic in $\epsilon^{\frac{1}{2}}$ for $|\epsilon| < r_1 r_2$ with*

$$S^{(g_1+g_2)}(x,y) = \begin{cases} S^{(g_a)}(x,y) + O(\epsilon), & x, y \in \widehat{\Sigma}^{(g_a)},\\ O(\epsilon^{\frac{1}{2}}), & x \in \widehat{\Sigma}^{(g_a)},\ y \in \widehat{\Sigma}^{(g_{\bar{a}})}.\end{cases}$$

[2]Note that we exclude those Riemann theta characteristics for which (32) exists but where either of the lower genus theta functions vanishes i.e. we assume that (31) exists for $a = 1, 2$.

Proof. Applying Theorem 3.1 to (8) we have

$$\vartheta\begin{bmatrix}\alpha^{(g_1+g_2)}\\ \beta^{(g_1+g_2)}\end{bmatrix}\left(z^{(g_1+g_2)}|\Omega^{(g_1+g_2)}\right) = \vartheta\begin{bmatrix}\alpha^{(g_1)}\\ \beta^{(g_1)}\end{bmatrix}\left(z^{(g_1)}|\Omega^{(g_1)}\right)\vartheta\begin{bmatrix}\alpha^{(g_2)}\\ \beta^{(g_2)}\end{bmatrix}\left(z^{(g_2)}|\Omega^{(g_2)}\right) + O(\epsilon), \tag{33}$$

with $(\alpha^{(g_1+g_2)}) = (\alpha_1^{(g_1)}, \ldots \alpha_{g_1}^{(g_1)}, \alpha_1^{(g_2)}, \ldots \alpha_{g_2}^{(g_2)})$ etc. We firstly show that the genus $g_1 + g_2$ prime form obeys

$$E^{(g_1+g_2)}(x,y) = \begin{cases} E^{(g_a)}(x,y) + O(\epsilon), & x, y \in \widehat{\Sigma}^{(g_a)}, \\ O(\epsilon^{-\frac{1}{2}}), & x \in \widehat{\Sigma}^{(g_a)},\ y \in \widehat{\Sigma}^{(g_{\bar{a}})}. \end{cases} \tag{34}$$

For the genus $g_1 + g_2$ odd characteristic of (10) we find from (33) that either $\vartheta\begin{bmatrix}\gamma^{(g_1)}\\ \delta^{(g_1)}\end{bmatrix}(0) \neq 0$ or $\vartheta\begin{bmatrix}\gamma^{(g_2)}\\ \delta^{(g_2)}\end{bmatrix}(0) \neq 0$ on the lower genus surfaces. Hence it follows that $\zeta^{(g_1+g_2)}(x)\zeta^{(g_1+g_2)}(y) = O(\epsilon)$ for $x \in \widehat{\Sigma}^{(g_a)}$, $y \in \widehat{\Sigma}^{(g_{\bar{a}})}$ for the 1-form (11). We also note that

$$\int_y^x \nu_{s_b}^{(g_1+g_2)} = \begin{cases} \int_y^x \nu_{s_a}^{(g_a)} + O(\epsilon), & x, y \in \widehat{\Sigma}^{(g_a)}, \\ \delta_{ab}\int_{p_a}^x \nu_{s_a}^{(g_a)} + \delta_{\bar{a}b}\int_y^{p_{\bar{a}}} \nu_{s_{\bar{a}}}^{(g_{\bar{a}})} + O(\epsilon), & x \in \widehat{\Sigma}^{(g_a)},\ y \in \widehat{\Sigma}^{(g_{\bar{a}})}, \end{cases} \tag{35}$$

from which it follows that $E^{(g_1+g_2)}(x,y) = O(\epsilon^{-\frac{1}{2}})$ for $x \in \widehat{\Sigma}^{(g_a)}$ and $y \in \widehat{\Sigma}^{(g_{\bar{a}})}$.

We next determine $E^{(g_1+g_2)}(x,y)$ for $x, y \in \widehat{\Sigma}^{(g_a)}$. The differential $\omega^{(g_1+g_2)}$ for $x, y \in \widehat{\Sigma}^{(g_a)}$ obeys

$$\omega^{(g_1+g_2)}(x,y) - \omega^{(g_a)}(x,y) = a_a(x)X_{\bar{a}\bar{a}}a_a^T(y) = O(\epsilon), \tag{36}$$

where $a_a(x)X_{\bar{a}\bar{a}}a_a^T(y) = \sum_{k,l\geq 1} a_a(x,k)X_{\bar{a}\bar{a}}(k,l)a_a(y,l)$ with $a_a(x,k)$ a certain 1-form on $\widehat{\Sigma}^{(g_a)}$ and $X_{\bar{a}\bar{a}}(k,l)$ an infinite matrix determined from genus g_1 and g_2 data (see [MT1] for details). It follows from (15) that

$$E^{(g_1+g_2)}(x,y) = E^{(g_a)}(x,y)e^{-\frac{1}{2}b_a X_{\bar{a}\bar{a}} b_a^T} = E^{(g_a)}(x,y) + O(\epsilon),$$

where $b_a(k) = \int_y^x a_a(\cdot, k)$. Thus (34) holds. We then apply Theorem 3.1, (33), (34) and (35) to (16) to prove the result. □

We next remark that for $x, z_a \in \widehat{\Sigma}^{(g_a)}$ then $S^{(g_a)}(x, z_a)S^{(g_1+g_2)}(z_a, y)$ is a meromorphic 1-form (cf. [HS]) in z_a periodic on the $\Sigma^{(g_a)}$ cycles (cf. (19))

with simple poles described by (18) where

$$\begin{aligned} S^{(g_a)}(x,z_a)S^{(g_1+g_2)}(z_a,y) &\sim \frac{dz_a}{x-z_a}S^{(g_1+g_2)}(x,y) \text{ for } z_a\sim x,\\ S^{(g_a)}(x,z_a)S^{(g_1+g_2)}(z_a,y) &\sim \frac{dz_a}{z_a-y}S^{(g_a)}(x,y) \text{ for } z_a\sim y \text{ if } y\in\widehat{\Sigma}^{(g_a)}. \end{aligned} \quad (37)$$

A similar behavior holds for $S^{(g_1+g_2)}(x,z_b)S^{(g_b)}(z_b,y)$ as a meromorphic 1-form in z_b. This allows us to determine the following integral equations

Proposition 3.3 *The Szegö kernel on $\Sigma^{(g_1+g_2)}$ is given by*

$$\begin{aligned} S^{(g_1+g_2)}(x,y) &= \delta_{ab}S^{(g_a)}(x,y) - \frac{1}{2\pi i}\oint_{\mathcal{C}_a(z_a)} S^{(g_a)}(x,z_a)S^{(g_1+g_2)}(z_a,y), \quad (38)\\ &= \delta_{ab}S^{(g_a)}(x,y) + \frac{1}{2\pi i}\oint_{\mathcal{C}_b(z_b)} S^{(g_1+g_2)}(x,z_b)S^{(g_b)}(z_b,y), \quad (39) \end{aligned}$$

for $x\in\widehat{\Sigma}^{(g_a)}$, $y\in\widehat{\Sigma}^{(g_b)}$ for $a,b=1,2$ and where $\mathcal{C}_a(z_a)\subset\mathcal{A}_a$ denotes a closed anti-clockwise oriented contour parameterized by z_a surrounding the puncture at $z_a=0$ on $\widehat{\Sigma}^{(g_a)}$.

Proof. Let σ_a be a contour on $\widehat{\Sigma}^{(g_a)}$ surrounding $\mathcal{A}_a$ and the given points x (and y, if $a=b$) on $\widehat{\Sigma}^{(g_a)}$ (see Fig. 2).[3]

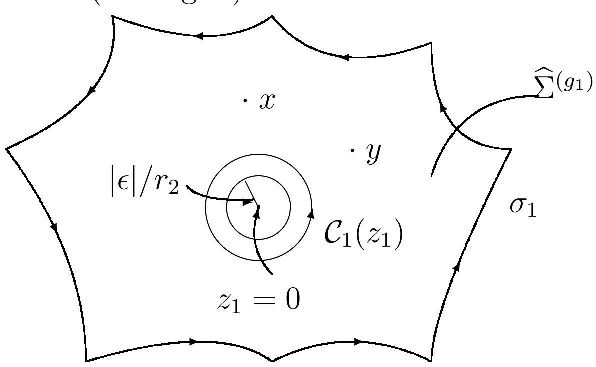

Fig. 2: Example with $x,y\in\widehat{\Sigma}^{(g_1)}$

[3] σ_a may be construed as being the boundary of the simple-connected covering space for $\Sigma^{(g_a)}$ as illustrated in Fig. 2 for a genus two surface.

From Cauchy's theorem $\oint_{\sigma_a} S^{(g_a)}(x,z_a)S^{(g_1+g_2)}(z_a,y) = 0$ and hence (37) gives

$$0 = -S^{(g_1+g_2)}(x,y) + \delta_{ab}S^{(g_a)}(x,y) + \frac{1}{2\pi i}\oint_{\mathcal{C}_a(z_a)} S^{(g_a)}(x,z_a)S^{(g_1+g_2)}(z_a,y),$$

giving (38). Considering $S^{(g_1+g_2)}(x,z_b)S^{(g_b)}(z_b,y)$ leads to (39). □

Similarly to [MT1] we define weighted moments for $S^{(g_1+g_2)}$ by

$$\begin{aligned} X_{ab}(k,l,\epsilon) = \quad & X_{ab}\begin{bmatrix}\theta^{(g_1+g_2)}\\ \phi^{(g_1+g_2)}\end{bmatrix}(k,l,\epsilon) \\ = \quad & \frac{\epsilon^{\frac{1}{2}(k+l-1)}}{(2\pi i)^2}\oint_{\mathcal{C}_a(x)}\oint_{\mathcal{C}_b(y)} x^{-k}y^{-l}S^{(g_1+g_2)}(x,y)dx^{\frac{1}{2}}dy^{\frac{1}{2}}, \end{aligned} \quad (40)$$

for $k,l\geq 1$. From (19) it follows that

$$X_{ab}\begin{bmatrix}\theta^{(g_1+g_2)}\\ \phi^{(g_1+g_2)}\end{bmatrix}(k,l,\epsilon) = -X_{ba}\begin{bmatrix}(\theta^{(g_1+g_2)})^{-1}\\ (\phi^{(g_1+g_2)^{-1}})\end{bmatrix}(l,k,\epsilon). \quad (41)$$

We denote by $X_{ab} = (X_{ab}(k,l,\epsilon))$ the infinite matrix indexed by $k,l\geq 1$.

We also define various moments for $S^{(g_a)}(x,y)$. These provide the data used to construct $S^{(g_1+g_2)}(x,y)$. Define holomorphic $\frac{1}{2}$-forms on $\widehat{\Sigma}^{(g_a)}$ by

$$h_a(k,x,\epsilon) = \quad h_a\begin{bmatrix}\theta^{(g_a)}\\ \phi^{(g_a)}\end{bmatrix}(k,x,\epsilon) = \frac{\epsilon^{\frac{k}{2}-\frac{1}{4}}}{2\pi i}\oint_{\mathcal{C}_a(z_a)} S^{(g_a)}(x,z_a)z_a^{-k}dz_a^{\frac{1}{2}}, \quad (42)$$

$$\bar{h}_a(k,y,\epsilon) = \quad \bar{h}_a\begin{bmatrix}\theta^{(g_a)}\\ \phi^{(g_a)}\end{bmatrix}(k,y,\epsilon) = \frac{\epsilon^{\frac{k}{2}-\frac{1}{4}}}{2\pi i}\oint_{\mathcal{C}_a(z_a)} S^{(g_a)}(z_a,y)z_a^{-k}dz_a^{\frac{1}{2}}, \quad (43)$$

and introduce infinite row vectors $h_a(x) = (h_a(k,x))$, $\bar{h}_a(x) = (\bar{h}_a(k,x))$ indexed by $k\geq 1$. From (19) it follows that

$$\bar{h}_a\begin{bmatrix}\theta^{(g_a)}\\ \phi^{(g_a)}\end{bmatrix}(k,x,\epsilon) = -h_a\begin{bmatrix}(\theta^{(g_a)})^{-1}\\ (\phi^{(g_a)})^{-1}\end{bmatrix}(k,x,\epsilon). \quad (44)$$

Finally, we define the moment matrix

$$
\begin{aligned}
F_a(k,l,\epsilon) = \quad & F_a \begin{bmatrix} \theta^{(g_a)} \\ \phi^{(g_a)} \end{bmatrix} (k,l,\epsilon) \\
= \quad & \frac{\epsilon^{\frac{1}{2}(k+l-1)}}{(2\pi i)^2} \oint_{\mathcal{C}_a(x)} \oint_{\mathcal{C}_a(y)} x^{-k} y^{-l} S^{(g_a)}(x,y) dx^{\frac{1}{2}} dy^{\frac{1}{2}} \\
= \quad & \frac{\epsilon^{\frac{k}{2}-\frac{1}{4}}}{2\pi i} \oint_{\mathcal{C}_a(x)} x^{-k} h_a(l,x) dx^{\frac{1}{2}} = \frac{\epsilon^{\frac{l}{2}-\frac{1}{4}}}{2\pi i} \oint_{\mathcal{C}_a(y)} y^{-l} \bar{h}_a(k,y) dy^{\frac{1}{2}}. \quad (45)
\end{aligned}
$$

$F_a(k,l,\epsilon)$ obeys a skew-symmetry property from (19) similar to (41). We may invert (42)-(45) using (18) to find for $x, y \in \widehat{\Sigma}^{(g_a)}$ that

$$
\begin{aligned}
S^{(g_a)}(x,y) &= [\frac{1}{x-y} + \sum_{k,l\geq 1} \epsilon^{-\frac{1}{2}(k+l-1)} F_a(k,l,\epsilon) x^{k-1} y^{l-1}] dx^{\frac{1}{2}} dy^{\frac{1}{2}} \quad (46) \\
&= \sum_{k\geq 1} \epsilon^{-\frac{k}{2}+\frac{1}{4}} h_a(k,x) y^{k-1} dy^{\frac{1}{2}} \quad (47) \\
&= \sum_{l\geq 1} \epsilon^{-\frac{l}{2}+\frac{1}{4}} x^{l-1} \bar{h}_a(l,y) dx^{\frac{1}{2}}. \quad (48)
\end{aligned}
$$

We are now in a position to express $S^{(g_1+g_2)}(x,y)$ in terms of the lower genus data. From the sewing relation (30) we have $dz_a = -\epsilon \frac{dz_{\bar{a}}}{z_{\bar{a}}^2}$ so that

$$
dz_a^{\frac{1}{2}} = (-1)^{\bar{a}} \xi \epsilon^{\frac{1}{2}} \frac{dz_{\bar{a}}^{\frac{1}{2}}}{z_{\bar{a}}}, \quad (49)
$$

where $\xi \in \{\pm\sqrt{-1}\}$ determines the square root branch chosen. We then find

Proposition 3.4 *$S^{(g_1+g_2)}(x,y)$ is given by*

$$
S^{(g_1+g_2)}(x,y) = \begin{cases} S^{(g_a)}(x,y) + h_a(x) X_{\bar{a}\bar{a}} \bar{h}_a^T(y), & x,y \in \widehat{\Sigma}^{(g_a)}, \\ h_a(x) \left(\xi(-1)^{\bar{a}} I - X_{\bar{a}a}\right) \bar{h}_{\bar{a}}^T(y), & x \in \widehat{\Sigma}^{(g_a)},\ y \in \widehat{\Sigma}^{(g_{\bar{a}})}, \end{cases} \quad (50)
$$

where I denotes the infinite identity matrix and T the transpose.

Proof. Consider $x, y \in \widehat{\Sigma}^{(g_1)}$. Noting that $\mathcal{C}_1(z_1)$ may be deformed to $-\mathcal{C}_2(z_2)$ on $\mathcal{A}$ via (30) we find

$$
\begin{aligned}
& S^{(g_1+g_2)}(x,y) - S^{(g_1)}(x,y) = -\frac{1}{2\pi i} \oint\limits_{\mathcal{C}_1(z_1)} S^{(g_1)}(x,z_1) S^{(g_1+g_2)}(z_1,y) \\
= \; & -\sum_{k\geq 1} h_1(k,x) \frac{\epsilon^{-\frac{k}{2}+\frac{1}{4}}}{2\pi i} \oint\limits_{\mathcal{C}_1(z_1)} S^{(g_1+g_2)}(z_1,y) z_1^{k-1} dz_1^{\frac{1}{2}} \\
= \; & \xi \sum_{k\geq 1} h_1(k,x) \frac{\epsilon^{\frac{k}{2}-\frac{1}{4}}}{2\pi i} \oint\limits_{\mathcal{C}_2(z_2)} S^{(g_1+g_2)}(z_2,y) z_2^{-k} dz_2^{\frac{1}{2}} \\
= \; & \xi \sum_{k\geq 1} h_1(k,x) \frac{\epsilon^{\frac{k}{2}-\frac{1}{4}}}{(2\pi i)^2} \oint\limits_{\mathcal{C}_2(z_2)} \oint\limits_{\mathcal{C}_1(u_1)} S^{(g_1+g_2)}(z_2,u_1) S^{(g_1)}(u_1,y) z_2^{-k} dz_2^{\frac{1}{2}} \\
= \; & -\xi^2 \sum_{k,l\geq 1} h_1(k,x) \bar{h}_1(l,y) \frac{\epsilon^{\frac{1}{2}(k+l-1)}}{(2\pi i)^2} \oint\limits_{\mathcal{C}_2(z_2)} \oint\limits_{\mathcal{C}_2(u_2)} S^{(g_1+g_2)}(z_2,u_2) z_2^{-k} u_2^{-l} dz_2^{\frac{1}{2}} du_2^{\frac{1}{2}} \\
= \; & h_1(x) X_{22} \bar{h}_1^T(y),
\end{aligned}
$$

using (38), (47), (49), (48), (39) and (49) again, respectively. Thus we recover the first line of (50) for $a = b = 1$. A similar analysis holds for $a = b = 2$.

For $x \in \widehat{\Sigma}^{(g_1)}$, $y \in \widehat{\Sigma}^{(g_2)}$ we find that

$$S^{(g_1+g_2)}(x,y) = -\frac{1}{2\pi i} \oint\limits_{\mathcal{C}_1(z_1)} S^{(g_1)}(x,z_1) S^{(g_1+g_2)}(z_1,y)$$

$$= \xi \sum_{k\geq 1} \frac{\epsilon^{\frac{k}{2}-\frac{1}{4}}}{2\pi i} \oint\limits_{\mathcal{C}_2(z_2)} z_2^{-k} dz_2^{\frac{1}{2}} h_1(k,x)$$

$$\cdot \left(S^{(g_2)}(z_2,y) + \frac{1}{2\pi i} \oint\limits_{\mathcal{C}_2(u_2)} S^{(g_1+g_2)}(z_2,u_2) S^{(g_1)}(u_2,y) \right)$$

$$= \xi \sum_{k\geq 1} h_1(k,x) \bar{h}_2(k,y)$$

$$+\xi^2 \sum_{k,l\geq 1} h_1(k,x)\bar{h}_2(l,y) \frac{\epsilon^{\frac{1}{2}(k+l-1)}}{(2\pi i)^2} \oint\limits_{\mathcal{C}_2(z_2)} \oint\limits_{\mathcal{C}_1(u_1)} S^{(g_1+g_2)}(z_2,u_1) z_2^{-k} dz_2^{\frac{1}{2}} u_1^{-l} du_1^{\frac{1}{2}}$$

$$= h_1(x) \left(\xi I - X_{21}\right) \bar{h}_2^T(y).$$

A similar result holds for $x \in \widehat{\Sigma}^{(g_2)}$, $y \in \widehat{\Sigma}^{(g_1)}$. □

We next compute the explicit form of the moment matrix X_{ab} in terms of the moments F_a of $S^{(g_a)}(x,y)$. It is useful to introduce infinite block matrices

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \qquad F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix},$$

$$\Xi = \begin{pmatrix} 0 & \xi I \\ -\xi I & 0 \end{pmatrix}, \qquad Q = F\Xi = \begin{pmatrix} 0 & \xi F_1 \\ -\xi F_2 & 0 \end{pmatrix}. \tag{51}$$

Then one finds:

Proposition 3.5 *X is given by*

$$X = (I-Q)^{-1} F, \tag{52}$$

where $(I-Q)^{-1} = \sum_{n\geq 0} Q^n$ is convergent for $|\epsilon| < |r_1 r_2|$.

Proof. Using (38) we find $X_{11}(k,l) - F_1(k,l)$ is given by

$$-\frac{\epsilon^{\frac{1}{2}(k+l-1)}}{(2\pi i)^3} \oint\limits_{\mathcal{C}_1(x)} \oint\limits_{\mathcal{C}_1(z_1)} S^{(g_1)}(x,z_1) x^{-k} dx^{\frac{1}{2}} \oint\limits_{\mathcal{C}_1(y)} S^{(g_1+g_2)}(z_1,y) y^{-l} dy^{\frac{1}{2}}$$
$$= -\frac{\epsilon^{\frac{1}{2}(k+l-1)}}{(2\pi i)^3} \sum_{m\geq 0} \left(\epsilon^{-\frac{m}{2}+\frac{1}{4}} \oint\limits_{\mathcal{C}_1(x)} h_1(m,x) x^{-k} dx^{\frac{1}{2}} \right.$$
$$\left. \cdot \oint\limits_{\mathcal{C}_1(z_1)} \oint\limits_{\mathcal{C}_1(y)} S^{(g_1+g_2)}(z_1,y) z_1^{m-1} y^{-l} dy^{\frac{1}{2}} \right)$$
$$= \xi \sum_{m\geq 0} \left(\frac{\epsilon^{\frac{k}{2}-\frac{1}{4}}}{2\pi i} \oint\limits_{\mathcal{C}_1(x)} h_1(m,x) x^{-k} dx^{\frac{1}{2}} \right.$$
$$\left. \cdot \frac{\epsilon^{\frac{1}{2}(m+l-1)}}{(2\pi i)^2} \oint\limits_{\mathcal{C}_2(z_2)} \oint\limits_{\mathcal{C}_1(y)} S^{(g_1+g_2)}(z_2,y) z_2^{-m} y^{-l} dy^{\frac{1}{2}} \right)$$
$$= \xi \left(F_1 X_{21}\right)(k,l),$$

using (47) and (49). Similarly we find $X_{22} = F_2 - \xi F_2 X_{12}$ so that $X_{aa} = (F+QX)_{aa}$ using (51). A similar calculation of X_{12} and X_{21} leads to $X_{a\bar{a}} = (QX)_{a\bar{a}}$. These combine to give $(I-Q)X = F$ which implies (52) provided $(I-Q)^{-1} = \sum_{n\geq 0} Q^n$ converges. But (52) can be rewritten

$$X = \sum_{n\geq 1} Q^n \Xi. \tag{53}$$

By Theorem 3.2, $X_{ab}(k,l)$ has a convergent series expansion in $\epsilon^{\frac{1}{2}}$ for $|\epsilon| < r_1 r_2$. But $\sum_{n=1}^{N} Q^n = O(\epsilon^{\frac{1}{2}N})$ so that (53) holds to all orders in $\epsilon^{\frac{1}{2}}$. Hence $(I-Q)^{-1}$ converges for $|\epsilon| < r_1 r_2$ and the proposition holds. □

Propositions 3.4 and 3.5 imply

Theorem 3.6 *$S^{(g_1+g_2)}(x,y)$ is given by*

$$S^{(g_1+g_2)}(x,y) = \delta_{ab} S^{(g_a)}(x,y) + h_a(x) \left(\Xi (I-Q)^{-1}\right)_{ab} \bar{h}_b^T(y),$$

for $x \in \widehat{\Sigma}^{(g_a)}, y \in \widehat{\Sigma}^{(g_b)}$. *Equivalently,*

$$S^{(g_1+g_2)}(x,y) = \begin{cases} S^{(g_a)}(x,y) + h_a(x)\left(I - F_{\bar{a}}F_a\right)^{-1} F_{\bar{a}}\bar{h}_a^T(y), & x, y \in \widehat{\Sigma}^{(g_a)}, \\ \xi(-1)^{\bar{a}} h_a(x)\left(I - F_{\bar{a}}F_a\right)^{-1} \bar{h}_{\bar{a}}^T(y), & x \in \widehat{\Sigma}^{(g_a)},\ y \in \widehat{\Sigma}^{(g_{\bar{a}})}. \end{cases} \quad \square$$

Remark 3.7 *Note that* $S^{(g_1+g_2)}(x,y)$ *is even (odd) in* $\epsilon^{\frac{1}{2}}$ *for* $x, y \in \widehat{\Sigma}^{(g_a)}$ *(respectively, for* $x \in \widehat{\Sigma}^{(g_a)}$, $y \in \widehat{\Sigma}^{(g_{\bar{a}})}$*). Thus* $S^{(g_1+g_2)}(x,y)$ *is invariant under a Dehn twist* $\epsilon \to e^{2\pi i}\epsilon$ *with* $\xi \to -\xi$ *from (49).*

Similarly to ref. [MT1] we define the determinant of $I - Q$ as a formal power series in $\epsilon^{\frac{1}{2}}$ by

$$\log\det(I - Q) = \operatorname{Tr}\log(I - Q) = -\sum_{n \geq 1} \frac{1}{n}\operatorname{Tr}(Q^n).$$

Clearly $\operatorname{Tr}(Q^{2k}) = 2\operatorname{Tr}\left((F_1F_2)^k\right)$ for $k \geq 0$ whereas $\operatorname{Tr}(Q^n) = 0$ for n odd. Furthermore, from (45) the diagonal terms $(F_1F_2)^k$ have integral power series in ϵ. Thus it follows that

Lemma 3.8 $\det(I - Q) = \det(I - F_1F_2)$ *and is a formal power series in* ϵ.

The determinant has the following holomorphic properties:

Theorem 3.9 $\det(I - Q)$ *is non-vanishing and holomorphic in* ϵ *for* $|\epsilon| < r_1r_2$.

Proof. The proof follows a similar argument to Theorem 2 of ref. [MT1]. Let $S^{(g_1+g_2)}(z_1,z_2) = f(z_1,z_2,\epsilon)dz_1^{\frac{1}{2}}dz_2^{\frac{1}{2}}$ for $|z_a| \leq r_a$ where $f(z_1,z_2,\epsilon)$ is holomorphic in $\epsilon^{\frac{1}{2}}$ for $|\epsilon| \leq r$ for $r < r_1r_2$ from Theorem 3.2. Apply Cauchy's inequality to the coefficients of $f(z_1,z_2,\epsilon) = \sum_{n\geq 0} f_n(z_1,z_2)\epsilon^{\frac{n}{2}}$ to find

$$|f_n(z_1,z_2)| \leq Mr^{-\frac{n}{2}}, \tag{54}$$

for $M = \sup_{|z_a|\leq r_a, |\epsilon|\leq r} |f(z_1,z_2,\epsilon)|$. Consider

$$\mathcal{I} = \frac{1}{(2\pi i)^2}\oint_{\mathcal{C}_{r_1}(z_1)}\oint_{\mathcal{C}_{r_2}(z_2)} S^{(g_1+g_2)}(z_1,z_2)\left(1 - \frac{\epsilon}{z_1z_2}\right)^{-1} dz_1^{\frac{1}{2}}dz_2^{\frac{1}{2}}, \tag{55}$$

for $\mathcal{C}_{r_a}(z_a)$ the contour with $|z_a| = r_a$. Then using (54) we find

$$|\mathcal{I}| \leq M. \sum_{n\geq 0} \left(\frac{|\epsilon|}{r}\right)^{\frac{n}{2}} . \left|1 - \frac{|\epsilon|}{r_1 r_2}\right|^{-1} .r_1 r_2,$$

i.e. $\mathcal{I}$ is absolutely convergent and thus holomorphic in $\epsilon^{\frac{1}{2}}$ for $|\epsilon| < r < r_1 r_2$. Since $|z_1 z_2| = r_1 r_2$ we may alternatively expand in $\epsilon/z_1 z_2$ to obtain

$$\begin{aligned}
\mathcal{I} &= \sum_{k\geq 1} \epsilon^k \frac{1}{(2\pi i)^2} \oint_{\mathcal{C}_{r_1}(z_1)} \oint_{\mathcal{C}_{r_2}(z_2)} S^{(g_1+g_2)}(z_1, z_2) z_1^{-k} z_2^{-k} dz_1^{\frac{1}{2}} dz_2^{\frac{1}{2}} \\
&= \epsilon^{\frac{1}{2}} \mathrm{Tr} X_{12},
\end{aligned}$$

where $\mathrm{Tr}X_{12} = \sum_{k\geq 1} X_{12}(k, k)$. But (52) implies

$$\mathrm{Tr}X_{12} = \xi \sum_{n\geq 1} \mathrm{Tr}((F_1 F_2)^n),$$

which is absolutely convergent for $|\epsilon| < r_1 r_2$. Hence we find

$$\mathrm{Tr}\log(I - F_1 F_2) = -\sum_{n\geq 1} \frac{1}{n} \mathrm{Tr}((F_1 F_2)^n),$$

is also absolutely convergent for $|\epsilon| < r_1 r_2$. Thus $\det(I - Q) = \det(I - F_1 F_2)$ is non-vanishing and holomorphic for $|\epsilon| < r_1 r_2$. □

3.3 Sewing Two Tori

Consider the genus two surface formed by sewing two oriented tori $\Sigma_a^{(1)} = \mathbb{C}/\Lambda_a$ for $a = 1, 2$, and lattice $\Lambda_a = 2\pi i(\mathbb{Z}\tau_a \oplus \mathbb{Z})$ for $\tau_a \in \mathbb{H}_1$. This is discussed at length in [MT1]. For local coordinate $z_a \in \mathbb{C}/\Lambda_a$ consider the closed disk $|z_a| \leq r_a$ which is contained in $\Sigma_a^{(1)}$ provided $r_a < \frac{1}{2}D(q_a)$ where

$$D(q_a) = \min_{\lambda\in\Lambda_a, \lambda\neq 0} |\lambda|,$$

is the minimal lattice distance. From Subsection 3.1 we obtain a genus two Riemann surface $\Sigma^{(2)}$ parameterized by the domain

$$\mathcal{D}^\epsilon = \{(\tau_1, \tau_2, \epsilon) \in \mathbb{H}_1 \times \mathbb{H}_1 \times \mathbb{C} \mid |\epsilon| < \frac{1}{4} D(q_1) D(q_2)\}. \tag{56}$$

$\mathcal{D}^\epsilon$ is preserved under the action of $G \simeq (SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})) \rtimes \mathbb{Z}_2$, the direct product of the left and right torus modular groups, which are interchanged upon conjugation by an involution β as follows

$$\begin{aligned}
\gamma_1(\tau_1,\tau_2,\epsilon) &= (\frac{a_1\tau_1+b_1}{c_1\tau_1+d_1},\tau_2,\frac{\epsilon}{c_1\tau_1+d_1}),\\
\gamma_2(\tau_1,\tau_2,\epsilon) &= (\tau_1,\frac{a_2\tau_2+b_2}{c_2\tau_2+d_2},\frac{\epsilon}{c_2\tau_2+d_2}),\\
\beta(\tau_1,\tau_2,\epsilon) &= (\tau_2,\tau_1,\epsilon),
\end{aligned} \tag{57}$$

for $(\gamma_1,\gamma_2) \in SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ with $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$.

There is a natural injection $G \to Sp(4,\mathbb{Z})$ in which the two $SL(2,\mathbb{Z})$ subgroups are mapped to

$$\Gamma_1 = \left\{ \begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & 1 & 0 & 0 \\ c_1 & 0 & d_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}, \; \Gamma_2 = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & b_2 \\ 0 & 0 & 1 & 0 \\ 0 & c_2 & 0 & d_2 \end{bmatrix} \right\}, \tag{58}$$

and the involution is mapped to

$$\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{59}$$

G also has a natural action on $\mathbb{H}_2$ as given in (5) which is compatible with respect to $\Omega^{(2)}$ as a function of (τ_1,τ_2,ϵ) [MT1].

The Szegö kernel on the torus $\Sigma_a^{(1)}$ is given by

$$S^{(1)}\begin{bmatrix}\theta_a \\ \phi_a\end{bmatrix}(x,y|\tau_a) = P_1\begin{bmatrix}\theta_a \\ \phi_a\end{bmatrix}(x-y,\tau_a)dx^{\frac{1}{2}}dy^{\frac{1}{2}},$$

from (23). It is straightforward to compute the moment matrix F_a of (45). Using the Laurant expansion (28) we find

$$P_1\begin{bmatrix}\theta \\ \phi\end{bmatrix}(x-y,\tau) = \frac{1}{x-y} + \sum_{k,l\geq 1} C\begin{bmatrix}\theta \\ \phi\end{bmatrix}(k,l)x^{k-1}y^{l-1}, \tag{60}$$

where for $k, l \geq 1$ we define

$$C\begin{bmatrix}\theta\\ \phi\end{bmatrix}(k,l,\tau) = (-1)^l \binom{k+l-2}{k-1} E_{k+l-1}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(\tau), \tag{61}$$

for twisted Eisenstein series (29). Then it follows that

$$F_a\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(k,l,\tau_a,\epsilon) = \epsilon^{\frac{1}{2}(k+l-1)} C\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(k,l,\tau_a). \tag{62}$$

We also have the analytic expansion

$$P_1\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x-y,\tau) \quad = \quad \sum_{k\geq 0} P_k\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x,\tau)y^{k-1}, \tag{63}$$

for $P_k\begin{bmatrix}\theta\\ \phi\end{bmatrix}(z,\tau) = \frac{(-1)^{k-1}}{(k-1)!}\partial_z^{k-1} P_1\begin{bmatrix}\theta\\ \phi\end{bmatrix}(z,\tau)$. Then we find

$$h_a\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(k,x,\tau_a,\epsilon) = \epsilon^{\frac{k}{2}-\frac{1}{4}} P_k\begin{bmatrix}\theta_a\\ \phi_a\end{bmatrix}(x,\tau_a)dx^{\frac{1}{2}}. \tag{64}$$

Using these results we may therefore determine the explicit form for $S^{(2)}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}$ on $\mathcal{D}^\epsilon$ via Theorem 3.6.

One may also confirm that $S^{(2)}$ satisfies the modular invariance property of (20) under the group G generated by γ_i, β of (58) and (59) with

$$S^{(2)}\left(\gamma\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}\right)(\gamma x, \gamma y|\gamma(\tau_1,\tau_2,\epsilon)) = S^{(2)}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x,y|\tau_1,\tau_2,\epsilon), \tag{65}$$

where

$$\gamma_1\begin{bmatrix}\theta_1\\ \theta_2\\ \phi_1\\ \phi_2\end{bmatrix} = \begin{bmatrix}\theta_1^{a_1}\phi_1^{b_1}\\ \theta_2\\ \theta_1^{c_1}\phi_1^{d_1}\\ \phi_2\end{bmatrix}, \quad \gamma_2\begin{bmatrix}\theta_1\\ \theta_2\\ \phi_1\\ \phi_2\end{bmatrix} = \begin{bmatrix}\theta_1\\ \theta_2^{a_2}\phi_2^{b_2}\\ \phi_1\\ \theta_2^{c_2}\phi_2^{d_2}\end{bmatrix}, \quad \beta\begin{bmatrix}\theta_1\\ \theta_2\\ \phi_1\\ \phi_2\end{bmatrix} = \begin{bmatrix}\theta_2\\ \theta_1\\ \phi_2\\ \phi_1\end{bmatrix},$$

and

$$\gamma_a x \quad = \quad \begin{cases} \frac{x}{c_a\tau_a+d_a} & \text{for } x\in\widehat{\Sigma}_a^{(1)},\\ x & \text{for } x\in\widehat{\Sigma}_{\bar{a}}^{(1)},\end{cases}$$

and where for $x = 2\pi i\,(u+v\tau_a) \in \widehat{\Sigma}_a^{(1)}$ with $0 \leq u, v < 1$ we define $\beta x = 2\pi i\,(u+v\tau_{\bar{a}})$. Finally, we note that $\det(I-Q) = \det\left(I - F_1\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix} F_2\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}\right)$ is also G invariant.

4 The Szegö kernel on a Self-Sewn Riemann Surface

4.1 The ρ-Formalism Sewing Scheme

We now consider the construction of the Szegö kernel on a Riemann surface $\Sigma^{(g+1)}$ formed by self-sewing a handle to a Riemann surface $\Sigma^{(g)}$ of genus g. We begin by reviewing the Yamada formalism [Y] in this scheme which, following [MT1], we refer to as the ρ-formalism. Consider a Riemann surface $\Sigma^{(g)}$ of genus g and let z_1, z_2 be local coordinates in the neighborhood of two separated points p_1 and p_2. Consider two disks $|z_a| \leq r_a$, for $r_a > 0$ and $a = 1, 2$. Note that r_1, r_2 must be sufficiently small to ensure that the disks do not intersect. Introduce a complex parameter ρ where $|\rho| \leq r_1 r_2$ and excise the disks

$$\{z_a, |z_a| < |\rho| r_{\bar{a}}^{-1}\} \subset \Sigma^{(g)},$$

to form a twice-punctured surface

$$\widehat{\Sigma}^{(g)} = \Sigma^{(g)} \backslash \bigcup_{a=1,2} \{z_a, |z_a| < |\rho| r_{\bar{a}}^{-1}\}.$$

As before, we use the convention $\bar{1} = 2$, $\bar{2} = 1$. We define annular regions $\mathcal{A}_a \subset \widehat{\Sigma}^{(g)}$ with $\mathcal{A}_a = \{z_a, |\rho| r_{\bar{a}}^{-1} \leq |z_a| \leq r_a\}$ and identify them as a single region $\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{A}_2$ via the sewing relation

$$z_1 z_2 = \rho, \tag{66}$$

to form a compact Riemann surface $\Sigma^{(g+1)} = \widehat{\Sigma}^{(g)} \backslash \{\mathcal{A}_1 \cup \mathcal{A}_2\} \cup \mathcal{A}$ of genus $g+1$. The sewing relation (66) can be considered to be a parameterization of a cylinder connecting the punctured Riemann surface to itself.

In the ρ-formalism we define a standard basis of cycles $\{a_1, b_1, \ldots a_{g+1}, b_{g+1}\}$ on $\Sigma^{(g+1)}$ where the set $\{a_1, b_1, \ldots a_g, b_g\}$ is the original basis on $\Sigma^{(g)}$. Let $\mathcal{C}_a(z_a) \subset \mathcal{A}_a$ denote a closed anti-clockwise contour parameterized by z_a surrounding the puncture at $z_a = 0$. Clearly $\mathcal{C}_2(z_2) \sim -\mathcal{C}_1(z_1)$ on applying the sewing relation (66). We then define the cycle a_{g+1} to be $\mathcal{C}_2(z_2)$ and define the cycle b_{g+1} to be a path chosen in $\widehat{\Sigma}^{(g)}$ between identified points $z_1 = z_0$ and $z_2 = \rho/z_0$ on the sewn surface.

As in the ϵ-formalism, the normalized differential of the second kind $\omega^{(g+1)}$, the holomorphic 1-forms $\nu_i^{(g+1)}$ and the period matrix $\Omega^{(g+1)}$ can be computed in terms of data coming from $\Sigma^{(g)}$ [Y, MT1] to find

Theorem 4.1 *$\omega^{(g+1)}$, $\nu_i^{(g+1)}$ and $\Omega_{ij}^{(g+1)}$ for $(i,j) \neq (g+1, g+1)$ are holomorphic in ρ for $|\rho| < r_1 r_2$ with*

$$\omega^{(g+1)}(x,y) = \omega^{(g)}(x,y) + O(\rho), \tag{67}$$
$$\nu_i^{(g+1)}(x) = \nu_i^{(g)}(x) + O(\rho), \quad i = 1 \ldots g \tag{68}$$
$$\nu_{g+1}^{(g+1)}(x) = \omega_{p_2-p_1}^{(g)}(x) + O(\rho), \tag{69}$$
$$\Omega_{ij}^{(g+1)} = \Omega_{ij}^{(g)} + O(\rho), \quad i,j = 1 \ldots g \tag{70}$$
$$\Omega_{i,g+1}^{(g+1)} = \frac{1}{2\pi i}\int_{p_1}^{p_2} \nu_i^{(g)} + O(\rho), \quad i = 1 \ldots g, \tag{71}$$

for $x, y \in \widehat{\Sigma}^{(g)}$. $e^{2\pi i \Omega_{g+1,g+1}^{(g+1)}}$ is holomorphic in ρ for $|\rho| < r_1 r_2$ with

$$e^{2\pi i \Omega_{g+1,g+1}^{(g+1)}} = -\frac{\rho}{K_0^2}\left(1 + O(\rho)\right), \tag{72}$$

where $K_0 = K^{(g)}(z_1 = 0, z_2 = 0)$ for $E^{(g)}(z_1, z_2) = K^{(g)}(z_1, z_2) dz_1^{-\frac{1}{2}} dz_2^{-\frac{1}{2}}$ expressed in terms of the local coordinates z_1, z_2. □

4.2 Szegö Kernel in the ρ-Formalism

We now determine the Szegö kernel $S^{(g+1)}(x,y) = S^{(g+1)}\begin{bmatrix}\theta^{(g+1)} \\ \phi^{(g+1)}\end{bmatrix}(x,y)$ on the sewn Riemann surface $\Sigma^{(g+1)}$ in terms of genus g data together with the multiplier parameters associated with the handle cycles. The $S^{(g+1)}$ multipliers (17) on the cycles a_i, b_i for $i = 1, \ldots g$ are determined by the multipliers of $S^{(g)}$ with $\phi_i^{(g+1)} = \phi_i^{(g)}$ and $\theta_i^{(g+1)} = \theta_i^{(g)}$ i.e. $\alpha_i^{(g+1)} = \alpha_i^{(g)}$ and $\beta_i^{(g+1)} = \beta_i^{(g)}$. The remaining two multipliers associated with the cycles a_{g+1} and b_{g+1}

$$\phi_{g+1} = \phi_{g+1}^{(g+1)} = -e^{2\pi i \alpha_{g+1}^{(g+1)}}, \tag{73}$$
$$\theta_{g+1} = \theta_{g+1}^{(g+1)} = -e^{-2\pi i \beta_{g+1}^{(g+1)}}, \tag{74}$$

must be additionally specified so that

$$S^{(g+1)}(e^{2\pi i} x_a, y) = -\phi_{g+1}^{a-\bar{a}}\, S^{(g+1)}(x_a, y), \tag{75}$$
$$S^{(g+1)}(x_a, y) = -\theta_{g+1}^{a-\bar{a}}\, S^{(g+1)}(x_{\bar{a}}, y), \tag{76}$$

for $x_a \in \mathcal{A}_a$ and $x_{\bar{a}} \in \mathcal{A}_{\bar{a}}$.

We next consider the analogue of Theorem 3.2 concerning the holomorphicity of $S^{(g+1)}$ as a function of ρ. It is convenient to define $\kappa \in \left[-\frac{1}{2}, \frac{1}{2}\right)$ by $\phi_{g+1} = -e^{2\pi i\kappa}$ i.e. $\kappa = \alpha_{g+1}^{(g+1)} \mod 1$. We then find

Theorem 4.2 *$S^{(g+1)}$ is holomorphic in ρ for $|\rho| < r_1 r_2$ with*

$$S^{(g+1)}(x, y) = S_\kappa^{(g)}(x, y) + O(\rho), \tag{77}$$

for $x, y \in \widehat{\Sigma}^{(g)}$ where $S_\kappa^{(g)}(x, y)$ is defined as follows: For $\kappa \neq -\frac{1}{2}$

$$S_\kappa^{(g)}(x, y) = \frac{U(x, y)^\kappa \vartheta \begin{bmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{bmatrix} \left(\int_y^x \nu^{(g)} + \kappa z_{p_1,p_2} | \Omega^{(g)} \right)}{E^{(g)}(x, y) \vartheta \begin{bmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{bmatrix} \left(\kappa z_{p_1,p_2} | \Omega^{(g)} \right)}, \tag{78}$$

where

$$U(x, y) = \frac{E^{(g)}(x, p_2) E^{(g)}(y, p_1)}{E^{(g)}(x, p_1) E^{(g)}(y, p_2)}, \tag{79}$$

for prime form $E^{(g)}$ and where

$$z_{p_1,p_2} = \int_{p_1}^{p_2} \nu^{(g)}, \tag{80}$$

for holomorphic 1-forms $\nu^{(g)}$. For $\kappa = -\frac{1}{2}$ then $S_{-\frac{1}{2}}^{(g)}(x, y)$ is given by

$$\begin{aligned} &\Bigg(\tfrac{U(x,y)^{\frac{1}{2}}}{E^{(g)}(x,y)} \vartheta \begin{bmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{bmatrix} \left(\int_y^x \nu^{(g)} + \tfrac{1}{2} z_{p_1,p_2} | \Omega^{(g)} \right) \\ &- \theta_{g+1} \tfrac{U(x,y)^{-\frac{1}{2}}}{E^{(g)}(x,y)} \vartheta \begin{bmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{bmatrix} \left(\int_y^x \nu^{(g)} - \tfrac{1}{2} z_{p_1,p_2} | \Omega^{(g)} \right) \Bigg) . \\ &\left(\vartheta \begin{bmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{bmatrix} \left(\tfrac{1}{2} z_{p_1,p_2} | \Omega^{(g)} \right) - \theta_{g+1} \vartheta \begin{bmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{bmatrix} \left(-\tfrac{1}{2} z_{p_1,p_2} | \Omega^{(g)} \right) \right)^{-1} . \end{aligned} \tag{81}$$

Proof. We firstly note that from (15) it follows that

$$E^{(g+1)}(x, y) = E^{(g)}(x, y) + O(\rho). \tag{82}$$

From Theorem 4.1 we may expand the genus $g + 1$ theta series to leading order in ρ for $|\rho| < r_1 r_2$ as follows

$$\vartheta \begin{bmatrix} \alpha^{(g+1)} \\ \beta^{(g+1)} \end{bmatrix} \left(\int_y^x \nu^{(g+1)} | \Omega^{(g+1)} \right) = \sum_{m\in\mathbb{Z}^g} \sum_{n\in\mathbb{Z}} \left(-\frac{\rho}{K_0^2} \right)^{\frac{1}{2}(n+\alpha_{g+1}^{(g+1)})^2} .$$
$$\exp \Bigg(i\pi(m+\alpha^{(g)}).\Omega^{(g)}.(m+\alpha^{(g)}) + (m+\alpha^{(g)}).(\int_y^x \nu^{(g)} + 2\pi i \beta^{(g)}) +$$
$$(n+\alpha_{g+1}^{(g+1)}) \left[(m+\alpha^{(g)}). \int_{p_1}^{p_2} \nu^{(g)} + \int_y^x \omega_{p_2-p_1}^{(g)} + 2\pi i \beta_{g+1}^{(g+1)} \right] \Bigg) .$$
$$(1+O(\rho)),$$

Clearly $|n+\alpha_{g+1}^{(g+1)}| \geq |\kappa|$. For $\kappa \neq -\frac{1}{2}$ it follows that this lower bound is satisfied for one value of n so that

$$\vartheta \begin{bmatrix} \alpha^{(g+1)} \\ \beta^{(g+1)} \end{bmatrix} \left(\int_y^x \nu^{(g+1)} | \Omega^{(g+1)} \right) = \left(-\frac{\rho}{K_0^2} \right)^{\frac{1}{2}\kappa^2} (-\theta_{g+1})^{-\kappa} U(x,y)^{\kappa}$$
$$\sum_{m\in\mathbb{Z}^g} \exp \big(i\pi(m+\alpha^{(g)}).\Omega^{(g)}.(m+\alpha^{(g)})$$
$$+ (m+\alpha^{(g)}).(\int_y^x \nu^{(g)} + \kappa z_{p_1,p_2} + 2\pi i \beta^{(g)}) \Big) .(1+O(\rho)),$$

for z_{p_1,p_2} of (80) and where from (13)

$$\int_y^x \omega_{p_2-p_1}^{(g)} = \int_y^x \int_{p_1}^{p_2} \omega^{(g)}(\cdot,\cdot) = \log U(x,y),$$

for $U(x,y)$ of (79). Therefore

$$\vartheta \begin{bmatrix} \alpha^{(g+1)} \\ \beta^{(g+1)} \end{bmatrix} \left(\int_y^x \nu^{(g+1)} | \Omega^{(g+1)} \right) =$$
$$\left(-\frac{\rho}{K_0^2} \right)^{\frac{1}{2}\kappa^2} (-\theta_{g+1})^{-\kappa} U(x,y)^{\kappa} \vartheta \begin{bmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{bmatrix} \left(\int_y^x \nu^{(g)} + \kappa z_{p_1,p_2} | \Omega^{(g)} \right) (1+O(\rho)).$$

Since $U(x,x) = 1$ we find that for $\kappa \neq -\frac{1}{2}$

$$\frac{\vartheta \begin{bmatrix} \alpha^{(g+1)} \\ \beta^{(g+1)} \end{bmatrix} \left(\int_y^x \nu^{(g+1)} | \Omega^{(g+1)} \right)}{\vartheta \begin{bmatrix} \alpha^{(g+1)} \\ \beta^{(g+1)} \end{bmatrix} (0|\Omega^{(g+1)})} = U(x,y)^{\kappa} \frac{\vartheta \begin{bmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{bmatrix} \left(\int_y^x \nu^{(g)} + \kappa z_{p_1,p_2} | \Omega^{(g)} \right)}{\vartheta \begin{bmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{bmatrix} (\kappa z_{p_1,p_2} | \Omega^{(g)})} (1+O(\rho)),$$

is holomorphic in ρ for $|\rho| < r_1 r_2$. Combining this result with (82) we immediately find (78) using the definition of the Szegö kernel (16).

For $\kappa = -\frac{1}{2}$ the lower bound on $|n + \alpha^{(g+1)}_{g+1}| = |\kappa|$ is satisfied for two values of n so that

$$\vartheta \begin{bmatrix} \alpha^{(g+1)} \\ \beta^{(g+1)} \end{bmatrix} \left(\int_y^x \nu^{(g+1)} | \Omega^{(g+1)} \right) =$$
$$\left(-\frac{\rho}{K_0^2}\right)^{\frac{1}{8}} \left[(-\theta_{g+1})^{-\frac{1}{2}} U(x,y)^{\frac{1}{2}} \vartheta \begin{bmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{bmatrix} \left(\int_y^x \nu^{(g)} + \frac{1}{2} z_{p_1,p_2} | \Omega^{(g)} \right) \right.$$
$$\left. + (-\theta_{g+1})^{\frac{1}{2}} U(x,y)^{-\frac{1}{2}} \vartheta \begin{bmatrix} \alpha^{(g)} \\ \beta^{(g)} \end{bmatrix} \left(\int_y^x \nu^{(g)} - \frac{1}{2} z_{p_1,p_2} | \Omega^{(g)} \right) \right] (1 + O(\rho)).$$

which eventually leads to (81). □

We next note that, similarly to (37), $S^{(g)}_\kappa(x, z_a) S^{(g+1)}(z_a, y)$ is a meromorphic 1-form in z_a periodic on the $\Sigma^{(g)}$ cycles a_i, b_i for $i = 1 \ldots g$ with simple poles

$$\begin{aligned} S^{(g)}_\kappa(x, z_a) S^{(g+1)}(z_a, y) &\sim \frac{dz_a}{x - z_a} S^{(g+1)}(x, y) \text{ for } z_a \sim x, \\ S^{(g)}_\kappa(x, z_a) S^{(g+1)}(z_a, y) &\sim \frac{dz_a}{z_a - y} S^{(g)}_\kappa(x, y) \text{ for } z_a \sim y. \end{aligned} \quad (83)$$

Furthermore, $S^{(g)}_\kappa(x, z_a) S^{(g+1)}(z_a, y)$ is also periodic on the a_{g+1} cycle defined by $\mathcal{C}_2(z_2) \sim -\mathcal{C}_1(z_1)$. This follows from applying (75) to (77) so that

$$S^{(g)}_\kappa(x, e^{2\pi i} z_a) = e^{2\pi i \kappa(\bar{a} - a)} S^{(g)}_\kappa(x, z_a), \quad (84)$$

(or alternatively we may apply $U(x, e^{2\pi i} z_a)^\kappa = e^{2\pi i \kappa(\bar{a}-a)} U(x, z_a)^\kappa$). Similar properties hold for $S^{(g)}_\kappa(x, z_a) S^{(g+1)}(z_a, y)$. This leads to the following analogue of Proposition 3.3

Proposition 4.3 *The Szegö kernel on a genus $g+1$ Riemann surface in the ρ-formalism for $x, y \in \widehat{\Sigma}^{(g)}$ is given by*

$$\begin{aligned} S^{(g+1)}(x, y) &= S^{(g)}_\kappa(x, y) + \sum_{a=1,2} \frac{1}{2\pi i} \oint_{\mathcal{C}_a(z_a)} S^{(g)}_\kappa(x, z_a) S^{(g+1)}(z_a, y), \quad (85) \\ &= S^{(g)}_\kappa(x, y) - \sum_{a=1,2} \frac{1}{2\pi i} \oint_{\mathcal{C}_a(z_a)} S^{(g+1)}(x, z_a) S^{(g)}_\kappa(z_a, y). \quad (86) \end{aligned}$$

Proof. The proof follows along the same lines as Proposition 3.3. Let σ be a contour on $\widehat{\Sigma}^{(g)}$ surrounding $\mathcal{A}_a$ and the given points $x, y \in \widehat{\Sigma}^{(g)}$ as shown in Fig. 3.

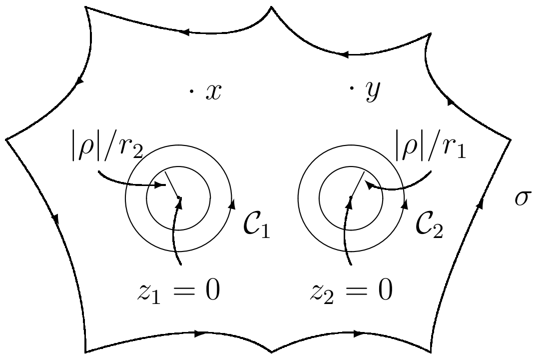

Fig. 3: Contour σ

Cauchy's Theorem and (83) imply

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\oint_{\sigma} S_{\kappa}^{(g)}(x,z)S^{(g+1)}(z,y) \\ &= -S^{(g+1)}(x,y) + S_{\kappa}^{(g)}(x,y) + \sum_{a=1,2}\frac{1}{2\pi i}\oint\limits_{\mathcal{C}_a(z_a)} S_{\kappa}^{(g)}(x,z_a)S^{(g+1)}(z_a,y), \end{aligned}$$

recalling that $S_{\kappa}^{(g)}(x,z_a)S^{(g+1)}(z_a,y)$ is periodic on $\mathcal{C}_a$. Thus (85) follows. A similar argument holds for (86). $\square$

We next define weighted moments of $S^{(g+1)}(x,y)$. Let

$$k_a = k + (-1)^{\bar{a}}\kappa,$$

for $a = 1, 2$ and integer $k \geq 1$ and define

$$\begin{aligned} Y_{ab}(k,l) &= Y_{ab}\begin{bmatrix}\theta^{(g+1)} \\ \phi^{(g+1)}\end{bmatrix}(k,l) \\ &= \frac{\rho^{\frac{1}{2}(k_a+l_b-1)}}{(2\pi i)^2}\oint_{\mathcal{C}_{\bar{a}}(x_{\bar{a}})}\oint_{\mathcal{C}_b(y_b)}(x_{\bar{a}})^{-k_a}(y_b)^{-l_b}S^{(g+1)}(x_{\bar{a}},y_b)dx_{\bar{a}}^{\frac{1}{2}}dy_b^{\frac{1}{2}}. \quad (87) \end{aligned}$$

We define $Y = (Y_{ab}(k,l))$ to be the infinite matrix indexed by a, k and b, l. From (19) we note the skew-symmetry property

$$Y_{ab}\begin{bmatrix}\theta^{(g+1)} \\ \phi^{(g+1)}\end{bmatrix}(k,l) = -Y_{\bar{b}\bar{a}}\begin{bmatrix}(\theta^{(g+1)})^{-1} \\ (\phi^{(g+1)})^{-1}\end{bmatrix}(k,l). \quad (88)$$

We also introduce moments for $S^{(g)}_\kappa(x,y)$

$$G_{ab}(k,l) = G_{ab}\begin{bmatrix}\theta^{(g)}\\ \phi^{(g)}\end{bmatrix}(\kappa;k,l)$$
$$= \frac{\rho^{\frac{1}{2}(k_a+l_b-1)}}{(2\pi i)^2}\oint_{\mathcal{C}_{\bar{a}}(x_{\bar{a}})}\oint_{\mathcal{C}_b(y_b)}(x_{\bar{a}})^{-k_a}(y_b)^{-l_b}S^{(g)}_\kappa(x_{\bar{a}},y_b)dx_{\bar{a}}^{\frac{1}{2}}dy_b^{\frac{1}{2}}, \quad (89)$$

with associated infinite matrix $G=(G_{ab}(k,l))$. This also satisfies a skew-symmetry property

$$G_{ab}\begin{bmatrix}\theta^{(g)}\\ \phi^{(g)}\end{bmatrix}(\kappa;k,l) = -G_{\bar{b}\bar{a}}\begin{bmatrix}(\theta^{(g)})^{-1}\\ (\phi^{(g)})^{-1}\end{bmatrix}(-\kappa;k,l). \quad (90)$$

Finally we define half-order differentials

$$h_a(k,x) = h_a\begin{bmatrix}\theta^{(g)}\\ \phi^{(g)}\end{bmatrix}(\kappa;k,x) = \frac{\rho^{\frac{1}{2}(k_a-\frac{1}{2})}}{2\pi i}\oint_{\mathcal{C}_a(y_a)}y_a^{-k_a}S^{(g)}_\kappa(x,y_a)dy_a^{\frac{1}{2}}, \quad (91)$$
$$\bar{h}_a(k,y) = \bar{h}_a\begin{bmatrix}\theta^{(g)}\\ \phi^{(g)}\end{bmatrix}(\kappa;k,y) = \frac{\rho^{\frac{1}{2}(k_a-\frac{1}{2})}}{2\pi i}\oint_{\mathcal{C}_{\bar{a}}(x_{\bar{a}})}x_{\bar{a}}^{-k_a}S^{(g)}_\kappa(x_{\bar{a}},y)dx_{\bar{a}}^{\frac{1}{2}}, \quad (92)$$

and let $h(x)=(h_a(k,x))$ and $\bar{h}(y)=(\bar{h}_a(k,y))$ denote the infinite row vectors indexed by a,k. These are related by skew-symmetry with

$$h_a\begin{bmatrix}\theta^{(g)}\\ \phi^{(g)}\end{bmatrix}(\kappa;k,x) = -\bar{h}_{\bar{a}}\begin{bmatrix}(\theta^{(g)})^{-1}\\ (\phi^{(g)})^{-1}\end{bmatrix}(-\kappa;k,x). \quad (93)$$

These moments can be inverted to obtain

$$S^{(g)}_\kappa(x,y_a) = \sum_{k\geq 1}\rho^{-\frac{k_a}{2}+\frac{1}{4}}h_a(k,x)y_a^{k_a-1}dy_a^{\frac{1}{2}} \quad (94)$$
$$S^{(g)}_\kappa(x_{\bar{a}},y) = \sum_{k\geq 1}\rho^{-\frac{k_a}{2}+\frac{1}{4}}x_{\bar{a}}^{k_a-1}\bar{h}_a(k,y)dx_{\bar{a}}^{\frac{1}{2}}. \quad (95)$$

From the sewing relation (66) we have

$$dz_a^{\frac{1}{2}} = (-1)^{\bar{a}}\xi\rho^{\frac{1}{2}}\frac{dz_{\bar{a}}^{\frac{1}{2}}}{z_{\bar{a}}}, \quad (96)$$

for $\xi\in\{\pm\sqrt{-1}\}$. We then find in a similar way to Proposition 3.4 that

Proposition 4.4 *For $x, y \in \widehat{\Sigma}^{(g)}$ then $S^{(g+1)}(x,y)$ is given by*

$$S^{(g+1)}(x,y) = S^{(g)}_{\kappa}(x,y) + \xi h(x) D^{\theta}\left(I + \xi Y D^{\theta}\right) \bar{h}(y)^T, \tag{97}$$

for infinite diagonal matrix $D^{\theta}(k,l) = \begin{bmatrix} \theta_{g+1}^{-1} & 0 \\ 0 & -\theta_{g+1} \end{bmatrix} \delta(k,l)$.

Proof. From (85) of Proposition 4.3 we find

$$S^{(g+1)}(x,y) - S^{(g)}_{\kappa}(x,y) = \sum_{a=1,2} \frac{1}{2\pi i} \oint_{\mathcal{C}_a(z_a)} S^{(g)}_{\kappa}(x,z_a) S^{(g+1)}(z_a,y)$$

$$= \sum_{a=1,2} \sum_{k \geq 1} h_a(k,x) \frac{\rho^{-\frac{k_a}{2}+\frac{1}{4}}}{2\pi i} \oint_{\mathcal{C}_a(z_a)} z_a^{k_a - 1} S^{(g+1)}(z_a,y) dz_a^{\frac{1}{2}}$$

$$= \xi \sum_{a,k} h_a(k,x) D^{\theta}_{aa}(k,k) \frac{\rho^{\frac{k_a}{2}-\frac{1}{4}}}{2\pi i} \oint_{\mathcal{C}_{\bar{a}}(z_{\bar{a}})} z_{\bar{a}}^{-k_a} S^{(g+1)}(z_{\bar{a}},y) dz_{\bar{a}}^{\frac{1}{2}},$$

using respectively (94), (76) and (96). Applying (86) it follows that $S^{(g+1)}(x,y) - S^{(g)}_{\kappa}(x,y)$ is given by

$$\xi \sum_{a,k} h_a(k,x) D^{\theta}_{aa}(k,k) \frac{\rho^{\frac{k_a}{2}-\frac{1}{4}}}{2\pi i} \oint_{\mathcal{C}_{\bar{a}}(z_{\bar{a}})} z_{\bar{a}}^{-k_a} S^{(g)}_{\kappa}(z_{\bar{a}},y) dz_{\bar{a}}^{\frac{1}{2}}$$

$$-\xi \sum_{a,k} h_a(k,x) D^{\theta}_{aa}(k,k) \frac{\rho^{\frac{k_a}{2}-\frac{1}{4}}}{(2\pi i)^2} \oint_{\mathcal{C}_{\bar{a}}(z_{\bar{a}})} z_{\bar{a}}^{-k_a}.$$

$$\sum_{b=1,2} \oint_{\mathcal{C}_{\bar{b}}(z_{\bar{b}})} S^{(g+1)}(z_{\bar{a}},w_{\bar{b}}) S^{(g)}_{\kappa}(w_{\bar{b}},y) dz_{\bar{a}}^{\frac{1}{2}}$$

$$= \xi h(x) D^{\theta} \bar{h}(y)^T - \xi \sum_{a,b,k,l} h_a(k,x) D^{\theta}_{aa}(k,k).$$

$$\frac{\rho^{\frac{1}{2}(k_a - l_b)}}{(2\pi i)^2} \oint_{\mathcal{C}_{\bar{a}}(z_{\bar{a}})} \oint_{\mathcal{C}_{\bar{b}}(w_{\bar{b}})} z_{\bar{a}}^{-k_a} w_{\bar{b}}^{l_b - 1} S^{(g+1)}(z_{\bar{a}},w_{\bar{b}}) dz_{\bar{a}}^{\frac{1}{2}} dw_{\bar{b}}^{\frac{1}{2}} \, \bar{h}_b(l,y)$$

$$= \xi h(x) D^{\theta} \bar{h}(y)^T - h(x) D^{\theta} Y D^{\theta} \bar{h}(y)^T$$

on applying (95), (76) and (96). Hence the result follows. □

We next compute the explicit form of Y in terms of the weighted moment matrix G for $S^{(g)}_{\kappa}$. In particular it is convenient to define $T = \xi G D^{\theta}$. From Proposition 4.4 it follows on taking moments that

$$Y = G + \xi G D^{\theta}(I + \xi Y D^{\theta})G.$$

This can be solved recursively to obtain $Y = \sum_{n\geq 0} T^n G$. Following a similar argument to that given for Proposition 3.5 we then find

Proposition 4.5 $Y = (I-T)^{-1}G$ *where* $(I-T)^{-1} = \sum_{n\geq 0} T^n$ *is convergent for* $|\rho| < r_1 r_2$. $\square$

This result together with Proposition 4.4 implies

Theorem 4.6 $S^{(g+1)}(x,y)$ *is given by*

$$S^{(g+1)}(x,y) = S^{(g)}_{\kappa}(x,y) + \xi h(x) D^{\theta}(I-T)^{-1}\bar{h}^T(y). \quad \square$$

Finally, similarly to Theorem 3.9 we may define $\det(I-T)$ and find

Theorem 4.7 $\det(I-T)$ *is non-vanishing and holomorphic in* ρ *for* $|\rho| < r_1 r_2$. $\square$

4.3 Self-Sewing a Sphere

We consider the example of sewing the Riemann sphere $\Sigma^{(0)} = \mathbb{C} \cup \{\infty\}$ to itself to form a torus. Choose local coordinates $z_2 = z \in \mathbb{C}$ in the neighborhood of the origin $p_2 = 0$, and $z_1 = 1/z'$ for z' in the neighborhood of the point at infinity $p_1 = \infty$. Identify the annular regions $|q| r_{\bar{a}}^{-1} \leq |z_a| \leq r_a$ for a complex sewing parameter $\rho = q$ obeying $|q| \leq r_1 r_2$, via the sewing relation

$$z = q z'.$$

These annular regions do not intersect on the sphere provided $r_1 r_2 < 1$ so that $|q| < 1$. Furthermore, the sewing relation implies $\log z = \log z' + 2\pi i \tau + 2\pi i k$ for integer k where $q = e^{2\pi i \tau}$. This is the standard parameterization of the torus with periods $2\pi i \tau$ and $2\pi i$ and modular parameter $\tau \in \mathbb{H}_1$.

We now show that the results of the previous subsection allow us to recover the genus one Szegö kernel (23) from the genus zero one. For $x, y \in \mathbb{C}$ the genus zero prime form and Szegö kernel are given by

$$E^{(0)}(x,y) = (x-y) dx^{-\frac{1}{2}} dy^{-\frac{1}{2}}, \quad (98)$$

$$S^{(0)}(x,y) = \frac{1}{x-y} dx^{\frac{1}{2}} dy^{\frac{1}{2}}. \quad (99)$$

Let $\theta = \theta^{(1)}$ and $\phi = \phi^{(1)} = -e^{2\pi i\kappa}$ denote the multipliers on the torus cycles. Then since $p_1 = \infty$ and $p_2 = 0$ we find $U(x,y) = x/y$ so that (78) and (81) imply

$$S_\kappa^{(0)}(x,y) = \frac{x^\kappa y^{-\kappa}}{x-y} dx^{\frac{1}{2}} dy^{\frac{1}{2}} + \frac{\theta}{1-\theta} \frac{dx^{\frac{1}{2}} dy^{\frac{1}{2}}}{x^{\frac{1}{2}} y^{\frac{1}{2}}} \delta_{\kappa,-\frac{1}{2}}. \qquad (100)$$

Computing moments one finds that for $\kappa \neq -\frac{1}{2}$ the half-differentials are

$$\begin{aligned} h_1(k,x) &= -\xi q^{\frac{1}{2}(k+\kappa-\frac{1}{2})} x^{k+\kappa-1} dx^{\frac{1}{2}}, \\ h_2(k,x) &= q^{\frac{1}{2}(k-\kappa-\frac{1}{2})} x^{-k+\kappa} dx^{\frac{1}{2}}, \\ \bar{h}_1(k,y) &= -q^{\frac{1}{2}(k+\kappa-\frac{1}{2})} y^{-k-\kappa} dy^{\frac{1}{2}}, \\ \bar{h}_2(k,y) &= \xi q^{\frac{1}{2}(k-\kappa-\frac{1}{2})} y^{k-\kappa-1} dy^{\frac{1}{2}}, \end{aligned}$$

for $x, y \in \widehat{\Sigma}^{(0)}$ and the moment matrix $T = \xi G D^\theta$ is diagonal with

$$T_{ab}(k,l) = \theta^{a-\bar{a}} q^{k_a - \frac{1}{2}} \delta_{ab} \delta(k,l).$$

Altogether we find from Theorem 4.6 that for $\kappa \neq -\frac{1}{2}$ and $x, y \in \widehat{\Sigma}^{(0)}$

$$\begin{aligned} S^{(1)}(x,y) &= S_\kappa^{(0)}(x,y) + \xi h(x) D^\theta (I-T)^{-1} \bar{h}^T(y) \\ &= \left[-\frac{\left(\frac{x}{y}\right)^{\kappa+\frac{1}{2}}}{1-\frac{x}{y}} - \sum_{k\geq 1} \frac{\theta^{-1} q^{k+\kappa-\frac{1}{2}}}{1-\theta^{-1} q^{k+\kappa-\frac{1}{2}}} \left(\frac{x}{y}\right)^{k+\kappa-\frac{1}{2}} \right. \\ &\qquad \left. + \sum_{k\geq 1} \frac{\theta q^{k-\kappa-\frac{1}{2}}}{1-\theta q^{k-\kappa-\frac{1}{2}}} \left(\frac{y}{x}\right)^{k-\kappa-\frac{1}{2}} \right] \frac{dx^{\frac{1}{2}} dy^{\frac{1}{2}}}{x^{\frac{1}{2}} y^{\frac{1}{2}}}. \end{aligned}$$

Denoting $q_u = e^u$ for any u we define X, Y by $x = q_X$, $y = q_Y$ and let $Z = X - Y$. We also define $\lambda = \kappa + \frac{1}{2}$ with $0 < \lambda < 1$ and obtain

$$\begin{aligned} S^{(1)}(X,Y) &= \left[-\frac{q_Z^\lambda}{1-q_Z} - \sum_{k\geq 0} \frac{\theta^{-1} q^{k+\lambda}}{1-\theta^{-1} q^{k+\kappa+\frac{1}{2}}} q_Z^{k+\lambda} \right. \\ &\qquad \left. + \sum_{k\leq -1} \frac{\theta q^{-k-\lambda}}{1-\theta q^{-k-\lambda}} q_Z^{k+\lambda} \right] dX^{\frac{1}{2}} dY^{\frac{1}{2}} \\ &= -\sum_{k\in\mathbb{Z}} \frac{q_Z^{k+\lambda}}{1-\theta^{-1} q^{k+\lambda}} dX^{\frac{1}{2}} dY^{\frac{1}{2}} = P_1(Z,q) dX^{\frac{1}{2}} dY^{\frac{1}{2}}, \end{aligned}$$

from (24). A similar result also holds for $\kappa = -\frac{1}{2}$ for $\theta \neq 0$ i.e. $(\theta, \phi) \neq (0, 0)$.

Lastly, we note that $(I - T)^{-1}$ is convergent for $|q| < 1$ and that furthermore

$$\det(I - T) = \prod_{k \geq 1} \left(1 - \theta^{-1} q^{k+\kappa-\frac{1}{2}}\right) \left(1 - \theta\, q^{k-\kappa-\frac{1}{2}}\right), \tag{101}$$

is holomorphic for $|q| < 1$ from Theorem 4.7. In vertex operator algebra theory, $\det(I - T)$ is related to the genus one partition function for a continuous orbfolding of a rank two free fermion system e.g. [MTZ]. Furthermore, the infinite product (101) is part of that arising in the Jacobi triple identity on applying the bosonic decomposition of this theory.

4.4 Self-Sewing a Torus

We next consider the example of self-sewing an oriented torus $\Sigma^{(1)} = \mathbb{C}/\Lambda$ for lattice $\Lambda = 2\pi i(\mathbb{Z}\tau \oplus \mathbb{Z})$ and $\tau \in \mathbb{H}_1$. This is discussed in detail in ref. [MT1]. Define annuli $\mathcal{A}_a, a = 1, 2$ centered at $p_1 = 0$ and $p_2 = w$ of $\Sigma^{(1)}$ with local coordinates $z_1 = z$ and $z_2 = z - w$ respectively. Take the outer radius of $\mathcal{A}_a$ to be $r_a < \frac{1}{2}D(q)$ for $D(q) = \min_{\lambda \in \Lambda, \lambda \neq 0} |\lambda|$ and the inner radius to be $|\rho|/r_{\bar{a}}$, with $|\rho| \leq r_1 r_2$. Identifying the annuli via (66) we obtain a compact genus two Riemann surface $\Sigma^{(2)}$ parameterized by

$$\mathcal{D}^\rho = \{(\tau, w, \rho) \in \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C} \mid |w - \lambda| > 2|\rho|^{\frac{1}{2}} > 0,\ \lambda \in \Lambda\}. \tag{102}$$

For $x, y \in \Sigma^{(1)}$ the genus one prime form and Szegö kernel with multipliers $\theta_1 = -e^{-2\pi i \beta_1}$ and $\phi_1 = -e^{2\pi i \alpha_1}$ are given by (22) and (23). Let $\theta_2 = -e^{-2\pi i \beta_2}$ and $\phi_2 = -e^{2\pi i \alpha_2} = -e^{2\pi i \kappa}$ denote the multipliers on a_2, b_2 cycles. Then, in this case

$$U(x, y) = \frac{\vartheta_1(x - w, \tau)\vartheta_1(y, \tau)}{\vartheta_1(x, \tau)\vartheta_1(y - w, \tau)},$$

and $z_{0,w} = \kappa w$ so that for $\kappa \neq -\frac{1}{2}$

$$S_\kappa^{(1)} \begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix} (x, y|\tau, w) = \left(\frac{\vartheta_1(x - w, \tau)\vartheta_1(y, \tau)}{\vartheta_1(x, \tau)\vartheta_1(y - w, \tau)}\right)^\kappa \frac{\vartheta \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} (x - y + \kappa w, \tau)}{\vartheta \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} (\kappa w, \tau)\, K(x - y, \tau)} dx^{\frac{1}{2}} dy^{\frac{1}{2}},$$

with a similar result for $\kappa = -\frac{1}{2}$. We take $\kappa \neq -\frac{1}{2}$ from now on.

It is straightforward to see that

$$S^{(1)}_{\kappa}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x,y|\tau,w)=S^{(1)}_{-\kappa}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x-w,y-w|\tau,-w). \qquad (103)$$

Computing moments and using (93) and (103) the half-differentials (91), (92) for $x\in\widehat{\Sigma}^{(1)}$ and $\kappa\neq -\frac{1}{2}$ are given by

$$\begin{aligned}
h_1\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(\kappa;k,x|\tau,w,\rho) &= \frac{\rho^{\frac{1}{2}(k+\kappa-\frac{1}{2})}}{2\pi i}\left(\frac{\vartheta_1(x-w,\tau)}{\vartheta_1(x,\tau)}\right)^{\kappa}\frac{dx^{\frac{1}{2}}}{\vartheta\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(\kappa w,\tau)}\\
&\quad \oint_{\mathcal{C}_1(y)} y^{-k-\kappa}\left(\frac{\vartheta_1(y,\tau)}{\vartheta_1(y-w,\tau)}\right)^{\kappa}\frac{\vartheta\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(x-y+\kappa w,\tau)}{K(x-y,\tau)}dy,\\
h_2\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(\kappa;k,x|\tau,w,\rho) &= h_1\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(-\kappa;k,x-w|\tau,-w,\rho),\\
\bar{h}_1\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(\kappa;k,x|\tau,w,\rho) &= -h_1\begin{bmatrix}\theta_1^{-1}\\ \phi_1^{-1}\end{bmatrix}(\kappa;k,x-w|\tau,-w,\rho),\\
\bar{h}_2\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(\kappa;k,x|\tau,w,\rho) &= -h_1\begin{bmatrix}\theta_1^{-1}\\ \phi_1^{-1}\end{bmatrix}(-\kappa;k,x|\tau,w,\rho).
\end{aligned} \qquad (104)$$

Similarly, using (103), the moment matrix (89) is given by

$$\begin{aligned}
G_{11}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(\kappa;k,l|\tau,w,\rho) &= \frac{\rho^{\kappa+\frac{1}{2}(k+l-1)}}{(2\pi i)^2}\oint_{\mathcal{C}_2(x_2)}\oint_{\mathcal{C}_1(y_1)} {x_2}^{-k-\kappa}{y_1}^{-l-\kappa}\\
&\quad S^{(1)}_{\kappa}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x_2,y_1|\tau,w)dx_2^{\frac{1}{2}}dy_1^{\frac{1}{2}},\\
&= G_{22}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(-\kappa;k,l|\tau,-w,\rho),\\
G_{21}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(\kappa;k,l|\tau,w,\rho) &= \frac{\rho^{\frac{1}{2}(k+l-1)}}{(2\pi i)^2}\oint_{\mathcal{C}_1(x_1)}\oint_{\mathcal{C}_1(y_1)} {x_1}^{-k+\kappa}{y_1}^{-l-\kappa}\\
&\quad S^{(1)}_{\kappa}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x_1,y_1|\tau,w)dx_1^{\frac{1}{2}}dy_1^{\frac{1}{2}},\\
&= G_{12}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(-\kappa;k,l|\tau,-w,\rho).
\end{aligned} \qquad (105)$$

The genus two Szego kernel is determined for $T = \xi G \begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix} D^{\theta_2}$ by (97)

$$S^{(2)} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} (x, y|\tau, w, \rho) =$$
$$S_\kappa^{(1)} \begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix} (x, y|\tau, w) + \xi h \begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix} (x) D^{\theta_2} (I - T)^{-1} \bar{h}^T \begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix} (y). \quad (106)$$

4.4.1 Modular Invariance

We now consider the modular invariance of (106) under the action of a particular subgroup $L \subset Sp(4, \mathbb{Z})$ and verify that (20) holds. We define L as follows [MT1]. Consider $\hat{H} \subset Sp(4, \mathbb{Z})$ with elements

$$\mu(a, b, c) = \begin{pmatrix} 1 & 0 & 0 & b \\ a & 1 & b & c \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (107)$$

$\hat{H}$ is generated by $A = \mu(1, 0, 0)$, $B = \mu(0, 1, 0)$ and $C = \mu(0, 0, 1)$ with relations $[A, B]C^{-2} = [A, C] = [B, C] = 1$. We also define $\Gamma_1 \subset Sp(4, \mathbb{Z})$ where $\Gamma_1 \cong SL(2, \mathbb{Z})$ with elements

$$\gamma_1 = \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & 1 & 0 & 0 \\ c_1 & 0 & d_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad a_1 d_1 - b_1 c_1 = 1. \quad (108)$$

Together these groups generate $L = \hat{H} \rtimes \Gamma_1 \subset Sp(4, \mathbb{Z})$ with center $Z(L) = \langle C \rangle$ where $J = L/Z(L) \cong \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ is the Jacobi group.

From Lemma 15 of [MT1] we find that L acts on the domain $\mathcal{D}^\rho$ of (102) as follows:

$$\mu(a, b, c).(\tau, w, \rho) = (\tau, w + 2\pi i a\tau + 2\pi i b, \rho), \quad (109)$$
$$\gamma_1.(\tau, w, \rho) = \left(\frac{a_1\tau + b_1}{c_1\tau + d_1}, \frac{w}{c_1\tau + d_1}, \frac{\rho}{(c_1\tau + d_1)^2} \right). \quad (110)$$

The kernel of the action is $Z(L)$, so that the effective action is that of J. However, this action is lifted to L when considering the covering space $\widehat{\mathcal{D}}^\rho$ of $\mathcal{D}^\rho$ for which $\Omega^{(g+1)}_{g+1,g+1}$ of (72) is single-valued (Theorems 10, 11 of [MT1]). In particular, one finds that C acts as

$$C.(\tau, w, \rho) = (\tau, w, e^{2\pi i}\rho), \quad (111)$$

which has a non-trivial action on $\widehat{\mathcal{D}}^\rho$.

Let us now consider the action of L on $S^{(2)}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x,y|\tau,w,\rho)$. This is partly determined by the action of J on $S^{(1)}_\kappa\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x,y|w,\tau)$. For $\gamma_1\in\Gamma_1$ it is clear from (25) that

$$S^{(1)}_\kappa\begin{bmatrix}{\theta_1}^a\phi_1^b\\ {\theta_1}^c\phi_1^d\end{bmatrix}(\gamma_1 x,\gamma_1 y|\gamma_1(\tau,w))=S^{(1)}_\kappa\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x,y|\tau,w).$$

$h_a\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(\kappa;k,x|\tau,w,\rho)$ and $G_{ab}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(\kappa;k,l|\tau,w,\rho)$ are similarly Γ_1 invariant so that $S^{(2)}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x,y|\tau,w,\rho)$ is Γ_1 invariant in a similar fashion to (65).

Next we consider the action of the generators A, B and C. We firstly note that (21) implies

$$A\begin{bmatrix}\theta_1\\ \theta_2\\ \phi_1\\ \phi_2\end{bmatrix}=\begin{bmatrix}\theta_1\\ -\theta_2\theta_1\\ -\phi_1\phi_2^{-1}\\ \phi_2\end{bmatrix},\quad B\begin{bmatrix}\theta_1\\ \theta_2\\ \phi_1\\ \phi_2\end{bmatrix}=\begin{bmatrix}-\theta_1\phi_2\\ -\theta_2\phi_1\\ \phi_1\\ \phi_2\end{bmatrix},\quad C\begin{bmatrix}\theta_1\\ \theta_2\\ \phi_1\\ \phi_2\end{bmatrix}=\begin{bmatrix}\theta_1\\ -\theta_2\phi_2\\ \phi_1\\ \phi_2\end{bmatrix}. \quad (112)$$

Using (8) and recalling that $\phi_2=-e^{2\pi i\kappa}$ we find

$$\begin{aligned} S^{(1)}_\kappa\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(x,y|\tau,w) &= S^{(1)}_\kappa\begin{bmatrix}\theta_1\\ -\phi_1\phi_2^{-1}\end{bmatrix}(x,y|\tau,w+2\pi i\tau) &(113)\\ &= S^{(1)}_\kappa\begin{bmatrix}-\theta_1\phi_2\\ \phi_1\end{bmatrix}(x,y|\tau,w+2\pi i), &(114)\end{aligned}$$

where the multipliers comply with those of (112) for A and B respectively. Define infinite diagonal matrices

$$E^\alpha(k,l)=\begin{bmatrix}1&0\\0&-\alpha\end{bmatrix}\delta(k,l),\quad F^\alpha(k,l)=\begin{bmatrix}-\alpha^{-1}&0\\0&1\end{bmatrix}\delta(k,l), \quad (115)$$

for $\alpha \in U(1)$. Then (104), (113) and (114) imply

$$
\begin{aligned}
h\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(\kappa;x|\tau,w,\rho) &= h\begin{bmatrix}\theta_1\\ -\phi_1\phi_2^{-1}\end{bmatrix}(\kappa;x|\tau,w+2\pi i\tau,\rho)E^{\theta_1}\\
&= h\begin{bmatrix}-\theta_1\phi_2\\ \phi_1\end{bmatrix}(\kappa;x|\tau,w+2\pi i,\rho)E^{\phi_1}\\
&= e^{-i\pi\kappa}h\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(\kappa;x|\tau,w,e^{2\pi i}\rho)E^{\phi_2},\\
\bar{h}\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(\kappa;x|\tau,w,\rho) &= h\begin{bmatrix}\theta_1\\ -\phi_1\phi_2^{-1}\end{bmatrix}(\kappa;x|\tau,w+2\pi i\tau,\rho)F^{\theta_1}\\
&= h\begin{bmatrix}-\theta_1\phi_2\\ \phi_1\end{bmatrix}(\kappa;x|\tau,w+2\pi i,\rho)F^{\phi_1}\\
&= e^{i\pi\kappa}h\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(\kappa;x|\tau,w,e^{2\pi i}\rho)F^{\phi_2}.
\end{aligned}
$$

Similarly, from (105) we find

$$
\begin{aligned}
G\begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}(\kappa|\tau,w,\rho) &= F^{\theta_1}G\begin{bmatrix}\theta_1\\ -\phi_1\phi_2^{-1}\end{bmatrix}(\kappa|\tau,w+2\pi i\tau,\rho)E^{\theta_1}\\
&= F^{\phi_1}G\begin{bmatrix}-\theta_1\phi_2\\ \phi_1\end{bmatrix}(\kappa|\tau,w+2\pi i,\rho)E^{\phi_1}\\
&= F^{\phi_2}G\begin{bmatrix}-\theta_1\\ \phi_1\end{bmatrix}(\kappa|\tau,w,e^{2\pi i}\rho)E^{\phi_2}.
\end{aligned}
$$

Noting that $E^{\alpha}D^{\theta_2}F^{\alpha} = D^{-\alpha\theta_2}$ for $\alpha = \theta_1, \phi_1$ and ϕ_2 we may then easily confirm that $S^{(2)}\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right](x,y|\tau,w,\rho)$ is invariant under the generators A, B and C respectively. Therefore $S^{(2)}\left[\begin{smallmatrix}\theta^{(2)}\\ \phi^{(2)}\end{smallmatrix}\right](x,y|\tau,w,\rho)$ is modular invariant under L. Furthermore, since $\det(E^{\alpha}F^{\alpha}) = 1$ it follows that $\det(I-T)$ is also L invariant.

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